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ZERO TEMPERATURE LIMITS FOR QUOTIENTS OF POTENTIALS ON COUNTABLE MARKOV SHIFTS

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Introduction

The present work is framed in the context of ergodic theory. Especifically, in the study of zero temperature limits for potentials defined over Markov shifts with countable alphabet.

Zero temperature limits theorems constitute, in turn, an intersection point for two theories of great relevance in ergodic theory. Namely, ergodic optimization and thermodynamic formalism.

According to [**JMU2**], for a dynamical system (X,T) and $f : X \to \mathbb{R}$ a continuous function, we refer as ergodic optimization to the circle of problems relating the search of orbits which maximize (or minimize) the time averages of f and T-invariant probabilities which optimize the spatial means of f. In particular, a T-invariant probability measure μ is called *f-maximizing* if it verifies

$$\int f d\mu = \max\left\{\int f d\nu : \nu \text{ is a } T\text{-invariant probability measure}\right\}.$$

On the other hand, thermodyamic formalism is concerned with the existence, uniqueness and ergodic properties of Borel $T-$invariant probability measures that maximize the *Free energy*,

$$\sup\left\{h(\mu) + \int f\, d\mu : \mu \text{ is a } T-\text{invariant probability measure}\right\}, \tag{1}$$

where $h(\mu)$ is the Kolmogorov-Sinai entropy. Measures attaining the supremum in (1) are called *equilibrium measures* for f. It was shown during the 1970s that if the space X is compact, the system T has some strong form of hyperbolicity and the function f is regular enough, then equilibrium measures do exist, they are unique and also satisfy the Gibbs property (see [**VO**, Section 12]).

It turns out that these two theories are related. Consider the one-parameter family of functions tf, with $t \in \mathbb{R}$. Assume that for every $t > 0$ there exists an equilibrium measure μ_t for tf. In statistical mechanics the parameter t is interpreted as the inverse of the temperature, thus as t converges to infinity we may say that the *temperature* tends to zero. If the space X is compact and the map T continuous then the space of invariant probability measures, endowed with the weak-* topology, is compact. Therefore, $(\mu_t)_t$ has an accumulation point μ as t tends to infinity. We say that μ is a *zero temperature limit* of the equilibrium measures. As t grows the relative importance of the entropy in (1) diminishes and it has been proven (see for instance, [**J**, Thm. 4.1]) that μ is actually a maximizing measure for f.

The literature features diverse extensions of these results and related problems to the non-compact spaces corresponding to Markov shifts with a countable alphabet, which will be denoted as (Σ_A, σ) (see [**FV, JMU, Mo**]). Moreover, these results even admit adaptations for almost-additive sequences of potentials [**IY2**].

In these cases, the sole existence of accumulation points for the family of equilibrium measures μ_t as $t \to \infty$ is not trivial, since the compactness hypothesis is no longer available.

The main purpose of this work is to develop zero temperature limits theorems for quotients of potentials over a countable Markov shift. This means that we extend the scope of the zero temperature limits to a to a framework which considers the relation between two potentials f, g in a countable Markov shift by means of their quotient. That is, given two potentials $f, g : \Sigma_A \to \mathbb{R}$, we intend to describe, if they exist, the σ-invariant probability measures μ for which

$$\frac{\int f d\mu}{\int g d\mu} = \sup \left\{ \frac{\int f d\nu}{\int g d\nu} : \nu \text{ is } \sigma\text{-invariant}, \int g d\nu \neq 0, \int g d\nu \neq \infty \right\} \tag{2}$$

as an accumulation point of the equilibrium measures of a family of potentials related to f and g.

Posing the problem in this manner entails several obstacles. To be more precise with these concepts, let us consider a topologically mixing countable Markov shift with the BIP property (Σ_A, σ) and two continuous, positive potentials $f, g : \Sigma_A \to \mathbb{R}$. In the first place, it is necessary to redefine the framework in order to the related concepts to allow the simultaneous description of the behaviour of two potentials. For instance, we have to redefine what is meant by a maximizing measure when studying quotients and, instead of considering potentials of the form tf to make $t \to \infty$, we now have to make an appropriate choice of a family of potentials relating f, g to describe what is meant by a zero temperature limit involving two potentials. This choice is not obvious since the equilibrium measures of the potentials from this family must have an accumulation point which in turn is expected to be a maximizing measure.

The difficulties of this adaptation not only come from the fact of considering quotients, but also from the fact that the CMS are not compact spaces. This means that the space of $\sigma-$invariant probability measures is not compact either in the weak-* topology. In general, a sequence of measures in Σ_A does not necessarily have accumulation points and since the space is not compact, we now allow potentials to be unbounded, meaning that the convergence of sequences of measures in the weak-* topology does not imply the convergence of the associated integrals with respect to a potential. Additionally, since the studied shift spaces are defined over a countable alphabet, the invariant measures on Σ_A may have infinite entropy, therefore the pressure of the involved potentials is not necessarily finite.

To provide a general overview of the results from this work, let us first describe the implied concepts.

Let f, g be positive potentials in Σ_A. If $\mathcal{M}_\sigma(-g)$ denotes the set of σ-invariant probability measures ν such that $\int g d\nu < \infty$, we say that a measure $\mu \in \mathcal{M}_\sigma(-g)$ which attains the supremum from (10) is (f, g)-*maximizing*. On the other hand, in order to determine what is meant by a zero temperature limit for quotients of potentials we will be interested in families of potentials of the form $tf - sg$ with $t, s \in \mathbb{R}^+$. To fix the appropriate dependence between the parameters t and s we prove that under certain assumptions, for every sufficiently big value of t there exists a unique real number $O(t)$ such that the free energy of $tf - O(t)g$ is zero. We call the map $t \to O(t)$ *zero-pressure map* and prove that there exists a real number $t^* \in \mathbb{R}$ such that it is real analytic in (t^*, ∞). We also prove that $O(t)$ has a lower bound denoted by a number s_∞ which in turn has the property of making

(s_∞, ∞) the maximal interval such that the map $t \mapsto P(-tg)$ is real-analytic. Our main result (see Theorem 5) establishes that the zero temperature accumulation points of the equilibrium measures for $tf - O(t)g$ are $(f, g)-$maximizing measures. In order for our argument to work we require the function g to dominate f, see equation (12), and that Gibbs-equilibrium measures for $tf - O(t)g$ do exist (see condition (9)). This discussion is developed in Chapter 2 and an application of how the theorem can be applied to describe zero temperature limits of suspension flows is also exhibited. The results from this Chapter were submitted for publication and they appear in [**P**].

Additionally, in [**IY2**], the theorems about zero temperature limits were extended in a different direction. Instead of considering a potential $f : \Sigma_A \to \mathbb{R}$, a sequence $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ of potentials is considered. This class of sequences satisfy a property called *almost additivity* and it consists of the existence of a constant $C_{\mathcal{F}}$ such that given $n, m \in \mathbb{N}$ we have, for every $x \in \Sigma_A$

$$e^{-c} f_n(x) f_m \circ \sigma^n(x) \leq f_{n+m} \leq f_n(x) f_m \circ \sigma^n(x) e^c.$$

Thermodynamic formalism for these sequences was introduced by Barreira and Mummert [**Ba, Mu**] for compact dynamical systems and later on it was generalized for almost-additive sequences over a CMS [**IY1**]. These sequences arise naturally when studying products of matrices and they generalize the classic additive case since the Birkhoff sums of every potential $f : \Sigma_A \to \mathbb{R}$ form an almost-additive sequence whose thermodynamic formalism replicates the one from f. Results from [**IY2**] provide a zero temperature limits theorem for almost-additive sequences of potentials was developed. If these notions can be extended from the additive to the almost additive framework, it makes sense to study if the development of zero-temperature limits for quotients of potentials makes sense in this new context. Existing literature shows several analogies between the behaviors of additive and almost-additive potentials. The main challenge of developing these results consists of translating the elements from additive thermodynamic formalism in terms that are suitable for the almost-additive context, that is, involving sequences of functions instead of a single potential, and establish these results in a way that recovers the original additive setting. This topic is addressed in Chapter 3 and we obtain a slightly weaker result. The zero-pressure map in this frame is not proven to be real-analytic, but it is shown that it is differentiable everywhere for big enough values of t, except by at most a countable set. The conclusion remains true provided that the accumulation points of the equilibrium measures as the temperature drops to zero are reached by subsequences $\{t_k\}_{k \in \mathbb{N}}$ such that $O(t_k)$ is differentiable for every $k \in \mathbb{N}$ (see Theorem 9). An example of these result applied to describe the invariant measures that maximize the ratio between expected values of the top Lyapunov exponents for linear cocycles is also provided in this section.

CHAPTER 1

Preliminaries

The concepts and results contained in this chapter constitute the foundations of our future discussion. There are three main topics mentioned: In the first place, we approach countable Markov shifts, which form the family of dynamical systems over which we set all the relevant results of this work. The other two topics are thermodynamic formalism and ergodic optimization. Both of these are related to ergodic theory and therefore we also remark the relevance of some important classes of Borel probability measures over Countable Markov shifts.

1. Countable Markov shifts

Let us start by setting up the dynamical systems over which we develop our work. First, we fix a countable set $\mathcal{A}$ that will be called an *alphabet*, without loss of generality we will consider $\mathcal{A} = \mathbb{N}$, the set consisting of the positive integers and we will refer to the elements of $\mathcal{A}$ as *symbols*.

Definition 1. *Let $A : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$ be a function. Suppose that for every $j \in \mathbb{N}$ there exist $i, k \in \mathbb{N}$ such that $A(i, j) = A(j, k) = 1$. Let*

$$\Sigma_A = \{(i_1, i_2, \cdots, i_n, \cdots) \in \mathbb{N}^{\mathbb{N}} : A(i_j, i_{j+1}) = 1, \text{ for every } j \in \mathbb{N}\}.$$

We define the shift map *$\sigma : \Sigma_A \to \Sigma_A$ by*

$$\sigma(i_1, i_2, \cdots) = (i_2, i_3, \cdots).$$

The dynamical system (Σ_A, σ) is said to be a Countable Markov Shift (CMS) *and A is called its* Transition Matrix.

Remark 1. When A is the function $A(i, j) \equiv 1$ for every $i, j \in \mathbb{N}$, (Σ_A, σ) is called the *full shift* over $\mathbb{N}$.

The elements of Σ_A can be interpreted as the states of a system that changes over discrete time. The symbols i_n can be thought of as the state of the system at the instant n. We can also interpret A as an indicator function of the allowed changes of state in the system and σ as the map which represents the time passing to the next instant.

Let us now fix an arbitrary number $\theta \in (0, 1)$. If $x = (x_n)_{n\in\mathbb{N}}, y = (y_n)_{n\in\mathbb{N}} \in \Sigma_A$, the map defined as

$$d_\theta(x, y) = \theta^{\min\{n\in\mathbb{N}: x_n \neq y_n\}},$$

with the convention $d_\theta(x, x) = 0$ is a metric over the space Σ_A. The following definition characterizes the balls that generate the topology for (Σ_A, d_θ).

Definition 2. *Let $n \in \mathbb{N}$. A set of the form*

$$[i_1, \cdots, i_n] := \{x = (x_1, x_2, ...) \in \Sigma_A : x_1 = i_1, \cdots, x_n = i_n\}.$$

with $i_1, i_2, \cdots, i_n \in \mathbb{N}$ *is called a* cylinder set *of length* n.

Remark 2. If $x = (x_1, x_2, \cdots) \in \Sigma_A$, we denote $C_n(x) = [x_1, x_2, \cdots, x_n]$, for any $n \in \mathbb{N}$. This is the subset of Σ_A whose elements agree with x in the first n coordinates.

Remark 3. If $\delta \in [\theta^{n+1}, \theta^n)$, the open ball $B(x, \delta)$ in (Σ_A, d_θ) agrees with $C_n(x)$. Thus, a function $f : \Sigma_A \to \Sigma_A$ is continuous if $f^{-1}(C_n(x))$ can be written as a (possibly empty) union of cylinder sets for every $x \in \Sigma_A$ and every $n \in \mathbb{N}$.

We now observe that for any $x \in \Sigma_A$ and $n \in \mathbb{N}$ we have

$$\begin{aligned}
\sigma^{-1}(C_n(x)) &= \{y \in \Sigma_A : \sigma(y) \in C_n(x)\} \\
&= \{y \in \Sigma_A : (y_2, y_3, \cdots) \in C_n(x)\} \\
&= \{y \in \Sigma_A : y_2 = x_1, y_3 = x_2, \cdots, y_{n+1} = x_n\} \\
&= \bigcup_{i \in \mathbb{N}} \{y \in \Sigma_A : y_1 = i, y_2 = x_1, y_3 = x_2, \cdots y_{n+1} = x_n\} \\
&= \bigcup_{i \in \mathbb{N}} [i, x_1, x_2, \cdots, x_n].
\end{aligned}$$

This shows that σ is a continuous function.

From now on, we assume that the system (Σ_A, σ) is topologically mixing, that is, for every $i, j \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have $\sigma^{-n}([i]) \cap [j] \neq \varnothing$.

We stress that Countable Markov shifts differ from subshifts of finite type in their topological features. If the alphabet $\mathcal{A}$ is a finite set, then Σ_A is a compact set. This is not necessarily the case when the alphabet is a countable infinite subset (w.l.o.g. $\mathcal{A} = \mathbb{N}$). For instance, given any sequence $\{n_k\}_k \in \mathbb{N}$ such that $[n_k] \neq \varnothing$ for every $k \in \mathbb{N}$, we can define a sequence $(x_k)_{k \in \mathbb{N}}$ in Σ_A such that $x_k \in [n_k]$ for every k. No sequence obtained from this construction has a convergent subsequence.

2. The space of σ-invariant probability measures in a CMS

Let $\mathcal{M}_\sigma$ denote the space of $\sigma-$invariant Borel probability measures in Σ_A. This space is endowed with a topology called the *weak-** (read as "weak star") topology. The convergence in the weak-* topology is characterized by the Portmanteau Theorem (see, for instance, [**D**]).

Theorem 1. *Let* $\{\mu_n\}_{n=1}^\infty$ *be a sequence in* $\mathcal{M}_\sigma$*. Given* $\mu \in \mathcal{M}_\sigma$*, the following statements are equivalent:*

(a) *For every bounded continuous function* f*,* $\int f d\mu_n \to \int f d\mu$ *when* $n \to \infty$*.*
(b) *For every upper semi-continuous, bounded from above function* f*, we have* $\limsup_{n\to\infty} \int f d\mu_n \leq \int f d\mu$*.*
(c) *For every lower semi-continuous function* f*, bounded from below, we have* $\liminf_{n\to\infty} \int f d\mu_n \geq \int f d\mu$*.*

If any of the conditions from Theorem 1 holds for a sequence $\{\mu_n\}_{n=1}^\infty$ and a measure $\mu \in \mathcal{M}_\sigma$ we say that μ_n *converges to* μ *(*$\mu_n \to \mu$*) in the weak-* topology.*

Let $\mathcal{N} \subseteq \mathcal{M}_\sigma$ be a family of σ-invariant Borel probability measures in Σ_A. A measure μ is called an *accumulation point* in the weak-* topology for $\mathcal{N}$ if there exists a sequence $\{\mu_n\}_{n=1}^\infty$ in $\mathcal{N}$ such that $\mu_n \to \mu$ in the weak-* topology.

REMARK 4. Since no compactness hypotheses have been set, the existence of accumulation points for an arbitrary family $\mathcal{N} \subseteq \mathcal{M}_\sigma$ is not granted. In general, $\mathcal{M}_\sigma$ endowed with the weak-* topology is not a compact space. For instance, consider a sequence of measures μ_n in $\mathcal{M}_\sigma$ such that $\mathrm{supp}(\mu_n) \subseteq \bigcup_{j \geq n}[j]$ for every $n \in \mathbb{N}$. Now, for every $m \in \mathbb{N}$, define $\chi_{[m]} : \Sigma_A \to \mathbb{R}$ as the locally constant function given by

$$\chi_{[m]}(x) = \begin{cases} 1, & x \in [m] \\ 0, & x \notin [m] \end{cases} .$$

If there is an accumulation point μ of μ_n, there is also a subsequence μ_{n_k} such that μ_{n_k} converges to μ in the weak-* topology when $k \to \infty$. For every $m \in \mathbb{N}$ it follows that for sufficiently big $k \in \mathbb{N}$ we have that $\chi_{[m]}(x) = 0$, for every $x \in \mathrm{supp}(\mu_{n_k})$. Therefore,

$$\mu([m]) = \int \chi_{[m]} d\mu = \lim_{k \to \infty} \int \chi_{[m]} d\mu_{n_k} = 0.$$

Since $\sum_{m \in \mathbb{N}} \mu([m])$ must be 1, such measure μ cannot exist.

The following definition describes a property that ensures the existence of accumulation points.

DEFINITION 3. *A family $\mathcal{N} \subseteq \mathcal{M}_\sigma$ is called* tight *if for every $\varepsilon > 0$ there exists a compact subset K of Σ_A such that $\mu(\Sigma_A \setminus K) < \varepsilon$ for every $\mu \in \mathcal{N}$.*

The link between this definition and the existence of accumulation points is given by Prohorov's Theorem(see, for instance, [**VO**]).

PROPOSITION 1. *A family $\mathcal{F} \subseteq \mathcal{M}_\sigma$ is tight if and only if every sequence in $\mathcal{F}$ admits a subsequence that converges in $\mathcal{M}_\sigma$ with the weak-* topology.*

3. Thermodynamic formalism over countable Markov Shifts

We now develop the main results related to thermodynamic formalism over countable Markov shifts. Our first step towards this direction is to establish the family of functions that will be studied to this end. If A is a transition matrix, any continuous function $f : \Sigma_A \to \mathbb{R}$ will be called a *potential* in Σ_A.

DEFINITION 4. *Let f be a potential in Σ_A. The n-*th variation of f*, denoted by $V_n(f)$, is defined as*

$$V_n(f) := \sup\{|f(x) - f(y)| : x, y \in \Sigma_A, C_n(x) = C_n(y)\}.$$

Moreover, we say that f has summable variations *provided*

$$\sum_{n=1}^{\infty} V_n(f) < \infty.$$

REMARK 5. We define Locally Hölder potentials in terms of their variations. Indeed, $f : \Sigma_A \to \mathbb{R}$ is said to be a *locally Hölder* potential if and only if there exist $C > 0$,$\beta \in (0, 1)$ such that for any $n \geq 1$ we have $V_n(f) < C\beta^n$. As a straightforward consequence, we have that every locally Hölder potential in Σ_A has summable variations.

REMARK 6. Let $K \subseteq \Sigma_A$. We denote by $\chi_K : \Sigma_A \to \mathbb{R}$ the characteristic function

$$\chi_K(x) := \begin{cases} 1, & x \in K \\ 0, & x \notin K \end{cases} .$$

The following definition extends the notion of pressure from compact dynamical systems to countable Markov shifts. It was introduced by Sarig [**S**] based on the previous work of Gurevich [**G1, G2**].

DEFINITION 5. *Let $f : \Sigma_A \to \mathbb{R}$ be a potential with summable variations. The* (Gurevich) pressure *of f, denoted $P_\sigma(f)$ is defined as*

$$P_\sigma(f) = \lim_{n\to\infty} \frac{1}{n} \sum_{\sigma^n(x)=x} \exp\left(\sum_{i=0}^{n-1} f(\sigma^i x)\right) \chi_{[i_0]}(x), \tag{3}$$

where $i_0 \in \mathbb{N}$ is arbitrary.

It can be shown [**S**, Thm.1] that the expression in the right side of (3) is well defined and does not depend on the choice of i_0.

REMARK 7. A potential $f : \Sigma_A \to \mathbb{R}$ satisfies $P(f) < \infty$ if and only if the condition

$$\sum_{i\in\mathbb{N}} \exp(\sup f|_{[i]}) < \infty \tag{4}$$

is held [**MU**, Prop. 2.1.9].

We can approximate the pressure of a potential by considering increasing sequences of invariant compact subsets of Σ_A. The following is called the approximation property [**S**, Thm. 2].

THEOREM 2. *Let $f : \Sigma_A \to \mathbb{R}$ be a potential with summable variations and let $\mathcal{K}$ denote the family consisting of all the non-empty σ-invariant compact subsets of Σ_A. For every $K \in \mathcal{K}$ denote by $P_K(f)$ the pressure of $f|_K$. Then*

$$P(f) = \sup_{K\in\mathcal{K}} P_K(f).$$

There also exists a version of the variational principle for Countable Markov shifts [**MU**], which provides a simpler characterization of the pressure of a potential. We denote by $\mathcal{M}_\sigma$ the set of $\sigma-$invariant probability measures on Σ_A and for every $\nu \in \mathcal{M}_\sigma$ we denote by $h(\nu)$ its entropy.

THEOREM 3. *Let f be a potential in Σ_A with summable variations. Then*

$$P(f) = \sup\left\{h(\nu) + \int f d\nu : \nu \in \mathcal{M}_\sigma \text{ and } -\int f d\nu < \infty\right\}. \tag{5}$$

REMARK 8. For any potential $f : \Sigma_A \to \mathbb{R}$, we can denote by $\mathcal{M}_\sigma(f)$ the set of measures $\nu \in \mathcal{M}_\sigma$ such that $\int f d\nu > -\infty$. This way, the notation from equation (5) can be simplified to

$$P(f) = \sup_{\nu\in\mathcal{M}_\sigma(f)} h(\nu) + \int f d\nu. \tag{6}$$

If the supremum in (5) is attained by some measure $\mu \in \mathcal{M}_\sigma$, we say that μ is an *equilibrium measure* for the potential f. i.e. μ is an equilibrium measure for f if and only if

$$P(f) = h(\mu) + \int f d\mu.$$

REMARK 9. According to [**PU**, Thm. 4.1], under the hypotheses of Theorem 2, we have, for every $t \in \mathbb{R}$, and K a compact $\sigma-$invariant subset of Σ_A

$$\frac{d}{dt}P(tf|_K)\bigg|_{t=t_0} = \int f_K d\mu_{t_0},$$

where μ_{t_0} is an equilibrium measure for the potential $t_0 f|_K$ in the standard compact setting.

Gibbs Measures for countable Markov Shifts are defined in a similar way to the finite alphabet framework.

