PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE
ESCUELA DE INGENIERÍA

A MULTIFACTOR STOCHASTIC VOLATILITY MODEL OF COMMODITY PRICES

MATÍAS FRANCISCO LÓPEZ ABUKALIL

Thesis submitted to the Office of Research and Graduate Studies
in partial fulfillment of the requirements for the degree of
Master of Science in Engineering

Advisor:
GONZALO CORTÁZAR

Santiago de Chile, April 2015

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Members of the Committee:
GONZALO CORTÁZAR
ROLANDO REBOLLEDO
TOMÁS REYES
LORENZO NARANJO
GUSTAVO LAGOS

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To my parents.

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In first place I must express my deepest gratitude to professors Gonzalo Cortázar and Lorenzo Naranjo for their advice, support, guidance and excellent willingness throughout this thesis. This has been a tremendous life experience. I've grown academically, professionally and personally. All these achievements would not have been possible without them.

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ABSTRACT

We propose a novel representation of commodity spot prices in which the cost-of-carry and the spot price volatility are both driven by an arbitrary number of risk factors, nesting many existing specifications. The model exhibits unspanned stochastic volatility, provides simple closed-form expressions of commodity futures, and yields analytic formulas of European options on futures. The model is estimated using oil futures and options data, and find that the pricing of observed contracts is accurate for a wide range of maturities and strike prices. The results suggest that at least three risk factors in the spot price volatility are needed to fit accurately the volatility surface of options on oil futures, highlighting the importance of using general multifactor models in pricing commodity contingent claims.

Keywords: Commodities; Multifactor Models; Stochastic Volatility; Derivatives; Asset pricing.

RESUMEN

Nosotros proponemos una novedosa representación de los precios spot de commodities en la cual el cost-of-carry y la volatilidad del precio spot son ambas explicadas por un nmero arbitrario de factores de riesgo, anidando así muchas de las ya existentes especificaciones. El modelo exhibe unspanned stochastic volatility, provee simples y cerradas expresiones para los precios futuros y entrega frmulas analíticas para opciones europeas sobre futuros. El modelo es estimado utilizando datos de futuros y opciones sobre petrleo, encontrando que la valorizacin de los contratos observados es precisa para un amplio rango de madureces y precios de ejercicio. Los resultados sugieren que al menos tres factores de riesgo en la volatilidad del precio spot son necesarios para ajustar correctamente la superficie de volatilidad presente en las opciones sobre futuros de petrleo, destacando así la importancia de usar modelos generales y multifactoriales en la valorizacin de derivados de commodities.

Palabras Claves: Commodities; Modelos Multifactoriales; Volatilidad Estocstica; Derivados; Valorizacin de Activos.

1. ARTICLE BACKGROUND

1.1. Introduction

Since the beginning of this century, the commodity derivatives markets has shown remarkable growth, both in number of traded contracts and their notional value. This increase is due not only to producers and consumers hedging their risk exposures but also to a rise of speculative activity, a phenomenon known as financialization. This phenomenon has generated a renewed interest in understanding the stochastic behavior of commodities spot prices and derivative contracts observed in these markets.

In order to achieve this comprehension, several commodity prices models have been proposed by practitioners and academics. In particular, they have focused most of their attention into two specific contracts: futures and options. However, since these contracts exhibit linear and convex payoff structure respectively, the relevant variables included in each model might change dramatically.

Given its linear payoff structure, commodity futures prices show an almost exclusive dependence on the cost-of-carry (the difference between the risk-free interest rate and the convenience yield), turning its dynamics to one of the crucial points for any model that seeks to deliver an accurate fit of the observed futures prices term-structure (E. S. Schwartz, 1997). The Cortazar and Naranjo (2006) N-factor model is a remarkable example since, under its framework, futures prices are driven by an arbitrary number of factors which is crucial to achieve an excellent fit of the term-structure.

The valuation of derivatives contracts with convex payoffs, such as options, requires a far more complex model (Cortazar, Gutierrez, & Ortega, 2015). In this case, the attention is primarily drawn to the dispersion and asymmetry of the price returns distribution. This has lead researchers to develop elaborated models which accounts for sophisticated features such as asymmetric shocks, stochastic volatility and price jumps, where the price of the contract is computed through an inverse Fourier transform of an expression involving the characteristic function of the price returns distribution (Heston, 1993; Hughen, 2010; Larsson & Nossman, 2011; Richter & Sørensen, 2002; Trolle & Schwartz, 2009b).

Classical models that consider the aforementioned features often belongs to the affine diffusion framework introduced by Dai and Singleton (2000). However, due to the complexity of the formulas involved in pricing equations, empirical implementation of these models over an extensive data panel of futures and options is quite difficult with the current statistical methods and computational capabilities. Trolle and Schwartz (2009b) are able to perform a practical implementation of their model but it belongs to the Heath, Jarrow, and Morton (1992) (HJM) framework. The main reason that allowed them to do so and one of their greatest contributions is to realize that commodity derivative markets exhibit a phenomenon known as unspanned stochastic volatility (USV), which implies that futures and other commodity linear contracts are unable to hedge spot price volatility risk, making options over futures non-redundant assets.

The rest of this chapter is structured as follows: section 1.2 states the main objectives pursued in this work, section 1.3 presents a literature review of the main theoretical framework and models for commodity derivatives pricing, section 1.4 expose the main conclusions of this work and section 1.5 trace possible paths for future research. Following this, chapter 2 contains the main article of this thesis. Within this, section 2.1 presents a brief review on different commodity prices models, section 2.2 introduces the theoretical framework for USV in affine diffusion models and deepens the review early presented, section 2.3 explains the model and its features, section 2.4 presents the details of the empirical implementation of the affine diffusion model, while its results are reported and discussed in section 2.5. Section 2.6 finally concludes. All proofs, details on the numerical estimation, figures and tables are presented in the appendix.

1.2. Main Objectives

The main goal of this thesis is to present a novel model for pricing commodity contingent claims that belongs to the classical affine diffusion framework, with a naturally risen multifactor structure that exhibits USV. The model may be interpreted as a generalization of the N-factor model by including a rich multifactor structure to the spot price volatility which is driven by M additional factors.

In order to demonstrate the contributions of this model, the article has two main objectives. The first objective is to develop the condition to be met by an affine diffusion model in order to exhibit USV. As it will be shown, the condition is quite simple but has powerful consequences. In this case, it yields closed-form formulas for futures prices, quasi-analytical expressions for option prices and separates the contracts exposure to the factors. The latter means that futures prices depend solely on the first N factors while options prices are explained by the remaining M factors.

The second objective is to show explicitly the great fit of futures and options prices that may be fulfilled by the model. To achieve this, an empirical implementation is carried out using the Extended Kalman Filter and considering different numbers of factors and using an extensive panel of WTI oil futures and options ranging from January 2006 to December 2014. Within this objective, this application will also include a discussion about the parameters estimated values, an interpretation for the volatility factors, an analysis of the model's robustness through the consideration of and in-sample and out-of-sample results and the study of the implied volatility skewness which constitutes a proxy for the third moment of the spot price returns distribution.

1.3. Literature Review

Since the seminal work of Black and Scholes (1973), on the valuation of equity contingent claims, the modern asset pricing theory is based on the assumption that, in an equilibrium state, assets are priced in such a way that arbitrage opportunities are ruled out.

This implies the existence of a probability measure $\mathbb{Q}$, equivalent to the physical probability measure $\mathbb{P}$, under which all assets have the same expected rate of return, the risk-free rate. A direct implication of this is that any asset may be priced as the discounted future cash flows at the risk-free rate and due to this, the probability measure $\mathbb{Q}$ is known as risk-neutral measure.

Equilibrium commodity models often specify an affine process to model spot price dynamics, under which the instantaneous return (the instantaneous difference of the logarithm of the spot) follows a gaussian distribution. The variance of the spot price is commonly model as the linear combination a set of variables following square-root process that allows them to remain positive and revert to a mean which may be also stochastic; while the risk-neutral expected instantaneous return, the cost-of-carry, has been classically modeled as the linear combination of a another set of variables that may or may not be related with the variance variables.

However, there are some models that specify the entire forward convenience yield, cost of carry or interest rate curve instead of their spot counterparts. This has the advantage of exactly fit the initial forward curve, but at expenses of deriving an non-arbitrage condition. Moreover, to empirically implement this models, it must be cast into a non-intuitive state variable space that exhibits an affine structure. Models developed under this assumption are usually stated as being developed under the HJM framework.

1.3.1. Constant Volatility Models of Commodity Prices

Earlier commodity models found in the literature, regardless the number of factors considered, often assume that each of these exhibits a constant volatility. For example, Brennan and Schwartz (1985) consider a simple process where the spot price follows a geometric brownian motion along with constant drift and volatility. Since the only source of uncertainty corresponds to the spot price, it is said that this is a 1-factor model. E. S. Schwartz (1997) also proposes a similar 1-factor model but where the spot price logarithm follows an Ornstein-Uhlenbeck process.

Gibson and Schwartz (1990) present a 2-factor model where the first factor corresponds to the spot price which also follows a geometric brownian motion. The second factor follows an mean-reverting process which is correlated with the spot price. E. Schwartz and Smith (2000) also present a 2-factor model where the spot price logarithm is the sum of an arithmetic brownian motion, representing the persistent variations, and a zero-mean reverting process, capturing the short-term deviations.

More sophisticated commodity models with constant volatility often assume 3-factor specifications which extends the aforementioned models including a stochastic process for interest rates (Casassus & Collin-Dufresne, 2005; Hilliard & Reis, 1998; E. S. Schwartz, 1997).

As mentioned earlier, Cortazar and Naranjo (2006) is a remarkable example among the constant volatility models. The authors develop a canonical model with N factors (where N is an arbitrary number) which belongs to the affine diffusion framework presented in the seminal work of Dai and Singleton (2000). In this model, the spot price logarithm is the sum of N factors following a multivariate mean-reversion process. As it is shown in the article, the N-factor model is equivalent to, up to a rotation, to one where the spot price follows a geometric brownian motion where its drift corresponds to the sum of $N - 1$ factors following a multivariate mean-reversion process. This model also nest several models commonly found in the literature and, due to its arbitrary number of factors, achieves an excellent fit of the futures prices term-structure.

1.3.2. Stochastic Volatility Models of Commodity Prices

Despite of the evidence rejecting the constant volatility assumption (Duffie, Gray, & Hoang, 1999; Larsson & Nossman, 2011), it does not have a major impact in the pricing performance of futures contract due to the presence of USV in commodity derivative markets. However, removing this hypothesis is crucial for valuation of more complex derivatives with a convex payoff structure.

The seminal work of Heston (1993) establishes the main framework to price options under the stochastic volatility assumption. He proposes a 2-factor model composed by the spot price and the variance factor. The first follows a geometric brownian motion whose volatility is given by the square root of the variance factor, which follows a correlated square root process. The price of an option contract is computed by finding the partial differential equation solved by the characteristic function of the spot price and then applying the Fourier inversion theorem.

Richter and Sørensen (2002) develop a 3-factor model to value options on agricultural commodities including seasonality an stochastic volatility under an affine framework. Hughen (2010) also presents a 3-factor model but, in this case, it is maximal under the affine diffusion framework. More recently, Chiang, Hughen, and Sagi (2015) explore a 4-factor model where the convenience yield is driven by two factors.

Yan (2002) introduces a 4-factor model with spot price jumps where the risk-free interest rate and the convenience yield are separately modeled as an square-root process and a mean-reverting process, respectively.

Trolle and Schwartz (2009b) develop a model under the HJM framework with two volatility factors following a multivariate square root process. They interpret their results as that the first volatility factor captures the short-term deviations while the second captures the more persistent ones. They also test their specification using only one volatility factor which results in a worst fit of options prices. Their model exhibits USV since, as they mention, it arises naturally under the HJM framework.

Finally, some authors have explored models that allow for at least some dependence of futures prices on stochastic volatility. Nielsen and Schwartz (2004) proposes a 2-factor model which extends the Gibson and Schwartz (1990) model by letting the volatility to be proportional to the convenience yield. Liu and Tang (2011) generalizes the previous specification by modeling the risk-free interest rate and the convenience yield similarly as Yan (2002), but in this case, both follows an square-root process.

1.4. Main Conclusions

The financialization of commodity derivatives markets has led researchers to seek a better understanding of the stochastic behavior of prices observed in the market. Consequently, different models have been proposed with the aim of providing an explanation for such behavior.

This thesis presents a novel model under the affine diffusion framework of Dai and Singleton (2000). The model develops a multifactor specification for the cost-of-carry and instantaneous volatility of spot price, and nests many existing models commonly found in literature. The model also exhibits USV, yielding closed-form formulas for futures prices, and quasi-analytical expressions for option prices.

An empirical implementation of the model is conducted using the an extensive panel of futures and options data ranging from January 2006 to December 2014 while considering a different number of volatility factors. The results of this implementation suggest that the multifactor structure is crucial to achieve an accurate fit of futures and options prices. Moreover, for the WTI oil case, at least two cost-of-carry and three volatility factors are required to obtain accurate valuations.

Additional results of the empirical implementation also suggest that increasing the number of factors not only provide a better fit of the observed prices but also delivers higher robustness of the estimation and offers a better fit of the price return distribution, for example, through its third moment.

1.5. Further Research

Given the generality, intuitiveness and robustness of its multifactorial specification, the model presented in this paper is placed as a very attractive element for further research.

Since the model belongs to the affine diffusion framework, it may be easily extended to the affine jump-diffusion framework of Duffie, Pan, and Singleton (2000). The inclusion of jumps provides might be relevant in capturing the dynamics of commodity returns and when pricing short-term in-the-money options. However, at this time, is not clear how to perform an empirical implementation of this model with such extensive panel as the one used in this work.

Finally, the model presented in this article allows to characterize the different commodity markets in terms of how many factors are required to perform an accurate fit of the futures and options prices observed. For example, do the copper implied volatility term-structure also need at least three factors to be properly explained? This one interesting question may be answered directly by the model implementation.

2. A MULTIFACTOR STOCHASTIC VOLATILITY MODEL OF COMMODITY PRICES

2.1. Introduction

Commodity contingent claims play a key role in modern financial markets. Commodity producers and consumers actively use futures and options contracts to hedge their exposures to unpredictable price swings. At the same time, speculative activity in these markets has increased over time, leading to large investment flows from institutional investors and wealthy individuals into commodities, a phenomenon commonly known as financialization (Tang & Xiong, 2012). On the public policy side, there has been increasing pressure to understand whether demand for commodity related contracts affects the behavior of underlying prices (Masters, 2008, 2009). All these factors have created a renewed interest in understanding the dynamics and stochastic behavior of spot prices, and the associated derivative contracts traded in these markets.

In this paper we propose a novel representation of commodity prices that generalizes and nests many models commonly found in the literature, such as Casassus and Collin-Dufresne (2005); E. Schwartz and Smith (2000); E. S. Schwartz (1997) and Cortazar and Naranjo (2006), among many others. In our model, we allow for both the cost-of-carry and the spot price volatility to be driven each by an arbitrary number of risk factors, in a way that is simple and straightforward to implement. Empirically, the model performs well when applied to oil futures and options data, yielding accurate valuations of observed contracts for a wide range of maturities and strike prices. As a consequence, the model is able to explain well-known empirical regularities in option markets such as the dynamics of volatility smiles, as well as the skew in risk-neutral distributions.

Early models of commodity prices such as E. S. Schwartz (1997) propose multifactor representations of the convenience yield, but leave the volatility of the spot price constant. While providing a good fit to the observed term-structure of futures prices, these models usually perform poorly when applied to options (Cortazar et al., 2015). As a result, recent studies in the commodities literature have focused in incorporating stochastic volatility into the dynamics of spot prices (see e.g. Chiang et al., 2015; Trolle & Schwartz, 2009b).

Our model generalizes several recent stochastic volatility models such as Chiang et al. (2015) by adding a rich multifactor structure to the spot price variance. Specifically, in our model futures prices are driven by N factors (one factor corresponding to the logarithm of the spot price and the remaining $N-1$ factors modeling its cost-of-carry) while options prices are driven by M additional volatility factors. Our specification builds on the general affine diffusion framework of Dai and Singleton (2000), and exhibits unspanned stochastic volatility (USV),[1] providing simple closed-form expressions of commodity futures, and yielding easy-to-compute analytic formulas of European options on futures.

We estimate model parameters using quasi-maximum likelihood and the Extended Kalman Filter (EKF) on a sample of daily WTI oil futures and options from January 2006 until December 2014. Our results reveal that the model achieves an accurate fit of the term-structure of futures prices, the implied volatility surface, and the implied volatility skewness. We test the robustness of the model by comparing in- and out-of-sample calibrations.

Our results suggest that the multifactor structure of the model is crucial in pricing accurately futures and options contracts alike. Empirically, we confirm that only the cost-of-carry factors and the spot price are used to fit futures prices, while the volatility factors only affect the pricing of options contracts, consistent with the USV nature of our model. Finally, our analysis reveals that at least two cost-of-carry and three volatility factors are required to obtain accurate futures and options valuations. Adding a fourth volatility factor improves the pricing of options in periods of market stress.

[1]The phenomenon of USV was initially studied in fixed-income markets (Collin-Dufresne & Goldstein, 2002; Li & Zhao, 2006), where it refers to the fact that bonds alone are unable to hedge interest-rate volatility risk, making interest-rate options non-redundant assets. For commodities, USV implies that futures and other commodity linear contracts are unable to hedge spot price volatility risk, making options on futures non-redundant assets.

There is recent literature on stochastic volatility models for commodity prices that we survey in Section 2.2. Within this literature, the papers closest to ours are Chiang et al. (2015) and Trolle and Schwartz (2009b). We believe that we add to their work. For example, we show in Section 2.3.3 that the model studied in Chiang et al. (2015) is a restricted version of a specification of ours in which we use three factors to model futures contracts and one factor to explain option prices. Trolle and Schwartz (2009b) propose an USV multifactor model of commodity prices within the Heath et al. (1992) (HJM) framework. We show that we can obtain tractable and general results within the widely used affine-diffusion class of models of the spot price, allowing us to generalize a large body of existing literature by embedding an arbitrary multifactor structure in the stochastic behavior of the variance. Furthermore, we provide simple sufficient conditions that deliver USV in multifactor models of the spot price.

