FACULTAD DE MATEMÁTICAS

ON ACCUMULATION POINTS OF VOLUMES OF STABLE SURFACES WITH ONE CYCLIC QUOTIENT SINGULARITY

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To my nephew Nicolás

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Contents

1 Introduction

This work is about studying the behavior of K^2 for complex stable surfaces with particular singularities. Stable surfaces are the surfaces used by Kollár–Shepherd-Barron [KSB88] and Alexeev [Ale94] to give a natural compactification to the moduli space of surfaces of general type. But the interest on them goes beyond that compactification. A current topic of study is the distribution of volumes K^2 in the set of positive rational numbers. A fundamental result is the *Descending Chain Condition* (DCC for short) for $\{K^2\}$ (the set of all K^2 of stable surfaces), which is due to Alexeev. (Its most general version for log stable surfaces can be found in [Ale94].) In particular, the DCC property implies the existence of a minimum for $\{K^2\}$. It is still an open problem to know its value. Knowing the exact lower bound for K^2 can be used, for example, to explicitly bound the automorphism group for surfaces of general type. (See e.g. [Ale94] and [Kol94] for more motivation.) Various authors have found low values for K^2 (see e.g. [Bla95], [UYn17], [Liu17], [AL19a], [AL19c]). On the other hand, upper bounds are not possible even if we fix the geometric genus [UU19, Thm. 1.9], contrary to what happens for smooth projective surfaces of general type.

It turns out that we can also have accumulation points for the set of volumes of (log) stable surfaces. There has been a recent interest on understanding better the set of accumulation points $\mathrm{Acc}(\{K^2\})$, see e.g. [Kol94], [Bla95], [UYn17], [AL19b], [AL19c]. In early times, Blache [Bla95] showed that $1 \in \mathrm{Acc}(\{K^2\})$. It was done by constructing a family of stable surfaces with ten cyclic quotient singularities. Blache conjectured that $\mathbb{N} \subseteq \mathrm{Acc}(\{K^2\}) \subseteq \mathbb{Q}$. In [AL19b] (see also [UYn17]), it is shown that all natural numbers are accumulation points, and that iterated accumulation points can have arbitrary complexity in unbounded regions. Additionally, Alexeev and Liu [AL19b] proved general results about volumes of log canonical surfaces, which has several implications. One of them is to solve the conjecture of Blache about the closure of $\{K^2\}$, which is indeed in $\mathbb{Q}$. Although it is not known if the set of K^2 is closed (for empty boundary). A full description of $\{K^2\}$ and $\mathrm{Acc}(\{K^2\})$ is still missing. In the case of smooth varieties of general type, there is a positive lower bound for the volume which depends only on the dimension (see e.g [HM06], [HMX13]). However, no optimal bounds are known for dimensions greater than 2. Recently, Totaro and Wang studied the asymptotic behavior of the volumes for smooth varieties in [TW21].

This work aims to describe how accumulation points of volumes of stable surfaces are formed, in the case of surfaces with only one cyclic quotient singularity. We find the following numerical constraints which optimally bound singularities[1] when we restrict to specific situations, such as T-singularities (recovering [RU19, Thm. 1.1] for example) or generalized T-singularities (see Lemma 6.4). Of course one cannot expect to bound all cyclic quotient singularities because of the existence of accumulation points, but this theorem gives a way to detect them. All notations will be introduced in the body of the work.

Theorem 1.1. *Let W be a stable surface with only one cyclic quotient singularity of type $\frac{1}{n}(1,q)$ at $P \in W$. Let*

$$C = C_1 + \ldots + C_r$$

be the chain of the exceptional curves in the minimal resolution of P, and let $[b_1, \ldots, b_r]$ be its Hirzebruch-Jung continued fraction. Let X be the minimal resolution of W, and let $\pi\colon X \to S$ be a minimal model of X. Then

$$\sum_{j=1}^{r} \big(b_j - 2\big) \leq 2(K_W^2 - K_S^2) + 2\left(\frac{2(n-1) - q - q'}{n}\right) + \delta - \pi^* K_S \cdot C, \quad (1)$$

and

$$r \leq 13K_W^2 - 2K_S^2 + 38 - \left(\frac{2 + q + q'}{n}\right) + \delta - \pi^* K_S \cdot C, \quad (2)$$

where $0 < q' < n$ with $qq' \equiv 1 \ (mod\, n)$, and δ is the positive number computed in Lemma 5.13 for distinct geometric situations.

We point out that the core of Theorem 1.1 relies on finding explicit δ's and a classification of all possible geometric realizations. Bounding of δ for a sequence of stable surfaces with one cyclic quotient singularity is directly related to the existence of accumulation points. We will use Bogomolov-Miyaoka-Yau inequality for proving the bound in (2) in Theorem 1.1. However, the bound in (1), and the computation of δ in Lemma 5.13, remain valid in any characteristic.

Theorem 1.1 directly implies the following results about boundedness and accumulation points.

[1] We refer to boundedness in this context to describe the possible cyclic quotient singularities which may occur on a surface with bounded invariants K^2 and χ.

Corollary 1.2. *Let $c > 0$, and let $\mathcal{S}$ be a set of stable surfaces W with one cyclic quotient singularity, K_S nef, and $K_W^2 \leq c$. Let $\mathrm{Sing}(\mathcal{S})$ be the set of singularities of the surfaces in $\mathcal{S}$. Then $\mathrm{Sing}(\mathcal{S})$ is finite if and only if the number of 2's at the extremes of every $[b_1, \ldots, b_r] \in \mathrm{Sing}(\mathcal{S})$ is bounded.*

Corollary 1.3. *Let $\{W_k\}$ be a sequence of stable surfaces with only one cyclic quotient singularity. Assume that for every k the minimal model S_k of W_k has canonical class nef, and that $K_{W_k}^2 \leq c$ for a positive number c. If the cases (A),(B.2) or (D.3) in Lemma 5.13 hold except for a finite number of indices k, then $Acc(\{K_{W_k}^2\}) = \emptyset$.*

Next we introduce the set-up that will be used to define and work with generalized T-singularities.

Definition 1.4 (see e.g. [OW77])**.** Let $\{a_1, \ldots, a_s\}$ be an ordered set of positive natural numbers. Let $p_{-1} = 0$, $p_0 = 1$, $q_0 = 0$, $q_1 = 1$, and for $i \geq 1$,

$$p_{i+1} = a_{i+1}p_i + p_{i-1} \quad , \quad q_{i+1} = a_{i+1}q_i + q_{i-1}.$$

We say that $\{a_1, \ldots, a_s\}$ is *admissible* if $p_i > 0$ for $i = 0, \ldots, s-1$.

It is a straightforward calculation to show that if $\{a_1, \ldots, a_s\}$ is admissible, then the Hirzebruch-Jung continued fraction $[a_1, \ldots, a_s]$ is well-defined (see Definition 3.14). As an example, we know that the sequence $\{5, 1, 5\}$ is admissible but $\{5, 1, 1, 5\}$ is not. Note that if we have $a_i \geq 2$ for every i, then $\{a_1, \ldots, a_s\}$ is admissible.

Definition 1.5. Let $[b_1, \ldots, b_r]$ be a Hirzebruch-Jung continued fraction with $b_i \geq 2$ for all i. We say that $[b_1, \ldots, b_r]$ is *admissible for chains* if

$$\{b_1, \ldots, b_r, 1, b_1, \ldots, b_r, 1, \ldots, 1, b_1, \ldots, b_r\} \tag{3}$$

is admissible for any number of inserted 1's.

As for regular Hirzebruch-Jung continued fractions, we think geometrically of $\{b_1, \ldots, b_r, 1, b_1, \ldots, b_r, 1, \ldots, 1, b_1, \ldots, b_r\}$ as a chain of $\mathbb{P}^1$'s, where we have (-1)-curves inserted between some minimal resolution chains of the cyclic quotient singularity associated to $[b_1, \ldots, b_r]$. For example, we have that $[4]$ is admissible for chains, and it gives all the initial chains to construct all the T-singularities [KSB88, Prop.3.11]. We take this to define generalized T-singularities.

Definition 1.6. Let $\{a_1, \ldots, a_s\}$ be an admissible set. Its *reduced* Hirzebruch-Jung continued fraction is the continued fraction obtained after contracting all (-1)-curves in $\{a_1, \ldots, a_s\}$, and all the new (-1)-curves after that.

Notation 1.7. The reduced Hirzebruch- Jung continued fraction of

$$\{b_1, \ldots, b_r, 1, b_1, \ldots, b_r, 1, \ldots, 1, b_1, \ldots, b_r\},$$

where u is the number of inserted 1's, will be denoted by $[b_1^u, \ldots, b_{r_u}^u]$. Also, we will write $[b_1^0, \ldots, b_{r_0}^0]$ to refer to $[b_1, \ldots, b_r]$. We write the singularity $[a_1, \ldots, a_s]$ to refer to the cyclic singularity associated to this continued fraction.

Definition 1.8. Let $[b_1, \ldots, b_r]$ be a Hirzebruch-Jung continued fraction which is admissible for chains. We define the class of *generalized T-singularity* of center $[b_1, \ldots, b_r]$ inductively in the following way

(i) The singularities $[b_1^u, \ldots, b_{r_u}^u]$ for every $u \geq 0$ are generalized T-singularities.

(ii) If $[a_1, \ldots, a_s]$ is a generalized T-singularity, then so are

$$[2, a_1, \ldots, a_{s-1}, a_s + 1] \quad \text{and} \quad [a_1 + 1, a_2, \ldots, a_s, 2].$$

(iii) Every generalized T-singularity of center $[b_1, \ldots, b_r]$ is obtained by starting with one of the singularities described in (i) and iterating the steps described in (ii).

We say that we apply the *T-chain algorithm* if we apply iterations of (ii).

It is clear that T-singularities are the generalized T-singularities of center [4]. Rana and Urzúa in [RU19] showed an optimal bound of T-singularities for stable surfaces with one singularity. A natural question is whether that result remains valid for generalized T-singularities. We answer this question in the following theorem by describing how the accumulation points of K^2 on stable surfaces with one generalized T-singularity of fixed center are formed.

Theorem 1.9. *Let $\{W_k\}$ be a sequence of stable surfaces such that any W_k has only one generalized T-singularity with a fixed center $[b_1, \ldots, b_r]$, say at $P_k \in W_k$. Suppose that the minimal model S_k of the minimal resolution of W_k has canonical class nef. Then $\{K^2_{W_k}\}$ has accumulation points if and only if $\{K^2_{W_k}\}$ satisfy the property (*) (see Definition 6.8.)*

It is shown in Proposition 6.12 that every accumulation point which is coming from a sequence as one described in Theorem 1.9, can be constructed by blowing up a particular configuration of curves in a smooth surface and then contracting the new configuration obtained.

Notation

Let S be a smooth surface. We recall the following notation:

- $\mathcal{O}_S(D)$ is the invertible sheaf corresponding to a divisor D.
- $Pic(S)$ is the group of isomorphism classes of invertible sheaves on S.
- $NS(S)$ is the Néron-Severi group of S.
- $q(S) = h^1(\mathcal{O}_S)$ is the irregularity of S.
- $p_g(S) = h^2(\mathcal{O}_S)$ is the geometric genus of S.
- $g(C) = p_a(C) = h^1(\mathcal{O}_C)$ is the arithmetic genus of a smooth curve C.
- $e(S)$ is the topological Euler-Poincaré characteristic of S.
- $\chi(S)$ is the algebraic Euler characteristic of S.

2 General facts on nonsingular surfaces

We start by giving some definitions and basic properties of the objects that are related to this work (see e.g. [Har77], [Bea96]). A projective variety over a field k is a subset of the projective space $\mathbb{P}^n$, which is the zero locus of a finite number of polynomials with coefficients in k. Throughout this work, the field k will be the field $\mathbb{C}$ of complex numbers. A projective surface is a projective variety of dimension two. A non-singular point of a variety S is a point P, which has a regular local ring $\mathcal{O}_P$. A variety with non-singular points is called smooth or non-singular.

A variety is called normal if its local rings are integrally closed domains. The set of singular points of a normal variety has codimension at least two, so we can say that a normal variety has 'simpler' singularities than the ones arising in non-normal varieties. In particular, a normal curve does not have singular points, and every singular point in a normal surface is isolated.

Divisors on a variety X are generalizations of subvarieties of codimension one, and they are used to obtain properties of X. There are several different ways of defining divisors. A *Weil divisor* is a formal sum with multiplicities of subvarieties of codimension one. We say that a Weil divisor is principal if it is the divisor of a rational function.

A *Cartier divisor* D is a global section of the sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$, where $\mathcal{K}_X$ is the sheaf of rational functions on X. D can be described as $D = \{(f_i, U_i)\}$, where $\{U_i\}$ is an open cover of X, $f_i \in \mathcal{K}_X^*(U_i)$, and $f_i f_j^{-1} \in \mathcal{O}_X^*$. We say that a Cartier divisor is principal if $D = \{(f, X)\}$. The set of Cartier divisors forms a group with the multiplication in each open set U_i of the functions f_i. We have two important equivalence relations among the divisors, which are linear equivalence and numerical equivalence. Indeed, we say that two Cartier divisors D, and D' are *linearly equivalent* if their difference is principal, and in such a case, it is denoted by $D \sim D'$. We say that two Cartier divisors D, D' are *numerically equivalent* if their intersections with any curve are equal, and in this case, it is denoted by $D \equiv D'$.

Furthermore, there is a one to one correspondence between Cartier divisors on X and invertible subsheaves of $\mathcal{K}$, which respects linear equivalence. That is, given a divisor $D = \{(f_i, U_i)\}$, we have the invertible sheaf $\mathcal{O}_X(D)$, where $\mathcal{O}_X(D)|_{U_i} = f_i^{-1}\mathcal{O}_{U_i}$. If $D \sim D'$, then $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$. The *Picard group* is the group of isomorphism classes of invertibles sheaves on X, it is denoted by $Pic(X)$. Also, we have the *Néron-Severi group*, which is the group of elements in the Picard group modulo numerical equivalence, it is denoted by $NS(X)$.

The definition of Weil divisor is the same as Cartier divisor in the case where X is a smooth variety. If X is normal, then all Cartier divisors are Weil divisors. Nevertheless, in general, both definitions are not the same.

In addition, we can identify a Cartier divisor with a line bundle. So, when we have a smooth variety, we will refer to the class of a Cartier divisor D, or to the corresponding line bundle $\mathcal{L}_D$, or to the corresponding invertible sheaf $\mathcal{O}_X(D)$, as the same object without distinction.

A *Rational map* $f : X \longrightarrow X'$, is an equivalence class of pairs (f, U), where $f : U \subseteq X \longrightarrow W$ is a morphism and U is an open subset. Two pairs (f, U) and (f', U') are considered equivalent if their restrictions are equal in $U \cap U'$. Given a divisor D, we can associate to it a rational map, to say $\phi_{|D|}$, by choosing a basis of the global section in $\mathcal{O}_X(D)$. Locally in an open set U, the map is the evaluation of the basis at points of U.

2.1 Intersection theory

In this section we list some of the basic results about intersection theory. It can be found in [Bea96] and [Har77].

Definition 2.1. (Intersection Multiplicity) (see e.g [Bea96] Definition I.2). Let C, C' be two distinct irreducible curves on a surface S, $x \in C \cap C'$, $\mathcal{O}_x$ the local ring of S at x. If f (resp. g) is an equation of C (resp. C') in $\mathcal{O}_x$, the *intersection multiplicity* of C and C' at x is defined to be

$$m_x(C \cap C') = \dim_{\mathbb{C}} \mathcal{O}_x/(f, g).$$

Definition 2.2. (Intersection Number) (see e.g. [Bea96] Definition I.3). If C, C' are two distinct irreducible curves on S, the intersection number $(C \cdot C')$ is defined by:

$$(C \cdot C') = \sum_{x \in C \cap C'} m_x(C \cap C').$$

Theorem 2.3. *(Intersection Form) For L, L' in $Pic(S)$, define*

$$(L \cdot L') = \chi(\mathcal{O}_S) - \chi(L^{-1}) - \chi(L'^{-1}) + \chi(L^{-1} \otimes L'^{-1}).$$

then $(\cdot)$ is a symmetric bilinear form on $Pic(S)$, such that if C and C' are two distinct irreducible curves on S then

$$(\mathcal{O}_S(C) \cdot \mathcal{O}_S(C')) = (C \cdot C').$$

Proof. See [Bea96], Theorem I.4. ☐

Theorem 2.4 (Adjunction Formula)**.** *Let D be a smooth divisor in a smooth variety X over $\mathbb{C}$. Then $K_D \sim (K_X + D)|_D$.*

Proof. See [Har77, II], Proposition 8.20. ☐

In what follows S will be a complex smooth projective surface. By taking the divisor D as a smooth curve C on S, in Theorem 2.4, we obtain the Genus Formula

$$2g(C) - 2 = C^2 + C \cdot K_S.$$

It also can be obtained by using the Riemann-Roch Theorem.

Theorem 2.5 (Riemann-Roch Theorem)**.** *For all $D \in Pic(S)$,*

$$\chi(D) = \chi(\mathcal{O}_S) + \frac{1}{2}(D^2 - D \cdot K_S).$$

Proof. See [Bea96], Theorem I.12. ☐

Theorem 2.6 (Noether's Formula)**.**

$$\chi(\mathcal{O}_S) = \frac{1}{12}(K_S^2 + e(S))$$

Theorem 2.7 (Nakai-Moishezon Criterion)**.** *A divisor D on the surface X is ample if and only if $D^2 > 0$ and $D \cdot C > 0$ for all irreducible curves C in X.*

Proof. See [Har77, pp. 365], Theorem 1.10. ☐

2.2 Enriques classification

The Enriques classification consists of associating to each algebraic complex surface a minimal model (up to birational equivalence). We say that two varieties are *birational* if there exists a rational map between them, whose inverse is also a rational map. This definition is a generalization of the isomorphism, in the sense that two birational varieties have open subsets isomorphism, but the birational map is not necessarily defined in all points on the variety. In particular, the blow-up of a point $p \in S$ is a birational map ϵ, which replaces p with the projective tangent space at that point. Some of its properties are contained in the next proposition.

