Pontificia Universidad Católica de Chile
Faculty of Mathematics

Essential Minimum in Families

Marcos Morales

Supervisor: Ricardo Menares Valencia

This thesis is presented to the Faculty of Mathematics of
the Pontificia Universidad Católica de Chile
to obtain the Phd degree.

Committee:

-Jan Beno Kiwi Krauskopf (Pontificia Universidad Católica de Chile)
-Martín Sombra (ICREA and Universitat de Barcelona)

March 8, 2023

Santiago, Chile

Acknowledgments

I want to thank to my Ph.D. advisor Ricardo Menares and the maths department of the Pontifical Catholic University of Chile. Also, I would like to thank to Kiwi Krauskopf and Martín Sombra for all the comments they provided to this thesis.

Special thanks to ANID for all the financial support. Award number: 21180907.

Contents

1 Introduction

1.1 Historical context and definitions

The study of heights of points in arithmetic varieties plays an important role in questions of arithmetic geometry. It is related to interesting problems where there have been a lot of development in the past 30 years. For instance, it plays a central role in the proof of finiteness results on integral and rational points on curves an Abelian varieties like the theorems of Siegel, Mordell-Weil and Faltings (see for instance [4]). It is also very useful in transcendence theory and it is an essential tool in Diophantine Geometry (see [13]). Given a height h in a variety X over $\overline{\mathbb{Q}}$ it has been of special interest to understand the threshold where the points with bounded height begin to be Zariski dense.

Let X be an algebraic variety over $\mathbb{Q}$. For a function $f : X(\overline{\mathbb{Q}}) \to \mathbb{R}$, the essential minimum $\mu^{ess}(f)$, is defined by

$$\mu^{ess}(f) = \inf\{\theta \in \mathbb{R} : \{\alpha \in X(\overline{\mathbb{Q}})/f(\alpha) \leq \theta\} \text{ is Zariski dense}\} .$$

When f is a height function, $\mu^{ess}(f)$ plays a role in the phenomenon of equidistribution of small points (e.g see [7], [20], and [21]). However, it is notoriously difficult to compute this quantity in general situations.

The exact value of the essential minimum is well known when X is related to a group variety endowed with a height that behaves well with respect to the group structure and contains sufficiently many torsion points. In fact, we have that the height associated to such group variety is zero at any torsion point (see [23], section 6). Therefore the essential minimum of that height is zero. More generally, in 2015 J. Burgos Gil, P. Philippon and M. Sombra computed the exact value of the essential minimum for toric varieties endowed with toric heights (see [8] Theorem A and Theorem B for more details).

For concreteness, we proceed to review the case of the Zhang-Zagier height, which is related to the line $x + y = 1$. In this work we treat the non-toric situations.

To recall the definition of the Zhang-Zagier height, given $\alpha \in \overline{\mathbb{Q}}$, we will consider a number field K containing α which is a Galois extension of degree d. For any valuation v we denote the local degree $d_v = [K_v : \mathbb{Q}_v]$, where K_v and $\mathbb{Q}_v$ are the completions of K and $\mathbb{Q}$ with respect to v. We denote M_K the set of all places of K normalized in such a way that they satisfy the product formula, this means that for each arquimedean place v we consider $|x|_v = |x|^{d_v/d}$ and for each non-arquimedean place w we consider $|x|_w = p^{-d_w/d}$, where p is the unique prime such that $|p|_w < 1$. The Weil height $h : \overline{\mathbb{Q}} \to \mathbb{R}$, is given by

$$h(\alpha) = \frac{1}{[K : \mathbb{Q}]} \sum_{\nu \in M_K} \log^+ |\alpha|_\nu .$$

It is well known that $h(\alpha)$ is independent of the field containing α (e.g see [4] Lemma 1.5.2). Also, $h(\zeta) = 0$ for all roots of unity ζ. Hence, we have that $\mu^{ess}(h) = 0$.

The Zhang-Zagier height $h_Z : \overline{\mathbb{Q}} \to \mathbb{R}$ is defined by $h_Z(\alpha) := h(\alpha) + h(1 - \alpha)$. Let $X \subset \mathbb{C}^2$, a proper subvariety defined over $\mathbb{Q}$. Then, we can define the height $h_X : X(\overline{\mathbb{Q}}) \to \mathbb{R}$, by $h_X(x, y) = h(x) + h(y)$.

Let $T \subset \mathbb{C}^2$ be a subvariety, we have that T is a torsion subvariety if and only if there are $n, m \in \mathbb{Z} \setminus \{0\}$ and ζ a root of unity, such that

$$T = \{(x, y) : x^n y^m = \zeta\}.$$

We state a particular case of a Theorem of S.-W. Zhang.

Theorem.([23]): *Let $X \subset \mathbb{C}^2$ defined over $\mathbb{Q}$ be a proper subvariety. Then, $\mu^{ess}(h_X) = 0$ if and only if X contains a torsion subvariety.*

From this theorem we can conclude that $\mu^{ess}(h_Z) > 0$. The question now, is how to compute the essential minimum of heights like h_Z. However, the answer remains unknown. Despite of this, there have been several attempts to approach this number. In 1993 Zagier proved the following theorem

Theorem.([22]): *For all $\alpha \in \overline{\mathbb{Q}}$ such that $\alpha \notin \{0, 1, e^{i\pi/3}, e^{-i\pi/3}\}$, we have*

$$h_Z(\alpha) \geq \frac{1}{2} \log \left(\frac{1 + \sqrt{5}}{2} \right) \approx 0.2406059....$$

With equality if and only if α or $1 - \alpha$ is a primitive 10th root of unity

This theorem tells us that $\mu^{ess}(h_Z) \geq 0.240606$. In 2001 Doche improved this result and proved the following theorem

Theorem.([10], [9]): *Let α be an algebraic number different from the roots of $(z^2 - z)(z^2 - z + 1)\phi_{10}(z)\phi_{10}(1 - z)$. Then*

$$h(\alpha) \geq 0.2482474.$$

Furthemore, the smallest limit point of $\mathcal{V} = \{h(\alpha) : \alpha \in \overline{\mathbb{Q}}\}$ is less than 0.25443678

Note that the smallest limit point of $\mathcal{V}$ is greater or equal than $\mu^{ess}(h_Z)$. In fact, we have that

$$\mu^{ess}(h_Z) = \inf\{\theta \in \mathbb{R} : \{\alpha \in \overline{\mathbb{Q}}/h_Z(\alpha) \leq \theta\} \text{ is infinite }\}.$$

Therefore, if x is a limit point of $\mathcal{V}$, there exists a sequence of algebraic numbers $\{\alpha_n\}_{n\in\mathbb{N}}$, such that $h_Z(\alpha_n) \to x$. Therefore, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that if $n > n_0$, then $|h_Z(\alpha_n)-x| < \epsilon$. Then, we have that $h_Z(\alpha_n) < x+\epsilon$ for all $n > n_0$, we conclude that $\mu^{ess}(h_Z) \leq x+\epsilon$, since ϵ is arbitrary, we have that $\mu^{ess}(h_Z) \leq x$. We conclude that $\mu^{ess}(h_Z)$ is less than or equal to the smallest limit point of $\mathcal{V}$

In 2018 V. Flammang improved the lower bound.

Theorem.([12]): *If α is an algebraic integer different from the roots of $(z^2 - z)(z^2 - z + 1)\phi_{10}(z)\phi_{10}(1 - z)$, then*

$$h(\alpha) \geq 0.248744.$$

These results show that $0.248744 \leq \mu^{ess}(h_Z) \leq 0.25443678$. In 2021 F. Ballaÿ proved that the essential minimum is equal to another quantity called the asymptotic minimal slope (see [2] definition 5.2), however there is no an effectively computable method yet to obtain the exact value of this number.

1.2 Methods and main Results

In this work we consider heights attached to families of lines and obtain upper and lower bounds for the essential minimum of these functions. Also we determine, in some cases, explicit intervals where the image of the height function is dense. Our main tool to find upper bounds and intervals of density, is a refinement of the classical Fekete-Szegö theorem due to Burgos Gil, Philippon, Rivera-Letelier and Sombra ([7]). To obtain lower bounds we adapt the techniques used in [6] by J. Burgos Gil, J. Rivera-Letelier and R. Menares to this specific case.

We consider the case in which X is a non-vertical line. Let $a, b \in \overline{\mathbb{Q}}$, $a \neq 0$, we consider the subvariety $X = L_{a,b} = \{(x, y) \in \mathbb{C}^2 : y = ax + b\}$, and we denote $h_{a,b} = h_{L_{a,b}}$. Note that the Zhang-Zagier height, corresponds to the variety $L_{-1,1}$, and $h_Z = h_{-1,1}$. The reason why we study this case first is because we apply a slightly different method which make the results and computations simpler.

Before proceeding, we introduce some notations. Let $a, b \in \overline{\mathbb{Q}}$, we write K_a the Galois closure of $\mathbb{Q}(a)/\mathbb{Q}$ and $K_{a,b}$ the Galois closure of the field generated by a and b over $\mathbb{Q}$. We also write $G(a)$ =Gal$(K_a/\mathbb{Q})$, $G(a, b)$ =Gal$(K_{a,b}/\mathbb{Q})$, $\deg(a) = [\mathbb{Q}(a) : \mathbb{Q}]$ and $\mathrm{Gal}(a) = \{\sigma(a) : \sigma \in G(a)\}$.

Let $\sigma \in G(a, b)$, we define, $\Psi^{\sigma}_{a,b} : \mathbb{R} \to \mathbb{R}$, $\varphi : \mathbb{R} \to \mathbb{R}$, $\Delta : \overline{\mathbb{Q}} \times \overline{\mathbb{Q}} \to \mathbb{R}$ and $\Omega_{a,b} : \mathbb{R} \to \mathbb{R}$, given by

$$\Psi^{\sigma}_{a,b}(t) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| e^{i\theta} + \frac{\sigma(b)}{\sigma(a)} + t \right| d\theta, \tag{1}$$

$$\varphi(t) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |e^{i\theta} + t| d\theta, \tag{2}$$

$$\Delta(a, b) = \sum_{\substack{p \\ \text{prime}}} \sum_{\sigma \in G(a,b)} \log^+ \max(|\sigma(a)|_p, |\sigma(b)|_p) + \sum_{\sigma \in G(a,b)} \log^+ |\sigma(a)|, \tag{3}$$

$$\Omega_{a,b}(t) = \Delta(a, b) + \varphi(t) \ + \sum_{\sigma \in G(a,b)} \Psi^{\sigma}_{a,b}(t). \tag{4}$$

Now, we can state the following theorem

Theorem A: *Let $a, b \in \overline{\mathbb{Q}}$ and $t \in \mathbb{R}$. Then,*

$$\mu^{ess}(h_{a,b}) \le \Omega_{a,b}(t)$$

Furthermore, assume that one of the following properties holds

i) *There exists $\sigma_0 \in G(a, b)$, such that, $||\sigma_0(a)| - |\sigma_0(b)|| > 1$.*

ii) *There exists $\sigma_0 \in G(a, b)$, such that, $||\sigma_0(a)| - |\sigma_0(b)|| < 1$ and $|\sigma_0(a)| + |\sigma_0(b)| < 1$.*

iii) *The minimum value of $h_{a,b}$ is only achieved in a finite set of algebraic numbers.*

Then, there exists an effectively computable number $\mathcal{K}(a, b) > 0$, such that

$$\mu^{ess}(h_{a,b}) \ge \mathcal{K}(a, b)$$

(See section 4, definition 4.1.2 for the definition of $\mathcal{K}(a, b)$).

We can compare the lower bound in Theorem A results with the results in [18] by W. Schmidt and the results in [3] by F. Beukers and D. Zagier which where generalized by C. Samuels in [17]. These works use a similar method. For instance, if we take $a = 1$ and $b = 4$, then our lower bound is better than the one given by W. Schmidt or F. Beukers and D. Zagier, in our case the lower bound is $\mathcal{K}(1, 4) = \log(3)$ and in the other cases is $1/52$ and 0.090087 respectively. However, our method fails for $a = 0.5$ and $b = 1$, and the lower bounds given by M. Schmidt or F. Beukers and D. Zagier are positive. In the case of the Zhang-Zagier height, our lower bound is the same given by Zagier in [22]

In order to construct intervals where the image of $h_{a,b}$ is dense, we need to restrict the values of a and b to $\mathbb{Q}$, also we need some definitions. Let $a, b \in \mathbb{Q}$, $a \neq 0$. If $b \neq 0$, we can write $a = a_1/a_2$, $b = b_1/b_2$ and $(a_1, a_2) = 1$, $(b_1, b_2) = 1$. We define $S_{a,b} = \{p \text{ prime} : p|a_2 \vee p|b_2\}$. If $b = 0$, then $S_{a,0} = S_a = \{p : p|a_2\}$. We also write $s = |S_{a,b}|$, then $S = \{p_i : 1 \le i \le s\}$. Using this notation, we can define the function $\Gamma_{a,b} : \mathbb{R}^{s+1} \to \mathbb{R}$

$$\Gamma_{a,b}(x,r_1,r_2,...,r_s)=\sum_{i=1}^{s}\log^+|r_i|+\log^+\max(|a|_{p_i}r_i,|b|_{p_i})$$
$$+\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{e^{i\theta}}{r_1r_2...r_s}+x\right|+\log^+\left|\frac{ae^{i\theta}}{r_1r_2...r_s}+b+x\right|d\theta.$$

Theorem B: *Let $a,b\in\mathbb{Q}$. Then, for each $x,r_1,r_2,...,r_s\in\mathbb{R}$, the image of $h_{a,b}$ is dense in the interval $[\Gamma_{a,b}(x,r_1,r_2,...,r_s),\infty)$, in particular*

$$\mu^{ess}(h_{a,b})\leq\Gamma_{a,b}(x,r_1,r_2,...,r_s).$$

The following corollary is a direct consequence of Theorem A and Theorem B by setting $t=-1/2$

Corollary C: *Let h_Z be the Zhang-Zagier height. Then, the image of h_Z is dense in the interval $[0.31944909,\infty)$. In particular*

$$\mu^{ess}(h_Z)\leq 0.31944909.$$

This is also a direct consequence of Theorem 1 in [15].

In 2003, P. Dresden proved that the image of h_Z is dense in the interval $[0.39678,\infty)$ (see [11]), our result is slightly better, however it is not clear that our intervals of density are better in all cases.

Note that Corollary C does not improve the upper bound given by C. Doche, however it gives intervals where the image of h_Z is dense. The same way, Theorem B does not improve the upper bound given in Theorem A, however, it does give intervals of density for $a,b\in\mathbb{Q}$.

For instance, if we take $a=1$ and $b=2$, we find that $\mathcal{K}(1,2)=\log(\sqrt{3})$ and $\Omega_{1,2}(0)\leq 0.6461599$ (see Theorem 2.9 and Corollary 3.9). Therefore, Theorem B gives us that, $\log(\sqrt{3})=0.5493061...\leq\mu^{ess}(h_{1,2})\leq 0.6461599$ and the image of $h_{1,2}$, is dense in the interval $[0.6461599,\infty]$. We can compare this result with the one given in [23] by Zhang, we can see that our bounds are slightly better, the lower bound and upper bound given by Zhang are 0.50146 and 0.680367 respectively. However the result given by Zhang is more general and can be used in cases where our lower bound fails, also the intervals of density can not be found by the method given by Zhang. In general, it is not clear when our lower bounds and upper bounds are better than the ones given by Zhang, if we take for instance $a=0.1$ and $b=1.01$, we have that both bounds are worse, our lower bound is zero because our method fails and our upper bound is greater than 1, meanwhile the lower bound given by Zhang is positive and the upper bound is lower that 1 .

Let $v\in M_{\mathbb{Q}}$, we denote by $\mathbb{Q}_v$ the completion of $\mathbb{Q}$ at v. Let $\overline{\mathbb{Q}}_v$, the algebraic closure of $\mathbb{Q}_v$, and let $\mathbb{C}_v$ denote the completion of $\overline{\mathbb{Q}}_v$. It is well known that $\mathbb{C}_v$ is algebraically closed, we also define $D_v(a,r)=\{z\in\mathbb{C}_v:|z-a|_v\leq r\}$, sometimes we will write $D_v(0,1)=\mathcal{O}_v$.

Now, let $q\in\overline{\mathbb{Q}}(t)$, $q(t)=q_1(t)/q_2(t)$, we write $q_1(t)=a_nt^n+...+a_1t+a_0$ and $q_2(t)=b_mt^m+...+b_1t+b_0$. Let $q,r\in\overline{\mathbb{Q}}(t)$, we consider $X_{q,r}=\{(q(t),r(t))\in\mathbb{C}^2:t\in\mathbb{C}\}$ and write $h_{X_{q,r}}=h_{q,r}$. Let $\sigma\in\mathrm{G}(a_1,...,a_n,b_1,....,b_m)=:\mathrm{G}(q)$, we call $q_\sigma(t)=\sigma(a_n)t^n+...+\sigma(a_1)t+\sigma(a_0)$. We also denote K_q, the Galois closure of the field generated by $a_1,...,a_n,b_1,....,b_m$ over $\mathbb{Q}$. Now, we can define a function $\Psi_q^\sigma:\mathbb{R}\to\mathbb{R}$ and two functions $\Delta:\overline{\mathbb{Q}}[t]\to\mathbb{R}$ and $\Omega_{q,r}:\mathbb{R}\to\mathbb{R}$, given by

$$\Psi_q^\sigma(t)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|q_\sigma(e^{i\theta}+t)|d\theta. \tag{5}$$

$$\Delta(q)=\sum_{\substack{p\\ \text{prime}}}\sum_{\sigma\in\mathrm{G}(q)}\log^+\max(|\sigma(a_1)|_p,...,|\sigma(a_n)|_p). \tag{6}$$

$$\Omega_{q,r}(t) = \Delta(q) + \Delta(r) + \sum_{\sigma \in \mathrm{G}(q)} \Psi_q^{\sigma}(t) + \sum_{\sigma \in \mathrm{G}(r)} \Psi_r^{\sigma}(t). \tag{7}$$

Now, we can state the following theorem analogous to Theorem B

Theorem D: *Let $q, r \in \overline{\mathbb{Q}}(t)$ and $t \in \mathbb{R}$. Then,*

$$\mu^{ess}(h_{q,r}) \leq \Omega_{q,r}\,(t)$$

Furthermore, assume that $q(t) = t$, $r(t) = at^n + c$, where $n \in \mathbb{N}$, $a, c \in \overline{\mathbb{Q}}$ and there exists $\sigma_0 \in G(a, c)$ such that $|\sigma_0(c)| - |\sigma_0(a)| > 1$. Then, there exists an effectively computable number $\mathcal{K}(q, r) > 0$, such that.

$$\mu^{ess}(h_{q,r}) \geq \mathcal{K}(q, r).$$

(See section 6, Definition 6.2.2, for the definition of $\mathcal{K}(q, r)$).