The remainder of the article is organized as follows. Section 2.2 describes our model in its most general form, studies broad sufficient conditions that deliver USV, and reviews the literature. Section 2.3 explains the affine diffusion implementation of our model, and derives formulas for pricing commodity contingent claims. Section 2.4 presents the empirical methodology, while results are reported and discussed in Section 2.5. Section 2.6 finally concludes. All proofs and details on the numerical estimation are presented in the Appendix.

2.2. General USV Model Formulation

We present our model of commodity prices in its most general form, and identify simple, although broad, sufficient conditions that deliver USV. For commodities, such models yield simple valuation formulas for futures prices, while at the same time allowing for arbitrarily complex dynamics in the volatility, which is relevant when pricing options and other derivatives with convex payoffs. On the other hand, there is a large body of literature that has studied general multifactor models of commodity prices while either leaving the volatility constant, or allowing the variance to be driven by a simple univariate process. We show how to naturally embed these well-known models of commodity prices within the more general USV class.

Throughout this paper we consider a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a filtration $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$ generated by standard $\mathbb{P}$-Wiener processes $(\mathbf{W}_t)_{t\geq 0}$ in $\mathbb{R}^{N+2M}$ and satisfying the usual conditions (see e.g. Protter, 2005). The spot price S is described by the process:

$$\frac{dS_t}{S_t} = (y_t + \pi_t)dt + \sigma_S dB_t + \sqrt{v_t} dZ_t, \tag{2.1}$$

where $(B_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$ are standard $\mathbb{P}$-Wiener processes in $\mathbb{R}$ spanned by $(\mathbf{W}_t)_{t\geq 0}$, y represents the cost-of-carry, π designates the commodity risk-premium, and σ_S denotes the constant component of the variance while v denotes its stochastic component. Under the pricing measure $\mathbb{Q}$, equivalent to the physical measure $\mathbb{P}$, the spot price S is described by the process:

$$\frac{dS_t}{S_t} = y_t dt + \sigma_S dB_t^{\mathbb{Q}} + \sqrt{v_t} dZ_t^{\mathbb{Q}}, \tag{2.2}$$

where $B^{\mathbb{Q}}$ and $Z^{\mathbb{Q}}$ are standard $\mathbb{F}$-adapted $\mathbb{Q}$-Wiener processes in $\mathbb{R}$.

Equations (2.1) and (2.2) capture the essence of our modeling approach. In the next section we show how to operationalize the model and write the $\mathbb{F}$-adapted processes $(y_t)_{t\geq 0}$, $(\pi_t)_{t\geq 0}$ and $(v_t)_{t\geq 0}$ as multifactor affine diffusions, but for the moment we leave them unspecified. Nevertheless, it will prove useful in our analysis to put some restrictions on the statistical relation between y, v, $B^{\mathbb{Q}}$ and $Z^{\mathbb{Q}}$ in order to (i) obtain simple futures and option valuation formulas, and (ii) separate the problem of fitting futures and option prices. Since these two objectives are achieved when the model exhibits USV, we introduce the following sufficient (although not necessary) assumption that yields the result.

ASSUMPTION 1. The $\mathbb{F}$-adapted processes $\left\{(y_t)_{t\geq 0}, \left(B_t^{\mathbb{Q}}\right)_{t\geq 0}\right\}$ are $\mathbb{Q}$-independent of $\left\{(v_t)_{t\geq 0}, \left(Z_t^{\mathbb{Q}}\right)_{t\geq 0}\right\}$.

Notwithstanding its generality and simplicity, we must note that Assumption 1 is not necessary to obtain USV. In Appendix A we present an example of a model that exhibits

USV but in which Assumption 1 is violated since the convenience yield is correlated with the stochastic component of the variance. However, as will be shown later in our empirical analysis, the model written using Assumtion 1 is already flexible enough to fit futures and option prices well. Hence, we do not find necessary to complicate the analysis further.

Consider now a futures contract $F_{t,\tau}$ at instant t with delivery at time $T = t + \tau$. It is well known that $F_{t,\tau} = \mathbb{E}_t^{\mathbb{Q}}[S_T]$ (Duffie, 2001; Pozdnyakov & Steele, 2004). A direct application of It$\bar{\text{o}}$'s Lemma allows us to write:

$$S_T = S_t \exp\left\{\int_t^T \left(y_u - \frac{1}{2}\sigma_S^2\right) du + \int_t^T \sigma_S dB_u^{\mathbb{Q}}\right\} \exp\left\{\int_t^T \left(-\frac{1}{2}v_u\right) du + \int_t^T \sqrt{v_u} dZ_u^{\mathbb{Q}}\right\}, \tag{2.3}$$

which implies that

$$F_{t,\tau} = S_t \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left\{\int_t^T \left(y_u - \frac{1}{2}\sigma_S^2\right) du + \int_t^T \sigma_S dB_u^{\mathbb{Q}}\right\}\right] \tag{2.4}$$

since both exponentials in (2.3) are $\mathbb{Q}$-independent under Assumption 1, and the second exponential is a $\mathbb{Q}$-martingale.

Equation (2.4) shows that futures prices in this model are unable to hedge the volatility risk, i.e. the stochastic component of the variance remains unspecified. Moreover, futures prices can be computed "as if" the spot price follows the simpler homoskedastic process under $\mathbb{Q}$:

$$\frac{dS_t}{S_t} = y_t dt + \sigma_S dB_t^{\mathbb{Q}}. \tag{2.5}$$

Notice that in Equation (2.5) the cost-of-carry y can be correlated with the spot price S, a stylized feature of many constant-variance multifactor models of commodity prices (see e.g. E. S. Schwartz, 1997). Some recent USV models such as Chiang et al. (2015) do not allow for such dependence. Our modeling approach is able to combine both views.

Models in which futures prices are characterized as in (2.4) have been widely studied in the literature. For example, Gibson and Schwartz (1990), E. S. Schwartz (1997), Hilliard and Reis (1998), E. Schwartz and Smith (2000), Casassus and Collin-Dufresne (2005), and Cortazar and Naranjo (2006) model the cost-of-carry y as a linear combination of correlated Gaussian processes that are also correlated with S. Using a Gaussian process for the cost-of-carry has the advantage of delivering simple closed-form expressions for futures prices that are suitable for empirical implementations, even with an arbitrary number of risk factors and large data sets (Cortazar & Naranjo, 2006). In all the aforementioned studies the volatility of spot returns remains constant.

The assumption of constant volatility, though, contrasts with the evidence that the variance of many commodities is stochastic and clusters in times of economic stress (see e.g. Du, Yu, & Hayes, 2011; Larsson & Nossman, 2011; Nazlioglu, Erdem, & Soytas, 2013). Also, stochastic volatility models perform better than constant volatility ones when pricing commodity options (Cortazar et al., 2015).

The most common way to introduce stochastic volatility in (2.1) is to assume that v_t follows a square-root process as in Heston (1993). Within the USV class, Larsson and Nossman (2011) explore the time-series properties of a model in which the cost-of-carry is constant and the variance follows a square-root process. Yan (2002) introduces a model in which the cost-of-carry at instant t is represented as $y_t = r_t - \delta_t$, where in interest rate r and the variance v follow square-root processes, and the convenience yield δ follows a Ornstein-Uhlenbeck process. Chiang et al. (2015) explore the macro-economic implications of a model in which the cost-of-carry y is driven by two correlated Gaussian processes, and the variance v follows an square-root process.

Even though USV provides simple futures valuation formulas, some authors have explored models that allow for at least some dependence of futures prices on stochastic volatility. Nielsen and Schwartz (2004) propose a model in which the cost-of-carry at instant t is $y_t = r - \delta_t$, where the convenience yield δ follows a square-root process that also drives the variance v. Liu and Tang (2011) generalize the previous specification by modeling the cost-of-carry as $y_t = r_t - \delta_t$, where both r and δ follow independent square-root processes, and the variance v depends on r and δ. Richter and Sørensen (2002) and

Hughen (2010) explore more general three-factor models in which the variance v is allowed to follow an independent square-root process. The model of Hughen (2010) is maximal within the affine $\mathbb{A}_1(3)$ class of Dai and Singleton (2000) and allows for USV under suitable parameter restrictions. Notwithstanding the greater generality of these models, USV allows for simpler futures and options valuation formulas as we show in later in the paper, which is useful when estimating the model using a large panel of futures and option prices. Moreover, we show that a multifactor USV structure is rich enough to fit the cross-section of futures and options prices accurately.

A somewhat different strand of literature explores the pricing of commodity derivatives within the HJM framework (Amin, Ng, & Pirrong, 1999; Cortazar & Schwartz, 1994; Miltersen, 2003; Miltersen & Schwartz, 1998; Trolle & Schwartz, 2009b). With the exception of Trolle and Schwartz (2009b), these papers do not allow for stochastic volatility. By mimicking the reasoning in the fixed-income literature, Trolle and Schwartz (2009b) conclude that USV arises naturally under the HJM framework, whereas in multifactor models of the spot price like (2.2) the volatility is almost invariably completely spanned by the futures contracts. We show that Assumption 1 provides a natural and simple sufficient condition to obtain USV in multifactor models of commodity spot prices. This allows us to generalize a large body of existing literature by embedding an arbitrary multifactor structure in the stochastic behavior of the variance.

We finalize this section by noting that some of the previous models also allow for jumps in the spot price (Casassus & Collin-Dufresne, 2005; Hilliard & Reis, 1998; Larsson & Nossman, 2011; Yan, 2002). Existing studies in equity markets show that jumps are important in explaining the time-series of equity returns (Eraker, Johannes, & Polson, 2003; Johannes, Polson, & Stroud, 2009). Larsson and Nossman (2011) find similar evidence for oil prices. Moreover, several studies have shown that jumps are also relevant in explaining short-term maturity options (Bates, 2000; Eraker, 2004; Pan, 2002). Our results suggest, however, that our model is able to explain futures and option prices well without resorting to jumps since we allow for many factors in the variance. Moreover, in unreported results, we find that including jumps in our model leaves the pricing ability of the model almost unchanged, specially for specifications with three or four factors in the variance.

2.3. USV Affine Diffusion Model Formulation

In this section we show how to model the cost-of-carry y, the risk-premium π, and the stochastic variance v presented in (2.1) and (2.2) using the affine diffusion (AD) framework introduced by Dai and Singleton (2000). Specifically, in our model futures prices are driven by N factors (one factor corresponding to the logarithm of the spot price and the remaining $N-1$ factors modeling its cost-of-carry) while options prices are driven by M additional volatility factors.

Our specification, however, satisfies Assumption 1 and hence exhibits USV. Furthermore, we use the drift normalized specification for the volatility proposed by Joslin (2014) which allows for richer formulations in the instantaneous variance of the spot price.[2] The proposed model yields simple futures and option pricing formulas, and nests many factor models commonly found in the commodities literature (e.g. Casassus & Collin-Dufresne, 2005; Cortazar & Naranjo, 2006; E. Schwartz & Smith, 2000; E. S. Schwartz, 1997). Hence, our USV-AD formulation is a particular case of the general AD class $\mathbb{A}_M(N+M)$ introduced by Dai and Singleton (2000) where we restrict the volatility of the cost-of-carry factors to be constant in order to ensure USV. In the rest of the article we refer to our restricted USD-AD specification by $\mathbb{A}_M^{\mathrm{U}}(N+M)$.

[2]Cheridito, Filipović, and Kimmel (2010) provide examples of affine diffusion processes with state space in $\mathbb{R}^2 \times \mathbb{R}_+^2$ that are not contained in the Dai and Singleton (2000) representation. Joslin (2014) shows that the Dai and Singleton (2000) representation for affine processes in the state space in $\mathbb{R}^N \times \mathbb{R}_+^M$ is exhaustive only when $N \leq 1$ or $M \leq 1$.

2.3.1. State Vector Dynamics

We consider a state vector at instant t, $\mathbf{X}_t$, which belongs to the state space $\mathbb{R}^N \times \mathbb{R}^M_+$ and is denoted as

$$\mathbf{X}_t = \begin{pmatrix} \mathbf{Y}_t \\ \log S_t \\ \mathbf{V}_t \end{pmatrix} \tag{2.6}$$

where $\mathbf{Y}_t = (x_{1,t}, x_{2,t}, \ldots, x_{N-1,t})'$ is a state vector in $\mathbb{R}^{N-1}$ driving the cost-of-carry, S_t denotes the commodity spot price where $\log S_t = x_{N,t}$, and $\mathbf{V}_t = (x_{N+1,t}, x_{N+2,t}, \ldots, x_{N+M,t})'$ is a state vector in $\mathbb{R}^M_+$ characterizing the stochastic nature of the spot price volatility.

The dynamics of the state vector $\mathbf{X}$ under the physical measure $\mathbb{P}$ are characterized by the following AD specification:

$$d\mathbf{X}_t = (\mathbf{\Theta} - \mathbf{K}\mathbf{X}_t)\, dt + \mathbf{\Sigma}_t d\mathbf{W}_t, \tag{2.7}$$

where $\mathbf{\Theta} \in \mathbb{R}^{N+M}$, $\mathbf{K} \in \mathbb{R}^{(N+M)\times(N+M)}$, $\mathbf{\Sigma}_t \in \mathbb{R}^{(N+M)\times(N+2M)}$, and $\mathbf{W}_t \in \mathbb{R}^{N+2M}$ are such that

$$\mathbf{\Theta} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \theta_N \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} \kappa_1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \kappa_{N-1} & 0 & 0 & 0 & \cdots & 0 \\ 1 & \cdots & 1 & 0 & \vartheta_1 & \vartheta_2 & \cdots & \vartheta_M \\ 0 & \cdots & 0 & 0 & \kappa_{N+1} & \kappa_{N+1,N+2} & \cdots & \kappa_{N+1,N+M} \\ 0 & \cdots & 0 & 0 & \kappa_{N+2,N+1} & \kappa_{N+2} & \cdots & \kappa_{N+2,N+M} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \kappa_{N+M,N+1} & \kappa_{N+M,N+2} & \cdots & \kappa_{N+M} \end{pmatrix},$$

with $\kappa_1, \ldots, \kappa_{N+M} > 0$, and $\kappa_{N+i,N+j} < 0$ for $i, j \in \{1, \ldots, M\}, i \neq j$.[3] The matrix $\mathbf{\Sigma}_t$ is chosen such that the instantaneous covariance matrix $\mathbf{H}_t = \mathbf{\Sigma}_t \mathbf{\Sigma}_t'$ satisfies

$$\mathbf{H}_t = \mathbf{H}_0 + \mathbf{H}_1 x_{N+1,t} + \ldots + \mathbf{H}_M x_{N+M,t}, \tag{2.8}$$

where $\mathbf{H}_0, \mathbf{H}_1 \ldots, \mathbf{H}_M \in \mathbb{R}^{(N+M)\times(N+M)}$ are given by

$$\mathbf{H}_0 = \begin{pmatrix} \mathbf{\Gamma}_0 & \mathbf{0}_{N\times M} \\ \mathbf{0}_{M\times N} & \mathbf{0}_{M\times M} \end{pmatrix}, \mathbf{H}_i = \begin{pmatrix} \mathbf{0}_{(N-1)\times(N-1)} & \mathbf{0}_{(N-1)\times(M+1)} \\ \mathbf{0}_{(M+1)\times(N-1)} & \mathbf{\Gamma}_i \end{pmatrix} \text{ for } i \in \{1, \ldots, M\},$$

and $\mathbf{W}_t$ is a vector of standard $\mathbb{P}$-Wiener processes. The matrix $\mathbf{\Gamma}_0 \in \mathbb{R}^{N\times N}$ is positive definite and specified as

$$\mathbf{\Gamma}_0 = \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{12} & \cdots & \sigma_1\sigma_N\rho_{1N} \\ \sigma_1\sigma_2\rho_{12} & \sigma_2^2 & \cdots & \sigma_2\sigma_N\rho_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_1\sigma_N\rho_{1N} & \sigma_2\sigma_N\rho_{2N} & \cdots & \sigma_N^2 \end{pmatrix}$$

with $\sigma_1, \ldots, \sigma_N > 0$ and $|\rho_{ij}| \leq 1$ for $i, j \in \{1, \ldots, N\}, i \neq j$, whereas matrices $\mathbf{\Gamma}_1, \ldots, \mathbf{\Gamma}_M \in \mathbb{R}^{(M+1)\times(M+1)}$ are also positive-definite and given by

$$\mathbf{\Gamma}_1 = \begin{pmatrix} \gamma_1^2 & \gamma_1\varsigma_1\varrho_1 & \cdots & 0 \\ \gamma_1\varsigma_1\varrho_1 & \varsigma_1^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \ldots, \mathbf{\Gamma}_M = \begin{pmatrix} \gamma_M^2 & 0 & \cdots & \gamma_M\varsigma_M\varrho_M \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_M\varsigma_M\varrho_M & 0 & \cdots & \varsigma_M^2 \end{pmatrix}$$

[3]These technical conditions are necessary for the existence of such process (Dai & Singleton, 2000) since they ensure that the drifts of the volatility factors are positive as they approach zero.

where $\gamma_1, \ldots, \gamma_M, \varsigma_1, \ldots, \varsigma_M > 0$ and $|\varrho_1|, \ldots, |\varrho_M| \leq 1$. Hence, we can express the instantaneous volatility matrix $\mathbf{\Sigma}_t$ as

$$\mathbf{\Sigma}_t = \left(\begin{array}{c|cccccccc} \mathbf{\Upsilon}_0 & \multicolumn{8}{c}{\mathbf{0}_{(N-1)\times 2M}} \\ \cline{2-9} & \tilde{\gamma}_{1,t} & \tilde{\gamma}_{2,t} & \ldots & \tilde{\gamma}_{M,t} & 0 & 0 & \ldots & 0 \\ \hline & \tilde{\varsigma}_{1,t}\varrho_1 & 0 & \cdots & 0 & \tilde{\varsigma}_{1,t}\sqrt{1-\varrho_1^2} & 0 & \cdots & 0 \\ & 0 & \tilde{\varsigma}_{2,t}\varrho_2 & \cdots & 0 & 0 & \tilde{\varsigma}_{2,t}\sqrt{1-\varrho_2^2} & \cdots & 0 \\ \mathbf{0}_{M\times N} & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ & 0 & 0 & \cdots & \tilde{\varsigma}_{M,t}\varrho_M & 0 & 0 & \cdots & \tilde{\varsigma}_{M,t}\sqrt{1-\varrho_M^2} \end{array}\right), \tag{2.9}$$

where $\mathbf{\Upsilon}_0$ denotes the Cholesky decomposition of $\mathbf{\Gamma}_0$, i.e. $\mathbf{\Upsilon}_0\mathbf{\Upsilon}_0' = \mathbf{\Gamma}_0$, and $\tilde{\gamma}_{m,t} = \gamma_m\sqrt{x_{N+m,t}}$, $\tilde{\varsigma}_{m,t} = \varsigma_m\sqrt{x_{N+m,t}}$ for $m \in \{1, \ldots, M\}$.