Proposition 2.8. *Let S be a surface and $p \in S$. Then there exist a surface $\hat{S}$ and a morphism $\epsilon : \hat{S} \longrightarrow S$, which are unique up to isomorphism, such that*

(i) The restriction of ϵ to $\epsilon^{-1}(S \setminus \{p\})$ is an isomorphism onto $S \setminus \{p\}$.

(ii) $\epsilon^{-1}(p) = E$, is isomorphic to $\mathbb{P}^1$. It is called the exceptional curve of the blow-up.

(iii) Let C be a curve passing through p, then $\epsilon^* C = \hat{C} + mE$, where $\hat{C}$ is the strict transform of the curve C, and m is the multiplicity of P in C.

(iv) We have the isomorphisms of groups $Pic(\hat{S}) \cong Pic(S) \oplus \mathbb{Z}$, and $NS(\hat{S}) \cong NS(S) \oplus \mathbb{Z}[E]$.

(v) Let D, and D' be divisors on S. Then $(\epsilon^* D) \cdot (\epsilon^* D') = D \cdot D'$, $E \cdot (\epsilon^* D) = 0$, and $E^2 = -1$.

(vi) $K_{\hat{S}} = \epsilon^* K_S + E$.

Proof. See [Bea96, pp.11-12]. □

We recall that a curve $E \subseteq S$ is a (-1)-curve if it is the exceptional curve of a blow-up. The birational maps between surfaces are completely determined by blow-ups, as the following theorem shows.

Theorem 2.9. *Let $f : S \longrightarrow S_0$ be a birational morphism of surfaces. Then there is a sequence of blow ups $\epsilon_k : S_k \longrightarrow S_{k-1}$ $(k = 1, \ldots, n)$ and an isomorphism $u : S \longrightarrow S_n$ such that $f = \epsilon_1 \circ \cdots \circ \epsilon_n \circ u$.*

Proof. See [Bea96], Theorem II.11. □

We denote by $B(S)$ the set of isomorphism classes of surfaces birationally equivalent to S. If S, $S' \in B(S)$, then S is said to dominate S' if there is a birational morphism $S \longrightarrow S'$. A surface is called *minimal surface* (or minimal model) if its class in $B(S)$ is minimal. We would like to have a representative of each class by a minimal surface.

Proposition 2.10. *Every surface dominates a minimal surface.*

Proof. See [Bea96], Theorem II.16. □

By Theorem 2.9, we have that a surface without (-1)-curves is a minimal surface. The following theorem gives us a numerical characterization of (-1)-curves and allows us to construct the minimal model of a given smooth algebraic surface.

Theorem 2.11. *(Castelnuovo's contractibility criterion). Let X be a surface and $E \subseteq X$ a curve isomorphic to $\mathbb{P}^1$ with $E^2 = -1$. Then E is an exceptional curve on X.*

Proof. See [Bea96], Theorem II.17. □

We say that two surfaces are in the same class if they are birational. What follows is a distinguished birational invariant for every variety, which is crucial to develop the classification.

Definition 2.12 (Kodaira dimension)**.** Let X be a smooth projective variety, let K be the canonical divisor of X, and let ϕ_{nK} be the rational map from X to the projective space associated with the system $|nK|$. The *Kodaira dimension* of X, written $\kappa(X)$, is the maximum dimension of the images $\phi_{nK}(X)$, for $n \geq 1$.

In general, given a variety X of dimension n, $\kappa(X)$ can assume the values $-\infty, 0, \ldots, n$. In the case of algebraic curves we have the following classification.

κ	g	Minimal model
$-\infty$	0	The projective line $\mathbb{P}^1$
0	1	Elliptic curves
1	≥ 2	Curves of general type

Table 1: Table of algebraic curves classification.

In the case of surfaces, by starting from the classification by Kodaira dimension, we obtain four different classes. The surfaces in the classes $\kappa(X) = -\infty, 0, 1$ can be classified in much more detail. In fact, there are eight different classes which correspond to algebraic surfaces.

κ	K^2	q	p_g	Minimal model
$-\infty$	8 or 9	0	0	Rational surfaces: $\mathbb{P}^2$,$\mathbb{F}_n$, $\mathbb{P}^1 \times \mathbb{P}^1$
	$8(1-g)$	> 0	0	Ruled surfaces of genus $g \geq 1$: $\mathbb{P}_C(\mathcal{E})$, $C \neq \mathbb{P}^1$,
				where $\mathcal{E}$ is a rank 2 vector bundle over a curve C.
0	0	0	1	K3 surfaces
		0	0	Enriques surfaces
		1	0	Hyperelliptic (Bielliptic) surfaces
		2	1	Abelian surfaces
1	0	≥ 0	≥ 0	Properly elliptic surfaces
2	> 0	≥ 0	≥ 0	Surfaces of general type

Table 2: Table of Enriques classification for algebraic surfaces.

We recall that minimal models are unique (up to isomorphism), with only one exception, rational surfaces, which have infinitely many minimal models. Next, we show some known examples of surfaces with different κ.

Example 2.13. (Complete intersection). Let $S_{d_1,\ldots,d_r}$ be a surface in $\mathbb{P}^{r+2}$ which is the complete intersection of r hypersurfaces of degrees $d_1, \ldots, d_r$, and let K be its canonical class. By the adjunction formula we obtain that $K = (\sum d_i - r - 3)H$, where H is a hyperplane section of the surface (see Theorem 2.4). Thus, for $\sum d_i < r + 3$ we obtain the surfaces S_2 ($\cong \mathbb{P}^1 \times \mathbb{P}^1$), S_3 (del Pezzo surface), and $S_{2,2}$ (del Pezzo surface) which have $\kappa = -\infty$. So they are rational surfaces. For $d_i = r + 3$ we obtain the surfaces $S_4, S_{2,3}$, and $S_{2,2,2}$ with $\kappa = 0$ and $K \equiv 0$. One can check that these surfaces are $K3$ surfaces. For $d_i > r + 3$ we obtain that K is ample, and so the surfaces $S_{d_1,\ldots,d_r}$ have $\kappa = 2$. So they are surfaces of general type.

Example 2.14. (Godeaux surface). A Godeaux surface is a minimal surface of general type with $p_g = q = 0$ and $K^2 = 1$. Let $S = \{x_0^5 + x_1^5 + x_2^5 + x_3^5 = 0\}$ be a quintic surface on $\mathbb{P}^3$. Let $G\colon = \mathbb{Z}/5\mathbb{Z}$, and let ξ be a primitive 5-th root of 1. We define the following action on S

$$\sigma \cdot [x_0, x_1, x_2, x_3] := [x_0, \xi x_1, \xi^2 x_2, \xi^3 x_3],$$

one can check directly that σ is a automorphism without fixed points. So the surface $S'\colon = S/G$ is smooth. It is known as the Godeaux surface. Now, since $K_S \sim \mathcal{O}_S(1)$ (see Theorem 2.4) then $K_S^2 = 5$. By the Riemann-Hurwitz formula we know that $e(S) = 5e(S')$ and $\chi(S) = 5\chi(S')$. So, we obtain that $p_g(S) = 4$, $\chi(S) = 1 + 4 = 5$, and $q(S) = 0$. It follows that $q(S') = 0$,

$\chi(S') = 1$, and $p_g(S') = 0$. Putting all together in Noether's formula for S (see Theorem 2.6) we obtain $12 = 1 + e(S')$, and so by Noether's formula for S' we have that $K^2_{S'} = 1$. Thus, the surface S' is of general type with $p_g = q = 0$, and $K^2_{S'} = 1$.

Example 2.15. (Enriques surface) An Enriques surface is a surface such that the irregularity $q = 0$ and the canonical line bundle K is non-trivial but has trivial square. Let $S_{2,2,2}$ be a complete intersection of 3 quadrics in $\mathbb{P}^5$ (see Example 2.13). Say, the quadrics are $Q_i(x_0, x_1, x_2) + Q'_i(x_3, x_4, x_5)$. We know that $K \equiv 0$, and one can compute that $q(S_{2,2,2}) = 0$. That is, $S_{2,2,2}$ is a $K3$ surface. The involution σ of $\mathbb{P}^5$ defined by

$$\sigma \cdot (x_0, x_1, x_2, x_3, x_4) := (x_0, x_1, x_2, -x_3, -x_4, -x_5),$$

takes X to itself. By choosing generic quadrics Q_1, Q_2, Q_3 (resp. Q'_1, Q'_2, Q'_3) such that they have no points in common in the planes $x_0 = x_1 = x_2 = 0$ (resp. $x_3 = x_4 = x_5 = 0$), we obtain that σ acts on $S_{2,2,2}$ without fixed points, and then the quotient $X := S_{2,2,2}/\sigma$ is smooth. Let $\pi\colon S_{2,2,2} \to X$ the quotient map. Since, we have that $K_X \equiv \pi^* K_{S_{2,2,2}} \equiv 0$ we obtain that $2K_X \equiv \pi_* \pi^* K_{S_{2,2,2}} \equiv 0$ and then X is an Enriques surface.

Theorem 2.16. *(Noether inequality). Let X be a minimal surface of general type. Then*

$$p_g(X) \leq \frac{1}{2} K^2_X + 2.$$

Proof. See [BHPVdV04], Theorem VII.3.1. □

Theorem 2.17. *(Bogomolov-Miyaoka-Yau Inequality) For every surface of general type X the inequality $K^2_X \leq 3e(X)$ holds.*

Proof. See [BHPVdV04], Theorem VII.4.1.

□

3 Singularities and stable surfaces

3.1 General facts on singular surfaces

We list below some definitions and basic facts about singular surfaces. Some of them can be found in [Mum61], [KM08], and [Har77].

Definition 3.1. Let X be a surface. X is *nonsingular* at a point $P \in X$ if the local ring $\mathcal{O}_{P,X}$ is a regular local ring ($\dim_k \mathfrak{m}_P/\mathfrak{m}_P^2 = \dim_{krull} \mathcal{O}_{P,X}$). Otherwise, X is *singular* at P.

Definition 3.2. Let W be a normal surface. A *resolution* of W is a proper birational morphism $\phi\colon X \longrightarrow W$ such that X is smooth.

For the next definition, we consider a normal algebraic surface W with a singularity $P \in W$. Let $\phi\colon X \to W$ be the minimal resolution of P, and let A, B be divisors through P on W. Let $E_1, \ldots, E_n$ be the exceptional divisors which are contracted by ϕ to the point P.

Definition 3.3. (Intersection Theory) (See [Mum61, pp. 241]). The *total transform* A' of A is defined to be

$$A' := A'_0 + \sum r_i E_i$$

where A'_0 is the proper transform of A, and the vector $(r_1, \ldots, r_n)$ is the unique solution of the linear system $(A'_0 \cdot E_j) + \sum_i r_i(E_i \cdot E_j)$ for $j = 1, \ldots, n$. We also define the *intersection number* of A, B at P to be the number such that

$$\begin{aligned} i(A \cdot B, P) &:= \sum_{P' \text{ over } P} \left[i(A'_0 \cdot B'_0, P') + \sum r_i i(E_i \cdot B'_0, P') \right] \\ &= \sum_{P' \text{ over } P} \left[i(A'_0 \cdot B'_0, P') + \sum s_i i(A'_0 \cdot E_i, P') \right] \end{aligned}$$

where $A' := A'_0 + \sum r_i E_i$, and $B' := B'_0 + \sum s_i E_i$ are the total transforms of A, B respectively.

Proposition 3.4. *Under the previous notation, we have the following properties:*

(i) If $A = (f)$ in W then $A' = (f)$ in X; hence $A \sim B$ implies $A' \sim B'$.

(ii) Let A be an effective divisor, then all r_i are positive.

(iii) $i(A \cdot B, P)$ is symmetric and distributive.

(iv) Let A and B be effective divisors, then $i(A \cdot B, P)$ is greater than 0.

(v) $i(A \cdot B, P)$ is independent of the choice of X.

Proof. See [Mum61, pp. 241]. $\square$

Theorem 3.5 (Nakai-Moishezon Criterion)**.** *Let D be a Cartier divisor on a projective surface W. Then D is ample on W if and only if $D^2 > 0$ and $D \cdot C > 0$ for all irreducible curves C in W.*

Proof. See e.g. [Har77, pp. 434], Theorem 5.1.

$\square$

Corollary 3.6. *(Asymptotic Riemann-Roch) Let W be an irreducible projective surface, and let D be a nef divisor on W. Then*

$$h^0\big(W, \mathcal{O}_W(mD)\big) = \frac{(D^2)}{2}m^2 + O(m).$$

More generally,

$$h^0\big(W, \mathcal{F} \otimes \mathcal{O}_W(mD)\big) = rank(\mathcal{F})\frac{(D^2)}{2}m^2 + O(m).$$

for any coherent sheaf $\mathcal{F}$ on W.

Proof. See e.g [Laz17, 1.4.41] $\square$

Definition 3.7 (Volume of a line bundle)**.** (See e.g [Laz17, 2.2.31]). Let W be an irreducible projective surface, and let L be a line bundle on W. The *volume* of L is defined to be the non-negative real number

$$vol(L) = vol_W(L) = \limsup_{m\to\infty} \frac{h^0(X, L^{\otimes m})}{m^2/2}.$$

The volume is an interesting invariant of a big divisor D that measures the asymptotic growth of the linear series $|mD|$ for $m >> 0$.

Definition 3.8. (Contractible) (See e.g [Art62] Section 0). Let X be a complete algebraic surface and let C be a connected curve on X. Assume that X is nonsingular along C. We say that C is *contractible* if there exists a map $\pi\colon X \to W$ onto a surface W which is an isomorphism at every point of $X \setminus C$ and such that $\pi(C)$ is a single point P.

We remark that if C is contractible then π is determined uniquely by the condition that P be a normal point of W.

Theorem 3.9. *(Artin's Contractibility Theorem) Let X be a complete algebraic surface and let $C = \bigcup C_i$ be a connected curve. The following are equivalent:*

(a) C is contractible and if $\pi\colon X \to W$ is the contraction of C then $\chi(W) = \chi(X)$.

(b) i. The intersection matrix $||(C_i \cdot C_j)||$ is negative definite.

ii. For every cycle Z with support on C, $p_a(Z) \leq 0$.

Moreover, if X is a normal projective surface and (a) holds, then W is also projective.

Proof. See [Art62], Theorem 2.3.

□

Definition 3.10. (See e.g [TZ92]). Let X be a smooth projective surface and let $B = \sum B_i$ be a reduced effective divisor on X with only simple normal crossings. The pair (X, B) is called *minimal* if $K_X + B$ has a decomposition into a sum of $\mathbb{Q}$-divisors:

$$K_X + B = \big(K_X + \sum \alpha_i B_i\big) + \sum (1 - \alpha_i) B_i,$$

where

- $0 \leq \alpha_i \leq 1$ and $\alpha_i \in \mathbb{Q}$
- $K_X + \sum \alpha_i B_i$ is numerically effective,
- the intersection matrix of $\sum (1 - \alpha_i) B_i$ is negative definite, and
- if $\alpha_j \neq 1$ then $\big(K_X + \sum \alpha_i \cdot B_j\big) = 0$ and $B_j^2 \leq -2$.

This section ends with two known inequalities that we will use in the proof of Proposition 5.4, and Theorem 1.1 for bounding the length of a singularity only in terms of K_W^2.

Remark 3.11. Let W be a stable surface. Let $\phi\colon X \to W$ be the minimal resolution of W. Assume that $K_X+B \sim \phi^* K_W$, and that (X, B) is a minimal pair. Then we have the generalized Noether's inequality shown in [TZ92],

$$\chi(\mathcal{O}_W) \leq K_W^2 + 3.$$

Assume that W has only isolated quotient singularities. The Bogomolov-Miyaoka-Yau inequality for orbifolds is (see e.g. [Lan03])

$$K_W^2 \leq 3e_{orb}(W),$$

where $e_{orb}(W)$ is the orbifold Euler number of a quasiprojective surface W with only isolated cyclic quotient singularities. That is defined as

$$e_{orb}(W) = e(W) - \sum_{w \in \mathrm{Sing(W)}} \left(1 - \frac{1}{|\pi_1(L_w)|}\right),$$

where L_w is the link of $w \in \mathrm{Sing(W)}$, and π_1 denotes the fundamental group.

3.2 Cyclic quotient singularities

The results listed below can be found in [Ful93, pp.31-50]. We will only consider stable surfaces W with one cyclic quotient singularity P. We denote its minimal resolution by X, and a minimal model of X by S (i.e. S has no (-1)-curves).

Definition 3.12. A two dimensional cyclic quotient singularity is the germ at the origin of the quotient of $\mathbb{C}^2$ by $\mathbb{Z}/n$. It is denoted by $\frac{1}{n}(1, q)$, where $\xi \cdot (x, y) \mapsto (\xi x, \xi^q y)$ is the action of $\mathbb{Z}/n$ on $\mathbb{C}^2$, ξ is a primitive n-root of 1, and $\gcd(q, n) = 1$.

Example 3.13. Toric Construction of $\frac{1}{2}(1, 1)$. Let σ be the cone over $\mathbb{Z}^2$ generated by $e_2 = (0, 1)$ and $2e_1 - e_2 = (2, -1)$. The cone dual $\sigma*$ over $\mathbb{R}^2$ determines a commutative semigroup S_σ which is generated by $e_1^*, e_1^* + e_2^*, e_1^* + 2e_2^*$. Then, we have an finitely generated algebra

$$\mathbb{C}[S_\sigma] := \mathbb{C}[X, XY, XY^2]$$

where $X = \chi^{e_1^*}, Y = \chi^{e_2^*}$. Note that

$$\mathbb{C}[X, XY, XY^2] = \mathbb{C}[u, v, w]/(v^2 - uw).$$

It coincides with the quotient of $\mathbb{C}^2$ by $\mathbb{Z}/2$, where $\xi \cdot (X, Y) = (-X, -Y)$ is the action.

Every cyclic quotient singularity $\frac{1}{n}(1, q)$ can be constructed as a singularity of a toric surface by starting from the cone generated by $e_2 = (0, 1)$ and $ne_1 - qe_2 = (n, -q)$. This construction can be used to obtain an explicit resolution of the singularity, which is entirely determined by the numbers n and q as is described in Proposition 3.16.

Definition 3.14. Let q, n be integers such that $0 < q < n$ and $gcd(q, n) = 1$. We define a *Hirzebruch-Jung continued fraction* of n/q as the expression

$$\frac{n}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_r}}}, \tag{4}$$

where $b_j \geq 1$ are integers, and none of the denominators in (4) are equal to zero. It is denoted by $[b_1, \ldots, b_r]$.

