FACULTAD DE MATEMÁTICAS

ON THE GEOGRAPHY OF 3-FOLDS VIA ASYMPTOTIC BEHAVIOR OF INVARIANTS

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Abstract

Roughly speaking, the problem of geography asks for the existence of varieties of general type after we fix some invariants. In dimension 1, where we fix the genus, the geography question is trivial, but already in dimension 2 it becomes a hard problem in general. In higher dimensions, this problem is essentially wide open. In this paper, we focus on geography in dimension 3. We generalize the techniques which compare the geography of surfaces with the geography of arrangements of curves via asymptotic constructions. In dimension 2 this involves resolutions of cyclic quotient singularities and a certain asymptotic behavior of the associated Dedekind sums and continued fractions. We discuss the general situation with emphasis in dimension 3, analyzing the singularities and various resolutions that show up, and proving results about the asymptotic behavior of the invariants we fix.

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Contents

Chapter 1

Introduction

1.1 Geography problem

We work with normal projective varieties X over the complex numbers $\mathbb{C}$. As usual, when $\dim X = 1$, 2, or $d \geq 3$ we say that X is a curve, a surface, or a d-fold respectively. A central problem in algebraic geometry is to classify varieties of general type, i.e., varieties with positive *canonical volume* $\mathrm{vol}(X)$. By definition

$$\mathrm{vol}(X) = \limsup_{m \to \infty} \frac{h^0(X, mK_X)}{m^d/d!},$$

where K_X is the Weil divisor associated with the dualizing sheaf and $d = \dim X$. It is called the *canonical divisor* of X.

Nowadays this classification problem is guided as follows. Given a non-singular variety X of general type, we first identify a good birational model that represents X. For that, we run the minimal model program to find a variety X' birational to X that has at most terminal singularities, and nef canonical class. After that, we consider its (unique) canonical model X_{can}. This process always works due to [BCHM10]. The numerical biregular invariants of X_{can} become invariants for the birational class. One could consider the top-intersection of $K_{X_{\mathrm{can}}}$, the analytic Euler characteristic $\chi(\mathcal{O}_{X_{\mathrm{can}}})$, the topological Euler characteristic $e(X_{\mathrm{can}})$, etc. Then, after we agree on numerical invariants, the *geography problem* asks: *Given a collection of numbers, is there a canonical variety of general type whose numerical invariants are equal to that collection of numbers?* Often this set of invariants $\mathcal{C}$ are the labels for the corresponding moduli space $M_{\mathcal{C}}$. That moduli space geometrically collects all canonical varieties of general type whose invariants are equal to $\mathcal{C}$. In this way, the geography problem asks whether $M_{\mathcal{C}}$ is not empty for a given $\mathcal{C}$. It turns out that typically $M_{\mathcal{C}}$ is an open variety, but there is a geometric compactification $\overline{M}_{\mathcal{C}}$ [Kol23]. In this compactification, which may be bigger than the closure of $M_{\mathcal{C}}$, we encounter varieties with controlled singularities. Then it is interesting to know whether $\overline{M}_{\mathcal{C}}$ is not empty for a given $\mathcal{C}$, although this does not imply the question for $M_{\mathcal{C}}$ [Rol21]. Hence in this paper, we are interested in a wider geography problem for varieties of general type with (log) terminal/canonical singularities with nef/ample canonical bundle.

In dimension one, the main invariant is the genus g. If X is a non-singular curve of general type, then we have

$$\mathrm{vol}(X) = \deg K_X = -2\chi(\mathcal{O}_X) = -e(X) = 2g - 2 > 0 \Leftrightarrow g \geq 2.$$

Thus, the geography problem asks: *Given an integer $g \geq 2$, is there a curve of genus g?* This can be trivially answered. Indeed, one constructs curves for any $g \geq 2$ as double coverings branched at $2g + 2$ points on $\mathbb{P}^1$. Moreover, there exists a quasi-projective variety M_g of dimension $3g - 3$ which parametrizes all isomorphism classes of curves of genus g. The variety M_g is called the moduli space of curves of genus g. It has a natural compactification due to Deligne and Mumford [DM69], which is an irreducible projective variety $\overline{M}_g$ parametrizing nodal curves of arithmetic genus g and ample dualizing sheaf.

In dimension two the problem is much harder. In order to choose invariants, the Riemann-Roch theorem suggests that, for any surface X, one can pick the top-intersection of canonical class K_X^2, and the analytical Euler characteristic $\chi(\mathcal{O}_X)$. When X is non-singular, we have the Noether's formula $K_X^2 + e(X) = 12\chi(\mathcal{O}_X)$ [Noe75]. Then, these two invariants $K_X^2, \chi(\mathcal{O}_X)$ are equivalent to the two Chern numbers $c_1^2(X) = K_X^2$, and $c_2(X) = e(X)$.

Any pair of birational non-singular curves are isomorphic. In contrast, we have infinitely many birational non-singular models in higher dimensions, for example, by means of blow-ups at points. However, as we said above, we can run the MMP for the canonical class, obtaining a minimal model, which is unique in dimension two. If we fix our attention to minimal surfaces of general type, then we have well-known geographical constraints:

$$c_1^2, c_2 > 0, \quad \text{General type inequality,} \tag{1.1}$$
$$5c_1^2 - c_2 + 36 \geq 0, \quad \text{Noether's inequality [Noe75],} \tag{1.2}$$
$$c_1^2 \leq 3c_2, \quad \text{BMY-inequality [Bog78, Miy77, Yau77].} \tag{1.3}$$

They all are well-known in the theory of surfaces [BHPV04, Ch.VII].

Through the GIT strategy, Gieseker [Gie77] proved that there exists a moduli space $M_{K^2,\chi}$ of surfaces of general type with fixed invariants K^2, χ. It is a quasi-projective scheme. The space $M_{K^2,\chi}$ parametrizes the canonical models of the minimal models, so they may be singular. (We recall that it is a classical theorem of Mumford that the canonical models exist [Zar62].) They have at most ADE singularities, which are precisely the canonical singularities in dimension two. This moduli space $M_{K^2,\chi}$ could be highly singular, and with many irreducible components. It actually satisfies Murphy's law in algebraic geometry [Vak06]. As in the case of curves, there exists a geometric compactification $\overline{M}_{K^2,\chi}$ of $M_{K^2,\chi}$ due to Kollár–Shepherd-Barron [KSB88] and Alexeev [Ale94], but it is constructed now via MMP. These are called *KSBA compactifications*, and they are projective schemes and parametrize singular surfaces with only semi-log-canonical singularities and ample dualizing sheaf. Hence normal surfaces X with log-canonical singularities and K_X ample are KSBA surfaces.