DEFINITION 6. *Let Σ_A be a countable Markov shift and $f : \Sigma_A \to \mathbb{R}$ be a potential in Σ_A. A measure μ is called a* Gibbs measure *for f when there exist constants $K_1, K_2 > 0$ such that for any $n \in \mathbb{N}$ and $x \in \Sigma_A$, μ satisfies the inequalities*

$$K_1 < \frac{\mu(C_n(x))}{\exp\left(-nP(f) + \sum_{i=0}^{n-1} f \circ \sigma^i(x)\right)} < K_2. \tag{7}$$

REMARK 10. In particular, if we pick $n = 1$ this yields, for any $i \in \mathbb{N}$ and for every $x \in [i]$:

$$\mu[i] \leq K_2 \exp\left(f(x) - P(f)\right). \tag{8}$$

REMARK 11. The constant K_2 can be chosen as $\exp(4V(f))$, where $V(f) := \sum_{n=1}^{\infty} V_n(f)$ (see Definition 4)

Existence of Gibbs measures depends upon the following combinatorial property of the countable Markov shift:

DEFINITION 7. *Let Σ_A be a countable Markov Shift over $\mathbb{N}$. We say that Σ_A satisfies the* big images and preimages property (BIP property) *if there exists a finite subset $S = \{b_1, \cdots, b_n\} \subset \mathbb{N}$ such that for every symbol $a \in \mathbb{N}$ there exist $b_i, b_j \in S$ satisfying $A(b_i, a)A(a, b_j) = 1$.*

If (Σ_A, σ) is topologically mixing, the following condition also holds (See, for instance [**ILY**, Lemma 2.1]).

DEFINITION 8. *Let (Σ_A, σ) be a countable Markov shift. The transition matrix A is said to be* finitely primitive *if there exist $k \in \mathbb{N}$ and a finite family $W \subset \mathbb{N}^k$ such that for every $a, b \in \mathbb{N}$, the cylinder $[a, w_1, \cdots, w_k, b]$ is non-empty.*

The following proposition [**IJ**, Thm 2.3]] summarizes some of the generalizations developed in [**S, S2, MU**] of the classic results.

PROPOSITION 2. *Let (Σ_A, σ) be a topologically mixing countable Markov shift satisfying the BIP property. Let $g : \Sigma_A \to \mathbb{R}$ be a locally Hölder, positive potential such that $P(-sg) < \infty$ for some $s \in \mathbb{R}^+$. Then, there exists $s_\infty \geq 0$ such that:*

(a) *$P(-sg) = \infty$ for every $s < s_\infty$.*
(b) *The map $t \mapsto P(-sg)$ is real-analytic in $(s_\infty, +\infty)$.*
(c) *for every $s \in (s_\infty, +\infty)$, the potential $-sg$ has a unique equilibrium measure.*
(d) *The potential $-sg$ has a unique Gibbs measure for every $s \in (s_\infty, +\infty)$.*

In the context of Proposition 2, Gibbs measures μ_G and equilibrium measures μ_{eq} for a potential tf agree whenever $\int f d\mu_G > -\infty$. The following is an equivalent condition that doesn't depend explicitly on the Gibbs measure μ_G and also implies that Gibbs measures are equilibrium measures as well.

PROPOSITION 3. [**MU**, Lemma 2.2.8, Theo 2.2.9] *Let (Σ,σ) be a CMS satisfying the BIP property. Let g be a locally Hölder potential with summable variations. Then, the condition*

$$\sum_{n=1}^{\infty}\inf(-g|_{[i]})\exp(\inf g|_{[i]})<\infty \tag{9}$$

implies that the only Gibbs measure of g is an equilibrium measure as well.

4. Maximizing measures and zero-temperature limits

Ergodic optimization deals with problems related with points whose orbits maximize the time average and probability measures that maximize the spatial average of their potential. Since we are concerned with the study of the accumulation points of the equilibrium measures of potentials, we will only focus on the latter case. So, the measures of interest are the σ-invariant measures that maximize the spatial average of a potential $f:\Sigma_A\to\mathbb{R}$. This topic has been broadly studied in compact spaces and we refer the reader to [**Bo, J**] for a general overview. However, since we are dealing with a non-compact setting, we will adopt the approach from [**JMU**].

DEFINITION 9. *Let (Σ_A,σ) be a countable Markov shift satisfying the BIP property. Let f be a potential in Σ_A. A $\sigma-$invariant measure μ is called an f-*maximizing measure *if*

$$\int f d\mu=\sup_{\nu\in\mathcal{M}_\sigma}\int f d\nu.$$

Under our combinatorial assumptions, it has been proven by Bissacot and Garibaldi that when the potential f is bounded from above, maximizing measures do exist and they are supported over $\sigma-$invariant subshifts.

PROPOSITION 4. [**BG**, Thm.1] *Let (Σ_A,σ) be a Topologically mixing countable Markov Shift satisfying the BIP property. Let f be a bounded from above Locally Hölder potential on Σ_A satisfying $\lim\limits_{i\to\infty}\sup_{[i]}f=-\infty$. Then, f has a maximizing measure μ. Moreover, there exists a σ-invariant compact subset $\Omega\subseteq\Sigma_A$ such that $supp(\mu)\subseteq\Omega$.*

The following result from [**JMU**] relates maximizing measures of a potential f in Σ_A with the equilibrium measures of the potentials tf, for $t\in\mathbb{R}$.

PROPOSITION 5. [**JMU**, Thm.1] *Let Σ_A be a countable Markov shift satisfying the BIP property. Let f be a locally Hölder potential satisfying (9) and let $t^*\in\mathbb{R}$ such that $t\mapsto P(tf)$ is real-analytic in (t^*,∞). Then, the family $(m_{tf})_{t>t^*}$ of equilibrium measures for tf has a weak-* accumulation point $m\in\mathcal{M}_\sigma$ as $t\to\infty$. Moreover, m is an f-maximizing measure and verifies $\lim\limits_{t\to\infty}\int f dm_t=\int f dm$.*

Proposition states the existence of accumulation points for the measures m_{tf} as $t\to\infty$. The existence of a limit for these measures is not granted, even for potentials in compact settings. Indeed, it was proven in [**CH**] that there exists Lipchitz potentials f where the limits fail to exist

PROPOSITION 6. [**CH**, Thm 1.1] *There exist subshifts $X\{0,1\}^{\mathbb{N}}$ such that the potential $f(y)=-d(y,X)$, the sequence m_{tf} of equilibrium states of tf does not converge in the weak-* topology as $t\to\infty$.*

A stronger statement was shown by Coronel Rivera-Letelier for shifts with finite alphabets. They proved that there exists a Lipschitz potential f_0 and complementary open subsets U^+, U^- such that we can find arbitrarily small perturbations f of f_0 satisfying that every sequence $\{t_k\}_{k=1}^{\infty}$ with $t_k \to \infty$ as $k \to \infty$ the sequence of equilibrium states for the potentials $t_k f$ accumulates simultaneously on a measure supported in U^+ and a measure supported in U^- (See [**CR**, Thm. B]). Notice that since there exist two different accumulation points with disjoint supports, the limit cannot exist. However, if we narrow down the scope to locally constant functions, we indeed have the convergence of the equilibrium measures.

DEFINITION 10. *Let Σ_A be a Markov subshift over a finite alphabet. A potential $f : \Sigma_A \to \mathbb{R}$ is called* locally constant *if there exists $n \in \mathbb{N}$ such that for every cylinder set $C := [i_1, \cdots i_n]$, the restriction $f|_C$ is a constant.*

PROPOSITION 7. [**Br**, Thm.2.1] *Let f be a locally constant potential over a Markov subshift of finite type and for every $t \in \mathbb{R}$ denote by m_{tf} the equilibrium measure of tf. Then, there exists an invariant measure μ such that $m = \lim_{t\to\infty} m_{tf}$ in the weak-* topology.*

CHAPTER 2

A zero-temperature limit result for quotients of potentials

Let (Σ_A, σ) be a topologically mixing countable Markov shift that satisfies the BIP property. Proposition 4 presents a link between two important theories: On the first place, it involves thermodynamic formalism since it considers the family of equilibrium measures of a family of potentials $(tf)_{t\geq 1}$, whilst on the other hand it deals with ergodic optimization since it characterizes the accumulation points of those equilibrium measures as a maximizing measure for the potential f. Our main goal in this chapter is to develop a generalization of this result to a framework that considers the relation between two potentials at the same time by means of their quotient. That is, given two potentials $f, g : \Sigma_A \to \mathbb{R}$, we intend to describe, if they exist, the σ-invariant probability measures μ for which

$$\frac{\int f d\mu}{\int g d\mu} = \sup_{\nu \in \mathcal{M}_\sigma(-g)} \frac{\int f d\nu}{\int g d\nu} \tag{10}$$

as an accumulation point of the equilibrium measures of a family of potentials related to f and g. This generalization allows, for instance, the study of zero temperature limits for suspension flows since the invariant measures for a suspension space are strongly related to quotients between an auxiliary function and their roof function (see section 3.1 for details). The first challenge in order to achieve our goal consists of making clear what is meant by a zero-temperature limit for two potentials. That is, we have to set a proper family of potentials of the form $(tf - sg)_{t,s}$ and then describe such family with a single parameter by setting a dependence between the parameters s and t. This family must be chosen in a way that their equilibrium measures μ_t have an accumulation point (in the weak-* topology) which, at the same time, achieves the supremum from (10). Notice as well that since Σ_A is not compact, neither is the space of $\sigma-$invariant measures $\mathcal{M}_\sigma$. Therefore, the existence of the accumulation points for a family of equilibrium measures is not granted a priori. At the light of these facts, we first introduce a function $O(t)$, named *zero-pressure map*, which sets the dependence between t and s and therefore establishing the relevance of the potentials of the form $(tf - O(t)g)_{t\geq 1}$. Section 1 introduces the zero-pressure map and its properties. In section 2 the main result is presented and proved. It is first proven that the family of equilibrium measures μ_t for the potentials $tf - O(t)g$ has an accumulation point when $t \to \infty$ in the weak-* topology. Then, it is shown that accumulation points for this family of measures achieve the maximum from (10). We finish this chapter by presenting two applications of this result in section 3. We first show how zero-temperature limits for suspension flows can be described as a zero-temperature limit for a quotient. We also exhibit an example where the main result is applied to maximize the ratio between the integrals of two functions related to the continuous fraction expansion of an irrational number.

1. Zero-pressure map and its properties

The first step into extending proposition 4 to describe zero temperature limits of equilibrium measures of certain potentials as maximizing measures for their quotient consists of introducing the meaning of these concepts when they involve a second potential. The following definition formalizes what is meant by a maximizing measure in this case.

Definition 11. *Let Σ_A be a topollogically mixing countable Markov shift with the BIP property. Let f, g be positive, Locally Hölder potentials in Σ_A satisfying* (12),(13) *and such that g is bounded away from zero. A measure $\mu \in \mathcal{M}_\sigma$ is called an (f,g)*-maximizing measure *if*

$$\frac{\int f d\mu}{\int g d\mu} = \max_{\nu \in \mathcal{M}_\sigma(-g)} \frac{\int f d\nu}{\int g d\nu}. \tag{11}$$

In Proposition 4, maximizing measures for a potential f were described as zero-temperature limits for potentials tf, when $t \in \mathbb{R}$. In order to describe maximizing measures for quotients as zero-temperature limits, we need to define a family of potentials that relates two potentials f and g in a way that thermodynamic formalism properties remain valid. If we consider potentials of the form $tf + sg$ with $t, s \in \mathbb{R}$, we need to find a dependence $s = O(t)$ in order to obtain a family of the form $tf + O(t)g$ whose equilibrium measures have accumulation points when $t \to \infty$ in the weak-* topology which in turn are $(f,g)-$maximizing. In this section we define such function $O(t)$, and prove some properties to develop a zero-temperature limits theorem for quotients later on.

Such family of potentials cannot be described without developing results that ensure that it exists and it is well defined. So, let us set some conditions on the potentials.

Let $f, g : \Sigma_A \to \mathbb{R}$ be positive, locally Hölder potentials in Σ_A such that g is bounded away from zero. Assume as well that f, g satisfy

$$\lim_{n \to \infty} \frac{\sup\{f(x) : x \in [n]\}}{\inf\{g(x) : x \in [n]\}} = 0. \tag{12}$$

We also assume g to be such that the potential $-sg$ has finite pressure for some $s \in \mathbb{R}$. According to Proposition 2, this means that there exists s_∞ such that

$$P(-sg) = \begin{cases} \text{finite}, & s > s_\infty \\ \infty, & s < s_\infty \end{cases} \tag{13}$$

Definition 12. *Let (Σ_A, σ) be a topologically mixing countable Markov shift with the BIP property. Let f, g be two potentials in Σ_A such that g is bounded away from zero and* (12),(13) *hold. The* zero-pressure map *for f and g is defined as the map $O : \mathbb{R}^+ \to \mathbb{R}^+$ given by*

$$O(t) := \inf\{s \in \mathbb{R} : P(tf - sg) \leq 0\}. \tag{14}$$

The following result establishes, under our assumptions, the well-definiteness (see Lemma 1) and regularity properties for the zero-pressure map. Its proof also implies some useful properties of $O(t)$ as Corollaries.

Theorem 4. *Let (Σ_A, σ) be a countable Markov shift with the BIP property. Let $f, g : \Sigma_A \to \mathbb{R}$ be positive, locally Hölder functions such that* (12), (13) *hold and g is bounded away from zero. Then, zero-pressure map, $O(t)$ for f and g is finite and real-analytic in $\mathbb{R}^+$.*

The proof of Theorem 4 is decomposed in several lemmas.

Lemma 1. *For every $t \in \mathbb{R}^+$, we have $O(t) < \infty$.*

Proof. Let $\varepsilon > 0$. According to (12), there exists $N \in \mathbb{N}$ such that for every $n \geq N$,

$$\frac{\sup\{f(x) : x \in [n]\}}{\inf\{g(x) : x \in [n]\}} < \varepsilon.$$

Besides, since f and g are locally Hölder functions, their restrictions to cylinders are bounded. Therefore, for every $i \in \{1, \cdots, N-1\}$ there exists K_i such that for $x \in [i]$ we have $\frac{f(x)}{g(x)} < K_i$. Letting $K = \max\{K_1, \cdots, K_{N-1}, \varepsilon\}$ we get

$$f(x) < Kg(x), \text{ for every } x \in \Sigma.$$

Therefore, for any $\nu \in \mathcal{M}_\sigma$ we have $\int f(x)d\nu < K \int g(x)d\nu$. Hence,

$$\frac{\int f(x)d\nu}{\int g(x)d\nu} < K. \tag{15}$$

Thus, given $s \in \mathbb{R}$ we have

$$\begin{aligned} P(tf - sg) &= \sup_{\nu \in \mathcal{M}_\sigma(tf-sg)} h(\nu) + t\int f d\nu - s\int g d\nu \\ &< \sup_{\nu \in \mathcal{M}_\sigma(tf-sg)} h(\nu) + t\left(K\int g d\nu\right) - s\int g d\nu \\ &= \sup_{\nu \in \mathcal{M}_\sigma(tf-sg)} h(\nu) + (Kt - s)\int g d\nu. \end{aligned}$$

Notice that since $f, g > 0$ we have $\mathcal{M}_\sigma(tf - sg) = \mathcal{M}_\sigma(-g) = \mathcal{M}_\sigma((s - Kt)g)$. Thus, from the previous inequality, we obtain $P(tf - sg) < P((Kt - s)g)$. So, setting $s > tK + s_\infty$, we obtain $s - tK < s_\infty$ and therefore, $P(tf - sg) < \infty$. Now, recall that g is bounded away from zero. Therefore there exists $C > 0$ such that $g(x) > C$ for every $x \in \Sigma_A$. Thus, by picking $\hat{s}, s$ such that $tK + s_\infty < \hat{s} < s$ and denoting by $\mu_{t,s}$ the equilibrium measure for $tf - sg$, it follows

$$\begin{aligned} P(tf - sg) &= h(\mu_{t,s}) + t\int f d\mu_{t,s} - s\int g d\mu_{t,s} \\ &= h(\mu_{t,s}) + \int (tf - \hat{s}g) d\mu_{t,s} - (s - \hat{s})\int g d\mu_{t,s}. \end{aligned}$$

Since $\hat{s} > tK + s_\infty$, it follows $P(tf - \hat{s}g) < \infty$ and $\mathcal{M}_\sigma(tf - \hat{s}g) = \mathcal{M}_\sigma(tf - sg) = \mathcal{M}_\sigma(-g)$, we have $\mu_{t,s} \in \mathcal{M}_\sigma(-tf - \hat{s}g)$. This implies

$$h(\mu_{t,s}) + \int tf - \hat{s}g d\mu_{t,s} \leq P(tf - \hat{s}g),$$

and therefore we have

$$P(tf - sg) \leq P(tf - \hat{s}g) - (s - \hat{s}) \int g d\mu_{t,s}$$
$$< P(tf - \hat{s}g) - (s - \hat{s}) \int C d\mu_{t,s} = P(tf - \hat{s}g) - (s - \hat{s})C.$$

From this inequality, it follows that $P(tf-sg) \to \infty$ when $s \to \infty$. In particular, $P(tf - sg) \leq 0$ for sufficiently big s. Hence $O(t) = \inf\{s \in \mathbb{R} : P(tf - sg) \leq 0\} < \infty$. $\square$

The following technical Lemma is inspired from [**IRV**, Thm. 3.7]. Its proof is analogous and constitutes an adaptation of that result to the frame of quotients of potentials.

LEMMA 2. *There exists a sequence of $\sigma-$invariant probability measures $\{\nu_n\}_{n\in\mathbb{N}}$ such that $\lim\limits_{n\to\infty} \int g d\nu_n = \infty$ and $\liminf\limits_{n\to\infty} \frac{h(\nu_n)}{\int g d\nu_n} \geq s_\infty$.*

PROOF. Let $n \in \mathbb{N}$. By (13) we have that

$$P\left(-\left(s_\infty - \frac{1}{n}\right) g\right) = \infty \text{ and } P\left(-\left(s_\infty + \frac{1}{n}\right) g\right) < \infty.$$

Therefore

$$P\left(-\left(s_\infty - \frac{1}{n}\right) g\right) - P\left(-\left(s_\infty + \frac{1}{n}\right) g\right) = \infty.$$

Theorem 2 implies that there is a $\sigma-$invariant compact set K_n verifying

$$P_{K_n}\left(-\left(s_\infty - \frac{1}{n}\right) g\right) > 2n + P\left(-\left(s_\infty + \frac{1}{n}\right) g\right)$$
$$> 2n + P_{K_n}\left(-\left(s_\infty + \frac{1}{n}\right) g\right)$$

Then,

$$P_{K_n}\left(-\left(s_\infty - \frac{1}{n}\right) g\right) - P_{K_n}\left(-\left(s_\infty + \frac{1}{n}\right) g\right) > 2n,$$

i.e.,

$$n^2 < \frac{P_{K_n}\left(-\left(s_\infty - \frac{1}{n}\right) g\right) - P_{K_n}\left(-\left(s_\infty + \frac{1}{n}\right) g\right)}{\frac{2}{n}}.$$

Since K_n is compact, the map $t \mapsto P_{K_n}(tg)$ is differentiable and its derivative is $\int_{K_n} g d\nu_n$, where ν_n is an equilibrium measure for $tg|_{K_n}$ (see Remark 9). Therefore the Mean Value Theorem implies the existence of $t_n \in [s_\infty - \frac{1}{n}, s_\infty + \frac{1}{n}]$ such that $n^2 < \int_{K_n} g d\nu_n$. Since ν_n is supported in K_n, it follows $\int_{K_n} g d\nu_n = \int g d\nu_n$. This shows that $\lim\limits_{n\to\infty} \int g d\nu_n = \infty$, which proves the first part of the Lemma. Lastly, since $t_n < s_\infty + \frac{1}{n}$, we notice

$$P_{K_n}(-(s_\infty + 1)g) < P_{K_n}(-t_n g) = h(\nu_n) - t_n \int_{K_n} g d\nu_n,$$

i.e.,

$$\frac{h(\nu_n)}{\int g d\nu_n} > \frac{t_n \int_{K_n} g d\nu_n}{\int g d\nu_n} + \frac{P_{K_n}(-(s_\infty + 1)g)}{\int g d\nu_n}.$$

Since $t_n \in [s_\infty - \frac{1}{n}, s_\infty + \frac{1}{n}]$, we have $\lim_{n\to\infty} t_n = s_\infty$. Letting $n \to \infty$, and recalling that $\mathrm{supp}(\nu_n) \subseteq K_n$, and $\lim_{n\to\infty} \int g d\nu_n \to \infty$, we obtain

$$\liminf_{n\to\infty} \frac{h(\nu_n)}{\int g d\nu_n} \geq \lim_{n\to\infty} \left(\frac{t_n \int g d\nu_n}{\int g d\nu_n} + \frac{P_{K_n}(-(s_\infty + 1)g)}{\int g d\nu_n} \right) = s_\infty.$$

□

Lemma 3. *For every $t \in \mathbb{R}^+$ we have that*

$$P(tf - sg) = \begin{cases} \infty, & \text{if } s < s_\infty; \\ \text{finite}, & \text{if } s > s_\infty. \end{cases}$$

Proof. We first claim that given a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ in $\mathcal{M}_\sigma(-g)$ such that $\int g d\mu_n \to \infty$, we have $\limsup_{n\to\infty} \frac{h(\mu_n)}{\int g d\mu_n} \leq s_\infty$ and $\lim_{n\to\infty} \frac{\int f d\mu_n}{\int g d\mu_n} = 0$. Let us first set $\tilde{s} > s_\infty$ and notice that:

$$h(\mu_n) - \tilde{s} \int g d\mu_n \leq P(-\tilde{s} g) < \infty,$$

then

$$\frac{h(\mu_n)}{\int g d\mu_n} \leq \tilde{s} + \frac{P(-\tilde{s}g)}{\int g d\mu_n}.$$

Letting $n \to \infty$ and recalling that $\tilde{s} > s_\infty$ is arbitrary, it follows that

$$\limsup_{n\to\infty} \frac{h(\mu_n)}{\int g d\mu_n} \leq s_\infty, \tag{16}$$

proving first part of the claim. Now, notice from (12) that for every $\varepsilon > 0$ there exists N such that for every $x \in \bigcup_{j=N}^{\infty} [j]$:

$$\frac{f(x)}{g(x)} < \varepsilon.$$

So, denoting $A = \bigcup_{j=1}^{N-1} [j], B = \bigcup_{j=N}^{\infty} [j]$, we can write

$$\frac{\int f d\mu_n}{\int g d\mu_n} = \frac{\int_A f d\mu_n}{\int g d\mu_n} + \frac{\int_B f \mu_n}{\int g d\mu_n} \leq \frac{\int_A f d\mu_n}{\int g d\mu_n} + \frac{\int_B f \mu_n}{\int_B g d\mu_n} \leq \frac{\max f|_A}{\int g d\mu_n} + \varepsilon.$$

By taking lim sup in this inequality, and recalling that $\int g d\mu_n \to \infty$, it results

$$\limsup_{n\to\infty} \frac{\int f d\mu_n}{\int g d\mu_n} \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and $f, g > 0$, we conclude

$$\lim \frac{\int f d\mu_n}{\int g d\mu_n} = 0, \tag{17}$$

as claimed.