We adopt the completely-affine risk-premia specification of Dai and Singleton (2000) and define the market price of risk vector $\mathbf{\Lambda}_t \in \mathbb{R}^{N+2M}$ as

$$\mathbf{\Lambda}_t = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_N \\ \lambda_{N+1}\sqrt{x_{N+1,t}} \\ \vdots \\ \lambda_{N+M}\sqrt{x_{N+M,t}} \\ \lambda_{N+M+1}\sqrt{x_{N+1,t}} \\ \vdots \\ \lambda_{N+2M}\sqrt{x_{N+M,t}} \end{pmatrix}, \tag{2.10}$$

which implies that dynamics of the state vector under the risk-neutral measure $\mathbb{Q}$ are

$$d\mathbf{X}_t = \left(\mathbf{\Theta}^{\mathbb{Q}} - \mathbf{K}^{\mathbb{Q}}\mathbf{X}_t\right)dt + \mathbf{\Sigma}_t d\mathbf{W}_t^{\mathbb{Q}}, \tag{2.11}$$

where $\mathbf{\Theta}^{\mathbb{Q}} \in \mathbb{R}^{N+M}, \mathbf{K}^{\mathbb{Q}} \in \mathbb{R}^{(N+M)\times(N+M)}$ are such that

$$\mathbf{\Theta}^{\mathbb{Q}} = \begin{pmatrix} \theta_1^{\mathbb{Q}} \\ \vdots \\ \theta_{N-1}^{\mathbb{Q}} \\ \theta_N^{\mathbb{Q}} \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \quad \mathbf{K}^{\mathbb{Q}} = \begin{pmatrix} \kappa_1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \kappa_{N-1} & 0 & 0 & 0 & \cdots & 0 \\ 1 & \cdots & 1 & 0 & \frac{1}{2}\gamma_1^2 & \frac{1}{2}\gamma_2^2 & \cdots & \frac{1}{2}\gamma_M^2 \\ 0 & \cdots & 0 & 0 & \kappa_{N+1}^{\mathbb{Q}} & \kappa_{N+1,N+2} & \cdots & \kappa_{N+1,N+M} \\ 0 & \cdots & 0 & 0 & \kappa_{N+2,N+1} & \kappa_{N+2}^{\mathbb{Q}} & \cdots & \kappa_{N+2,N+M} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \kappa_{N+M,N+1} & \kappa_{N+M,N+2} & \cdots & \kappa_{N+M}^{\mathbb{Q}} \end{pmatrix},$$

with

$$\begin{aligned} \theta_n^{\mathbb{Q}} &= -\left(\lambda_1(\Upsilon_0)_{n,1} + \ldots + \lambda_i(\Upsilon_0)_{n,n}\right) \qquad \text{for } n \in \{1,\ldots,N-1\}, \\ \theta_N^{\mathbb{Q}} &= \theta_N - \left(\lambda_1(\Upsilon_0)_{N,1} + \lambda_2(\Upsilon_0)_{N,2} + \ldots + \lambda_N(\Upsilon_0)_{N,N}\right), \\ \kappa_{N+m}^{\mathbb{Q}} &= \kappa_{N+m} + \varsigma_i\left(\lambda_{N+m}\varrho_m + \lambda_{N+M+m}\sqrt{1-\varrho_m^2}\right) > 0 \qquad \text{for } m \in \{1,\ldots,M\}. \end{aligned}$$

There are several comments to make on our AD specification. First, it is apparent from (2.11) that the cost-of-carry

$$y_t = \theta_N^{\mathbb{Q}} - x_{1,t} - x_{2,t} - \ldots - x_{N-1,t}$$

follows a Gaussian process, as in the models of E. S. Schwartz (1997), E. Schwartz and Smith (2000), Casassus and Collin-Dufresne (2005), and Cortazar and Naranjo (2006), among others.

Second, the econometric identification of the model in (2.7) is guaranteed by the fact that we impose $\theta_{N+m} = 1$ for all $m \in \{1,\ldots,M\}$. Even though we could have set $\gamma_m = 1$ to achieve the same objective, we prefer the former alternative since it normalizes the volatility dynamics while keeping their values and related parameters of the same order of magnitude.

Third, the model presented in (2.7) is nested by the general specification (2.1) since we have the following natural correspondances:

$$
\begin{aligned}
\pi_t &= \lambda_1(\Upsilon_0)_{N,1} + \lambda_2(\Upsilon_0)_{N,2} + \ldots + \lambda_N(\Upsilon_0)_{N,N} \\
&\qquad + \lambda_{N+1}\gamma_1 x_{N+1,t} + \lambda_{N+2}\gamma_2 x_{N+2,t} + \ldots + \lambda_{N+M}\gamma_M x_{N+M,t}, \\
\sigma_S &= \sigma_N, \\
\sqrt{v_t} &= \sqrt{\gamma_1^2 x_{N+1,t} + \gamma_2^2 x_{N+2,t} + \ldots + \gamma_M^2 x_{N+M,t}}, \\
B_t &= \frac{(\Upsilon_0)_{N,1} W_{1,t} + (\Upsilon_0)_{N,2} W_{2,t} + \ldots + (\Upsilon_0)_{N,N} W_{N,t}}{\sigma_N}, \\
Z_t &= \frac{\gamma_1\sqrt{x_{N+1,t}}W_{N+1} + \gamma_2\sqrt{x_{N+2,t}}W_{N+2,t} + \ldots + \gamma_M\sqrt{x_{N+M,t}}W_{N+M,t}}{\sqrt{\gamma_1^2 x_{N+1,t} + \gamma_2^2 x_{N+2,t} + \ldots + \gamma_M^2 x_{N+M,t}}}.
\end{aligned}
$$

Hence, innovations to the log-spot price are driven by constant and stochastic volatility shocks, and the log-spot price exhibits correlation with both the cost-of-carry and the stochastic component of the variance.

Fourth, our model allows for a rich correlation structure among the cost-of-carry factors, among the stochastic variance factors, and between the cost-of-carry and stochastic variance factors with the spot price. First, if we denote by

$$
(\Xi_t)_{i,j} = \frac{\mathbb{C}\mathrm{ov}_t\left[dx_{i,t}, dx_{j,t}\right]}{\sqrt{\mathbb{V}_t\left[dx_{i,t}\right]}\sqrt{\mathbb{V}_t\left[dx_{j,t}\right]}} = \frac{(H_t)_{i,j}}{\sqrt{(H_t)_{i,i}}\sqrt{(H_t)_{j,j}}}
$$

the instantaneous correlation between $x_{i,t}$ and $x_{j,t}$,[4] a direct computation reveals that

$$
(\Xi_t)_{i,j} = \rho_{ij} \text{ for } i, j = 1, \ldots, N.
$$

Hence, the instantaneous correlation among the factors driving the cost-of-carry, and between those factors and the spot price, is constant and determined by the parameters ρ_{ij} in the full matrix $\boldsymbol{\Gamma}_0$. Furthermore, for $m \in \{1, \ldots, M\}$ we have that

$$
(\Xi_t)_{N,N+m} = \frac{\varrho_m \gamma_m \sqrt{x_{N+m,t}}}{\sqrt{\sigma_N^2 + \gamma_1^2 x_{N+1,t} + \gamma_2^2 x_{N+2,t} + \ldots + \gamma_M^2 x_{N+M,t}}},
$$

[4]Here $(\Xi_t)_{i,j}$ denotes the element (i, j) of matrix Ξ_t.

revealing that the instantaneous correlation between the factors driving the stochastic variance and the spot price is itself time-varying, and bounded in absolute value by $|\varrho_m|$. In contrast, the instantaneous correlation among the factors driving the stochastic variance is zero, but these factors are indirectly correlated through the parameters $\kappa_{N+i,N+j}$ in the full block inside $\mathbf{K}$.

Fifth, the risk-neutral specification presented in (2.11) is nested by (2.2) since we also have

$$B_t^{\mathbb{Q}} = \frac{(\Upsilon_0)_{N,1} W_{1,t}^{\mathbb{Q}} + (\Upsilon_0)_{N,2} W_{2,t}^{\mathbb{Q}} + \ldots + (\Upsilon_0)_{N,N} W_{N,t}^{\mathbb{Q}}}{\sigma_N},$$

$$Z_t^{\mathbb{Q}} = \frac{\gamma_1 \sqrt{x_{N+1,t}} W_{N+1}^{\mathbb{Q}} + \gamma_2 \sqrt{x_{N+2,t}} W_{N+2,t}^{\mathbb{Q}} + \ldots + \gamma_M \sqrt{x_{N+M,t}} W_{N+M,t}^{\mathbb{Q}}}{\sqrt{\gamma_1^2 x_{N+1,t} + \gamma_2^2 x_{N+2,t} + \ldots + \gamma_M^2 x_{N+M,t}}}.$$

Assumption 1 is then trivially satisfied because the structure of $\mathbf{K}^{\mathbb{Q}}$ implies that under the risk-neutral measure $\mathbb{Q}$ the processes $\mathbf{Y}$ and $\mathbf{V}$ do not have common terms in their drifts, and as may be seen from the expression for $\mathbf{\Sigma}_t$ in (2.9), their diffusive noises are uncorrelated. Hence, we can conclude that the AD model presented in (2.11) exhibits USV. This characterization will be explicitly verified when we compute closed-form expressions for futures prices.

2.3.2. Pricing of Commodity Derivatives

Our USV-AD model yields closed-form expressions for futures prices and simple quasi-analytical formulas for the European option prices. We follow Duffie et al. (2000) and use the joint moment generating function of the state vector $\mathbf{X}$. Given $\boldsymbol{\phi} \in \mathbb{C}^{N+M}$, consider the transform

$$\Psi_{t,\tau}(\boldsymbol{\phi}) = \mathbb{E}_t^{\mathbb{Q}} \left[e^{\boldsymbol{\phi}' \mathbf{X}_T} \right] \tag{2.12}$$

where $T = t + \tau$ and $\mathbb{E}_t^{\mathbb{Q}}[\cdot]$ denotes the time-t conditional expectation under the risk-neutral measure $\mathbb{Q}$. This transform has an exponentially affine solution as shown by proposition 1.

PROPOSITION 1. *The transform* Ψ *defined in* (2.12) *is given by*

$$\log \Psi_{t,\tau}(\boldsymbol{\phi}) = \alpha(\tau; \boldsymbol{\phi}) + \boldsymbol{\beta}(\tau; \boldsymbol{\phi})' \mathbf{X}_t \tag{2.13}$$

where α *and* $\boldsymbol{\beta}$ *satisfy the complex-valued ODEs*

$$\dot{\alpha} = \boldsymbol{\Theta}^{\mathbb{Q}\prime} \boldsymbol{\beta} + \frac{1}{2} \boldsymbol{\beta}' \mathbf{H}_0 \boldsymbol{\beta} \tag{2.14}$$

$$\dot{\boldsymbol{\beta}} = -\mathbf{K}^{\mathbb{Q}\prime} \boldsymbol{\beta} + \sum_{m=1}^{M} \frac{1}{2} \boldsymbol{\beta}' \mathbf{H}_m \boldsymbol{\beta} \mathbf{e}_{N+m} \tag{2.15}$$

with initial conditions $\alpha(0; \boldsymbol{\phi}) = 0$ *and* $\boldsymbol{\beta}(0; \boldsymbol{\phi}) = \boldsymbol{\phi}$*, and* $\mathbf{e}_n$ *denotes a basis vector in* $\mathbb{R}^{N+M}$ *whose* n*-th element is equal to one, and zero otherwise.*

PROOF. See Appendix B.1. □

Furthermore, due to the structure of the matrix $\mathbf{K}^{\mathbb{Q}}$, it is possible to compute closed-form expressions for $\beta_1, \ldots, \beta_N$ in (2.14):

$$\beta_n = \begin{cases} e^{-\kappa_n^{\mathbb{Q}} \tau} \phi_n + \dfrac{e^{-\kappa_n^{\mathbb{Q}} \tau} - 1}{\kappa_n^{\mathbb{Q}}} \phi_N & \text{if } n < N, \\ \phi_N & \text{if } n = N, \end{cases} \tag{2.16}$$

which yields closed-form expressions for futures prices as shown by Proposition 2.

2.3.2.1. Futures Prices

Let $F_{t,\tau}$ denote the futures price at instant t with delivery at time $T = t + \tau$. It is a well known fact (e.g. Duffie, 2001; Pozdnyakov & Steele, 2004) that the futures price corresponds to the risk-neutral expectation of the future spot price

$$F_{t,\tau} = \mathbb{E}_t^{\mathbb{Q}} \left[S_T \right] = \Psi_{t,\tau}(\mathbf{e}_N).$$

Hence, futures price can be obtained directly from the transform Ψ introduced in (2.12).

PROPOSITION 2. *The futures price at instant t with delivery at time $T = t + \tau$ is given by*

$$\log F_{t,\tau} = \alpha^F(\tau) + \boldsymbol{\beta}^F(\tau)'\mathbf{X}_t$$

where α^F and $\boldsymbol{\beta}^F$ are given by

$$\alpha^F(\tau) = \theta_N^{\mathbb{Q}}\tau + \sum_{n=1}^{N-1}\frac{\theta_n^{\mathbb{Q}}}{\kappa_n^{\mathbb{Q}}}\left(\frac{1-e^{-\kappa_n^{\mathbb{Q}}\tau}}{\kappa_n^{\mathbb{Q}}} - \tau\right) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i\sigma_j\rho_{ij}\zeta_{ij}(\tau)$$

$$\beta_n^F(\tau) = \begin{cases} \frac{1}{\kappa_n^{\mathbb{Q}}}\left(e^{-\kappa_n^{\mathbb{Q}}\tau} - 1\right) & \textit{if } n < N, \\ 1 & \textit{if } n = N, \\ 0 & \textit{if } n > N, \end{cases}$$

and $\zeta_{ij}(\tau) = \zeta_{ji}(\tau)$ where

$$\zeta_{ij}(\tau) = \begin{cases} \frac{1}{\kappa_i^{\mathbb{Q}}\kappa_j^{\mathbb{Q}}}\left(\tau - \frac{1-e^{-\kappa_i^{\mathbb{Q}}\tau}}{\kappa_i^{\mathbb{Q}}} - \frac{1-e^{-\kappa_j^{\mathbb{Q}}\tau}}{\kappa_j^{\mathbb{Q}}} + \frac{1-e^{-(\kappa_i^{\mathbb{Q}}+\kappa_j^{\mathbb{Q}})\tau}}{\kappa_i^{\mathbb{Q}}+\kappa_j^{\mathbb{Q}}}\right) & \textit{if } i, j < N, \\ \frac{1}{\kappa_i^{\mathbb{Q}}}\left(\frac{1-e^{-\kappa_i^{\mathbb{Q}}\tau}}{\kappa_i^{\mathbb{Q}}} - \tau\right) & \textit{if } i < j = N, \\ \tau & \textit{if } i = j = N. \end{cases}$$

PROOF. See Appendix B.2. □

We note that in Proposition 2 the functions $\beta_{N+1}^F, \ldots, \beta_{N+M}^F$ are equal to zero, which verifies that the model presented in (2.11) indeed exhibits USV.

2.3.2.2. European Options Prices

Let $P_{t,\tau_0,\tau_1}(K)$ denote the price at time t of an European put option expiring at time $T_0 = t + \tau_0$ with strike K written on a futures contract expiring at time $T_1 = T_0 + \tau_1$. The option price is then given by

$$P_{t,\tau_0,\tau_1}(K) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_0} r_s ds}\left(K - F_{T_0,\tau_1}\right)1_{\left\{F_{T_0,\tau_1} < K\right\}}\right],$$

where 1_A denotes the indicator function of the set A and r denotes the instantaneous risk-free rate. For the purpose of simplifying valuation formulas, we assume that $(r_t)_{t\geq 0}$ is independent of $(\mathbf{X}_t)_{t\geq 0}$, which gives a very accurate approximation of the true price of short-term or medium-term options (Trolle & Schwartz, 2009b).

We first define the auxiliary transform

$$\Psi^F_{t,\tau_0,\tau_1}(u) = \mathbb{E}^{\mathbb{Q}}_t\left[e^{u\log F_{T_0,\tau_1}}\right], \tag{2.17}$$

for some $u \in \mathbb{C}$. In our USV-AD setting, Ψ^F can be computed in closed-form as shown in the following proposition.

PROPOSITION 3. *The transform Ψ^F defined in* (2.17) *is given by*

$$\begin{aligned}\log\Psi^F_{t,\tau_0,\tau_1}(u) = u\left(\alpha^F(\tau_1) - \alpha^F(\tau_0+\tau_1)\right) + \alpha\left(\tau_0; u\boldsymbol{\beta}^F(\tau_1)\right) \\ + u\log F_{t,\tau_0+\tau_1} + \sum_{m=1}^{M}\beta_{N+m}\left(\tau_0; u\boldsymbol{\beta}^F(\tau_1)\right)v_{m,t}.\end{aligned} \tag{2.18}$$

PROOF. See Appendix B.3. □

We can then apply this result to obtain a quasi-analytic formula for P_{t,τ_0,τ_1}.