Remark 3.15. If we have that $b_j \geq 2$ then we obtain a one-to-one correspondence between the numbers n/q and the Hirzebruch-Jung continued fraction $[b_1, \ldots, b_r]$. Otherwise, we can have different continued fractions for a number n/q. For instance, we know that $15/4 = [4, 4]$, and $15/4 = [5, 1, 5]$.

Proposition 3.16. *Let W be a surface with a singularity $\frac{1}{n}(1, q)$. Then, the minimal resolution $\phi \colon X \to W$ contains a chain C of exceptional curves $C_1, \ldots, C_r$ such that $C_j \simeq \mathbb{P}^1$, and*

$$C_i \cdot C_j = \begin{cases} 1 & \text{if } i = j \pm 1 \\ -b_j & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

where $[b_1, \ldots, b_r]$ is the Hirzebruch-Jung continued fraction of $\frac{n}{q}$. We say that this singularity has length r.

Given the chain $C = C_1 + \cdots + C_r$, its dual graph is defined as in Figure 1, where the i-th vertex corresponds to the curve C_i, and the edge between the curves C_j and C_{j+1} corresponds to the point in the intersection between them.

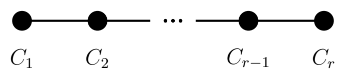

Figure 1: The dual graph of $\frac{1}{n}(1,q)$.

In this case, we have the following numerical equivalence

$$K_X \equiv \phi^* K_W + \sum_{j=1}^{r} a_j C_j \tag{6}$$

where the coefficients a_j are rational numbers $a_j \in]-1,0]$ called discrepancies. We call (6) the *canonical class formula*.

Remark 3.17. The vector of discrepancies is the solution of the following linear system

$$A = \left(\begin{array}{cccccc|c} -b_1 & 1 & 0 & 0 & \cdots & 0 & b_1 - 2 \\ 1 & -b_2 & 1 & 0 & \cdots & 0 & b_2 - 2 \\ 0 & 1 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 & 0 & b_{r-2} - 2 \\ 0 & \cdots & 0 & 1 & -b_{r-1} & 1 & b_{r-1} - 2 \\ 0 & \cdots & 0 & 0 & 1 & -b_r & b_r - 2 \end{array}\right)$$

That linear system can be solved using the tridiagonal matrix algorithm because we have that $b_i \geq 2$ (see e.g. [HJ12]). Then, we obtain the discrepancies from the formulas $a_r = d_r$, and $a_j = d_j - c_j a_{j+1}$ for $j = 2, \ldots r$, where c_j, d_j are auxiliary coefficients defined as follows:

- $c_1 = \dfrac{1}{-b_1}$, and $c_j = \dfrac{1}{-b_j - c_{j-1}}$ for $j = 2, \ldots, r-1$.
- $d_1 = \dfrac{b_1 - 2}{-b_1}$, and $d_j = \dfrac{1}{-b_j - c_{j-1}}$ for $j = 2, \ldots, r$.

Following Proposition 3.16, we denote by $[b_1, \ldots, b_r]$ the continued fraction of P. Also, we denote by q' the inverse of q modulo n, that is, the unique integer $0 < q' < n$ such that $qq' \equiv 1(mod\ n)$.

Proposition 3.18. *Let W be a normal projective surface with only one singularity and of type $\frac{1}{n}(1, q)$. Let $\phi\colon X \to W$ be the minimal resolution of W. Then we have*

$$K_X^2 = K_W^2 + \sum_{j=1}^{r}(2 - b_j) + \frac{2(n-1) - q - q'}{n}.$$

Proof. See e.g. the proof of Proposition 3.4 in [Urz10]. □

3.3 Stable surfaces

This section introduce two essential concepts that appear in the minimal model program (MMP for short), and the moduli theory. The MMP aims to study geometric and cohomological properties of algebraic varieties by constructing a birational model that is as simple as possible. It starts with Mori for varieties of higher dimension (see [Mor87]). In the case of dimension two, this program agrees with the theory of minimal models of smooth surfaces developed by Castelnuovo and Enriques. Contrary to the surfaces, a minimal model of a smooth variety of higher dimensions is usually a singular variety.

On the other hand, moduli theory aims to construct and describe a suitable space parameterizing surfaces of general type with K^2 and χ fixed. In compactifying such a space, Kollár and Shepherd-Barron also obtained the same class of singularities of the MMP. Both theories include not only varieties X but pairs of the form (X, B), where B is a suitable divisor.

Definition 3.19. Let X be a smooth variety and $D \subset X$ a divisor. We say that D is a *simple normal crossing* divisor, snc divisor for short, if every irreducible component of D is smooth and all intersections are transverse.

Definition 3.20. Let X be a normal variety and $B = \sum b_i B_i$ a $\mathbb{Q}$-divisor on X such that $K_X + B$ is $\mathbb{Q}$-Cartier. A *log resolution* of (X, B) is a proper birational morphism $\phi : Y \longrightarrow X$ such that Y is smooth and $Exc(\phi) \cup \phi_*^{-1}(B)$ has simple normal crossing support.

We can write

$$K_Y \equiv \phi^*(K_X + B) + \sum a(E_i, X, B) E_i$$

where E_i are the divisors in $Exc(\phi)$ and $\phi_*^{-1}(B)$. The numbers a_i are called the discrepancies of E_i with respect to (X, B).

In the following table, we find some of the singularities needed to run the minimal model program. We say that (X, B) is:

Terminal	if $a_i > 0$
Canonical	if $a_i \geq 0$
Kawamata log terminal (klt)	if $a_i > -1$ and $\lfloor B \rfloor = 0$
Log canonical (lc)	if $a_i \geq -1$

where $\phi : Y \longrightarrow X$ is a resolution of (X, B). One can prove that the above definitions are independent of the choice of the resolution. Assuming $B = 0$, terminal singularities are the smallest class needed for running the MMP, starting with smooth varieties. Canonical singularities appear on canonical models of varieties of general type. In the case of surfaces, they have been studied since antiquity, and they are known as Du Val singularities (see e.g [Dur79]). The terminal and canonical singularity log version appear when we run the MMP for pairs (X, B). Also, there are other classes of singularities which are defined in [KM08] for technical purposes, namely purely log terminal singularities (plt for short) and divisorial log terminal singularities (dlt for short). Both of them agree with klt singularities when $B = 0$, hence they are known as *log terminal* singularities.

Theorem 3.21. *The normal surface singularity $P \in W$ is log terminal if and only if it is a quotient singularity.*

Proof. See [Kaw88]. □

In the case of having a non-normal variety, we also have a similar notion of a log canonical pair. It was introduced by Kollár and Shepherd-Barron in [KSB88, 4.17], and it is known as *semi log canonical* pair, slc for short. Since, we will only work with normal surfaces with only one cyclic quotient singularity, we will not introduce technical details about slc singularities, but these can be found in [KSB88]. Scl singularities appear on stable degenerations of smooth varieties of general type. Hence, slc singularities play an important role in moduli theory.

A moduli space is a geometric space whose points represent isomorphism classes of some geometric objects. For instance, $\mathcal{M}_g$ is the moduli space of smooth projective curves of genus $g \geq 2$ together with their isomorphisms.

Because there are smooth curves which can be degenerated into singular curves, we have that $\mathcal{M}_g$ is not a complete space. One of the ways for constructing a compactification of $\mathcal{M}_g$ is to add more curves that maybe resolve the new moduli problem. That curves are called Deligne-Mumford moduli-stable curves. They are defined as connected and complete reduced curves X with ordinary nodes only, such that the automorphism group $Aut(X)$ is finite, and the dualizing sheaf K_X is ample. This construction is supported into the invariant geometric theory, and it involves the minimal model program for surfaces.

Similarly to the construction of $\mathcal{M}_g$, Gieseker in [Gie77] showed the existence of a quasi-projective scheme parameterizing surfaces with at worst Du Val singularities, ample canonical class K, and K^2 fixed. As well as in the case of curves, this space is not complete. In order to obtain a complete such a space, Kollár–Shepherd-Barron enlarge the parameter space to one that parameterizes smoothable stable surfaces with K^2. They constructed such a space as a separated algebraic space. It is denoted by $\mathcal{M}^{sm}_{K^2}$. (See [KSB88]).

Definition 3.22. A *stable surface* is a projective surface S such that S has only semi-log-canonical singularities, and $\omega_S^{[k]}$ is locally free and ample for some $k > 0$.

We can find a more general version of Definition 3.22 in [Kol90, Definition 5.2].

Definition 3.23. A singularity (X, x) is called *smoothable* if there exists a one-parametric deformation $\phi\colon (\mathscr{X}, x) \to (\mathbb{C}, 0)$ of (X, x) such that for $t \in \mathbb{C} \setminus \{0\}$ sufficiently close to 0 the fibre $\mathscr{X}_t = \phi^{-1}(t)$ is smooth.

Furthermore, Kollár proved that if $\mathcal{M}^{sm}_{K^2}$ is bounded, then it is coarsely represented by a projective algebraic scheme (See [Kol90, Corollary 5.6]), and that boundedness was proved by Alexeev in [Ale94]. The moduli space $\mathcal{M}_{(K+B)^2,\chi}$ for stable pairs (X, B) and its (KSBA) compactification can be found in [Ale96].

3.4 Some facts about volumes of stable surfaces

In this section, we list some of the known results about the set $\mathbb{K}^2(\mathcal{C})$. As part of the work carried out for the compactification of the moduli space in

[Ale94], Alexeev proved the *Descending Chain Condition*, D.C.C. for short, for the set

$$\mathbb{K}^2(\mathcal{C}) := \{(K_X + B)^2 : (X, B) \in \mathcal{S}(\mathcal{C}), K_X + B \text{ is ample}\},$$

where $\mathcal{S}(\mathcal{C})$ is the set of log-canonical projective surfaces (X, B) such that the coefficients of B belong to $\mathcal{C}$.

Definition 3.24. A set $\mathcal{C} \subset \mathbb{R}$ is called a D.C.C set if it satisfies the descending chain condition: a decreasing subsequence of $\mathcal{C}$ eventually stabilizes.

Theorem 3.25 ([Ale94],8.2)**.** *Fix a D.C.C set $\mathcal{C}$. Consider all log canonical projective surfaces X with an $\mathbb{R}$-divisor $B = \sum b_i B_i$ such that $K_X + B$ is ample, and $b_i \in \mathcal{C}$. Then the set $\{(K_X + B)^2\}$ is a D.C.C set.*

Remark 3.26. The three most commonly used sets of coefficients are $\mathcal{C}_0 = \emptyset$, $\mathcal{C}_1 = \{1\}$, and $\mathcal{C}_2 = \{1 - \frac{1}{n} | n \in \mathbb{N}\} \cup \{1\}$.

The D.C.C. is an important property of $\mathrm{Acc}(\mathbb{K}^2(\mathcal{C}))$, the set of accumulation points of $\mathbb{K}^2(\mathcal{C})$. The problem of describing this set remains still open. The following result shows that $\mathbb{K}^2(\emptyset)$ is unbounded even if we fix the geometric genus.

Theorem 3.27. *[UU19, Thm. 1.9] Given integers $g \geq 0$ and N, there exists a normal projective surface X over $\mathbb{C}$ with the following properties:*

1. *X has geometric genus $p_g = g$.*
2. *X has only one singular point, which is log-terminal.*
3. *K_X is $\mathbb{Q}$-Cartier and ample.*
4. *$K_X^2 > N$.*

Theorem 3.28. *[AL19b, Theorem 1.1] Suppose that $\mathcal{C} \subset (0, 1]$ satisfies the descending chain condition. Then $v_\infty \in \mathbb{R}_{>0}$ is an accumulation point of $\mathbb{K}^2(\mathcal{C})$ if and only if there exists a log canonical surface $(W, B) \in \mathcal{S}(\overline{\mathcal{C}} \cup \{1\})$ such that*

1. *$K_W + B$ is ample, and $v_\infty = (K_W + B)^2$.*
2. *One of the following conditions is satisfied:*

(a) *The set of codiscrepancies of divisors* over W *with respect to* (W, B) *contains an accumulation point of* $\mathcal{C}$.

(b) (W, B) *has an accessible nklt center.*

3. *If* 1 *is not in the closure* $\overline{\mathcal{C}}$ *of* $\mathcal{C}$ *then each irreducible component of* $\lfloor B \rfloor$ *has geometric genus at most 1.*

Corollary 3.29. *[AL19b, Corollary 1.3] Let* $\mathcal{C} \subset (0, 1]$ *be a D.C.C set. Then* $Acc(\mathbb{K}^2(\mathcal{C})) \subset Acc(\mathbb{K}^2(\bar{\mathcal{C}} \cup \{1\}))$. *In particular, if* $\bar{\mathcal{C}} \subset \mathbb{Q}$ *then* $Acc(\mathbb{K}^2(\mathcal{C})) \subset \mathbb{Q}$.

Theorem 3.30. *[AL19b, Theorem 1.7] The following is true:*

(i) *For* $\mathcal{C} = \mathcal{C}_0$, $\mathcal{C}_1$ *and* $\mathcal{C}_2$, *ones has* $\min\ \mathrm{Acc}(\mathbb{K}^2(\mathcal{C})) = \min \mathbb{K}^2_{nklt}(\mathcal{C})$.

(ii) *One has* $\frac{1}{86436} = \frac{1}{7^2 \cdot 42^2} \leq \min \mathbb{K}^2_{nklt}(\mathcal{C}_2) \leq \frac{1}{1764} = \frac{1}{42^2}$.

(iii) *For* $\mathcal{C} = \mathcal{C}_0$ *or* $\mathcal{C}_1$, *one has* $\frac{1}{86436} = \frac{1}{7^2 \cdot 42^2} \leq \min \mathbb{K}^2_{nklt}(\mathcal{C}) \leq \frac{1}{462} = \frac{1}{11 \cdot 42}$.

4 Generalized T-singularities

4.1 T-singularities

Singularities which admit a one-parameter $\mathbb{Q}$-Gorenstein smoothing are important because only surfaces with these singularities can appear in the boundary of $\overline{\mathcal{M}}_{K^2,\chi}$ as stable limits of families of surfaces in $\mathcal{M}_{K^2,\chi}$. The facts that we list below can be found in [KSB88].

Definition 4.1. ([KSB88], Definition 3.1). A normal surface W is $\mathbb{Q}$-Gorenstein if some nonzero integral multiple mK_W of the canonical divisor K_W is Cartier.

Definition 4.2. ([KSB88], Definition 3.7). A normal surface singularity is of class T if it is a quotient singularity and admits a one-parameter $\mathbb{Q}$-Gorenstein smoothing. We also called it a T-singularity.

The next proposition gives us a characterization of the T-singularities which are not rational double points, RPD for short.

Proposition 4.3. *(i) The cyclic quotient singularities associated to* $[4]$ *and* $[3,2,\ldots,2,3]$ *are T-singularities.*

(ii) If the singularity associated to $[b_1,\ldots,b_r]$ *is a T-singularity, then so are* $[2,b_1,\ldots,b_r+1]$ *and* $[b_1+1,\ldots,b_r,2]$*.*

(iii) Every T-singularity that is not an RDP can be obtained by starting with one of the singularities described in (i)and iterating the steps described in (ii).

Proof. See [KSB88], Proposition 3.11. □

It is follows from Proposition 4.3 that any T-singularity that is not an RDP is a cyclic quotient singularity of type $\frac{1}{dn^2}(1, dna-1)$ where $(n,a)=1$, and d is square-free. In particular, when $d=1$ we have the following definition.

Definition 4.4. A *Wahl singularity* is a cyclic quotient singularity $\frac{1}{n^2}(1, na-1)$ where $(n,a)=1$.

Wahl singularities are the cyclic quotient singularities which admit only one one-parameter $\mathbb{Q}$-Gorenstein smoothing. Proposition 4.3 was proved by Wahl in [Wah81] in the case of Wahl Singularities.

4.2 Admissibility

This section describes the Hirzebruch-Jung continued fractions that are admissible for chains (see Definition 1.5). Given a singularity $[b_1, \ldots, b_r]$, we say that a coefficient b_i does not contract if the curve associated with b_i does not.

Lemma 4.5. *If $[b_1, \ldots, b_r]$ is admissible for chains, then there are b_i, b_j with $i \leq j$ such that*

$$[b_1, \ldots, b_r] - 1 - [b_1, \ldots, b_r] - 1 - [b_1, \ldots, b_r]$$

does not contract b_i and b_j.

Proof. Otherwise we would have eventually inside of the contraction a situation $[1, 1]$, and that makes the chain not admissible. In order to prove that, we will use induction on r.

We first compute the base case for $r = 1$. Here, to contract b_1 we must have either $b_1 = 2$ or $b_1 = 3$, and so we obtain either the situation $[1, 1, 1, 2]$ or $[1, 1]$, respectively.

Let us suppose that for every $[\mathbf{a}] = [a_1, \ldots, a_k]$, and $k < r$ if a_i are contracted for every i, then we obtain eventually the situation $[1, 1]$ inside of the contraction $[\mathbf{a}] - 1 - [\mathbf{a}] - 1 - [\mathbf{a}]$. Now, let $k = r$. We must have that $b_1 = 2$ or $b_r = 2$. Assume, without loss of generality (we could flip the order), that $b_r = 2$. Then, after contracting b_r, we have

$$1 + [b_1 - 1, \ldots, b_{r-1}, 1, b_1 - 1, \ldots, b_{r-1}, 1, b_1 - 1, \ldots, b_{r-1}, 2],$$

and so, we obtain inside of it the situation $[\mathbf{a}] - 1 - [\mathbf{a}] - 1 - [\mathbf{a}]$, where $[\mathbf{a}] = [b_1 - 1, b_2, \ldots, b_{r-1}]$. By the inductive hypothesis, we conclude that the situation $[1, 1]$ will appear inside of $[\mathbf{a}] - 1 - [\mathbf{a}] - 1 - [\mathbf{a}]$, and so inside of $[b_1, \ldots, b_r] - 1 - [b_1, \ldots, b_r] - 1 - [b_1, \ldots, b_r]$. $\square$

The $[b_i, \ldots, b_j]$ is a sort of core which is necessary for the property admissible for chains.