We now prove that for every $t > 0$, $s < s_\infty$ we have $P(tf - sg) = \infty$. Recall Lemma 2 and define a sequence $\{\nu_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \int g d\nu_n = \infty$ and $\liminf_{n\to\infty} \frac{h(\nu_n)}{\int g d\nu_n} \geq s_\infty$. These properties along with our claim imply

$$\lim_{n\to\infty} \frac{h(\nu_n)}{\int g d\nu_n} = s_\infty.$$

Now, from the variational principle it follows

$$P(tf-sg) \geq \lim_{n\to\infty}\left(h(\nu_n)+t\int fd\nu_n - s\int gd\nu_n\right)$$
$$= \lim_{n\to\infty}\int gd\nu_n\left(\frac{h(\nu_n)}{\int gd\nu_n}+t\frac{\int fd\nu_n}{\int gd\nu_n}-s\right).$$

Letting $n\to\infty$, it follows from (16) and (17) that $\int gd\nu_n\to\infty$ and

$$\left(\frac{h(\nu_n)}{\int gd\nu_n}+t\frac{\int fd\nu_n}{\int gd\nu_n}-s\right)\to(s_\infty-s)>0.$$

Hence, $P(tf-sg)=\infty$ if $s<s_\infty$.

Otherwise, if $s>s_\infty$, let us suppose to obtain a contradiction that $P(tf-sg)=\infty$, i.e., that there is a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ in $\mathcal{M}_\sigma(-g)$ such that

$$\lim_{n\to\infty}\left(h(\mu_n)+t\int fd\mu_n - s\int gd\mu_n\right)=\infty. \tag{18}$$

Since $g>0$, it follows from this equation that $\lim\limits_{n\to\infty}h(\mu_n)=\infty$ or $\lim\limits_{n\to\infty}\int fd\mu_n=\infty$.

If $h(\mu_n)\to\infty$, since $\infty>P(-sg)\geq h(\mu_n)-s\int gd\mu_n$, it follows that $\int gd\mu_n\to\infty$. On the other hand, if $\int fd\mu_n\to\infty$, from (15) we have $\int gd\mu_n\to\infty$ since

$$\frac{\int fd\mu_n}{K}<\int gd\mu_n.$$

Since in both cases we have $\lim\limits_{n\to\infty}\int gd\mu_n=\infty$, relations (16) and (17) hold.

Now, (18) shows that for big values of n

$$h(\mu_n)+t\int fd\mu_n - s\int gd\mu_n>0,$$

so

$$\frac{h(\mu_n)}{\int gd\mu_n}>s-t\frac{\int fd\mu_n}{\int gd\mu_n}.$$

By letting $n\to\infty$ and recalling (17) we obtain, $\limsup\limits_{n\to\infty}\frac{h(\mu_n)}{\int gd\mu_n}>s>s_\infty$. This is a contradiction with (16) and proves that $P(tf-sg)<\infty$ when $s>s_\infty$. □

LEMMA 4. *For every $s>s_\infty$ there exists $t^*(s)$ in $\mathbb{R}^+$ such that for every $t>t^*$ we have $P(tf-sg)>0$.*

PROOF. Let $\nu\in\mathcal{M}_\sigma(-g)$. Pick $s>s_\infty$. Since $\int gd\nu<\infty$ and $\int fd\nu>0$, there exists t^* such that

$$t^*\int fd\nu>s\int gd\nu-h(\nu),$$

hence, if $t>t^*$

$$t\int fd\nu>s\int gd\nu-h(\nu),$$

that is

$$h(\nu)+t\int fd\nu-s\int gd\nu>0.$$

Taking supremum over $\nu\in\mathcal{M}(-g)$ we obtain

$$P(tf-sg)>0.$$

□

LEMMA 5. *The zero-pressure map $O(t)$ is real-analytic in (t^*, ∞) for some $t^* \in \mathbb{R}^+$.*

PROOF. Choose $s_+ > s_\infty$ and denote $t^* = t^*(s_+)$ as in Lemma 4. According to Lemma 3, the map $\varphi : (t^*, \infty) \times (s_\infty, \infty) \to \mathbb{R}$ defined by $\varphi(t, s) = P(tf - sg)$ is finite and therefore, by proposition 2, φ is real-analytic in both variables s and t. Now, if $t \in (t^*, \infty)$, then the function $s \mapsto \varphi(t, s)$ is decreasing and $\lim_{s\to\infty} \varphi(t, s) = -\infty$. Moreover, $\varphi(t, s_+) > 0$. Thus, there exists a unique $s_0 > s_\infty$ such that $\varphi(t, s_0) = 0$ and $\frac{\partial \varphi}{\partial s}(s_0) < 0$. This implies that $O(t) = s_0$. So, the implicit function theorem ensures that $O(t)$ is real-analytic in A. $\square$

REMARK 12. Since the choice of t^* in Lemma 4 can be replaced by any $t^{*\prime} > t^*$, we can assume without loss of generality the existence of the derivatives of $O(t^*)$ of any order. Therefore, we can always pick t^* such that $O(t)$ is a C^∞ function in the interval $[t^*, \infty)$.

The following corollaries are direct consequences of the equality $O(t) = s_0$ in the previous lemma.

COROLLARY 1. *For every $t \geq t^*$, the condition $P(tf - O(t)g) = 0$ holds.*

COROLLARY 2. *For every $t \geq t^*$, we have $O(t) > s_\infty$.*

The following lemma shows an explicit form of the first order derivative of $O(t)$.

LEMMA 6. *For any $t \geq t^*$, $O'(t) = \frac{\int f d\mu_t}{\int g d\mu_t}$ where μ_t is the Gibbs-equilibrium measure for $tf - O(t)g$.*

PROOF. Let $\varepsilon > 0$. From the variational principle and Corollary 1 we have

$$0 = P(tf - O(t)g) = h(\mu_t) + t \int_{\Sigma_A} f d\mu_t - O(t) \int_{\Sigma_A} g d\mu_t$$

and

$$0 = P\left((t+\varepsilon)f - O(t+\varepsilon)g\right) \geq h(\mu_t) + (t+\varepsilon) \int_{\Sigma_A} f d\mu_t - O(t+\varepsilon) \int_{\Sigma_A} g d\mu_t.$$

Subtracting these relations and dividing by ε yields

$$0 \geq \int_{\Sigma_A} f d\mu_t - \frac{O(t+\varepsilon) - O(t)}{\varepsilon} \int_{\Sigma_A} g d\mu_t.$$

Let O'_+, O'_- respectively denote the right and the left derivative for O. If $\varepsilon \to 0^+$, then

$$\int_{\Sigma_A} f d\mu_t \leq O'_+(t) \lim_{n\to\infty} \int_{\Sigma_A} g d\mu_t.$$

Similarly, if we pick $\varepsilon < 0$ we obtain

$$0 \leq \int_{\Sigma_A} f d\mu_t - \frac{O(t+\varepsilon) - O(t)}{\varepsilon} \int_{\Sigma_A} g d\mu_t,$$

and then, by letting $\varepsilon \to 0^-$, it follows

$$O'_-(t) \int_{\Sigma_A} g d\mu_t \leq \int_{\Sigma_A} f d\mu_t.$$

Since $O(t)$ is differentiable, $O'_{+}(t) = O'_{-}(t) = O'(t)$. Hence

$$\int_{\Sigma_A} f d\mu_t \leq O'(t) \int_{\Sigma_A} g d\mu_t \leq \int_{\Sigma_A} f d\mu_t,$$

i.e.,

$$O'(t) \int_{\Sigma_A} g d\mu_t = \int_{\Sigma_A} f d\mu_t.$$

Hence,

$$O'(t) = \frac{\int f d\mu_t}{\int g d\mu_t}.$$

□

LEMMA 7. *The zero-pressure map $O(t)$ is a convex function in $[t^*, \infty)$.*

PROOF. It suffices to show that $O(t)$ is the maximum of a set of affine functions, whence convexity is an immediate consequence. We claim that

$$O(t) = \max_{\nu \in \mathcal{M}_\sigma(tf - O(t)g)} \left\{ \frac{h(\nu)}{\int g d\nu} + t \frac{\int f d\nu}{\int g d\nu} \right\}. \tag{19}$$

Given $\nu \in \mathcal{M}_\sigma(tf - O(t)g)$, by Theorem 3 and Corollary 1, we obtain

$$0 = P(tf - O(t)g) \geq h(\nu) + t \int_{\Sigma_A} f d\nu - O(t) \int_{\Sigma_A} g d\nu$$

with equality if and only if $\nu = \mu_t$. Therefore

$$O(t) \geq \frac{h(\nu)}{\int g d\nu} + t \frac{\int f d\nu}{\int g d\nu}, \tag{20}$$

with equality if and only if $\nu = \mu_t$. Since the equality is attained, Equation (19) follows. □

COROLLARY 3. *The function $O'(t) = \frac{\int f d\mu_t}{\int g d\mu_t}$ is increasing in the interval $[t^*, \infty)$.*

We finish this section with two relations between t and $O(t)$ which allow us to bound the growth rate of the zero-pressure map. We first state a technical lemma which will be used to compare the entropy of the equilibrium states of the potentials $tf - O(t)g$.

LEMMA 8. *The equilibrium states $(\mu_t)_{t > t^*}$ for $tf - O(t)g$ satisfy that the sequence*

$$\left(\frac{h(\mu_t)}{\int g d\mu_t} \right)_{t \geq t^*}$$

is decreasing.

PROOF. Since μ_t is an equilibrium measure for $tf - O(t)g$, we have

$$0 = P(tf - O(t)g) = h(\mu_t) + t \int f d\mu_t - O(t) \int g d\mu_t$$

and for any $\nu \in \mathcal{M}(-g)$:

$$0 \geq h(\nu) + t \int f d\nu - O(t) \int g d\nu.$$

Solving for $O(t)$ in both relations we obtain

$$\frac{h(\mu_t)}{\int gd\mu_t} + t\frac{\int fd\mu_t}{\int gd\mu_t} \geq \frac{h(\nu)}{\int gd\nu} + t\frac{\int fd\nu}{\int gd\nu}. \tag{21}$$

Set $t_2 > t_1 > t^*$ and define the affine functions

$$\ell_1(t) := \frac{h(\mu_{t_1})}{\int gd\mu_{t_1}} + t\frac{\int fd\mu_{t_1}}{\int gd\mu_{t_1}}$$

and

$$\ell_2(t) := \frac{h(\mu_{t_2})}{\int gd\mu_{t_2}} + t\frac{\int fd\mu_{t_2}}{\int gd\mu_{t_2}}.$$

Let us assume, in order to obtain a contradiction, that

$$\frac{h(\mu_{t_1})}{\int gd\mu_{t_1}} < \frac{h(\mu_{t_2})}{\int gd\mu_{t_2}},$$

i.e., $(\ell_2 - \ell_1)(0) > 0$. Notice that $(\ell_2 - \ell_1)$ is also an affine function, therefore, it must be monotonous. On the other hand, from (21), we have $\ell_1(t_1) \geq \ell_2(t_1)$ and therefore $(\ell_2 - \ell_1(t_1)) \leq 0$, whence $(\ell_2 - \ell_1)$ is a decreasing function. Nevertheless, from (21) it also can be seen that $\ell_1(t_2) \leq \ell_2(t_2)$, which implies that $(\ell_2 - \ell_1)(t_2) \geq 0$, which contradicts the monotonicity of $(\ell_2 - \ell_1)$ since we now have $(\ell_2 - \ell_1)(0) > (\ell_2 - \ell_1)(t_1) \geq (\ell_2 - \ell_1)(t_2)$. Hence,

$$\frac{h(\mu_{t_1})}{\int gd\mu_{t_1}} \geq \frac{h(\mu_{t_2})}{\int gd\mu_{t_2}},$$

which proves the lemma. □

COROLLARY 4. *For every $t > t^*$ let μ_t be an equilibrium measure for the potential $tf - O(t)g$. Then the limit*

$$\lim_{t\to\infty} \frac{h(\mu_t)}{\int gd\mu_t}$$

exists in $\mathbb{R}$.

LEMMA 9. *There exist constants $\alpha, \gamma > 0$ such that*

$$\gamma^{-1}t \leq O(t) \leq \alpha t$$

for every $t \geq t^*$.

PROOF. Let $t \geq t^*$ and let μ_t be an equilibrium measure for $tf - O(t)g$. It follows that

$$P(tf - O(t)g) = 0 = h(\mu_t) + t\int fd\mu_t - O(t)\int gd\mu_t.$$

Therefore

$$\frac{O(t)}{t} = \frac{h(\mu_t)}{t\int gd\mu_t} + \frac{\int fd\mu_t}{\int gd\mu_t}.$$

Pick K as in inequality (15) and set $\alpha = \frac{h(\mu_{t^*})}{t^*\int gd\mu_{t^*}} + K$. It follows from the definition of K and Lemma 8 that $\frac{O(t)}{t} \leq \alpha$. Therefore

$$O(t) \leq \alpha t. \tag{22}$$

On the other hand. We also have

$$O(t) = \frac{h(\mu_t)}{\int g d\mu_t} + t\frac{\int f d\mu_t}{\int g d\mu_t} \geq t\frac{\int f d\mu_t}{\int g d\mu_t} = tO'(t).$$

From Corollary 3, it follows $O(t) \geq tO'(t^*)$. Therefore, by picking $\gamma^{-1} = O'(t^*)$, we obtain

$$O(t) \geq \gamma^{-1} t. \tag{23}$$

□

2. Zero-temperature limits for quotients of potentials

As it was stated at the beginning of this chapter, we now develop a zero-temperature limit for quotients. Notice that thanks to the properties of the zero-pressure map, we can now describe a family of potentials $\{tf - O(t)g\}$ which relates both potentials, f and g, but depends only on one parameter.

THEOREM 5. *Let (Σ_A, σ) be a topologically mixing countable Markov shift satisfying the BIP property. Suppose that $f, g : \Sigma_A \to \mathbb{R}$ are positive, locally Hölder functions satisfying (12) and that g is bounded away from zero. Suppose that $P(-sg) < \infty$ for some $s \geq 0$ and pick $t^* > 0$ such that the zero-pressure map $O(t)$ is analytic in (t^*, ∞). Then,*

(a) *For each $t > t^*$ there exists a unique Gibbs-equilibrium measure μ_t for the function $tf - O(t)g$.*
(b) *the family $(\mu_t)_{t>t^*}$ has an accumulation point μ in the weak-* topology.*
(c) *μ is an (f, g)-maximizing measure.*

This section is devoted to prove Theorem 5, which relates thermodynamic formalism with ergodic optimization for quotients of potentials. We first prove the existence and then study the maximizing property of the accumulation points of the family of equilibrium measures μ_t for $tf - O(t)g$. The proof of this Theorem is splitted into several lemmas that lead to the conclusion of each of its statements.

By Prohorov's Theorem, the following lemma suffices to prove (b). Its proof follows the ideas from [**JMU**].

LEMMA 10. *The family $(\mu_t)_{t>t^*}$ is tight.*

PROOF. Let $\varepsilon > 0, t > t^*$. For any $i \in \mathbb{N}$, since μ_t is a Gibbs measure, inequality (8) and Remark 11 imply

$$\begin{aligned}
\mu_t[i] &\leq K_2 \exp(\sup\{tf - O(t)g|_{[i]}\} - P(tf - O(t)g)) \\
&= \exp(4V(tf - O(t)g)) \exp(\sup\{tf - O(t)g|_{[i]}\}) \\
&\leq \exp(4(tV(f) + O(t)V(g)) \exp(\sup\{tf - O(t)g|_{[i]}\}).
\end{aligned}$$

Now, apply (23) in this inequality to obtain

$$\begin{aligned}
\mu_t[i] &\leq \exp(4\gamma O(t)V(f) + 4O(t)V(g) + \sup\{\gamma(O(t)f - O(t)g)|_{[i]}\} \\
&= \exp(O(t)(4\gamma V(f) + 4V(g) + \sup\{(\gamma f - g)|_{[i]}\})).
\end{aligned} \tag{24}$$

From (12), it follows that $\sup(-g)|_{[i]} \to -\infty$ as $i \to \infty$. Therefore, there exists $J \in \mathbb{N}$ such that for every $i \geq J$ we have

$$\sup(\gamma f - g)|_{[i]} < -4\gamma V(f) - 4V(g)$$

and therefore

$$4\gamma V(f) + 4V(g) + \sup(\gamma f - g)|_{[i]} < 0.$$

Since $O(t)$ is increasing, we have $O(t^*) \leq O(t)$, hence

$$O(t)(4\gamma V(f) + 4V(g) + \sup(\gamma f - g)|_{[i]}) \leq O(t^*)(4\gamma V(f) + 4V(g) + \sup(\gamma f - g)|_{[i]}).$$

From this condition and inequality (24) we deduce that for every $t > t^*$ and $i \geq J$ we have

$$\begin{aligned}\mu_t[i] &\leq \exp(O(t^*)(4\gamma V(f) + 4V(g) + \sup(\gamma f - g)|_{[i]}) \\ &= \hat{K} \exp(\sup(\gamma O(t^*) f - O(t^*) g)|_{[i]}\}),\end{aligned}$$

where $\hat{K} := \exp(O(t^*)(4\gamma V(f) + 4V(g)))$.

This way, for every $n \geq J$:

$$\sum_{i=n}^{\infty} \mu_t[i] \leq \hat{K} \sum_{i=n}^{\infty} \exp(\sup(\gamma O(t^*) f - O(t^*) g)|_{[i]}). \tag{25}$$

Recall Lemma 3 and Corollary 2 to deduce

$$P(\gamma O(t^*) f - O(t^*) g) < \infty.$$

From (4), we obtain that $\sum_{i=J}^{\infty} \exp(\sup(\gamma O(t^*) f - O(t^*) g)|_{[i]}) < \infty$. So, for every $k \in \mathbb{N}$ there is $n_k \geq J$ such that

$$\sum_{i=n_k}^{\infty} \exp(\sup(\gamma O(t^*) f - O(t^*) g)|_{[i]}) < \frac{\varepsilon}{2^k \hat{K}}.$$

An application of this inequality in (25) with $n = n_k$ yields the relation

$$\sum_{i=n_k}^{\infty} \mu_t[i] \leq \frac{\varepsilon}{2^k}.$$

Let us now define the compact set $K := \{x \in \Sigma_A : 1 \leq x_k \leq n_k, \forall k \in \mathbb{N}\}$. It follows that

$$\begin{aligned}\mu_t(K) = \mu_t\left(\Sigma_A \setminus \bigcup_{k=1}^{\infty} \{x \in \Sigma_A : x_k > n_k\}\right) &\geq 1 - \sum_{k=1}^{\infty} \mu_t\left(\{x \in \Sigma_A : x_k > n_k\}\right) \\ &= 1 - \sum_{k=1}^{\infty} \sum_{i=n_k+1}^{\infty} \mu_t(\pi_k^{-1}([i])) = 1 - \sum_{k=1}^{\infty} \sum_{i=n_k+1}^{\infty} \mu_t[i] \\ &> 1 - \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = 1 - \varepsilon.\end{aligned}$$

Hence, $(\mu_t)_{t>t^*}$ is a tight family of probability measures. $\square$

LEMMA 11. *The identity*

$$\lim_{t\to\infty} \frac{\int f d\mu_t}{\int g d\mu_t} = \frac{\int f d\mu}{\int g d\mu}$$

holds.