PROPOSITION 4. *The price at instant t of a European put option expiring at time $T_0 = t + \tau_0$ with strike K written on a futures contract expiring at time $T_1 = T_0 + \tau_1$ is given by*

$$P_{t,\tau_0,\tau_1}(K) = B_{t,\tau_0}\left(KG_{0,1}\left(\tau_0,\tau_1;\log K\right) - G_{1,1}\left(\tau_0,\tau_1;\log K\right)\right)$$

where B_{t,τ_0} denotes the price at instant t of a zero-coupon bond maturing in τ_0, and

$$G_{a,b}(\tau_0,\tau_1;k) = \frac{\Psi^F_{t,\tau_0,\tau_1}(a)}{2} - \frac{1}{\pi}\int_0^\infty \frac{\operatorname{Im}\left[\Psi^F_{t,\tau_0,\tau_1}(a+\mathtt{i}ub)e^{-\mathtt{i}uk}\right]}{u}du,$$

with $\operatorname{Im}[c]$ denoting the imaginary part of $c \in \mathbb{C}$ and $\mathtt{i} = \sqrt{-1}$ corresponding to the imaginary unit.

PROOF. See Appendix B.4. □

Following an analogous procedure, we can also price a European call option with the same characteristics as shown in the following proposition.

PROPOSITION 5. *The price at instant t of a European call option expiring at time $T_0 = t + \tau_0$ with strike K written on a futures contract expiring at time $T_1 = T_0 + \tau_1$ is given by*

$$C_{t,\tau_0,\tau_1}(K) = B_{t,\tau_0} \left(G_{1,-1} \left(\tau_0, \tau_1; -\log K \right) - K G_{0,-1} \left(\tau_0, \tau_1; -\log K \right) \right)$$

where B_{t,τ_0} and $G_{a,b}(\tau_0, \tau_1; k)$ are defined in Proposition 4.

PROOF. See Appendix B.5. □

2.3.3. Nested Models

Due to its general structure, our model nests several models commonly found in the commodities literature. Some of these models, however, are written in a different although equivalent form. We denote by $\boldsymbol{\xi}$ the state vector of a different model, and say that this model is equivalent to ours if there exists an affine transformation

$$\mathbf{X}_t = \mathbf{L}\boldsymbol{\xi}_t + \boldsymbol{\varphi} \tag{2.19}$$

such that the matrix $\mathbf{L}$ is invertible and the state vector spaces are preserved (Dai & Singleton, 2000). The last condition is trivially satisfied if $\mathbf{L}$ has the form

$$\mathbf{L} = \begin{pmatrix} \mathbf{L}_{DD} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_{DV} \end{pmatrix}$$

where $\mathbf{L}_{DV} \in \mathbb{R}^{M \times M}$ is a diagonal matrix with positive entries, and $\varphi_{N+m} = 0$ for each $m \in \{1, \ldots, M\}$. In that case, $\boldsymbol{\xi}$ also follows an AD processes as in (2.11) with parameters

$\bar{\boldsymbol{\Theta}}^{\mathbb{Q}}$, $\bar{\mathbf{K}}^{\mathbb{Q}}$, $\bar{\boldsymbol{\Sigma}}_t$, and $\bar{\mathbf{H}}_i$ for each $i \in \{0, 1, \ldots, M\}$, that relate to the original model by

$$\boldsymbol{\Theta}^{\mathbb{Q}} = \mathbf{L}\bar{\boldsymbol{\Theta}}^{\mathbb{Q}} + \mathbf{L}\bar{\mathbf{K}}^{\mathbb{Q}}\mathbf{L}^{-1}\boldsymbol{\varphi},$$

$$\mathbf{K}^{\mathbb{Q}} = \mathbf{L}\bar{\mathbf{K}}^{\mathbb{Q}}\mathbf{L}^{-1},$$

$$\boldsymbol{\Sigma}_t = \mathbf{L}\bar{\boldsymbol{\Sigma}}_t,$$

$$\mathbf{H}_0 = \mathbf{L}\bar{\mathbf{H}}_0\mathbf{L}',$$

$$\mathbf{H}_m = \frac{1}{(\mathbf{L})_{N+m,N+m}}\mathbf{L}\bar{\mathbf{H}}_m\mathbf{L}', \quad m \in \{1, \ldots, M\}.$$

Note that market prices of risk are invariant to affine transformations.

For example, the model studied in Cortazar and Naranjo (2006) introduces a state vector $\boldsymbol{\xi}_t$ at instant t which belongs to the state space $\mathbb{R}^N$, where the spot price S at instant t is defined as

$$\log S_t = \xi_{1,t} + \ldots + \xi_{N-1,t} + \xi_{N,t}.$$

In their model, the state vector dynamics under $\mathbb{Q}$ are determined by

$$\bar{\boldsymbol{\Theta}}^{\mathbb{Q}} = \begin{pmatrix} \bar{\theta}_1 \\ \vdots \\ \bar{\theta}_{N-1} \\ \bar{\theta}_N \end{pmatrix}, \quad \bar{\mathbf{K}}^{\mathbb{Q}} = \begin{pmatrix} \bar{\kappa}_1 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & \bar{\kappa}_{N-1} & 0 \\ 0 & \ldots & 0 & 0 \end{pmatrix},$$

where the covariance matrix is constant and given by

$$\bar{\boldsymbol{\Sigma}}_t\bar{\boldsymbol{\Sigma}}_t' = \begin{pmatrix} \bar{\sigma}_1^2 & \bar{\rho}_{12}\bar{\sigma}_1\bar{\sigma}_2 & \ldots & \bar{\rho}_{1N}\bar{\sigma}_1\bar{\sigma}_N \\ \bar{\rho}_{12}\bar{\sigma}_1\bar{\sigma}_2 & \bar{\sigma}_2^2 & \ldots & \bar{\rho}_{2N}\bar{\sigma}_2\bar{\sigma}_N \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\rho}_{1N}\bar{\sigma}_1\bar{\sigma}_N & \bar{\rho}_{2N}\bar{\sigma}_2\bar{\sigma}_N & \ldots & \bar{\sigma}_N^2 \end{pmatrix}.$$

Consider the affine transformation (2.19) parameterized by

$$\mathbf{L} = \begin{pmatrix} \bar{\kappa}_1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \bar{\kappa}_{N-1} & 0 \\ 1 & \dots & 1 & 1 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{\varphi} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

Observe that, under this transformation, the spot price is given by $\log S_t = x_{N,t}$. The resulting state vector $\mathbf{X}_t$ satisfies (2.11) where

$$\boldsymbol{\Theta}^{\mathbb{Q}} = \begin{pmatrix} \bar{\kappa}_1 \bar{\theta}_1 \\ \vdots \\ \bar{\kappa}_{N-1} \bar{\theta}_{N-1} \\ \bar{\theta}_1 + \dots + \bar{\theta}_{N-1} + \bar{\theta}_N \end{pmatrix}, \quad \mathbf{K}^{\mathbb{Q}} = \begin{pmatrix} \bar{\kappa}_1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \bar{\kappa}_{N-1} & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{\Sigma}_t \boldsymbol{\Sigma}_t' = \begin{pmatrix} \bar{\kappa}_1^2 \bar{\sigma}_1^2 & \bar{\rho}_{12} \bar{\kappa}_1 \bar{\sigma}_1 \bar{\kappa}_2 \bar{\sigma}_2 & \dots & \rho_{1N} \bar{\kappa}_1 \bar{\sigma}_1 \sigma_N \\ \bar{\rho}_{12} \bar{\kappa}_1 \bar{\sigma}_1 \bar{\kappa}_2 \bar{\sigma}_2 & \bar{\kappa}_2^2 \bar{\sigma}_2^2 & \dots & \rho_{2N} \bar{\kappa}_2 \bar{\sigma}_2 \sigma_N \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1N} \bar{\kappa}_1 \bar{\sigma}_1 \sigma_N & \rho_{2N} \bar{\kappa}_2 \bar{\sigma}_2 \sigma_N & \dots & \sigma_N^2 \end{pmatrix},$$

$$\sigma_N = \sqrt{\sum_{i=1}^{N} \bar{\sigma}_i^2 + 2 \sum_{i=1}^{N} \sum_{j=i+1}^{N} \bar{\rho}_{ij} \bar{\sigma}_i \bar{\sigma}_j} \quad \text{and} \quad \rho_{iN} = \frac{\bar{\sigma}_1 \bar{\rho}_{1i} + \bar{\sigma}_2 \bar{\rho}_{2i} \dots + \bar{\sigma}_N \bar{\rho}_{iN}}{\sigma_N}.$$

Since $\mathbf{L}$ is invertible, we conclude that the model presented by Cortazar and Naranjo (2006), which in turn nests the models of Cortazar and Schwartz (2003); Gibson and Schwartz (1990); E. Schwartz and Smith (2000); E. S. Schwartz (1997), is equivalent to our $\mathbb{A}_0^{\mathrm{U}}(N)$ specification. Also, note that by using a more general market price of risk specification our $\mathbb{A}_0^{\mathrm{U}}(3)$ model is also equivalent to the one presented in Casassus and Collin-Dufresne (2005).

The model introduced by Hughen (2010) with USV restrictions is equivalent to our $\mathbb{A}_1^{\mathrm{U}}(3)$ specification. Hughen (2010) includes a parameter ν to denote the minimum level of the instantaneous spot variance, a role performed by σ_N^2 in our model.

The model presented in Chiang et al. (2015) is nested by our $\mathbb{A}_1^{\mathrm{U}}(4)$ specification. In their model, the log-spot price corresponds to the sum of a mean-reverting and a persistent factor, where only the later exhibits stochastic volatility. The authors introduce a state vector $\boldsymbol{\xi}_t$ at instant t which belongs to the state space $\mathbb{R}^3 \times \mathbb{R}_+$, and define the spot price S at instant t as

$$\log S_t = \xi_{1,t} + \xi_{3,t}.$$

The state vector dynamics under $\mathbb{Q}$ are determined by

$$\bar{\boldsymbol{\Theta}}^{\mathbb{Q}} = \begin{pmatrix} 0 \\ \bar{\theta}_1 \\ 0 \\ \bar{\theta}_2 \end{pmatrix}, \quad \bar{\mathbf{K}}^{\mathbb{Q}} = \begin{pmatrix} \bar{\kappa}_1 & 0 & 0 & 0 \\ 0 & \bar{\kappa}_2 & 0 & 0 \\ 0 & -1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \bar{\kappa}_3 \end{pmatrix},$$

while the covariance matrix is given by

$$\bar{\boldsymbol{\Sigma}}_t \bar{\boldsymbol{\Sigma}}_t' = \begin{pmatrix} \bar{\sigma}_1^2 & \bar{\rho}_1\bar{\sigma}_1\bar{\sigma}_2 & -\bar{\sigma}_1^2 & 0 \\ \bar{\rho}_1\bar{\sigma}_1\bar{\sigma}_2 & \bar{\sigma}_2^2 & -\bar{\rho}_1\bar{\sigma}_1\bar{\sigma}_2 & 0 \\ -\bar{\sigma}_1^2 & -\bar{\rho}_1\bar{\sigma}_1\bar{\sigma}_2 & \bar{\sigma}_1^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \bar{\rho}_2\bar{\sigma}_3 \\ 0 & 0 & \bar{\rho}_2\bar{\sigma}_3 & \bar{\sigma}_3^2 \end{pmatrix} \xi_{4,t}.$$

Consider the affine transformation (2.19) parameterized by

$$\mathbf{L} = \begin{pmatrix} \bar{\kappa}_1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\varphi} = \begin{pmatrix} 0 \\ \frac{\bar{\theta}_1}{\bar{\kappa}_2} \\ 0 \\ 0 \end{pmatrix}.$$

Observe that, under this transformation, the spot price is given by $\log S_t = x_{3,t}$ while its instantaneous volatility is given by $x_{4,t}$. The resulting state vector $\mathbf{X}_t$ satisfies the affine diffusion (2.11) where

$$\mathbf{\Theta}^{\mathbb{Q}} = \begin{pmatrix} 0 \\ 0 \\ \frac{\bar{\theta}_1}{\bar{\kappa}_2} \\ \bar{\theta}_2 \end{pmatrix}, \quad \mathbf{K}^{\mathbb{Q}} = \begin{pmatrix} \bar{\kappa}_1 & 0 & 0 & 0 \\ 0 & \bar{\kappa}_2 & 0 & 0 \\ 1 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \bar{\kappa}_3 \end{pmatrix},$$

while the covariance matrix is given by

$$\mathbf{\Sigma}_t \mathbf{\Sigma}_t' = \begin{pmatrix} \bar{\kappa}_1^2 \bar{\sigma}_1^2 & -\bar{\rho}_1 \bar{\kappa}_1 \bar{\sigma}_1 \bar{\sigma}_2 & \boxed{0} & 0 \\ -\bar{\rho}_1 \bar{\kappa}_1 \bar{\sigma}_1 \bar{\sigma}_2 & \bar{\sigma}_2^2 & \boxed{0} & 0 \\ \boxed{0} & \boxed{0} & \boxed{0} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \bar{\rho}_2 \bar{\sigma}_3 \\ 0 & 0 & \bar{\rho}_2 \bar{\sigma}_3 & \bar{\sigma}_3^2 \end{pmatrix} x_{4,t}.$$

Since $\mathbf{L}$ is invertible, we conclude that the model presented by Chiang et al. (2015) is nested by our $\mathbb{A}_1^{\mathrm{U}}(4)$ specification. The two models, though, are not equivalent since the authors set the constant volatility parameter to zero which forces the spot price to be uncorrelated with the cost-of-carry factors. As a consequence, all the coefficients inside the box in the previous expression are equal to zero.

2.4. Data and Estimation Procedure

2.4.1. Data

Our dataset includes daily WTI futures and option prices from January, 3rd 2006 until December, 31st 2014 traded at NYMEX, which corresponds to 9 years of data. In what follows, we apply the same filters and data definitions as Trolle and Schwartz (2009b).

For futures, we select twelve generic contracts: the first 6-month futures (M1–M6), the following two contracts with expiration either in March, June, September and December (Q1–Q2), and the next four contracts with expiration in December (Y1–Y4). We discard all futures with fourteen or fewer days to expiration since they report a significant reduction in their open interest.

Figure D.1 presents the evolution of the term-structure of futures prices for the sample period. If we use the closest-to-maturity contract as a proxy for the spot price, the figure displays strong variability of oil prices reaching almost USD 150 per barrel at the end of 2008, and dropping below USD 40 per barrel short after. The figure also shows that the term-structure changes its shape continuously during the sample period, going from contango to backwardation, and vice-versa. Finally, the figure displays a sustained drop in prices at the end of 2014.

For options, moneyness is defined as the strike price divided by the price of the underlying futures contract. For each maturity we split the range of moneyness into eleven intervals: 0.78–0.82, 0.82–0.86, 0.86–0.90, 0.90–0.94, 0.94–0.98, 0.98–1.02, 1.02–1.06, 1.06–1.10, 1.10–1.14, 1.14–1.18, and 1.18–1.22. For a given maturity and moneyness, we select the contract that is closest to the medium-point of the interval.

Since options on oil futures are American, we use equivalent European option prices obtained by first computing implied volatilities using the approximation detailed in Barone-Adesi and Whaley (1987), and then computing the corresponding European price using the method of Black (1976).

We only include out-of-the-money (OTM) and at-the-money (ATM) options, which has the advantage of minimizing the errors of our early-exercise approximation. Moreover, OTM options tend to be more liquid than in-the-money (ITM) options.[5] Furthermore, we only include contracts with open interest greater than 100, price over 10 cents, and written over the first eight futures (M1–Q2) to minimize the impact of the early-exercise approximation for longer dated contracts.

[5]Given a certain maturity, it may occur that there is a put and a call with moneyness exactly to one, or two options with the same moneyness distance to the medium-point of their interval. In such cases we keep both contracts.

Figure D.2 displays the implied volatility for ATM options, that moves significantly during the sample period. Similar to futures prices, the term-structure of implied volatilities also changes its shape through time. We can observe a large increase in implied volatilities at the end of 2008 and more recently at the end of 2014.

After applying the filters to the data, we are left with 27 156 futures and 175 938 options over a period of 2 263 trading days.

2.4.2. Estimation Procedure

We estimate the unobserved state-vector $\mathbf{X}$ using the Extended Kalman filter (EKF), and model parameters through quasi maximum-likelihood.[6] The Euler-discretized transition equation is given by

$$\mathbf{X}_t = \mathbf{A}\mathbf{X}_{t-1} + \mathbf{b} + \mathbf{w}_t,$$

where $\mathbf{A}$ and $\mathbf{b}$ may be computed directly from (2.7), and $(\mathbf{w}_t)_{t \geq 0}$ are random independent Gaussian variables such that $\mathbb{E}_{t-1}[\mathbf{w}_t] = 0$ and $\mathbb{V}_{t-1}[\mathbf{w}_t] = \mathbf{\Sigma}_t \mathbf{\Sigma}_t' \Delta t$.[7]

Measurement equations are given by:

$$\boldsymbol{\eta}_t = c_t(\mathbf{X}_t) + \boldsymbol{\epsilon}_t,$$

where $\boldsymbol{\eta}_t$ denotes the vector of observations, $\mathbf{c}_t$ is a differentiable function, and $(\boldsymbol{\epsilon}_t)_{t \geq 0} \overset{\text{(iid)}}{\sim} \mathrm{N}(0, \mathbf{R}_t)$ denotes measurement errors which are cross-sectionally uncorrelated with constant variance that is different for futures and options. In order to update the prediction of the state vector $\mathbf{X}$, the EKF uses the first order approximation around $\hat{\mathbf{X}}_{t|t-1}$:

$$\boldsymbol{\eta}_t = \underbrace{\partial_{\mathbf{x}} \mathbf{c}_t \left(\hat{\mathbf{X}}_{t|t-1}\right)}_{\mathbf{C}_t} \mathbf{X}_t + \underbrace{c_t \left(\hat{\mathbf{X}}_{t|t-1}\right) - \partial_{\mathbf{x}} \mathbf{c}_t \left(\hat{\mathbf{X}}_{t|t-1}\right) \hat{\mathbf{X}}_{t|t-1}}_{\mathbf{d}_t} + \boldsymbol{\epsilon}_t,$$

[6] Trolle and Schwartz (2009a) study the small-sample properties of estimated parameters using this approach in a multi-factor term-structure model of interest rates under stochastic volatility, and find negligible biases in the estimates.

[7] Since our database consist of daily observations, we approximate the time discretized interval as $\Delta t = 1/252$.

where $\partial_{\mathbf{x}} \mathbf{c}_t (\mathbf{x}_0)$ denotes the jacobian matrix of $\mathbf{c}_t$ evaluated at $\mathbf{x}_0$.