The geography problem for surfaces asks: *Given $(a,b) \in \mathbb{Z}^2$ satisfying the restrictions in (1.1, 1.2, 1.3) and the Noether's formula, is there a minimal surface X such that $c_1^2(X) = a$, and $c_2(X) = b$?*

A pioneer in this question (and the creator of the term "geography") was Ulf Persson [Per81]. He proved the existence of minimal surfaces of general type X with $c_2(X) = a$, $c_1^2(X) = b$ satisfying $\frac{a-36}{5} \leq b \leq 2a$ with $2b \neq a - k$, where $k = 2$, or k is odd with $1 \leq k \leq 15$ or $k = 19$. A more general result was obtained in [Che87] and [Che91]. Putting all together, for every pair (a, b), the region bounded by $b \leq 3a$ and $5b - a + 36 \geq 0$ is realized by the Chern numbers of a minimal surface of general type, except maybe for points in finitely many lines $b - 3a + 4k = 0$ for $0 \leq k \leq 347$. In Figure 1.1, is sketched a map that shows the geographical picture. The black points represent places where such surfaces exist. (Unfortunately, the author is aware that this may not be a complete map.) The red zone is delimited by the inequalities above. The gray lines are defined by $c_1^2 + c_2 = 12k, k \in \mathbb{Z}_{>0}$. The dotted lines have equations $b - 3a + 4k = 0$ for $k = 1, \ldots, 8$, and so there may be infinitely many points with no representative.

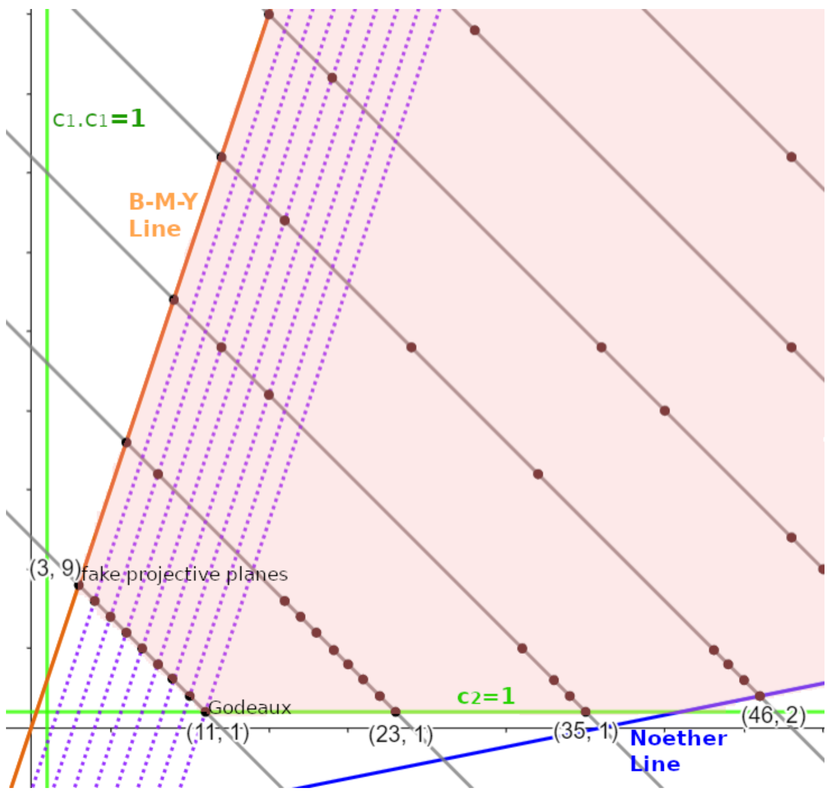


Figure 1.1: A sketched map in coordinates (c_2, c_1^2) for surfaces of general type.

As we just saw, working on the geography problem for pairs (a, b) could be cumbersome. We can weaken the problem considering *Chern slopes* c_1^2/c_2. Somesse [Som84] proves that any rational number in $[1/5, 3]$ is realized as $c_1^2(X)/c_2(X)$ by a minimal surface of general type X, answering a question asked by Hirzebruch in [Hir83, (3.3)]. In particular, there are no "not-allowed-holes" in $[1/5, 3]$. The surfaces constructed by Sommese have no control over other invariants, for example, their fundamental group. One could ask:

How do the slopes of minimal surfaces of general type with fixed fundamental group distribute in $[1/5, 3]$*?* This was not a trivial question even for simply-connected surfaces. In [RU15] it was proved that Chern slopes are actually dense in $[1/5, 3]$ for simply-connected surfaces. The corresponding geography problem for pairs (a, b) is wide open. For the case of other fundamental groups, it was recently proved [TU22] that if G is the fundamental group of a surface, then the Chern slopes of minimal surfaces of general type with π_1 isomorphic to G is dense in $[1, 3]$.

In dimensions greater than or equal to 3, we know dramatically less about their geography. For each dimension, at least we have the existence of moduli spaces for KSB-stable models. A comprehensive treatment can be found in [Kol23], where the following theorem is stated.

Theorem. *The moduli functor $\overline{\mathcal{M}}_{d,K^d}$ of KSB-stable families of dimension d and volumen K^d has a projective coarse moduli space $\overline{M}_{d,K^d}$. In this way, the moduli space M_{d,K^d} of canonical models of dimension d and volume K^d admits a compactification.*

Since in higher dimensions, our minimal models may admit singularities, we do not have directly the notion of Chern numbers. If X is a non-singular 3-fold, then we have Chern numbers

$$c_1^3(X) = -K_X^3, \quad c_1c_2(X) = 24\chi(\mathcal{O}_X), \quad c_3(X) = e(X).$$

For X minimal non-singular 3-fold we have a Noether type inequality [CH06],

$$c_1^3(X) \leq \frac{1}{27}c_1c_2(X) + \frac{10}{3}. \tag{1.4}$$

When X is singular, we consider as invariants for our geography problem: K_X^3, $\chi(\mathcal{O}_X)$, and $e(X)$. As in the case of surfaces, we also have some inequalities for them. If X is minimal of general type, we have $K_X^3 > 0$. In [Hu18], was proved the Noether type inequality for Gorenstein minimal 3-folds of general type:

$$-K_X^3 \leq \frac{4}{3}\chi(\mathcal{O}_X) + 2.$$

We have the Miyaoka-Yau inequality for Gorenstein minimal 3-folds of general type [Miy87],

$$-K_X^3 \geq 72\chi(\mathcal{O}_X),$$

with equality if and only if X is a ball quotient. In [GKPT19] was proved a $\mathbb{Q}$-version of Miyaoka-Yau inequality, i.e., a similar inequality when X is a projective Kawamata log-terminal 3-fold of general type with nef canonical divisor. (In the previous two inequalities above, the results were written in terms of K_X^3 and $\chi(\omega_X)$. However, since we are restricting ourselves to the Gorenstein case, we have $\chi(\omega_X) = -\chi(\mathcal{O}_X)$.)

From the birational geometry of 3-folds, the triple $(K_X^3, \chi(\mathcal{O}_X), e(X)) \in \mathbb{Q} \times \mathbb{Z}^2$ is invariant between minimal models of the same birational class. This due to the following observations:

- The volume K_X^3 is invariant on its birational class because for each other minimal model representative $X' \in [X]$, the birational map relating them induces an isomorphism $X' - B' \cong X - B$, where B, B' are of codimensión at least two [Mat10, 12.1.2]. Then X and X' have the same divisor class group, therefore $K_X^3 = K_{X'}^3$.