Proof. Since $O(t)$ is convex, $O'(t)$ is non decreasing. Therefore, from (15):

$$\lim_{t\to\infty} O'(t) = \lim_{t\to\infty} \frac{\int f d\mu_t}{\int g d\mu_t} < \infty.$$

As a particular case of (20) we have

$$O(t) \geq \frac{h(\mu)}{\int g d\mu} + t\frac{\int f d\mu}{\int g d\mu}.$$

Since $O(t)$ is a convex function, we can compare asymptotic derivatives to conclude

$$\lim_{t\to\infty} O'(t) = \lim_{t\to\infty} \frac{\int f d\mu_t}{\int g d\mu_t} \geq \frac{\int f d\mu}{\int g d\mu}. \tag{26}$$

Let us now prove the opposite inequality. For each $k \in \mathbb{N}$, let f_k, g_k be bounded functions satisfying $f_k \uparrow f$ and $g_k \uparrow g$. from (15), it follows that there exists $I \in \mathbb{R}$ such that $I > \frac{\int f d\mu}{\int g d\mu}$. Applying the monotonous convergence theorem, we obtain

$$\lim_{k\to\infty} \frac{\int f_k d\mu}{\int g_k d\mu} = \frac{\lim\limits_{k\to\infty} \int f_k d\mu}{\lim\limits_{k\to\infty} \int g_k d\mu} = \frac{\int f d\mu}{\int g d\mu} < I.$$

Therefore, for sufficiently big values of k, we have

$$\frac{\int f_k d\mu}{\int g_k d\mu} < I. \tag{27}$$

Set $\varepsilon > 0$ arbitrarily and pick any $k \in \mathbb{N}$ that simultaneously satisfies $f - f_k < \varepsilon$ and (27). Since $\mu_t \to \mu$ when $t \to \infty$ in the weak-* topology and f_k, g_k are bounded, for big enough values of t, we have $\int f_k d\mu > \int f_k d\mu_t - \varepsilon$ and $\int g_k d\mu < g_k d\mu_t + \varepsilon$. Now, since $g_k < g$, we have

$$\begin{aligned}\frac{\int f_k d\mu_t - \varepsilon}{\int g_k d\mu_t + \varepsilon} > \frac{\int f_k d\mu_t - \varepsilon}{\int g d\mu_t + \varepsilon} &= \frac{\int f d\mu_t - \varepsilon}{\int g d\mu_t + \varepsilon} - \frac{\int (f - f_k) d\mu_t}{\int g d\mu_t + \varepsilon} \\ &> \frac{\int f d\mu_t - \varepsilon}{\int g d\mu_t + \varepsilon} - \frac{\varepsilon}{\int g d\mu_t + \varepsilon} = \frac{\int f d\mu_t - 2\varepsilon}{\int g d\mu_t + \varepsilon}.\end{aligned}$$

Summarizing, the following inequalities hold:

$$I > \frac{\int f_k d\mu}{\int g_k d\mu} > \frac{\int f_k d\mu_t - \varepsilon}{\int g_k d\mu_t + \varepsilon} > \frac{\int f d\mu_t - 2\varepsilon}{\int g d\mu_t + \varepsilon}.$$

Since $I > \frac{\int f d\mu}{\int g d\mu}$ and $\varepsilon > 0$ were arbitrary:

$$\frac{\int f d\mu}{\int g d\mu} \geq \frac{\int f d\mu_t}{\int g d\mu_t}.$$

Therefore, if $t \to \infty$ we obtain

$$\lim_{t\to\infty} \frac{\int f d\mu_t}{\int g d\mu_t} \leq \frac{\int f d\mu}{\int g d\mu}.$$

This inequality, along with (26), completes the proof. $\square$

We finally prove part (c) of Theorem 5.

LEMMA 12. *The measure μ is an $(f,g)-$maximizing probability measure.*

PROOF. Let $\nu \in \mathcal{M}_\sigma(-g)$. Since we have $\mathcal{M}_\sigma(tf - O(t)g) = \mathcal{M}_\sigma(-g)$, we obtain $\nu \in \mathcal{M}_\sigma(tf - O(t)g)$. From Lemma 7, it follows that we can compare asymptotic derivatives in (20) to obtain

$$\lim_{t\to\infty} O'(t) \geq \frac{\int f d\nu}{\int g d\nu}.$$

From Lemmas 6 and 11, it follows

$$\frac{\int f d\mu}{\int g d\mu} \geq \frac{\int f d\nu}{\int g d\nu}.$$

Hence, μ is (f,g)-maximizing. □

2.1. An alternative proof for Lemma 12. Lemma 12 consists of part c) from Theorem 5, that is, it states that zero temperature limits for quotients of potentials are indeed maximizing measures. We present a second proof of this result, which instead of comparing asymptotic derivatives with the zero pressure map, relies upon the existence of the limit of the ratio between the entropy of the equilibrium measures μ_t of $tf - O(t)g$ and the corresponding integral $\int g d\mu$ (see Lemma 8).

ALTERNATIVE PROOF OF LEMMA 12. Let $\nu \in \mathcal{M}(-g)$. From (21), it follows

$$\frac{h(\mu_t)}{t\int g d\mu_t} + \frac{\int f d\mu_t}{\int g d\mu_t} \geq \frac{h(\nu)}{t\int g d\nu} + \frac{\int f d\nu}{\int g d\nu}.$$

Recalling Corollary 4 and $\int g d\nu < \infty$, it follows that by letting $t \to \infty$ we obtain

$$\frac{\int f d\mu}{\int g d\mu} \geq \frac{\int f d\nu}{\int g d\nu},$$

which proves that μ is an $(f,g)-$maximizing measure. □

3. Applications and examples

3.1. Suspension flows. Let (Σ, σ) be a topologically mixing countable Markov Shift having the BIP property. Let $\tau : \Sigma \to (0, \infty)$ be a locally Hölder, bounded away from zero function satisfying the hypotheses for g in Theorem 5. The *suspension space* Σ_τ is defined as

$$\Sigma_\tau = (\Sigma \times \mathbb{R})/\sim,$$

where $\sim$ is the equivalence relation generated by matching all the pairs of the form $(x,y) \sim (\sigma(x), y - \tau(x)), x \in \Sigma$.

DEFINITION 13. *The* suspension flow σ_τ *is the flow defined in* Σ_τ *as*

$$\sigma_\tau^t(x,y) = (x, y+t).$$

A thorough study of suspension flows over countable Markov shifts and its corresponding thermodynamic formalism can be found in [**BI**].

REMARK 13. For every σ-invariant probability measure μ in Σ satisfying $\int \tau d\mu < \infty$ we can establish a corresponding σ_τ^t-invariant probability measure μ_τ in Σ_τ. That correspondence is given by

$$\mu_\tau = \frac{\mu \times \lambda}{\int_\Sigma \tau(x) d\mu}, \tag{28}$$

where λ denotes the Lebesgue measure. Moreover, if $\mathcal{M}_{\sigma_\tau}$ denotes the set of σ_τ-invariant measures in Σ_τ, this correspondence is a bijection between $\mathcal{M}_\sigma(-\tau)$ and $\mathcal{M}_{\sigma_\tau}$.

The entropy of a measure μ_τ in Σ_τ is related to the one of its corresponding measure μ by Abramov's Formula

$$h(\mu_\tau) = \frac{h(\mu)}{\int \tau d\mu}.$$

Let $G : X_\tau \to \mathbb{R}$ be a continuous function such that the map f defined as $f(x) = \int_0^{\tau(x)} G(x,y)d\lambda(y)$ is locally Hölder and positive for every $x \in \Sigma_A$. Assume that

$$\lim_{n\to\infty} \frac{\sup f|_{[n]}}{\inf \tau|_{[n]}} = 0.$$

Let us define the subset $\mathcal{M}_{\sigma_\tau}(G)$ as the family of measures in $\mathcal{M}_{\sigma_\tau}$ verifying $\int G d\mu_\tau > -\infty$. We define the *pressure* of G as

$$P(G) = \sup_{\mu_\tau \in \mathcal{M}_{\sigma_\tau}(G)} \left\{ h_{\mu_\tau}(\sigma_\tau) + \int G d\mu_\tau \right\}, \tag{29}$$

and, as usual, μ_τ will be called an *equilibrium measure* of G if μ_τ attains the supremum in (29).

Let us fix $g(x) = \tau(x)$ and adopt the notation from Theorem 5. We obtain from part (c) that μ satisfies

$$\frac{\int_\Sigma \int_0^{\tau(x)} G(x,y)dyd\mu}{\int_\Sigma \tau(x)d\mu} = \max_{\nu\in\mathcal{M}} \frac{\int_\Sigma \int_0^{\tau(x)} G(x,y)dyd\nu}{\int_\Sigma \tau(x)d\nu}, \tag{30}$$

and in virtue of Remark 13, this is equivalent to

$$\int_{\Sigma_\tau} G(x,y)d\mu_\tau = \max_{\nu_\tau\in\mathcal{M}_{\sigma_\tau}} \int_{\Sigma_\tau} G d\nu_\tau,$$

i.e. the weak-* accumulation points μ of the equilibrium measures μ_t when $t \to \infty$ induce maximizing measures for the potential G in the suspension flow Σ_τ. These observations can be summarized as shown in the following proposition

PROPOSITION 8. *Let (Σ, σ) be a topologically mixing countable Markov shift satisfying the BIP property. Set $\tau : \Sigma \to (0, \infty)$ a locally Hölder, bounded away from zero function such that $P_\sigma(-s\tau) < \infty$ for some $s \in \mathbb{R}^+$. Let $G : \Sigma_\tau \to \mathbb{R}$ be a continuous potential in the suspension flow Σ_τ. Set $f(x) = \int_0^{\tau(x)} G(x,y)dy$. If f is a positive locally Hölder function such that*

$$\lim_{n\to\infty} \frac{\sup\{f(x) : x \in [n]\}}{\inf\{g(x) : x \in [n]\}} = 0,$$

then for every sufficiently big value of t, there exists a unique equilibrium measure for the potential tG. Moreover, These equilibrium measures have an accumulation point μ_τ in the weak- topology as $t \to \infty$ which in turn maximizes G, i.e.,*

$$\int_{\Sigma_\tau} G d\mu_\tau = \max_{\nu_\tau\in\mathcal{M}_{\sigma_\tau}} \int_{\Sigma_\tau} G d\nu_\tau.$$

Related results have been obtained in [**MSV, RV**].

3.2. An example about continuous fractions. Our main result can also be applied to infinite continuous fractions. First, set an irrational number $x_0 \in (0,1) \setminus \mathbb{Q}$ and consider its continuous fraction expansion

$$x_0 = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\cdots}}}},$$

where $a_1, a_2 \cdots, a_n, \cdots \in \mathbb{N}$. Notice that $(a_1, a_2, \cdots)$ can be thought of as an element from the full shift Σ. Indeed, the *Gauss map*

$$G(x) := \frac{1}{x} - \left[\frac{1}{x}\right]$$

is such that if x_0 is coded as $(a_1, a_2, \cdots)$ then $G(x_0)$ is identified with $(a_2, a_3, \cdots)$. Therefore, there is a topological equivalence between (Σ, σ) and $((0,1) \setminus \mathbb{Q}, G)$.

Let us define $f(x_0) = \log a_1$ and $g(x_0) = a_1$. Since these functions are locally constant, it is clear that they are locally Hölder. Since $\frac{\log(a_1)}{a_1} \to 0$ when $a_1 \to \infty$, condition (12) is also met. On the other hand, for every $s > 0$ we have $\sum_{n\in} e^{-sn}, \sum_{n\in} sne^{-sn} < \infty$, so (4) and (9) hold for $-sg$. The full shift is topologically mixing and has the BIP property. Therefore, since pressure is preserved by topological equivalence, we can establish that for every $x_0 \in (0,1) \setminus \mathbb{Q}$ there is $t^* > 0$ such that if $O(t)$ denotes the zero-pressure map of f and g, the family $(\mu_t)_{t>t^*}$ of equilibrium measures of the potentials $tf - O(t)g$ has an accumulation point μ which satisfies

$$\frac{\int \log a_1 d\mu}{\int a_1 d\mu} = \max_{\nu \in \mathcal{M}} \frac{\int \log a_1 d\nu}{\int a_1 d\nu}.$$

We stress that this example is different than applying the main result from [**JMU**] to the potential $\phi(x) = \frac{\log a_1}{a_1} = \log a_1^{a_1}$, since in the latter case, the accumulation points of the equilibrium measures of $t\phi$ maximize the integral $\int \frac{\log a_1}{a_1} d\nu$, whilst our approach gives a measure μ maximizes the ratio between the integrals of f and g.

CHAPTER 3

Quotients of almost additive sequences

Thermodynamic formalism has been extended from additive potentials to several families of functions. When we deal with non-additive potentials, we consider sequences of functions $\mathcal{F} = \{\varphi_n\}_{n=1}^{\infty}$ instead of a single potential. A remarkable family of admissible potentials to develop this theory is constituted by the almost-additive sequences of functions. These sequences have the form $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$, where $f_n : \Sigma \to \mathbb{R}^+$ are continuous functions which satisfy some restrictions that relate their terms (see Definition 14). This has been done in the compact setting by Barreira [**Ba**], and for almost-additive sequences in Countable Markov shifts the thermodynamic formalism has been developed by Iommi and Yayama in [**IY1**]. Moreover, results from [**JMU**] about zero-temperature limits for an additive potential have been extended to this new framework [**IY2**]. If these notions can be extended from the additive to the almost additive framework, it makes sense to study if the development of zero-temperature limits for quotients of potentials makes sense in this new context. Existing literature shows several analogies between the behaviors of additive and almost-additive potentials. The main challenge of developing these results consists of translating the elements from additive thermodynamic formalism in terms that are suitable for the almost-additive context, that is, involving sequences of functions instead of a single potential, and establish these results in a way that recovers the original additive setting.

The purposes of this chapter can be summarized into two main goals. In the first place, we present a brief review of Almost-additive thermodynamic formalism (mostly results from [**IY1**]) to show the reader the analogous concepts and hypotheses that will be used to develop the zero-temperature limits result for quotients later on. The second goal is to state and prove an almost-additive version of Theorem 5. As before, the main result (see Theorem 9) is framed in a topologically mixing Countable Markov shift with the BIP property. The result requires the development of an almost-additive version of the zero-pressure map $O(t)$ (see section 2). It describes the hypotheses under which, given two almost-additive sequences $\mathcal{F}, \mathcal{G}$, we have accumulation points of the equilibrium measures μ_t of the potentials $t\mathcal{F} - = (t)\mathcal{G}$ as $(\mathcal{F}, \mathcal{G})$ maximizing measures.

Notice that this result means that we have to redefine both the maximizing measures as well as the zero-pressure map in order to make sense from this statement.

In contrast to the additive setting, we remark that we cannot assert the differentiability of the pressure function for almost-additive sequences by replicating the arguments from the previous chapter. This is due to the fact that in the additive setting the real-analiticity of the pressure relied heavily on the properties of the transfer operator, which doesn't have an analogous version in the almost additive setting. In our context, this implies that we cannot state the real-analiticity of the zero-pressure map either. Since the differentiability of the zero-pressure map played a crucial role in the previous chapter, we choose the accumulation points in a way that the zero pressure map is still differentiable by studying the accumulation points of the equilibrium measures along sequences $\{t_k\}$ that make the zero-pressure map $O(t_k)$ to be differentiable along these points. Another difference with the additive case is that since the condition (12) relates the values of f and g for only two functions, it is necessary to develop some arguments to link this relation with the behaviour of the whole sequences of almost-additive functions. The proof of the existence of accumulation points for the family of equilibrium measures of the sequences when the temperature drops to zero also relies on Prohorov's Theorem, but the argument to prove the tightness of this family differs from the one presented in the previous chapter, since in the almost additive setting we work with Bowen sequences instead of Locally Hölder potentials. The argument for the tightness almost additive sequences also on the BIP property, but introduces a technical lemma in order to bound the constants from the definition of Gibbs measures. The first section establishes the preliminary concepts of almost-additive sequences. Some important results from [**IY1**] are mentioned, along with the development of some properties that will be needed to extend the main theorem to the almost-additive sequences framework.

In section 2, the zero-pressure map is defined and their properties are stated and proven. Since the differentiability of the zero pressure map is no longer guaranteed. The properties of $O(t)$ will be derived from its convexity. Section 3 presents the main result (see Theorem 9). This time we study sequences of equilibrium measures along points that make the zero-pressure map differentiable, in contrast to the previous chapter, where we took any accumulation point.

1. Preliminary concepts

As we stated before, non-additive thermodynamic formalism is concerned with different families of functions that replace the potentials. We aim to develop zero-temperature limits to a particular class of non-additive potential that is defined as follows.

DEFINITION 14. *Let (Σ, σ) be a topologically mixing countable Markov shift satisfying the BIP property. For every $n \in \mathbb{N}$ let $f_n : \Sigma_A \to \mathbb{R}^+$ be a continuous function. The sequence $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ is called* almost-additive *if there exists $C_{\mathcal{F}} \geq 0$ such that for every $m, n \in \mathbb{N}, x \in \Sigma$ we have*

$$f_n(x) f_m(\sigma^n x) e^{-C_{\mathcal{F}}} \leq f_{n+m}(x), \tag{31}$$

and

$$f_{n+m}(x) \leq f_n(x) f_m(\sigma^n x) e^{C_{\mathcal{F}}}. \tag{32}$$

Equivalently, equations (31) and (32) can be expressed respectively as

$$\log f_n(x) + \log f_m(\sigma^m x) - C_{\mathcal{F}} \leq \log f_{n+m}(x), \tag{33}$$

and

$$\log f_{n+m} \leq \log f_n(x) + \log f_m(\sigma^n(x)) + C_{\mathcal{F}}. \tag{34}$$

REMARK 14. Let $f : \Sigma_A \to \mathbb{R}$ be continuous. For every $n \in \mathbb{N}$, set $\log f_n(x) = \sum_{j=0}^{n-1} f \circ \sigma^j(x)$. If we define $\mathcal{F} := \{\log f_n\}_{n \in \mathbb{N}}$ we obtain that $\mathcal{F}$ is additive. In particular, if we pick $C_{\mathcal{F}} = 0$, we conclude that $\mathcal{F}$ is an almost-additive sequence. If we identify every potential f on Σ with the sequence $\mathcal{F}$ constructed in this fashion, we obtain that almost-additive sequences are a more general class than continuous potentials on Σ.

Remark 15. Applying (33) and (34) inductively, we respectively obtain, for every $n \in \mathbb{N}$:

$$\sum_{j=0}^{n-1} \log f_1 \circ \sigma^j - (n-1)C_{\mathcal{F}} \leq \log f_n$$

and

$$\log f_n \leq \sum_{j=0}^{n-1} \log f_1 \circ \sigma^j + (n-1)C_{\mathcal{F}}.$$

Therefore, for every $\mu \in \mathcal{M}_\sigma$ we have

$$n \int \log f_1 d\mu - (n-1)C_{\mathcal{F}} \leq \int \log f_n d\mu$$

and

$$\int \log f_n d\mu \leq n \int \log f_1 d\mu + (n-1)C_{\mathcal{F}}.$$

Therefore

$$\int \log f_1 d\mu - C_{\mathcal{F}} \leq \lim_{n\to\infty} \frac{1}{n} \int \log f_n d\mu \leq \int \log f_1 d\mu + C_{\mathcal{F}}. \tag{35}$$

The development of thermodynamic formalism for almost-additive sequences of functions in countable Markov shifts presented in [**IY1**] requires that the sequences satisfy the following regularity condition.

Definition 15. *Let (Σ_A, σ) be a countable Markov shift satisfying the BIP property. For every $n \in \mathbb{N}$, let $f_n : \Sigma_A \to \mathbb{R}^+$ be a continuous function. The sequence $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ is called a* Bowen sequence *if there exists $M \in \mathbb{R}^+$ such that*

$$\sup\{A_n : n \in \mathbb{N}\} \leq M,$$

where

$$A_n = \sup\left\{ \frac{f_n(x)}{f_n(y)} : x, y \in \Sigma_A, C_n(x) = C_n(y) \right\}.$$

All the thermodynamic formalism results for almost-additive sequences are stated in terms of Bowen sequences. Since we aim to study zero-temperature limits relating two potentials, the following result shows that this hypothesis remains valid when dealing with linear combinations of Bowen sequences of functions.

Lemma 13. *Let $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ and $\mathcal{G} = \{\log g_n\}_{n=1}^{\infty}$ be almost-additive Bowen sequences. Then, for every $t, s > 0$, the sequence $\mathcal{H} := t\mathcal{F} - s\mathcal{G}$ is also an almost-additive Bowen sequence.*

Proof. Notice that

$$\mathcal{H} = t\mathcal{F} - s\mathcal{G} = \{t \log f_n - s \log g_n\}_{n=1}^{\infty} = \left\{ \log\left(\frac{f_n^t}{g_n^s} \right) \right\}_{n=1}^{\infty}.$$

Denoting $h_n := \frac{f_n^t}{g_n^s}$ for every $n \in \mathbb{N}$ we have $\mathcal{H} = \{\log h_n\}_{n=1}^{\infty}$.

Now, denote by $C_{\mathcal{F}}, C_{\mathcal{G}}$ the almost-additivity constants for $\mathcal{F}$ and $\mathcal{G}$ respectively. Define $C_{\mathcal{H}} := tC_{\mathcal{F}} + sC_{\mathcal{G}} > 0$. According to equations (31) and (32), we have for every $x \in \Sigma_A$:

$$\begin{aligned} h_{n+m}(x) &= \frac{f_{n+m}^t(x)}{g_{n+m}^s(x)} \\ &\leq \frac{(f_n(x)f_m(\sigma^n(x))e^{C_{\mathcal{F}}})^t}{(g_n(x)g_m(\sigma^n(x))e^{-C_{\mathcal{G}}})^s} \\ &= \frac{f_n^t(x)}{g_n^s(x)} \cdot \frac{f_m^t(\sigma^n(x))}{g_m^s(\sigma^n(x))} \cdot e^{tC_{\mathcal{F}}+sC_{\mathcal{G}}} \\ &= h_n(x)h_m(\sigma^n(x))e^{C_{\mathcal{H}}}. \end{aligned}$$

Similarly, $h_{n+m}(x) \geq h_n(x)h_m(\sigma^n(x))e^{C_{\mathcal{H}}}$. Whence, $\mathcal{H}$ is an almost-additive sequence.

Now, let us denote by $M_{\mathcal{F}}, M_{\mathcal{G}}$ the constant M from definition 15 for $\mathcal{F}$ and $\mathcal{G}$ respectively. Since both, $M_{\mathcal{F}}$ and $M_{\mathcal{G}}$ are positive, we have

$$M_{\mathcal{H}} := M_{\mathcal{F}}^t M_{\mathcal{G}}^s > 0.$$

Now, set $n \in \mathbb{N}$ and $x, y \in \Sigma_A$ such that $C_n(x) = C_n(y)$. Then

$$\begin{aligned} \frac{h_n(x)}{h_n(y)} &= \frac{f_n^t(x)}{g_n^s(x)} \cdot \frac{g_n^s(y)}{f_n^t(y)} \\ &= \left(\frac{f_n(x)}{f_n(y)}\right)^t \cdot \left(\frac{g_n(y)}{g_n(x)}\right)^s \\ &\leq M_{\mathcal{F}}^t M_{\mathcal{G}}^s \\ &= M_{\mathcal{H}}. \end{aligned}$$

This shows that $\mathcal{H}$ is also a Bowen sequence. □

The Gurevich pressure for almost-additive Bowen sequences is defined as follows.

Definition 16. *Let $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be an almost-additive Bowen sequence in a topologically mixing countable Markov shift with the BIP property. The* Gurevich pressure *of $\mathcal{F}$, denoted $P(\mathcal{F})$ is defined as*

$$P(\mathcal{F}) = \lim_{n\to\infty} \frac{1}{n} \log\left(\sum_{x:\sigma^n x = x} f_n(x)\chi_{[a]}(x)\right),$$

where $\chi_{[a]}(x)$ is the characteristic function of the cylinder set $[a]$ for $a \in \mathbb{N}$.