For futures, Proposition 2 shows that log-prices are linear in the state-vector, implying that:

$$\log F_{t,\tau} = \alpha^F(\tau) + \boldsymbol{\beta}^F(\tau)'\mathbf{X}_t + \epsilon_{F,t},$$

where $\epsilon_{F,t}$ denotes the measurement error with variance σ_F^2. In this case no-linearization is required.

For options, we scale prices by $\mathcal{V}$, their Black (1976) vegas,

$$\frac{O_{t,\tau_0,\tau_1}}{\mathcal{V}_{t,\tau_0}} = \frac{\hat{O}_{t,\tau_0,\tau_1}}{\mathcal{V}_{t,\tau_0}} + \epsilon_{O,t},$$

where $\epsilon_{O,t}$ denotes the measurement error with variance σ_O^2. Note that $\epsilon_{O,t}$ approximately represents the measurement error in implied volatilities since the vega corresponds to the derivative of the option premium with respect to its implied volatility, yielding

$$\epsilon_{O,t} = \frac{O_{t,\tau_0,\tau_1} - \hat{O}_{t,\tau_0,\tau_1}}{\mathcal{V}_{t,\tau_0}} \approx \sigma_{t,\tau_0} - \hat{\sigma}_{t,\tau_0}.$$

The discount factor B_{t,τ_0} in Propositions 4 and 5 was obtained by fitting the Nelson and Siegel (1987) curve each trading day to 1W, 1M, 3M, 6M, 9M, and 12M LIBOR rates, and the 2Y LIBOR SWAP rate. We avoid using overnight (O/N) LIBOR rates since they display strong credit-risk effects in the later part of 2008. Hence, our anchor for the interpolation is the 1W LIBOR rate.

The formulas presented in Propositions 4 and 5 allow us to use futures prices directly, generating measurement equations that are driven exclusively by the volatility factors. Therefore, the measurement matrix $\mathbf{C}_t$ is block-diagonal.

2.5. Results

We estimate the model presented in Section 2.3 using the data and estimation procedure of Section 2.4. We analyze several models in which we vary the number of volatility factors while keeping the number of factors driving the cost-of-carry constant. We do this since it is well known in the commodities literature (Cortazar & Naranjo, 2006; E. S. Schwartz, 1997) that models with two factors driving the cost-of-carry usually perform well in pricing the term-structure of futures prices. Hence, it is more interesting to know how many factors are needed to accurately price observed options for different strikes and maturities, knowing that futures are already accurately priced. Therefore, we focus our study in analyzing the models $\mathbb{A}_M^{\mathrm{U}}(3+M)$ as M varies over $\{0,1,2,3,4\}$.

2.5.1. Parameter Estimates and Interpretation of Volatility Factors

Tables E.1 and E.2 report parameter estimates using the entire sample period from 2006 to 2014. Standard errors computed using the outer-product of the log-likelihood gradient are reported in parenthesis.

The standard deviation of the measurement error for futures prices (σ_F) is practically the same across all models (39 bp), confirming that adding volatility factors does not improve the pricing of futures contracts. This behavior is expected since our model exhibits USV, making the pricing of futures and options independent from each other. On the other hand, adding extra volatility factors does improve significantly the pricing of options as can be seen by looking at σ_O. This parameter reveals that the model of Cortazar and Naranjo (2006), which corresponds to our $\mathbb{A}_0^{\mathrm{U}}(3)$ specification, produces pricing errors in implied volatility (IV) of 1 117 bp. Adding one stochastic volatility factor reduces this number to 235 bp, while introducing a second factor yields an error of 136 bp. Using four factors in the variance reduces this number by two, which is highly economically significant since reducing the error in IVs by 50 bp reduces the percentage option pricing error by 500 bp if the Vega is 10.[8] These estimates are consistent with the analysis of the pricing performance of the model that we perform in Section 2.5.3.

The parameters that describe the dynamics of $\mathbf{Y}$, i.e. the mean-reverting parameters κ_1 and κ_2, the volatility parameters σ_1 and σ_2, the instantaneous correlation between the two processes ρ_{12}, and the risk-neutral drift parameters $\theta_1^{\mathbb{Q}}$ and $\theta_2^{\mathbb{Q}}$, present similar estimates across models. This is expected since we keep the number of factors in $\mathbf{Y}$ fixed across models, and they do not affect the measurement equations of option contracts. Since $\kappa_1 > \kappa_2$ and $\sigma_1 > \sigma_2$, we interpret x_1 as the factor capturing short-term shocks in the cost-of-carry, while x_2 captures more persistent ones. Finally, the correlation among these two factors is negative and ranges between -0.50 and -0.30.

The addition of the first volatility factor decreases sharply the value of σ_3. Indeed, whereas in the model of Cortazar and Naranjo (2006) this parameter measures the average spot price volatility, in our model it represents a lower bound for the stochastic spot price volatility. As a consequence, the correlation parameters ρ_{12}, ρ_{13} and ρ_{23} also change when adding stochastic volatility factors.

The estimates of the parameters that describe the dynamics of $\mathbf{V}$ suggest the following interpretation for the volatility factors under $\mathbb{Q}$. Given its high mean-reversion coefficient ($\kappa_4^{\mathbb{Q}}$) and its high volatility (ς_1), we interpret the first volatility factor x_4 as the one capturing short-term shocks to spot volatility. With the exception of the $\mathbb{A}_2^{\mathrm{U}}(5)$ model, this factor displays negative correlation (ϱ_1) with the spot price, allowing spot returns to exhibit negative skewness.

The second volatility factor x_5 presents the lowest mean-reversion ($\kappa_5^{\mathbb{Q}}$) and volatility (ς_2), leading us to interpret it as the one capturing long-term shocks. Moreover, since κ_{54} is statistically non-significant and κ_{45} has a similar order of magnitude than $\kappa_4^{\mathbb{Q}}$ (except for the $\mathbb{A}_4^{\mathrm{U}}(7)$ model, see below), we may interpret x_5 as a long-term mean for x_4 which

[8] In our sample, Vega varies from 0.7 to 52.6 with an average value of 16.2.

is consistent with the volatility process suggested by Duffie et al. (2000), and the findings presented in Trolle and Schwartz (2009b).

The third volatility factor x_6 displays intermediate values of mean-reversion ($\kappa_6^{\mathbb{Q}}$) and volatility (ς_3) compared with those presented before. It is important to observe that it adds flexibility to the model since it presents positive correlation with the spot (ϱ_3). In the $\mathbb{A}_2^{\mathrm{U}}(5)$ model, due to the absence of additional volatility factors, this task is performed by the first factor. Adding a third volatility factor allows x_1 to maintain its negative correlation with the spot price. This feature is important in order to achieve a good fit to option prices in periods where the overall skewness is positive.[9]

Lastly, the fourth volatility factor x_7 captures medium-term perturbations since it presents a mean-reversion ($\kappa_7^{\mathbb{Q}}$) and volatility (ς_4) coefficients in between the ones obtained for second and first factors, i.e. $\kappa_5^{\mathbb{Q}} < \kappa_7^{\mathbb{Q}} < \kappa_4^{\mathbb{Q}}$ and $\varsigma_2 < \varsigma_4 < \varsigma_1$. Also, by analyzing the values for κ_{47} and κ_{57}, we conclude that the addition of the fourth factor modifies the behavior of x_5 as it becomes the long-term mean for x_7, abandoning its role as long-term mean for x_4 which is now performed by the extra factor.

The risk-premia parameters are in general difficult to estimate. The drift parameters for the cost-of-carry under the risk-neutral measure ($\theta_1^{\mathbb{Q}}$ and $\theta_2^{\mathbb{Q}}$) and the mean reverting parameters of the volatility factors under the physical measure ($\kappa_4, \ldots, \kappa_7$), present comparatively higher estimation errors than other parameters. Nevertheless, the completely affine risk-premia specification yields statistically significant estimates, which is consistent with Pan (2002) and Broadie, Chernov, and Johannes (2007).

2.5.2. Volatility Comparison

We would like to highlight that the instantaneous spot volatility filtered by the EKF agrees with other standard methods used in practice to estimate stochastic volatility.

[9]See section 2.5.4 for more details.

Figure D.3 plots the volatility of oil spot returns filtered using the EKF methodology, and compares it with the volatility estimated using a GARCH(1,1) model with Gaussian errors using the simple residues of the spot price extracted from the $\mathbb{A}_4^{\mathrm{U}}(7)$ model. The figure shows that our estimate of the spot returns volatility follows closely the volatility estimate obtained from the GARCH model. The correlation between the two time-series is 0.9458, which confirms the results observed from the graph. Moreover, the correlations for the $\mathbb{A}_1^{\mathrm{U}}(4)$, $\mathbb{A}_2^{\mathrm{U}}(5)$ and $\mathbb{A}_3^{\mathrm{U}}(6)$ models are also high and equal to 0.9090, 0.9533 and 0.9584, respectively.

2.5.3. Pricing Performance

We compute theoretical futures and option prices from the filtered state variables of the $\mathbb{A}_M^{\mathrm{U}}(3+M)$ model for each $M \in \{0,1,2,3,4\}$, and compare them with observed transactions.

Table E.3 reports the percentage root mean square errors (RMSEs) for futures contracts computed as the differences in logarithms of the fitted and actual futures prices. Consistent with previous studies (e.g. Casassus & Collin-Dufresne, 2005; Cortazar & Naranjo, 2006; E. S. Schwartz, 1997), the term-structure of futures prices is well explained by the first three factors of our model: the first two driving the dynamics of the cost-of-carry and the third representing the spot price. The average pricing errors for different futures contracts are relatively low, ranging from 19 to 66 bp, yielding an overall average error of 35 bp. Consistent with the USV nature of our model, the number of factors driving the variance has negligible influence in the pricing of futures contracts.

The top panel in Figure D.4 plots the time series of daily percentage RMSEs for futures prices and for all models. As can be observed from the figure, the pricing performance of all models is stable through our sample period except between the end of 2008 and the beginning of 2009, just after the collapse of Lehman Brothers. As can be observed from Figure D.1, during the crisis period of 2008-2009 oil prices drop significantly, and the term-structure of futures contracts displays a pronounced contango shape. A similar phenomenon seems to be occurring at the end of 2014.

Table E.4 reports the RMSEs for options computed as the differences between fitted and actual Black (1976) IVs (in percentages). In this table, options are grouped by maturity. As expected (see e.g. Cortazar et al., 2015), the $\mathbb{A}_0^{\mathrm{U}}(3)$ model performs very poorly due to its constant volatility specification. Also, it is apparent that introducing additional volatility factors reduces monotonically the RMSE for all maturities. The table reveals that at least three volatility factors are needed to achieve an overall RMSE below 100 bp. Also, the table shows that $\mathbb{A}_4^{\mathrm{U}}(7)$ model achieves a pricing performance that is quite homogeneous across the term, except for the shortest maturity contracts.

The bottom panel in Figure D.4 plots the time-series of the overall option RMSE for all models. We omit in the graph the RMSE plot for the $\mathbb{A}_0^{\mathrm{U}}(3)$ model since it is an order of magnitude larger than for the other models. As for futures contracts, the $\mathbb{A}_1^{\mathrm{U}}(4)$ model produces larger errors during the 2008–2009 crisis period. Introducing additional volatility factors substantially reduces the pricing error during this period. For the $\mathbb{A}_2^{\mathrm{U}}(5)$, $\mathbb{A}_3^{\mathrm{U}}(6)$, and $\mathbb{A}_4^{\mathrm{U}}(7)$, however, the larger pricing errors are observed during the year 2011.

Tables E.5 and E.6 report RMSEs for options by maturity and moneyness for all models. Remember that we only use OTM options, implying that contracts with moneyness below one represent puts, and options with moneyness greater than one describe calls. For short-term options, the $\mathbb{A}_0^{\mathrm{U}}(3)$ model produces higher errors when pricing OTM put options, whereas the $\mathbb{A}_4^{\mathrm{U}}(7)$ struggles more with OTM call options. For all models, ATM contracts are priced more accurately than OTM options. For long-term options, all models perform similarly across different maturities.

Figure D.5 plots average IVs by moneyness, for different maturities and for all models, and compares them with the average observed IV. First, it is interesting to note that in our sample, short-term options display a characteristic smile, whereas medium- and long-term contracts exhibit a pronounced skew. Note however that the graphs have different scales, so the magnitude of the smile for short-term options is more significant than the skew for medium- and long-term contracts. Second, it is also apparent than adding stochastic volatility factors significantly improves the fit of all models compared to the constant volatility one. Finally, the figure also confirms than using three or four volatility factors significantly improves the pricing of short-term options.

2.5.4. Risk-Neutral Skewness

A different approach to analyze the pricing performance of the model is to look at how well it can replicate the skew of the risk-neutral distribution. In the literature, this quantity is usually proxied by the implicit volatility skewness (IVSkew), which is a measure of the difference between the tails of the implied volatility smile. Bakshi, Kapadia, and Madan (2003) conclude that IVSkew measures are good proxies of the risk-neutral skewness.

The literature proposes several measures of IVSkew. Mixon (2011) surveys many of them, and concludes that the normalized 25Δ risk-reversal on 3-month options is a good proxy for the risk-neutral skewness:

$$\text{IVSkew} = \frac{25\Delta \text{ call IV} - 25\Delta \text{ put IV}}{50\Delta \text{ IV}}. \tag{2.20}$$

Due to our particular sorting of data, we implement (2.20) using the IV on M3 contracts as

$$\text{IVSkew} = \frac{1.2 \text{ moneyness call IV} - 0.8 \text{ moneyness put IV}}{\text{ATM IV}}.[10] \tag{2.21}$$

Note that the sign of our measure yields an intuitively positive correlation with the skewness of returns, as mentioned in Bali, Hu, and Murray (2014). A negative IVSkew means that the probability density is skewed to the left, whereas a positive value represents skewness to the right.

Figure D.6 displays the IVSkew of traded contracts, and compares it with the IVSkew generated by the $\mathbb{A}_4^{\text{U}}(7)$ model. First, the figure documents that the IVSkew of oil returns

[10] When a moneyness interval contains two observations, we compute the representative implicit volatility as the average of both.

is in general negative, although it becomes positive in 2006 and 2011. Second, the IVSkew measure varies significantly during our sample period, going from -0.5 to 0.2. Third, the figure confirms the $\mathbb{A}_4^{\mathrm{U}}(7)$ model is able to accurately replicate the dynamics of the risk-neutral skew during our sample period.

2.5.5. Robustness

We check the robustness of our model by checking whether parameters estimated in different samples can price futures and option contracts in- and out-of-sample. For this exercise, we define three panels: Panel A corresponds to the entire sample period from January 3rd, 2006 to December 31st, 2014; Panel B goes from January 3rd, 2006 to December 31st, 2010; Panel C comprises the period from January 3rd, 2011 to December 31st, 2014.

Table E.7 reports RMSEs for futures contracts computed on each panel and for all models, using parameters that were calibrated using each of the different panels. For example, Calibration B - Panel C displays the errors generated by each model calibrated using the data on Panel B, when pricing all contracts included in Panel C. In this example, Panel B contains the in-sample data while Panel C represents out-of-sample data. As expected, RMSEs in the example are larger than the ones obtained in Calibration C - Panel C (in-sample), but are nevertheless small. Similar patterns are observed for other cases, showing that all models are reasonably stable when pricing futures contracts.

Table E.8 is similar to Table E.7 but reports RMSEs for option contracts. The table shows that all models are also robust when pricing options. In particular, the $\mathbb{A}_4^{\mathrm{U}}(7)$ model achieves in- and out-of-sample RMSEs ranging from 60 to 111 bp.

2.6. Conclusions

This article proposes a general affine diffusion model for commodity prices in the spirit of Dai and Singleton (2000). Our model develops a multifactor specification for the cost-of-carry and the instantaneous volatility of the spot price, and nests many existing models commonly found in literature. The model also exhibits USV, yielding closed-form formulas for futures prices, and quasi-analytical expressions for option prices.

We implement our model using WTI futures and options contracts from January 3rd, 2006 to December 31st, 2014. The model is estimated using quasi-maximum likelihood and the Extended Kalman Filter.

Our results suggest that the multifactor structure of the model is crucial in pricing accurately futures and options contracts alike. Its USV nature and the way we derive our option pricing formulas guarantee that only the cost-of-carry factors and the spot price are used to fit futures prices, while the volatility factors only affect the pricing of options contracts. We conclude from our analysis that at least two cost-of-carry ($N \geq 3$) and three volatility factors ($M \geq 3$) are required to obtain accurate futures and options valuations. Adding a fourth volatility factor improves the pricing of options in periods of market stress.