Finally, let $q, r \in \mathbb{Q}[t]$, $q(t) = a_n t^n + ... + a_0$ and $r(t) = b_m t^m ... + b_0$, we consider $S_q = \{p \text{ prime } : p|a_0 \vee p|a_1 \vee ... \vee p|a_n\} = \{p_1, p_2, ..., p_{s_q}\}$, $S_r = \{p \text{ prime } : p|b_0 \vee p|b_1 \vee ... \vee p|b_m\} = \{p_{s_q+1}, p_{s_q+2}, ..., p_{s_q+s_r}\}$ and $\mathbb{Q}^+ = \{c \in \mathbb{Q} : c > 0\}$. Then, we define $\Gamma_{q,r} : \mathbb{R} \times (\mathbb{Q}^+)^{s_q+s_r} \to \mathbb{R}$, given by

$$\begin{aligned}\Gamma_{q,r}(x, r_1, r_2, ..., r_{s_q+s_r}) &= \sum_{i=1}^{s_q} \log^+ \max_{|z|_{p_i} \leq r_i} |q(z)|_{p_i} + \sum_{i=s_q+1}^{s_q+s_r} \log^+ \max_{|z|_{p_i} \leq r_i} |r(z)|_{p_i} \\ &+ \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| q\left(\frac{e^{i\theta}}{r_1 r_2 ... r_{s_q+s_r}} + x\right)\right| + \log^+ \left| r\left(\frac{e^{i\theta}}{r_1 r_2 ... r_{s_q+s_r}} + x\right)\right| d\theta.\end{aligned}$$

Theorem E: *Let $q, r \in \mathbb{Q}[t]$. Then, for each $x, r_1, r_2, ..., r_{s_q+s_r} \in \mathbb{R}$, the image of $h_{q,r}$ is dense in the interval $[\Gamma_{a,b}(x, r_1, r_2, ..., r_{s_q+s_r}), \infty)$, in particular*

$$\mu^{ess}(h_{q,r}) \leq \Gamma_{q,r}(x, r_1, r_2, ..., r_{s_q+s_r}).$$

We will start in section 2 by giving a brief introduction to Berkovich spaces and capacity theory. Secondly in section 3 we will begin to study lines by determining upper bounds for the essential minimum. Then, in section 4 we will continue the case of lines, compute lower bounds and prove Theorem A using the methods outlined in [6]. In section 5 we will determine intervals of density for the image of $h_{a,b}$ for $a, b \in \mathbb{Q}$ and prove Theorem B. Finally in section 6 we will compute upper bounds, lower bounds and intervals of density for the essential minimum of $h_{q,r}$, where $q, r \in \overline{\mathbb{Q}}[t]$ and prove Theorem D and Theorem E.

2 Capacity theory

2.1 Classical Capacity theory

In this section we will give a brief review about the classical Capacity theory, all the definitions and notations have been taken from [16].

Let $D \subset \mathbb{R}^n$, $n \in \mathbb{N}$ equipped with the induced topology, and let $C_c^0(D, \mathbb{R})$ the space of all continuous functions with compact support over D. A Radon measure over D is a continuous linear functional in $C_c^0(D, \mathbb{R})$. If μ is a Radon measure over D and $f \in C_c^0(D, \mathbb{R})$ is a continuous function with compact support, we use the functional notation $\mu(f)$ as well as the integral notation

$$\mu(f) = \int_D f d\mu = \int_D f(x) d\mu(x).$$

A *positive measure* is a Radon measure μ such that for all $f \in C_c^0(D, \mathbb{R})$ with $f(z) \geq 0$, $\forall z \in D$, we have $\mu(f) \geq 0$.

If D is a compact set and μ is a Radon measure, we define the mass of μ by

$$\mu(D) = \int_D 1 d\mu(z)$$

Let D be a compact set, a *probability measure* in D is a positive Radon measure μ such that $\mu(D) = 1$. The set of all probability measures in D is denoted by $\mathcal{M}(D)$.

Now, let $E \subset \mathbb{C}$ a compact set, we define the energy integral of μ by $I_E : \mathcal{M}(D) \to \mathbb{R}$, given by

$$I_E(\nu) = \int\int_{E \times E} \log \frac{1}{|z - t|} d\mu(z) d\mu(t)$$

The Robin constant of a compact set $E \subset \mathbb{C}$ is defined by

$$V_E = \inf_{\mu \in \mathcal{M}(E)} I(\mu)$$

Finally, we define the capacity of E by

$$\mathrm{Cap}(E) = e^{-V_E}$$

It is well known that the value of V_E is achieved at a measure $\mu \in \mathcal{M}(E)$. Furthermore, if $\mathrm{Cap}(E) > 0$, then this measure is unique and it is denoted by μ_E, we call this the *equilibrium measure* of E.

2.2 The Berkovich unit disk

Now, we will give a brief review about the Berkovich unit disc, all the definitions and notations have been taken from [1]. Let $v \in M_{\mathbb{Q}}$ be a nonarchimedean place. We define $\mathcal{A}_v = \mathbb{C}_v\langle t \rangle$, the ring of all power series with coefficients in $\mathbb{C}_v$, converging on $D_v(0, 1)$. That is, $\mathcal{A}_v$ is the ring of all power series $f(t) = \sum_{i=0}^{\infty} a_i t^i \in K[[t]]$, such that $\lim_{i \to \infty} |a_i|_v = 0$. Equipped with the Gauss norm $||.||_v$ defined by $||f||_v = \max_i(|a_i|_v)$, $\mathcal{A}_v$ becomes a Banach algebra over $\mathbb{C}_v$

Definition 2.1 : A multiplicative seminorm (m.s) on $\mathcal{A}_v$ is a function $[.]_x : \mathcal{A}_v \to \mathbb{R}_{\geq 0}$ such that $[0]_x = 0$, $[1]_x = 1$, $[fg]_x = [f]_x[g]_x$ and $[f+g]_x \leq [f]_x + [g]_x$, for all $f, g \in \mathcal{A}_v$. It is a norm provided that $[f]_x = 0$ if and only if $f = 0$

A m.s $[.]_x$ is called bounded if there is a constant C_x such that $[f]_x \leq C_x ||f||_v$, for all $f \in \mathcal{A}_v$

Definition 2.2 : The Berkovich unit disk $\mathcal{D}_v(0,1)$ is the set of all bounded m.s on $\mathcal{A}_v$

We will write $x \in \mathcal{D}_v(0,1)$ instead of $[.]_x$. We denote $\zeta_{\text{Gauss}} = ||.||_v$, the Gauss's norm. Note that $\zeta_{\text{Gauss}} \in \mathcal{D}(0,1)$, furthermore, for all $x \in \mathcal{D}_v(0,1)$ and all $f \in \mathcal{A}_v$, $[f]_x \leq [f]_{\zeta_{\text{Gauss}}}$.

The topology on $\mathcal{D}_v(0,1)$ is taken to be the Gelfand topology, it is the weakest topology such that for all $f \in \mathcal{A}_v$ and $\alpha \in \mathbb{R}$, the sets

$$U(f,\alpha) = \{x \in \mathcal{D}_v(0,1) : [f]_x < \alpha\}$$

$$V(f,\alpha) = \{x \in \mathcal{D}_v(0,1) : [f]_x > \alpha\}$$

are open. This topology makes $\mathcal{D}_v(0,1)$ into a compact Hausdorff space. For each $E_v \subset \mathcal{D}_v(0,1)$, we call $\overline{E}_v$, the closure of E_v under this topology. The space $\mathcal{D}_v(0,1)$ is connected and path-connected.

Let $a \in D_v(0,1)$, we can define the evaluation seminorm

$$[f]_a = |f(a)|_v$$

It is clear that $[f]_a$ is a m.s, so we can identify elements in $a \in D_v(0,1)$ with elements in $\mathcal{D}_v(0,1)$ through the map $I : D_v(0,1) \to \mathcal{D}_v(0,1)$, $I(a) = [f]_a$. Also, for each subdisc $D_v(a,r) \subset D_v(0,1)$, we have the supremum norm

$$[f]_{D_v(a,r)} = \sup_{z \in D_v(a,r)} |f(z)|_v.$$

Since $|.|_v$ is a non-archimedean absolute value, then this norm is multiplicative. More generally, for any decreasing sequence of discs $x = \{D_v(a_i,r_i)\}_{i\geq 1}$, one can consider the limit seminorm

$$[f]_x = \lim_{i\to\infty} [f]_{D_v(a_i,r_i)}$$

Every $x \in \mathcal{D}_v(0,1)$ can be realized in this form (See [1] Theorem 1.2). Moreover, we have the Berkovich classification theorem, which says that every $x \in \mathcal{D}_v(0,1)$ is one of the following types

Type I : $x = [.]_a$ $a \in D_v(0,1)$ (classical points)

Type II : $x = [.]_{D_v(a,r)}$ $a \in D_v(0,1)$ and $r \in |\mathbb{C}_v^*|$ the value group of $\mathbb{C}_v$ (rational points)

Type III : $x = [.]_{D_v(a,r)}$ $a \in D_v(0,1)$ and $r \notin |\mathbb{C}_v^*|$ (irrational points)

Type IV : x is a point corresponding to the sequence $\{D_v(a_i,r_i)\}_{i\geq 1}$ with empty intersection (necessarily $\lim r_i > 0$).

Points of type I and points of type II are dense in $\mathcal{D}_v(0,1)$ (See [1] Lemma 1.7 and Lemma 1.8).

Let $x, y \in \mathcal{D}_v(0,1)$, we say that $x \preceq y$ if $[f]_x \leq [f]_y$ for all $f \in \mathbb{C}_v\langle t\rangle$. We have that $(\mathcal{D}_v(0,1), \preceq)$ is a partially ordered set and ζ_{Gauss} is the unique maximal point with respect this partial order. Let $D_v(a_i,r_i)$ and $D_v(a'_i,r'_i)$ be two strictly decreasing sequences of discs corresponding to x and y, respectively. Then $x \preceq y$ if and only if for each natural number k there exist $m, n > k$ such that $D_v(a_m,r_m) \subseteq D_v(a'_n,r'_n)$.

Let $(T, \leq)$ be a partially ordered set satisfying the following two axioms:

(P1) T has a unique maximal element ζ called the root of T.

(P2) For each $x \in T$, the set $S_x = \{z \in T : z \geq x\}$ is totally ordered.

We say that T is a parametrized rooted tree if there is a function $\alpha : T \to \mathbb{R}_{\geq 0}$, such that:

(P4) $\alpha(\zeta) = 0$.
(P2) α is order-reversing, in the sense that $x \leq y$ implies $\alpha(x) \geq \alpha(y)$.
(P2) The restriction of α to any full totally ordered subset of T gives a bijection onto a real interval.

Let $\operatorname{diam} : \mathcal{D}_v(0,1) \to \mathbb{R}_{\geq 0}$ be the function defined in the following way, if $x \in \mathcal{D}_v(0,1)$ corresponds to a sequence of nested discs $\{D_v(a_i, r_i)\}$ and $r = \lim_{i\to\infty} r_i$, then $\operatorname{diam}(x) = r$. It can be proved that $(\mathcal{D}_v(0,1), \preceq)$ provided with the function $\alpha : \mathcal{D}_v(0,1) \to \mathbb{R}_{\geq 0}$ given by $\alpha(x) = 1 - \operatorname{diam}(x)$, is a parametrized rooted tree. Moreover, let $x, y \in \mathcal{D}_v(0,1)$, we can define $x \vee y$ to be the unique point belonging to

$$S_x \cap \alpha^{-1}\left(\sup_{z\in S_y\cap S_x} \alpha(z)\right)$$

We have that $x \preceq x \vee y$, $y \preceq x \vee y$, and that if $z \in \mathcal{D}_v(0,1)$ is any point with $x \preceq z$ and $y \preceq z$, then $x \vee y \preceq z$. We call $x \vee y$ the least upper bound of x and y. Now we can define the metric $d : \mathcal{D}_v(0,1) \to \mathbb{R}$

$$d(x, y) = 2\operatorname{diam}(x \vee y) - \operatorname{diam}(x) - \operatorname{diam}(y)$$

2.3 Capacity on the Berkovich unit disc

The metric d defied in the previous section makes $\mathcal{D}_v(0,1)$ into a metric space such that for any two points $x, y \in \mathcal{D}_v(0,1)$ there is a unique arc $[x, y] = \{z \in \mathcal{D}_v(0,1) : x \preceq z \preceq x \vee y\} \cup \{z \in \mathcal{D}_v(0,1) : y \preceq z \preceq x \vee y\}$ in $\mathcal{D}_v(0,1)$ joining x to y, and this arc is a geodesic segment. We also have that $\mathcal{D}_v(0,1)$ is uniquely path-connected. Let $x, y \in \mathcal{D}_v(0,1)$, we define the function $\delta : \mathcal{D}_v(0,1)^2 \to \mathbb{R}_{\geq 0}$ given by

$$\delta(x, y) = \operatorname{diam}(x \vee y).$$

If x corresponds to a sequence of nested discs $\{D_v(a_i, r_i)\}$ and y $\{D_v(b_i, s_i)\}$, then

$$\delta(x, y) = \lim_{n\to\infty} \max(r_i, s_i, |a_i - b_i|).$$

Given a probability measure ν with support contained in $E_v \subset \mathcal{D}_v(0,1)$, define the energy integral

$$I_{E_v}(\nu) = \int\int_{E_v\times E_v} -\log \delta(x, y) d\nu(x) d\nu(y).$$

let ν vary over probability measures with support contained in E, and define the Robin constant

$$V(E_v) = \inf_{\nu} I_{E_v}(\nu)$$

Finally, we define the capacity of E

$$\operatorname{Cap}(E_v) = e^{-V(E_v)}$$

It is well known that $\operatorname{Cap}(\mathcal{D}_v(0,1)) = 1$ and it is achieved when $\nu = \delta_{\zeta_{\text{Gauss}}}$, this is, the linear functional such that for every polynomial f, we have $\nu(f)(\zeta_{\text{Gauss}}) = \sup_{|z|_p\leq 1} |f(z)|_p$.

All previous results are extensible to any disc $\mathcal{D}_v(a, r)$, with $a, b \in \mathbb{R}$, the results are analogous.

3 Upper Bounds

3.1 Main tools

Our purpose is to give a good upper bound for the essential minimum of each element in the family of heights $\{h_{a,b}\}_{a,b\in\overline{\mathbb{Q}}}$. Since the case $a=0$ is trivial, we will assume henceforth $a\neq 0$.

Let p be a prime and let $|.|_p$ be the standard $p-$adic value on $\mathbb{Q}_p$. It is well known that $|.|_p$ can be uniquely extended to $\overline{\mathbb{Q}}_p$. We fix an embedding $\iota_0:\overline{\mathbb{Q}}\to\overline{\mathbb{Q}}_p$, and for $\alpha\in\overline{\mathbb{Q}}$, we define $|\alpha|_p=|\iota_0(\alpha)|_p$. Before proceeding, we need the following lemmas.

Lemma 3.1.1: *Let $\alpha\in\overline{\mathbb{Q}}$, and K number field containing α, which is a Galois extension. Then*

$$h(\alpha)=\frac{1}{[K:\mathbb{Q}]}\left(\sum_{\substack{p\\ \text{prime}}}\sum_{\sigma\in\mathrm{Gal}(K/\mathbb{Q})}\log^+|\sigma(\alpha)|_p+\sum_{\sigma\in\mathrm{Gal}(K/\mathbb{Q})}\log^+|\sigma(\alpha)|_\infty\right).$$

Proof: Using [4], Corollary 1.3.5 and its proof, we have that, for each place $\nu\in M_K$, there exists a $\sigma\in\mathrm{Gal}(K/\mathbb{Q})$ and a place $|.|_w$ of $\mathbb{Q}$, such that $|.|_\nu=|.|_w\circ\sigma$, this proves the lemma. □

Now, we define the function $U_a^b:\overline{\mathbb{Q}}\to\mathbb{R}$, given by

$$U_a^b(\alpha)=\frac{1}{\deg(\alpha)}\sum_{\substack{\beta\in\mathrm{Gal}(\alpha)\\ \sigma\in G(a,b)}}\log^+\left|\beta+\frac{\sigma(b)}{\sigma(a)}\right|.$$

Let $a,b\in\overline{\mathbb{Q}}$, $a\neq 0$, we recall the definition of $\Delta(a,b)$ given in section 1 (3)

$$\Delta(a,b)=\sum_{\substack{p\\ \text{prime}}}\sum_{\sigma\in G(a,b)}\log^+\max(|\sigma(a)|_p,|\sigma(b)|_p)+\sum_{\sigma\in G(a,b)}\log^+|\sigma(a)|.$$

Lemma 3.1.2: *Let $a,b\in\overline{\mathbb{Q}}$ and $\alpha\in\overline{\mathbb{Z}}$, then*

$$h(a\alpha+b)\leq U_a^b(\alpha)+\Delta(a,b).$$

Proof: We consider K the Galois closure of the field generated by a, b and α. It is clear that $K_a\subseteq K_{a,b}\subset K$ and $\deg(\alpha)\leq[K_\alpha:\mathbb{Q}]\leq[K:\mathbb{Q}]$. Then, using Lemma 3.1.1, we have that

$$\begin{aligned}h(a\alpha+b)&=\frac{1}{[K:\mathbb{Q}]}\left(\sum_{\substack{p\\ \text{prime}}}\sum_{\sigma\in\mathrm{Gal}(K/\mathbb{Q})}\log^+|\sigma(a\alpha+b)|_p+\sum_{\sigma\in\mathrm{Gal}(K/\mathbb{Q})}\log^+|\sigma(a\alpha+b)|\right)\\&\leq\frac{1}{[K:\mathbb{Q}]}\left(\sum_{\substack{p\\ \text{prime}}}\sum_{\substack{\delta\in G(\alpha)\\ \sigma\in G(a,b)}}\log^+|\sigma(a)\delta(\alpha)+\sigma(b)|_p\right)+\frac{1}{[K_\alpha:\mathbb{Q}]}\left(\sum_{\substack{\delta\in G(\alpha)\\ \sigma\in G(a,b)}}\log^+|\sigma(a)\delta(\alpha)+\sigma(b)|\right).\end{aligned}$$

In the last inequality we have used that $[K_\alpha:\mathbb{Q}]\leq[K:\mathbb{Q}]$ and that the number of elements in $\mathrm{Gal}(K/\mathbb{Q})$ is less than or equal to the number of pairs (σ,δ), with $\sigma\in G(\alpha)$ and $\delta\in G(a,b)$. Using the fact that $\alpha\in\overline{\mathbb{Z}}$, we conclude that for every p prime, $\sigma\in G(a,b)$ and $\delta\in G(\alpha)$, we have $|\sigma(a)\delta(\alpha)+\sigma(b)|_p\leq\max(|\sigma(a)|_p,|\sigma(b)|_p)$. Therefore

$$
\begin{aligned}
h(a\alpha+b) &\leq \sum_{\substack{p\\ \text{prime}}} \sum_{\sigma\in G(a,b)} \log^+ \max(|\sigma(a)|_p, |\sigma(b)|_p) + \frac{1}{[K_\alpha:\mathbb{Q}]}\left(\sum_{\substack{\delta\in G(\alpha)\\ \sigma\in G(a,b)}} \log^+ |\sigma(a)| + \log^+ \left|\delta(\alpha)+\frac{\sigma(b)}{\sigma(a)}\right| \right)\\
&\leq \Delta(a,b) + \frac{1}{[K_\alpha:\mathbb{Q}]} \sum_{\substack{\delta\in G(\alpha)\\ \sigma\in G(a,b)}} \log^+ \left|\delta(\alpha)+\frac{\sigma(b)}{\sigma(a)}\right|\\
&= \Delta(a,b) + \frac{1}{\deg(\alpha)} \sum_{\substack{\beta\in \mathrm{Gal}(\alpha)\\ \sigma\in G(a,b)}} \log^+ \left|\beta+\frac{\sigma(b)}{\sigma(a)}\right|\\
&= U_a^b(\alpha) + \Delta(a,b).
\end{aligned}
$$

This concludes the proof of the lemma. $\square$

Let $z\in\mathbb{C}$, $E\subset\mathbb{C}$ a compact set, and $r>0$, we define $d(z,E)=\inf_{a\in E}|z-a|$, and $B(E,r)=\{z\in\mathbb{C}: d(z,E)<r\}$.