- It is well-known that $\chi(X)$ is a birational invariant between non-singular varieties, but not always birational in the singular case. For example [Maa20]. It is known that minimal models in the same birational class are connected by flops [Mat10, Cf. Th.12.1.8.]. They are surgeries reversing the orientation of some K-trivial rational curves. Then let X be a minimal model representative of a birational class, and
$$\begin{tikzcd} C \subset X \arrow[rr, dashed] \arrow[dr, "f"'] & & C^+ \subset X^+ \arrow[dl, "f^+"] \\ & Y & \end{tikzcd}$$
be a flop. It is known that Y has rational singularities, and we have $\mathcal{O}_Y = f_*\mathcal{O}_X = {f^+}_*\mathcal{O}_{X^+}$ since f/f^+ are contractions of C/C^+. Then for each i,
$$H^i(X, \mathcal{O}_X) \cong H^i(Y, f_*\mathcal{O}_X) \cong H^i(Y, f_*^+\mathcal{O}_{X^+}) \cong H^i(X^+, \mathcal{O}_{X^+}),$$
from we get $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{X^+})$.

- For the topological characteristic $e(X)$ we follow [Kol91, Th. 3.2.2.] to sketch that flops leave invariant the Betti numbers $h^i(X)$ for all $i \geq 0$. Let $C \subset X \to Y \leftarrow X^+ \supset C^+$ be a flop as before. Set $A = X \setminus C$, and $A^+ = X^+ \setminus C^+$. Take U, U^+ analytical neighbourhoods of C and C^+ respectively which retract to these rational curves, and set

$B = U \setminus C$ and $B^+ = U^+ \setminus C^+$. Using Mayer-Vietoris for $X = A \cup U$ and $X^+ = A^+ \cup U^+$, we get long exact sequences between singular cohomology groups

$$... \to H^i(B) \to H^i(A) \oplus H^i(U) \to H^i(X) \to H^{i+1}(B) \to ...$$

$$... \to H^i(B^+) \to H^i(A^+) \oplus H^i(U^+) \to H^i(X^+) \to H^{i+1}(B^+) \to ...$$

Since U and U^+ retract to rational curves, we have $H^i(U) = H^i(U^+) = 0$ for all i except for $i = 0, 2$. Also, $H^i(B) = H^i(B^+)$ and $H^i(A) = H^i(A^+)$ for $i \geq 3$, and $H^2(X) = H^2(X^+)$ [Kol91, 2.2.9]. We get $\text{rank}H^i(X) = \text{rank}H^i(X^+)$, i.e., the same Betti numbers. So, we have $e(X) = e(X^+)$.

The main purpose of this thesis is to study the geography problem for 3-folds of general type. We can state the geography problem as follows: *Given a triple $(a, b, c) \in \mathbb{Q} \times \mathbb{Z}^2$ satisfying the restrictions above, is there a 3-fold of general type X such that $(-K_X^3, \chi(\mathcal{O}_X), e(X)) = (a, b, c)$?* We can distinguish two cases:

(C) Geography for minimal 3-folds with terminal singularities, in the realm of canonical models.

(LC) Geography for minimal 3-folds with log-terminal singularities, in the realm of KSB-stable models.

For minimal 3-folds X of general type, $\chi(\mathcal{O}_X)$ and $e(X)$ could be negative, positive or zero. This suggest to take quotients over c_1^3, however this put a limit in our geography of slopes if we want go out of the realm of general type varieties. In [Hun89] was proposed the study of slopes

$$[c_1^3, c_1c_2, c_3] \in \mathbb{P}^2_{\mathbb{Q}}$$

for non-singular minimal models, extending naturally to the singular case by taking slopes $[-K_X^3, \chi(\mathcal{O}_X), e(X)] \in \mathbb{P}^2_{\mathbb{Q}}$. This, allows us to work by charts. Indeed, in [Hun89], the author studied slopes of non-singular 3-folds with ample canonical class in the chart $c_1c_2 \neq 0$. Moreover, since $\chi(\mathcal{O}_X) < 0$ for Gorenstein minimal models of general type, we can extend the zone of study in that chart for now.

Finally, a first big question that one can ask is: *Those slopes lie in a bounded region?*. Observe that all inequalities above just bound the coordinate c_1^3/c_1c_2. In [CL01] is proved that the Chern slopes $[c_1^3, c_1c_2, c_3]$ of non-singular 3-folds with ample canonical bundle defines a bounded region by finding inequalities for the coordinate c_3/c_1c_2. However, this bound is not optimal. In a more general set-up, let $N(d)$ be the number of partitions of d. In [DS22] is proved that Chern slopes $[c_1^d, \dots, c_d] \in \mathbb{P}_{\mathbb{Q}}^{N(d)-1}$ of non-singular d-folds with ample canonical bundle define a bounded region.

To illustrate the case of 3-folds, in Figure 1.2 we see the map in chart $c_1c_2 \neq 0$ for minimal non-singular 3-folds of general type. In the rest, we describe the zones on that map.

- **Allowed zone:** This is the region bounded by the Miyaoka-Yau inequality $c_1^3/c_1c_2 \leq 8/3$ and $c_1^3/c_1c_2 > 0$. They are sketched by black lines.

- **Noether's line:** Since $\chi(\mathcal{O}_X) < 0$ is an integer, we can avoid small values of this invariant. Thus, from the Noether's inequality (1.4), 3-folds with $|c_1c_2| \gg 0$ must satisfy $c_1^3/c_1c_2 \geq 1/27$. Thus, the *Noether's line* is the **red line** defined by $c_1^3/c_1c_2 = 1/27$.

- **Cartesian Product Zone (CP line):** The zone CP collects 3-folds of the form $S \times C$ for surface S and a curve C both with ample canonical bundle [Hun89, Subsection 7.2.3] (also Example 2.2.5). Hunt proved that the Chern slopes $(c_1^3/c_1c_2, c_3/c_1c_2)$ of these 3-folds distribute densely along the line in $\mathbb{A}^2$ connecting the points $(9/4, 1/4)$ and $(1/2, 5/6)$. This is the **purple line** on the map.

- **3-folds in $\mathbb{P}^4$ and $\mathbb{P}^5$.** The minimal non-singular singular 3-folds of general type in $\mathbb{P}^4$ are hypersurfaces of degree $d > 5$. Thus, they have well-known Chern classes (see Section 5.2). The Chern numbers have the form
$$\begin{aligned} c_1^3 &= -d(d^3 - 15d^2 + 75d - 125), \\ c_1c_2 &= -d(d^3 - 10d^2 + 35d - 50), \\ c_3 &= -d(d^3 - 5d^2 + 10d - 10). \end{aligned}$$
So, their slopes $(c_1^3/c_1c_2, c_3/c_1c_2)$ are asymptotically near to $(1, 1)$ as d grows. On the map, they are sketched with **blue points**.