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APPENDIX

A. A MORE GENERAL CASE OF USV

Under the risk-neutral measure $\mathbb{Q}$, consider the affine diffusion model given by

$$d\mathbf{X}_t = \left(\mathbf{\Theta}^{\mathbb{Q}} - \mathbf{K}^{\mathbb{Q}}\mathbf{X}_t\right) dt + \mathbf{\Sigma}_t d\mathbf{W}_t^{\mathbb{Q}} \tag{A.1}$$

where the state vector $\mathbf{X}_t$ belongs to $\mathbb{R}^3 \times \mathbb{R}_+^1$ and

$$\bar{\mathbf{\Theta}}^{\mathbb{Q}} = \begin{pmatrix} \theta_1^{\mathbb{Q}} \\ \theta_2^{\mathbb{Q}} \\ \theta_3^{\mathbb{Q}} \\ 1 \end{pmatrix} \quad , \quad \bar{\mathbf{K}}^{\mathbb{Q}} = \begin{pmatrix} \kappa_1^{\mathbb{Q}} & 0 & 0 & \kappa_{14}^{\mathbb{Q}} \\ 0 & \kappa_2^{\mathbb{Q}} & 0 & \kappa_{24}^{\mathbb{Q}} \\ 1 & 1 & 0 & \frac{1}{2}\gamma_1^2 \\ 0 & 0 & 0 & \kappa_4^{\mathbb{Q}} \end{pmatrix},$$

while the covariance matrix is given by

$$\bar{\mathbf{\Sigma}}_t\bar{\mathbf{\Sigma}}_t' = \begin{pmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \rho_{13}\sigma_1\sigma_3 & 0 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & \rho_{23}\sigma_2\sigma_3 & 0 \\ \rho_{13}\sigma_1\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \sigma_3^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} *_1 & 0 & 0 & *_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_1^2 & \varrho_1\gamma_1\varsigma_1 \\ *_{14} & 0 & \varrho_1\gamma_1\varsigma_1 & \varsigma_1^2 \end{pmatrix} x_{4,t}.$$

In this case, the components of $\boldsymbol{\beta}^F$ satisfies the ODEs

$$\begin{aligned}
\dot{\beta}_1^F &= -\kappa_1^{\mathbb{Q}}\beta_1^F - \beta_3^F \\
\dot{\beta}_2^F &= -\kappa_2^{\mathbb{Q}}\beta_2^F - \beta_3^F \\
\dot{\beta}_3^F &= 0 \\
\dot{\beta}_4^F &= -\kappa_{14}^{\mathbb{Q}}\beta_1^F - \kappa_{24}^{\mathbb{Q}}\beta_2^F - \frac{1}{2}\gamma_1^2\beta_3^F - \kappa_4^{\mathbb{Q}}\beta_4^F \\
&\quad + \frac{1}{2}\left(*_1{\beta_1^F}^2 + 2 *_{14} \beta_1^F\beta_4^F + \gamma_1^2{\beta_3^F}^2 + 2\varrho_1\gamma_1\varsigma_1\beta_3\beta_4 + \varsigma_1^2{\beta_4^F}^2\right)
\end{aligned}$$

with the initial condition $\boldsymbol{\beta}^F(0) = \mathbf{e}_3$. The solution for the first three coordinates is given by

$$\beta_1^F = \frac{1}{\kappa_1^{\mathbb{Q}}}\left(e^{-\kappa_1^{\mathbb{Q}}\tau} - 1\right)$$

$$\beta_2^F = \frac{1}{\kappa_2^{\mathbb{Q}}}\left(e^{-\kappa_2^{\mathbb{Q}}\tau} - 1\right)$$
$$\beta_3^F = 1$$

We replace the solutions on the fourth ODE and observe that, due to the uniqueness and existence theorem for ODEs, the model (A.1) exhibits USV if and only if

$$0 = -\frac{\kappa_{14}^{\mathbb{Q}}}{\kappa_1^{\mathbb{Q}}}\left(e^{-\kappa_1^{\mathbb{Q}}\tau} - 1\right) - \frac{\kappa_{24}^{\mathbb{Q}}}{\kappa_2^{\mathbb{Q}}}\left(e^{-\kappa_2^{\mathbb{Q}}\tau} - 1\right) + \frac{*_1}{2\kappa_1^{\mathbb{Q}2}}\left(e^{-2\kappa_1^{\mathbb{Q}}\tau} - 2e^{-\kappa_1^{\mathbb{Q}}\tau} + 1\right). \qquad \text{(A.2)}$$

If we assume

$$\kappa_2^{\mathbb{Q}} = 2\kappa_1^{\mathbb{Q}}$$

then (A.2) may be written as

$$0 = \left(\frac{*_1}{2\kappa_1^{\mathbb{Q}2}} - \frac{\kappa_{24}^{\mathbb{Q}}}{2\kappa_1^{\mathbb{Q}}}\right)e^{-2\kappa_1^{\mathbb{Q}}\tau} - \left(\frac{*_1}{\kappa_1^{\mathbb{Q}2}} + \frac{\kappa_{14}^{\mathbb{Q}}}{\kappa_1^{\mathbb{Q}}}\right) + \left(\frac{*_1}{2\kappa_1^{\mathbb{Q}2}} + \frac{\kappa_{14}^{\mathbb{Q}}}{\kappa_1^{\mathbb{Q}}} + \frac{\kappa_{24}^{\mathbb{Q}}}{2\kappa_1^{\mathbb{Q}}}\right)$$

So (A.2) is equivalent to impose the following restrictions

$$\frac{*_1}{\kappa_1^{\mathbb{Q}}} - \kappa_{24}^{\mathbb{Q}} = 0$$
$$\frac{*_1}{\kappa_1^{\mathbb{Q}}} + \kappa_{14}^{\mathbb{Q}} = 0$$
$$\frac{*_1}{2\kappa_1^{\mathbb{Q}}} + \kappa_{14}^{\mathbb{Q}} + \frac{\kappa_{24}^{\mathbb{Q}}}{2} = 0$$

which results in

$$\kappa_{14}^{\mathbb{Q}} = -\kappa_{24}^{\mathbb{Q}} = -\frac{*_1}{\kappa_1^{\mathbb{Q}}}$$

making the last restriction redundant.

We note that in this specification, the first of the two cost-of-carry factors can exhibit stochastic volatility by simply restricting the parameters $\kappa_{24}^{\mathbb{Q}}$ and $\kappa_{14}^{\mathbb{Q}}$ as we just showed (supposing that we let $\kappa_1^{\mathbb{Q}}$ and $*_1$ free). Hence, by allowing similar restrictions on the mean-reverting coefficients, each new volatility factor that we add to this model allows us to add a new pair of cost-of-carry factors where one of them exhibits stochastic volatility and such that the model admits USV.

B. PROOFS

B.1. Proof of Proposition 1

In this proof we follow the Proposition 1 in Duffie et al. (2000). Assume that Ψ is of the form (2.13). By the law of iterated expectation, Ψ is a $\mathbb{Q}$-martingale and, from a direct application of the Itō's Lemma, we conclude that α and $\boldsymbol{\beta}$ must satisfy the complex-valued ODEs

$$\dot{\alpha} = \boldsymbol{\Theta}^{\mathbb{Q}\prime}\boldsymbol{\beta} + \frac{1}{2}\boldsymbol{\beta}'\mathbf{H}_0\boldsymbol{\beta}$$
$$\dot{\boldsymbol{\beta}} = -\mathbf{K}^{\mathbb{Q}\prime}\boldsymbol{\beta} + \sum_{m=1}^{M}\frac{1}{2}\boldsymbol{\beta}'\mathbf{H}_m\boldsymbol{\beta}\mathbf{e}_{N+m}$$

with initial conditions $\alpha(0;\boldsymbol{\phi}) = 0$ and $\boldsymbol{\beta}(0;\boldsymbol{\phi}) = \boldsymbol{\phi}$ and where $\mathbf{e}_{N+m}$ denotes a basis vector in $\mathbb{R}^{N+M}$.

B.2. Proof of Proposition 2

Since $F_{t,\tau} = \mathbb{E}_t^{\mathbb{Q}}[S_T] = \Psi_{t,\tau}(\mathbf{e}_N)$, the result follows from Proposition 1 if we show that $\alpha^F(\tau) = \alpha(\tau;\mathbf{e}_N)$ and $\boldsymbol{\beta}^F(\tau) = \boldsymbol{\beta}(\tau;\mathbf{e}_N)$. For $n \in \{1,\ldots,N\}$, the last statement is a consequence of the closed-form solution given in (2.16). Also, from the ODEs in (2.14), it is easy to see that $\beta_{N+m}(\tau) = 0$ for each $m \in \{1,\ldots,M\}$ is a unique solution to the system. Replacing these results in the ODE for α and integrating finishes the proof.

B.3. Proof of Proposition 3

From Proposition 2, notice that

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{u\log F_{T_0,\tau_1}}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[e^{u\left(\alpha^F(\tau_1)+\boldsymbol{\beta}^F(\tau_1)'\mathbf{X}_{T_0}\right)}\right]$$

$$= e^{u\alpha^F(\tau_1)}\mathbb{E}_t^{\mathbb{Q}}\left[e^{u\boldsymbol{\beta}^F(\tau_1)'\mathbf{X}_{T_0}}\right]$$
$$= e^{u\alpha^F(\tau_1)+\alpha\left(\tau_0;u\boldsymbol{\beta}^F(\tau_1)\right)+\boldsymbol{\beta}\left(\tau_0;u\boldsymbol{\beta}^F(\tau_1)\right)'\mathbf{X}_t},$$

where Proposition 1 has been used in the last step.

Using the closed-form expression in (2.16) and Proposition 2, for $n < N$ we have that

$$\begin{aligned}
\beta_n\left(\tau_0; u\boldsymbol{\beta}^F(\tau_1)\right) &= e^{-\kappa_n^{\mathbb{Q}}\tau_0} u\beta_n^F(\tau_1) + \left(\frac{e^{-\kappa_n^{\mathbb{Q}}\tau_0}-1}{\kappa_n^{\mathbb{Q}}}\right) u\beta_N^F(\tau_1) \\
&= u\left[e^{-\kappa_n^{\mathbb{Q}}\tau_0}\left(\frac{e^{-\kappa_n^{\mathbb{Q}}\tau_1}-1}{\kappa_n^{\mathbb{Q}}}\right) + \left(\frac{e^{-\kappa_n^{\mathbb{Q}}\tau_0}-1}{\kappa_n^{\mathbb{Q}}}\right)\right] \\
&= u\left(\frac{e^{-\kappa_n^{\mathbb{Q}}(\tau_0+\tau_1)}-1}{\kappa_n^{\mathbb{Q}}}\right) \\
&= u\beta_n^F\left(\tau_0+\tau_1\right).
\end{aligned}$$

The result is finally obtained by replacing the expression for $\log F_{t,\tau_0+\tau_1}$ obtained in Proposition 2.

B.4. Proof of Proposition 4

We follow Duffie et al. (2000). Assuming that the risk-free interest rate and the futures prices are uncorrelated under the $\mathbb{Q}$ measure, the price at time t of an European put option expiring at time $T_0 = t+\tau_0$ with strike K on a futures contract expiring at time $T_1 = T_0+\tau_1$ is given by

$$\begin{aligned}
P_{t,\tau_0,\tau_1}(K) &= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_0} r_s ds}\left(K - F_{T_0,\tau_1}\right)1_{\left\{F_{T_0,\tau_1}<K\right\}}\right] \\
&= B_{t,\tau_0}\left(K\mathbb{E}_t^{\mathbb{Q}}\left[1_{\left\{\log F_{T_0,\tau_1}<\log K\right\}}\right] - \mathbb{E}_t^{\mathbb{Q}}\left[e^{\log F_{T_0,\tau_1}}1_{\left\{\log F_{T_0,\tau_1}<\log K\right\}}\right]\right) \\
&= B_{t,\tau_0}\left(KG_{0,1}\left(\tau_0,\tau_1;\log K\right) - G_{1,1}\left(\tau_0,\tau_1;\log K\right)\right)
\end{aligned}$$

where, as shown in Proposition 2 of Duffie et al. (2000),

$$G_{a,b}(\tau_0,\tau_1;k) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{a\log F_{T_0,\tau_1}} 1_{\left\{b\log F_{T_0,\tau_1}<y\right\}}\right]$$
$$= \frac{\Psi^F_{t,\tau_0,\tau_1}(a)}{2} - \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\left[\Psi^F_{t,\tau_0,\tau_1}(a+\mathtt{i}ub)e^{-\mathtt{i}uk}\right]}{u}du,$$

with $\mathrm{Im}[c]$ denoting the imaginary part of $c \in \mathbb{C}$ and $\mathtt{i} = \sqrt{-1}$ corresponding to the imaginary unit.

B.5. Proof of Proposition 5

In an analogous way to Appendix B.4, the price at time t of an European call option expiring at time $T_0 = t+\tau_0$ with strike K on a futures contract expiring at time $T_1 = T_0+\tau_1$ is given by

$$\begin{aligned} C_{t,\tau_0,\tau_1}(K) =& \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_0} r_s ds}\left(F_{T_0,\tau_1} - K\right) 1_{\left\{F_{T_0,\tau_1}>K\right\}}\right] \\ =& B_{t,\tau_0}\left(\mathbb{E}_t^{\mathbb{Q}}\left[e^{\log F_{T_0,\tau_1}} 1_{\left\{\log F_{T_0,\tau_1}>\log K\right\}}\right] - K\mathbb{E}_t^{\mathbb{Q}}\left[1_{\left\{\log F_{T_0,\tau_1}>\log K\right\}}\right]\right) \\ =& B_{t,\tau_0}\left(G_{1,-1}\left(\tau_0,\tau_1;-\log K\right) - KG_{0,-1}\left(\tau_0,\tau_1;-\log K\right)\right) \end{aligned}$$

where $G_{a,b}(\tau_0,\tau_1;y)$ is defined as in Appendix B.4.

C. NUMERICAL ISSUES

The system of ODEs presented in Proposition 1 is solved using the classical Runge-Kutta fourth order method[11] and, following Trolle and Schwartz (2009b), the Fourier inversion integrals in Proposition 4 and Proposition 5 are evaluated using the Gauss-Legendre quadrature formula with 30 points: 15 for the [0,50] interval and 15 for the [50,400] interval.

[11] Specifically, we used the MATLAB `ode45` routine.

In order to speed our algorithm, we first solve numerically α and $\boldsymbol{\beta}$ for different initial conditions, and only then compute the EKF iteration. Using an Intel Core I5-2500 processor with 8 Gb of RAM, a log-likelihood evaluation for the complete sample takes between 3.5 and 6.5 seconds, depending on the number of volatility factors.

The maximization of the log-likelihood function is performed by switching between Nelder-Mead and semi-quadratic programming algorithms.[12]

The gradients involved in the computation of the estimation errors were calculated using an adaptive Jacobian routine based on finite-differences and Romberg extrapolation.[13]

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[12] These algorithms are implemented in MATLAB's Optimization Toolbox inside the `fminsearch` and `fmincon` routines, respectively.

[13] Specifically, we used the `jacobianest` routine which was published in MATLAB Central under Adaptive Robust Numerical Differentiation by John D'Errico (`woodchips@rochester.rr.com`).

D. FIGURES

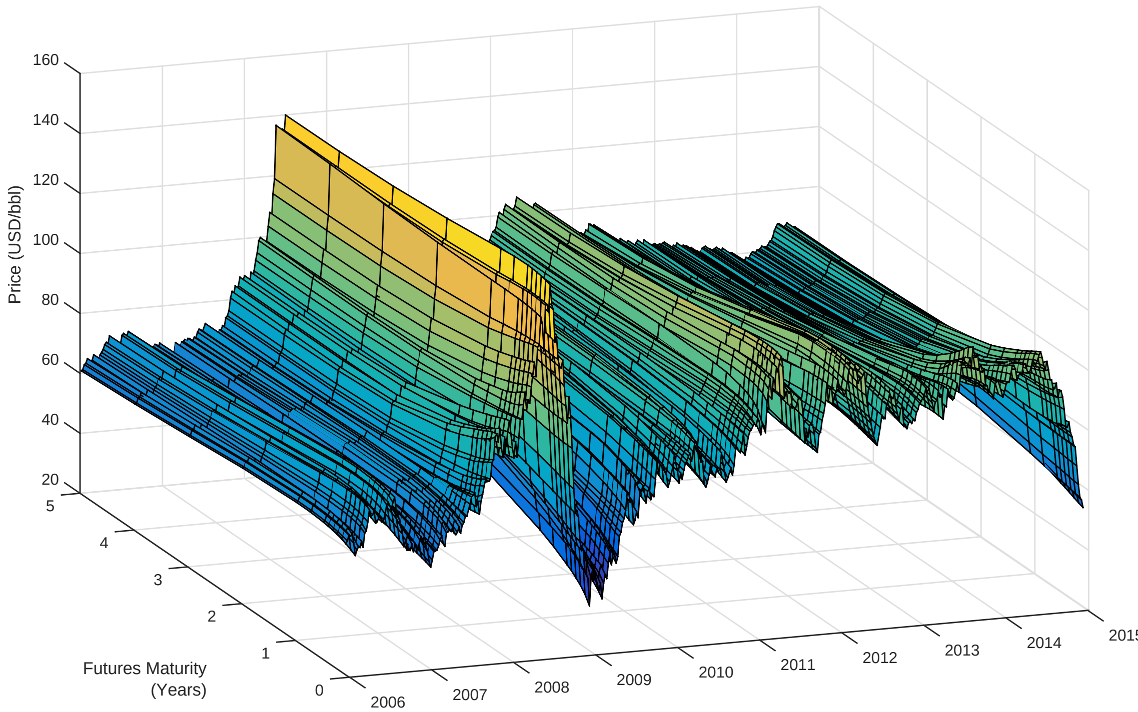

Figure D.1. WTI futures term-structures spanned by the M1, M2, M3, M4, M5, M6, Q1, Q2, Y1, Y2, Y3, and Y4 contracts from January 3rd, 2006 to December 31st, 2014. To avoid cluttering the figure, we display only the data on Wednesdays.

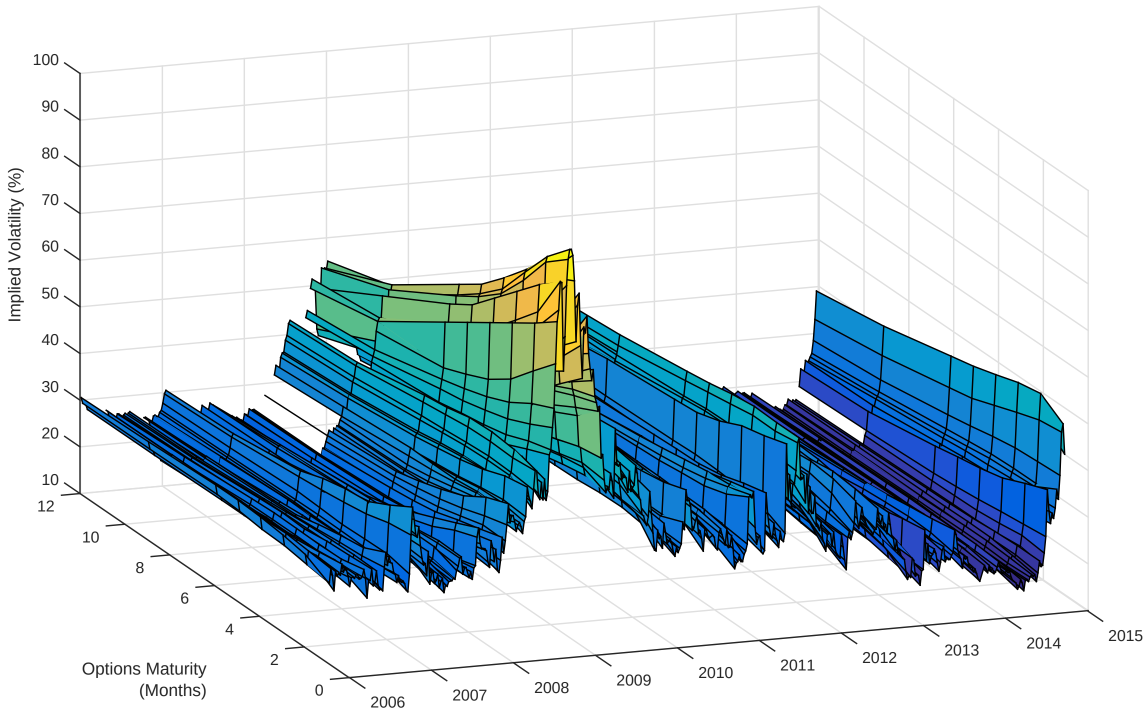

Figure D.2. WTI implied volatility term-structures of at-the-money options written on the M1, M2, M3, M4, M5, M6, Q1, and Q2 futures contracts from January 3rd, 2006 to December 31st, 2014. To avoid cluttering the figure, we display only the data on Wednesdays.