According to [**IY1**, Prop. 2.1, Lemma 2.5] the limit from the right side exists and it is independent of the choice of a.

We now cite the most remarkable properties of the Gurevich pressure. In the first place, Gurevich pressure of a Bowen sequence $\mathcal{F}$ can be approximated by the pressure of its restrictions to $\sigma-$invariant compact subsets of Σ_A.

Theorem 6. *Let (Σ_A, σ) be a topologically mixing countable Markov shift and $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be an almost-additive Bowen sequence on Σ_A. Then*

$$P(\mathcal{F}) = \sup\{P(\mathcal{F}|_K) : K \subseteq \Sigma \text{ compact and } \sigma^{-1}(K) = K\}.$$

A version of the variational principle for almost-additive sequences was proven in [**IY1**].

THEOREM 7. *Let (Σ_A, σ) be a topologically mixing countable Markov shift and $\mathcal{F}$ be an almost-additive Bowen sequence on Σ_A, with* $\sup f_1 < \infty$. *Then*

$$\begin{aligned} P(\mathcal{F}) &= \sup\left\{h(\mu) + \lim_{n\to\infty}\frac{1}{n}\int \log f_n d\mu : \mu \in \mathcal{M}_\sigma \text{ and } \lim_{n\to\infty}\frac{1}{n}\int \log f_n d\mu > -\infty\right\} \\ &= \sup\left\{h(\mu) + \int \lim_{n\to\infty}\frac{1}{n}\log f_n d\mu : \mu \in \mathcal{M}_\sigma \text{ and } \int \lim_{n\to\infty}\frac{1}{n}\log f_n d\mu > -\infty\right\}. \end{aligned}$$

In order to simplify notation we define, for every almost-additive Bowen sequence $\mathcal{F}$ on Σ and $\mu \in \mathcal{M}_\sigma$

$$I_{\mathcal{F}}(\mu) := \lim_{n\to\infty}\frac{1}{n}\int \log f_n d\mu,$$

and

$$\mathcal{M}_\sigma(\mathcal{F}) = \{\mu \in \mathcal{M}_\sigma : I_{\mathcal{F}}(\mu) > -\infty\}$$

This way, the first equality from Theorem 7 can be rewritten as

$$P(\mathcal{F}) = \sup_{\mu\in\mathcal{M}_\sigma(\mathcal{F})} \left(h(\mu) + I_{\mathcal{F}}(\mu)\right).$$

Notice that $I_{\mathcal{F}}(\mu)$ plays an analogous role as the integral of the additive potential from the classic version of the variational principle. This way, we define equilibrium measures for almost-additive Bowen sequences resembling the definition from the additive framework.

DEFINITION 17. *Let (Σ_A, σ) be a topologically mixing countable Markov shift and for every $n \in \mathbb{N}$ let $f_n : \Sigma_A \to \mathbb{R}^+$. Assume that $\mathcal{F} = \{\log f_n\}$ is a Bowen almost-additive sequence. A Borel probability measure $\mu \in \mathcal{M}_\sigma$ is called an* equilibrium measure *for $\mathcal{F}$ if*

$$P(\mathcal{F}) = h(\mu) + I_{\mathcal{F}}(\mu).$$

REMARK 16. When $\mathcal{F}$ satisfies $\sup \log f_1 < \infty$, we can study the pressure of the sequence $t\mathcal{F} = \{t \log f_n\}_{n=1}^{\infty}$, for $t > 0$. In this case,

$$P(t\mathcal{F}) = \sup_{\mu\in\mathcal{M}_\sigma(\mathcal{F})} \left(h(\mu) + tI_{\mathcal{F}}(\mu)\right).$$

Since for every $t > 0$, we have that $P(t\mathcal{F})$ is the supremum of a family of affine functions, we obtain that $t \mapsto P(t\mathcal{F})$ is a convex function.

REMARK 17. Since the map $t \mapsto P(t\mathcal{F})$ is convex in $\mathbb{R}^+$, it follows that it is differentiable everywhere, except at most in a countable set of points. Moreover, if t_0 is a point where the derivative exists, the derivative of $P(t\mathcal{F})$ is given by the equality

$$\left.\frac{d}{dt}P(t\mathcal{F})\right|_{t=t_0} = I_{\mathcal{F}}(\mu),$$

where μ is the equilibrium measure for $t_0\mathcal{F}$ (see [**IY2**, Lemma 4.4]) .

When Remark 14 is taken into account, we expect that Gibbs measures for a sequence of Birkhoff sums for a potential f agree with the (additive) Gibbs measures of the potential itself. This way, we say that $\mu \in \mathcal{M}_\sigma$ is a *Gibbs measure* for an almost-additive Bowen sequence $\mathcal{F} = \{\log f_n\}_{n\in\mathbb{N}}$ if there exists a constant $K_0 > 0$ such that for every $n \in \mathbb{N}$ and $x \in \Sigma_A$, we have

$$K_0^{-1} \leq \frac{\mu(C_n(x))}{\exp\left(-nP(\mathcal{F})\right) f_n(x)} \leq K_0. \tag{36}$$

When Σ_A has the BIP property, Gibbs measures do exist and agree with equilibrium measures whenever they have finite entropy. The following result (see [**IY1**, Thm. 4.1]) states this phenomenon.

Theorem 8. *Let (Σ_A, σ) be a topologically mixing countable Markov shift with the BIP property. Let $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ be an almost-additive Bowen sequence on Σ with $\sum_{a\in\mathbb{N}} \sup f_1|_{[a]} < \infty$. Then there is a Gibbs measure μ for $\mathcal{F}$ and it is mixing. Moreover, if $h(\mu) < \infty$, then μ is the unique equilibrium measure for $\mathcal{F}$.*

Remark 18. As stated in [**IY2**, Prop. 3.1], under the hypotheses of Theorem 8 the condition $h(\mu) < \infty$ can be substituted for the equivalent condition

$$\sum_{i=1}^{\infty} \sup\left(\log f_1|_{[i]}\right) \sup\left(f_1|_{[i]}\right) > -\infty. \tag{37}$$

Therefore, if a Bowen almost-additive sequence satisfies (37), then it has a unique Equilibrium-Gibbs measure.

2. Zero-pressure map

The first step towards developing a zero-temperature limits result for quotients of almost-additive sequences should be the construction of an almost-additive version of the zero-pressure map. To achieve this goal we first have to set the hypotheses that we are going to impose over the involved potentials.

For every $n \in \mathbb{N}$ let $f_n, g_n : \Sigma_A \to (1, \infty)$ be continuous functions. Set $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}, \mathcal{G} = \{\log g_n\}_{n=1}^{\infty}$ and let us assume that $\mathcal{F}$ and $\mathcal{G}$ are almost-additive Bowen sequences. Notice that we have set the codomain of the functions f_n, g_n in order to obtain $\log f_n, \log g_n$ to be positive functions. Moreover, we set the sequence $\mathcal{F}$ to satisfy

$$\lim_{n\to\infty} \frac{1}{n} \log f_n(x) > 0, \qquad \text{for every } x \in \Sigma_A.$$

As before, for $\mathcal{G}$ we need a stronger condition in order to have a uniform lower bound when working with denominators.

Definition 18. *An almost-additive potential $\mathcal{G}$ is called* bounded away from zero *whenever there exists a constant $C > 0$ verifying*

$$\lim_{n\to\infty} \frac{1}{n} \log g_n > C, \qquad \textit{for every } x \in \Sigma_A.$$

Remark 19. The previous conditions imply $I_{\mathcal{F}}(\mu) > 0$ and $I_{\mathcal{G}}(\mu) > C$ for every $\mu \in \mathcal{M}_\sigma$.

Suppose that $\mathcal{G}$ is bounded away from zero and that $\mathcal{F}$ and $\mathcal{G}$ are related by the condition

$$\lim_{n\to\infty} \frac{\sup\{\log f_1|_{[n]}\}}{\inf\{\log g_1|_{[n]}\}} = 0. \tag{38}$$

REMARK 20. As a consequence of condition (38), we have that

$$\sup_{\nu\in\mathcal{M}_\sigma(-\mathcal{G})} \frac{\int \log f_1 d\nu}{\int \log g_1 d\nu} < \infty.$$

PROOF. The proof is similar to the argument given in lemma 1. Indeed, for every $\varepsilon > 0$, there exists $J \in \mathbb{N}$ such that for every $x \in \bigcup_{j>J}[j]$ we have

$$\frac{\log f_1(x)}{\log g_1(x)} < \varepsilon \Rightarrow \log f_1(x) < \varepsilon \log g_1.$$

On the other hand, for each $j < J$ pick any $x_j \in [j]$. Since $\mathcal{F}, \mathcal{G}$ are Bowen sequences, we have, that there exist constants $M_\mathcal{F}, M_\mathcal{G}$ such that for $x \in [j]$

$$\log f_1(x) \leq M_\mathcal{F} \log f_1(x_j) \text{ and } \log g_1(x_j) \leq M_\mathcal{G} \log g_1(x),$$

therefore

$$\frac{\log f_1(x)}{\log g_1(x)} \leq M_\mathcal{F} M_\mathcal{G} \frac{f(x_j)}{g(x_j)}.$$

Setting $M_j := M_\mathcal{F} M_\mathcal{G} \frac{f(x_j)}{g(x_j)}$, we obtain $\log f_1(x) \leq M_j \log g_1(x)$.

Set $M := \max\{\varepsilon, M_1, \cdots M_J\}$. This way we have

$$\log f_1(x) \leq M \log g_1(x) \text{ for every } x \in \Sigma_A.$$

Hence, for every $\nu \in \mathcal{M}_\sigma(-\mathcal{G})$:

$$\int \log f_1(x) d\nu \leq M \int \log g_1(x) d\nu. \tag{39}$$

Equivalently

$$\frac{\int \log f_1(x) d\nu}{\int \log g_1(x) d\nu} < M < \infty.$$

$\square$

REMARK 21. Every measure $\nu \in \mathcal{M}_\sigma(-\mathcal{G})$ is also in the set $\nu \in \mathcal{M}_\sigma(-\mathcal{F})$. Indeed, from (35) and (39) it follows that

$$I_\mathcal{F}(\nu) \leq \int \log f_1 d\nu + C_\mathcal{F} \leq M \int \log g_1 d\nu + C_\mathcal{F} \leq M(I_\mathcal{G}(\nu) + C_\mathcal{G}) + C_\mathcal{F} < \infty.$$

As before, we also assume the existence of a constant s_∞ with the property

$$P(-s\mathcal{G}) = \begin{cases} \infty, & \text{if } s < s_\infty \\ \text{finite}, & \text{if } s > s_\infty \end{cases}. \tag{40}$$

REMARK 22. When $s > s_\infty$ equilibrium measures agree with Gibbs measures for $-s\mathcal{G}$. Indeed, if μ is an equilibrium measure for this sequence, it follows that $I_{-s\mathcal{G}}(\mu)$ corresponds to the (possibly lateral) derivative of $P(-s\mathcal{G})$. Since $P(-s\mathcal{G}) < \infty$ in a neighborhood of s, this derivative cannot be infinite. From Theorem 7, we have $h(\mu) < \infty$. Therefore, Theorem 8 shows that μ is a Gibbs-equilibrium measure for $-s\mathcal{G}$.

Under these conditions, we define the almost-additive version of the zero-pressure map.

Definition 19. *Let $\mathcal{F}, \mathcal{G}$ be two Almost-additive Bowen sequences in a topologically mixing countable Markov shift (Σ_A, σ) satisfying the BIP property. Assume that $\mathcal{G}$ is bounded away from zero and that conditions* (38) *and* (40) *are held. The* zero-pressure map *$O(t)$ for $\mathcal{F}$ and $\mathcal{G}$ is defined as*

$$O(t) := \inf\{s \in \mathbb{R} : P(t\mathcal{F} - s\mathcal{G}) \leq 0\}. \tag{41}$$

The arguments used in the previous chapter to establish the real-analiticity of the zero-pressure map do not work in the almost-additive setting since the pressure map $P(t\mathcal{F})$ itself is no longer necessarily real-analytic. In other words, Proposition 2, which was a key point in the argument, does not have an analogous statement for almost-additive potentials. Nevertheless, each of the other properties proven in the additive setting can be adapted to almost-additive Bowen sequences. Let us first prove that under our assumptions, the zero pressure map $O(t)$ is well-defined.

Lemma 14. *For every $t \in \mathbb{R}^+$ we have $O(t) < \infty$.*

Proof. Let C be the constant from definition 18, and recall Remark 20 to set $M := \sup_{\nu \in \mathcal{M}} \frac{\int \log f_1 d\nu}{\int \log g_1 d\nu}$ and notice from (35) that for every $\nu \in \mathcal{M}_\sigma$,

$$\begin{aligned}
\frac{I_{\mathcal{F}}(\nu)}{I_{\mathcal{G}}(\nu)} &\leq \frac{\int \log f_1 d\nu + C_{\mathcal{F}}}{I_{\mathcal{G}}(\nu)} \\
&= \frac{\int \log f_1 d\nu}{\int \log g_1 d\nu} \cdot \frac{\int \log g_1 d\nu}{I_{\mathcal{G}}(\nu)} + \frac{C_{\mathcal{F}}}{I_{\mathcal{G}}(\nu)} \\
&= \frac{\int \log f_1 d\nu}{\int \log g_1 d\nu} \cdot \frac{\left(\int \log g_1 d\nu - C_{\mathcal{G}}\right) + C_{\mathcal{G}}}{I_{\mathcal{G}}(\nu)} + \frac{C_{\mathcal{F}}}{I_{\mathcal{G}}(\nu)} \\
&\leq \frac{\int \log f_1 d\nu}{\int \log g_1 d\nu} \cdot \frac{I_{\mathcal{G}}(\nu) + C_{\mathcal{G}}}{I_{\mathcal{G}}(\nu)} + \frac{C_{\mathcal{F}}}{I_{\mathcal{G}}(\nu)} \\
&\leq M \cdot \left(1 + \frac{C_{\mathcal{G}}}{I_{\mathcal{G}}(\nu)}\right) + \frac{C_{\mathcal{F}}}{I_{\mathcal{G}}(\nu)} \\
&< M \cdot \left(1 + \frac{C_{\mathcal{G}}}{C}\right) + \frac{C_{\mathcal{F}}}{C}.
\end{aligned}$$

This proves that $\frac{I_{\mathcal{F}}}{I_{\mathcal{G}}}$ is bounded. Let us denote by K an upper bound for this quantity. Now, pick $s \in \mathbb{R}$. This yields

$$\begin{aligned}
P(t\mathcal{F} - s\mathcal{G}) &= \sup_{\nu \in \mathcal{M}_\sigma(t\mathcal{F} - s\mathcal{G})} h(\nu) + tI_{\mathcal{F}}(\nu) - sI_{\mathcal{G}}(\nu) \\
&< \sup_{\nu \in \mathcal{M}_\sigma(t\mathcal{F} - s\mathcal{G})} h(\nu) + (tK - s)I_{\mathcal{G}}(\nu) \\
&= \sup_{\nu \in \mathcal{M}_\sigma(-\mathcal{G})} h(\nu) + (tK - s)I_{\mathcal{G}}(\nu) \\
&= P((tK - s)\mathcal{G}).
\end{aligned}$$

By choosing $s > tK + s_\infty$, it results

$$P(t\mathcal{F} - s\mathcal{G}) < \infty.$$

Now, pick $s, \hat{s} \in \mathbb{R}^+$ such that $tK + s_\infty < \hat{s} < s$ and denote by μ_s the equilibrium measure of $t\mathcal{F} - s\mathcal{G}$. It follows

$$\begin{aligned}P(t\mathcal{F}-s\mathcal{G}) &= h(\mu_s)+\lim_{n\to\infty}\frac{1}{n}\int(t\log f_n - s\log g_n)\,d\mu\\ &= \left[h(\mu_s)+\lim_{n\to\infty}\frac{1}{n}\int(t\log f_n-\hat{s}\log g_n)\,d\mu_s\right]+(\hat{s}-s)\lim_{n\to\infty}\frac{1}{n}\int g_n d\mu_s\\ &\leq P(t\mathcal{F}-\hat{s}\mathcal{G})+(\hat{s}-s)I_{\mathcal{G}}(\mu_s)\leq P(t\mathcal{F}-\hat{s}\mathcal{G})+(\hat{s}-s)C.\end{aligned}$$

The last inequality follows from the fact that $\mathcal{G}$ is bounded away from zero. This shows that

$$\lim_{s\to\infty}P(t\mathcal{F}-s\mathcal{G})=-\infty. \tag{42}$$

In particular, $P(t\mathcal{F}-s\mathcal{G})\leq 0$ for sufficiently big s. Hence,

$$O(t)=\inf\{s\in\mathbb{R}:P(t\mathcal{F}-s\mathcal{G})\leq 0\}<\infty.$$

□

Remark 23. The proof of the preceding lemma also shows that there exists $K>0$ such that

$$\sup_{\nu\in\mathcal{M}_\sigma(-\mathcal{G})}\frac{I_{\mathcal{F}}(\nu)}{I_{\mathcal{G}}(\nu)}\leq K.$$

The following is the almost-additive version of Lemma 2. As before, it is a technical result that will be useful to prove Lemma 16 later on.

Lemma 15. *There exists a sequence of $\sigma-$invariant measures $\{\nu_N\}_{N\in\mathbb{N}}$ such that $\lim_{N\to\infty}I_{\mathcal{G}}(\nu_N)=\infty$ and $\liminf_{N\to\infty}\frac{h(\nu_N)}{I_{\mathcal{G}}(\nu_N)}\geq s_\infty$.*

Proof. Let $N\in\mathbb{N}$ and let s,s_2 be such that $s_\infty-\frac{1}{N}<s<s_\infty<s_2<s_\infty+\frac{1}{N}$. According to (40), we have

$$P(-s\mathcal{G})=\infty,\ P(-s_2\mathcal{G})<\infty.$$

For each $N\in\mathbb{N}$ the approximation property (see theorem 6) implies the existence of a $\sigma-$invariant compact set K_N such that

$$\begin{aligned}P_{K_N}(-s\mathcal{G}) &\geq N+P(-s_2\mathcal{G})\\ &\geq N+P_{K_N}(-s_2\mathcal{G})\end{aligned}\tag{43}$$

Since the map $s\mapsto P_{K_N}(-s\mathcal{G})$ is convex (see Remark 16), we have that this function is differentiable everywhere except for a countable set of values for s. Therefore, there exists $s_1\in(s_\infty-\frac{1}{N},s)$ such that the function $s\mapsto P_{K_N}(-s\mathcal{G})$ is differentiable in s_1. Acording to Remark 17, if $\tilde{\nu}_N$ is the equilibrium measure for $-s_1\mathcal{G}$, we have

$$\frac{d}{ds}P_{K_N}(-s\mathcal{G})\bigg|_{s=s_1}=-I_{\mathcal{G}}(\tilde{\nu}_N).$$

Now, from the monotonicity of the pressure and (43) we follow

$$P_{K_N}(-s_1\mathcal{G})\geq N+P_{K_N}(-s_2\mathcal{G}),$$

equivalently,

$$\frac{P_{K_N}(-s_1\mathcal{G})-P_{K_N}(-s_2\mathcal{G})}{s_1-s_2}\leq-\frac{N}{s_2-s_1}.$$

From this inequality and the convexity of the pressure function we obtain

$$-I_{\mathcal{G}}(\tilde{\nu}_N)=\frac{d}{ds}P_{K_N}(-s\mathcal{G})\bigg|_{s=s_1}$$

$$\begin{aligned}
&\le \frac{P_{K_N}(-s_1\mathcal{G}) - P_{K_N}(-s_2\mathcal{G})}{s_1 - s_2} \\
&\le -\frac{N}{s_2 - s_1} \le -\frac{N}{(s_\infty + \frac{1}{N}) - (s_\infty - \frac{1}{N})} \\
&= -\frac{N}{\frac{2}{N}} = -\frac{N^2}{2},
\end{aligned}$$

i.e.,

$$I_{\mathcal{G}}(\tilde{\nu}_N) \ge \frac{N^2}{2}.$$

As usual, we extend $\tilde{\nu}_N$ to a measure $\nu \in \sigma$ by defining $\nu(E) = \tilde{\nu}_N(E \cap K_N)$ for every measurable set E. This way

$$\lim_{N\to\infty} I_{\mathcal{G}}(\nu_N) = \infty.$$

We are now going to prove that $\liminf_{N\to\infty} \frac{h(\nu_N)}{I_{\mathcal{G}}(\nu_N)} \ge s_\infty$. Notice that the choice of s_1 depends on N. So, let us reset the notation to denote for each $N \in \mathbb{N}$, $s_N := s_1(N)$. We have $s_N \in (s_\infty - \frac{1}{N}, s_\infty)$. Therefore

$$\begin{aligned}
P_{K_N}(-(s_\infty + 1)\mathcal{G}) &< P_{K_N}(-s_N\mathcal{G}) \\
&= h(\tilde{\nu}_N) - s_N I_{\mathcal{G}}(\tilde{\nu}_N) \\
&= h(\nu_N) - s_N I_{\mathcal{G}}(\nu_N),
\end{aligned}$$

then

$$\frac{h(\nu_N)}{I_{\mathcal{G}}(\nu_N)} > s_N + \frac{P_{K_N}(-(s_\infty + 1)\mathcal{G})}{I_{\mathcal{G}}(\nu_N)}.$$

Since $s_N \in (s_\infty - \frac{1}{N}, s_\infty)$ and $I_{\mathcal{G}}(\nu_N) \to \infty$, by letting $N \to \infty$, we obtain

$$\liminf_{N\to\infty} \frac{h(\nu_N)}{I_{\mathcal{G}}(\nu_N)} \ge s_\infty.$$

□

The following result links the definition of s_∞ and the relation between $\mathcal{F}$ and $\mathcal{G}$ (namely, equation (38)) to establish a half-plane of points in $(t, s) \in \mathbb{R}^2$ where the pressure $P(t\mathcal{F} - s\mathcal{G})$ is finite.