Lemma 4.6. *Let $0 < a < n$ be coprime integers, let $\frac{n}{a} = [x_1, \ldots, x_f]$ and $\frac{n}{n-a} = [y_1, \ldots, y_g]$. Then*

$$[x_1, \ldots, x_f, 1, y_g, \ldots, y_1] = 0.$$

Lemma 4.6 is well-known, and it is the justification for the Riemenschneider's dot diagram. For example, if $\frac{n}{n-a} = [2, \ldots, 2, y_i, \ldots, y_g]$ where $y_i > 2$, then $x_1 = i - 1 + 2 = i + 1$.

Definition 4.7. A *core* is a Hirzebruch-Jung continued fraction $[e_1, \ldots, e_s]$ such that $e_i > 1$ for all i and either

(1) $s = 1$ and $e_1 \geq 4$

(2) $s \neq 1$ and $e_1 \geq 3$ and $e_s \geq 3$.

In this way, the limit cases $[4]$ and $[3, e_2, \ldots, e_{s-1}, 3]$ are cores, and with $e_i = 2$ for $i = 2, \ldots, s-1$, they are exactly the cores of T-chains. One can check by a direct computation that every core is admissible for chains. The remarkable fact is that all $[b_1, \ldots, b_r]$ admissible for chains are constructed from a core following the formation rule of T-chains.

Theorem 4.8. *Let $[b_1, \ldots, b_r]$ be an admissible for chains continued fraction. Then there is a unique core $[e_1, \ldots, e_s]$ such that $[b_1, \ldots, b_r]$ is obtained by applying the T-chain algorithm to $[e_1, \ldots, e_s]$.*

Proof. Consider the center $[b_i, \ldots, b_j]$ of $[b_1, \ldots, b_r]$ as shown in Lemma 4.5, adding the condition that i is the minimal index such that b_i is not contracted. Similarly, we ask for j to be the maximal index such that b_j is not contracted.

First, we assume that $i = 1$, and $j = r$. In this case, we obtain directly that $[b_1, \ldots, b_r]$ is a core. So, we take $[e_1, \ldots, e_s] = [b_1, \ldots, b_r]$.

In what follows, we will suppose that $1 < i$ or $j < r$. Let us write

$$[b_1, \ldots, b_r] = [a_1, \ldots, a_u, b_i, \ldots, b_j, c_1, \ldots, c_v],$$

of course keeping the position of $[b_i, \ldots, b_j]$. Note that the initial conditions over i, j imply that $[c_1, \ldots, c_v, 1, a_1, \ldots, a_u]$ will disappear. Assume, without loss of generality (we could flip the order), that a_u is the last curve that disappears. In particular, we have that $1 < i$. Now, we should treat $j = r$ and $j < r$ separately.

Case A. Say $j = r$. Observe that $[a_1, \ldots, a_u] = [2, \ldots, 2]$, and that

$$[u + 1, 1, a_1, \ldots, a_u] = 0.$$

As b_i, b_j have to survive, we obtain that $b_i \geq 3$ and $b_j \geq u + 3$.

In this case, we have that $[b_1,\dots,b_r]$ is obtained by applying the T-chain algorithm to $[e_1,\dots,e_s]:=[b_i,\dots,b_{j-1},b_j-u]$. We observe that if $i<j$ then $[e_1,\dots,e_s]$ is a core. However, if $i=j$ we have to prove that $b_j-u\geq 4$. On the contrary, let us suppose that $b_j-u=3$. Then, we obtain the situation $[a_1,\dots,a_u,b_j-(u+1),b_j-(u+2),b_j-1]=0$ inside of

$$[b_1,\dots,b_r]-1-[b_1,\dots,b_r]-1-[b_1,\dots,b_r]. \tag{7}$$

But, this is impossible because $b_j=u+3$ must survive. So, we obtain that $b_j\geq 4$. Thus, we know that $[e_1,\dots,e_s]$ is a core.

Case B. Say $j<r$. Let $[a_1,\dots,a_u]=[a_1,\dots,a_w,2,\dots,2]$ with $a_w>2$ and say that the number of 2's at the end is l. Then, one can check that

$$[l+2,c_1,\dots,c_v,1,a_1,\dots,a_w,2,\dots,2]=0.$$

As b_i,b_j have to survive, we know that $b_i\geq 3$ and $b_j\geq l+4$. Let $[e_1,\dots,e_s]=[b_i,\dots,b_{j-1},b_j-(l+1)]$. The Riemenschneider's dot diagram will then give the algorithm from T-chains.

On the other hand, we note that $[e_1,\dots,e_s]$ is a core if $i<j$. Similarly to Case A, if $i=j$ and $b_j=l+4$ then we obtain $[a_1,\dots,a_u,b_j-(u+2),b_j-(u+3),b_j-1,c_1,\dots,c_v]=0$ inside of (7). But, this is impossible. So, if $i=j$ then $b_j\geq l+5$. Thus, we have that $[e_1,\dots,e_s]$ is a core. Due to the choice of $[b_i,\dots,b_j]$, we conclude that $[e_1,\dots,e_s]$ is unique. $\square$

Definition 4.9. Let $[e_1,\dots,e_s]$ be a core, we say that $[e_1,\dots,e_s]$ is *minimal* if it cannot be obtained from another core $[b_1,\dots,b_r]$ by inserting 1's (see Definition 1.8, (i)).

Remark 4.10. Let $[b_1,\dots,b_r]$ be an admissible for chains continued fraction, and let $[e_1,\dots,e_s]$ be its associated core (see Theorem 4.8). It is immediate that the set of generalized T-singularities of center $[b_1,\dots,b_r]$, is contained in the set of generalized T-singularities of center $[e_1,\dots,e_s]$. In particular, we have that fact if the core is minimal. For instance, the family of T-singularities is obtained by starting with the minimal core [4]. We classify the minimal cores in Proposition 4.11.

Proposition 4.11. *A core $[e_1,\dots,e_s]$ is minimal if and only if one of the following cases holds:*

(i) s is a prime number and $[e_1,\dots,e_s]\neq[e_1,e_1-1,\dots,e_1-1,e_1]$.

(ii) s is not prime and for every $1 < u < s$ divisor of s (say $s = ur$), either

- *there exist $2 \le i < r$, and $1 \le j < u$ such that $e_i \ne e_{i+jr}$,*
- *there exists $1 \le j < u$ such that $e_{1+jr} + 1 \ne e_1$, or*
- *there exists $1 \le j < u$ such that $e_{r+jr} + 1 \ne e_s$.*

Proof. Let $[b_1, \dots, b_r]$ be a core, and let $[c_1, \dots, c_{kr}]$ be the continued fraction $[b_1, \dots, b_r, 1, b_1, \dots, b_r, 1, \dots, 1, b_1, \dots, b_r]$, where $k-1$ is the number of inserted 1's. We start by analysing the coefficients of the new continued fraction. Indeed, we obtain the following:

- For every $2 \le i < r$, and $1 \le j < k$ we have that $c_i = c_{i+jr} = b_i$.
- For every $1 \le j < k$ we have that $c_{1+jr} = c_1 - 1 = b_1 - 1$.
- For every $1 \le j < k$ we have that $c_{r+jr} = c_{kr} - 1 = b_r - 1$.

In particular, if we fix a divisor $1 < u < kr$ of k then $c_{1+jvr} = c_1 - 1$, and $c_{r+jvr} = c_r - 1$ for every $1 \le j < u$. Also, we have that for every $2 \le i < vr$, and $1 \le j < u$ we have that $c_i = c_{i+jvr}$. Thus, we can also obtain $[c_1, \dots, c_{kr}]$ from a core $[a_1, \dots, a_{vr}]$ by inserting $u-1$ 1's, where $k = uv$. However, it may not be valid if we choose a divisor u of r.

Now, let $[e_1, \dots, e_s]$ be a core. Assume that s is not a prime number. Then, we have that the core $[e_1, \dots, e_s]$ is not minimal if and only if it fulfills the conditions above for some divisor $u > 1$ of s.

Say s is prime. By the conditions shown above, we have that $[e_1, \dots, e_s]$ is a minimal core if and only if $[e_1, \dots, e_s] = [e_1, e_1 - 1, \dots, e_1 - 1, e_1]$ (it is obtained by starting in $[e_1 + 1]$).

□

Proposition 4.12. *Let $[b_1, \dots, b_s] = n/q$ be a continued fraction. Let q' be the inverse of q modulo n with $0 < q' < n$, and let m be the integer such that $qq' = 1 + mn$. Assume that $n > 2$. Then, we have that $[b_1 + 1, b_2, \dots, b_s, 2] = N/Q$, where $N = 2q - m + 2n - q'$, and $Q = 2q - m$. Moreover, we have that $[2, b_s, \dots, b_2, b_1 + 1] = N/Q'$, where $Q' = q + n$.*

Proof. We know that $\frac{n}{q} = [b_1, \dots, b_s]$ implies $[b_1, \dots, b_s] = \frac{n}{q'}$, where q' is the inverse of q modulo n. So, we obtain that $[2, b_s, \dots, b_1] = \frac{2n-q'}{n}$. Now, we would like to find n' the inverse of n modulo $2n - q'$. We observe that $n(2q -$

$m) = q(2n - q') + 1$, so $n' \equiv (2q - m)\ mod\ (2n - q')$. Thus, $[b_1, \dots, b_s, 2] = \frac{2n-q'}{n'}$.

We will prove that $n' = 2q - m$. Observe that $m < q$. Otherwise, if we have that $m > q$, then $mn+1 \geq qn > qq'$, but $qq' = mn+1$. Also, if we have $m = q$, then $q(q' - n) = 1$. But this is impossible. So we know that $m < q$. In the same way, we obtain that $m < q'$. So, we have that $2q - m > 0$. In addition, because $m + 1 \leq q'$, and $2n - q' > 2$ (if $n > 2$), then we have that $2 + m(2n - q') < q'(2n - q')$. So, we obtain that $(2q - m) < (2n - q')$ by using $q' > 0$, and $qq' = 1 + mn$. Thus, $n' = 2q - m$.

Therefore, we have that $[b_1 + 1, b_2, \dots, b_s, 2] = \frac{N}{Q}$, where $N = 2q - m + 2n - q'$ and $Q = 2q - m$.

Finally, we show that the inverse of Q modulo N is $q + n$. Indeed, note that $(2q - m)(q + n) \equiv 1\ mod(n)$. Also, by using that $m < q$ we obtain that $N = q + n + (q + n - m - q') > q + n > 0$. Thus, we obtain that $[2, b_s, \dots, b_2, b_1 + 1] = N/Q'$, where $Q' = q + n$. $\square$

5 Bounding the case with one cyclic quotient singularity P

5.1 Type of diagrams for exceptional divisors with $E \cdot C = 1$

We start with some notation. Let us consider the diagram

$$\begin{tikzcd} & X \arrow[dl, "\pi"'] \arrow[dr, "\phi"] & \\ S & & W \end{tikzcd}$$

where $\phi : X \longrightarrow W$ is the minimal resolution of W, and $\pi \colon X \to S$ is a birational morphism to the minimal model S. Thus it is a composition of blow ups, each of which contracts a single (-1)-curve $F_i \subset X_i$ to a point $x_{i-1} \in X_{i-1}$. In this way we have the diagram:

$$X = X_m \stackrel{\pi_m}{\to} X_{m-1} \stackrel{\pi_{m-1}}{\to} \cdots \stackrel{\pi_2}{\to} X_1 \stackrel{\pi_1}{\to} X_0 = S$$

Let us define $E_m := F_m$, and for each $i \in \{1, \ldots, m-1\}$

$$E_i := (\pi_{i+1} \circ \pi_{i+2} \circ \cdots \circ \pi_m)^*(F_i) \subset X. \tag{8}$$

It follows from the definition that $E_i^2 = -1$ and $E_i \cdot E_j = 0$ whenever $i \neq j$. Furthermore, we have that each E_i is not necessarily reduced, and its support is a tree of smooth rational curves. Assuming that $m > 0$, each E_i contains at least one (-1)-curve, and their irreducible components intersect transversally at most once. Of course we have

$$K_W^2 - K_S^2 = \sum_{j=1}^{r} (b_j - 2) - m - \left(\frac{2(n-1) - q - q'}{n} \right). \tag{9}$$

Lemma 5.1. *We have $\left(\sum_{i=1}^m E_i \right) \cdot C = \sum_{j=1}^r (b_j - 2) - \lambda$, where $\lambda = \pi^* K_S \cdot C$.*

Proof. It follows directly from $K_X \cdot C = \sum\limits_{j=1}^{r} (b_j - 2)$, and $\sum_{i=1}^m E_i = K_X - \pi^* K_S$. $\square$

In order to describe the behavior of the accumulations points of volumes, we will find a suitable lower bound for the intersection between C and $\sum_{i=1}^{m} E_i$. We first introduce a graph Γ_{E_i} for each exceptional divisor, as it was done in [Ran17, pp.9]. It is constructed by replacing the j-th vertex in the dual graph of C, by a box if $C_i \subset E_i$. For instance, if we have C_1, C_5 belonging to E_i, the Γ_{E_i} is as in Figure 2.

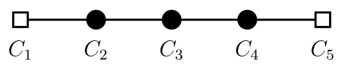

Figure 2: Example of the graph of E_i.

As a way of example, it follows from Figure 2 that there are at least two points in the intersection of curves in C not in E_i and E_i, which correspond to the two extreme edges of the graph.

Lemma 5.2. *For any i, we have $E_i \cdot C \geq 1$.*

Proof. First we observe that if $C_j \subset E_i$, then $E_i \cdot C_j = -1$ is only possible for one j. Otherwise, we have $E_i \cdot C_j = 0$. Since, there is a (-1)-curve $F \subset E_i$, and because of ampleness of K_W then we have that $F \cdot C \geq 2$. Hence, we have that E_i intersects with $C \setminus E_i$ in at least 2. Thus, we conclude that $E_i \cdot C \geq 1$. □

Remark 5.3. As we saw in the proof, we remark that for any (-1)-curve F in X we must have $F \cdot C \geq 2$. (This is because K_W is ample.) Similarly, any (-2)-curve in X must intersect the chain C positively. In addition, note that we have $\sum_{i=1}^{m} E_i \cdot C \geq m + 1$.

In this section, we consider a normal stable surface W with only one cyclic quotient singularity P, following the notation used previously. The goal is to show optimal bounds for the continued fraction associated to P. To start, we can easily see that if every exceptional divisor E_i satisfies $E_i \cdot C \geq 2$, then we obtain the following bounds.

Proposition 5.4. *Assume that $E_i \cdot C \geq 2$ for all i. Then*

$$\sum_{j=1}^{r} \big(b_j - 2\big) \leq 2(K_W^2 - K_S^2) + 2\left(\frac{2(n-1) - q - q'}{n}\right) - \pi^* K_S \cdot C, \qquad (10)$$

and

$$r \leq 13K_W^2 - 2K_S^2 + 38 - \left(\frac{2+q+q'}{n}\right) - \pi^* K_S \cdot C. \tag{11}$$

Proof. This corresponds to have $\delta = 0$ in Theorem 1.1. □

In this way, if K_S is nef, then we can bound singularities for all such W with bounded K_W^2.

Remark 5.5. In the particular case when $m = 0$, we have that

$$\sum_{j=1}^{r} \big(b_j - 2\big) = (K_W^2 - K_S^2) + 2 - \left(\frac{2+q+q'}{n}\right),$$

and then, by using (18) we have

$$r \leq 12K_W^2 - K_S^2 + 36.$$

So, if $K_W \leq c$ for some positive number c, and K_S is nef, we obtain finitely many options for $b_1, \ldots, b_r$, and r. Thus, in what follows we will assume that $m > 0$.

Therefore, the critical case is when there exist an exceptional divisors E_i such that

$$E_i \cdot C = 1.$$

Remark 5.6. Using the same strategies as in the proof of Lemma 5.2, we can see that if there are at least three points in the intersection of curves in $C \setminus E_i$ and curves in E_i then we have that $E_i \cdot \big(\sum_{j=1}^{r} C_j\big) \geq 2$. Thus, if we have $E_i \cdot \big(\sum_{j=1}^{r} C_j\big) = 1$ then there are two or fewer points on this intersection, and Γ_{E_i} must be one of the following (see [RU19, pp. 6]):

Figure 3: Case A.

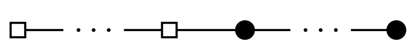

Figure 4: Case B.

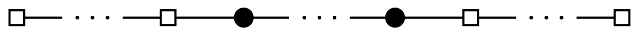

Figure 5: Case C.

In order to describe more in detail the behavior of Γ_{E_i}, where E_i is an exceptional divisor such that $E_i \cdot C = 1$, we will introduce the following definition.

Definition 5.7. Let k, l be positive integers. We say that E_i has a *long diagram* if Γ_{E_i} is a diagram of type (i), (ii), (iii), (iv), and there is a (-1)-curve F as shown in the following figures.

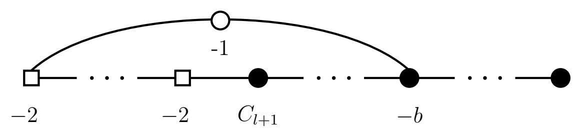

Figure 6: Diagram of type (i).

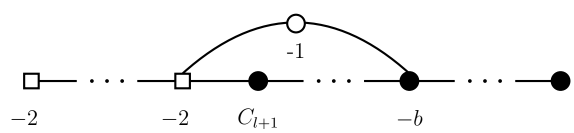

Figure 7: Diagram of type (ii).

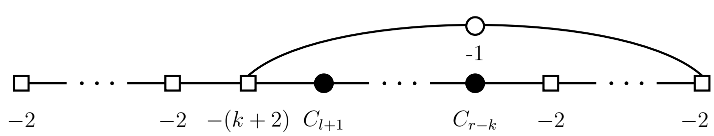

Figure 8: Diagram of type (iii).

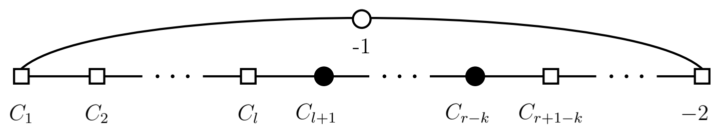

Figure 9: Diagram of type (iv).