Let $E=\{a_1,a_2,...,a_k\}\subset\mathbb{C}$ a finite set, we define the probability measure $\delta(E): C_c^0(E,\mathbb{R})\to\mathbb{R}$, by

$$\delta(E)(f)=\frac{1}{k}\sum_{n=1}^{k} f(a_n).$$

Now, we can set the following proposition

Proposition 3.1.3: *Let $E\subset\mathbb{C}$ a compact set with $Cap(E)=1$ and invariant under complex conjugation, then, there exists a sequence of algebraic integers $\{\alpha_n\}_{n\in\mathbb{N}}$, such that $Gal(\alpha_n)\subset B\left(E,\frac{1}{n}\right)$, and*

$$\delta(Gal(\alpha_n)) \xrightarrow{*} \mu_E.$$

Proof: Using [7], Proposition 7.4 and Proposition 7.3, with $E_{|.|_\infty}=E$ and $E_{|.|_p}=\mathcal{O}_p=\{z\in\mathbb{C}_p: |z|_p\leq 1\}$, we conclude the proof. $\square$

Proposition 3.1.4: *Let $a,b\in\overline{\mathbb{Q}}$, and $a\neq 0$. Then, for each $t\in\mathbb{R}$ we have that*

$$\mu^{ess}(h_{a,b})\leq \Omega_{a,b}(t).$$

Here, $\Omega_{a,b}$ is the function defined in (4) given by

$$\Omega_{a,b}(t)=\Delta(a,b)+\varphi(t)\ +\frac{1}{2\pi}\sum_{\sigma\in G(a,b)}\int_0^{2\pi}\log^+\left|e^{i\theta}+\frac{\sigma(b)}{\sigma(a)}+t\right| d\theta.$$

Proof: By Lemma 3.1.2, given $\alpha\in\overline{\mathbb{Z}}$, we have that

$$
\begin{aligned}
h_{a,b}(\alpha) &= h(\alpha)+h(a\alpha+b),\\
&\leq h(\alpha)+U_a^b(\alpha)+\Delta(a,b) =: \eta_{a,b}(\alpha).
\end{aligned}
$$

For Proposition 3.1.3, given $E\subseteq\mathbb{C}$, a compact set with $\mathrm{Cap}(E)=1$ and invariant under complex conjugation, there exists a sequence of algebraic integers $\alpha_n\in\overline{\mathbb{Z}}$, such that $\mathrm{Gal}(\alpha_n)\subset B\left(E,\frac{1}{n}\right)$ and

$$h(\alpha_n) + U_a^b(\alpha_n) \underset{n\to\infty}{\longrightarrow} \int_E \log^+ |x|\, d\mu_E(x) + \sum_{\sigma\in G(a,b)} \int_E \log^+ \left| x + \frac{\sigma(b)}{\sigma(a)} \right| d\mu_E(x) =: M_E.$$

Therefore

$$h_{a,b}(\alpha_n) \le \eta_{a,b}(\alpha_n) \underset{n\to\infty}{\longrightarrow} \Delta(a,b) + M_E =: J_E.$$

We claim that $J_E < \infty$. In fact, we have that the only possible unbounded term in the definition could be M_E. Let the functions, $f : \mathbb{R} \to \mathbb{R}$, and for each $\sigma \in G(a,b)$ the function $g_\sigma : \mathbb{R} \to \mathbb{R}$, be defined by $f(t) = \log^+ |t|$ and $g_\sigma(t) = \log^+ |t + \sigma(b)/\sigma(a)|$. These functions are continuous and E is a compact set, therefore, $M_E < \infty$. Since $J_E < \infty$, the sequence $\{h(\alpha_n)\}_{n\in\mathbb{N}}$ is bounded, we conclude that there is a subsequence which is convergent, we call it $\{\beta_n\}_{n\in\mathbb{N}}$.

If $h_{a,b}(\beta_n) \underset{n\to\infty}{\longrightarrow} Z$, then, by definition of limit, given $\varepsilon > 0$ the set $\{\beta_n \in \overline{\mathbb{Z}} : h_{a,b}(\beta_n) \le Z + \varepsilon\}$ is infinite, therefore for every $\varepsilon > 0$ we have $\mu^{ess}(h_{a,b}) \le Z + \varepsilon$. Taking $\varepsilon \to 0$ we get $\mu^{ess}(h_{a,b}) \le Z$. Since $Z \le J_E$, we conclude that $\mu^{ess}(h_{a,b}) \le J_E$. Given $t \in \mathbb{R}$ we use $E = S_t = S_1 + t$, where $S_1 = \{z \in \mathbb{C} : |z| = 1\}$. Then, μ_{S_t} is the natural translation of the measure $\mu_{S_1} = \frac{d\theta}{2\pi}$. We deduce that

$$\mu^{ess}(h_{a,b}) \le J_{S_t} = \Omega_{a,b}(t).$$

This concludes the proof of the proposition. □

Let $a, b \in \mathbb{Q}$, $a \neq 0$, then

$$\Omega_{a,b}(t) = \Delta(a,b) + \varphi(t) + \varphi(t + b/a). \tag{8}$$

We want to find a point for which $\Omega_{a,b}$ achieves its minimum value and compute a power series for $\Omega_{a,b}$ at that point. Before proceeding, we will prove the following two lemmas

Lemma 3.1.5: *The function φ satisfies the following properties*

i) *For each $t \in \mathbb{R}$, $\varphi(t) = \varphi(-t)$,*

ii) *for $|t| \ge 2$, $\varphi(t) = \log|t|$.*

Proof: Let $t \in \mathbb{R}$, we have

$$\begin{aligned}
\varphi(-t) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|e^{i\theta} - t\right| d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|e^{i(\theta+\pi)} + t\right| d\theta \\
&= \frac{1}{2\pi}\int_\pi^{3\pi} \log^+ \left|e^{i\theta} + t\right| d\theta = \frac{1}{2\pi}\left(\int_\pi^{2\pi} \log^+ \left|e^{i\theta} + t\right| d\theta + \int_{2\pi}^{3\pi} \log^+ \left|e^{i\theta} + t\right| d\theta\right) \\
&= \frac{1}{2\pi}\left(\int_\pi^{2\pi} \log^+ \left|e^{i\theta} + t\right| d\theta + \int_0^{\pi} \log^+ \left|e^{i\theta} + t\right| d\theta\right) \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|e^{i\theta} + t\right| d\theta = \varphi(t).
\end{aligned}$$

This proves (i). Now, if $|t| \ge 2$, then for each $\theta \in \mathbb{R}$, $|e^{i\theta} + t| > |1 - |t|| = |t| - 1 \ge 1$, hence, $\log^+ |e^{i\theta} + t| = \log |e^{i\theta} + t|$, we conclude that

$$
\begin{aligned}
\varphi(-t) &= \frac{1}{2\pi}\int_0^{2\pi} \log\left|t - e^{i\theta}\right| d\theta \\
&= \frac{1}{2\pi} Re\left(\int_0^{2\pi} \log\left(t - e^{i\theta}\right) d\theta\right) \\
&= \frac{1}{2\pi} Re\left(\int_{S_1} \frac{1}{iz} \log\left(t - z\right) dz\right) \\
&= \frac{1}{2\pi} Re\left(2\pi i \frac{1}{i} \log\left(t\right)\right) \\
&= \log|t|
\end{aligned}
$$

This proves (ii). $\square$

Lemma 3.1.6: *Let* $f : \overline{\mathbb{Q}} \to \mathbb{R}$, $\gamma, \delta \in \overline{\mathbb{Q}}$ *and* $\gamma \neq 0$, *define* $f^{\gamma,\delta} : \overline{\mathbb{Q}} \to \mathbb{R}$ *by* $f^{\gamma,\delta}(\alpha) = f(\gamma\alpha + \delta)$, *then*

$$\mu^{ess}(f) = \mu^{ess}(f^{\gamma,\delta}).$$

Proof: In fact, if $\mu^{ess}(f) = A$, then there exist a sequence $\{\alpha_n\}_{n\in\mathbb{N}} \subseteq \overline{\mathbb{Q}}$ such that $f(\alpha_n) \to A$. Let $\beta_n = (\alpha_n - \delta)/\gamma \in \overline{\mathbb{Q}}$, then $f^{\gamma,\delta}(\beta_n) = f(\alpha_n) \to A$, therefore $\mu^{ess}(f^{\gamma,\delta}) \leq \mu^{ess}(f)$. Using the fact that $1/\gamma, -\delta \in \overline{\mathbb{Q}}^*$ and the same argument, we can show the other inequality and the lemma is proved. $\square$

Corollary 3.1.7: *Let* $a, b \in \mathbb{Q}$, *then*

$$\mu^{ess}(h_{a,b}) = \mu^{ess}(h_{|a|,|b|}).$$

Proof: We have that $h_{-a,b}(\alpha) = h(\alpha) + h(-a\alpha + b) = h(-\alpha) + h(a(-\alpha) + b) = h_{a,b}(-\alpha)$. Using Lemma 3.1.6 with $\gamma = -1$ and $\delta = 0$, we conclude that $\mu^{ess}(h_{-a,b}) = \mu^{ess}(h_{a,b})$. We also note that $h_{a,-b}(\alpha) = h(\alpha) + h(a\alpha - b) = h(-\alpha) + h(-a\alpha + b) = h_{-a,b}(\alpha) = h_{a,b}(-\alpha)$. Hence, we have $\mu^{ess}(h_{-a,b}) = \mu^{ess}(h_{a,-b}) = \mu^{ess}(h_{a,b})$. Combining this two equalities we can conclude all the others, this completes the proof of the corollary. $\square$

3.2 Explicit upper bounds

Let $c \in \mathbb{R}$, we define the function $\varepsilon_c(n,t) : \mathbb{N} \times \mathbb{R} \to \mathbb{C}$, given by

$$\varepsilon_c(n,t) = \int_c^{\pi} \left(e^{i\theta} - e^{it}\right)^n d\theta\ ;$$

Expanding the binomial and integrating, we get

$$\varepsilon_c(n,t) = (-1)^n(\pi - c)e^{itn} + \sum_{k=0}^{n-1} \frac{1}{i(n-k)}\binom{n}{k}(-1)^k(e^{i\pi(n-k)+itk} - e^{ic(n-k)+itk})\ .$$

Using this notation, we have the following proposition.

Proposition 3.2.1: Let $a, b \in \mathbb{Q}$ with $a > 0$ and $b \geq 0$.

i) *If* $b = 0$, *then*

$$\Omega_{a,0}(0) = h(a).$$

ii) *If* $0 < b/a < 4$, *then*

$$\Omega_{a,b}\left(-\frac{b}{2a}\right) = \Delta(a,b) + \frac{2}{\pi} Re\left(\log\left(\frac{b}{2a} - e^{i\frac{\pi+\alpha_{a,b}}{2}}\right)(\pi - \alpha_{a,b}) - \sum_{n=1}^{\infty} \frac{(2a)^n}{n\left(b - 2ae^{i\frac{\pi+\alpha_{a,b}}{2}}\right)^n} \varepsilon_{\alpha_{a,b}}\left(n, \frac{\pi+\alpha_{a,b}}{2}\right)\right).$$

Where $\alpha_{a,b} = \arctan\left(\frac{\sqrt{16a^2-b^2}}{b}\right)$ *and* $\log$ *is the main branch of the logarithm.*

iii) *If* $b/a \geq 4$, *then*

$$\Omega_{a,b}(0) = \Delta(a,b) + \log\left(\frac{b}{a}\right).$$

Proof: If $b = 0$, using (8) and $\varphi(0)$=0, we get that $\Omega_{a,b}(0) = h(a)$. Now, assume $0 < b/a < 4$. Using (4) and Lemma 3.1.5 (i), we have

$$\Omega_{a,b}\left(-\frac{b}{2a}\right) = \Delta(a,b) + 2\varphi\left(-\frac{b}{2a}\right) = \Delta(a,b) + \frac{1}{\pi}\int_0^{2\pi} \log^+\left|\frac{b}{2a} - e^{i\theta}\right| d\theta$$

Let $\alpha_{a,b}$ be the argument of the complex number given by the intersection of the two circumferences $S_{\frac{b}{2a}}$ and S_1 in the first quadrant. Then, we have that $\log^+\left|\frac{b}{2a} - e^{i\theta}\right| > 0$ if and only if $\alpha_{a,b} < \theta < 2\pi - \alpha_{a,b}$. Since the circle is symmetric we have that

$$\Omega_{a,b}\left(-\frac{b}{2a}\right) = \Delta(a,b) + \frac{1}{\pi}\int_{\alpha_{a,b}}^{2\pi-\alpha_{a,b}} \log\left|\frac{b}{2a} - e^{i\theta}\right| d\theta = \Delta(a,b) + \frac{2}{\pi}\int_{\alpha_{a,b}}^{\pi} \log\left|\frac{b}{2a} - e^{i\theta}\right| d\theta. \quad (9)$$

The intersection between the circles in the first quadrant occur when $x^2+y^2 = 1$, $(x-b/a)^2+y^2 = 1$, $x > 0$ and $y > 0$. Solving these two equations, we obtain $x = b/4a$ and $y = \sqrt{16a^2 - b^2}/4a$. Therefore $\alpha_{a,b} = \arctan(x/y) = \arctan\left(\frac{\sqrt{16a^2-b^2}}{b}\right)$.

Let $f : \mathbb{C} \setminus \left[\frac{b}{2a}, +\infty\right) \to \mathbb{C}$, defined by $f(z) = \log\left(\frac{b}{2a} - z\right)$, where log is the main branch of the logarithm, then the power series of f around $z = e^{i\frac{\pi+\alpha_{a,b}}{2}}$, is given by

$$f(z) = \log\left(\frac{b}{2a} - e^{i\frac{\pi+\alpha_{a,b}}{2}}\right) + \sum_{n=1}^{\infty} \frac{-(2a)^n}{n\left(b - 2ae^{i\frac{\pi+\alpha_{a,b}}{2}}\right)^n}\left(z - e^{i\frac{\pi+\alpha_{a,b}}{2}}\right)^n.$$

The convergence radius of this series is

$$\frac{1}{r} = \lim_{n\to\infty}\left|\frac{2na}{(n+1)\left(b - 2ae^{i\frac{\pi+\alpha_{a,b}}{2}}\right)}\right| = \frac{2a}{\left|b - 2ae^{i\frac{\pi+\alpha_{a,b}}{2}}\right|},$$

$$r = \left|\frac{b}{2a} - e^{i\frac{\pi+\alpha_{a,b}}{2}}\right|.$$

On the other hand, we have

$$\int_{\alpha_{a,b}}^{\pi} \log\left|\frac{b}{2a} - e^{i\theta}\right| d\theta = Re\left(\int_{\alpha_{a,b}}^{\pi} \log\left(\frac{b}{2a} - e^{i\theta}\right) d\theta\right)$$

$$= Re\left(\int_{\alpha_{a,b}}^{\pi} \log\left(\frac{b}{2a} - e^{i\frac{\pi+\alpha_{a,b}}{2}}\right) + \sum_{n=1}^{\infty} \frac{-(2a)^n}{n\left(b - 2ae^{i\frac{\pi+\alpha_b}{2}}\right)^n}\left(e^{i\theta} - e^{i\frac{\pi+\alpha_{a,b}}{2}}\right)^n d\theta\right).$$

The maximum value of $\left|e^{i\theta} - e^{i\frac{\pi+\alpha_{a,b}}{2}}\right|$ for $\theta \in [\alpha_{a,b}, \pi]$ is achieved when $\theta = \pi$, therefore for $\theta \in [\alpha_{a,b}, \pi]$, we have

$$\left|e^{i\theta} - e^{i\frac{\pi+\alpha_{a,b}}{2}}\right| \leq \left|1 + e^{i\frac{\pi+\alpha_{a,b}}{2}}\right|.$$

Note that $0 < \alpha_{a,b} \leq \pi/2$, therefore $\pi/2 < (\pi + \alpha_{a,b})/2 \leq 3\pi/4$, it follows that

$$-\frac{\sqrt{2}}{2} \leq \cos\left(\frac{\pi+\alpha_{a,b}}{2}\right) < 0.$$

From this last equation, we conclude that

$$\left|e^{i\theta} - e^{i\frac{\pi+\alpha_{a,b}}{2}}\right| \leq \left|1 + e^{i\frac{\pi+\alpha_{a,b}}{2}}\right| < \left|\frac{b}{2a} - e^{i\frac{\pi+\alpha_{a,b}}{2}}\right| = r. \tag{10}$$

Since the convergence is uniform we can exchange the integral with the series and we get obtain

$$\Omega_{a,b}\left(-\frac{b}{2a}\right) = \Delta(a,b) + \frac{2}{\pi}Re\left(\log\left(\frac{b}{2a} - e^{i\frac{\pi+\alpha_{a,b}}{2}}\right)(\pi - \alpha_{a,b}) - \sum_{n=1}^{\infty}\frac{(2a)^n}{n\left(b - 2ae^{i\frac{\pi+\alpha_{a,b}}{2}}\right)^n}\varepsilon_{\alpha_{a,b}}\left(n, \frac{\pi+\alpha_{a,b}}{2}\right)\right).$$