In [Cha99] was proved that 3-folds embedded in $\mathbb{P}^5$ whose canonical class K satisfy $K^iH^j > 0$ for a hyperplane section, distribute along the line defined by

$$x + y = 2, \quad 1 \leq x \leq 2.$$

We sketch it in the map with a **blue line**.

- **Smooth Complete Intersection Zone (SCI):** The zone SCI collects non-singular complete intersections with ample canonical sheaf. In [Hun89] was studied the case of X being a complete intersection in $\mathbb{P}^{r+3}$ of n hypersurfaces of degree $a_1+1, ..., a_r+1$ with $a_i \geq 1$. Observe that

$$K_X = \left(\sum_{i=1}^{r} a_i - 4\right) H|_X,$$

so in case $r \geq 5$ we are talking about smooth minimal 3-folds of general type. Moreover, we have

$$\frac{c_1^3(X)}{c_1c_2(X)} = \frac{2(s_1-4)^2}{s_1^2+s_2-6s_1+12},$$

$$\frac{c_3(X)}{c_1c_2(X)} = \frac{(s_1^3+3s_1s_2+2s_3)/3 - 2(s_1^2+s_2)+6s_1-8}{(s_1^2+s_2-6s_1+12)(s_1-4)},$$

where $s_j = \sum a_i^j$. In [Cha97] it is proved the existence of an optimal region for which we have density.

Theorem 1.1.1. *Let $R \subset \mathbb{Q}^2$ be the region between the points $(1,1)$ and $(2,1/3)$, and closed between the curves*

$$U: \quad y + \frac{x}{3} - 1 = \frac{(2-x)^{3/2}}{3\sqrt{x}}$$

and the piece-wise function

$$L: \quad y = 1 - \frac{x}{3} + \frac{2-x}{n} - \frac{2x}{3n^2} + \frac{(n-2)x}{3n^2\sqrt{n-1}}\left(\frac{2n}{x} - n - 1\right)^{3/2},$$

where $2 - 2/n < x < 2 - 2/(n+1)$ for each $n \geq 2$. Every point $p \in R$ realizes a smooth complete intersection with given Chern slopes $(c_1^3/c_1c_2, c_3/c_1c_2) = p$.

This region is sketched with orange color on the map. The problem of this region is that it not contains all SCI 3-folds. The remainder are distributed discretely outside the region R. However, in [SXZ14] was proved the following.

Theorem 1.1.2. *The convex closure of the slopes* $(c_1^3/c_1c_2, c_3/c_1c_2)$ *for SCI* 3*-folds is a rational polyhedron with infinitely many faces. The vertices of these polyhedron are*

$$p_1 = (1/16, 43/8), \quad p_2 = (1/10, 19/5),$$
$$p_3 = (1/8, 13/4), \quad p_4 = (1/3, 23/12),$$
$$p_n = \left(\frac{2(n-4)^2}{n^2-5n+12}, \frac{n^3-3n^2+14n-24}{n^2-5n+12}\right), \quad n \geq 5,$$
$$p_\infty = (2, 1/3).$$

The region described in this theorem is sketched with the color yellow on the map.

- **Zone Fermat:** Generalizing techniques that Sommese developed for the case of surfaces using the basic construction of Fermat covers, in [Hun89, Th. 8.2.1 and Th.8.2.2] was proved that there exists triangles ΔABC in $\mathbb{Q}^2$ such that for every $(a, b) \in \Delta ABC$ there exists a smooth projective 3-fold X with ample canonical bundle such that,

$$\left(\frac{c_1^3(X)}{c_1c_2(X)}, \frac{c_3(X)}{c_1c_2(X)}\right) = (a, b).$$

Indeed, their construction is based on pivotal points on the CP line. For example,

$$A = (12/11, 1/11), B = (6/5, 3/5), C = (41/33, 19/33),$$
$$A = (1, -2/5), B = (1, 2/3), C = (32/29, 55/87).$$

They are sketched by **light green** triangles on the map. The disadvantage is that the above triangles are the larger found. Other attempts by Hunt just construct *microscopic triangles* that can not be sketched in the map. This technique was improved in [Liu97]. The author used other pivotal points to construct another dense region. Indeed, it is constructed by infinitely many Hunt's triangles. The regions constructed are sketched with **green**.

- **Isolated points:** The isolated **gray points** show minimal non-singular 3-folds constructed by Fermat over arrangements of hyperplanes, and desingularizations of complete intersection. In this case, we have examples for $c_3 > 0$ [Hun89, Section 7.3]. We sketch two of these points

$$(1.4, -0.22) \qquad \text{Fermat cover over } CEVA^3(2),$$

$$(0.83, -0.125), \qquad \text{Desingularization of complete intersection } (5,5) \text{ in } \mathbb{P}^5\ .$$

- **Ball quotient point:** Among all isolated points there is one special. The Miyaoka-Yau inequality asserts that every non-singular 3-fold with ample canonical bundle satisfying $c_1^3 = \frac{8}{3}c_1c_2$ is a smooth ball quotient. By Hirzebruch proportionality [Hir58] it is known that this ball quotient must satisfy $c_3/c_1c_2 = \frac{1}{6}$. This implies that all non-singular varieties with ample canonical bundle and equating the Miyaoka-Yau inequality lies in only **one point**, i.e., $(8/3, 1/6)$.

- **Unknown Zone:** Hunt conjectured that over the dense zone constructed in [Cha97] is empty. However, the result of [SXZ14] expands this region for discrete points in it. Thus we redefine the *unknown zone* as the region over the smooth complete intersection and over the extension of the CP line from the point $(2, 1/3)$. In this new region, we do not know about the existence of minimal non-singular 3-folds with such slopes. This is the *unknown zone*. See Section 6.3.

1.2 About this thesis

When working with slopes, it turns out that we can apply the tool of n-th root coverings to investigate their behavior. Although it is not a straightforward result for dimensions 2 or more, in dimension 1 it is easy to see this phenomenon.

Consider the following data. Let Z be a non-singular projective variety of dimension d, and let $D_1, \ldots, D_r$ be distinct prime divisors in Z. Assume that $D_{red} := D_1 + \ldots + D_r$ is a simple normal crossing divisor (SNC). Let $n > 1$ be a positive integer, and let $0 < \nu_i < n$ be a collection of r integers