Now, we order the set of exceptional divisors according to their graph. Indeed, we say that Γ_{E_i} is a subtree of Γ_{E_j} if every $\square$-vertex of Γ_{E_i} is a $\square$-vertex of Γ_{E_j}. Note that the set of the graphs Γ_{E_i} is a partially ordered set with the following order:

$$\Gamma_{E_i} \leq \Gamma_{E_j} \iff \Gamma_{E_i} \text{ is a subtree of } \Gamma_{E_j}.$$

Definition 5.8. We say that the graph Γ_{E_i} is called *maximal* if it is a maximal element with respect $\leq$, and $E_i \cdot C = 1$.

By adding the discarded cases on Lemma 2.7 in [RU19], which is valid in the context of T-singularities, we obtain the following result to the general case of cyclic quotient singularities.

Lemma 5.9. *Suppose that $E_i \cdot C = 1$ for some i, then E_i has a long diagram. Moreover, if E_i has a diagram of type (iv), and Γ_{E_i} is maximal, then there exists a sequence $\{m_1, \ldots, m_s\}$ of natural numbers such that C has continued fraction*

$$[\ldots, 2, 2+m_3, \underbrace{2, \ldots, 2}_{m_2-1}, 2+m_1, a_1, \ldots, a_t, \underbrace{2, \ldots, 2}_{m_1-1}, 2+m_2, 2, \ldots],$$

where $-a_1, \ldots, -a_t$ correspond to the self-intersection of the $\bullet$ curves in Γ_{E_i}, and one of the ends is 2 and the other one is $2+m_s$.

Proof. We divide this proof into the three cases of Remark 5.6. In the first two cases, the argument is the one used in Lemma 2.7 in [RU19] for T-singularities.

Case (1). Assume that E_i has the diagram shown in Figure 3. Because of the ampleness of K_W we obtain that there is a (-1)-curve F in E_i which intersects C twice (see Remark 5.3). In this situation, we would obtain either a loop in E_i or a third point of intersection with $C \setminus E_i$. But these cannot happen because E_i is a tree of rational curves and $E_i \cdot C = 1$.

For the next case, we will denote by $C_1, \ldots, C_l$ the $\square$ curves on the left side in the diagrams shown in Figure 4 and 5.

Case (2). Suppose that E_i has the diagram shown in Figure 4. By the same argument done in Case (1), there exists a (-1)-curve F in E_i which intersects a $\square$ curve C_j, and a $\bullet$ curve $C_{j'}$, in both cases transversally. Note that there are no more intersections of F and curves in E_i, because otherwise we will have a loop in E_i.

In what follows, we will prove that the (-1)-curve F must intersect C as is shown in Figure 6 or in Figure 7, and that the $\square$ curves are (-2)-curves. Indeed, we first claim that $C_j^2 = -2$. Otherwise, we would need other (-1)-curve disjoint to F to continue contracting E_i, but this situation gives from the beginning either a cycle in E_i or a third point of intersection with $C \setminus E_i$. So, we have $C_j^2 = -2$.

Now, we note that if C_j had two $\square$ neighbors, then F would have multiplicity at least 2 in E_i, which violates that $E_i \cdot C = 1$. So we have that $C_j = C_1$ or $C_j = C_l$ (see Figure 6 and Figure 7).

On the other hand, note that after contracting C_j, we will have the same situation above for the curve C_2 or C_{l-1} respectively. Thus, applying the same argument above, we obtain that all curves $C_1, \ldots, C_l$ are (-2)-curves. So, we conclude that E_i either has a diagram of type (i) or (ii) (see Definition 5.7).

For the last case, we will denote by $C_1, \ldots, C_l$ the $\square$ curves on the left side of Γ_{E_i}, and $C_{r-k+1}, \ldots, C_r$ the $\square$ curves on the right side.

Case (3). Suppose that E_i has the diagram shown in Figure 5. By Remark 5.3, we have that a (-1)-curve F in E_i intersects C twice. In this case, the curve F must intersect one $\square$ curve C_j on the left, and one $\square$ curve $C_{j'}$ on the right (see Figure 10).

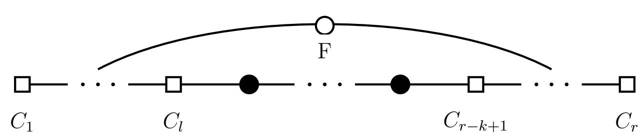

Figure 10: Case (3), and (-1)-curve F.

We first claim that $C_j^2 = -2$ or $C_{j'}^2 = -2$. On the contrary, we would need another (-1)-curve to contract them, but this would give either a loop in E_i or a third point of intersection with C.

Let us say $C_{j'}^2 = -2$, then we must have that $C_{j'} = C_r$. Indeed, if we suppose that $C_{j'}$ has two $\square$ neighbors, then after contracting F and $C_{j'}$, we will have a triple point in some E_j. But, it is not possible because E_j is a normal simple crossings tree of rational curves. Thus, we have that C_j' is one of the curves C_{r-k+1} or C_r. Now, if we had $C_{j'} = C_{r-k+1}$ then C_{r-k+1} would have multiplicity at least 2 in E_i, which contradicts the fact that $E_i \cdot C = 1$. Thus, we obtain that $C_{j'} = C_r$.

On the other hand, we will prove that C_j must be either C_1 or C_l. Otherwise, assume that C_j has two $\square$ neighbors. Let i' be a index such that $C_{r-i'+1}^2 = \cdots = C_r^2 = -2$, and $C_{r-i'} < -2$. We know that C_{r-k} is not a curve in E_i, then we have $C_{r-k}^2 < -2$, and $i' \leq k$. Assume that after blowing down $F, C_r, \ldots, C_{r-i'+1}$, we have that C_j becomes a (-1)-curve. If $i' = k$, then $C_{r-i'+1}$ would has multiplicity at least two in E_i, but then $E_i \cdot C > 1$. If instead $i' < k$, then contracting those curves and C_j would give a triple point, which is not possible. Thus, we have that C_j does not become a (-1)-curve. In this situation, we must need another (-1)-curve F' to contract C_j. If F' is disjoint of $F, C_r, \ldots, C_{r-i'+1}$, then F' must intersect a $\bullet$ curve. But this would impliy that $E_i \cdot C > 1$. So, F' must intersect some of the $C_r, \ldots, C_{r-i'+1}$ in E_i, which is not possible because E_i does not have loops. Thus, the unique possible case is that $i' = k$. But this implies that C_{r-k+1} would have multiplicity at least two in E_i, which violates that $E_i \cdot C = 1$. Thus, we obtain that C_j cannot have two $\square$ neighbors. So, we know that $C_j = C_1$ or $C_j = C_l$. In this situation, we have that E_i has a diagram of type (iv) if $C_j = C_1$. (See Definition 5.7). We recall that diagrams of type (iv) were discarded on Lemma 2.7 in [RU19] for T-singularities, because of the ampleness of K_W^2, we cannot have a (-1)-curve intersecting both ends in a T-configuration.

Assume that $C_j = C_l$. We want to show that E_i has a diagram of type (iii). Indeed, let i' be the maximal number such that $C_r^2 = \cdots = C_{r-i'+1}^2 = -2$, and $C_{r-i'}^2 < -2$. As we did before, we know that $i' \leq k$ because $C_{r-k}^2 < -2$. Let us first suppose that $i' < k$. Note that if after blowing down F and those (-2)-curves the curve C_l becomes a (-1)-curve, then C_l would have multiplicity at least 2 in E_i because $i' < k$. So, C_l does not became a (-1)-curve. Then, we must need another (-1)-curve F' to contract C_l. If F' is a (-1)-curve at the beginning, this would imply either a loop in E_i or a third point of intersection of E_i with C, none of which is possible. So, we have that F' must intersect some curve in $C_r, \dots, C_{r-i'+1}$. It implies that C_{r-k+1} would has multiplicity at least two in E_i which violates the fact that $E_i \cdot C = 1$.

Thus we have that $i' = k$, that is $C^2_{r-k+1} = \cdots = C^2_r = -2$. In this case, after blowing down F and those (-2)-curves, the curve C_l must become a (-1)-curve. On the contrary, we need another (-1)-curve intersecting C_l. That curve must be a (-2)- curve in the beginning in the process of contracting E_i and it intersects the curve C_{r+1-k}, which implies that C_{r+1-k} has at least multiplicity two in E_i. But this contradicts that $E_i \cdot C = 1$. So, we have $C^2_l = -(k+2)$. By using a similar argument, it is shown that $C^2_1 = \cdots C^2_{l-1} = -2$. Therefore, we have shown that E_i has a diagram of type (iii). We recall that this case was discarded on Lemma 2.7 in [RU19] for T-singularities, because we cannot have a T-configuration with (-2)-curves in both ends.

For the last part of the proof, let us assume that E_i has a diagram of type (iv), and that Γ_{E_i} is maximal. Let m_s be the maximal number such that $C^2_{r-m_s} < -2$, and $C^2_r = \cdots = C^2_{r-m_s+1} = -2$. If after blowing down the curves $F, C_r, \dots, C_{r-m_s+1}$ the curve C_1 does not became a (-1)-curve then we must need another (-1)-curve F' in E_i to contract C_1. If we have that F' is a (-1)-curve at the beginning, it would imply either a loop in E_i or a third point of intersection of E_i with C. So, F' must intersect the curves $F, C_r, \dots, C_{r-m_s+1}$, and then C_{r-k+1} would has multiplicity at least two in E_i which violates the fact that $E_i \cdot C = 1$.

Thus, we have shown that C_1 is contracted after blowing down the curves $F, C_r, \dots, C_{r-m_s+1}$, and then $C^2_1 = -(m_s + 2)$, where $0 < m_s \leq k-1$. We also note that after contracting the curves $F, C_r, \dots, C_{r-m_s+1}$ we obtain the same situation for the remaining curves in C, and then we can apply the same analysis. Therefore, we obtain that C has continued fraction:

$$[\dots, 2, 2+m_3, \underbrace{2, \dots, 2}_{m_2-1}, 2+m_1, a_1, \dots, a_t, \underbrace{2, \dots, 2}_{m_1-1}, 2+m_2, 2, \dots],$$

where $-a_1, \dots, -a_t$ correspond to the self-intersection of the $\bullet$ curves in Γ_{E_i}, and $\{m_1, \dots, m_s\}$ is a fixed sequence of natural numbers. $\square$

Remark 5.10. Let E_i be an exceptional divisor with diagram of type (iv) such that Γ_{E_i} is maximal. Assume that E is a pullback of a curve in E_i. We associate to E the sub sequence of $\{m_1, \dots, m_s\}$ which corresponds to the curves C_{m_j} with $C^2_{m_j} = -(2+m_j)$ that are contracted in E.

Remark 5.11. We remark that only one of the following situations can happen.

- We have that $E_i \cdot C \geq 2$ for all i.
- There is a unique exceptional divisor E_i such that its graph is maximal.
- There are two exceptional divisors E_i, E_j such that their graphs are maximal. Moreover, we have that Γ_{E_i}, Γ_{E_j} must be of type (i) or (ii).

5.2 Optimal bounds for the Hirzebruch-Jung continued fraction associated to P

Notation 5.12. The number of exceptional divisors E_j such that $E_j \cdot C = 1$ will be denoted by δ.

Lemma 5.13. *Under conditions of Theorem 1.1. Let $E_1, \ldots, E_m$ be the exceptional divisors defined after Diagram* (8). *(They satisfy $E_i^2 = -1$ and $E_i \cdot E_j = 0$.) We have that one of the following cases holds:*

(A) For every i we have $E_i \cdot C \geq 2$. In this case $\delta = 0$.

(B) There is a unique Γ_{E_i} maximal graph. Assume that E_i contains only the curves $C_1, \ldots, C_l$ in C. Then

- *(B.1) $\delta = l$ if E_i has a diagram of type (i).*
- *(B.2) $\delta = 1$ if E_i has a diagram of type (ii).*

(C) There is a unique Γ_{E_i} maximal graph. Assume that E_i contains only the curves $C_1, \ldots, C_l, C_{r+1-k}, \ldots, C_r$ in C. Then

- *(C.1) $\delta = k + 1$ if E_i has a diagram of type (iii).*
- *(C.2) $\delta = k + l$ if E_i has a diagram of type (iv).*

(D) There exist two maximal graphs $\Gamma_{E_i}, \Gamma_{E_{i'}}$. Assume that E_i only contains the curves $C_1, \ldots, C_l$, and that $E_{i'}$ only contains the curves $C_{r+1-k}, \ldots, C_r$ in C. Then

- *(D.1) $\delta = l + k$ if E_i and $E_{i'}$ have diagrams of type (i).*
- *(D.2) $\delta = l + 1$ if E_i has a diagram of type (i), and $E_{i'}$ has a diagram of type (ii).*

(D.3) $\delta = 2$ *if* E_i *and* $E_{i'}$ *have diagrams of type* (ii).

Proof. We divide the proof into the cases of the statement.

If there are not exceptional divisor with a long diagram, then by Lemma 5.2, and Lemma 5.9 we have that $E_i \cdot C \geq 2$ for every i. Thus, we obtain the case **(A)** which was proven in Proposition 5.4. In what follows, we will suppose that there are exceptional divisors with a long diagram.

(B) Suppose that there is a unique Γ_{E_i} which is maximal. Assume that E_i contains only the curves $C_1, \ldots, C_l$ in C. So, by this assumption and Lemma 5.9 we obtain that the graph Γ_{E_i} is of type (i) or (ii), $C_j^2 = -2$ for every $1 \leq j \leq l$, and $C_{l+1}^2 \leq -3$. Without loss of generality, assume that π starts by blowing down F, where F is the (-1)-curve in E_i, that is $E_m = F$.

Let E be an exceptional divisor such that $E \cdot C = 1$. By Lemma 5.9, we have that E has a long diagram. Since Γ_{E_i} is the unique maximal graph, then E has a diagram of type (i) or (ii), and it must have as components some of the (-2)-curves $\{C_1, \ldots, C_l\}$ or maybe all of them; otherwise we would obtain another maximal graph. Note that if the (-1)-curve in the diagram of E is not F, then we have either a loop in E or $E \cdot C \geq 2$, thus $F \subseteq E$, and hence E has a diagram of the same type as E_i.

Let us write $E = c_1F + c_1C_1 + c_2C_2 + \cdots + c_lC_l + D$, where $c_1 \geq 1$, $c_i \geq 0$ for $i > 1$, and D is an effective divisor which has no components of C in its support. By using $1 = E \cdot C = c_1 + D \cdot C$, we obtain that $c_1 = 1$, and $D \cdot C = 0$. But if $D > 0$, we have that to contract D, it must exist another curve (-1) disjoint from C, which contradicts the condition K_W ample, and then $D = 0$. Thus,

$$E = c_1F + c_1C_1 + c_2C_2 + \cdots + c_lC_l$$

where $c_1 \geq 1$, $c_i \geq 0$ for $i > 1$.

At the same time, in the process of contracting $F, C_1, \ldots, C_l$, we obtain the following exceptional divisors:

$$E_{m-j} = \begin{cases} F + C_1 + \cdots + C_j & if \quad \Gamma_{E_i} \ is \ of \ type \ (i), 1 \leq j \leq l. \\ F + C_l + \cdots + C_{l+1-j} & if \quad \Gamma_{E_i} \ is \ of \ type \ (ii), 1 \leq j \leq l. \end{cases}$$

With this notation we have that $E_i = E_{m-l}$. If Γ_{E_i} is of type (i), analyzing the graph of E_{m-j}, we obtain that only for $0 < j \leq l$ we could have $E_{m-j} \cdot C = 1$. In the case that Γ_{E_i} is of type (ii), we have that E_i is the only divisor such that $E_{m-j} \cdot C = 1$ in the list $0 \leq j \leq l$. For case (i), we have $E_{m-j} \cdot C = 1$ for all j.

Therefore, if E_i has a diagram of type (i) then $\delta = l$, and if E_i has a diagram of type (ii) then $\delta = 1$. (See [RU19, Lemma 2.10]).

(C) Suppose that there is a unique Γ_{E_i} which is maximal. Assume that E_i contains only the curves $C_1, \ldots, C_l, C_{r+1-k}, \ldots, C_r$ in C. By Lemma 5.9, we have that E_i can only have diagram of type (iii) or (iv). We assume that π starts by blowing down the (-1)-curve F in E_i, that is $E_m = F$. We recall that $C_1^2 = -2$ or $C_r^2 = -2$ (see proof of Lemma 5.9). Let us say that $C_r^2 = -2$.

(C.1) Say that E_i has a diagram of type (iii). Let E be a exceptional divisor such that $E \cdot C = 1$, by Lemma 5.9 we have that E has a long diagram. So, because Γ_{E_i} is maximal, then E must have some of $C_1, \ldots, C_l, C_{r-k+1}, \ldots, C_r$ (or maybe all of them) as components; otherwise we would obtain another maximal graph. We also note that the (-1)-curve in the diagram of E is F. Otherwise, we would have either a loop in E_i or $E_i \cdot C > 1$, but this is not possible. So, we can write E as follows.

$$E = c_1C_1 + \cdots + c_lC_l + c_{r-k+1}C_{r-k+1} + \cdots + c_rC_r + (c_l + c_r)F + D,$$

where $c_j \geq 0$, $c_r > 0$, and D is an effective divisor which has no components of C in its support. Since, we have that $C_j^2 = -2$ for $j \in \{1, \ldots, l-1, r-k+1, \ldots, r\}$, and $C_l^2 = -(k+2)$ (see Figure 8). Then,

$$1 = E \cdot C = c_r - (c_1 + (k-2)c_l) + D \cdot C. \tag{12}$$

Now, we prove that $D \cdot C = 0$. On the contrary, suppose that $D \cdot C > 0$. We first note that if $c_1 = 0$ then $c_2 = \cdots = c_l = 0$, since otherwise E would not have a long diagram, and so $E \cdot C > 1$. In this case, because D is effective then by (12) we obtain that $D \cdot C = 0$. If instead $c_1 > 0$ then $c_2, \ldots, c_l > 0$ because E has a long diagram. Observe that D can only intersect one component C_j of E, otherwise after contracting D, we would obtain a loop in E which is not possible. Also, we note that D does not intersect the curves C_l or C_{r-k+1}, since otherwise we would have c_l or $c_{r-k+1} > 1$, and then $E \cdot C > 1$. In addition, if D intersects a curve C_j for some $1 \leq j \leq l-1$, then $c_l > 1$ which violates the fact that $E \cdot C = 1$. Thus, the divisor D could only intersect a component C_j of E for some $j = r-k+2, \ldots, r$. Then, contracting D does not affect the curves $C_1, \ldots, C_l$, and so we obtain $c_1 = \cdots = c_l = 1$. Finally, because $c_{r-k+1} = 1$ we obtain that $c_r \geq k$. Then, by (12), we obtain $D \cdot C = k - c_r \leq 0$. In both cases, we conclude that $D \cdot C = 0$.