On the other hand, if $|b/a| \geq 4$, then $|b/a| > 2$, using (ii) from lemma 3.1.5, we conclude that, $\Omega_{a,b}(0) = \Delta(a,b) + \varphi(0) + \varphi(b/a) = \Delta(a,b) + \log(b/a)$, this completes the proof of the proposition.
□

Using Proposition 3.1.4 and Proposition 3.2.1, we can prove the following theorem

Theorem 3.2.2: *Let $a, b \in \mathbb{Q}$. If $b = 0$, we have that*

$$\mu^{ess}(h_{a,0}) \leq \Omega_{|a|,0}(0) = h(a).$$

For $|b/a| = 1$, we have that

$$\mu^{ess}(h_{a,b}) \leq \Omega_{|a|,|b|}\left(-\tfrac{1}{2}\right) \leq \Delta(a,b) + 0.3194490869562.$$

For $|b/a| = 2$,

$$\mu^{ess}(h_{a,b}) \leq \Omega_{|a|,|b|}(-1) \leq \Delta(a,b) + 0.6461598436469.$$

For $|b/a| = 3$,

$$\mu^{ess}(h_{a,b}) \leq \Omega_{|a|,|b|}\left(-\tfrac{3}{2}\right) \leq \Delta(a,b) + 0.9909205628144.$$

For $|b/a| \geq 4$,

$$\mu^{ess}(h_{a,b}) \leq \Omega_{|a|,|b|}(0) = \Delta(a,b) + \log\left(\frac{b}{a}\right).$$

Proof: Using Corollary 3.1.7, we can assume that $a > 0$ and $b \geq 0$. Assume first that, $b/a = 1$. Let

$$\begin{aligned}T_1 &:= \frac{2}{\pi}Re\left(\log\left(\frac{1}{2} - e^{i\frac{\pi+\alpha_{a,b}}{2}}\right)(\pi - \alpha_{a,b}) - \sum_{n=1}^{20}\frac{2^n}{n\left(1 - 2e^{i\frac{\pi+\alpha_{a,b}}{2}}\right)^n}\varepsilon_{\alpha_{a,b}}\left(n, \frac{\pi+\alpha_{a,b}}{2}\right)\right)\\ &= 0.3194345111561... \leq 0.3194345111562.\end{aligned}$$

Let $R_1 = \Omega_{a,b}\left(-\frac{1}{2}\right) - T_1 - \Delta(a,b)$. Using Proposition 3.2.1, we have that

$$|R_1| \leq \frac{2}{\pi} \sum_{n=21}^{\infty} \frac{2^n}{n\left|1-2e^{i\frac{\pi+\alpha_{a,b}}{2}}\right|^n} \int_{\alpha_{a,b}}^{\pi} \left|e^{i\theta} - e^{i\frac{\pi+\alpha_{a,b}}{2}}\right|^n d\theta$$

Note that, since $b/a = 1$, we have $\alpha_{a,b} = \arctan(\sqrt{15})$, therefore $\left|1-2e^{i\frac{\pi+\alpha_{a,b}}{2}}\right| = \sqrt{5+\sqrt{6}}$. Moreover, using (10), we conclude that, for each $\theta \in [\alpha_{a,b}, \pi]$, $\left|e^{i\theta} - e^{i\frac{\pi+\alpha_{a,b}}{2}}\right| \leq \sqrt{2-\sqrt{\frac{3}{2}}}$, therefore

$$\begin{aligned}
|R_1| &\leq \frac{2}{\pi} \sum_{n=21}^{\infty} \frac{2^n}{n\left(\sqrt{5+\sqrt{6}}\right)^n} \left(\sqrt{2-\sqrt{\frac{3}{2}}}\right)^n (\pi - \alpha_{a,b}) \\
&\leq \frac{2(\pi-\alpha_{a,b})}{\pi} \sum_{n=21}^{\infty} \frac{1}{n} \left(\frac{2}{\sqrt{5+\sqrt{6}}}\left(\sqrt{2-\sqrt{\frac{3}{2}}}\right)\right)^n
\end{aligned}$$

Using the fact that, for $|a| < 1$, $\sum_{n=1}^{\infty} \frac{a^n}{n} = -\log(1-a)$, we obtain

$$\begin{aligned}
|R_1| &\leq \frac{2(\pi-\alpha_{a,b})}{\pi} \left(-\log\left(1-\left(\frac{2}{\sqrt{5+\sqrt{6}}}\left(\sqrt{2-\sqrt{\frac{3}{2}}}\right)\right)\right) - \sum_{n=1}^{20} \frac{1}{n}\left(\frac{2}{\sqrt{5+\sqrt{6}}}\left(\sqrt{2-\sqrt{\frac{3}{2}}}\right)\right)^n\right) \\
&= 0.0000145757... \\
&\leq 0.0000145758.
\end{aligned}$$

Therefore, since $\Omega_{a,b}(-1/2) \geq 0$, we have

$$\begin{aligned}
\Omega_{a,b}\left(-\frac{1}{2}\right) &= \left|\Omega_{a,b}\left(-\frac{1}{2}\right)\right| = |\Delta(a,b) + T_1 + R_1| \\
&\leq \Delta(a,b) + |T_1| + |R_1| = \Delta(a,b) + 0.3194490869562.
\end{aligned}$$

Taking $t = -1/2$ in Proposition 3.1.4, we conclude the case $|b/a| = 1$. For $|b/a| = 2$, we only need to expand the series given by Proposition 3.2.1 until 15 terms, then, taking $t = -1$ in Proposition 3.1.4, we obtain the upper bound required. For $|b/a| = 3$, the process is exactly the same but we only need to expand the series until 7 terms and take $t = -3/2$ in Proposition 3.1.4. Finally, the cases $|b/a| \geq 4$ and $b = 0$ follow directly from Proposition 3.2.1 This completes the proof of the theorem. $\square$

3.3 The case $\mathbb{Q}[i]$

Theorem 3.2.2 gives us rigorous upper bounds for specific values of $|b/a|$. For instance, if we take $a = 7/15$, $b = 250/36$, then $b/a > 4$. Therefore

$$\mu^{ess}(h_{a,b}) \leq \log 5 + \log 9 + \log 4 + \log(7/15) + \log(3750/252) = \log(1250).$$

If we take the Zhang-Zagier height $h_Z = h_{1,-1}$, we have $|b/a| = 1$, therefore, Theorem 3.8 gives us the upper bound

$$\mu^{ess}(h_Z) \leq 0.31944909.$$

Other cases like $|b/a| = 1/2$ must be treated individually using Proposition 3.2.1

Assume now that $a, b \in \mathbb{Q}[i] \setminus \mathbb{Q}$, we can prove the following theorem

Theorem 3.3.1: *Let $a, b \in \mathbb{Q}[i] \setminus \mathbb{Q}$, then*

$$\mu^{ess}(h_{a,b}) \leq \Delta(a,b) + \varphi\left(\mathrm{Re}\left(\frac{b}{a}\right)\right) + 2\varphi\left(\mathrm{Im}\left(\frac{b}{a}\right)\right).$$

Proof: Using Proposition 3.1.4, we obtain that $\mu^{ess}(h_{a,b})$ is less or equal than

$$\Omega_{a,b}(t) = \Delta(a,b) + \varphi(t) + \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|e^{i\theta} + \frac{b}{a} + t\right| + \log^+\left|e^{i\theta} + \frac{\overline{b}}{\overline{a}} + t\right| d\theta.$$

Since

$$\begin{aligned}\int_0^{2\pi} \log^+\left|e^{i\theta} + \frac{\overline{b}}{\overline{a}} + t\right| d\theta &= \int_0^{2\pi} \log^+\left|\overline{e^{-i\theta} + \frac{b}{a} + t}\right| d\theta = \int_0^{2\pi} \log^+\left|e^{-i\theta} + \frac{b}{a} + t\right| d\theta \\ &= \int_{-2\pi}^{0} \log^+\left|e^{i\theta} + \frac{b}{a} + t\right| d\theta = \int_0^{2\pi} \log^+\left|e^{i\theta} + \frac{b}{a} + t\right| d\theta.\end{aligned}$$

We conclude that

$$\Omega_{a,b}(t) = \Delta(a,b) + \varphi(t) + 2\left(\frac{1}{2\pi}\int_0^{2\pi} \log^+\left|e^{i\theta} + \frac{b}{a} + t\right| d\theta\right).$$

Now, we evaluate the function at $t = -\mathrm{Re}(b/a)$

$$\Omega_{a,b}\left(-\mathrm{Re}\left(\frac{b}{a}\right)\right) = 2\log|a| + \varphi\left(-\mathrm{Re}\left(\frac{b}{a}\right)\right) + 2\left(\frac{1}{2\pi}\int_0^{2\pi} \log^+\left|e^{i\theta} + i\mathrm{Im}\left(\frac{b}{a}\right)\right| d\theta\right).$$

Note that

$$\begin{aligned}\int_0^{2\pi} \log^+\left|e^{i\theta} + i\mathrm{Im}\left(\frac{b}{a}\right)\right| d\theta &= \int_0^{2\pi} \log^+\left|e^{i(\theta+\frac{3\pi}{2})} + \mathrm{Im}\left(\frac{b}{a}\right)\right| d\theta = \int_{3\pi/2}^{2\pi+3\pi/2} \log^+\left|e^{i\theta} + \mathrm{Im}\left(\frac{b}{a}\right)\right| d\theta \\ &= \int_{3\pi/2}^{2\pi} \log^+\left|e^{i\theta} + \mathrm{Im}\left(\frac{b}{a}\right)\right| d\theta + \int_{2\pi}^{2\pi+3\pi/2} \log^+\left|e^{i\theta} + \mathrm{Im}\left(\frac{b}{a}\right)\right| d\theta \\ &= \int_{3\pi/2}^{2\pi} \log^+\left|e^{i\theta} + \mathrm{Im}\left(\frac{b}{a}\right)\right| d\theta + \int_0^{3\pi/2} \log^+\left|e^{i\theta} + i\mathrm{Im}\left(\frac{b}{a}\right)\right| d\theta = 2\pi\varphi\left(\mathrm{Im}\left(\frac{b}{a}\right)\right).\end{aligned}$$

Finally, using Lemma 3.1.5, we conclude the theorem. $\square$

Remark: *The specific values of t used in the various applications of Proposition 3.1.4, were suggested by numerical experiments.*

4 Lower Bounds

4.1 The method to obtain lower bounds

In this section we will compute lower bounds for $\mu^{ess}(h_{a,b})$ for $a,b \in \overline{\mathbb{Q}}$ and $a \neq 0$. We use the method described in [6] section 2.2, [3] and [18]. For every $\sigma \in G(a,b)$, we consider the real-valued functions $g_\sigma, f_\sigma, G_\sigma$, given by

$$\begin{aligned} g_\sigma(z) &= \log^+|z| + \log^+|\sigma(a)z+\sigma(b)|, \\ f_\sigma(z) &= \log^+|z| + \log^+\left|\frac{1}{\sigma(a)z+\sigma(b)}\right|, \\ G_\sigma(z) &= \log^+|z| + \log^+\left|\frac{\sigma(a)+\sigma(b)z}{z}\right|. \end{aligned}$$

We have that these functions go to ∞ when $|z| \to \infty$. Furthermore, $f_\sigma \to \infty$ when $z \to -\sigma(b)/\sigma(a)$ and $G_\sigma \to \infty$ when $z \to 0$, and they are continuous elsewhere, so they attain their minimum values. We denote by $\min(g_\sigma), \min(f_\sigma)$ and $\min(G_\sigma)$, the minimum value of each of these functions respectively. Then, we define $g^{\min} = \sum_{\sigma\in G(a,b)} \min(g_\sigma)$, $f^{\min} = \sum_{\sigma\in G(a,b)} \min(f_\sigma)$ and $G^{\min} = \sum_{\sigma\in G(a,b)} \min(G_\sigma)$. Finally, we define

$$\mathcal{L}(a,b) = \frac{1}{[K_{a,b}:\mathbb{Q}]} \max\{g^{\min}, f^{\min}, G^{\min}\}.$$

Assume now that the minimum value of $h_{a,b}$ is achieved only at a finite non empty set of algebraic numbers. Let $\alpha_1, \alpha_2, ..., \alpha_k$ be the algebraic numbers where $h_{a,b}$ is equal to the minimum. We consider $\{f_1, f_2, ..., f_r\}$ a set monic irreducible polynomials with rational coefficients, such that their combined roots are $\{\alpha_1, \alpha_2, ..., \alpha_k\}$. Now, let $A_1, A_2, ..., A_r \in \mathbb{R}_{\geq 0}$ be such that, $A_1 \deg(f_1) A_2 \deg(f_2) ... A_r \deg(f_r) < 2$ and for every $\alpha \in \overline{\mathbb{Q}} \backslash \bigcup_{i=1}^k \mathrm{Gal}(\alpha_i)$, and every non-archimedean place ν in $M_{\overline{\mathbb{Q}}}$, we have

$$\log^+|\alpha|_\nu + \log^+|a\alpha+b|_\nu \geq \sum_{i=1}^r A_i \log|f_i(\alpha)|_\nu. \tag{11}$$

We call P the set of all $(A_1, A_2, ..., A_r) \in \mathbb{R}^r$, such that these conditions hold. Since $(0,0,0,...,0) \in P$, we have that $P \neq \emptyset$.

Lemma 4.1.1: *The set P defined before is bounded*

Proof: Firstly, we have that $A_i \geq 0$, therefore each A_i is bounded from below. We fix $\nu_0 \in M_{\overline{\mathbb{Q}}}$ and consider $\alpha \in \overline{\mathbb{Q}} \setminus \bigcup_{i=1}^k \mathrm{Gal}(\alpha_i)$, such that, $\log^+|a\alpha+b|_{\nu_0} = \log|a\alpha|_{\nu_0}$ and for each $1 \leq i \leq r$, $|f_i(\alpha)|_{\nu_0} = |\alpha^{\deg(f_i)}|_{\nu_0} \geq 1$. Therefore, from (11) we have $\log|a\alpha^2|_{\nu_0} \geq \sum_{i=1}^r A_i \log|\alpha^{\deg(f_i)}|_{\nu_0} \geq A_t \log|\alpha^{\deg(f_t)}|_{\nu_0} = \log|\alpha^{\deg(f_t)A_t}|_{\nu_0}$, for each $t \in \{1,2,...,r\}$. For $|\alpha|_{\nu_0}$ large enough, this equality holds only if $A_t \leq 2/\deg(f_t)$. This completes the proof of the lemma. $\square$

Now, for each $\sigma \in G(a,b)$ we define the real valued function $g_{A_1,...,A_r,\sigma}$ by

$$g_{A_1,...,A_r,\sigma}(z) = \log^+|z| + \log^+|\sigma(a)z+\sigma(b)| - \sum_{i=1}^r A_i \log|f_i(z)|.$$

Since $A_i \geq 0$ for each $i \in \{1,2,...,r\}$, and $A_1 \deg(f_1) A_2 \deg(f_2) ... A_r \deg(f_r) < 2$, we have that $g_{A_1,...,A_k,\sigma} \to \infty$ when $|z| \to \infty$ or $z \to x$, where $x \in \bigcup_{i=1}^k \mathrm{Gal}(\alpha_i)$, and it is continuous elsewhere, so it attains its minimum value. Now, we consider the function $H_\sigma : P \to \mathbb{R}$, given by

$$H_\sigma(A_1, A_2, ..., A_r) = \inf g_{A_1,...,A_r,\sigma}.$$

We define

$$\tau(a,b) = \frac{1}{[K_{a,b}:\mathbb{Q}]} \sup_{(A_1,...,A_r)\in P} \sum_{\sigma\in G(a,b)} H_\sigma(A_1, ..., A_r).$$

Definition 4.1.2: *Let $a, b \in \overline{\mathbb{Q}}$. If the minimum value of $h_{a,b}$ is achieved only at a finite set of algebraic numbers, we define $\mathcal{K}(a,b) = \tau(a,b)$. Otherwise $\mathcal{K}(a,b) = \mathcal{L}(a,b)$*

Now, we can state the following theorem.

Theorem 4.1.3: *Let $a, b \in \overline{\mathbb{Q}}$ with $a \neq 0$. Then,*

$$\mu^{ess}(h_{a,b}) \geq \mathcal{L}(a,b).$$

Moreover, if the minimum value is achieved only at a finite non empty set of algebraic numbers, then

$$\mu^{ess}(h_{a,b}) \geq \tau(a,b).$$

Before proving the theorem we need the following lemma

Lemma 4.1.4: *Let $f : \overline{\mathbb{Q}} \to \mathbb{R}$. Define $f^{in} : \overline{\mathbb{Q}}^* \to \mathbb{R}$ by $f^{in}(\alpha) = f(1/\alpha)$. Then*

$$\mu^{ess}(f) = \mu^{ess}(f^{in}).$$

Proof: We write, $\mu^{ess}(f) = M$. Then, there exists a sequence of distinct algebraic numbers $\{\gamma_n\}_{n\in\mathbb{N}}$ such that $f(\gamma_n) \to M$. In particular, we may assume $\gamma_n \neq 0$ for all $n \in \mathbb{N}$. Then $\{\beta_n\}_{n\in\mathbb{N}}$ given by $\beta_n = 1/\gamma_n$ is a sequence of algebraic numbers, and $f^{in}(\beta_n) = f(\gamma_n) \to M$, therefore $\mu^{ess}(f^{in}) \leq M$. The other inequality is similar. □

Proof of Theorem 4.1.3: Let $K/\mathbb{Q}$ be a Galois extension such that $a, b, \alpha \in K$. If $\nu \in M_K$ extends $v \in M_\mathbb{Q}$ we will write $\nu|v$. Note that

$$\begin{aligned}
h_{a,b}(\alpha) &= \frac{1}{[K:\mathbb{Q}]}\left(\sum_p \sum_{\nu||.|_p} \log^+|\alpha|_\nu + \log^+|a\alpha+b|_\nu + \sum_{\nu||.|} \log^+|\alpha|_\nu + \log^+|a\alpha+b|_\nu\right) \\
&\geq \frac{1}{[K:\mathbb{Q}]} \sum_{\nu||.|} \log^+|\alpha|_\nu + \log^+|a\alpha+b|_\nu \\
&= \frac{1}{[K:\mathbb{Q}]} \sum_{\sigma\in G(a,b)} \sum_{\substack{\tau\in\mathrm{Gal}(K\backslash\mathbb{Q}) \\ \tau|\sigma}} \log^+|\tau(\alpha)| + \log^+|\sigma(a)\tau(\alpha)+\sigma(b)| \\
&\geq \frac{1}{[K_{a,b}:\mathbb{Q}]} \sum_{\sigma\in G(a,b)} \inf g_\sigma \\
&= g^{\min}.
\end{aligned}$$

Therefore, $g^{\min} \leq h_{a,b}(\alpha)$ for all α, we conclude that $g^{\min} \leq \mu^{ess}(h_{a,b})$. Similarly, we can conclude that $f^{\min} \leq \mu^{ess}(h_{a,b})$. In fact, we can consider the real-valued function $j_{a,b}$, given by $j_{a,b}(\alpha) = h(\alpha) + h(1/(a\alpha+b))$. Since $h(\alpha) = h(1/\alpha)$, we have $j_{a,b} = h_{a,b}$. However, the Archimedean parts of them are different, so, we can use the same inequalities as before and conclude that $f^{\min} \leq \mu^{ess}(j_{a,b}) = \mu^{ess}(h_{a,b})$. Finally, let the real valued function $n_{a,b}$, be given by $n_{a,b}(\alpha) = h(\alpha) + h((a+b\alpha)/\alpha) = h^{in}_{a,b}$. Using Lemma 4.1.4 and the same method used before, we conclude that $G^{\min} \leq \mu^{ess}(n_{a,b}) = \mu^{ess}(h^{in}_{a,b}) = \mu^{ess}(h_{a,b})$. Therefore, $\mathcal{L}(a,b) \leq \mu^{ess}(h_{a,b})$.