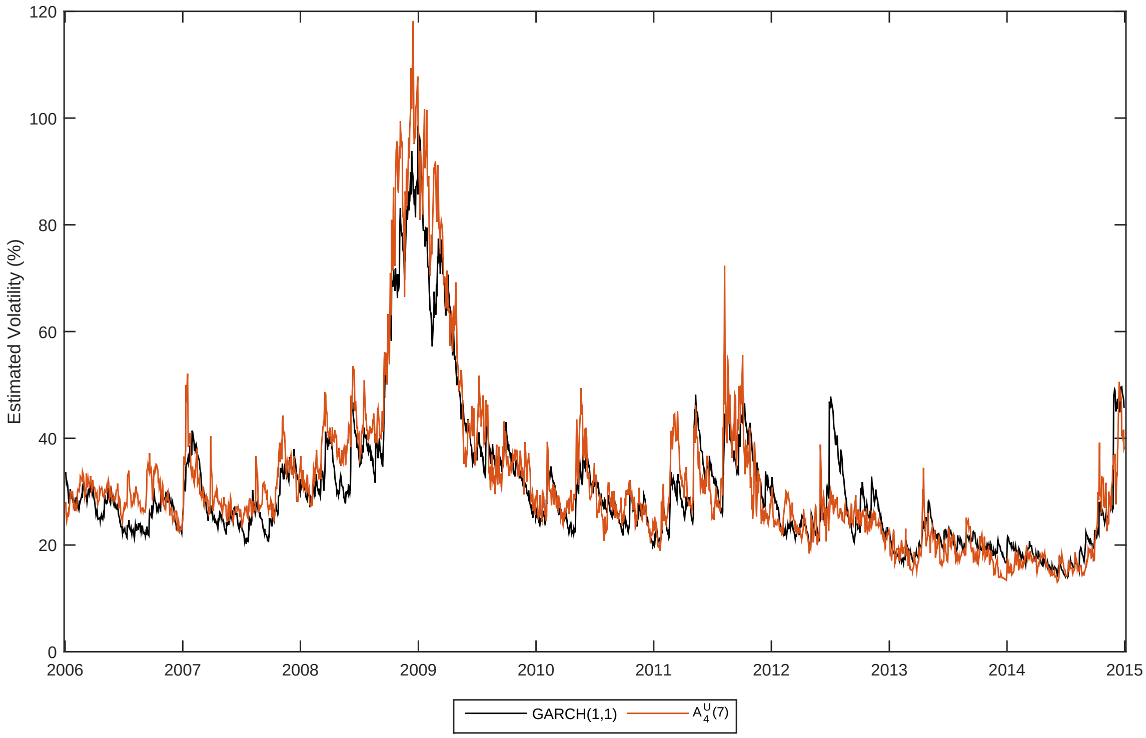

Figure D.3. Time-series of the daily instantaneous WTI spot volatility filtered from January 3rd, 2006 to December 31st, 2014 using the $\mathbb{A}_4^{\mathrm{U}}(7)$ model. The GARCH(1,1) model is estimated using the daily residuals of the filtered spot price.

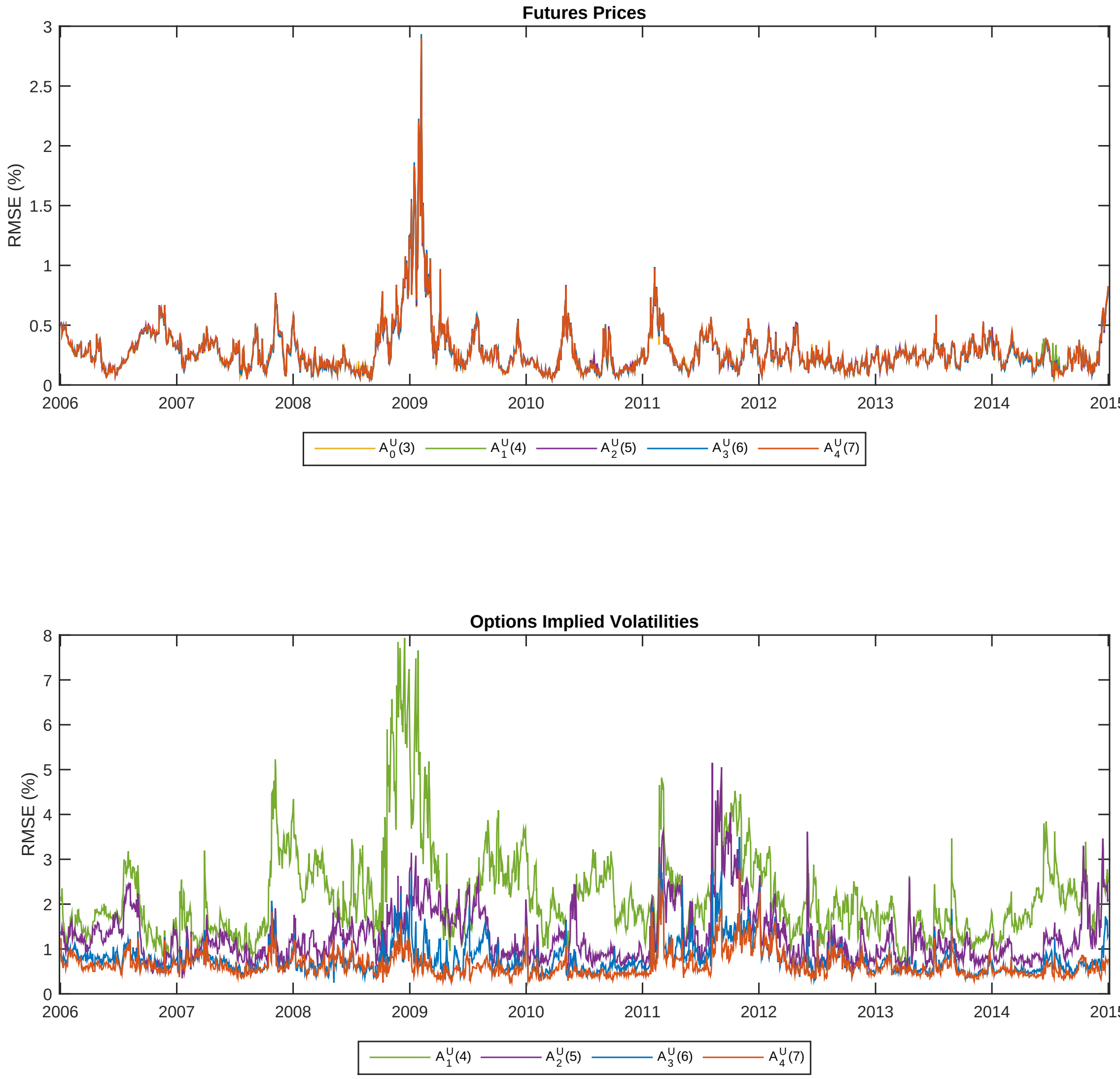

Figure D.4. Time-series of daily root mean squared errors (RMSEs) obtained by different models on WTI futures and options from January 3rd, 2006 to December 31st, 2014. The top panel presents the RMSE on futures defined as the difference between the logarithms of fitted and observed futures prices. The bottom panel shows the RMSE on implied option's volatility defined as the difference between fitted and observed implied volatilities. We avoid reporting the RMSE of the $\mathbb{A}_0^{\mathrm{U}}(3)$ model since it is an order of magnitude larger than the other stochastic volatility models.

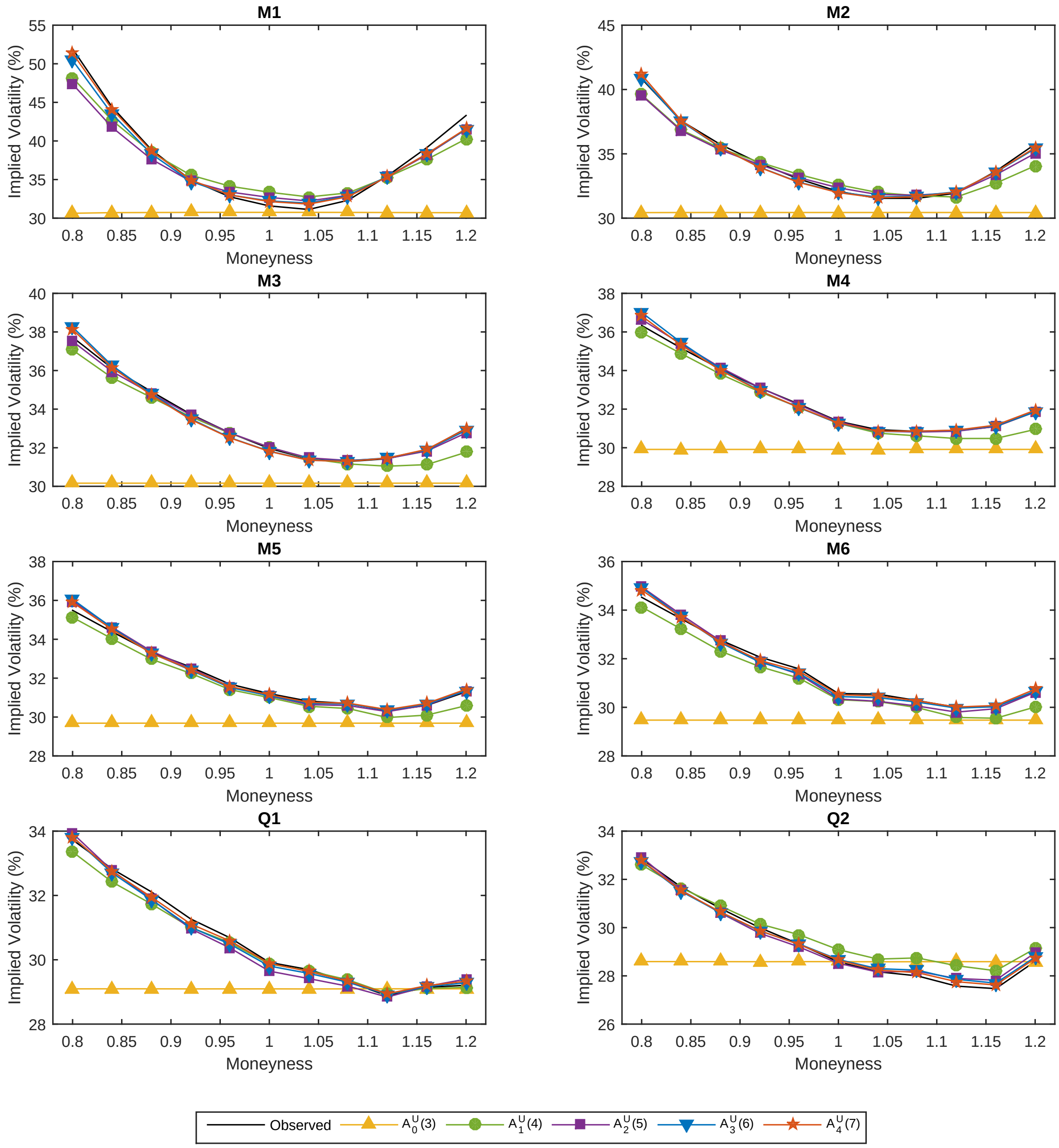

Figure D.5. Average fit to WTI implied volatility smiles across different moneyness for options written on M1, M2, M3, M4, M5, M6, Q1 and Q2 contracts from January 3rd, 2006 to December 31st, 2014. Moneyness is defined as the option strike divided by the price of the underlying futures contract.

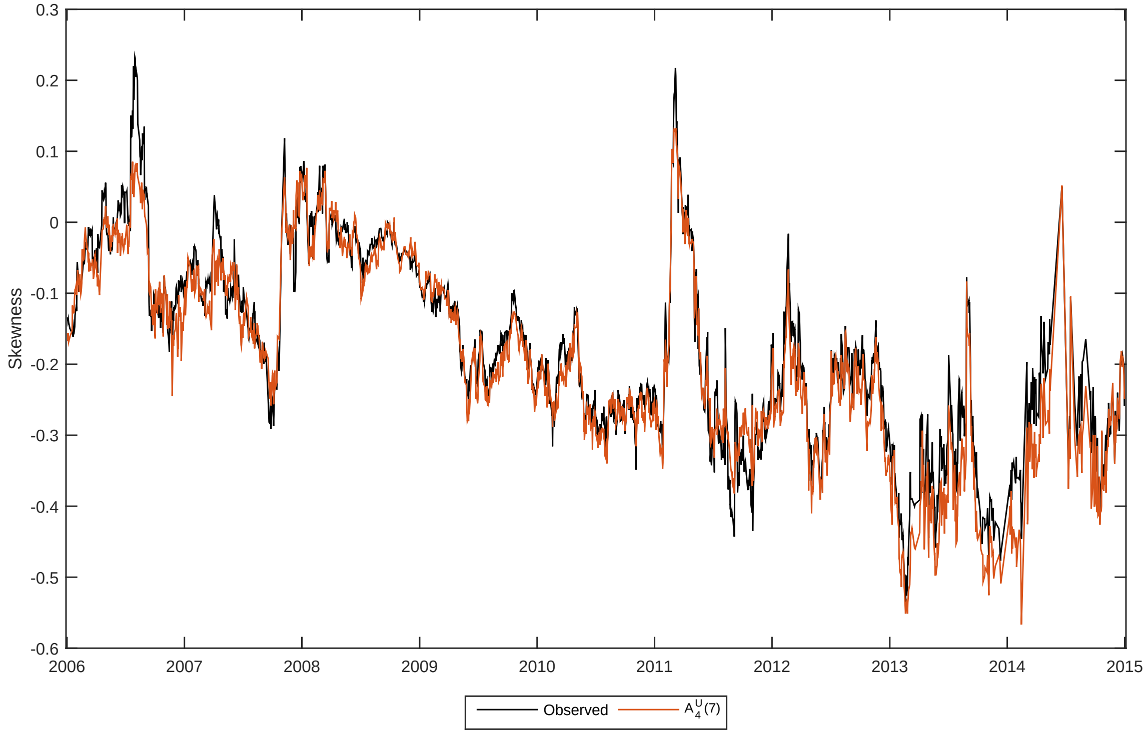

Figure D.6. Implicit volatility skewness (IVSkew) for WTI options from January 3rd, 2006 to December 31st, 2014, where IVSkew $= (1.2$ moneyness call IV $- 0.8$ moneyness put IV$)/$ATM IV is computed using options written on the M3 contract.

E. TABLES

Table E.1. Maximum-likelihood estimates for $\mathbb{A}_M^{\mathrm{U}}(3+M)$ models for different M. Outer-product standard errors are reported in parentheses.

	$\mathbb{A}_0^{\mathrm{U}}(3)$		$\mathbb{A}_1^{\mathrm{U}}(4)$		$\mathbb{A}_2^{\mathrm{U}}(5)$		$\mathbb{A}_3^{\mathrm{U}}(6)$		$\mathbb{A}_4^{\mathrm{U}}(7)$	
θ_3	0.0311	(0.0691)	0.1097	(0.0660)	-0.0961	(0.0821)	-0.1665	(0.0823)	-0.0347	(0.1063)
ϑ_1			0.1067	(0.2110)	-0.0638	(0.0484)	-0.0117	(0.0356)	-0.0309	(0.0227)
ϑ_2					0.0081	(0.0390)	0.0469	(0.0574)	0.0591	(0.0168)
ϑ_3							-0.0140	(0.0080)	-0.0010	(0.0061)
ϑ_4									0.0233	(0.0133)
κ_1	1.7297	(0.0047)	1.7343	(0.0051)	1.7809	(0.0051)	1.7148	(0.0053)	1.7620	(0.0051)
κ_2	0.3753	(0.0011)	0.3783	(0.0012)	0.3792	(0.0012)	0.3802	(0.0013)	0.3698	(0.0013)
κ_4			0.7810	(0.4695)	1.0817	(0.5411)	8.6556	(0.1720)	8.5772	(0.6156)
κ_5					1.3722	(0.5826)	0.8388	(0.1787)	0.7774	(0.2913)
κ_6							2.6045	(0.3333)	0.2658	(0.1292)
κ_7									3.0888	(0.2755)
κ_{45}					-8.4807	(0.4290)	-13.9706	(0.8442)	-0.0002	(0.0765)
κ_{46}					-0.0004	(0.0024)	-0.0003	(0.0037)	-0.0002	(0.0242)
κ_{56}							-0.0075	(0.0011)	0.0000	(0.0001)
κ_{47}							0.0000	(0.0034)	-15.3240	(11.5165)
κ_{57}							-0.0060	(0.0896)	-0.0051	(0.0165)
κ_{67}							-0.1147	(0.1756)	-0.0059	(0.0832)
κ_{54}									-0.0021	(0.0149)
κ_{64}									0.0000	(0.1154)
κ_{74}									-1.2676	(0.9543)
κ_{65}									-3.9571	(1.1150)
κ_{75}									-2.6596	(1.0285)
κ_{76}									-0.0033	(0.0053)

Table E.2. This table is the continuation of Table E.1.