Using the fact that $D \cdot C = 0$, we will prove that $D = 0$. Indeed, if we had that $D > 0$, then to contract D it must exist another (-1)-curve disjoint from C (that is, a (-1)-curve from the beginning in E) because D does not intersect C. But, this contradicts the condition K_W ample, and then $D = 0$.

Therefore, the divisor E shows up in the process of contracting E_i. Note that in that process, we obtain the exceptional divisors $E_{m-(j+1)} = F + C_r + \cdots + C_{r-j}$ where $0 \leq j \leq k-1$, and that $E_{m-(k+j+1)} = (k+1)F + kC_r + \cdots + C_{r-k+1} + C_l + \cdots + C_{l-j}$ where $0 \leq j \leq l-1$. Here, because Γ_{E_i} is the unique maximal graph, then $E_i = E_{m-(k+l)}$. By analyzing the graph of E_{m-j} and by using (12), we obtain that $E \cdot C = 1$ only for E_i and $E_{m-(j+1)}$, where $0 \leq j \leq k-1$. Thus, we have $\delta = k+1$.

(C.2) Say that E_i has a diagram of type (iv). Let E be an exceptional divisor such that $E \cdot C = 1$. As in the Case (C.1), we have that E must have some of $C_1, \ldots, C_l, C_{r-k+1}, \ldots, C_r$ (or maybe all of them) as components, and that the (-1)-curve in the diagram of E is F. Thus, we can write E as follows.

$$E = c_1C_1 + \cdots + c_lC_l + c_{r-k+1}C_{r-k+1} + \cdots + c_rC_r + (c_1 + c_r)F + D,$$

where $c_j \geq 0$, $c_r > 0$, and D is an effective divisor which has no components of C in its support. Now, we prove that $D \cdot C = 0$. On the contrary, suppose that $D \cdot C > 0$. Note that D can only intersect one component C_j of E, otherwise after contracting D, we would obtain a loop in E which is not possible. In addition, we have that D must be contracted after blowing down the curves C_j in E. Otherwise, we would obtain a loop in E. Let $j \leq l$ the maximal number such that $c_j > 0$, and let $j' \leq r$ the minimal number such that $c_{j'} > 0$. If we have that D intersect one component of C in E, then we would have that $c_j > 0$ or $c_{j'} > 0$, neither of which is possible. (both imply $E \cdot C > 1$).

Therefore, we have that D can only intersect one curve in C which is not in E. Then to contract D it must exist another (-1)-curve disjoint from the curves of C in E or a (-2)-curve intersecting C_j or $C_{j'}$. But this is not possible because K_W is ample and $E \cdot C = 1$. Thus, $D \cdot C = 0$.

As we proved in case (C.1), the fact $D \cdot C = 0$ implies $D = 0$. Thus, E is a pullback of a curve in E_i. So, E has a diagram of type (i), and a

(-1)-curve intersecting both ends of the chain or it has a diagram of type (iv).

Now, we will show that $\delta = k + l$. Indeed, by following the notation described in Lemma 5.9, let $\{m_1, \ldots, m_s\}$ be the sequence associated to E_i. Let us denote by C_{m_j} to the curve with self-intersection $-(2+m_j)$ in C, and by c_{m_j} their multiplicities in E_i. In addition, let us write $c_{m_{s+1}} := c_r$, and $C_{m_{s+1}} := C_r$. With this notation, we have $c_1 = c_{m_s}$, and then

$$E_i \cdot C = c_{m_s}(1 - m_s) + c_{m_{s+1}} + \sum_{j=1}^{s-1} c_{m_j}(-m_j). \tag{13}$$

We first show that $\delta = k+l$ in the case when $s = 1$ (see Figure 11). Note that in this case we have that $l = 1$, and $k = m_1$.

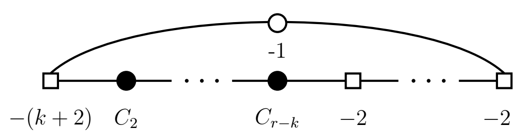

Figure 11: The case when $s = 1$.

Note that in this case, E_i has also a diagram of type (iii), and then $\delta = k+1$ as we proved in case (C.1). Here we obtain that $c_{m_1} = 1$, $c_{m_2} = m_1$.

Assume $s > 1$. We claim that $c_{m_j} = c_{m_{j-2}} + m_{j-1}c_{m_{j-1}}$ for $2 < j \leq s+1$. Indeed, by Lemma 5.9, we have that $m_{j-1}-1$ is the number of the (-2)-curves between C_{m_j} and $C_{m_{j-1}}$ in C. After contracting C_{m_j} and such (-2)-curves, we obtain a SNC situation between $C_{m_{j-1}}$ and $C_{m_{j-2}}$. It follows from the form of contracting this curves, that $c_{m_j} = c_{m_{j-2}} + m_{j-1}c_{m_{j-1}}$.

Moreover, if we assume that

$$c_{m_{s-2}}(1 - m_{s-2}) + c_{m_{s-1}} + \sum_{j=1}^{s-3} c_{m_j}(-m_j) = 1. \tag{14}$$

Then, by plugging $c_{m_j} = c_{m_{j-2}} + m_{j-1}c_{m_{j-1}}$ for $j = s, s+1$, and (14) in (13) , we obtain

$$E_i \cdot C = c_{m_s}(1 - m_s) + c_{m_{s+1}} + \sum_{j=1}^{s-1} c_{m_j}(-m_j) = 1. \tag{15}$$

Note that for E_i we have that $c_{m_1} = 1$, and $c_{m_2} = m_1$. However, the analysis above does not depend on the initial values of c_{m_j}. But, it only depends on the form of contracting the curves in diagrams of type (iv). For instance, one can have $c_{m_1} = \cdots = c_{m'_j} = 0$ for some $j' \leq s$. Let $1 \leq j' \leq s+1$ be the minimal number such that $C_{m'_j}$ is contracted in E. This implies that $c_{m'_j} > 0$, and $c_{m_1} = \cdots = c_{m_{j'-1}} = 0$. Since, we know that E is a pullback of a curve in E_i, then by the form of contracting the curves in E we obtain that $c_{m_j} > 0$ for every $j \geq j'$. By using the form of the contraction in E, we compute that $c_{m_{j'+1}} = m_{j'} c_{m_{j'}}$ Thus, we know that $c_{m_j} = c_{m_{j-2}} + m_{j-1} c_{m_{j-1}}$ for every $j \geq j' + 2$, and then we have that (13), and (15) remain valid for E. So, we obtain that $E \cdot C = 1$.

Therefore, because $k+l$ is the number of exceptional divisors E that show up as pullback of curves in E_i, we obtain that $\delta = k + l$.

By Remark 5.11, we have the following last case.

(D) Suppose that there exist two maximal graphs $\Gamma_{E_i}, \Gamma_{E_{i'}}$. Assume that $E_{i'}$ only contains the curves $C_{r+1-k}, \ldots, C_r$, and that E_i only contains the curves $C_1, \ldots, C_l$ in C. In this case, by Lemma 5.9 we have that E_i and E'_i can only have a diagram of type (i) or (ii), and we have that $C_1, \ldots, C_l, C_{r+1-k}, \ldots, C_r$ are (-2)-curves. Because E_i, and E'_i are maximal divisors such that their intersection with C is equal to 1, then we obtain that $C^2_{l+1} \leq -3$, and $C^2_{r-k} \leq -3$. As before, we assume that π starts by blowing down the (-1)-curve F in E_i. Let F' be the (-1)-curve in the diagram of E'_i. We must have that $F \neq F'$, otherwise, we could not contract the divisor $E_{i'}$. Thus, we can describe separately the process to contract the divisors E_i, and E'_i using Case (B). Combining the possible situations for E_i and $E_{i'}$, we obtain the values described for δ.

□

Proof of Theorem 1.1. We start by proving the initial inequality. It follows from the assumption that $\left(\sum_{i=1}^{m} E_i\right) \cdot C \geq 2m - \delta$. It follows from (9), and Lemma 5.1 that

$$\sum_{j=1}^{r} \big(b_j - 2\big) \leq 2(K_W^2 - K_S^2) + 2\left(\frac{2(n-1) - q - q'}{n}\right) + \delta - \pi^* K_S \cdot C. \quad (16)$$

In order to have the bound for the length we will use a generalization of the Bogomolov-Miyaoka-Yau inequality to orbifolds, log-BMY inequality for short (see e.g. [Lan03])

$$K_W^2 \leq 3e_{orb}(W),$$

where $e_{orb}(W)$ is the orbifold Euler number of a quasiprojective surface W, with only isolated cyclic quotient singularities. That is defined as

$$e_{orb}(W) = e(W) - \sum_{w \in \mathrm{Sing(W)}} \left(1 - \frac{1}{|\pi_1(L_w)|}\right),$$

where L_w is the link of $w \in \mathrm{Sing(W)}$, and π_1 denotes the fundamental group. We recall that the germ of $w \in \mathrm{Sing(W)}$ is topologically the cone over a 3-manifold S^3/G, where $G \subset U(2,\mathbb{C})$ is a subgroup acting without fixed points. The 3-manifold S^3/G is called the link of w.

In our case, the surface W has $\mathrm{Sing(W)} = \{\mathrm{P}\}$. Since the singularity P is defined as the germ at the origin of a quotient of $\mathbb{C}^2$ by the group $\mathbb{Z}/n\mathbb{Z}$, we have that $|\pi_1(L_w)| = |\mathbb{Z}/n| = n$. Therefore,

$$e_{orb}(W) = e(W) - \left(1 - \frac{1}{n}\right).$$

Plugging this formula together with $e(W) = e(X) - r$, and the Log-BMY inequality, we have

$$K_W^2 \leq 3e(X) - 3r - 3\left(1 - \frac{1}{n}\right).$$

By Proposition 3.18, and Noether's formula for X, we have that

$$12\chi(\mathcal{O}_X) = K_W^2 + A + e(X),$$

where $A = \sum_{j=1}^r \left(2 - b_j\right) + \dfrac{2(n-1) - q - q'}{n}$. Putting these formulas together and using the fact that $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_W)$ (W has a rational singularity), we have that

$$12\chi(\mathcal{O}_W) \leq 4e(X) - 3r - 3\left(1 - \frac{1}{n}\right) + A. \tag{17}$$

Also, by the Noether's formula for X, we have $4e(X) = 48\chi(\mathcal{O}_X) - 4K_X^2$. Replacing this in (17), and by Proposition 3.18, we obtain that

$$r \leq 12\chi(\mathcal{O}_W) - \frac{4}{3}K_W^2 - A - \left(1 - \frac{1}{n}\right). \tag{18}$$

Finally, using (18) together with the generalized Noether's inequality (see Remark 3.11), and the inequality for the sum $\sum_{j=1}^{r}\big(b_j-2\big)$ previously obtained, we have the boundedness for r. $\square$

Proof of Corollary 1.2. Assume that the number of 2's at the extremes of every $[b_1,\dots,b_r]\in \mathrm{Sing}(\mathcal{S})$ is bounded, say by a number $k>0$. Then, by Theorem 1.1 we have that $\delta\leq k+1$, and because $K_S\geq 0$ for every surface in $\mathcal{S}$, we obtain that r, and $\sum_{j=1}^{r} b_j$ are bounded. So, we conclude that $Sing(\mathcal{S})$ is finite. On the other hand, if the number of 2's at the extremes of every $[b_1,\dots,b_r]\in \mathrm{Sing}(\mathcal{S})$ is not bounded, then we can construct infinitely many sets $\mathcal{S}$ of stable surfaces with K_S nef, and $K_W^2\leq c$ such that $Sing(\mathcal{S})$ is infinite. (See e.g. Example 6.1 with $n_0\geq c$). $\square$

Proof of Corollary 1.3. It follows from the facts that K_S is nef, and that $\delta\leq 2$ in that cases. $\square$

6 Accumulation points of volumes for surfaces with one cyclic quotient singularity

6.1 Example

The following example shows a sequence of accumulation points of $\{K^2\}$ on stable surfaces with only one cyclic singularity. It is constructed in a similar way to the one shown in [Bla95].

Example 6.1. Let $S' \to \mathbb{P}^1$ be an elliptic fibration obtained by blowing up at the intersection points of two general cubic curves in $\mathbb{P}^2$. It has 12 nodal rational fibers (type I_1 according to Kodaira's notation). Now, let $n_0 > 0$ and let $f\colon S \to S'$ be the n_0-th cyclic cover (see e.g [Urz10]) branched along $F_1 + \cdots + F_{n_0}$, where F_i are general fibers on S'.

We have that $K_S^2 = 0$. Note that for every (-1)-curve β in S' the self-intersection of $f^*(\beta)$ is $-n_0$. Let us choose two nodal singular fibers F and F' in S'. After blowing up at the points on the nodes of F, and F', we obtain a smooth projective surface X_0, which has the configuration $[4, n_0, 4]$ shown in Figure 12.

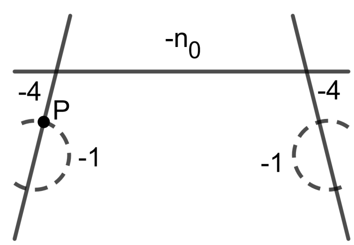

Figure 12: The configuration $[4, n_0, 4]$.

Let us denote by C_1, C_2 and C_3 the curves in Figure 12 with self-intersection $-4, -n_0$, and -4 respectively. We will construct a sequence of smooth projective surfaces $\{X_k\}$ by blowing up at points in $[4, n_0, 4]$. First, let X_1 be the surface obtained by blowing up at the point P in Figure 12, and let Γ the exceptional curve. The surface X_2 is obtained by blowing up at $C_1 \cap \Gamma$, where C_1 is the strict transform. We continue blowing up at the point in the intersection between the strict transform of C_1 and the last exceptional curve obtained. After k blow ups at points in the configuration, we obtain a smooth projective surface X_k, which has a chain of rational curves C. Each component C_j of C has a self-intersection in the sequence $\{-2,\dots,-2,-(4+k),-n_0,-4\}$, where k is the number of 2's on the left side. By the construction, there is a (-1)-curve intersecting C_1, and C_{k+1}. By Artin's contractibility Theorem [Art62, Thm. 2.3], we may contract C to obtain a normal projective surface W_k with only one cyclic quotient singularity. Note that there are $k+1$ divisors E_j corresponding to the pull-back of the (-1)-curves of the blow downs. The graph of Γ_{E_j} for $j=1,\dots,k$ is shown in Figure 13, where j is the number of 2's on the left side.

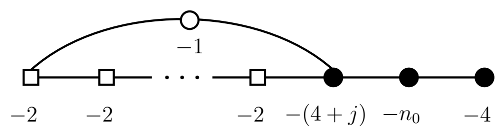

Figure 13: The graph of Γ_{E_j}.

By pulling back the canonical divisor K_{W_k}, we can directly write it as an effective sum of divisors, and then by the Nakai-Moishezon criterion, we obtain that K_{W_k} is ample. By the canonical formula and induction over k, we obtain that

$$\lim_{k\to\infty} K_{W_k}^2 = \frac{4n_0^2-8n_0+2}{4n_0-1},$$

which is a sequence of accumulation points tending to ∞.

6.2 Case of generalized T-singularities

Remark 6.2. In particular, by Theorem 4.8 we have that a generalized T-singularity $[b_1,\dots,b_r]$ fulfills the condition $b_1>2$ or $b_r>2$.

Corollary 6.3. *Let W be a stable surface with a unique generalized T-singularity of center $[b_1,\dots,b_r]$. Then, we have either $E_i\cdot C\geq 2$ for every exceptional divisor E_i or there is a unique exceptional divisor E_i such that Γ_{E_i} is maximal. In this case, we have that Γ_{E_i} cannot be a diagram of type (iii).*

Proof. By Theorem 4.8 we have that $b_1>2$ or $b_r>2$. So, we cannot have two maximal graphs nor a graph of type (iii). Then, by Remark 5.11, and by

Lemma 5.9 we conclude that either $E_i \cdot C \geq 2$ for every exceptional divisor or there is a unique exceptional divisor E_i with a maximal graph such that Γ_{E_i} is of type $(i), (ii)$ or (iv). $\square$

Lemma 6.4. *Let W be a stable surface with a unique generalized T-singularity $[a_1, \dots, a_s]$ of center $[b_1, \dots, b_r]$. Assume that the minimal model of the minimal resolution of W has canonical class nef. Suppose that the maximal exceptional divisor E_i has diagram of type (i), and there is not a (-1)-curve intersecting the ends of C, then*

$$2\delta \leq \sum_{j=1}^{s} \big(a_j - 2\big) - 2.$$

Proof. Let Γ be the curve in C which intersects the (-1)-curve in E_i. We have that $[a_1, \dots, a_s]$ is of the following form:

$$[2, \dots, 2, x_1, \dots, x_{s-l-1}, x_s + l]$$

where l is the number of 2's on the left side, and $x_s + l \geq 3$. Note that $x_1 \geq 3$ because of the admissibility of $[b_1, \dots, b_r]$.

The curve Γ cannot be a (-2)-curve on the left of the chain, because E_i does not have loops. Thus Γ is a curve C_{l+j} such that $C_{l+j}^2 = -x_j$ for some $1 \leq j \leq s - l - 1$.

By Remark 6.6, we have

$$\sum_{j=1}^{s} \big(a_j - 2\big) = \sum_{j=1}^{s-l-1} \big(x_j - 2\big) + l.$$

Let us suppose that $\Gamma = C_{l+1}$, that is $\Gamma^2 = -x_1$. After contracting the curves $F, C_1, \dots, C_l$, the curve Γ becomes a curve D which has self-intersection equal to $-x_1 + l + 2$. By the adjunction formula and because K_S is nef, we obtain that $x_1 - 2 \geq l + 2$. Due to the fact that $0 \leq x_j - 2$ for all j, we obtain

$$l + 2 \leq x_1 - 2 \leq \sum_{j=1}^{s} \big(a_j - 2\big) - l,$$

so $2\delta \leq \sum_{j=1}^{s} \big(a_j - 2\big) - 2$ because by Theorem 1.1 we have that $\delta = l$.