Assume now that the minimum value of $h_{a,b}$ is achieved only at a finite set $\{\alpha_1, \alpha_2, .., \alpha_k\}$. Let $\alpha \in \overline{\mathbb{Q}} \setminus \bigcup_{i=1}^{k} \mathrm{Gal}(\alpha_i)$ and $A_1, A_2, ..., A_r \in P$. The product formula gives us that for each $i \in \{1, 2, ..., r\}$, we have

$$\sum_{\nu \in M_K} A_i \log |f_i(\alpha)|_\nu = 0$$

Therefore

$$\begin{aligned}
h_{a,b}(\alpha) &= \frac{1}{[K:\mathbb{Q}]} \left(\sum_{p} \sum_{\nu || . |_p} \log^+ |\alpha|_\nu + \log^+ |a\alpha + b|_\nu - \sum_{i=1}^{r} A_i \log |f_i(\alpha)|_\nu \right) \\
&\quad + \frac{1}{[K:\mathbb{Q}]} \left(\sum_{\nu || . |} \log^+ |\alpha|_\nu + \log^+ |a\alpha + b|_\nu - \sum_{i=1}^{r} A_i \log |f_i(\alpha)|_\nu \right) \\
&\geq \frac{1}{[K:\mathbb{Q}]} \left(\sum_{\nu || . |} \log^+ |\alpha|_\nu + \log^+ |a\alpha + b|_\nu - \sum_{i=1}^{r} A_i \log |f_i(\alpha)|_\nu \right) \\
&= \frac{1}{[K:\mathbb{Q}]} \sum_{\sigma \in G(a,b)} \sum_{\substack{\tau \in \mathrm{Gal}(K/\mathbb{Q}) \\ \tau | \sigma}} g_{A_1, A_2, ..., A_r, \sigma}(\tau(\alpha)) \\
&\geq \frac{1}{[K_{a,b}:\mathbb{Q}]} \sum_{\sigma \in G(a,b)} H_\sigma(A_1, A_2, ..., A_r).
\end{aligned}$$

Since the last inequality holds for all $(A_1, A_2, ..., A_r) \in P$, we have that

$$\begin{aligned}
h_{a,b}(\alpha) &\geq \frac{1}{[K_{a,b}:\mathbb{Q}]} \sup_{(A_1, A_2, ..., A_k) \in P} \sum_{\sigma \in G(a,b)} H_\sigma(A_1, A_2, ..., A_r) \\
&= \tau(a,b).
\end{aligned}$$

Since the last inequality holds for all α except finitely many, we conclude that $\tau(a,b) \leq \mu^{ess}(h_{a,b})$. □

Corollary 4.1.5: *Let $a, b \in \overline{\mathbb{Q}}$ with $a \neq 0$. Then,*

$$\mu^{ess}(h_{a,b}) \geq \mathcal{K}(a,b).$$

Proof: This is a direct consequence of Theorem 4.1.4. □

4.2 Specific cases

The following observation will be useful to compute lower bounds. Since $g_{A_1,...,A_k,\sigma}$ is harmonic off the two sets $|z| = 1$ and $|\sigma(a)z + \sigma(b)| = 1$, then the minimum is achieved only on these sets. The same happens with f_σ and g_σ. For G_σ, the function is harmonic off the two sets $|z| = 1$ and $|(\sigma(a) + \sigma(b)z)/z| = 1$, therefore the minimum is achieved on these sets. Now we are ready to prove the following proposition

Proposition 4.2.1: *Let $a, b \in \overline{\mathbb{Q}}$ with $a \neq 0$. Assume that, there exists $\sigma_0 \in G(a,b)$, such that $|\sigma_0(b)| - |\sigma_0(a)| > 1$. Then*

$$\frac{1}{[K_{a,b} : \mathbb{Q}]} \min\left(\log(|\sigma_0(b)| - |\sigma_0(a)|), \log\left(\frac{|\sigma_0(b)| - 1}{|\sigma_0(a)|}\right)\right) \leq \mu^{ess}(h_{a,b}).$$

Proof: Consider the function

$$g_{\sigma_0}(z) = \log^+ |z| + \log^+ |\sigma_0(a)z + \sigma_0(b)|.$$

Firstly, we consider $z = e^{i\theta}$, with $0 \leq \theta \leq 2\pi$, $\arg(\sigma_0(a)) = \gamma$ and $\arg(\sigma_0(b)) = \beta$, note that $|\sigma_0(a)e^{i\theta} + \sigma_0(b)| > 1$, so we have

$$\begin{aligned} g_{\sigma_0}(e^{i\theta}) &= \log |\sigma_0(a)e^{i\theta} + \sigma_0(b)| \\ &= \frac{1}{2}\log(|\sigma_0(a)|^2 + |\sigma_0(b)|^2 + 2|\sigma_0(b)||\sigma_0(a)|\cos(\theta + \gamma - \beta)). \end{aligned}$$

The minimum value is achieved when $\cos(\theta + \gamma - \beta) = -1$, and the minimum value is $\log(|\sigma_0(b)| - |\sigma_0(a)|)$. On the other hand, note that $(|\sigma_0(b)| - 1)/|\sigma_0(a)| > 1$, therefore, $|e^{i\theta} - \sigma_0(b)|/|\sigma_0(a)| > 1$. We conclude that

$$\begin{aligned} g_{\sigma_0}\left(\frac{e^{i\theta} - \sigma_0(b)}{\sigma_0(a)}\right) &= \log\left|\frac{e^{i\theta} - \sigma_0(b)}{\sigma_0(a)}\right| \\ &= \frac{1}{2}\log\left(\frac{|\sigma_0(b)|^2 + 1 - 2|\sigma_0(b)|\cos(\theta - \beta)}{|\sigma_0(a)|^2}\right). \end{aligned}$$

The minimum is achieved when $\theta = \beta$, and its value is $\log((|\sigma_0(b)| - 1|)/|\sigma_0(a)|)$. Since, $\mathcal{L}(a,b) \geq \min g_{\sigma_0}/[K_{a,b} : \mathbb{Q}]$, using Theorem 4.1.3 we conclude the proof of the proposition. ☐

Proposition 4.2.2: *Let $a, b \in \overline{\mathbb{Q}}$, $b \neq 0$. Assume that there exist $\sigma_0 \in G(a,b)$, such that, $|\sigma_0(a)| - |\sigma_0(b)| > 1$. Then*

$$\frac{1}{[K_{a,b} : \mathbb{Q}]} \log\left(\frac{|\sigma_0(a)|}{|\sigma_0(b)| + 1}\right) \leq \mu^{ess}(h_{a,b}).$$

Proof: we consider

$$G_{\sigma_0}(z) = \log^+ |z| + \log^+ \left|\frac{\sigma_0(a) + \sigma_0(b)z}{z}\right|.$$

Let's consider, $z = e^{i\theta}$, and let $\arg(\sigma_0(a)) = \gamma$ and $\arg(\sigma_0(b)) = \beta$. Since $|\sigma_0(a)e^{i\theta} + \sigma_0(b)| > 1$,

$$\begin{aligned} G_{\sigma_0}(e^{i\theta}) &= \log |\sigma_0(b)e^{i\theta} + \sigma_0(a)| \\ &= \frac{1}{2}\log(|\sigma_0(a)|^2 + |\sigma_0(b)|^2 + 2|\sigma_0(b)||\sigma_0(a)|\cos(\theta + \beta - \gamma)). \end{aligned}$$

The minimum value is achieved when $\cos(\theta + \beta - \gamma) = -1$, an the value is $\log(|\sigma_0(a)| - |\sigma_0(b)|)$. On the other hand, note that $|\sigma_0(a)|/(|\sigma_0(b)| + 1) > 1$, therefore, $|\sigma_0(a)|/|e^{i\theta} - \sigma_0(b)| > 1$. We conclude that

$$G_{\sigma_0}\left(\frac{\sigma_0(a)}{e^{i\theta}-\sigma_0(b)}\right) = \log\left|\frac{\sigma_0(a)}{e^{i\theta}-\sigma_0(b)}\right|$$
$$= -\frac{1}{2}\log\left(\frac{|\sigma_0(b)|^2+1-2|\sigma_0(b)|\cos(\theta-\beta)}{|\sigma_0(a)|^2}\right).$$

The minimum is achieved when $\cos(\theta-\beta)=-1$, and its value is $\log(|\sigma_0(a)|/(|\sigma_0(b)|+1))$. Note that, since $|\sigma_0(a)|-|\sigma_0(b)|>1$, it is not hard to prove that $|\sigma_0(a)|-|\sigma_0(b)|>|\sigma_0(a)|/(|\sigma_0(b)|+1)$. Since, $\mathcal{L}(a,b)\geq G^{\min}\geq [K_{a,b}/\mathbb{Q}]\min G_{\sigma_0}$, using Theorem 4.1.3 we conclude the proof of the proposition. □

Proposition 4.2.3: *Let $a,b\in\overline{\mathbb{Q}}$, $a\neq 0$, $b\neq 0$. Assume that, there exist $\sigma_0\in G(a,b)$, such that, $0\leq ||\sigma_0(a)|-|\sigma_0(b)||\leq 1$ and $|\sigma_0(a)|+|\sigma_0(b)|<1$. Then*

$$\frac{1}{[K_{a,b}:\mathbb{Q}]}\log\left(\frac{1}{|\sigma_0(a)|+|\sigma_0(b)|}\right)\leq \mu^{ess}(h_{a,b}).$$

Proof: In this case, we consider

$$f_{\sigma_0}(z)=\log^+|z|+\log^+\left|\frac{1}{\sigma_0(a)z+\sigma_0(b)}\right|.$$

If $z=e^{i\theta}$, then $1/|\sigma_0(a)z+\sigma_0(b)|>1$, therefore

$$f_{\sigma_0}(e^{i\theta})=\log\left|\frac{1}{\sigma_0(a)e^{i\theta}+\sigma_0(b)}\right|.$$

The minimum value is $\log(1/||\sigma_0(a)|+|\sigma_0(b)||)$, now, assume $z=(1-\sigma_0(b)e^{i\theta})/(\sigma_0(a)e^{i\theta})$, we note that $|(1-\sigma_0(b)e^{i\theta})/(\sigma_0(a)e^{i\theta})|\geq (1-|\sigma_0(b)|)/|\sigma_0(a)|>1$, therefore

$$f_{\sigma_0}\left(\frac{1-\sigma_0(b)e^{i\theta}}{\sigma_0(a)e^{i\theta}}\right)=\log\left|\frac{1-\sigma_0(b)e^{i\theta}}{\sigma_0(a)e^{i\theta}}\right|.$$

The minimum value is $\log((1-|\sigma_0(b)|)/|\sigma_0(a)|)$. Since $|\sigma_0(a)|+|\sigma_0(b)|<1$, it is not hard to prove that $(1-|\sigma_0(b)|)/|\sigma_0(a)|>1/(|\sigma_0(a)|+|\sigma_0(b)|)$, we conclude that the minimum value of f_{σ_0} is $\log(1/(|\sigma_0(a)|+|\sigma_0(b)|))$. Since, $\mathcal{L}(a,b)\geq f^{\min}\geq [K_{a,b}/\mathbb{Q}]\min f_{\sigma_0}$, using Theorem 4.1.3 we conclude the proof of the proposition. □

4.3 Examples

Proof of Theorem A: Propositions 4.2.1, 4.2.2 and 4.2.3 give us non zero lower bounds for the cases $a,b\in\overline{\mathbb{Q}}$, such that there exists $\sigma_0\in G(a,b)$, such that $||\sigma_0(a)|-|\sigma_0(b)||>1$, $b\neq 0$, or the cases where $||\sigma_0(a)|-|\sigma_0(b)||\leq 1$ and $|\sigma_0(a)|+|\sigma_0(b)|<1$. Using Theorem 4.1.3 together with Propositions 3.1.4, 4.2.1, 4.2.2 and 4.2.3 we obtain Theorem A. □

Lemma 4.3.1: *Let $a,b\in\overline{\mathbb{Q}}$, the following sentences are equivalent*

i) *The minimum value of $h_{a,b}$ is achieved and* $\min(h_{a,b})=0$.

ii) $b=0$ *or b is a root of unity, or there exists a root of unity ζ, such that* $a\zeta+b=0$ *or* $a\zeta+b=\zeta_0$ *a root of unity.*

Furthermore, assume that any of this sentences hold, if $b\neq 0$, b is not a root of unity and $|a|\neq|b|$, then necessarily $||a|-|b||\leq 1$ and $|a|+|b|\geq 1$

Proof: (i)⇒(ii) Assume that $\min h_{a,b}$ is achieved at $\alpha \in \overline{\mathbb{Q}}$. Since, $h_{a,b}(\alpha) = h(\alpha) + h(a\alpha + b)$, we need $h(\alpha) = 0$ and $h(a\alpha + b) = 0$. The first equality implies $\alpha = 0$ or $\alpha = \zeta$ a root of unity. If $\alpha = 0$, we also need $h(b) = 0$, so there are two options, $b = 0$ or b is a root of unity. On the other hand, if $\alpha = \zeta$ is a root of unity, we also need $a\zeta + b = 0$ or $a\zeta + b$ a root of unity.

(ii)⇒(i) If $b = 0$ or a root of unity, we take $\alpha = 0$ and we obtain $h_{a,0}(\alpha) = 0$. On the other hand, if there exists a root of unity ζ such that $a\zeta + b = 0$ or $a\zeta + b = \zeta_0$ a root of unity, we take $\alpha = \zeta$, and we obtain $h_{a,b}(\alpha) = 0$. □

Now, assume that $b \neq 0$, b is not a root of unity and $|a| \neq |b|$, by (ii), we have that there exist two roots of unity ζ and ζ_0, such that $a\zeta + b = \zeta_0$, therefore $|a\zeta + b| = 1$, we conclude that $||a| - |b|| \leq |a\zeta + b| = 1$ and $1 = |a\zeta + b| \leq |a\zeta| + |b| = |a| + |b|$. □

Corollary 4.3.2: *Let $a \in \overline{\mathbb{Q}} \setminus \{0\}$. Then $\mu^{ess}(h_{a,0}) = 0$ if and only if a is a root of unity (see [23] for more details or [4] Theorem 1.5.9 for the proof). Moreover,*

$$\mu^{ess}(h_{a,0}) \geq h(a)$$

Proof: We consider $a \in \overline{\mathbb{Q}} \setminus \{0\}$ not a root of unity, then $h_{a,0}(0) = 0$, and it is zero only at 0, therefore $f_1(x) = x$. Firstly we need to determine the possible values of A_1. For each ν non-Archimedean and $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$, we need that

$$\log^+ |\alpha|_\nu + \log^+ |a\alpha|_\nu \geq A_1 \log |\alpha|_\nu.$$

if $|\alpha|_\nu \leq 1$, this inequality always occur. If $|\alpha|_\nu > 1$ and $|a\alpha|_\nu \leq 1$, we get the restriction $A_1 \leq 1$. On the other hand, if $|\alpha|_\nu < 1$ and $|a\alpha|_\nu < 1$, we get the already known restriction $0 \leq A_1$, therefore $0 \leq A_1 \leq 1$.

Now, we will consider $\sigma \in G(a)$. Then,

$$g_{A_1,\sigma}(z) = \log^+ |z| + \log^+ |\sigma(a)z| - A_1 \log |z|.$$

if $z = e^{i\theta}$, we have

$$g_{A_1,\sigma}(e^{i\theta}) = \log^+ |\sigma(a)|.$$

On the other hand, if $z = e^{i\theta}/\sigma(a)$, then

$$g_{A_1,\sigma}\left(\frac{e^{i\theta}}{\sigma(a)}\right) = \log^+ \frac{1}{|\sigma(a)|} - A_1 \log \frac{1}{|\sigma(a)|}.$$

Assume that $|\sigma(a)| \geq 1$. Then, the minimum value is $H_\sigma(A_1) = A_1 \log |\sigma(a)|$. On the other hand, if $|\sigma(a)| < 1$, then the minimum value is 0, therefore $\inf g_{A_1,\sigma} = 0$. Summarizing, $H_\sigma(A_1) = A_1 \log^+ |\sigma(a)|$, therefore

$$\begin{aligned}
\tau(a, 0) &= \frac{1}{[K_a : \mathbb{Q}]} \sup_{0 \leq A_1 \leq 1} \sum_{\sigma \in G(a)} A_1 \log^+ |\sigma(a)| \\
&= \frac{1}{[K_a : \mathbb{Q}]} \sum_{\sigma \in G(a)} \log^+ |\sigma(a)|
\end{aligned}$$

Furthermore, we can improve this result, note that if we go back to the proof of theorem 4.1.3, we can obtain that, a lower bound for $\mu^{ess}(h_a)$ is given by

$$\frac{1}{[K_a : \mathbb{Q}]} \sum_{\substack{p \\ \text{prime}}} \sum_{\sigma \in G(a)} \min_{\alpha \in \overline{\mathbb{Q}}} \left(\log^+ |\alpha|_p + \log^+ |\sigma(a)\alpha)|_p - \log |\alpha|_p\right) + \tau(a, 0)$$

Now, for each p prime and $\sigma \in G(a)$, we set the function $f_p^\sigma : \overline{\mathbb{Q}} \to \mathbb{R}_{\geq 0}$, defined by $f_p^\sigma(\alpha) = \log^+ |\alpha|_p + \log^+ |\sigma(a)\alpha|_p - \log|\alpha|_p$, we will find the minimum value of f_p^σ. Firstly, suppose that $|\sigma(a)|_p \leq 1$ then, we can take α a root of unity, and we get, $f_p^\sigma(\alpha) = 0$, since $f_p^\sigma \geq 0$, this will be the minimum value. On the other hand, if $|\sigma(a)|_p > 1$, let $\alpha \in \overline{\mathbb{Q}}$, if $|\alpha|_p \geq 1$, then $f_p^\sigma(\alpha) = \log|\sigma(a)\alpha|_p \geq \log|\sigma(a)|_p$. Assume now that, $|\alpha|_p < 1$, then $f_p^\sigma(\alpha) = \log^+ |\sigma(a)\alpha|_p - \log|\alpha|_p$, if $|\sigma(a)\alpha|_p \leq 1$, then $f_p^\sigma(\alpha) = \log 1/|\alpha|_p \geq \log|a|_p$. On the other hand, if $|\sigma(a)\alpha|_p > 1$, then $f_p^\sigma(\alpha) = \log|\sigma(a)\alpha|_p - \log|\alpha|_p = \log|\sigma(a)|_p$. Summarizing, we have that $\min(f_p^\sigma) = \log^+ |\sigma(a)|_p$. We conclude that

$$\mu^{ess}(h_{a,0}) \geq \frac{1}{[K_a : \mathbb{Q}]} \sum_{\substack{p \\ \text{prime}}} \sum_{\sigma \in G(a)} \log^+ |\sigma(a)|_p + \tau(a, 0) = h(a)$$

This concludes the proof of the corollary. $\square$

Note that, using (i) of Proposition 3.2.1, we obtain that $\mu^{ess}(h_{a,0}) \leq h(a)$, therefore, we have that $\mu^{ess}(h_{a,0}) = h(a)$.