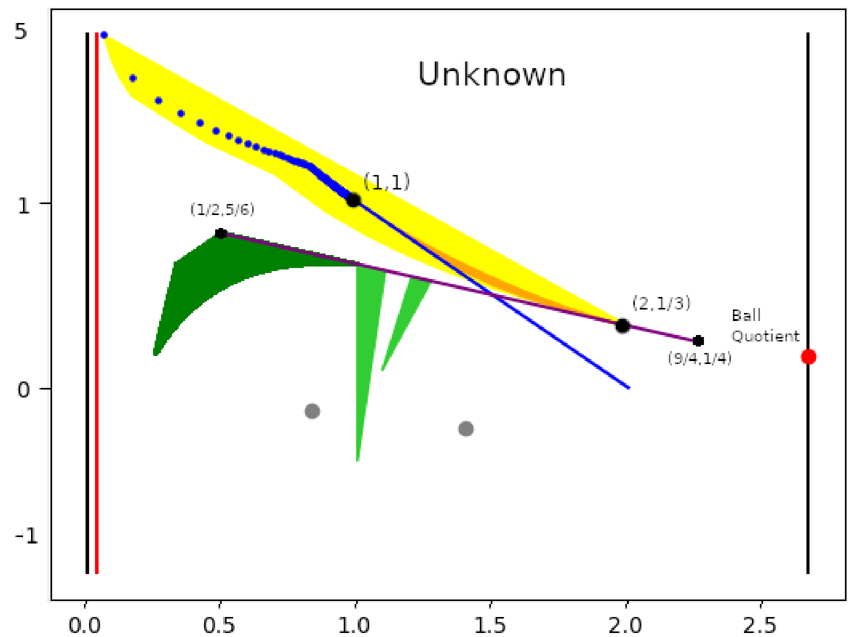


Figure 1.2: The map in coordinates $(c_1^3/c_1c_2, c_3/c_1c_2)$ for non-singular minimal 3-folds of general type. The y-axis is scaled to show the complete picture.

coprime to n. Assume that there exists a line bundle $\mathcal{L}$ on Z such that

$$\mathcal{L}^{\otimes n} \simeq \mathcal{O}_Z\left(\sum_{i=1}^{r} \nu_i D_i\right). \tag{1.5}$$

Then, there exists an n-th root cover $h_n \colon Y_n \to Z$ branched along D_{red}, where Y_n is a normal projective variety (Section 2.3). There is an action of $\mathbb{Z}/n\mathbb{Z}$ on Y_n such that $Z = Y_n/(\mathbb{Z}/n\mathbb{Z})$. These are the n-th root covers developed by Esnault and Viehweg in [Esn82], [Vie82] (Cf. [EV92]). For curves, the prime divisors $D_1, \ldots, D_r$ are distinct points on Z, and the existence of such $\mathcal{L}$ is equivalent to $\sum_{i=1}^{r} \nu_i \equiv 0$ modulo n. Moreover, since points are isolated, we have Y_n as a non-singular curve. By the Riemann-Hurwitz formula, we have

$$\frac{c_1(Y_n)}{n} = \frac{nc_1(Z) - (n-1)r}{n} = c_1(Z) - r + \frac{r}{n} = \bar{c}_1(Z, D_{red}) + \frac{r}{n}.$$

Here $\bar{c}_1(Z, D_{red})$ is the first logarithmic Chern number of the pair (Z, D_{red}) (see Section 2.2). Hence, if we fix the points D_{red} and we consider partitions

$\nu_1 + \ldots + \nu_r = n$ with $n \gg 0$ a prime number, then we asymptotically have $c_1(Y_n) \approx n\bar{c}_1(Z, D_{red})$.

Question 1.2.1. *Does this asymptotic phenomenon happen in higher dimensions?*

In Chapter 3, we prove that this phenomenon occurs in general when the branch locus is a disjoint collection of non-singular distinct prime divisors $D_1, \ldots, D_r$. In this case, as above, we have Y_n as a non-singular projective variety, and it is independent of the multiplicities ν_i.

Theorem 1. *Assume we have n-th root covers $h : Y_n \to Z$ branched at $D = \sum_j \nu_j D_j$, for n arbitrarily large. If D_{red} is non-singular, then for each partition $i_1 + \ldots + i_m = d$, the Chern numbers satisfy,*

$$\frac{c_{i_1} \ldots c_{i_m}(Y_n)}{n} \approx \bar{c}_{i_1} \ldots \bar{c}_{i_m}(Z, D),$$

for prime numbers $n \gg 0$.

However, this theorem is restrictive for us in terms of geography. It is difficult to get the necessary hypothesis to construct minimal d-fold of general type. See Section 3.2. On the other hand, we do not drop the possibility of having applications in other contexts. Also, it is a cornerstone in our research and open the following discussion.

For $\dim Z \geq 2$, if the branch divisor $D_{red} = D_1 + \cdots + D_r$ has singularities, then Y_n have rational singularities [Vie77]. In order to have well-behaved invariants, we can choose a (partial) resolution of singularities. For $\dim Z = 2$, the asymptoticity of Chern numbers was proved in [Urz09] for *random* n-th root covers. Let us explain briefly what random means (see Section 2.7 for more details). First, in dimension two, each singularity of Y_n is a cyclic surface singularity of type $\frac{1}{n}(q, 1)$ for some $0 < q < n$. Thus, we use the Hirzebruch-Jung algorithm (see Section 2.5) to resolve these singularities in a minimal way. For us, there are two important quantities, (1) the *length of the resolution*, i.e., the number of steps of the algorithm, and (2) the Dedekind sums,

$$d(q, 1, n) = \sum_{i=1}^{n-1} \left(\left(\frac{iq}{n}\right)\right) \left(\left(\frac{i}{n}\right)\right),$$

where $((\cdot)) : \mathbb{R} \to \mathbb{R}$ is the saw-tooth function (see Section 2.6). We get a resolution of singularities $X_n \to Y_n$. However, the Chern numbers c_1^2 and c_2 depend on the lengths and the Dedekind sums coming from all cyclic singularities resolved. To guarantee asymptoticity, we have to consider *asymptotic arrangements*. Indeed, for each prime number $n \geq 17$, there exists a set $O_n \subset \{1, \ldots, n\}$ (Section 2.7) such that for each $q \in O_n$, the lengths and Dedekind sums are bounded by $c\sqrt{n}$ for a constant $c > 0$. Thus we say that D_{red} is an asymptotic arrangement if satisfies :

- For prime numbers $n \gg 0$, there exists multiplicities $0 < \nu_j < n$ such that the singularity over $D_j \cap D_k$ of Y_n is of type $\frac{1}{n}(q_{jk}, 1)$ with $q_{jk} \in O_n$.
- For each n there are line bundles $\mathcal{L} \in \mathrm{Pic}(Z)$ such that

$$\mathcal{L}^{\otimes n} \simeq \mathcal{O}_Z \left(\sum_{j=1}^{r} \nu_j D_j \right).$$

The main set-up is when there exists $H \in \mathrm{Pic}(Z)$ such that $D_j \simeq H$ for each component of D_{red}. Observe that the condition (1.5) is satisfied as

$$D = \nu_1 D_1 + \ldots + \nu_r D_r \sim (\nu_1 + \ldots + \nu_r) H \sim nH,$$

thus we can construct n-th root covers. In [Urz09] was proved that for a random partition $\nu_1 + \ldots + \nu_r = n$, the probability of D being an asymptotic arrangement tends to 1 as n grows. In this way, for $n \gg 0$ we take random asymptotic partitions, and we get a family of random surfaces X_n with

$$c_1^2(X_n) \approx n\bar{c}_1^2(Z, D_{red}), \quad c_2(X_n) \approx n\bar{c}_2(Z, D_{red}).$$

Now we can discard the SNC property for the branch divisor. Consider $D_1, \ldots, D_r$ non-singular curves on Z, such that any two of them intersect transversally, and $D_{red} = D_1 + \ldots + D_r$ no necessarily SNC. Singularities of D_{red} are m-points, this is, points with exactly m curves passing through them. We consider a log *resolution* $\gamma \colon Z' \to Z$, i.e., a sequence of blow-ups over the m-points of D_{red} with $m > 2$, such that the reduced divisor defined by $\gamma^* D_{red}$ is SNC. We denote it by D'_{red} and observe that it contains the inverse preimage of each D_j, together with the exceptional data of h_n. In this case, we set $\bar{c}_i(Z, D_{red}) := \bar{c}_i(Z', D'_{red})$ for each i. If D'_{red} turns out to be an asymptotic arrangement, then we have the asymptotic result for Chern numbers as above. See Theorem 2.7.8 in Section 2.7 or [Urz16] in order to connect this result with minimal models.