On the other hand, if $\Gamma = C_j$ for $1 < j \leq s-l-1$, we obtain that the curve C_j becomes a curve D, which has $D^2 = -x_j + l + 1$. Because K_S is nef, we obtain that $x_j \geq l+3$. So we have

$$l+3-2+1 \leq l+3-2+x_1-2 \leq x_j-2+x_1-2 \leq \sum_{j=1}^{s} \big(a_j-2\big) - l,$$

so we obtain that $2\delta \leq \sum_{j=1}^{s} \big(a_j-2\big) - 2$. □

Notation 6.5. We denote by $\mathcal{B}([b_1,\dots,b_r])$ to the set formed for each iteration of (ii) (the T-chain algorithm) applied to $[b_1,\dots,b_r]$. (See Definition 1.8).

Remark 6.6. Let $[b_1,\dots,b_r]$ be a continued fraction. Let $[a_1,\dots,a_s]$ be an element of $\mathcal{B}([b_1,\dots,b_r])$. Then, by a direct computation, we obtain

$$\sum_{j=1}^{s}(a_j-2) = \sum_{j=1}^{r}(b_j-2) + (s-r). \tag{19}$$

Assume that $[b_1,\dots,b_r]$ is a core. Let us denoted by $[b_1^k,\dots,b_{r_k}^k]$ the resulting continued fraction $[b_1,\dots,b_r,1,b_1,\dots,b_r,1,\dots,1,b_1,\dots,b_r]$, where k is the number of inserted 1's. Then, $r_k = r(k+1)$ and

$$\sum_{j=1}^{r_k}(b_j^k-2) = (k+1)\sum_{j=1}^{r}(b_j-2) - 2k. \tag{20}$$

Therefore, given a generalized T-singularity $[a_1,\dots,a_s]$ of center $[b_1,\dots,b_r]$, we have

$$\sum_{j=1}^{s}(a_j-2) = (k+1)\sum_{j=1}^{r}(b_j-2) - 2k + (s-r_k), \tag{21}$$

where k is a number such that $[a_1,\dots,a_s]$ belongs to $\mathcal{B}([b_1^k,\dots,b_{r_k}^k])$.

Lemma 6.7. *Let W be a stable surface with only one generalized T-singularity with a fixed center $[b_1,\dots,b_r]$, say at $P \in W$. Suppose that the minimal model S of the minimal resolution of W has canonical class nef. Assume that $K_W^2 < c$ for some positive number c. Then, one of the following holds* *(i) Assume that there is not a (-1)-curve intersecting the ends of the chain that resolves P, or that we have $E_i \cdot C \geq 2$ for every exceptional divisor, then*

$$\sum_{j=1}^{s} \big(a_j - 2\big) < 4c + 6, \text{ and } s < 15c + 40.$$

(ii) Assume that there exists a (-1)-curve intersecting the ends of the chain that resolves P, then $P \in \mathcal{B}([b_1^u, \ldots, b_{r_u}^u])$ for some $u < 2c + 1$. Moreover, we have that there exists a non-negative number m' such that $m' + 1 \leq \sum_{j=1}^{r_u}(b_j^u - 2)$, and

$$K_W^2 = K_S^2 + \sum_{j=1}^{r_u}(b_j^u - 2) - (m' + 1) - \left(\frac{2(n-1) - q - q'}{n}\right).$$

Proof. We start by fixing some notation. Let $\frac{n}{q} = [a_1, \ldots, a_s]$ be the Hirzebruch-Jung continued fraction associated to $P \in W$. We first note that it can be assumed that $[b_1, \ldots, b_r]$ is a core. Otherwise, by Theorem 4.8, there exists a core $[e_1, \ldots, e_s]$ such that $[b_1, \ldots, b_r]$ belongs to $\mathcal{B}([e_1, \ldots, e_s])$, and that $[a_1, \ldots, a_s]$ is a generalized T-singularity of center $[e_1, \ldots, e_s]$.

Now, let $\phi \colon X \to W$ be the minimal resolution of P and let C be the chain of exceptional rational curves. Let $\pi \colon X \to S$ be a birational morphism to the minimal model S. By Corollary 6.3 we have the following two cases.

(1) We have that $E_i \cdot C \geq 2$ for every exceptional divisor of π. That is, $\delta = 0$. In this case, by Theorem 1.1 we have

$$\sum_{j=1}^{s} \big(a_j - 2\big) < 2c + 4, \text{ and } s < 13c + 38. \tag{22}$$

(2) There is a unique exceptional divisor E_i of π such that $E_i \cdot C = 1$, and its graph is maximal. By Corollary 6.3, we also have that Γ_{E_i} is of type (i), (ii) or (iv). Now, we divide this case into the following sub-cases.

(2.A) Suppose that E_i has a diagram of type (i), and there is not a (-1)-curve intersecting the chain C at both ends. Then, by putting together the bound for δ shown in Lemma 6.4, and Theorem 1.1 we obtain that

$$\sum_{j=1}^{s}(a_j - 2) < 4c + 6, \text{ and } s < 15c + 40. \tag{23}$$

(2.B) Assume that E_i has a diagram of type (ii). Then, by Theorem 1.1 we obtain that $\delta = 1$, and

$$\sum_{j=1}^{s}(a_j - 2) < 2c + 5, \text{ and } s < 13c + 39. \tag{24}$$

By putting cases (1), (2.A), and (2.B) together, we obtain the first part of the statement. By Corollary 6.3, the last case is the following.

(2.C) Suppose that E_i has a diagram of type (iv) or E_i has a diagram of type (i), and there exist (-1)-curve intersecting both ends of C. Let F be the (-1)-curve of E_i.

Claim 1. Let P be a T-singularity (a generalized T-singularities of center [4]). Then there is not a (-1)-curve intersecting both ends of C.

Proof. We recall that a T-singularity P can be expressed as $\frac{1}{dn^2}(1, dna - 1)$ for some natural numbers n, d, a such that $gcd(n, a) = 1$, and d is a square-free. Let us suppose that there exists a (-1)-curve F which intersects both ends of C. Then, we obtain that

$$\phi(F) \cdot K_W = -1 + 1 - \frac{dna - 1 + 1}{dn^2} + 1 - \frac{dn(n-a) - 1 + 1}{dn^2} = 0,$$

since the discrepancies of the ends of the chain are $-1 + \frac{dna-1+1}{dn^2}$ and $-1 + \frac{dn(n-a)-1+1}{dn^2}$. (See e.g. [Urz16, Section 2.1]). But, that violates the condition of being ample for K_W. $\square$

Therefore, we obtain that $[a_1, \ldots, a_s]$ cannot be a T-singularity (an usual T-singularity). Now, we know that $[a_1, \ldots, a_s] \in \mathcal{B}([b_1^u, \ldots, b_{r_u}^u])$ for some $u \geq 0$. (See Notation 1.7). Here, by Definition 4.7 we have that $[b_1^u, \ldots, b_{r_u}^u]$ is a core, and then $r_u = r(u+1)$ (see Remark 6.6).

We claim that $u < 2c + 1$. Indeed, we know that $b_1^u = b_1 > 2$, and $b_{r_u}^u = b_r > 2$, because $[b_1, \ldots, b_r]$ is a core. Also, by the formation rule of $[a_1, \ldots, a_s]$, we obtain that E_i contains exactly $(s - r_u)$ curves in C. They are contracted by starting at F. Thus, we know that $\delta = s - r_u$. (see Cases (B.1), and (C.2) in Theorem 1.1). So, by plugging the formula in (21) into the inequality (1), we obtain that

$$(u+1)\sum_{j=1}^{r}(b_j - 2) - 2u + (s - r_u) < 2c + 4 + s - r_u. \tag{25}$$

Due to the fact that $[a_1, \ldots, a_s]$ is not a T-singularity, we obtain that neither is $[b_1, \ldots, b_r]$. So, we have that $\sum_{i=1}^{r}(b_j - 2) \geq 3$. Thus, by (25) we conclude that $0 \leq u < 2c + 1$. Therefore, we have proved that

$$P \in \bigcup_{u=0}^{\lfloor 2c+1 \rfloor} \mathcal{B}([b_1^u, \ldots, b_{r_u}^u]),$$

and so $P \in \mathcal{B}([b_1^u, \ldots, b_{r_u}^u])$ for some $u < 2c + 1$.

In addition, by (20) in Remark 6.6, we obtain that $r_u < r(2c + 2)$ and

$$\sum_{j=1}^{r_u}(b_j^u - 2) < (2c + 2)\sum_{j=1}^{r}(b_j - 2). \tag{26}$$

On the other hand, by (19) in Remark 6.6, we know that

$$\sum_{j=1}^{s}(a_j - 2) = \sum_{j=1}^{r_u}(b_j^u - 2) + (s - r_u). \tag{27}$$

Let m be the number of blow downs necessary to reach the minimal model S from X. Note that by the formation rule of $[a_1, \ldots, a_s]$, we can write $m = (s - r_u + 1) + m'$ with $m' \geq 0$. By putting (27) in Equation (9), we obtain that

$$K_W^2 = K_S^2 + \sum_{j=1}^{r_u}(b_j^u - 2) - (m' + 1) - \left(\frac{2(n-1) - q - q'}{n}\right). \tag{28}$$

Note that by Lemma 5.1, and Remark 5.3 for m, we obtain $m' + 1 \leq \sum_{j=1}^{r_u}(b_j^u - 2)$. $\square$

Definition 6.8. Let $\{W_k\}$ be a sequence of stable surfaces with only one generalized T-singularity of center $[b_1, \ldots, b_r]$. We say that $\{K_{W_k}^2\}$ satisfy the property (*) if there exists an infinite set of indices J such that

- The self-intersection $K_{S_k}^2$ is constant for every $k \in J$.
- There exists a (-1)-curve intersecting the ends of the chain that resolves P_k for every $k \in J$.
- There exists a number $u \geq 0$ such that $P_k \in \mathcal{B}([b_1^u, \ldots, b_{r_u}^u])$ for every $k \in J$.

- The reduced Hirzebruch-Jung continued fraction of P_k is different for each $k \in J$.

Proof of Theorem 1.9. We start by fixing some notation. Let $\frac{n_k}{q_k} = [a_1, \ldots, a_s]$ be the Hirzebruch-Jung continued fraction associated to $P_k \in W_k$. As in the proof of Lemma 6.7, we can assume without loss of generality that $[b_1, \ldots, b_r]$ is a core. Let $\phi_k \colon X_k \to W_k$ be the minimal resolution of P_k and let C^k be the chain of exceptional rational curves. Let $\pi_k \colon X_k \to S_k$ be a birational morphism to the minimal model S_k. Let c be a positive but arbitrary real number.

Assume that $\{K^2_{W_k}\}$ has accumulation points. Then, there exists a positive number c such that $\{K^2_{W_k} : K^2_{W_k} < c\}$ has accumulation points. Let J' be the set of indices k such that $K^2_{W_k} < c$, and there exists a (-1)-curve intersecting both ends of C^k. We know that J' is an infinite set. Otherwise, by Lemma 6.7 we obtain bounds for $\sum_{j=1}^{s}(a_j - 2)$, and s which only depend on c. So, we would have that $\{K^2_{W_k} : K^2_{W_k} < c\}$ has no accumulation points. But this is impossible, so J' is an infinite set of indices. More precisely, we know that

$$\mathrm{Acc}\big(\{K^2_{W_k} : K^2_{W_k} < c\}\big) = \mathrm{Acc}\big(\{K^2_{W_k} : k \in J'\}\big). \tag{29}$$

Again, by Lemma 6.7 for each $k \in J'$ there exists $u < 2c + 1$ such that $P_k \in \mathcal{B}([b^u_1, \ldots, b^u_{r_u}])$, and so

$$K^2_{W_k} = K^2_{S_k} + \sum_{j=1}^{r_u}(b^u_j - 2) - (m'_k + 1) - \left(\frac{2(n_k - 1) - q_k - q'_k}{n_k}\right), \tag{30}$$

where $0 < m'_k + 1 \leq \sum_{j=1}^{r_u}(b^u_j - 2)$.

So, by replacing the bound for m'_k in (30), it follows that $K^2_{S_k} < c + 2$ for every $k \in J'$. Thus, because $K^2_{S_k}$ is an integer for every k, we obtain that $\{K^2_{S_k} : k \in J'\}$ is a finite set.

For each $u < 2c + 1$, let $J'_u \subseteq J'$ be the set of indices k such that $P_k \in \mathcal{B}([b^u_1, \ldots, b^u_{r_u}])$. By Lemma 6.7, we know that $J' = \bigcup_{u=0}^{\lfloor 2c+1 \rfloor} J'_u$. So, there exists at least one $u < 2c + 1$ such that J'_u is an infinite set of indices. Note that between the infinite sets J'_u we can choose one of them with the property that the reduced Hirzebruch-Jung continued fraction of P_k are different for each $k \in J'_u$ (except maybe for a finite set of J'_u). On the contrary, we would have by (30) that $\{K^2_{W_k} : k \in J'_u\}$ is a finite set for every u, and then $\{K^2_{W_k} : K^2_{W_k} < c\}$ would not have accumulation points. But this is not possible. Let J'_{u_0} such a set. Now, because we have that $\{K^2_{S_k} : k \in J'\}$ is a finite set then we may choose an infinite subset of indices $J \subseteq J'_{u_0}$ such that $K^2_{S_k}$ is constant for every $k \in J$.

Then, we know that J is the set with the desired properties of the statement in Theorem 1.9.

Conversely, let us suppose that there exists an infinite set of indices J such that

- We have that $K^2_{S_k}$ is constant for every $k \in J$.
- There exists a (-1)-curve intersecting the ends of the chain that resolves P_k for every $k \in J$.
- There exists a number $u \geq 0$ such that $P_k \in \mathcal{B}([b^u_1, \ldots, b^u_{r_u}])$ for every $k \in J$.
- The reduced Hirzebruch-Jung continued fraction of P_k is different for each $k \in J$.

By using those statements and Lemma 6.7, it follows that for every $k \in J$

$$K^2_{W_k} = K^2_{S_k} + \sum_{j=1}^{r_u}(b^u_j - 2) - (m'_k + 1) - \left(\frac{2(n_k - 1) - q_k - q'_k}{n_k}\right), \qquad (31)$$

where $0 < m'_k + 1 \leq \sum_{j=1}^{r_u}(b^u_j - 2)$. Let c' be a positive number such that $K^2_{S_k} = c'$ for every $k \in J$. Then, by (31) we obtain that

$$K^2_{W_k} < c' + \sum_{j=1}^{r_u}(b^u_j - 2) + 2,$$

for every $k \in J$. So, we have that $\{K^2_{W_k} : k \in J\}$ is a bounded set.

Now, we construct an infinite set of indices J' such that the continued fraction of $P_{k_{i+1}}$ is obtained by applying the T-chain algorithm (see Definition 1.8) to the continued fraction of P_{k_i} for every $k_i \in J'$. In fact, let us fix an integer $s \geq r$. By using the formation rule in $\mathcal{B}([b^u_1, \ldots, b^u_{r_u}])$ and the fact that P_k has different continued fraction for every $k \in J$, we know that there exist finitely many $k \in J$ such that P_k has a continued fraction of length s. Thus, we can choose $k_0 \in J$ such that the continued fraction of P_k is obtained by applying the T-chain algorithm to the continued fraction of P_{k_0} for infinitely many $k \in J$. Let $J_{k_0} \subseteq J$ be a subset of indices with such a property. In the same way, we can choose an index $k_1 \in J_{k_0}$ such that the continued fraction length of P_{k_1} is greater than the length of P_{k_0}, and that P_k is obtained by applying the T-chain algorithm to the continued fraction of P_{k_1} for infinitely many $k \in J$. Let $J_{k_1} \subseteq J_{k_0}$ be an infinite set of indices with such a property. By using an inductive argument, we may construct an infinite set of indices $J' = \{k_0, k_1, \ldots\}$ with the desired property.

Observe that the quotients $2(n_{k_i} - 1) - q_{k_i} - q'_{k_i}/n_{k_i}$ are different for each $k_i \in J'$. Indeed, by Proposition 4.12, one can compute directly that

$$\frac{2(n_{k_{i+1}} - 1) - q_{k_{i+1}} - q'_{k_{i+1}}}{n_{k_{i+1}}} < \frac{2(n_{k_i} - 1) - q_{k_i} - q'_{k_i}}{n_{k_i}}.$$

Therefore, by (31) we know that $\{K^2_{W_k} : k \in J\}$ is an infinite set which also is bounded. So, we conclude that $\{K^2_{W_k}\}$ has accumulation points.

□

Proposition 6.9. *Let W be a stable surface which has only one generalized T-singularity $P \in W$ with continued fraction $[b_1, \ldots, b_r]$. Assume that there exists a (-1)-curve intersecting both ends of the chain C associated to P. Then there exist a sequence $\{W_k\}$ of stable surfaces with only one generalized T-singularity P_k of center $[b_1, \ldots, b_r]$ such that $\{K^2_{W_k}\}$ has an accumulation point.*

Proof. Let us write $W_1 := W$, and let X_1 be the minimal resolution of P. Let F_1 be the (-1)-curve intersecting the ends C_1, and C_r of C.

Step 1. Let X_2 be the smooth surface obtained by blowing up at the point in $F_1 \cap C_1$. Let F_2 be the exceptional curve in X_2. Note that X_2 has a configuration of $(r+1)$-rational curves (with SNC), and that F_2 intersects the ends of the chain. The new configuration has continued fraction $[b_1 + 1, \ldots, b_{r-1}, b_r, 2]$.

Step k. Let us assume constructed the surfaces $X_1, \ldots, X_{k-1}$, inductively. Let F_{k-1} be the exceptional curve in X_{k-1}. In the same way of Step 1, we construct a smooth surface X_k which is obtained by blowing up at the point in $F_{k-1} \cap C_1$, where $C_1^2 = -(b_1 + k - 2)$. Here, we obtain a new configuration of rational curves with SNC, and that F_k (the exceptional curve in X_{k+1}) intersects the ends of the chain. That configuration has continued fraction $[b_1 + (k-1), b_2 \ldots, b_r, 2, \ldots, 2]$, where $k-1$ is the number of 2's on the right side. Thus, we obtain a sequence $\{X_k\}$ of smooth surfaces with a configuration $[b_1+(k-1), b_2 \dots, b_r, 2, \dots, 2]$, and a (-1)-curve intersecting the ends of the chain.