LEMMA 16. *For every $t \in \mathbb{R}^+$ we have*

$$P(t\mathcal{F} - s\mathcal{G}) = \begin{cases} \text{finite}, & \text{if } s < s_\infty \\ \infty, & \text{if } s > s_\infty \end{cases}.$$

PROOF. We begin the proof with the following claim: Every sequence $\{\mu_N\}_{N\in\mathbb{N}}$ in $\mathcal{M}_\sigma(-\mathcal{G})$ such that $\lim_{N\to\infty} I_{\mathcal{G}}(\mu_N) = \infty$ satisfies the conditions:

a) $\limsup_{N\to\infty} \frac{h(\mu_N)}{I_{\mathcal{G}}(\mu_N)} \le s_\infty$, and
b) $\lim_{N\to\infty} \frac{I_{\mathcal{F}}(\mu_N)}{I_{\mathcal{G}}(\mu_N)} = 0$.

Condition a) is easy to see: Let $\hat{s} > s_\infty$. From the variational principle

$$h(\mu_N) - \hat{s} I_{\mathcal{G}}(\mu_N) \le P(-\hat{s}\mathcal{G}) < \infty.$$

Then

$$\frac{h(\mu_N)}{I_{\mathcal{G}}(\mu_N)} \leq \hat{s} + \frac{P(-\hat{s}\mathcal{G})}{I_{\mathcal{G}}(\mu_N)}.$$

By letting $N \to \infty$ and recalling that $\hat{s} > s_\infty$ is arbitrary, we obtain

$$\limsup_{N\to\infty} \frac{h(\mu_N)}{I_{\mathcal{G}}(\mu_N)} \leq s_\infty. \tag{44}$$

To prove condition b), let $\varepsilon > 0$. From (38), there exists $J \in \mathbb{N}$ such that for every $x \in \bigcup_{j>J}[j]$ we have

$$\frac{\log f_1(x)}{\log g_1(x)} < \varepsilon.$$

Define the sets $A = \bigcup_{j\leq J}[j], B = \bigcup_{j>J}[j]$. Recalling (35), it can be seen that for each $N \in \mathbb{N}$ we have

$$\begin{aligned}
\frac{I_{\mathcal{F}}(\mu_N)}{I_{\mathcal{G}}(\mu_N)} &\leq \frac{\int \log f_1 d\mu_N + C_{\mathcal{F}}}{I_{\mathcal{G}}(\mu_N)} \\
&= \frac{\int_A \log f_1 d\mu_N}{I_{\mathcal{G}}(\mu_N)} + \frac{\int_B \log f_1 d\mu_N}{I_{\mathcal{G}}(\mu_N)} + \frac{C_{\mathcal{F}}}{I_{\mathcal{G}}(\mu_N)} \\
&\leq \frac{\sup \log f_1|_A}{I_{\mathcal{G}}(\mu_N)} + \varepsilon\frac{\int_B \log g_1 d\mu_N}{I_{\mathcal{G}}(\mu_N)} + \frac{C_{\mathcal{F}}}{I_{\mathcal{G}}(\mu_N)} \\
&= \frac{\sup \log f_1|_A}{I_{\mathcal{G}}(\mu_N)} + \varepsilon\frac{\int_B \log g_1 d\mu_N - C_{\mathcal{G}} + C_{\mathcal{G}}}{I_{\mathcal{G}}(\mu_N)} + \frac{C_{\mathcal{F}}}{I_{\mathcal{G}}(\mu_N)} \\
&\leq \frac{\sup \log f_1|_A}{I_{\mathcal{G}}(\mu_N)} + \varepsilon\frac{I_{\mathcal{G}}(\mu_N) + C_{\mathcal{G}}}{I_{\mathcal{G}}(\mu_N)} + \frac{C_{\mathcal{F}}}{I_{\mathcal{G}}(\mu_N)}.
\end{aligned}$$

By letting $N \to \infty$ we obtain

$$\limsup_{N\to\infty} \frac{I_{\mathcal{F}}(\mu_N)}{I_{\mathcal{G}}(\mu_N)} < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and $\mathcal{F}, \mathcal{G}$ consist only of positive terms, it follows

$$\lim_{N\to\infty} \frac{I_{\mathcal{F}}(\mu_N)}{I_{\mathcal{G}}(\mu_N)} = 0,$$

which proves the claim.

Let us now fix $t > 0$ and $s < s_\infty$. Set $\{\nu_N\}_{N\in\mathbb{N}}$ with the properties from Lemma 15. These, along with equation (44) yield

$$\lim_{N\to\infty} \frac{h(\nu_N)}{I_{\mathcal{G}}(\mu_N)} = s_\infty. \tag{45}$$

Now, for every $N \in \mathbb{N}$ we have

$$\begin{aligned}
P(t\mathcal{F} - s\mathcal{G}) &\geq h(\nu_N) + tI_{\mathcal{F}}(\nu_N) - sI_{\mathcal{G}}(\nu_N) \\
&= I_{\mathcal{G}}(\nu_N)\left(\frac{h(\nu_N)}{I_{\mathcal{G}}(\nu_N)} + t\frac{I_{\mathcal{F}}(\nu_N)}{I_{\mathcal{G}}(\nu_N)} - s\right).
\end{aligned} \tag{46}$$

From (45), and the properties of the sequence ν_N we have that if $N \to \infty$ $I_{\mathcal{G}}(\nu_N) \to \infty$ and

$$\frac{h(\nu_N)}{I_{\mathcal{G}}(\nu_N)} + t\frac{I_{\mathcal{F}}(\nu_N)}{I_{\mathcal{G}}(\nu_N)} - s \to s_\infty - s > 0.$$

Therefore, the quantity at the last equality from (46) is arbitrarily large. Therefore $P(t\mathcal{F} - s\mathcal{G}) = \infty$ whenever $s < s_\infty$.

Otherwise, if $s > s_\infty$ we suppose in order to get a contradiction, that $P(t\mathcal{F} - s\mathcal{G}) = \infty$. Therefore, we can set a sequence $\{\mu_N\}_{N\in\mathbb{N}}$ in $\mathcal{M}_\sigma(-\mathcal{G})$ verifying

$$\lim_{N\to\infty}(h(\mu_N) + tI_{\mathcal{F}}(\mu_N) - sI_{\mathcal{G}}(\mu_N)) = \infty. \tag{47}$$

This means that either $\lim\limits_{N\to\infty} h(\mu_N) = \infty$ or $\lim\limits_{N\to\infty} I_{\mathcal{F}}(\mu_N) = \infty$.

If $h(\mu_N) \to \infty$, since $\infty > P(-s\mathcal{G}) \geq h(\mu_N) - sI_{\mathcal{G}}(\mu_N)$ it follows $I_{\mathcal{G}}(\mu_N) \to \infty$. On the other hand, if $I_{\mathcal{F}}(\mu_N) \to \infty$, part b) from the claim yields $I_{\mathcal{G}}(\mu_N) \to \infty$.

For sufficiently big values of N we get from (47)

$$h(\mu_N) + tI_{\mathcal{F}}(\mu_N) - sI_{\mathcal{G}}(\mu_N) > 0.$$

Then

$$\frac{h(\mu_N)}{I_{\mathcal{G}}(\mu_N)} > s - t\frac{I_{\mathcal{F}}(\mu_N)}{I_{\mathcal{G}}(\mu_N)}.$$

Now, by letting $N \to \infty$:

$$\limsup_{N\to\infty} \frac{h(\mu_N)}{I_{\mathcal{G}}(\mu_N)} \geq s > s_\infty.$$

Since $I_{\mathcal{G}}(\mu_N) \to \infty$, this is a contradiction with part a) from the claim. Hence, $P(t\mathcal{F} - s\mathcal{G}) < \infty$ when $s > s_\infty$. □

Lemma 17. *For every $s > s_\infty$, there exists $t^*(s) \in \mathbb{R}^+$ which satisfies $P(t\mathcal{F} - s\mathcal{G}) > 0$ for every $t \geq t^*$.*

Proof. The proof of this result follows the same idea from Lemma 4. Set a measure $\nu \in \mathcal{M}_\sigma(-\mathcal{G})$ and $s > s_\infty$. We have $I_{\mathcal{G}}(\nu) < \infty$ and, from Remarks 19 and 21, we have that $0 < I_{\mathcal{F}}(\nu) < \infty$. Therefore, there is t^* such that for every $t \geq t^*$ we have

$$tI_{\mathcal{F}}(\nu) > sI_{\mathcal{G}}(\nu) - h(\nu).$$

Therefore,

$$h(\nu) + tI_{\mathcal{F}}(\nu) - sI_{\mathcal{G}}(\nu) > 0.$$

By taking supremum over $\nu \in \mathcal{M}_\sigma(-\mathcal{G})$ we obtain

$$P(t\mathcal{F} - s\mathcal{G}) > 0$$

for every $t \geq t^*$. □

When developing the properties of the zero-pressure map $O(t)$ in the additive setting, one of them was its real-analiticity. Since this fact came from the real-analiticity of the pressure function, this will no longer be true for almost-additive potentials. However, the other properties that were deduced from the proof of Theorem 4 remain valid.

Lemma 18. *For sufficiently big values of t, $P(t\mathcal{F} - O(t)\mathcal{G}) = 0$.*

Proof. Pick $s_+ > s_\infty$ and $t^* = t^*(s_+)$ as in Lemma 17. Define, for every $t \geq t^*$, and $s \geq s_+$: $\phi_t(s) = P(t\mathcal{F} - s\mathcal{G})$. By construction, we have $\phi_t(s_+) > 0$ and, since $\mathcal{G}$ is bounded away from zero, we have from (42) that $\lim\limits_{s\to\infty} \phi_t(s) = -\infty$. Also, notice that

$$\phi_t(s) = \sup_{\nu\in\mathcal{M}_\sigma(-\mathcal{G})} (h(\nu) + tI_{\mathcal{F}}(\nu) - sI_{\mathcal{G}}(\nu))$$

is the supreme of affine functions (depending on the variable s), therefore, $\phi_t(s)$ is a convex function, whence we conclude that $\phi_t(s)$ is continuous. An application of the intermediate value theorem establishes the existence of $s_0 > s_+$ such that

$$\varphi_t(s) = P(t\mathcal{F} - s_0\mathcal{G}) = 0.$$

Finally, note that $\varphi_t(s)$ is a decreasing function. Hence, $O(t) = s_0$, proving the Lemma. $\square$

COROLLARY 5. *If t is such that $P(t\mathcal{F} - O(t)\mathcal{G}) = 0$, then $O(t) > s_\infty$.*

As we have stated, we do not prove that $O(t)$ is real-analytic. However, we are able to establish the existence and a explicit description of a derivative for $O(t)$ in every point, except for a countable set of values of t.

To achieve this goal, we first prove the convexity of the zero-pressure map. The following result is the almost-additive version of Lemma 7. Its proof follows the same argument, substituting the integrals of f, g for the operators $I_\mathcal{F}, I_\mathcal{G}$.

LEMMA 19. *Set $s_+ > s_\infty$ and $t^* = t^*(s_+)$ as in Lemma 17. Then, $O(t)$ is a convex function in $[t^*, \infty)$.*

PROOF. As in the additive case, we will prove that $O(t)$ is the maximum of a set of affine functions, whence convexity is an immediate consequence. Specifically, we will prove that

$$O(t) = \max_{\nu \in \mathcal{M}_\sigma(-\mathcal{G})} \left\{ \frac{h(\nu)}{I_\mathcal{G}(\nu)} + t\frac{I_\mathcal{F}(\nu)}{I_\mathcal{G}(\nu)} \right\}. \tag{48}$$

To prove the claim, let $\nu \in \mathcal{M}_\sigma(-\mathcal{G})$, from Theorem 7 and Lemma 18, it follows, for every $t \geq t^*$

$$0 = P(t\mathcal{F} - O(t)\mathcal{G}) \geq h(\nu) + tI_\mathcal{F}(\nu) - O(t)I_\mathcal{G}(\nu)$$

and the equality is attained for $\nu = \mu_t$, where μ_t is the equilibrium measure of $t\mathcal{F} - O(t)\mathcal{G}$. Therefore, for every $\nu \in \mathcal{M}_\sigma(-\mathcal{G})$ we have

$$O(t) \geq \frac{h(\nu)}{I_\mathcal{G}(\nu)} + t\frac{I_\mathcal{F}(\nu)}{I_\mathcal{G}(\nu)}, \tag{49}$$

and for $\nu = \mu_t$ it follows

$$O(t) = \frac{h(\mu_t)}{I_\mathcal{G}(\mu_t)} + t\frac{I_\mathcal{F}(\mu_t)}{I_\mathcal{G}(\mu_t)}. \tag{50}$$

Equations (49) and (50) imply (48). Therefore, the zero pressure map is convex. $\square$

COROLLARY 6. *The zero pressure map $O(t)$ is differentiable for every $t \in (t^*, \infty)$, except, maybe, for a countable subset of (t^*, ∞).*

An explicit derivative for $O(t)$ can be described at every point where $O'(t)$ exists. Notice that even though the derivative of the zero-pressure map doesn't necessarily exist for every $t > t^*$, the proof of the following lemma follows the same idea from Lemma 6.

LEMMA 20. *For any $t > t^*$ such that $O'(t)$ exists, let μ_t be the Gibbs-equilibrium measure for the potential $t\mathcal{F} - O(t)\mathcal{G}$. Then, the equality $O'(t) = \frac{I_\mathcal{F}(\mu_t)}{I_\mathcal{G}(\mu_t)}$ holds.*

Proof. Let $\varepsilon > 0$ be arbitrary. From Lemma 18, since μ_t is an equilibrium measure for $t\mathcal{F} - O(t)\mathcal{G}$, we have

$$\begin{aligned} 0 &= P(t\mathcal{F} - O(t)\mathcal{G}) \\ &= h(\mu_t) + tI_{\mathcal{F}}(\mu_t) - O(t)I_{\mathcal{G}}(\mu_t) \end{aligned} \tag{51}$$

and

$$\begin{aligned} 0 &= P\left((t+\varepsilon)\mathcal{F} - O(t+\varepsilon)\mathcal{G}\right) \\ &\geq h(\mu_t) + (t+\varepsilon)I_{\mathcal{F}}(\mu_t) - O(t+\varepsilon)I_{\mathcal{G}}(\mu_t). \end{aligned} \tag{52}$$

From these two relations, we obtain

$$0 \geq I_{\mathcal{F}}(\mu_t) - \frac{O(t+\varepsilon) - O(t)}{\varepsilon} I_{\mathcal{G}}(\mu_t).$$

Since $O(t)$ is a convex function, it follows that the lateral derivatives of $O(t)$ exist for every $t \in (t^*, \infty)$. Denote by O'_+ and O'_- respectively the right and the left derivative for the zero pressure map. Now, letting $\varepsilon \to 0^+$:

$$I_{\mathcal{F}}(\mu_t) \leq O'_+(t) I_{\mathcal{G}}(\mu_t).$$

Similarly, if we pick $\varepsilon < 0$, equations (51) and (52) yield

$$0 \geq \varepsilon I_{\mathcal{F}}(\mu_t) - [O(t+\varepsilon) - O(t)] I_{\mathcal{G}}(\mu_t),$$

and since $\varepsilon < 0$, it follows

$$0 \leq I_{\mathcal{F}}(\mu_t) - \frac{O(t+\varepsilon) - O(t)}{\varepsilon} I_{\mathcal{G}}(\mu_t).$$

Now, letting $\varepsilon \to 0^-$ we obtain

$$O'_-(t) I_{\mathcal{G}}(\mu_t) \leq I_{\mathcal{F}}(\mu_t).$$

Recall that $O'(t)$ exists by hypothesis, therefore $O'_+(t) = O'_-(t) = O'(t)$. Therefore

$$O'(t) I_{\mathcal{G}}(\mu_t) = I_{\mathcal{F}}(\mu_t),$$

equivalently,

$$O'(t) = \frac{I_{\mathcal{F}}(\mu_t)}{I_{\mathcal{G}}(\mu_t)}.$$

□

We now intend to compare the growth rate of $O(t)$ with respect to t as it was done in Lemma 9 in the previous chapter. As before, we start by stating the almost-additive analogous version of Lemma 8.

Lemma 21. *The equilibrium states $(\mu_t)_{t \geq t^*}$ for $t\mathcal{F} - O(t)\mathcal{G}$ satisfy that the sequence*

$$\left(\frac{h(\mu_t)}{I_{\mathcal{G}}(\mu_t)} \right)_{t \geq t^*}$$

is decreasing.

Proof. Recall that μ_t is an equilibrium measure for $t\mathcal{F} - O(t)\mathcal{G}$ to obtain

$$0 = P(t\mathcal{F} - O(t)\mathcal{G}) = h(\mu_t) + tI_{\mathcal{F}}(\mu_t) - O(t)I_{\mathcal{G}}(\mu_t).$$

Besides, from the variational principle, it follows that for any $\nu \in \mathcal{M}(-\mathcal{G})$:

$$0 \geq h(\nu) + tI_{\mathcal{F}}(\nu) - O(t)I_{\mathcal{G}}(\nu).$$

By computing $O(t)$ in both relations, we obtain

$$\frac{h(\mu_t)}{I_{\mathcal{G}}(\mu_t)} + t\frac{I_{\mathcal{F}}(\mu_t)}{I_{\mathcal{G}}(\mu_t)} \geq \frac{h(\nu)}{I_{\mathcal{G}}(\nu)} + t\frac{I_{\mathcal{F}}(\nu)}{I_{\mathcal{G}}(\nu)}. \tag{53}$$

Now, fix $t_2 > t_1 > t^*$ and define the affine functions

$$\ell_1(t) := \frac{h(\mu_{t_1})}{I_{\mathcal{G}}(\mu_{t_1})} + t\frac{I_{\mathcal{F}}(\mu_{t_1})}{I_{\mathcal{G}}(\mu_{t_1})}$$

and

$$\ell_2(t) := \frac{h(\mu_{t_2})}{I_{\mathcal{G}}(\mu_{t_2})} + t\frac{I_{\mathcal{F}}(\mu_{t_2})}{I_{\mathcal{G}}(\mu_{t_2})}.$$

Set $\ell := \ell_2 - \ell_1$ and notice that ℓ is also an affine function.

From (53), it follows $\ell_1(t_1) \geq \ell_2(t_1)$ and $\ell_2(t_2) \geq \ell_1(t_2)$, i.e. $\ell(t_1) \leq 0$ and $\ell(t_2) \geq 0$. Since ℓ is an affine function, this implies that ℓ is an increasing function. Therefore $\ell(0) \leq 0$, from where it results that $\ell_2(0) \leq \ell_1(0)$, i.e.,

$$\frac{h(\mu_{t_2})}{I_{\mathcal{G}}(\mu_{t_2})} \leq \frac{h(\mu_{t_1})}{I_{\mathcal{G}}(\mu_{t_1})}.$$

□

COROLLARY 7. *For each $t > t^*$, let μ_t be an equilibrium measure for the potential $t\mathcal{F} - O(t)\mathcal{G}$. Then the limit*

$$\lim_{t\to\infty} \frac{h(\mu_t)}{I_{\mathcal{G}}(\mu_t)}$$

exists in $\mathbb{R}$.

In the additive setting, the zero-pressure map was differentiable and convex, therefore its derivative was increasing with respect to t. This monotonicity was recalled in order to set lower bound for the quotients of the spatial means of the potentials. The following lemma achieves the same conclusion without needing $O(t)$ to be differentiable.

LEMMA 22. *The map $t \mapsto \frac{I_{\mathcal{F}}(\mu_t)}{I_{\mathcal{G}}(\mu_t)}$ is increasing in $[t^*, \infty)$.*

PROOF. Let $t^* \leq t_1 \leq t_2$. Recall (50) to obtain

$$\begin{aligned} O(t_2) &= \frac{h(\mu_{t_2})}{I_{\mathcal{G}}(\mu_{t_2})} + t_2\frac{I_{\mathcal{F}}(\mu_{t_2})}{I_{\mathcal{G}}(\mu_{t_2})} \\ &= \sup_{\nu\in\mathcal{M}_\sigma(-\mathcal{G})} \left(\frac{h(\nu)}{I_{\mathcal{G}}(\nu)} + t_2\frac{I_{\mathcal{F}}(\nu)}{I_{\mathcal{G}}(\nu)}\right) \\ &\geq \frac{h(\mu_{t_1})}{I_{\mathcal{G}}(\mu_{t_1})} + t_2\frac{I_{\mathcal{F}}(\mu_{t_1})}{I_{\mathcal{G}}(\mu_{t_1})}. \end{aligned} \tag{54}$$

On the other hand, from lemma 21 we have

$$\frac{h(\mu_{t_2})}{I_{\mathcal{G}}(\mu_{t_2})} \leq \frac{h(\mu_{t_1})}{I_{\mathcal{G}}(\mu_{t_1})},$$

i.e.,

$$-\frac{h(\mu_{t_1})}{I_{\mathcal{G}}(\mu_{t_1})} \leq -\frac{h(\mu_{t_2})}{I_{\mathcal{G}}(\mu_{t_2})}. \tag{55}$$

By substracting (54) and (55) it follows

$$t_2\frac{I_{\mathcal{F}}(\mu_{t_1})}{I_{\mathcal{G}}(\mu_{t_1})} \leq t_2\frac{I_{\mathcal{F}}(\mu_{t_2})}{I_{\mathcal{G}}(\mu_{t_2})}.$$

Hence

$$\frac{I_{\mathcal{F}}(\mu_{t_1})}{I_{\mathcal{G}}(\mu_{t_1})} \leq \frac{I_{\mathcal{F}}(\mu_{t_2})}{I_{\mathcal{G}}(\mu_{t_2})}.$$

□

Let us now prove that the growth rate of the zero pressure map $O(t)$ can be compared by scalar multiples of t.