	$\mathbb{A}_0^{\mathrm{U}}\,(3)$		$\mathbb{A}_1^{\mathrm{U}}\,(4)$		$\mathbb{A}_2^{\mathrm{U}}\,(5)$		$\mathbb{A}_3^{\mathrm{U}}\,(6)$		$\mathbb{A}_4^{\mathrm{U}}\,(7)$	
σ_1	0.2477	(0.0041)	0.1806	(0.0030)	0.2304	(0.0042)	0.2040	(0.0038)	0.1681	(0.0027)
σ_2	0.0873	(0.0012)	0.0894	(0.0014)	0.0916	(0.0015)	0.0908	(0.0015)	0.0878	(0.0013)
σ_3	0.3117	(0.0004)	0.0892	(0.0007)	0.1054	(0.0006)	0.1313	(0.0003)	0.1194	(0.0005)
ρ_{12}	-0.3010	(0.0188)	-0.4090	(0.0195)	-0.5065	(0.0177)	-0.4807	(0.0179)	-0.3817	(0.0196)
ρ_{13}	0.1237	(0.0148)	0.2042	(0.0318)	-0.2003	(0.0269)	0.1189	(0.0188)	-0.0217	(0.0222)
ρ_{23}	0.7007	(0.0107)	0.7737	(0.0217)	0.9406	(0.0108)	0.8135	(0.0203)	0.8650	(0.0247)
γ_1			0.3628	(0.0008)	0.1654	(0.0042)	0.1449	(0.0036)	0.1086	(0.0305)
γ_2					0.1414	(0.0009)	0.0262	(0.0010)	0.0408	(0.0027)
γ_3							0.0553	(0.0037)	0.0616	(0.0056)
γ_4									0.0200	(0.0027)
ς_1			1.8995	(0.0083)	6.0216	(0.1607)	9.8926	(0.2479)	18.1935	(5.1151)
ς_2					3.6217	(0.0227)	1.1061	(0.0098)	2.4223	(0.1705)
ς_3							14.3025	(0.9328)	15.1783	(1.3829)
ς_4									3.3056	(0.4422)
ϱ_1			-0.3751	(0.0018)	0.0740	(0.0006)	-0.6981	(0.0014)	-0.9557	(0.0033)
ϱ_2					-0.9743	(0.0036)	-0.8136	(0.0508)	-0.9843	(0.0177)
ϱ_3							0.6089	(0.0045)	0.5391	(0.0034)
ϱ_4									-0.9474	(0.0442)
$\theta_1^{\mathbb{Q}}$	0.2151	(0.0916)	0.2124	(0.0644)	0.2203	(0.0967)	0.2186	(0.0845)	0.2206	(0.0650)
$\theta_2^{\mathbb{Q}}$	-0.0249	(0.0294)	-0.0095	(0.0384)	-0.0172	(0.0364)	-0.0188	(0.0357)	-0.0139	(0.0343)
$\theta_3^{\mathbb{Q}}$	0.0500	(0.0733)	0.1000	(0.0839)	0.0833	(0.0729)	0.0867	(0.0674)	0.0925	(0.0766)
$\kappa_4^{\mathbb{Q}}$			1.6640	(0.0016)	4.5265	(0.0079)	5.2207	(0.0101)	13.2371	(0.0430)
$\kappa_5^{\mathbb{Q}}$					1.5201	(0.0078)	1.5973	(0.0067)	1.3458	(0.0129)
$\kappa_6^{\mathbb{Q}}$							2.3160	(0.0210)	3.8454	(0.0157)
$\kappa_7^{\mathbb{Q}}$									4.7629	(0.0407)
σ_F	0.0039	(0.0000)	0.0039	(0.0000)	0.0039	(0.0000)	0.0039	(0.0000)	0.0039	(0.0000)
σ_O	0.1117	(0.0000)	0.0235	(0.0000)	0.0136	(0.0000)	0.0083	(0.0000)	0.0068	(0.0000)
$\log L$	425 059		695 930		788 718		872 380		903 740	

Table E.3. Root-mean-square pricing errors (RMSEs) for futures contracts obtained by the $\mathbb{A}_M^{\mathrm{U}}\,(3+M)$ models for different M. RMSE is defined as the difference between the logarithms of the fitted and observed futures prices, and is reported in percentage points.

	Contract												
Model	M1	M2	M3	M4	M5	M6	Q1	Q2	Y1	Y2	Y3	Y4	Overall
$\mathbb{A}_0^{\mathrm{U}}\,(3)$	0.65	0.19	0.32	0.32	0.27	0.23	0.22	0.30	0.41	0.40	0.19	0.43	0.35
$\mathbb{A}_1^{\mathrm{U}}\,(4)$	0.66	0.21	0.32	0.32	0.28	0.23	0.22	0.31	0.41	0.41	0.19	0.42	0.35
$\mathbb{A}_2^{\mathrm{U}}\,(5)$	0.65	0.20	0.32	0.32	0.27	0.22	0.22	0.30	0.41	0.41	0.19	0.43	0.35
$\mathbb{A}_3^{\mathrm{U}}\,(6)$	0.66	0.20	0.32	0.32	0.28	0.23	0.22	0.30	0.41	0.40	0.19	0.42	0.35
$\mathbb{A}_4^{\mathrm{U}}\,(7)$	0.66	0.21	0.32	0.32	0.27	0.23	0.23	0.31	0.42	0.41	0.19	0.43	0.36

Table E.4. Root-mean-square pricing errors (RMSEs) for options contracts obtained by the $\mathbb{A}_M^{\mathrm{U}}\,(3+M)$ models for different M. RMSE is defined as the difference between the fitted and observed implied volatility, and is reported in percentage points.

	Contract								
Model	M1	M2	M3	M4	M5	M6	Q1	Q2	Overall
$\mathbb{A}_0^{\mathrm{U}}\,(3)$	15.98	13.27	12.27	11.33	10.57	9.62	9.20	8.18	11.54
$\mathbb{A}_1^{\mathrm{U}}\,(4)$	4.30	2.46	1.66	1.42	1.57	1.79	2.17	2.85	2.38
$\mathbb{A}_2^{\mathrm{U}}\,(5)$	2.68	1.59	1.24	1.05	0.94	0.89	1.00	1.23	1.41
$\mathbb{A}_3^{\mathrm{U}}\,(6)$	1.63	0.82	0.72	0.66	0.59	0.53	0.58	0.84	0.84
$\mathbb{A}_4^{\mathrm{U}}\,(7)$	1.32	0.76	0.59	0.52	0.51	0.50	0.50	0.58	0.70

Table E.5. Root-mean-square pricing errors (RMSEs) for options contracts obtained by the $\mathbb{A}_M^{\mathrm{U}}\,(3+M)$ models for different M per contract and maturity. RMSE is defined as the difference between the fitted and observed implied volatility, and is reported in percentage points.

		Contract							
Moneyness	Model	M1	M2	M3	M4	M5	M6	Q1	Q2
0.78–0.82	$\mathbb{A}_0^{\mathrm{U}}\,(3)$	26.57	15.82	13.50	12.12	11.27	10.06	9.65	8.71
	$\mathbb{A}_1^{\mathrm{U}}\,(4)$	5.95	2.83	2.24	2.17	2.31	2.43	2.63	3.13
	$\mathbb{A}_2^{\mathrm{U}}\,(5)$	6.10	2.83	2.02	1.65	1.39	1.08	0.84	0.86
	$\mathbb{A}_3^{\mathrm{U}}\,(6)$	2.58	1.12	1.16	1.20	1.06	0.82	0.65	0.78
	$\mathbb{A}_4^{\mathrm{U}}\,(7)$	1.62	1.00	0.86	0.85	0.84	0.79	0.76	0.74
0.82–0.86	$\mathbb{A}_0^{\mathrm{U}}\,(3)$	20.08	13.92	12.93	11.87	10.95	9.70	9.26	8.38
	$\mathbb{A}_1^{\mathrm{U}}\,(4)$	4.35	2.36	1.84	1.81	2.03	2.19	2.44	3.00
	$\mathbb{A}_2^{\mathrm{U}}\,(5)$	4.29	2.28	1.66	1.31	1.05	0.78	0.65	0.84
	$\mathbb{A}_3^{\mathrm{U}}\,(6)$	1.78	0.84	0.81	0.81	0.70	0.56	0.52	0.72
	$\mathbb{A}_4^{\mathrm{U}}\,(7)$	1.07	0.74	0.54	0.50	0.54	0.56	0.59	0.62
0.86–0.90	$\mathbb{A}_0^{\mathrm{U}}\,(3)$	16.12	13.59	12.60	11.61	10.42	9.43	9.20	8.18
	$\mathbb{A}_1^{\mathrm{U}}\,(4)$	3.53	2.00	1.47	1.51	1.76	1.96	2.36	2.86
	$\mathbb{A}_2^{\mathrm{U}}\,(5)$	2.96	1.81	1.34	1.02	0.76	0.58	0.60	0.87
	$\mathbb{A}_3^{\mathrm{U}}\,(6)$	1.31	0.70	0.65	0.62	0.53	0.46	0.50	0.69
	$\mathbb{A}_4^{\mathrm{U}}\,(7)$	0.72	0.66	0.47	0.36	0.41	0.44	0.50	0.54
0.90–0.94	$\mathbb{A}_0^{\mathrm{U}}\,(3)$	14.44	13.29	12.29	11.28	10.49	9.73	9.07	7.87
	$\mathbb{A}_1^{\mathrm{U}}\,(4)$	3.42	1.77	1.13	1.21	1.52	1.80	2.20	2.77
	$\mathbb{A}_2^{\mathrm{U}}\,(5)$	2.22	1.41	1.04	0.78	0.59	0.54	0.65	0.94
	$\mathbb{A}_3^{\mathrm{U}}\,(6)$	1.07	0.66	0.61	0.56	0.49	0.49	0.50	0.70
	$\mathbb{A}_4^{\mathrm{U}}\,(7)$	0.65	0.69	0.51	0.39	0.39	0.43	0.44	0.49
0.94–0.98	$\mathbb{A}_0^{\mathrm{U}}\,(3)$	14.33	13.14	12.18	11.23	10.25	9.44	9.07	8.07
	$\mathbb{A}_1^{\mathrm{U}}\,(4)$	3.72	1.72	0.85	0.90	1.27	1.62	2.13	2.81
	$\mathbb{A}_2^{\mathrm{U}}\,(5)$	1.94	1.08	0.80	0.61	0.50	0.55	0.71	1.01
	$\mathbb{A}_3^{\mathrm{U}}\,(6)$	1.09	0.63	0.61	0.53	0.46	0.46	0.48	0.72
	$\mathbb{A}_4^{\mathrm{U}}\,(7)$	0.88	0.69	0.53	0.41	0.38	0.38	0.37	0.47

Table E.6. This table is the continuation of Table E.5.

Moneyness	Model	Contract							
		M1	M2	M3	M4	M5	M6	Q1	Q2
0.98–1.02	$\mathbb{A}_0^{\mathrm{U}}\,(3)$	14.25	12.88	12.02	10.94	10.47	9.46	8.98	7.89
	$\mathbb{A}_1^{\mathrm{U}}\,(4)$	4.00	1.83	0.77	0.65	1.09	1.46	1.98	2.70
	$\mathbb{A}_2^{\mathrm{U}}\,(5)$	1.79	0.81	0.62	0.51	0.52	0.62	0.81	1.09
	$\mathbb{A}_3^{\mathrm{U}}\,(6)$	1.28	0.61	0.57	0.47	0.41	0.42	0.48	0.76
	$\mathbb{A}_4^{\mathrm{U}}\,(7)$	1.12	0.67	0.53	0.40	0.35	0.34	0.32	0.46
1.02–1.06	$\mathbb{A}_0^{\mathrm{U}}\,(3)$	14.17	12.87	12.02	10.98	10.44	9.73	9.19	7.89
	$\mathbb{A}_1^{\mathrm{U}}\,(4)$	4.16	2.05	0.94	0.64	1.04	1.43	1.96	2.69
	$\mathbb{A}_2^{\mathrm{U}}\,(5)$	1.55	0.70	0.61	0.58	0.65	0.77	0.96	1.22
	$\mathbb{A}_3^{\mathrm{U}}\,(6)$	1.35	0.64	0.57	0.47	0.42	0.44	0.51	0.81
	$\mathbb{A}_4^{\mathrm{U}}\,(7)$	1.17	0.65	0.53	0.43	0.37	0.36	0.31	0.46
1.06–1.10	$\mathbb{A}_0^{\mathrm{U}}\,(3)$	13.87	12.89	12.10	11.22	10.63	9.77	9.37	8.38
	$\mathbb{A}_1^{\mathrm{U}}\,(4)$	4.21	2.37	1.33	0.92	1.15	1.46	1.98	2.82
	$\mathbb{A}_2^{\mathrm{U}}\,(5)$	1.31	0.79	0.79	0.79	0.84	0.94	1.12	1.41
	$\mathbb{A}_3^{\mathrm{U}}\,(6)$	1.27	0.66	0.59	0.50	0.47	0.48	0.58	0.91
	$\mathbb{A}_4^{\mathrm{U}}\,(7)$	1.05	0.59	0.53	0.47	0.45	0.43	0.39	0.53
1.10–1.14	$\mathbb{A}_0^{\mathrm{U}}\,(3)$	14.24	12.54	11.91	11.18	10.53	9.42	9.28	8.38
	$\mathbb{A}_1^{\mathrm{U}}\,(4)$	4.51	2.75	1.77	1.29	1.32	1.52	2.00	2.85
	$\mathbb{A}_2^{\mathrm{U}}\,(5)$	1.83	1.08	1.05	1.02	1.03	1.09	1.29	1.60
	$\mathbb{A}_3^{\mathrm{U}}\,(6)$	1.56	0.74	0.61	0.51	0.49	0.50	0.65	1.00
	$\mathbb{A}_4^{\mathrm{U}}\,(7)$	1.30	0.61	0.52	0.51	0.50	0.48	0.50	0.64
1.14–1.18	$\mathbb{A}_0^{\mathrm{U}}\,(3)$	16.03	12.42	11.77	11.11	10.42	9.49	9.15	8.27
	$\mathbb{A}_1^{\mathrm{U}}\,(4)$	5.12	3.20	2.23	1.66	1.55	1.64	2.02	2.85
	$\mathbb{A}_2^{\mathrm{U}}\,(5)$	2.76	1.53	1.33	1.25	1.21	1.22	1.41	1.66
	$\mathbb{A}_3^{\mathrm{U}}\,(6)$	2.26	0.96	0.70	0.57	0.56	0.53	0.69	1.01
	$\mathbb{A}_4^{\mathrm{U}}\,(7)$	2.02	0.80	0.59	0.57	0.57	0.53	0.56	0.66
1.18–1.22	$\mathbb{A}_0^{\mathrm{U}}\,(3)$	19.13	12.61	11.49	11.06	10.31	9.51	8.98	8.00
	$\mathbb{A}_1^{\mathrm{U}}\,(4)$	6.06	3.72	2.63	1.99	1.82	1.88	2.08	2.85
	$\mathbb{A}_2^{\mathrm{U}}\,(5)$	3.88	2.08	1.59	1.44	1.40	1.41	1.53	1.80
	$\mathbb{A}_3^{\mathrm{U}}\,(6)$	3.29	1.32	0.88	0.72	0.67	0.62	0.75	1.10
	$\mathbb{A}_4^{\mathrm{U}}\,(7)$	3.06	1.16	0.75	0.69	0.67	0.63	0.65	0.75

Table E.7. Root-mean-square pricing errors (RMSEs) for futures contracts obtained by the $\mathbb{A}_M^{\mathrm{U}}(3+M)$ models for different M and panels. In Calibration i - Panel j each model is estimated using the data included in Panel i, while the RMSE is computed using the data in Panel j, where $i, j \in \{A, B, C\}$. Panel A corresponds to the entire sample period from January 3rd, 2006 to December 31st, 2014; Panel B goes from January 3rd, 2006 to December 31st, 2010; Panel C comprises the period from January 3rd, 2011 to December 31st, 2014. RMSE is defined as the difference between the logarithms of the fitted and observed futures prices, and is reported in percentage points.

	Calibration A			Calibration B			Calibration C		
Model	Panel A	Panel B	Panel C	Panel A	Panel B	Panel C	Panel A	Panel B	Panel C
$\mathbb{A}_0^{\mathrm{U}}(3)$	0.35	0.40	0.28	0.36	0.39	0.32	0.37	0.43	0.26
$\mathbb{A}_1^{\mathrm{U}}(4)$	0.35	0.40	0.28	0.36	0.40	0.32	0.37	0.43	0.27
$\mathbb{A}_2^{\mathrm{U}}(5)$	0.35	0.40	0.28	0.36	0.40	0.31	0.36	0.42	0.27
$\mathbb{A}_3^{\mathrm{U}}(6)$	0.35	0.40	0.28	0.36	0.40	0.30	0.36	0.43	0.27
$\mathbb{A}_4^{\mathrm{U}}(7)$	0.36	0.40	0.28	0.36	0.40	0.31	0.37	0.44	0.27

Table E.8. Root-mean-square pricing errors (RMSEs) for options contracts obtained by the $\mathbb{A}_M^{\mathrm{U}}(3+M)$ models for different M and panels. In Calibration i - Panel j each model is estimated using the data included in Panel i, while the RMSE is computed using the data in Panel j, where $i, j \in \{A, B, C\}$. Panel A corresponds to the entire sample period from January 3rd, 2006 to December 31st, 2014; Panel B goes from January 3rd, 2006 to December 31st, 2010; Panel C comprises the period from January 3rd, 2011 to December 31st, 2014. RMSE is defined as the difference between the fitted and observed implied volatility, and is reported in percentage points.

	Calibration A			Calibration B			Calibration C		
Model	Panel A	Panel B	Panel C	Panel A	Panel B	Panel C	Panel A	Panel B	Panel C
$\mathbb{A}_0^{\mathrm{U}}(3)$	11.54	13.51	8.41	11.39	11.84	10.79	13.45	16.33	8.49
$\mathbb{A}_1^{\mathrm{U}}(4)$	2.38	2.53	2.18	3.58	2.34	4.70	2.88	3.48	1.85
$\mathbb{A}_2^{\mathrm{U}}(5)$	1.41	1.30	1.54	1.83	1.18	2.43	1.43	1.40	1.47
$\mathbb{A}_3^{\mathrm{U}}(6)$	0.84	0.79	0.91	1.09	0.74	1.41	0.89	0.94	0.82
$\mathbb{A}_4^{\mathrm{U}}(7)$	0.70	0.64	0.77	0.86	0.60	1.11	0.76	0.82	0.68