Corollary 4.3.3: $\mu^{ess}(h_{1,2}) \geq \log(\sqrt{3})$.

Proof: Note that, α and $\alpha + 2$ are both roots of unity only at $\alpha = -1$, therefore, $f_1(x) = x + 1$. Let $A_0, A_1 \in \mathbb{R}$, for each $\alpha \in \overline{\mathbb{Q}}$ and p prime, we need

$$\log^+ |\alpha|_p + \log^+ |\alpha + 2|_p \geq A_1 \log|\alpha + 1|_p.$$

If $|\alpha|_p \leq 1$, then we need $A_1 \geq 0$. On the other hand, if $|\alpha|_p > 1$, we need $A_1 \leq 2$, so we need $0 \leq A_1 < 2$. Now, we take the function

$$g_{A_1}(z) = \log^+ |z| + \log^+ |z + 2| - A_1 \log|z + 1|.$$

The minimum is achieved when $z = e^{i\theta}$ or $z = e^{i\theta} - 2$. If $z = e^{i\theta}$, we have

$$\begin{aligned} g_{A_1}(e^{i\theta}) &= \log|e^{i\theta} + 2| - A_1 \log|e^{i\theta} + 1| \\ &= \frac{1}{2}\log(5 + 4\cos(\theta)) - \frac{A_1}{2}\log(2 + 2\cos(\theta)). \end{aligned}$$

Taking the derivative and equalizing to zero, we get the following possible values of theta

$$\theta = 0, \text{ or } \cos(\theta) = \frac{A_1}{4(1 - A_1)} - 1.$$

On the other hand, if $z = e^{i\theta} - 2$, then

$$\begin{aligned} g_{A_1}(e^{i\theta} - 2) &= \log|e^{i\theta} - 2| - A_1 \log|e^{i\theta} - 1| \\ &= \frac{1}{2}\log(5 - 4\cos(\theta)) - \frac{A_1}{2}\log(2 - 2\cos(\theta)). \end{aligned}$$

Again, taking the derivative and equalizing to zero, we get the next possible values of theta

$$\theta = \pi, \text{ or } \cos(\theta) = 1 - \frac{A_1}{4(1 - A_1)}.$$

It is not hard to see that the minimum value of the function will be the same in both sets, so we can take anyone. In order to find the minimum vale, we have to know if the function is lower at $\theta = 0$ or at $\cos(\theta) = A_1/(4(1-A_1))-1$. Firstly, we will assume $0 \leq A_1 < 1$, evaluating at these two values we obtain $\log(9) - A_1 \log(4)$ and $\log(1/(1-A_1)) - A_1 \log(A_1/(2(1-A_1)))$, we can take these two values and consider the function $l : [0, 1) \to \mathbb{R}$, given by $l(x) = \log(1/(1-x)) - x \log(x/(2(1-x))) - (\log(9) - x \log(4))$. Taking the derivative of l, we obtain $l'(x) = -\log(x/(8(1-x)))$ and $l''(x) = -1/(x(1-x))$. Therefore, we conclude that, $x = 8/9$ is an absolute maximum of l. Since $l(8/9) = 0$, we conclude that the minimum value is $\log(1/(1-A_1)) - A_1 \log(A_1/(2(1-A_1)))$, and it is achieved at $\cos(\theta) = A_1/(1-A_1) - 1$. Now, we consider the function respect to A_1, $H_{id}(A_1) = \log(1/(1-A_1)) - A_1 \log(A_1/(2(1-A_1)))$, taking the derivative respect to A_1 and equalizing to zero, we obtain $\log(A_1/2(1-A_1)) = 0$, therefore $A_1/2(1-A_1) = 1$, and we conclude that the maximum value of $H_{id}(A_1)$ is achieved at $A_1 = 2/3$, replacing this, we obtain $H_{id}(2/3) = \log(\sqrt{3})$. On the other hand, if $1 \leq A_1 < 2$, then, there is no θ such that $\cos(\theta) = A_1/(4(1-A_1))-1$. We conclude that, the minimum value of g_{A_1} is $(1/2)(\log(9) - A_1 \log(4))$, the maximum value is achieved at $A_1 = 1$ and $H_{id}(1) = \log(3/2)$. Since $\sqrt{3} > 3/2$, we have $\tau(a, b) = \sup_{0 \leq A_1 < 2} H_{id}(A_1) = \sqrt{3}$, therefore $\log(\sqrt{3}) \leq \mu^{ess}(h_{1,2})$. $\square$

5 Intervals of density

5.1 The main result

A Galois invariant adelic set is a set of the form

$$\mathbb{E} = \prod_{v \in M_{\mathbb{Q}}} E_v.$$

Where $E_v = D_v(a,b)$ is a subset of $\mathbb{C}_v$ invariant under the action of the absolute v-adic Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_v/\mathbb{Q})$, for all v, and such that $E_v = \mathcal{O}_v$ for all but a finite number of v. The capacity of $\mathbb{E}$ is defined by $\mathrm{Cap}(\mathbb{E}) = \prod_v \mathrm{Cap}(\overline{E}_v)$, where $\overline{E}_v$ is closure of E_v in the Berkovich disc. It is well defined, because $\mathrm{Cap}(\overline{E}_v) = \mathrm{Cap}(\mathcal{D}_v(0,1)) = 1$ (see [16], section 4), for all but a finite number of v. Furthermore, the equilibrium measure of $\mathcal{D}_v(0,r)$ is the linear functional $\mu_{\mathcal{D}_v(0,r)} : C^0(\mathcal{D}_v(0,r), \mathbb{R}) \to \mathbb{R}$, such that, for every polynomial f, we have

$$\int_{\mathcal{D}_v(0,r)} f(t) d\mu_{\mathcal{D}_v(0,r)}(t) = \sup_{|z|_v \le r} |f(z)|_v$$

(see [16], section 4, for more details). The following proposition will be useful

Proposition 5.1.1: *Let $\mathbb{E} = \prod_{v \in M_{\mathbb{Q}}} E_v$ an adelic set with Cap$(\mathbb{E}) = 1$. Then, there exists a sequence $(x_l)_{l \in \mathbb{N}}$ of pairwise distinc points of $\overline{\mathbb{Q}}^{\times}$, with Gal$(x_l)_v \subset B(E_v, 1/l)$, for all $v \in M_{\mathbb{Q}}$. Furthermore, for all $v \in M_{\mathbb{Q}}$, the sequence of measures $(\delta(Gal(x_l, v)))_{l \in \mathbb{N}}$ converges to the equilibrium measure $\mu_{\overline{E}_v}$ weakly.*

Proof: Direct from [7] Proposition 7.4 and Proposition 7.3. □

Proof of Theorem B: We use Proposition 5.1.1, for $p_i \in S_{a,b}$ we take $E_{p_i} = D_{p_i}(0, r_i) = \{z \in \mathbb{C}_{p_i} : |z|_{p_i} < r_i\}$, where $r_i \in \mathbb{Q}^+$. For $p \notin S_{a,b}$ we take, $E_p = \mathcal{O}_p$. For $|.|_\nu = |.|_\infty$, we take $E_{|.|} = B(x, 1/(r_1 r_2 ... r_s))$, where $x \in \mathbb{R}$. We define, $\mathbb{E} = \prod_v E_v$, then $\mathbb{E}$ is an adelic set such that $\mathrm{Cap}(\mathbb{E}) = 1$. Therefore, by Proposition 5.1.1, there is a sequence $(\alpha_n)_{n \in \mathbb{N}}$ satisfying $\mathrm{Gal}(\alpha_{n,\nu}) \subset B(E_\nu, 1/n)$, for each $\nu \in M_{\mathbb{Q}}$, such that $\delta(\mathrm{Gal}(\alpha_n, \nu)) \xrightarrow{*} \mu_{\overline{E}_\nu}$, therefore

$$h_{a,b}(\alpha_n) \underset{n\to\infty}{\longrightarrow} \sum_{\nu \in M_{\mathbb{Q}}} \int_{\overline{E}_\nu} \log^+ |t|_\nu + \log^+ |at+b|_\nu d\mu_{\overline{E}_\nu}(t) := M_{E_x}. \tag{12}$$

We claim that $M_{E_x} < \infty$. Let p a prime number and $d \in \mathbb{R}$, then

$$\sup_{|z|_p \le d} \log^+ |az+b|_p \le \sup_{|z|_p \le d} \log^+ \max\{|a|_p |z|_p, |b|_p\} = \log^+ \max\{|a|_p d, |b|_p\}$$

If $|a|_p d \le |b|_p$, then, taking t_0, such that $|t_0|_p < d$, we obtain that $|at_0 + b|_p = |b|_p = \max\{|a|_p d, |b|_p\}$. If $|a|_p d > |b|_p$, we can take t_1, such that $|t_1|_p = d$, we obtain that $|at_1 + b|_p = |a|_p d$. Therefore $\sup_{|z|_p \le d} \log^+ |az+b|_p = \log^+ \max\{|a|_p d, |b|_p\}$. Now, note that for $p \notin S_{a,b}$, we have

$$\begin{aligned}
\int_{\overline{E}_p} \log^+ |t|_p + \log^+ |at+b|_p d\mu_{\overline{E}_p}(t) &= \int_{\mathcal{D}_p(0,1)} \log^+ |t|_p + \log^+ |at+b|_p d\mu_{\mathcal{D}_p(0,1)}(t) \\
&= \sup_{|z|_p \le 1} \log^+ |z|_p + \sup_{|z|_p \le 1} \log^+ |az+b|_p = \log^+ \max\{|a|_p, |b|_p\} \\
&= 0.
\end{aligned}$$

Therefore, there are finite many $v \in M_{\mathbb{Q}}$ in (12). Furthermore, since $\mu_{\overline{E}_v}(\overline{E}_v) = \mu_{\overline{E}_v}(\mathbb{1}_{\overline{E}_v}) < \infty$ (where $\mathbb{1}_{\overline{E}_v}$ is the continuous function $\mathbb{1}_{\overline{E}_v} : \overline{E}_v \to \mathbb{R}$, given by $\mathbb{1}_{\overline{E}_v}(t) = 1$ for each $t \in \overline{E}_v$) and

$f(t) = \log^+ |t|_v + \log^+ |at+b| \leq \log^+ r_i + \log^+ \max\{|a|_p r_i, |b|_p\}$, for each $t \in \overline{E}_v$, we conclude that $M_{E_x} < \infty$. Consequently, we have that

$$\begin{aligned}
\mu^{ess}(h_{a,b}) &\leq \sum_{\nu \in M_{\mathbb{Q}}} \int_{\overline{E}_\nu} \log^+ |t|_\nu + \log^+ |at+b|_\nu d\nu_{\overline{E}_\nu}(t) \\
&= \sum_{\nu \in M_{\mathbb{Q}}} \int_{\overline{E}_\nu} \log^+ |t|_\nu + \log^+ |at+b|_\nu d\nu_{\overline{E}_\nu}(t) \\
&= \sum_{i=1}^{s} \sup_{|z|_{p_i} \leq r_i} \log^+ |z|_{p_i} + \sup_{|z| \leq r_i} \log^+ |az+b|_{p_i} + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{e^{i\theta}}{r_1 r_2 ... r_s} + x \right| + \log^+ \left| \frac{ae^{i\theta}}{r_1 r_2 ... r_s} + b + ax \right| d\theta. \\
&= \sum_{i=1}^{s} \log^+ r_i + \sup_{|z| \leq r_i} \log^+ |az+b|_{p_i} + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{e^{i\theta}}{r_1 r_2 ... r_s} + x \right| + \log^+ \left| \frac{ae^{i\theta}}{r_1 r_2 ... r_s} + b + ax \right| d\theta \\
&= \sum_{i=1}^{s} \log^+ r_i + \log^+ \max\{|a|_{p_i} r_i, |b|_{p_i}\} + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{e^{i\theta}}{r_1 r_2 ... r_s} + x \right| + \log^+ \left| \frac{ae^{i\theta}}{r_1 r_2 ... r_s} + b + ax \right| d\theta. \\
&= \Gamma_{a,b}(x, r_1, r_2, ..., r_s).
\end{aligned}$$

Moreover, suppose that $r_1, r_2, ..., r_s \in \mathbb{Q}^+$ are fixed, and $j \in [\Gamma_{a,b}(x, r_1, r_2, ..., r_s), \infty)$. Since, $\Gamma_{a,b}$ is continuous and $\Gamma_{a,b}(x, r_1, r_2, ..., r_s) \to \infty$, when $x \to \infty$, we can find $g \in \mathbb{R}$, such that, $j = \Gamma_{a,b}(g, r_1, r_2, ..., r_s) = M_{E_g}$, and then, there is a sequence of algebraic numbers $(\alpha_n)_{n \in \mathbb{N}}$, such that, $h_{a,b}(\alpha_n) \to j$. Hence, for each $x \in \mathbb{R}$, the image of $h_{a,b}$ is dense in the interval $[\Gamma_{a,b}(x, r_1, ..., r_s), \infty)$. This concludes the proof of the theorem. □

5.2 Observations

Experimental results show that the minimal value of $\Gamma_{a,b}$ is achieved when $r_1 = r_2 = ... = r_s = 1$, so, in general we will always take these values, furthermore, if we take $|a| = 1$, then

$$\Gamma_{1,b}(x, 1, 1, ..., 1) = \Delta(1, b) + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| e^{i\theta} + x \right| + \log^+ \left| e^{i\theta} + b + x \right| d\theta = \Omega_{1,b}(x).$$

Proof of Corollary C: Using Theorem B together with Proposition 3.2.2, we obtain the proof of this corollary. □

Finally, using Proposition 3.2.1, Proposition 4.2.1 and Theorem B, we obtain the following result

Theorem 5.2.1: *Let $a, b \in \mathbb{Q}$, $|a| \geq 1$, $|b| - |a| > 1$, and $|b/a| \geq 4$, then*

$$\log\left(\frac{|b| - 1}{|a|}\right) \leq \mu^{ess}(h_{a,b}) \leq \log\left(\frac{|b|}{|a|}\right).$$

Furthermore, assume $|a| = 1$. Then, $h_{a,b}$ is dense in the interval $[\log\left(\frac{|b|}{|a|}\right), \infty)$.

Proof: This a direct consequence of Proposition 3.2.1, Proposition 4.2.1 and Theorem B. □

6 Parametrized curves

6.1 Generalization of upper bounds

In this section we will generalize all results to the situation when the variety is a parametrized curve. We will begin with upper bounds. Let $p, q \in \overline{\mathbb{Q}}(t)$, we will consider the notation introduced in section 1 and formulas (5), (6) and (7). Let $q \in \overline{\mathbb{Q}}(t)$, $q = q_1(t)/q_2(t)$, write $q_1(t) = a_n t^n + a_{n-1}t^{n-1} + ... + a_0$ and $q_2(t) = b_m t^m + b_{m-1}t^{m-1} + ... + b_0$. We denote $G(q) = G(a_0, ..., a_n, b_1, ..., b_m)$. We define $U_q : \overline{\mathbb{Q}} \to \mathbb{R}$, given by

$$U_q(\alpha) = \frac{1}{\deg(\alpha)} \sum_{\substack{\beta \in \mathrm{Gal}(\alpha) \\ \sigma \in G(q)}} \log^+ |q_\sigma(\beta)|$$

Lemma 6.1.1: *Let $q(t) = q_1(t)/q_2(t)$, $r(t) = r_1(t)/r_2(t)$, $q_1, r_1 \in \overline{\mathbb{Q}}[t]$, $q_2, r_2 \in \overline{\mathbb{Z}}[t]$ and $\alpha \in \overline{\mathbb{Z}}$ such that $q_2(\alpha) \neq 0$ and $r_2(\alpha) \neq 0$. Then,*

$$h_{q,r}(\alpha) \leq U_q(\alpha) + U_r(\alpha) + \Delta(q) + \Delta(r).$$

Here, $\Delta(q)$ is the number defined in (6) given by

$$\Delta(q) = \sum_{\substack{p \\ \text{prime}}} \sum_{\sigma \in \mathrm{G}(q)} \log^+ \max(|\sigma(a_1)|_p, ..., |\sigma(a_n)|_p). \tag{13}$$

Proof: We consider K the Galois closure of the field generated by $a_0, a_1, ..., a_n, b_0, b_1, ..., b_m$ and α, K_q the Galois closure of the field generated by $a_0, a_1, ..., a_n, b_0, b_1, ..., b_m$. It is clear that $\deg(\alpha) \leq [K_\alpha : \mathbb{Q}] \leq [K : \mathbb{Q}]$. We denote $z = q(\alpha)$. Then, using Lemma 3.1.1, we have that

$$\begin{aligned} h(z) &= \frac{1}{[K:\mathbb{Q}]} \left(\sum_{\substack{p \\ \text{prime}}} \sum_{\sigma \in \mathrm{Gal}(K/\mathbb{Q})} \log^+ |\sigma(q(\alpha))|_p + \sum_{\sigma \in \mathrm{Gal}(K/\mathbb{Q})} \log^+ |\sigma(q(\alpha))| \right) \\ &\leq \frac{1}{[K:\mathbb{Q}]} \left(\sum_{\substack{p \\ \text{prime}}} \sum_{\substack{\delta \in G(\alpha) \\ \sigma \in G(q)}} \log^+ |q_\sigma(\delta(\alpha))|_p \right) + \frac{1}{[K_\alpha:\mathbb{Q}]} \left(\sum_{\substack{\delta \in G(\alpha) \\ \sigma \in G(q)}} \log^+ |q_\sigma(\delta(\alpha))| \right). \end{aligned}$$