Example 1.2.2. *A direct application is a relation between Chern slopes of simply-connected surfaces of general type and Chern slopes of arrangements of lines. Let $D_1, \ldots, D_r$ be a collection of r lines in $Z = \mathbb{P}^2$. Let $D_{red} = D_1 + \ldots + D_r$, and denote by t_m be the number of m-points of D_{red}. To prove the asymptotic result on Chern slopes, we assume that $t_r = t_{r-1} = 0$ (those are trivial arrangements). Then, $D'_{red} = (\gamma^* D_{red})_{red}$ is an asymptotic arrangement. Moreover, since each D_i is equivalent to a given line, the multiplicities for D'_{red} depend on partitions $\nu_1 + \ldots + \nu_r = n$ [Urz09, Th. 6.1.]. Thus, the n-th root cover construction $X_n \to Y_n \to Z' \to Z$ produces (non-singular) surfaces of general type X_n satisfying*

$$\frac{c_1^2(X_n)}{c_2(X_n)} \to \frac{\bar{c}_1^2(\mathbb{P}^2, D_{red})}{\bar{c}_2(\mathbb{P}^2, D_{red})} = \frac{9 - 5r + \sum_{m\geq 2}(3m-4)t_m}{3 - 2r + \sum_{m\geq 2}(m-1)t_m}.$$

It turns out that the surfaces X_n are simply-connected. Then, the geography of line arrangements is translated into results about surfaces of general type. For complex line arrangements with $t_r = t_{r-1} = 0$, one can prove that

$$2 - \frac{2}{r-2} \leq \frac{\bar{c}_1^2}{\bar{c}_2} \leq 3 - \frac{1}{3},$$

and slopes are dense in $[2, 5/2]$ [EFU22]. (It is not known if there is any accumulation point in $(5/2, 8/3]$.)

In this way, in higher dimensions, we have several issues to achieve an analog asymptotic result. For example, the singularities of Y_n are not cyclic, and the choice of a right (partial) resolution of Y_n with good behavior as n grows is a challenging problem.

Question 1.2.3. *Is there an analog of the asymptotic results in dimension two for dimension three?*

For instance, if this question has a positive answer, then we would be able to study the geography of 3-folds using arrangements of planes in $\mathbb{P}^3$. The first result in this direction is in Section 4.1, where we find that the Chern number $c_1c_2 = 24\chi$ is *asymptotic* independents of the chosen resolution.

However, the volume and the topological characteristic depend on the chosen resolution. As above, we may be interested in a log-geography. In this way, the singularities of Y_n are log-terminal, however of multiplicity n^2, too big. This means that in order to connect with a non-singular model, at a bad choice of resolution we could have big exceptional data with respect to n, and so we would lose the asymptoticity of invariants at least topologically. In Section 4.2, we introduce a prototype of *first step*, i.e., by toric methods we construct a *local cyclic resolution* . In this way, we get singularities of multiplicity lower than n, and of cyclic quotient type. We are interested in cyclic quotient singularities since they are log-terminal and have a well-known algorithm to resolve them: the Fujiki-Oka continuous fraction [Ash19, Cf.]. In Section 4.3 we globalize this local cyclic resolution, and we get a cyclic resolution $X_n \to Y_n$. It is important to note that it is an embedded $\mathbb{Q}$-resolution in the language of [ABMMOG12], introduced as an efficient resolution without *useless data*. We summarized the work of that section in the next theorem.

Theorem 2. *Let Z be any non-singular projective 3-fold, and let $\{D_1, \dots, D_r\}$ be an asymptotic arrangement. For prime numbers $n \gg 0$ there are projective 3-folds $X_n \to Z$ with at most cyclic quotient singularities such that*

$$\frac{-K_{X_n}^3}{n}, \frac{24\chi(\mathcal{O}_{X_n})}{n}, \frac{e(X_n)}{n} \approx \bar{c}_1^3(Z,D), \bar{c}_1\bar{c}_2(Z,D), \bar{c}_3(Z,D),$$

where $\bar{c}_i(Z,D)$ are the Chern classes of the arrangement.

This result can be extended to an arbitrary collection of surfaces $\mathcal{A} = \{D_1, \dots, D_r\}$ having only simple crossing, i.e., each D_j is non-singular, pairwise intersections are transversal, and components of $D_{j_1} \cap \ldots \cap D_{j_e}$ are non-singular. In Proposition 2.7.7 we use a log-resolution $Z' \to Z$ to get a SNC arrangement which results to be asymptotic if $\mathcal{A}$ is. In Z', we denote the logarithmic Chern classes for the reduced total transform of D as $\bar{c}_i(Z,D)$. Therefore, we can apply again Theorem 2. For example, using Platonic arrangements (see Example 2.7.9), we get 3-folds with at most cyclic singularities whose slopes are arbitrarily close to those of the Platonic solids.

Name	$\bar{c}_1^3(Z,D)/\bar{c}_1\bar{c}_2(Z,D)$	$\bar{c}_3(Z,D)/\bar{c}_1\bar{c}_2(Z,D)$
Hexahedron	11/27	17/27
Octahedron	19/31	16/31
Dodecahedron	623/705	31/205
Icosahedron	2459/2345	25/469

Table 1.1: Logarithmic Slopes for Platonic Solids

Finally, with these constructions: *What can we say about geography of 3-folds of general type?* As an application of the above theorem to the geography of 3-folds, we have two results about the behavior of the slopes of invariants. In Chapter 5 we see our resolution in terms of pairs $(X_n, \tilde{D}_{red}) \to (Z, D_{red})$. As a corollary, we prove that for asymptotic arrangements of hyperplane sections on a minimal 3-fold of general type, the resolution preserves the bigness of the *log*-canonical divisor $K_{X_n} + \tilde{D}_{red}$.