LEMMA 23. *There exists $\alpha \in \mathbb{R}^+$ such that $O(t) \leq \alpha t$ for each $t \geq t^*$.*

PROOF. Let μ_t be an equilibrium measure for $t\mathcal{F} - O(t)\mathcal{G}$. From Lemma 18, we obtain

$$h(\mu_t) + tI_{\mathcal{F}}(\mu_t) - O(t)I_{\mathcal{G}}(\mu_t) = 0,$$

Whence

$$\frac{O(t)}{t} = \frac{h(\mu_t)}{tI_{\mathcal{G}}(\mu_t)} + \frac{I_{\mathcal{F}}(\mu_t)}{I_{\mathcal{G}}(\mu_t)}.$$

Since $\frac{1}{t}$ is a decreasing function with respect to t, from Lemma 21 we obtain $\frac{h(\mu_t)}{tI_{\mathcal{G}}(\mu_t)} \leq \frac{h(\mu_{t^*})}{tI_{\mathcal{G}}(\mu_{t^*})}$. Therefore, by choosing K as in Remark 20, we have

$$\frac{O(t)}{t} \leq \frac{h(\mu_{t^*})}{t^*I_{\mathcal{G}}(\mu_t^*)} + K.$$

Therefore, by setting $\alpha := \frac{h(\mu_{t^*})}{t^*I_{\mathcal{G}}(\mu_{t^*})} + K$, the inequality $O(t) \leq \alpha t$ is obtained. □

LEMMA 24. *There exists $\gamma > 0$ such that $t \leq \gamma O(t)$ for each $t \geq t^*$.*

PROOF. Let $t \geq t^*$ and set μ_t the equilibrium measure of $t\mathcal{F} - O(t)\mathcal{G}$. Let us apply Lemma 18 once more to obtain

$$h(\mu_t) + tI_{\mathcal{F}}(\mu_t) - O(t)I_{\mathcal{G}}(\mu_t) = 0,$$

and therefore, applying lemma 22 we obtain

$$O(t) = \frac{h(\mu_t)}{I_{\mathcal{G}}(\mu_t)} + t\frac{I_{\mathcal{F}}(\mu_t)}{I_{\mathcal{G}}(\mu_t)} \geq \frac{tI_{\mathcal{F}}(\mu_t)}{I_{\mathcal{G}}(\mu_t)} \geq \frac{tI_{\mathcal{F}}(\mu_{t^*})}{I_{\mathcal{G}}(\mu_{t^*})}.$$

Set $\gamma := \frac{I_{\mathcal{G}}(\mu_{t^*})}{I_{\mathcal{F}}(\mu_{t^*})}$ to obtain $O(t) \geq t\gamma^{-1}$, or equivalently, $t \leq \gamma O(t)$. □

3. Limits for Quotients of Almost additive sequences

Set $t \in \mathbb{R}^+$ and $\mathcal{F}, \mathcal{G}$ be almost-additive sequences. Emulating the situation from the additive setting, it would be desirable that, under aproppriate hypotheses the accumulation points of the equilibrium measures for $t\mathcal{F} - O(t)\mathcal{G}$ were $(\mathcal{F}, \mathcal{G})$-maximizing. The purpose of this section is to establish a formal statement of this fact and develop the arguments to prove it. The first step is to adapt the notion of maximizing measures to the almost-additive setting.

DEFINITION 20. *Let (Σ_A, σ) be a Countable Markov Shift. Let $\mathcal{F}, \mathcal{G}$ be positive almost-additive sequences such that $\mathcal{G}$ is bounded away from zero. A probability measure $\mu \in \mathcal{M}_\sigma$ is called an $(\mathcal{F}, \mathcal{G})$-maximizing measure if the equality*

$$\sup_{\nu \in \mathcal{M}_\sigma} \frac{I_{\mathcal{F}}(\nu)}{I_{\mathcal{G}}(\nu)} = \frac{I_{\mathcal{F}}(\mu)}{I_{\mathcal{G}}(\mu)}$$

holds.

The purpose of this section is to prove the following result.

THEOREM 9. *Let (Σ_A, σ) be a topologically mixing countable Markov shift satisfying the BIP property. Suppose that $\mathcal{F}, \mathcal{G}$ are Bowen almost-additive sequences of positive functions satisfying (38) and that $\mathcal{G}$ is bounded away from zero. Assume that there exists s_∞ satisfying* (40) *and that* (37) *holds for $-s\mathcal{G}$ whenever $s \in (s_\infty, \infty)$. Set t^* such that $P(t\mathcal{F} - O(t)\mathcal{G}) = 0$ for every $t > t^*$ and denote by μ_t the Gibbs-equilibrium measure for $t\mathcal{F} - O(t)\mathcal{G}$. Then, the family $(\mu_t)_{t>t^*}$ has an accumulation point μ in the weak-* topology when $t \to \infty$. Moreover, if μ is the accumulation point of a sequence μ_{t_k} such that $O'(t_k)$ exists for every $k \in \mathbb{N}$, then μ is an $(\mathcal{F}, \mathcal{G})$-maximizing measure.*

REMARK 24. It was proven in [**C**] that for every sequence of functions $\{\log f_n\}_{n=1}^{\infty}$ satisfying

$$\inf_{f \in C(X)} \limsup_{n \to \infty} \frac{1}{n} ||f_n - S_n f||_\infty = 0,$$

there exists $f \in C(X)$ achieving the infimum, i.e. $\limsup\limits_{n \to \infty} \frac{1}{n} ||f_n - S_n f||_\infty = 0$. Almost-additive potentials satisfy that condition and this implies that they can be written as the Birkhoff sum of an additive potential and a sublinear error term. Especifically, for $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ an almost-additive sequence in (Σ_A, σ), there exist $f : \Sigma_A \to \mathbb{R}$ and $\{u_n\}_{n \in \mathbb{N}}$ such that $\lim\limits_{n \to \infty} \frac{\sup |u_n|}{n} = 0$ and

$$\log f_n = \sum_{j=0}^{n-1} f \circ \sigma + u_n. \tag{56}$$

If we set $\mathcal{F}$ an almost-additive sequence and f a potential in Σ_A satisfying these conditions, we have that f and $\mathcal{F}$ share the same value for their respective Gurevich pressure, i.e.,

$$P(\mathcal{F}) = P(f).$$

Moreover, if we denote by μ the equilibrium measure for $\mathcal{F}$, we have

$$h(\mu) + I_{\mathcal{F}}(\mu) = h(\mu) + \int f d\mu.$$

This means that both f and $\mathcal{F}$ share the same Gurevich pressure and equilibrium measures. We even have that every $\sigma-$invariant measure ν in Σ_A satisfies $I_{\mathcal{F}}(\nu) = \int f d\nu$. This suggest that the main result from the previous section can be easily adapted to obtain an analogous result for quotients of almost-additive sequences of function by applying these results. Nevertheless, there are some features and regularity properties of f that have not been proven to be inherited from $\mathcal{F}$. For instance, Gibbs measures of $\mathcal{F}$ are not (in general) Gibbs measures for the potential f and if $\mathcal{F}$ is a Bowen sequence, the existence of a potential f satisfying equation (56) whilst $\{\sum_{j=0}^{n-1} f \circ \sigma^j\}_{n \in \mathbb{N}}$ is also a Bowen sequence is posed as an open question.

In chapter 2 we let f, g be locally Hölder potentials. This condition is not ensured to hold when we obtain an additive potential f from a Bowen almost-additive sequence $\mathcal{F}$. Since we claimed the potentials f, g to be Locally Hölder and the equilibrium measures of f, g to be Gibbs measures, our approach consists in dealing directly with the almost additive sequences $\mathcal{F}, \mathcal{G}$ instead of attempting to relate almost additive sequences with continuous potentials by means of the work from [**C**].

As in the additive potential setting, the following lemma, along with Prohorov's Theorem, suffices to prove the existence of the accumulation points for the equilibrium measures μ_{t_k} stated in Theorem 9. Nevertheless, the proof of the tightness of the family of equilibrium measures for additive potentials relied on remark 11, which related the constants of the Gibbs measures of a potential f with their variation $V(f)$. Since the almost additive setting is framed in terms of Bowen sequences instead of potentials with summable variations, the arguments are slightly different.

Recall that since μ_t is a Gibbs measure for $t\mathcal{F} - O(t)\mathcal{G}$ there exists $K_0(t)$ such that

$$\mu_t[i] \leq K_0(t) \exp(-P(t\mathcal{F} - O(t)\mathcal{G}) \sup \left(\left. \frac{f_1^t}{g_1^{O(t)}} \right|_{[i]} \right).$$

Since (Σ_A, σ) is a topologically mixing countable Markov shift which also satisfies the BIP property, we have that (Σ_A, σ) is finitely primitive (see Definition 8). Therefore, there exists $k \in \mathbb{N}$ and $W \subset \mathbb{N}^k$ a finite set such that for every $a, b \in \mathbb{N}$, there exists $w \in W$ such that the cylinder $[a, w_1, \cdots, w_k, b]$ is non empty. Now, we cite [**IY2**, Lemma 4.2] to set the following relation.

LEMMA 25. *Let (Σ, σ) be a countable Markov shift satisfying the BIP property. Set k a natural number and $W \subset \mathbb{N}^k$ a finite family satisfying that for every a, b there exists $w \in W$ such that the cylinder $[a, w_1, \cdots, w_k, b]$ is non-empty. Set $\mathcal{F} = \log\{f_n\}_{n=1}^{\infty}$ an almost additive Bowen sequence satisfying* (37)*. If we define*

$$N_{\mathcal{F}} = \min_{w \in W} \sup_{z \in [w_1, \cdots, w_k]} f_n(z),$$

then for every $t \geq 1$, the Gibbs-equilibrium states for $t\mathcal{F}$, $\mu_{t\mathcal{F}}$ satisfy for every cylinder set $[i_1, \cdots, i_n]$ and $x \in [i_1, \cdots, i_n]$:

$$\frac{\mu_{t\mathcal{F}}[i_1, \cdots, i_n]}{e^{-nP(t\mathcal{F})} f_n^t(x)} \leq \left(\frac{M_{\mathcal{F}} e^{6C_{\mathcal{F}}}}{D^5} \right)^t,$$

where $M_{\mathcal{F}}$ is defined as in Definition 15, $C_{\mathcal{F}}$ is defined as in Definition 14, and

$$D := \frac{N_{\mathcal{F}} e^{-3C_{\mathcal{F}}}}{M_{\mathcal{F}}^3 e^{(k-1)C_{\mathcal{F}}} \max\{\sum_{i \in \mathbb{N}} \sup f_1|_{[i]}, (\sum_{i \in \mathbb{N}} \sup f_1|_{[i]})^k\}}.$$

REMARK 25. Lemma 25 shows that the constant K_0 (see equation (36)) for the Gibbs measure of $t\mathcal{F}$ can be chosen as

$$K_0 = \left(\frac{M_{\mathcal{F}} e^{6C_{\mathcal{F}}}}{D^5} \right)^t.$$

Now, we apply Lemma 25 to state a result which allows us to consider $K(t)$ without regarding the dependence on t.

LEMMA 26. *For any $t \geq t^*$ Set $\mathcal{H} = t\mathcal{F} - O(t)\mathcal{G}$ and set $h_n = \frac{f_n^t}{g_n^{(O(t))}}$. Let μ_t be the Gibbs measure for $\mathcal{H}$ and $K_0(t)$ the associated constant. There exists $\hat{K}_0 > 0$ which does not depend on t satisfying the relation $K_0(t) \leq \hat{K}_0^t$.*

PROOF. Let us first apply 25 to $\mathcal{H} = 1 \cdot \mathcal{H}$. Therefore it follows from Remark 25 that

$$K_0(t) \leq \frac{M_{\mathcal{H}} e^{6C_{\mathcal{H}}}}{D^5}, \tag{57}$$

where

$$D := \frac{N_{\mathcal{H}} e^{-3C_{\mathcal{H}}}}{M_{\mathcal{H}}^3 e^{(k-1)C_{\mathcal{H}}} \max\{\sum_{i\in\mathbb{N}} \sup h_1|_{[i]}, (\sum_{i\in\mathbb{N}} \sup h_1|_{[i]})^k\}}.$$

Now, Notice that the constants involved in this relations can be bounded as follows:

Without loss of generality, we can assume $M_{\mathcal{F}}, M_{\mathcal{G}} > 1$. From Lemmas 13 and 23, we have

$$M_{\mathcal{H}} = M_{\mathcal{F}}^t M_{\mathcal{G}}^{O(t)} \leq (M_{\mathcal{F}} M_{\mathcal{G}}^{\alpha})^t.$$

Similarly

$$C_{\mathcal{H}} = tC_{\mathcal{F}} + O(t)C_{\mathcal{G}} \leq t(C_{\mathcal{F}} + \alpha C_{\mathcal{G}}).$$

On the other hand, notice that since for every n we have $\log g_n > 0$ it follows $g_n > 1$. Therefore,

$$N_{\mathcal{H}} = \min_{w\in W} \sup_{z\in[w_1,\cdots,w_k]} \frac{f_k^t(z)}{g_k^{O(t)} z} \geq \min_{w\in W} \sup_{z\in[w_1,\cdots,w_k]} f_k^t(z) = N_{\mathcal{F}}^t.$$

Now, define $\hat{K}_1 := \sum_{i\in\mathbb{N}} \sup_{[i]} h_1$ and notice that $f_1 > 1$ since $\log f_1 > 0$. Therefore, Lemma 23 implies

$$\begin{aligned}\hat{K}_1 =&\leq \sum_{i\in\mathbb{N}} \sup_{[i]} \frac{f_1^{\alpha O(t)}}{g_1^{O(t)}} = \sum_{i\in\mathbb{N}} \sup_{[i]} \left(\frac{f_1^{\alpha}}{g_1}\right)^{O(t)} = \sum_{i\in\mathbb{N}} \left(\sup_{[i]} \left(\frac{f_1^{\alpha}}{g_1}\right)^{O(t^*)}\right)^{\frac{O(t)}{O(t^*)}} \\ &\leq \left(\sum_{i\in\mathbb{N}} \sup_{[i]} \left(\frac{f_1^{\alpha}}{g_1}\right)^{O(t^*)}\right)^{\frac{O(t)}{O(t^*)}}.\end{aligned}$$

The convergence of the last sum is equivalent to the condition $P(\alpha O(t^*)\mathcal{F} - O(t^\star)\mathcal{G}) < \infty$, which follows from Lemma 16 and Corollary 5. Now, if we define

$$\beta := \begin{cases} \alpha, & \text{if } \left(\sum_{i\in\mathbb{N}} \sup_{[i]} \left(\frac{f_1^{\alpha}}{g_1}\right)^{O(t^*)}\right) \geq 1 \\ \gamma^{-1}, & \text{if } \left(\sum_{i\in\mathbb{N}} \sup_{[i]} \left(\frac{f_1^{\alpha}}{g_1}\right)^{O(t^*)}\right) < 1 \end{cases},$$

then Lemmas 23 and 24 imply

$$\hat{K}_1 \leq \left(\sum_{i\in\mathbb{N}} \sup_{[i]} \left(\frac{f_1^{\alpha}}{g_1}\right)^{O(t^*)}\right)^{\frac{\beta t}{O(t^*)}}.$$

Now, by denoting

$$\hat{K}_1' := \sum_{i\in\mathbb{N}} \sup_{[i]} \left(\frac{f_1^\alpha}{g_1}\right)^{O(t^*)},$$

we obtain

$$\hat{K}_1 \le (\hat{K}_1')^t.$$

Now, we apply these relations to inequality (57) to obtain

$$\begin{aligned}
K_0(t) \le \left(\frac{M_{\mathcal{H}}e^{6C_{\mathcal{H}}}}{D^5}\right) &= \frac{M_{\mathcal{H}}e^{6C_{\mathcal{H}}}M_{\mathcal{H}}^{15}e^{5(k-1)C_{\mathcal{H}}}\left(\max\{\hat{K}_1, \hat{K}_1^k\}\right)^5}{N_{\mathcal{H}}^5 e^{-15C_{\mathcal{H}}}} \\
&= \frac{M_{\mathcal{H}}^{16}e^{(5k+16)C_{\mathcal{H}}}}{N_{\mathcal{H}}^5}\left(\max\{\hat{K}_1, \hat{K}_1^k\}\right)^5 \\
&\le \frac{(M_{\mathcal{F}}^{16}M_{\mathcal{G}}^{16\alpha})^t e^{(5k+16)(C_{\mathcal{F}}+\alpha C_{\mathcal{G}})t}}{N_{\mathcal{F}}^{5t}}\left(\max\{(\hat{K'}_1)^t, (\hat{K'}_1)^{kt}\}\right)^5 \\
&= \left(\frac{M_{\mathcal{F}}^{16}M_{\mathcal{G}}^{16\alpha}e^{(5k+16)(C_{\mathcal{F}}+\alpha C_{\mathcal{G}})}}{N_{\mathcal{F}}^5}\left(\max\{(\hat{K'}_1), (\hat{K'}_1)^k\}\right)^5\right)^t.
\end{aligned}$$

Denote

$$\hat{K}_0 := \left(\frac{M_{\mathcal{F}}^{16}M_{\mathcal{G}}^{16\alpha}e^{(5k+16)(C_{\mathcal{F}}+\alpha C_{\mathcal{G}})}}{N_{\mathcal{F}}^5}\left(\max\{(\hat{K'}_1), (\hat{K'}_1)^k\}\right)^5\right)$$

to obtain $K_0(t) \le \hat{K}_0^t$, as stated. □

We can now prove that the equilibrium measures for $t\mathcal{F} - O(t)\mathcal{G}$ form a tight family of probability measures.

Lemma 27. *The family $(\mu_t)_{t>t^*}$ is tight.*

Proof. Pick arbitrary numbers $\varepsilon > 0, t > t^*$ Denote $\mathcal{H} = t\mathcal{F} - O(t)\mathcal{G}$ and $h_n := f_n^t/g_n^{O(t)}$(so $\mathcal{H} = \{\log h_n\}_{n=1}^\infty$). Since μ_t is a Gibbs measure for $\mathcal{H}$, we have for every cylinder set of the form $[i]$ and $x \in [i]$ there exists $K_0 > 0$ such that:

$$\mu_t[i] \le K_0 \exp(-P(\mathcal{H}))h_1(x).$$

Since $P(\mathcal{H}) = P(t\mathcal{F} - O(t)\mathcal{G}) = 0$ and $x \in [i]$ is arbitrary, it results:

$$\mu_t[i] \le K_0 \sup_{[i]} h_1.$$

Now, from Lemma 26, we have

$$\mu_t[i] \le \hat{K}_0^t \sup_{[i]}\left(\frac{f_1^t}{g_1^{O(t)}}\right)$$

Now, according to Lemmas 23 and 24. An appropriate choice of $\beta \in \{\alpha, \gamma^{-1}\}$ yields

$$\mu_t[i] \le \hat{K}_0^{\beta O(t)} \sup_{[i]}\left(\frac{f_1^\gamma}{g_1}\right)^{O(t)} = \left(\hat{K}_0^\beta \sup_{[i]}\left(\frac{f_1^\gamma}{g_1}\right)\right)^{O(t)}.$$

From (38), it follows that $\lim_{i\to\infty} \sup_{[i]} \frac{f_1^\gamma}{g_1} = 0$. Therefore, there exists $J \in \mathbb{N}$ such that for every $i \geq J$ we have $\sup_{[i]} \frac{f_1^\gamma}{g_1} < \frac{1}{\hat{K}_0^\beta}$, whence $\hat{K}_0^\beta \sup_{[i]} \left(\frac{f_1^\gamma}{g_1}\right) < 1$ and since $O(t)$ is increasing, we have, for every $i \geq J$:

$$\mu_t[i] \leq \left(\hat{K}_0^\beta \sup_{[i]} \left(\frac{f_1^\gamma}{g_1}\right)\right)^{O(t^*)} = K' \sup_{[i]} \left(\frac{f_1^\gamma O(t^*)}{g_1^O(t^*)}\right),$$

where $K' := \hat{K}_0^{\beta O(t^*)}$.

Therefore, for every $n \geq J$

$$\sum_{i=n}^{\infty} \mu_t[i] \leq K' \sum_{i=n}^{\infty} \sup_{[i]} \left(\frac{f_1^\gamma}{g_1}\right)^{O(t^*)}. \tag{58}$$

The latter sum in (58) converges since $\gamma O(t^*)\mathcal{F} - O(t^*)\mathcal{G}$ has finite pressure due to Lemma 16 and Corollary 5. So, for every $k \in \mathbb{N}$ there exists $n_k \geq J$ such that

$$\sum_{i=n}^{\infty} \sup_{[i]} \left(\frac{f_1^\gamma}{g_1}\right)^{O(t^*)} < \frac{\varepsilon}{2^k K'}. \tag{59}$$

From (58) and (59) we obtain

$$\sum_{i=n_k}^{\infty} \mu_t[i] < \frac{\varepsilon}{2^k}.$$

Now, let $K := \{x \in \Sigma_A : 1 \leq x_k \leq n_k, \forall k \in \mathbb{N}\}$ and note that K is a compact set. Then

$$\begin{aligned}
\mu_t(K) &= \mu_t\left(\Sigma_A \setminus \bigcup_{k=1}^{\infty} \{x \in \Sigma_A : x_k > n_k\}\right) \\
&\geq 1 - \sum_{k=1}^{\infty} \mu_t\left(\{x \in \Sigma_A : x_k > n_k\}\right) \\
&= 1 - \sum_{k=1}^{\infty} \sum_{i=n_k+1}^{\infty} \mu_t(\sigma^{-k}([i])) \\
&= 1 - \sum_{k=1}^{\infty} \sum_{i=n_k+1}^{\infty} \mu_t[i] > 1 - \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = 1 - \varepsilon.
\end{aligned}$$

Hence, $(\mu_t)_{t>t^*}$ is a tight family of probability measures. □

The following lemma states that the limit of the quotient of the operators $I_\mathcal{F}, I_\mathcal{G}$ is compatible with the one from the accumulation points of the equilibrium states μ_{t_k} as $k \to \infty$.

LEMMA 28. *Let $\{t_k\}_{k\in\mathbb{N}}$ be a sequence of real numbers such that $O'(t_k)$ exists for every $k \in \mathbb{N}$ and $t_k \to \infty$ when $k \to \infty$. Let μ be an accumulation point for $\{\mu_{t_k}\}_{k\in\mathbb{N}}$ when $k \to \infty$. The identity*

$$\lim_{k\to\infty} \frac{I_\mathcal{F}(\mu_{t_k})}{I_\mathcal{G}(\mu_{t_k})} = \frac{I_\mathcal{F}(\mu)}{I_\mathcal{G}(\mu)}$$

holds.