In the last inequality we have used that $[K_\alpha : \mathbb{Q}] \leq [K : \mathbb{Q}]$ and that the number of elements in $\mathrm{Gal}(K/\mathbb{Q})$ is less than or equal to the number of pairs (σ, δ), with $\delta \in G(\alpha)$ and $\sigma \in G(q)$. Using the fact that $\alpha, b_0, b_1, ..., b_m \in \overline{\mathbb{Z}}$, we conclude that for every p prime, $\sigma \in G(q)$ and $\delta \in G(\alpha)$, we have $|q_\sigma(\delta(\alpha))|_p \leq \max(|\sigma(a_n)|_p, |\sigma(a_{n-1})|_p, ..., |\sigma(a_0)|_p)$. Therefore

$$\begin{aligned} h(z) &\leq \sum_{\substack{p \\ \text{prime}}} \sum_{\sigma \in G(q)} \log^+ \max(|\sigma(a_n)|_p, |\sigma(a_{n-1})|_p, ..., |\sigma(a_0)|_p) + \frac{1}{[K_\alpha:\mathbb{Q}]} \left(\sum_{\substack{\delta \in G(\alpha) \\ \sigma \in G(q)}} \log^+ |q_\sigma(\delta(\alpha))| \right) \\ &= \Delta(q) + \frac{1}{\deg(\alpha)} \sum_{\substack{\beta \in \mathrm{Gal}(\alpha) \\ \sigma \in G(q)}} \log^+ |q_\sigma(\beta)| \\ &= U_q(\alpha) + \Delta(q). \end{aligned}$$

Using the same argument for $r(t)$ we conclude the proof of the lemma. $\square$

Proposition 6.1.2: *Let $q(t) = q_1(t)/q_2(t)$, $r(t) = r_1(t)/r_2(t)$, $q_1, r_1 \in \overline{\mathbb{Q}}[t]$, $q_2, r_2 \in \overline{\mathbb{Z}}[t]$. Then, for each $t \in \mathbb{R}$ we have that*

$$\mu^{ess}(h_{q,r}) \leq \Omega_{q,r}(t).$$

Here, $\Omega_{q,r}$ is the function defined in (7) *given by*

$$\Omega_{q,r}(t) = \Delta(q) + \Delta(r) + \sum_{\sigma \in \mathrm{G}(q)} \Psi_q^\sigma(t) + \sum_{\sigma \in \mathrm{G}(r)} \Psi_r^\sigma(t).$$

Where

$$\Psi_q^\sigma(t) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |q_\sigma(e^{i\theta} + t)| d\theta.$$

Proof: For Lemma 6.1.1, given $\alpha \in \overline{\mathbb{Z}}$, we have that

$$h_{q,r}(\alpha) \leq U_q(\alpha) + U_r(\alpha) + \Delta(q) + \Delta(r) =: \eta_{q,r}(\alpha).$$

For Proposition 3.1.3, given $E \subseteq \mathbb{C}$, a compact set with $\mathrm{Cap}(E) = 1$ and invariant under complex conjugation, there exists a sequence of algebraic integers $\alpha_n \in \overline{\mathbb{Z}}$, such that $\mathrm{Gal}(\alpha_n) \subset B\left(E, \frac{1}{n}\right)$ and

$$U_q(\alpha_n) + U_r(\alpha_n) \underset{n\to\infty}{\longrightarrow} \sum_{\sigma \in G(q)} \int_E \log^+ |q_\sigma(x)| \, d\mu_E(x) + \sum_{\sigma \in G(r)} \int_E \log^+ |r_\sigma(x)| \, d\mu_E(x) =: M_E.$$

Therefore

$$h_{q,r}(\alpha_n) \leq \eta_{q,r}(\alpha_n) \underset{n\to\infty}{\longrightarrow} \Delta(q) + \Delta(r) + M_E =: J_E.$$

We claim that $J_E < \infty$. In fact, we have that the only possible unbounded term in the definition could be M_E. For each $\delta \in G(q)$, let the function, $f_\delta : \mathbb{R} \to \mathbb{R}$, and for each $\sigma \in G(r)$ the function $g_\sigma : \mathbb{R} \to \mathbb{R}$, be defined by $f_\delta(t) = \log^+ |q_\delta(t)|$ and $g_\sigma(t) = \log^+ |r_\sigma(t)|$. These functions are continuous and E is a compact set, therefore, $M_E < \infty$. Since $J_E < \infty$, the sequence $\{h_{q,r}(\alpha_n)\}_{n\in\mathbb{N}}$ is bounded, we conclude that there is a subsequence which is convergent, we call it $\{h_{q,r}(\beta_n)\}_{n\in\mathbb{N}}$.

If $h_{q,r}(\beta_n) \underset{n\to\infty}{\longrightarrow} Z$, then, by definition of limit, given $\varepsilon > 0$ the set $\{\beta_n \in \overline{\mathbb{Z}} : h_{q,r}(\beta_n) \leq Z + \varepsilon\}$ is infinite, therefore for every $\varepsilon > 0$ we have $\mu^{ess}(h_{q,r}) \leq Z + \varepsilon$. Taking $\varepsilon \to 0$ we get $\mu^{ess}(h_{q,r}) \leq Z$. Since $Z \leq J_E$, we conclude that $\mu^{ess}(h_{q,r}) \leq J_E$. Given $t \in \mathbb{R}$ we use $E = S_t = S_1 + t$, where $S_1 = \{z \in \mathbb{C} : |z| = 1\}$. Then, μ_{S_t} is the natural translation of the measure $\mu_{S_1} = \frac{d\theta}{2\pi}$. We deduce that

$$\mu^{ess}(h_{q,r}) \leq J_{S_t} = \Omega_{q,r}(t).$$

This concludes the proof of the theorem $\square$

6.2 Generalization of lower bounds

Now, we proceed with lower bounds. We denote $K_{q,r}$ the field generated over $\mathbb{Q}$ by the coefficients of q_1, q_2, r_1 and r_2, and $G(q,r) = \mathrm{Gal}(K_{q,r}/\mathbb{Q})$. Let $\sigma \in G(q,r)$, we consider the real-valued functions given by

$$
\begin{aligned}
g_\sigma(z) &= \log^+ |q_\sigma(z)| + \log^+ |r_\sigma(z)|,\ f_\sigma^1(z) = \log^+ |q_\sigma(z)| + \log^+ \left|\frac{1}{r_\sigma(z)}\right| \\
f_\sigma^2(z) &= \log^+ \left|\frac{1}{q_\sigma(z)}\right| + \log^+ |r_\sigma(z)|, G_\sigma^1(z) = \log^+ |q_\sigma(z)| + \log^+ \left|r_\sigma\left(\frac{1}{z}\right)\right| \\
G_\sigma^2(z) &= \log^+ \left|q_\sigma\left(\frac{1}{z}\right)\right| + \log^+ |r_\sigma(z)|.
\end{aligned}
$$

We have that these functions tend to ∞ when $|z| \to \infty$. Furthermore, $f_\sigma^1 \to \infty$ when $r_\sigma(z) \to 0$, $f_\sigma^2 \to \infty$ when $q_\sigma(z) \to 0$ and $G_\sigma^1, G_\sigma^2 \to \infty$ when $z \to 0$, and they are continuous elsewhere, so they attain their minimum values. We denote by $\min(g_\sigma), \min(f_\sigma^1)$, $\min(f_\sigma^2)$, $\min(G_\sigma^1)$ and $\min(G_\sigma^2)$, the minimum value of each of these functions respectively. Then, we define $g^{\min} = (1/[K_{q,r} : \mathbb{Q}]) \sum_{\sigma \in G(q,r)} \min(g_\sigma)$, $f^{i,\min} = (1/[K_{q,r} : \mathbb{Q}]) \sum_{\sigma \in G(q,r)} \min(f_\sigma^i)$, and $G^{i,\min} = (1/[K_{q,r} : \mathbb{Q}]) \sum_{\sigma \in G(q,r)} \min(G_\sigma^i)$, $i \in \{1,2\}$. Finally, we define

$$\mathcal{L}(p,q) = \max\{g^{\min}, f^{1,\min}, f^{2,\min}, G^{1,\min}, G^{2,\min}\}.$$

Assume now that the minimum value of $h_{p,q}$ is achieved only at a finite non empty set of algebraic numbers. Let $\alpha_1, \alpha_2, ..., \alpha_k$ be the algebraic numbers where $h_{p,q}$ is equal to the minimum. We consider $\{f_1, f_2, ..., f_r\}$ a set monic irreducible polynomials, such that their combined roots are $\{\alpha_1, \alpha_2, ..., \alpha_k\}$. Now, let $A_1, A_2, ..., A_r \in \mathbb{R}_{\geq 0}$ be such that, $A_1 \deg(f_1) A_2 \deg(f_2) ... A_r \deg(f_r) < m + n$ and for every $\alpha \in \overline{\mathbb{Q}} \setminus \bigcup_{i=1}^k \mathrm{Gal}(\alpha_i)$, and every non-archimedean place ν in $M_{\overline{\mathbb{Q}}}$, we have

$$\log^+ |q(\alpha)|_\nu + \log^+ |r(\alpha)|_\nu \geq \sum_{i=1}^r A_i \log |f_i(\alpha)|_\nu. \tag{14}$$

We call P the set of all $(A_1, A_2, ..., A_r) \in \mathbb{R}^r$, such that these conditions hold. Since $(0,0,0,...,0) \in P$, we have that $P \neq \emptyset$.

Lemma 6.2.1: *The set P defined before is bounded*

Proof: Firstly, we have that $A_i \geq 0$, therefore each A_i is bounded from below. We fix $\nu_0 \in M_{\overline{\mathbb{Q}}}$ and consider $\alpha \in \overline{\mathbb{Q}} \setminus \bigcup_{i=1}^k \mathrm{Gal}(\alpha_i)$, such that, $\log^+ |q(\alpha)|_{\nu_0} = \log |a_n \alpha^n|_{\nu_0}$, $\log^+ |r(\alpha)|_{\nu_0} = \log |b_m \alpha^m|_{\nu_0}$ and for each $1 \leq i \leq r$, $|f_i(\alpha)|_{\nu_0} = |\alpha^{\deg(f_i)}|_{\nu_0} \geq 1$. Therefore, from (13) we have $\log |a_n b_m \alpha^{n+m}|_{\nu_0} \geq \sum_{i=1}^r A_i \log |\alpha^{\deg(f_i)}|_{\nu_0} \geq A_t \log |\alpha^{\deg(f_t)}|_{\nu_0} = \log |\alpha^{\deg(f_t) A_t}|_{\nu_0}$, for each $t \in \{1, 2, ..., r\}$. For $|\alpha|_{\nu_0}$ large enough, this equality holds only if $A_t \leq (n+m)/\deg(f_t)$. This completes the proof of the lemma. $\square$

Now, for each $\sigma \in G(p,q)$ we define the real valued function $g_{A_1,...,A_r,\sigma}$ by

$$g_{A_1,...,A_r,\sigma}(z) = \log^+ |q_\sigma(z)| + \log^+ |r_\sigma(z)| - \sum_{i=1}^r A_i \log |f_i(z)|.$$

Since $A_i \geq 0$ for each $i \in \{1, 2, ..., r\}$, and $A_1 \deg(f_1) A_2 \deg(f_2) ... A_r \deg(f_r) < n + m$, we have that $g_{A_1,...,A_k,\sigma} \to \infty$ when $|z| \to \infty$ or $z \to x$, where $x \in \bigcup_{i=1}^k \mathrm{Gal}(\alpha_i)$, and it is continuous elsewhere, so it attains its minimum value. Now, we consider the function $H_\sigma : P \to \mathbb{R}$, given by

$$H_\sigma(A_1, A_2, ..., A_r) = \inf g_{A_1,...,A_r,\sigma}.$$

We define

$$\tau(q,r) = \frac{1}{[K_{q,r}:\mathbb{Q}]} \sup_{(A_1,...,A_r)\in P} \sum_{\sigma\in G(q,r)} H_\sigma(A_1, ..., A_r).$$

Definition 6.2.2: *Let $q, r \in \overline{\mathbb{Q}}(t)$. If the minimum value of $h_{q,r}$ is achieved only at a finite set of algebraic numbers, we define $\mathcal{K}(q,r) = \tau(q,r)$. Otherwise $\mathcal{K}(q,r) = \mathcal{L}(q,r)$*

Now, we can state the following theorem.

Theorem 6.2.3.: *Let $q, r \in \overline{\mathbb{Q}}(t)$. Then,*

$$\mu^{ess}(h_{q,r}) \geq \mathcal{L}(q,r).$$

Moreover, if the minimum value is achieved only at a finite non empty set of algebraic numbers, then

$$\mu^{ess}(h_{q,r}) \geq \tau(q,r).$$

Proof: Let $K/\mathbb{Q}$ be a Galois extension such that $a_n, ..., a_0, b_m, ..., b_0, \alpha \in K$. Note that

$$\begin{aligned}
h_{q,r}(\alpha) &= \frac{1}{[K:\mathbb{Q}]}\left(\sum_p \sum_{\nu||.|_p} \log^+|q(\alpha)|_\nu + \log^+|r(\alpha)|_\nu + \sum_{\nu||.|} \log^+|q(\alpha)|_\nu + \log^+|r(\alpha)|_\nu\right) \\
&\geq \frac{1}{[K:\mathbb{Q}]} \sum_{\nu||.|} \log^+|q(\alpha)|_\nu + \log^+|r(\alpha)|_\nu \\
&= \frac{1}{[K:\mathbb{Q}]} \sum_{\sigma\in G(q,r)} \sum_{\substack{\tau\in \mathrm{Gal}(K\backslash\mathbb{Q}) \\ \tau|\sigma}} \log^+|q_\sigma(\tau(\alpha))| + \log^+|r_\sigma(\tau(\alpha))| \\
&\geq \frac{1}{[K_{q,r}:\mathbb{Q}]} \sum_{\sigma\in G(q,r)} \inf g_\sigma \\
&= g^{\min}.
\end{aligned}$$

Therefore, $g^{\min} \leq h_{q,r}(\alpha)$ for all α, we conclude that $g^{\min} \leq \mu^{ess}(h_{q,r})$. Similarly, we can conclude that $f^{1,\min} \leq \mu^{ess}(h_{q,r})$. In fact, we can consider the real-valued function $j_{q,r}$, given by $j_{q,r}(\alpha) = h(1/q(\alpha)) + h(r(\alpha))$. Since $h(\alpha) = h(1/\alpha)$, we have $j_{q,r} = h_{q,r}$. However, the Archimedean parts of them are different, so, we can use the same inequalities as before and conclude that $f^{1,\min} \leq \mu^{ess}(j_{q,r}) = \mu^{ess}(h_{q,r})$, the proof for $f^{2,\min}$ is the analogous. Finally, let the real valued function $n_{p,q}$, be given by $n_{p,q}(\alpha) = h(q(\alpha)) + h(r(1/\alpha)) = h^{in}_{p,q}$. Using Lemma 4.1.4 and the same method used before, we conclude that $G^{1,\min} \leq \mu^{ess}(n_{p,q}) = \mu^{ess}(h^{in}_{p,q}) = \mu^{ess}(h_{p,q})$, the proof for $G^{2,\min}$ is the analogous. Therefore, $\mathcal{L}(p,q) \leq \mu^{ess}(h_{p,q})$.

Assume now that the minimum value of $h_{p,q}$ is achieved only at a finite set $\{\alpha_1, \alpha_2, .., \alpha_k\}$. Let $\alpha \in \overline{\mathbb{Q}} \setminus \bigcup_{i=1}^k \mathrm{Gal}(\alpha_i)$ and $A_1, A_2, ..., A_r \in P$. The product formula gives us that for each $i \in \{1, 2, ..., r\}$, we have

$$\sum_{\nu\in M_K} A_i \log|f_i(\alpha)|_\nu = 0$$

Therefore

$$\begin{aligned}
h_{q,r}(\alpha) &= \frac{1}{[K:\mathbb{Q}]}\left(\sum_{p}\sum_{\nu||.|_p}\log^+|q(\alpha)|_\nu + \log^+|r(\alpha)|_\nu - \sum_{i=1}^{r} A_i \log|f_i(\alpha)|_\nu\right)\\
&+ \frac{1}{[K:\mathbb{Q}]}\left(\sum_{\nu||.|}\log^+|q(\alpha)|_\nu + \log^+|r(\alpha)|_\nu - \sum_{i=1}^{r} A_i \log|f_i(\alpha)|_\nu\right)\\
&\geq \frac{1}{[K:\mathbb{Q}]}\left(\sum_{\nu||.|}\log^+|q(\alpha)|_\nu + \log^+|r(\alpha)|_\nu - \sum_{i=1}^{r} A_i \log|f_i(\alpha)|_\nu\right)\\
&= \frac{1}{[K:\mathbb{Q}]}\sum_{\sigma\in G(q,r)}\sum_{\substack{\tau\in \mathrm{Gal}(K/\mathbb{Q})\\ \tau|\sigma}} g_{A_1,A_2,\ldots,A_r,\sigma}(\tau(\alpha))\\
&\geq \frac{1}{[K_{q,r}:\mathbb{Q}]}\sum_{\sigma\in G(q,r)} H_\sigma(A_1,A_2,...,A_r).
\end{aligned}$$

Since the last inequality holds for all $(A_1, A_2, ..., A_r) \in P$, we have that

$$\begin{aligned}
h_{q,r}(\alpha) &\geq \frac{1}{[K_{q,r}:\mathbb{Q}]}\sup_{(A_1,A_2,\ldots,A_k)\in P}\sum_{\sigma\in G(q,r)} H_\sigma(A_1,A_2,...,A_r)\\
&= \tau(q,r).
\end{aligned}$$

Since the last inequality holds for all α except finitely many, we conclude that $\tau(q,r) \leq \mu^{ess}(h_{q,r})$.
□