Corollary 1.2.4. *Let $Z \hookrightarrow \mathbb{P}^d$ be a minimal non-singular projective 3-fold of general type, and let $\{H_1, \ldots, H_r\}$ be a collection of hyperplane sections in general position. Then, for prime numbers $n \gg 0$ there are finite morphisms of degree n $(X_n, \tilde{D}_{red}) \to (Z, D_{red})$ such that:*

1. *X_n is of* log-*general type, i.e., $K_{X_n} + \tilde{D}_{red}$ is big and nef,*
2. *$K^3_{X_n} > 0$,*
3. *X_n has cyclic singularities (*log-*terminal) of order lower than n, and*
4. *the slopes $(-K^3/24\chi, e/24\chi)$ of X_n are arbitrarily near to $(2, 1/3)$.*

The importance of this theorem is: We are in the realm of log-minimal models, and the existence of a non-singular model depends on resolve cyclic quotient singularities of order n. We point out that $K^3_{X_n} > 0$, so in future work, if we are able to apply the MMP, we will have K_{X_n} nef, so X_n could be of general type. For us the goal is that the asymptotic behavior of the slopes of X_n coincides with the slopes of its minimal models (see Theorem 2.7.8). For this, we need a *good terminalization* of the cyclic singularities obtained. In Chapter 6 we discuss about what means the word *good*.

In Section 5.2, choosing a similar path as for Theorem 2, we construct minimal non-singular 3-fold of general type using as a base 3-folds $Z \hookrightarrow \mathbb{P}^4$ with 3 hyperplane sections. So, we get almost asymptoticity of the volume. This means that with respect to $n \gg 0$ the volume converges to $c_1^3(Z, D)$ plus another finite quantity. However, this allows us to prove.

Theorem 3. *For $d > 5$ and $n \gg 0$ there are projective minimal non-singular 3-folds X_n of general type over Z with slopes*

$$\frac{c_1^3}{c_1c_2} \approx \frac{(d-2)^3 - 1}{(d-2)(d-1)^2}, \quad \frac{c_3}{c_1c_2} \approx \frac{(d-5)(d^2+2d+6)}{(d-2)(d-1)^2}.$$

In particular, as the degree of Z grows, the slopes have limit point $(1, 1)$.

We obtain a new map in Figure 1.3. The reader can observe the position of the point $(2, 1/3)$ where cyclic resolutions along hyperplane section arrangements accumulate. Indeed, this is in the border of the Unknown Zone. See discussion in Section 6.3. The slopes of Theorem D are inside of a parametrized curve accumulating at $(1, 1)$ defining new points on the map. Finally, in Chapter 6 we discuss the future of this work.

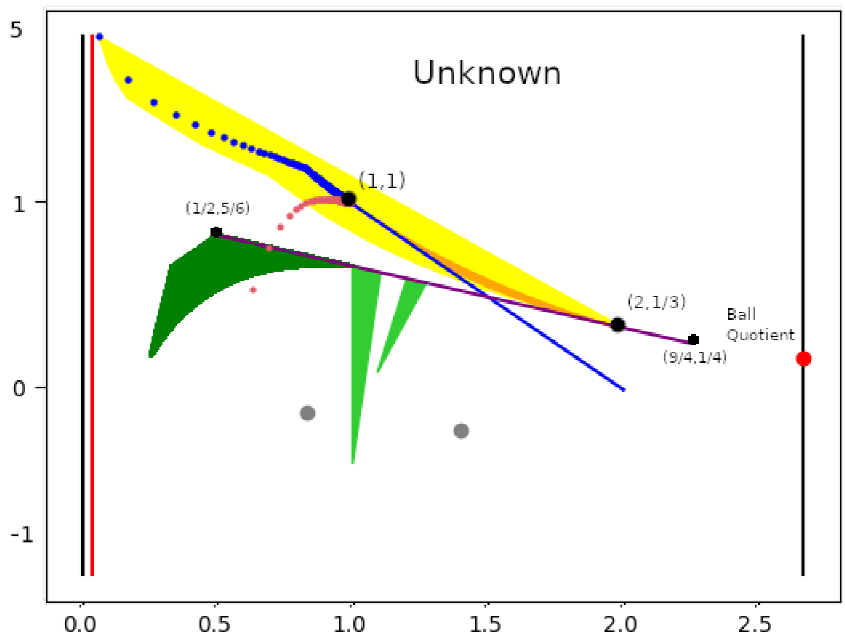


Figure 1.3: Map after this work. The parametrized curve of color contains minimal non-singular 3-folds of general type with slopes arbitrarily near to $(1, 1)$.

Chapter 2

Preliminaries

2.1 Intersection numbers

For us, the main reference for intersection theory will be [Ful98] and [Har77].

Let Z be a variety over $\mathbb{C}$ of dimension d. We have *Chow homology groups* $A_e(Z)$ of e-cycles, i.e., their elements are finite sums of the form $\sum_i n_i[V_i]$ where $n_i \in \mathbb{Z}$, and $[V_i]$ is the class of a subvariety $V_i \hookrightarrow Z$ of dimension e modulo rational equivalence. Any 0-cycle $\sum_i n_i[p_i]$ has a *degree* given by $\sum_i n_i$. Any e-cycle in $A_e(Z)$ can be intersected with a Cartier divisor D giving an element in $A_{e-1}(Z)$. If $[V]$ is the cycle defined by a closed subvariety $V \subset Z$ of dimension e, and $D_1, ..., D_e$ are Cartier divisors on Z, then we have *intersection numbers* well-defined as the degree of the intersection $D_1 \, \ldots \, D_e \, [V] \in A_0(Z)$. In this context, we will abuse the notation and denote these intersection numbers as $D_1 \, \ldots \, D_e \, V \in \mathbb{Z}$. For d divisors, we just denote

$$D_1 \, \ldots \, D_d := D_1 \, \ldots \, D_d \, Z \in \mathbb{Z},$$

this allow us to write $D^d := D^d \, Z$ for any Cartier divisor D.

Remark 2.1.1. *If Z is a normal projective variety, the dualizing sheaf defines a Weil divisor K_Z on Z which coincides with the canonical divisor if Z is non-singular. It will be important the case when K_Z is $\mathbb{Q}$-Cartier, so any closed curve C has a* curvature number *defined as*

$$K_Z \, C = \frac{1}{n}(nK_Z \, C) \in \mathbb{Q},$$

where n is the index of K_Z, i.e., the minimum positive integer such that nK_Z is Cartier. The word curvature arises since in the non-singular case it is well-known that the curvature of Z along C is measured by $-K_Z C$. We say that C is K-(trivial, positive or negative) if $K_Z C =, >, < 0$.

Remark 2.1.2. *As an extension of the above remark, we can define intersection numbers $D\,C \in \mathbb{Q}$ for any $\mathbb{Q}$-Cartier divisor D and $C \in A_1(Z)$, i.e.,*

$$D\,C = \frac{1}{n}(nD\,C),$$

where n is the index of D. Thus, if Z is $\mathbb{Q}$-factorial, we have well-defined intersection numbers on Z but now with values in $\mathbb{Q}$. In this way, we say that two divisors D_1 and D_2 are $\mathbb{Q}$-numerical equivalent *and denoted by $D_1 \sim_{\mathbb{Q}} D_2$ if for any curve C we have $D_1\,C = D_2\,C$.*

Let $f\colon Y \to Z$ be a proper morphism between varieties. There is an induced push-forward of cycles $f_*\colon A_e(Y) \to A_e(Z)$ given by

$$f_*[W] = \begin{cases} \deg(f|_W)[\overline{f(W)}] & \text{if } \dim f(W) = e, \\ 0 & \text{if } \dim f(W) < e. \end{cases}$$

Sometimes we can pull back a Cartier divisor, and we want to consider that aspect in our intersection numbers [Ful98, Sec. 2.2]. Let D be a Cartier divisor, then we have a pull-back sheaf $f^*\mathcal{O}_Z(D)$. If there exists a Cartier divisor E on Y such that $|E| \subset f^{-1}(|D|)$ and $f^*\mathcal{O}_Z(D) = \mathcal{O}_Y(E)$, we define $f^*D = E$ as the *pull-back* of D. In particular, if D is Cartier effective always we can pull it back.