PROOF. First, notice that the sequence described $\{t_k\}_{k\in\mathbb{N}}$ does indeed exist since the set of $t \in \mathbb{R}^+$ such that $O(t)$ is not differentiable is, at most, countable. Moreover, since $O(t)$ is a convex function, we have that $O'(t_k)$ is a non decreasing sequence, therefore, $\lim_{k\to\infty} O'(t_k)$ exists. The proof of Lemma 1 shows that $\frac{I_{\mathcal{F}}(\nu)}{I_{\mathcal{G}}(\nu)}$ is bounded for $\nu \in \mathcal{M}_\sigma$. This fact, along with Lemma 20 implies

$$\lim_{k\to\infty} O'(t_k) = \lim_{k\to\infty} \frac{I_{\mathcal{F}}(\mu_{t_k})}{I_{\mathcal{G}}(\mu_{t_k})} < \infty.$$

Now, recall (49) to obtain

$$O(t_k) \geq \frac{h(\mu)}{I_{\mathcal{G}}(\mu)} + t\frac{I_{\mathcal{F}}(\mu)}{I_{\mathcal{G}}(\mu)}.$$

The convexity of $O(t)$ allows the comparison of the asymptotic derivatives in this inequality. Therefore, we have

$$\lim_{k\to\infty} O'(t_k) = \lim_{k\to\infty} \frac{I_{\mathcal{F}}(\mu_{t_k})}{I_{\mathcal{G}}(\mu_{t_k})} \geq \frac{I_{\mathcal{F}}(\mu)}{I_{\mathcal{G}}(\mu)}. \tag{60}$$

We now aim to prove the opposite inequality: Define the sequences $\hat{\mathcal{F}} = \{\log \hat{f}_n\}_{n=1}^\infty$ and $\hat{\mathcal{G}} = \{\log \hat{g}_n\}_{n=1}^\infty$, where $\hat{f}_n = f_n e^{C_{\mathcal{F}}}$ and $\hat{g}_n = e^{C_{\mathcal{G}}}/g_n$.

Notice that since $\mathcal{F}$ is almost-additive, equation (32) gives

$$\begin{aligned}\log(\hat{f}_{n+m}) = \log(f_{n+m}e^{C_{\mathcal{F}}}) &\leq \log(f_n(f_m\circ\sigma^n)e^{2C_{\mathcal{F}}}) \\ &= \log(f_n e^{C_{\mathcal{F}}}) + \log(f_m\circ\sigma^n e^{C_{\mathcal{F}}}) \\ &= \log(\hat{f}_n) + \log(\hat{f}_m\circ\sigma^n),\end{aligned}$$

therefore, the sequence $\hat{\mathcal{F}}$ is subadditive.

Similarly, the almost-additivity of $\mathcal{G}$ and (31) imply

$$\begin{aligned}\log(\hat{g}_{n+m}) = \log\left(\frac{C_{\mathcal{G}}}{g_{n+m}}\right) &\leq \log\left(\frac{2C_{\mathcal{G}}}{g_n(g_m\circ\sigma^n)}\right) \\ &= \log\left(\frac{C_{\mathcal{G}}}{g_n}\right) + \log\left(\frac{C_{\mathcal{G}}}{g_m\circ\sigma^n}\right) \\ &= \log(\hat{g}_n) + \log(\hat{g}_m\circ\sigma^n),\end{aligned}$$

which shows that $\hat{\mathcal{G}}$ is subadditive.

Set $k \in \mathbb{N}$ and observe that

$$\int \log \hat{g_1} d\mu_{t_k} = C_{\mathcal{G}} - \int \log g_1 d\mu_{t_k} < C_{\mathcal{G}} < \infty,$$

therefore $\hat{\mathcal{G}}$ and μ_{t_k} satisfy the hypotheses from Kingman's ergodic subadditive Theorem. Now, notice from (35) that

$$\int \log \hat{f}_1 d\mu = \int \log f_1 d\mu_{t_k} + C_{\mathcal{F}} \leq I_{\mathcal{F}}(\mu_{t_k}) + 2C_{\mathcal{F}}.$$

Denote by B a bound for $\frac{I_{\mathcal{F}}(\nu)}{I_{\mathcal{G}}(\nu)}$, when $\nu \in \mathcal{M}_\sigma(-\mathcal{G})$. It follows

$$\int \log \hat{f}_1 d\mu \leq BI_{\mathcal{G}}(\mu_{t_k}) + 2C_{\mathcal{F}} < \infty. \tag{61}$$

The latter expression is finite since μ_{t_k} is an equilibrium measure for $t_k\mathcal{F}-O(t_k)\mathcal{G}$, whence $\mu_{t_k} \in \mathcal{M}_\sigma(-\mathcal{G})$. This shows that Kingman's ergodic subadditive Theorem also holds for $\hat{\mathcal{F}}$ and the measure μ_{t_k}, since from Remark 21 it follows that $\mu_{t_k} \in \mathcal{M}_\sigma(-\mathcal{F})$. This way we have, for any $j \in \mathbb{N}$:

$$\lim_{n\to\infty} \frac{1}{n}\int \log \hat{f}_n d\mu_{t_k} \leq \frac{1}{j}\int \log \hat{f}_j d\mu_{t_k},$$

that is

$$\lim_{n\to\infty} \left(\frac{1}{n}\int \log f_n d\mu_{t_k} + \frac{C_\mathcal{F}}{n}\right) \leq \frac{1}{j}\int \log f_j d\mu_{t_k} + \frac{C_\mathcal{F}}{j}.$$

Hence

$$I_\mathcal{F}(\mu_{t_k}) \leq \frac{1}{j}\int \log f_j d\mu_{t_k} + \frac{C_\mathcal{F}}{j}. \tag{62}$$

Similarly, for $\hat{\mathcal{G}}$ we have

$$\lim_{n\to\infty} \frac{1}{n}\int \log\left(\frac{e^{C_\mathcal{G}}}{g_n}\right) d\mu_{t_k} \leq \frac{1}{j}\int \log \frac{e^{C_\mathcal{G}}}{g_j} d\mu_{t_k},$$

or, equivalently,

$$\lim_{n\to\infty} \left(\frac{C_\mathcal{G}}{n} - \frac{1}{n}\int \log g_n d\mu_{t_k}\right) \leq \frac{C_\mathcal{G}}{j} - \frac{1}{j}\int \log g_j d\mu_{t_k}.$$

We conclude

$$I_\mathcal{G}(\mu_{t_k}) + \frac{C_\mathcal{G}}{j} \geq \frac{1}{j}\int \log g_j d\mu_{t_k} \tag{63}$$

From (62) and (63), we have

$$\frac{I_\mathcal{F}(\mu_{t_k})}{I_\mathcal{G}(\mu_{t_k}) + \frac{C_\mathcal{G}}{j}} \leq \frac{\int \log f_j d\mu_{t_k} + C_\mathcal{F}}{\int \log g_j d\mu_{t_k}}. \tag{64}$$

When $k \to \infty$, $\mu_{t_k} \to \mu$ in the weak-* topology. Since $\log g_j$ is positive (and therefore bounded from below) it follows that $\liminf_{k\to\infty} \int \log g_j d\mu_{t_k} \geq \int \log g_j d\mu$. We now claim that

$$\limsup_{k\to\infty} \int \log f_j d\mu_{t_k} \leq \int \log f_j d\mu. \tag{65}$$

Define an increasing sequence of bounded functions $\{f_{j,m}\}_{m=1}^\infty$ such that $f_{j,m}$ converges pointwise to f_j when $m \to \infty$ (for instance, $f_{j,m} := \min\{f_j, m\}$). For every $m \in \mathbb{N}$ we have:

$$\int \log f_{j,m} d\mu \leq \int \log f_j d\mu, \tag{66}$$

and, since $f_{j,m}$ is bounded, the convergence of μ_{t_k} in the weak-* topology establishes

$$\lim_{k\to\infty} \int \log f_{j,m} d\mu_{t_k} = \int \log f_{j,m} d\mu.$$

According to this, if we let $\varepsilon > 0$ be arbitrary, there is $k_0 \in \mathbb{N}$ such that for every $k > k_0$:

$$\int \log f_{j,m} d\mu_{t_k} - \varepsilon < \int \log f_{j,m} d\mu \tag{67}$$

On the other hand, the monotonous convergence theorem yields

$$\lim_{m\to\infty}\int \log f_{j,m} d\mu_{t_k} = \int \log f_j d\mu_{t_k}.$$

Notice that the same argument from (61) shows that $\int \log f_j d\mu_{t_k} < \infty$. So, there is $m_0 \in \mathbb{N}$ such that for every $m > m_0$ we have

$$\int \log f_j d\mu_{t_k} - \int \log f_{j,m} d\mu_{t_k} < \varepsilon. \tag{68}$$

Summarizing, for every $\varepsilon > 0, k > k_0, m > m_0$, relations (66), (67) and (68) yield the inequalities:

$$\begin{aligned}\int \log f_j d\mu_{t_k} - 2\varepsilon &< \int \log f_j d\mu_{t_k} - \left(\int \log f_j d\mu_{t_k} - \int \log f_j d\mu\right) - \varepsilon \\ &= \int \log f_j, m d\mu_{t_k} - \varepsilon < \int \log f_{j,m} d\mu \le \int \log f_j d\mu.\end{aligned}$$

Letting $k \to \infty$, it results

$$\limsup_{k\to\infty}\int f_j d\mu_{t_k} - 2\varepsilon \le \int f_j d\mu.$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\limsup_{k\to\infty}\int f_j d\mu_{t_k} \le \int f_j d\mu,$$

which is precisely what was claimed in (65).

Returning to (64), let $k \to \infty$ to obtain

$$\lim_{k\to\infty}\frac{I_{\mathcal{F}}(\mu_{t_k})}{I_{\mathcal{G}}(\mu_{t_k}) + \frac{C_{\mathcal{G}}}{j}} \le \frac{\limsup\limits_{k\to\infty}\int \log f_j d\mu_{t_k} + C_{\mathcal{F}}}{\liminf\limits_{k\to\infty}\int \log g_j d\mu_{t_k}} < \frac{\int \log f_j d\mu + C_{\mathcal{F}}}{\int \log g_j d\mu}.$$

Now, letting $j \to \infty$:

$$\lim_{k\to\infty}\frac{I_{\mathcal{F}}(\mu_{t_k})}{I_{\mathcal{G}}(\mu_{t_k})} \le \lim_{j\to\infty}\frac{\int \log f_j d\mu + C_{\mathcal{F}}}{\int \log g_j d\mu} = \lim_{j\to\infty}\frac{\frac{1}{j}\int \log f_j d\mu + \frac{C_{\mathcal{F}}}{j}}{\frac{1}{j}\int \log g_j d\mu} = \frac{I_{\mathcal{F}}(\mu)}{I_{\mathcal{G}}(\mu)}.$$

From this inequality and (60), we conclude that

$$\lim_{k\to\infty}\frac{I_{\mathcal{F}}(\mu_{t_k})}{I_{\mathcal{G}}(\mu_{t_k})} = \frac{I_{\mathcal{F}}(\mu)}{I_{\mathcal{G}}(\mu)}.$$

□

We now prove the last part of Theorem 9.

LEMMA 29. *Let $\{t_k\}_{k\in\mathbb{N}}$ be a sequence of real numbers such that $O'(t_k)$ exists for every $k \in \mathbb{N}$ and $t_k \to \infty$ when $k \to \infty$. Let μ be an accumulation point for $\{\mu_{t_k}\}_{k\in\mathbb{N}}$ when $k \to \infty$. Then, μ is $(\mathcal{F},\mathcal{G})$-maximizing*

PROOF. Let $\nu \in \mathcal{M}_\sigma(-\mathcal{G})$. As in the additive case, we compare asymptotic derivatives in (49) and conclude

$$\lim_{k\to\infty} O'(t_k) \ge \frac{I_{\mathcal{F}}(\nu)}{I_{\mathcal{G}}(\nu)}.$$

From Lemmas 20 and 28, we obtain

$$\frac{I_{\mathcal{F}}(\mu)}{I_{\mathcal{G}}(\mu)} \geq \frac{I_{\mathcal{F}}(\nu)}{I_{\mathcal{G}}(\nu)}.$$

Hence, μ is $(\mathcal{F}, \mathcal{G})$-maximizing. □

3.1. Another proof for Lemma 29. When dealing with additive potentials in the previous chapter, we presented two different proofs of the maximizing property of the accumulation points of the equilibrium measures of the potentials $tf - O(t)g$. The same arguments remain valid in the almost additive setting since they did not depend upon the differentiability of the pressure map. However, we still consider sequences $(t_k)_{k\in\mathbb{N}}$ such that $O(t_k)$ is differentiable since Lemma 28 is used when taking limits of the quotients of $I_{\mathcal{F}}$ and $I_{\mathcal{G}}$. The following results correspond to the almost-additive versions of those results and their proofs follow the same arguments from the additive setting, adapting them to the analogous concepts in the almost-additive setting.

Alternative proof of Lemma 29. Let $\nu \in \mathcal{M}(-\mathcal{G})$ and Let $(t_k)_{k\in\mathbb{N}}$ be such that $O(t)$ is differentiable at $t = t_k$ for every $k \in \mathbb{N}$ and $t_k \to \infty$ when $k \to \infty$. Set $t = t_k$ in (53) and then divide by t_k. It results

$$\frac{h(\mu_{t_k})}{t_k I_{\mathcal{G}}(\mu_{t_k})} + \frac{I_{\mathcal{F}}(\mu_{t_k})}{I_{\mathcal{G}}(\mu_{t_k})} \geq \frac{h(\nu)}{t_k I_{\mathcal{G}}(\nu)} + \frac{I_{\mathcal{F}}(\nu)}{I_{\mathcal{G}}(\nu)}.$$

Recalling Corollary 4 and noting that $I_{\mathcal{G}}(\nu) < \infty$, by letting $k \to \infty$ it results

$$\frac{I_{\mathcal{F}}(\mu)}{I_{\mathcal{G}}(\mu)} \geq \frac{I_{\mathcal{F}}(\nu)}{I_{\mathcal{G}}(\nu)},$$

which proves that μ is an $(\mathcal{F}, \mathcal{G})-$maximizing measure. □

4. An application to products of matrices

Theorem 9 consists of a generalization of Theorem 5. We now show an example about how this extension of its scope allows us to describe maximizing measures for ratios of expected values for Lyapunov exponents associated to linear cocycles.

Start by setting two sequences of matrices $\{A_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^{d\times d}, \{B_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^{d\times d}$ such that their entries $A_n(i,j), B_n(i,j)$ are positive for every $i, j \in \{1, \cdots d\}$. Assume that there exists $C > 0$ such that $B_n(i,j) > C$. Let U be the column vector

$$U = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}_{d\times 1}$$

and define the matrix norm

$$\|A\| = U^t A U.$$

Denote $x = (i_1, i_2, \cdots)$ and for every $n \in \mathbb{N}$, set $f_n, g_n : \Sigma \to \mathbb{R}$ as the functions

$$f_n(x) = \|A_{i_n} \cdots A_{i_1}\|$$

and

$$g_n(x) = \|B_{i_n} \cdots B_{i_1}\|.$$

Let us also define $\alpha, \beta : \Sigma_A \to \mathbb{R}$ by the equalities

$$\alpha(x) = A_{x_1}$$

and

$$\beta(x) = B_{x_1}.$$

The triples $(\Sigma_A, \sigma, \alpha)$ and $(\Sigma_A, \sigma, \beta)$ are called *linear cocycles.*

Define the sequences

$$\mathcal{F} = \{\log f_n\}_{n\in\mathbb{N}}, \mathcal{G} = \{\log g_n\}_{n\in\mathbb{N}}.$$

The following lemma states a condition in order of these sequences to be almost-additive. It has been stated in [**IY1**] and its proof replicates the arguments given by Feng [**F**] in the compact setting

LEMMA 30. *Assume that there exists $C_{\mathcal{F}} \in \mathbb{R}$ such that for every $n \in \mathbb{N}$, the inequality*

$$\frac{\min_{i,j} A_n(i,j)}{\max_{i,j} A_n(i,j)} \geq de^{-C_{\mathcal{F}}} \tag{69}$$

holds. Then, the sequence $\mathcal{F}$ is almost-additive.

PROOF. The norm $\|\cdot\|$ is submultiplicative, therefore, for any $x = (i_1, i_2, \cdots) \in \Sigma$:

$$\begin{aligned} f_{n+m}(x) = \|A_{i_{n+m}} \cdots A_{i_1}\| &= \|A_{i_{n+m}} \cdots A_{i_{n+1}}\| \cdot \|A_{i_n} \cdots A_{i_1}\| \\ &= f_m \circ \sigma^n(x) \cdot f_n(x) \leq f_n(x) \cdot f_m(x) \circ \sigma^n(x) \cdot e^{-C_{\mathcal{F}}}. \end{aligned}$$

On the other hand, equation (69) implies that for every n, i, j we have $A_n(i,j) \geq e^{-C_{\mathcal{F}}} dA_n(i,j)$. So, by denoting $E := UU^t$, the $d \times d$ matrix whose every entry is 1, we notice

$$e^{-C_{\mathcal{F}}} EA_n(i,j) = e^{-C_{\mathcal{F}}} E \sum_{r=1}^{d} A_n(r,j) \leq e^{-C_{\mathcal{F}}} \sum_{r=1}^{d} \frac{A_n(i,j)}{d - e^{C_{\mathcal{F}}}} = A_n(i,j).$$

Finally, we obtain

$$\begin{aligned} f_{n+m}(x) &= \|A_{i_{n+m}} \cdots A_{i_1}\| = U^t A_{i_{n+m}} \cdots A_{i_1} U \\ &\geq U^t A_{i_{n+m}} \cdots A_{n+1} e^{-C_{\mathcal{F}}} E A_{i_n} \cdots A_{i_1} U \\ &= U^t A_{i_{n+m}} \cdots A_{i_{n+1}} e^{-C_{\mathcal{F}}} UU^t A_{i_n} \cdots A_{i_1} U \\ &= e^{-C_{\mathcal{F}}} \|A_{i_{n+m}} \cdots A_{i_{n+1}}\| \cdot \|A_{i_n} \cdots A_{i_1}\| \\ &= e^{-C_{\mathcal{F}}} f_m \circ \sigma^n(x) f_n(x) \end{aligned}$$

This inequality shows that $\mathcal{F}$ is almost-additive. □

When $\mathcal{F}$ is almost-additive, we have that $\mathcal{F} + C_{\mathcal{F}}$ is a subadditive sequence. Therefore, Kingman's Subadditive theorem ensures that

$$\lim_{n\to\infty} \frac{1}{n}(\log f_n + C_{\mathcal{F}}) = \lim_{n\to\infty} \frac{1}{n} \log f_n$$

exist almost everywhere for each σ-invariant probability measure in Σ_A. So, we can define the *Top Lyapunov exponent* for the cocycles $(\Sigma_A, \sigma, \alpha)$ and $(\Sigma_B, \sigma, \beta)$ as

$$\lambda_\alpha(x) = \lim_{n\to\infty} \log ||A_{x_n} A_{x_{n-1}} \cdots A_{x_1}||$$

and

$$\lambda_\beta(x) = \lim_{n\to\infty} \log ||B_{x_n} B_{x_{n-1}} \cdots B_{x_1}||$$

respectively.

Let us now assume that condition (69) holds for both $\mathcal{F}$ and $\mathcal{G}$. Notice that both sequences are locally constant and therefore, they are Bowen sequences.

Since $B_n(i,j) > C$ for every n, i, j, we have

$$\frac{1}{n}\log g_n = \frac{1}{n}\log\|B_{i_n}\cdots B_{i_1}\| \geq \frac{1}{n}\log\|C^n E^n\| = C\|E^n\| \geq C\|E\|.$$

This inequalities show that $\mathcal{G}$ is bounded away from zero.

Finally, condition (38) is rewritten as

$$\lim_{n\to\infty}\frac{\log\|A_n\|}{\log\|B_n\|} = 0. \tag{70}$$

According to these arguments, we derive from Theorem 9 the following result:

PROPOSITION 9. *Let (Σ_A, σ) be a topologically mixing countable Markov Shift satisfying the BIP Property. Define $\{A_n\}_{n\in\mathbb{N}}, \{B_n\}_{n\in\mathbb{N}}$ two sequences of matrices in $\mathbb{R}^{d\times d}$ satisfying $A_n(i,j) > 0, B_n(i,j) > C$ for some $C > 0$, every $n \in \mathbb{N}$ and every $i, j \in \{0, \cdots, d\}$. Assume that both A_n and B_n satisfy conditions* (69), (70) *for every $n \in \mathbb{N}$. Define the sequences $\mathcal{F} = \{\log f_n\}_{n\in\mathbb{N}}$ and $\mathcal{G} = \{\log g_n\}_{n\in\mathbb{N}}$, where*

$$f_n(x) = \|A_{i_n}\cdots A_{i_1}\|, g_n(x) = \|B_{i_n}\cdots B_{i_1}||$$

for every $x = (i_1, i_2, \cdots) \in \Sigma_A$. If there exists $s_\infty \geq 0$ such that $P(-s\mathcal{G}) < \infty$ for every $s > s_\infty$, then the equilibrium measures μ_t of $t\mathcal{F} - O(t)\mathcal{G}$ have an accumulation point μ which satisfies

$$\frac{I_{\mathcal{F}}(\mu)}{I_{\mathcal{G}}(\mu)} = \max_{\nu\in\mathcal{M}_\sigma(-\mathcal{G})}\frac{I_{\mathcal{F}}(\nu)}{I_{\mathcal{G}}(\nu)} = \max_{\nu\in\mathcal{M}_\sigma(-\mathcal{G})}\lim_{n\to\infty}\frac{\int\log\|A_{i_n}\cdots A_{i_1}\|d\nu}{\int\log\|B_{i_n}\cdots B_{i_1}\|d\nu}.$$

in particular, the Top Lyapunov exponents $\lambda_\alpha, \lambda_\beta$ associated respectively to α and β satisfy

$$\frac{\int\lambda_\alpha d\mu}{\int\lambda_\beta d\mu} = \max_{\nu\in\mathcal{M}_\sigma(-\lambda_\beta)}\frac{\int\lambda_\alpha d\nu}{\int\lambda_\beta d\nu}.$$

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