Now, by Artin's contractibility Theorem [Art62, Thm. 2.3], we may contract the configuration in X_k for every k, to obtain a normal projective surface W_k with only one cyclic quotient singularity $P_k \in \mathcal{B}_{[b_1,\dots,b_r]}$. In particular, P_k is a generalized T-singularity of center $[b_1, \dots, b_r]$. Thus, to complete the proof, we must show that K_{W_k} is ample divisor for every k. In fact, let a_j be the discrepancies of $[b_1, \dots b_r]$ for $j = 1, \dots, r$, and let a'_j be the discrepancies of $[b_1+1, \dots, b_r, 2]$ for $i = 1, \dots, r+1$. We recall that $-1 < a_j, a'_j \leq 0$ for every j.

Claim 2. We have that $a_j > a'_j$ for every $j = 1, \dots, r$.

Proof. Let c_j, d_j (respectively c'_j, d'_j) be the auxiliary coefficients defined in Remark 3.17 for $[b_1, \dots, b_r]$ (respectively for $[b_1+1, \dots, b_r, 2]$). We recall that $c_1 = -1/b_1$, $c'_1 = -1/(b_1+1)$, $d_1 = (2-b_1)/b_1$, and $d'_1 = (1-b_1)/(b_1+1)$.

We have that $-1 < c'_j < c_j < 0$ for every $j = 1, \dots r-1$. Indeed, by a direct computation we know that $-1 < c'_1 < c_1 < 0$. Let us suppose the statement for $j-1$. Then,

$$c'_j = \frac{-1}{b_j + c'_{j-1}} < \frac{-1}{b_j + c_{j-1}} = c_j,$$

one can check directly that $-1 < c_j, c'_j < 0$ by using that $b_j \geq 2$.

Also, we obtain that $d'_j < d_j$ for every $j = 1, \dots, s$. In fact, we note that $d_1 - d'_1 = 2/(b_1(b_1+1)) > 0$. Assume the statement for $j-1$. Then,

$$d'_j - d_j \leq (b_j - 2 - d'_{j-1})c'_j - (b_j - 2 - d_j)c'_j = c'_j(d_{j-1} - d'_{j-1}),$$

by using $c'_j < 0$, and $d'_{j-1} < d_{j-1}$, we obtain that $d'_j < d_j$.

In addition, we note that because $-1 < c'_s < 0$ then

$$a_r - a'_r = d_r - d'_r(1-(c'_r)^2) > d_r - d'_r > 0.$$

Now, if we suppose that $a_{j+1} > a'_{j+1}$ then by using $c'_j < c_j < 0$ we obtain that

$$a_j - a'_j = (d_j - d'_j) + (c'_j a'_{j+1} - c_j a_{j+1}) > 0.$$

Thus, we obtain that $a_j > a'_j$ for every $j = 1, \dots, r$. $\square$

We will use Claim 2 to prove the ampleness. Let B_j the curves in the configuration on X_2, that is $B_j^2 = -b_j$, $B_1^2 = -(b_1+1)$, and $B_{r+1}^2 = -2$. Let $f\colon X_2 \to W_1$ be the map which contracts $F_2, B_1, \ldots, B_r$, and let $\phi\colon X_1 \to W_2$ be the map which contracts $B_1, \ldots, B_{r+1}$ (the minimal resolution of W_2). Then, one can check that

$$\phi^*(K_{W_2}) = f^*(K_{W_1}) + (a_1+1)F_2 + \sum_{j=1}^{r}(a_j - a_j')B_j - a_{r+1}'B_{r+1},$$

by Claim 2, we obtain K_{W_2} written as an effective sum of divisors. So, we only need to check that $K_{W_2}^2 > 0$ $\phi^*(K_2)\cdot F_2 > 0$ to prove the ampleness of K_{W_2}. We first recall that $-1-a_1-a_r = 1-(2+q+q')/n$, where $[b_1,\ldots,b_r] = n/q$. (See e.g. [Urz16, Section 2.1]).

Because of the ampleness of $K_{W_1}^2$, we obtain that $0 < -1-a_1-a_r$. So, we have that $2+q+q' < n$. Let N, Q be the integers such that $[b_1+1,\ldots,b_r,2] = N/Q$. By Proposition 4.12 we know that $N = 2q-m+2n-q'$, $Q = 2q-m$, and $Q' = q+n$. So,

$$\phi^*(K_2)\cdot F_2 = -1 - a_1' - a_{r+1}' = \frac{n(n-(2+q+q'))}{(n+q)(2n-q')+1}, \tag{32}$$

then, we obtain directly from (32) that $\phi^*(K_2)\cdot F_2 > 0$.

On the other hand, we have that

$$K_{W_2}^2 = K_{W_1}^2 + \frac{2+Q+Q'}{N} - \frac{2+q+q'}{n}, \tag{33}$$

and

$$(2+Q+Q')n-(2+q+q')N = \frac{(n-(2+q+q'))(n^2+2nq-nq'-qq'+1)}{n} > 0.$$

Thus, we obtain from (33) that $K_{W_2}^2 > 0$. Then, by the Nakai-Moishezon criterion we obtain that K_{W_2} is ample. Note that we only use the facts that K_{W_1} is ample, and the formation rule of the configuration in X_2 to prove that K_{W_2} is ample. So, we also proved that K_{W_k} ample implies $K_{W_{k+1}}$ ample for every k. Therefore, we have that $\{K_{W_k}^2\}$ has accumulation points. □

Remark 6.10. We recall some useful data from the proof of Proposition 6.9. Let W_1 be a stable surface which has only one generalized T-singularity $P_1 \in$

W_1 with continued fraction $[b_1, \dots, b_r]$. Let X_1 be the minimal resolution of P_1. Let X_2 be the surface obtained by blowing up X_1 as described in Step 1 (see proof of Proposition 6.9). As we saw in the proof, we have that W_2 has a generalized T-singularity P_2 with the continued fraction associated $[b_1 + 1, \dots, b_r, 2]$, and such that K_{W_2} is an ample divisor.

Example 6.11. Let W_1 be a stable surface which has only the singularity P with continued fraction $[n+1, 2] = 2n + 1/2$ for some $n \geq 5$. Note that the inverse of 2 modulo $2n+1$ is $n+1$. Assume that W fulfils the conditions of Proposition 6.9. Following the proof of Proposition 6.9, let $\{W_k\}$ be the sequence formed by blowing up the configuration of P such that W_k has only the singularity $[n+k, 2, \dots, 2]$, where k is the number of $2'$ on the right side. By Proposition 6.9 we know that W_k are stable surfaces and that $\{K^2_{W_k}\}$ has accumulation points.

Let $[n+k, 2, \dots, 2] = N_k/Q_k$, and let $0 < Q'_k < N_k$ be the inverse of Q_k modulo N_k. Let M_k the integer such that $Q_k Q'_k = 1 + M_k N_k$.

Now, by Proposition 4.12 and by induction, one can compute directly that $N_k = (k+1)n + k^2$, $Q_k = k+1$, $Q'_k = kn + k^2 - k + 1$, and $M_k = k$. Then,

$$\lim_{k\to\infty} K^2_{W_k} = \lim_{k\to\infty} K^2_{W_1} + \frac{2 + Q_k + Q'_k}{N_k} - \frac{n+5}{2n+1} = K^2_{W_1} + 1 - \frac{n+5}{2n+1}.$$

For the following proposition, let us consider the diagram

$$\begin{tikzcd}
 & X'_k \arrow[d, "\pi_k"'] \arrow[dr, "\phi_k"] & \\
 & X'_{k_1} \arrow[dl, "\pi"'] \arrow[dr, "\phi"] & W'_k \\
S_{k_1} & & W'_{k_1}
\end{tikzcd} \tag{34}$$

where the birational morphism $\phi\colon X'_{k_1} \to W'_{k_1}$ (Respectively $\phi_k\colon X'_k \to W'_k$) is the minimal resolution of W'_{k_1} (Respectively W'_k), and the surface S_{k_1} is the minimal model of X'_{k_1}, X'_k.

Proposition 6.12. *Under the assumptions of Theorem 1.9. Let $\nu_\infty \in \mathbb{Q}$ be an accumulation point of $\{K^2_{W_k}\}$. Then, there exists a sequence $\{W'_k\}$ of stable surfaces such that*

- *W'_k has only one generalized T-singularity P_k which is analytically the same singularity of W_k for every $k \in I$, where I is an infinite set of indices.*
- *There exist $k_1 \in I$ such that for every $k \in I$, the minimal resolution X'_k of W'_k is obtained by blowing up the minimal resolution X'_{k_1} of W'_{k_1}. (See Diagram 34).*
- *The limit of the sequence $K^2_{W'_k}$ is ν_∞.*

Proof. Let J be an infinite subset of indices as in Theorem 1.9 such that $\nu_\infty \in \mathrm{Acc}(\{K^2_{W_k} : k \in J\})$. Let us choose an infinite subset J' of J such that the sub sequence $\{K_{W_k}\}_{k\in J'}$ converges to ν_∞. As we saw in the proof of Theorem 1.9, we can choose an infinite set of indices $J'' \subseteq J'$ such that $K^2_{W_{k_i}}$ goes to ν_∞ when i goes to infinity, and also such that the continued fraction of P_{k_i} is obtained by applying the T-chain algorithm (see Definition 1.8) to the continued fraction of $P_{k_{i-1}}$.

By using Lemma 6.7, it follows that for every $k \in J''$

$$K^2_{W_k} = K^2_{S_k} + \sum_{j=1}^{r_u}(b^u_j - 2) - (m'_k + 1) - \left(\frac{2(n_k - 1) - q_k - q'_k}{n_k}\right), \tag{35}$$

where $0 < m'_k + 1 \leq \sum_{j=1}^{r_u}(b^u_j - 2)$. So, we may choose an infinite set of indices $I \subseteq J''$ such that m'_k is constant for every $k \in I$. After renaming the surfaces W_k we may suppose that $I = \mathbb{N}$.

Let us write $W'_1 := W_1$, and let $X'_1 := X_1$ be the minimal resolution of P_1. Now, let X'_2 be the surface obtained by blowing up the configuration C_1 associated to P_1 such that after contracting the new configuration C_2 in X'_2, we obtain a normal projective surface W'_2 with the generalized T-singularity P_2. We remark that is possible because the continued fraction of P_2 is obtained by applying the T-chain algorithm (see Definition 1.8) to the continued fraction of P_1. By using Remark 6.10 (maybe several times) we obtain that K_{W_2} is an ample divisor. Also, by using the facts that $K^2_{S_k}, m'_k$ are constants for every $k \in I$ in (35), we obtain that $K^2_{W'_2} = K^2_{W_2}$.

Finally, by using an inductive argument, we construct a sequence of stable surfaces $\{W'_k\}$ with the desired properties of the statement. $\square$

6.3 Open questions

To describe the behavior of the accumulation points for stable surfaces with one generalized T-singularity with a fixed center (Theorem 1.9), we used that the canonical class of S is nef. We do not know what happens otherwise. We note that Theorem 1.1 is still valid for K_S not nef, and so it could be used for some further analysis. Also, we are interested in finding properties for generalized T-singularities, like the one that motives the definition of T-singularities in [KSB88].

On the other hand, the general question on how accumulation points show up for stable surfaces with only one cyclic quotient singularity remains open.

Remark 6.13. Given a sequence as in Theorem 1.9, we saw in the proof of Theorem 1.9 that every accumulation point of a sequence $\{K^2_{W_k}\}$ can be obtained from a subsequence such that every W_k has only one singularity in the set $\mathcal{B}([b^u_1, \ldots, b^u_{r_u}])$ for a fixed $u \geq 0$. In that case, we recall that $K^2_{W_k}$ are related by the following formula

$$K^2_{W_k} = c + \sum_{j=1}^{r_u}(b^u_j - 2) - (m+1) - \left(\frac{2(n_k-1) - q_k - q'_k}{n_k}\right), \qquad (36)$$

where c, m are fixed numbers, and $c = K^2_{S_k}$. By Proposition 4.12, we know a recursive way of computing the quotients in (36). So, we have the following question

Question 6.14. *Let $\{W_k\}$ be a sequence of stable surfaces as in Theorem 1.9. What are the accumulation points of $\{K^2_{W_k}\}$?*

We saw in Proposition 6.12 that every accumulation point of stable surfaces with only one generalized T-singularity can be constructed by blowing up a certain configuration of curves in a smooth surface and then contracting the new configuration obtained. Following that idea, we want to finish with the following question concerning that topic.

Question 6.15. *Let $\{W_k\}$ be a sequence of stable surfaces such that any W_k has only one cyclic quotient singularity, say $P_k \in W_k$. Assume that singularities P_k are analytically different for every k. Let E^k be an exceptional divisor in the minimal resolution of W_k such that Γ_{E^k} is maximal (see Definition 5.8). Assume that E_k has only one type of diagram. Namely, a diagram of type (i), (iii) or (iv) (see Definition 5.7). Then, the set $\{K^2_{W_k}\}$ has accumulation points.*

References

[AL19a] Valery Alexeev and Wenfei Liu. Log surfaces of picard rank one from four lines in the plane. *European Journal of Mathematics*, 5(3):622–639, 2019.

[AL19b] Valery Alexeev and Wenfei Liu. On accumulation points of volumes of log surfaces. *Izvestiya: Mathematics*, 83(4):657, 2019.

[AL19c] Valery Alexeev and Wenfei Liu. Open surfaces of small volume. *Algebraic Geometry*, 6(3):312–327, 2019.

[Ale94] Valery Alexeev. Boundedness and k^2 for log surfaces. *International Journal of Mathematics*, 5(06):779–810, 1994.

[Ale96] Valery Alexeev. Moduli spaces mg, n (w) for surfaces, higher-dimensional complex varieties (trento, 1994), 1996.

[Art62] Michael Artin. Some numerical criteria for contractability of curves on algebraic surfaces. *American Journal of Mathematics*, 84(3):485–496, 1962.

[Bea96] Arnaud Beauville. *Complex algebraic surfaces*. Number 34. Cambridge University Press, 1996.

[BHPVdV04] W. P. Barth, K. Hulek, C. M. A. Peters, and A Van de Ven. *Compact complex surfaces*, volume 4. Springer, 2004.

[Bla95] Raimund Blache. An example concerning alexeev's boundedness results on log surfaces. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 118, pages 65–69. Cambridge University Press, 1995.

[Dur79] Alan Durfee. Fifteen characterizations of rational double points and simple critical points. *Enseign. Math*, 25(1-2):131–163, 1979.

[Ful93] William Fulton. *Introduction to toric varieties*. Number 131. Princeton University Press, 1993.

[Gie77] David Gieseker. Global moduli for surfaces of general type. *Inventiones mathematicae*, 43(3):233–282, 1977.

[Har77] Robin Hartshorne. *Algebraic Geometry*, volume 52. Springer-Verlag, 1977.

[HJ12] Roger A Horn and Charles R Johnson. *Matrix analysis*. Cambridge university press, 2012.

[HM06] Christopher D Hacon and James McKernan. Boundedness of pluricanonical maps of varieties of general type. *Inventiones mathematicae*, 166(1):1–25, 2006.

[HMX13] Christopher D Hacon, James McKernan, and Chenyang Xu. On the birational automorphisms of varieties of general type. *Annals of mathematics*, pages 1077–1111, 2013.

[Kaw88] Yujiro Kawamata. Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces. *Annals of Mathematics*, 127(1):93–103, 1988.

[KM08] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134. Cambridge university press, 2008.

[Kol90] János Kollár. Projectivity of complete moduli. *Journal of Differential Geometry*, 32(1):235–268, 1990.

[Kol94] János Kollár. Log surfaces of general type: some conjectures. *Contemporary Mathematics*, 162:261–261, 1994.

[KSB88] János Kollár and Nicholas Shepherd-Barron. Threefolds and deformations of surface singularities. *Inventiones mathematicae*, 91(2):299–338, 1988.

[Lan03] Adrian Langer. Logarithmic orbifold euler numbers of surfaces with applications. *Proceedings of the London Mathematical Society*, 86(2):358–396, 2003.

[Laz17] Robert K Lazarsfeld. *Positivity in algebraic geometry I: Classical setting: line bundles and linear series*, volume 48. Springer, 2017.

[Liu17] Wenfei Liu. The minimal volume of log surfaces of general type with positive geometric genus. *ArXiv*, 2017.

[Mor87] Shigefumi Mori. Classification of higher-dimensional varieties. In *Proc. Symp. Pure Math*, volume 46, pages 269–332, 1987.

[Mum61] David Mumford. The topology of normal singularities of an algebraic surface and a criterion for simplicity. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 9(1):5–22, 1961.

[OW77] Peter Orlik and Philip Wagreich. Algebraic surfaces withk*-action. *Acta Mathematica*, 138(1):43–81, 1977.

[Ran17] Julie Rana. A boundary divisor in the moduli spaces of stable quintic surface. *International Journal of Mathematics*, 28(4):1–61, 2017.

[RU19] Julie Rana and Giancarlo Urzúa. Optimal bounds for t-singularities in stable surfaces. *Advances in Mathematics*, 345:814–844, 2019.

[TW21] Burt Totaro and Chengxi Wang. Varieties of general type with small volume. *arXiv:2104.12200*, 2021.

[TZ92] Shuichiro Tsunoda and De-Qi Zhang. Noether's inequality for non-complete algebraic surfaces of general type. *Publications of the Research Institute for Mathematical Sciences*, 28(1):679–707, 1992.

[Urz10] Giancarlo Urzúa. Arrangements of curves and algebraic surfaces. *Journal of Algebraic Geometry*, 19(2):335–365, 2010.

[Urz16] Giancarlo Urzúa. Identifying neighbors of stable surfaces. *Annali della Scuola Normale Superiore di Pisa. Classe di scienze*, 16(4):1093–1122, 2016.

[UU19] Douglas Ulmer and Giancarlo Urzúa. Transversality of sections on elliptic surfaces with applications to elliptic divisibility sequences and geography of surfaces. *arXiv:1908.02208 [math.AG]*, 2019.

[UYn17] Giancarlo Urzúa and José Ignacio Yáñez. Notes on accumulation points of K^2. *Pre-print*, 2017.

[Wah81] Jonathan Wahl. Smoothings of normal surface singularities. *Topology*, 20(3):219–246, 1981.