6.3 Generalization of intervals of density

Now we are ready to prove Theorem E

Proof of Theorem E: We use Proposition 5.1.1, for $p_i \in S_q \bigcup S_r$ we take $E_{p_i} = D_{p_i}(0, r_i) = \{z \in \mathbb{C}_{p_i} : |z|_{p_i} < r_i\}$, where $r_i \in \mathbb{Q}^+$. For $p \notin S_q \bigcup S_r$ we take, $E_p = \mathcal{O}_p$. For $|.|_\nu = |.|_\infty$, we take $E_{|.|} = B(x, 1/(r_1 r_2 ... r_{s_q+s_r}))$, where $x \in \mathbb{R}$. We define, $\mathbb{E} = \prod_v E_v$, then $\mathbb{E}$ is an adelic set such that $\mathrm{Cap}(\mathbb{E}) = 1$. Therefore, by Proposition 5.1.1, there is a sequence $(\alpha_n)_{n\in\mathbb{N}}$ satisfying $\mathrm{Gal}(\alpha_{n,\nu}) \subset B(E_\nu, 1/n)$, for each $\nu \in M_\mathbb{Q}$, such that $\delta(\mathrm{Gal}(\alpha_n,\nu)) \xrightarrow{*} \mu_{\overline{E}_\nu}$ (see section 5 and section 2 for the definition of $\overline{E}_\nu$), therefore

$$h_{p,q}(\alpha_n) \underset{n\to\infty}{\longrightarrow} \sum_{\nu\in M_\mathbb{Q}}\int_{\overline{E}_\nu}\log^+|q(t)|_\nu + \log^+|r(t)|_\nu d\mu_{\overline{E}_\nu}(t) := M_{E_x}. \tag{15}$$

We claim that $M_{E_x} < \infty$. Let p a prime number and $d \in \mathbb{R}$. Then, using Gauss's Lemma (see [5] or [4] for an elementary proof)

$$\sup_{|z|_p\leq d}\log^+|q(z)|_p = \log^+\max_{0\leq k\leq n}|a_k|_p d^k$$

Let $\log^+ \max_{0\leq k\leq n} |a_k|_p d^k = |a_l|_p d^l$, if there are some other a, we can take t_1 such that $|t_1|_p < d$ and we obtain $|q(t_1)|_p = |a_l|_p d^l$. Now, note that for $p \notin S_q \bigcup S_r$, we have

$$
\begin{aligned}
\int_{\overline{E}_p} \log^+ |q(t)|_p + \log^+ |r(t)|_p d\mu_{\overline{E}_p}(t) &= \int_{\mathcal{D}_p(0,1)} \log^+ |q(t)|_p + \log^+ |r(t)|_p d\mu_{\mathcal{D}_p(0,1)}(t) \\
&= \sup_{|z|_p \leq 1} \log^+ |q(z)|_p + \sup_{|z|_p \leq 1} \log^+ |(z)|_p \\
&= \log^+ \max_{0 \leq k \leq n} |a_k|_p + \log^+ \max_{0 \leq k \leq m} |b_k|_p \\
&= 0
\end{aligned}
$$

Therefore, there are finite many $v \in M_{\mathbb{Q}}$ in (14). Furthermore, since $\mu_{\overline{E}_v}(\overline{E}_v) = \mu_{\overline{E}_v}(\mathbb{1}_{\overline{E}_v}) < \infty$ (where $\mathbb{1}_{\overline{E}_v}$ is the continuous function $\mathbb{1}_{\overline{E}_v} : \overline{E}_v \to \mathbb{R}$, given by $\mathbb{1}_{\overline{E}_v}(t) = 1$ for each $t \in \overline{E}_v$) and $f(t) = \log^+ |q(t)|_p + \log^+ |r(t)|_p \leq \log^+ \max_{0 \leq k \leq n} |a_k r_p^k|_v + \log^+ \max_{0 \leq k \leq m} |b_k r_p^k|_v$, for each $t \in \overline{E}_v$, we conclude that $M_{E_x} < \infty$. Consequently, we have that

$$
\begin{aligned}
\mu^{ess}(h_{q,r}) &\leq \sum_{\nu \in M_{\mathbb{Q}}} \int_{\overline{E}_\nu} \log^+ |q(t)|_\nu + \log^+ |r(t)|_\nu d\nu_{\overline{E}_\nu}(t) \\
&= \sum_{i=1}^{s_q} \log^+ \max_{|z|_{p_i} \leq r_i} |q(z)|_{p_i} + \sum_{i=s_q+1}^{s_q+s_r} \log^+ \max_{|z|_{p_i} \leq r_i} |r(z)|_{p_i} \\
&+ \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| q\left(\frac{e^{i\theta}}{r_1 r_2 \ldots r_{s_q+s_r}} + x \right) \right| + \log^+ \left| r\left(\frac{e^{i\theta}}{r_1 r_2 \ldots r_{s_q+s_r}} + x \right) \right| d\theta. \\
&= \Gamma_{q,r}(x, r_1, r_2, \ldots, r_{s_q+s_r}).
\end{aligned}
$$

Moreover, suppose that $r_1, r_2, \ldots, r_{s_q+s_r} \in \mathbb{Q}^+$ are fixed, and $j \in [\Gamma_{q,r}(x, r_1, r_2, \ldots, r_{s_q+s_r}), \infty)$. Since, $\Gamma_{q,r}$ is continuous and $\Gamma_{q,r}(x, r_1, r_2, \ldots, r_{s_q+s_r}) \to \infty$, when $x \to \infty$, we can find $g \in \mathbb{R}$, such that, $j = \Gamma_{q,r}(g, r_1, r_2, \ldots, r_{s_q+s_r}) = M_{E_g}$, and then, there is a sequence of algebraic numbers $(\alpha_n)_{n \in \mathbb{N}}$, such that, $h_{q,r}(\alpha_n) \to j$. Hence, for each $x \in \mathbb{R}$, the image of $h_{q,r}$ is dense in the interval $[\Gamma_{q,r}(x, r_1, \ldots, r_{s_q+s_r}), \infty)$. This concludes the proof of the theorem. $\square$

Experimental results show that the minimal value of Γ is achieved when $r_1 = r_2 = \ldots = r_{s+q} = 1$, so, in general we will always take these values. We have that

$$
\Gamma_{q,r}(x, 1, 1, \ldots, 1) = \Delta(q) + \Delta(r) + \sum_{\sigma \in \mathrm{Gal}(q)} \Psi_q^\sigma(t) + \sum_{\sigma \in \mathrm{Gal}(r)} \Psi_r^\sigma(t) = \Omega_{q,r}(x).
$$

6.4 Examples

Example 6.4.1: *Let $a \in \mathbb{Q}$, $|a| > 2$. We consider $Y : y^2 = x^3 + ax^2$. Then*

$$
\mu^{ess}(h_Y) \leq \log |a|.
$$

Moreover, the image of h_Y is dense in the interval $[\log |a|, \infty)$

Proof: We consider the parametrization $y = t^3 - at$, $x = t^2 - a$, therefore we have $q(t) = t^2 - a$ and $r(t) = t^3 - at$. Taking $t = 0$ in proposition 6.1.2, we have that $\mu^{ess}(h_Y) \leq \Omega_{q,r}(0)$. Then

$$\begin{aligned}\Omega_{q,r}(0) &= \frac{1}{2\pi}\int_0^{2\pi} \log|e^{2i\theta}-a| + \log|e^{3i\theta}-ae^{i\theta}|d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} 2\log|e^{2i\theta}-a|d\theta \\ &= \frac{1}{2\pi}Re\left(\int_0^{2\pi} 2\log(e^{2i\theta}-a)d\theta\right) \\ &= \frac{1}{2\pi}Re\left(\int_{S_1}\frac{1}{iz}\log|z^2-a|dz\right) \\ &= \log|a|.\end{aligned}$$

using the fact that $a \in \mathbb{Q}$ and Theorem E, we conclude the proof. □

Example 6.4.2: *Let $a, c \in \overline{\mathbb{Q}}$ with $a \neq 0$. We consider $Y : y = ax^2 + c$ and the parametrization $x = q(t) = t$ and $y = r(t) = at^2 + c$. Assume that $|a| - |c| > 1$. Then,*

$$\mu^{ess}(h_Y) \leq \Delta(a,c) + \frac{1}{2}\log|c|$$

Proof: Taking $t = 0$ in proposition 6.1.2, we have that $\mu^{ess}(h_Y) \leq \Omega_{q,r}(0)$. Then

$$\begin{aligned}\Omega_{q,r}(0) &= \Delta(r) + \frac{1}{2\pi}\int_0^{2\pi} \log|ae^{2i\theta}+c|d\theta \\ &= \Delta(r) + \frac{1}{2\pi}Re\left(\int_0^{2\pi} \log(ae^{2i\theta}+c)d\theta\right) \\ &= \Delta(r) + \frac{1}{2\pi}Re\left(\int_{S_1}\frac{1}{2iz}\log(az+c)dz\right) \\ &= \Delta(r) + \frac{1}{2\pi}Re\left(2\pi i\frac{1}{2i}\log(c)\right) \\ &= \Delta(a,c) + \frac{\log|c|}{2}\end{aligned}$$

□

Example 6.4.3: *Let $a, c \in \overline{\mathbb{Q}}$ with $a \neq 0$. We consider $Y : y = ax^n + c$ with $n \in \mathbb{N}$. Assume that, there exists $\sigma_0 \in G(a,c)$, such that $|\sigma_0(c)| - |\sigma_0(a)| > 1$. Then,*

$$\mu^{ess}(h_Y) \geq \frac{1}{[K_{a,c}:\mathbb{Q}]}\min\left\{\log\left(|\sigma_0(c)| - |\sigma_0(a)|\right), \log\left(\sqrt[n]{\frac{|\sigma_0(c)|-1}{|\sigma_0(a)|}}\right)\right\}.$$

Proof: We consider the parametrization $x = q(t) = t$ and $y = r(t) = at^n + c$. Then, we consider

$$g_{\sigma_0}(z) = \log^+|z| + \log^+|\sigma_0(a)z^n + \sigma_0(c)|$$

Firstly, we will take $z = e^{i\theta}$. Then,

$$\begin{aligned}g_{\sigma_0}(e^{i\theta}) &= \log^+|\sigma_0(a)e^{ni\theta} + \sigma_0(c)| \\ &\geq \log(|\sigma_0(c)| - |\sigma_0(a)|).\end{aligned}$$

On the other hand, if $\sigma_0(a)z^n + \sigma_0(c) = e^{i\theta}$, we have $z = \sqrt[n]{(e^{i\theta} - \sigma_0(c))/\sigma_0(a)}$. Then,

$$
\begin{aligned}
g_{\sigma_0}\left(\sqrt[n]{\frac{e^{i\theta} - \sigma_0(c)}{\sigma_0(a)}}\right) &= \log^+\left|\sqrt[n]{\frac{e^{i\theta} - \sigma_0(c)}{\sigma_0(a)}}\right| \\
&\geq \log\sqrt[n]{\frac{|\sigma_0\,(c)| - 1}{|\sigma_0(a)|}}.
\end{aligned}
$$

Both values are achieved. Then, by Proposition 6.2.3, we have that $\min g_{\sigma_0}/[K_{a,c} : \mathbb{Q}] \leq \mathcal{L}(q,r) \leq \mu^{ess}(h_Y)$. This concludes the proof. $\square$

Example 6.4.4: *We consider* $Y : x^2 + y^2 = 1$. *Then*

$$\mu^{ess}(h_Y) \leq \int_0^{2\pi} \log^+\left|\frac{1-(1+e^{i\theta})^2}{1+(1+e^{i\theta})^2}\right| + \log^+\left|\frac{2(1+e^{i\theta})}{1+(1+e^{i\theta})^2}\right| d\theta.$$

Proof: We will consider the parametrization

$$x = q(t) = \frac{1-t^2}{1+t^2} \text{ and } y = r(t) = \frac{2t}{1+t^2}$$

Using proposition 6.1.2 with $t = 1$, we conclude that

$$\mu^{ess}(h_{S_1}) \leq \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|\frac{1-(1+e^{i\theta})^2}{1+(1+e^{i\theta})^2}\right| + \log^+\left|\frac{2(1+e^{i\theta})}{1+(1+e^{i\theta})^2}\right| d\theta$$

In order to obtain a non-trivial upper bound, we have used $S_1 + 1$ instead of S_1. $\square$

Proof of Theorem D: This theorem is a direct consequence of Proposition 6.1.2, Theorem 6.2.3 and Example 6.4.3. $\square$

7 Further directions

Future investigations in this thesis will be related to the upper and lower bounds for $X = E$ an elliptic curve, in this case we will use the Néron-Tate Height $\hat{h}_E : E(\overline{\mathbb{Q}}) \to \mathbb{R}$, given by

$$\hat{h}_E(P) = \lim_{n\to\infty} \frac{h_E(nP)}{n^2}.$$

where h is a Weil height. This is well defined and it is zero if and only if P is a torsion point (see [23] section 6, [14] and [13] chapter B5). Since the set of torsion points is Zariski dense we have that $\mu^{ess}(\hat{h}_E) = 0$. Consider the height $H_E : E(\overline{\mathbb{Q}}) \times E(\overline{\mathbb{Q}}) \to \mathbb{R}$ given by

$$H_E(P,Q) = \hat{h}_E(P) + \hat{h}_E(Q).$$

Note that, given two torsion points $P, Q \in E(\overline{\mathbb{Q}})$, we have that $H_E(P,Q) = 0$. Assume that there is a non-torsion point $N \in E(\overline{\mathbb{Q}})$, then, we can consider the sub-variety of $E(\overline{\mathbb{Q}}) \times E(\overline{\mathbb{Q}})$ given by $V(N) = \{(P, P+N) : P \in E(\overline{\mathbb{Q}})\}$, and then consider $H_E^N = H_E|_{V(N)}$, note that, we can consider H_E^N as a one-variable function, $H_E^N : E(\overline{\mathbb{Q}}) \to \mathbb{R}$, given by, $H_E^N(P) = H_E(P) + H_E(P+N)$.

We have that $V(N)$ is a sub-variety of $E(\overline{\mathbb{Q}}) \times E(\overline{\mathbb{Q}})$ which is not a torsion subvariety. Now, using Zhang's theorem we have that $\mu^{ess}(H_E^N) > 0$, this is the height we are going to study. In order to study this height we will use the Fekete-Szegö theorem for elliptic curves, given in [16].

Let E be an elliptic curve, we consider E in its Weierstrass form, this means $E = E_{p,q} : y^2 = x^3 + px^2 + q$, where $p, q \in \overline{\mathbb{Q}}$. Now, assume that $rank(E) > 0$, then, there are non-torsion rational points in E, in order to find these points, we will use a result proved by Lutz and Nagell (see [19], Corollary 7.2), we consider a point $(x,y) \in E(\mathbb{Q})$, such that $(x,y) \notin E(\mathbb{Z})$; or $(x,y) \in E(\mathbb{Z})$, $2 * (x,y) \neq [0:1:0]$ and y^2 does not divide $\Delta(E) = -16(4p^3 + 27q^2)$, then (x,y) is a non-torsion point.

For instance, we can take $E : y^2 = x^3 + 17$, then, the point $P = \left(\frac{137}{64}, \frac{2651}{512}\right) \in E$ is a non-torsion rational point of E, therefore, the variety $V(P)$ is a non-torsion variety. We can also consider the point $Q = (2,5)$, in this case, $2Q \neq 0$ because the second coordinate of Q is not zero, and also, we have that 5^2 does not divide $\Delta(E) = -2^4 3^3 17^2$, therefore it is a non-torsion point and $V(Q)$ is a non-torsion variety.

We have that our method to obtain intervals of density in lines only works for $a, b \in \mathbb{Q}$, it is an open question if this result can be extended to any $a, b \in \overline{\mathbb{Q}}$.

References

[1] Matthew Baker and Robert Rumely. *Potential theory and dynamics on the Berkovich projective line*, volume 159 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2010.

[2] François Ballaÿ. Successive minima and asymptotic slopes in Arakelov geometry. *Compositio Mathematica*, 157(6):1302–1339, 2021.

[3] Frits Beukers and Don Zagier. Lower bounds of heights of points on hypersurfaces. *Acta Arithmetica*, 79(2):103–111, 1997.

[4] Enrico Bombieri and Walter Gubler. *Heights in Diophantine geometry*, volume 4 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2006.

[5] Siegfried Bosch, Ulrich Güntzer, and Reinhold Remmert. *Non-archimedean analysis*, volume 261. Springer Berlin, 1984.

[6] José Ignacio Burgos Gil, Ricardo Menares, and Juan Rivera-Letelier. On the essential minimum of Faltings' height. *Math. Comp.*, 87(313):2425–2459, 2018.

[7] José Ignacio Burgos Gil, Patrice Philippon, Juan Rivera-Letelier, and Martín Sombra. The distribution of Galois orbits of points of small height in toric varieties. *Amer. J. Math.*, 141(2):309–381, 2019.

[8] José Ignacio Burgos Gil, Patrice Philippon, and Martín Sombra. Successive minima of toric height functions. In *Annales de l'Institut Fourier*, volume 65, pages 2145–2197, 2015.

[9] Christophe Doche. On the spectrum of the Zhang-Zagier height. *Mathematics of computation*, 70(233):419–430, 2001.

[10] Christophe Doche. Zhang-Zagier heights of perturbed polynomials. volume 13, pages 103–110. 2001. 21st Journées Arithmétiques (Rome, 2001).

[11] Gregory Dresden. Sums of heights of algebraic numbers. *Mathematics of computation*, 72(243):1487–1499, 2003.

[12] Valérie Flammang. On the Zhang–Zagier measure. *International Journal of Number Theory*, 14(10):2663–2671, 2018.

[13] Marc Hindry and Joseph H Silverman. *Diophantine geometry: an introduction*, volume 201. Springer Science & Business Media, 2013.

[14] Serge Lang. *Fundamentals of Diophantine Geometry*. Springer Science & Business Media, 2013.

[15] Shenxiong Li. On the spectrum and essential minimum of heights in projective plane. 2019. http://www.shenxionghomepage.com/Research%20Page/1.%20Arithmetic%20Dynamics/On%20the%20Spectrum%20and%20Essential%20Minimum%20of%20Heights%20in%20Projective%20Plane.pdf.

[16] Robert S. Rumely. *Capacity theory on algebraic curves*, volume 1378 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1989.

[17] Charles L Samuels. Lower bounds on the projective heights of algebraic points. *Acta Arith.* 125 (2006)*, no.* 1*,* 41–50.

[18] Wolfgang Schmidt. Heights of algebraic points lying on curves or hypersurfaces. *Proceedings of the American Mathematical Society*, 124(10):3003–3013, 1996.

[19] Joseph H Silverman. *The arithmetic of elliptic curves*, volume 106. Springer, 2009.

[20] L. Szpiro, E. Ullmo, and S. Zhang. Équirépartition des petits points. *Invent. Math.*, 127(2):337–347, 1997.

[21] Xinyi Yuan. Big line bundles over arithmetic varieties. *Invent. Math.*, 173(3):603–649, 2008.

[22] D. Zagier. Algebraic numbers close to both 0 and 1. *Math. Comp.*, 61(203):485–491, 1993.

[23] Shouwu Zhang. Positive line bundles on arithmetic surfaces. *Ann. of Math. (2)*, 136(3):569–587, 1992.

Pontificia Universidad Católica de Chile
Email Address: mimorales4@uc.cl