Remark 2.1.3. *If f is flat, we can define a topological a pull-back $f^{-1}\colon A_e(Z) \to A_e(Y)$ given by $f^{-1}[V] = [f^{-1}(V)]$, where $f^{-1}(V)$ is defined as follows: If $\mathcal{I}_V$ is the ideal sheaf of V, then $f^{-1}(V)$ is the closed subvariety defined by the ideal sheaf $f^{-1}(\mathcal{I}_V)\mathcal{O}_Y$. In particular, it is identified with $Y \times_Z V$.*

Example 2.1.4. *Consider the finite flat morphism $f\colon \mathrm{Spec}(A) \to \mathbb{A}^d$, where $A = \mathbb{C}[x_1, \ldots, x_d, t]/(t^n - x_1 \cdots x_d)$, and the closed subvariety $V \hookrightarrow \mathbb{A}^d$ defined by $x_1 = 0$. Denote by D the Cartier divisor defined by V. We have $f^{-1}(V)$ as a closed subvariety of $\mathrm{Spec}(A)$ defined by the prime ideal $(t, x_1) \in \mathrm{Spec}(A)$, i.e., $f^{-1}[V]$ is the class of such closed subvariety. On the other hand, locally on $x_2 \cdots x_d \neq 0$, V is defined by the local equation $t^n = 0$. Let D' the Weil divisor corresponding to $f^{-1}(V)$, so we have $f^*D = nD'$. Observe that*

D' is $\mathbb{Q}$-Cartier. In general, if $A = \mathbb{C}[x_1, \dots, x_d, t]/(t^n - x_1^{\nu_1} \cdots x_d^{\nu_d})$ with $0 < \nu_j < n$, the same argument shows that V is defined by local equation $t^{n/\gcd(n,\nu_1)} = 0$, i.e.,

$$f^*D = \frac{n}{\gcd(n, \nu_1)} D'.$$

In the following, if a pull-back of a Cartier divisor f^*D appears, we assume that it exists.

Proposition 2.1.5. *We have the following properties.*

1. *The intersection numbers are local, i.e., if U is an open subscheme of Z which contains the support of $D_1, \dots, D_e$ and V, then*

$$D_1 \dots D_e V = D_1|_U \dots D_e|_U V|_U.$$

2. *$D_1 \dots D_e V$ is symmetric and multilinear in the D_j's.*

3. *If $Z \hookrightarrow \mathbb{P}^d$ and $\mathcal{O}_Z(H) = \mathcal{O}_Z(1)$, then $H^d = \deg Z$.*

4. *Projection formula: Let $W \hookrightarrow Y$ be a subvariety, then $f^*D\,[W] = D\,f_*[W]$ for any Cartier divisor D. As consequence,*

$$f^*D_1 \dots f^*D_e V = D_1 \dots D_e f_*[V].$$

5. *Push-pull formula: For Cartier divisors $D_1, \dots, D_d$ we have,*

$$f^*D_1 \dots f^*D_d = \deg(f) D_1 \dots D_d.$$

Proof. See [Ful98, Sec. 2.3 & 2.4]. □

Analogously, we can define the *Chow cohomology groups* $A^e(X)$ given by e-cycles of subvarieties of codimension e modulo rational equivalence [Har77, App. A.]. We can define

$$A_\bullet(Z) := \bigoplus_{e \geq 0} A_e(Z), \quad A^\bullet(Z) := \bigoplus_{e \geq 0} A^e(Z).$$

We have natural homomorphisms $\mathrm{Pic}(Z) \to A_{n-1}(Z)$ and $\mathrm{Pic}(Z) \to A^1(Z)$ which in general are not isomorphisms. However, if we assume Z non-singular, we have $A_1(Z) \cong A^{n-1}(Z) \cong \mathrm{Pic}(Z)$. Moreover, the intersection with divisors extends to the intersection between cycles giving a ring structure to both $A_\bullet(Z)$ and $A^\bullet(Z)$. Indeed, we have $A_\bullet(Z) \cong A^\bullet(Z)$ [Ful98, Ch. 19]. In this case, we call it the *Chow ring* of Z and we just denote it as $A(Z)$.

Remark 2.1.6. *For proper morphisms $f\colon Y \to Z$ between non-singular varieties, the pull-back $f^* : A(Z) \to A(Y)$ always exists. It coincides with both, f^* for Cartier divisor and f^{-1} from Remark 2.1.3. Indeed, it induce a morphism of rings $f^*\colon A(Z) \to A(Y)$.*

In the rest of this thesis, if we are working in a non-singular context, then we will use interchangeably A_{d-e} and A^e.

2.2 The Chern numbers

Let Z be a non-singular variety of dimension d, and let $\mathcal{E}$ be a vector bundle on Z of rank r. We have a non-singular projective bundle $\mathbb{P}(\mathcal{E}) \to Z$ with a well-known Chow ring [EH16, Ch. 9],

$$A(\mathbb{P}(\mathcal{E})) \cong \frac{A(Z)[\xi]}{(c_0(\mathcal{E})\xi^r + c_1(\mathcal{E})\xi^{r-1} + \ldots + c_r(\mathcal{E}))},$$

where ξ is the class of the tautological bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, and $c_e(\mathcal{E}) \in A^e(Z)$ with $c_0(\mathcal{E}) = 1$. The cycles $c_e(\mathcal{E})$ are the *Chern classes* of $\mathcal{E}$. We denote by $c_e(Z) \in A^e(Z)$, the Chern classes of the tangent bundle $\mathcal{T}_Z$, and we just call them the *Chern classes of* Z. We have that $c_1(Z) = -K_Z$, where K_Z is the canonical class. We define the *Chern numbers* of Z as the degree of the top-intersection of its Chern classes

$$c_{i_1}(Z)\ldots c_{i_m}(Z) \in A^d(Z), \quad i_1 + \ldots + i_m = d.$$

In the following, if the context is understood, we abuse notation using the symbol $c_{i_1}\ldots c_{i_m}(Z)$ for Chern numbers. Also, we may use the notation $c_{i_1}\ldots c_{i_m}$.

We have $c_1^d = (-1)^d K_Z^d$, and since we work over $\mathbb{C}$, it is well-known that $c_d = e(Z)$, the topological Euler characteristic. These numbers are codified into the Todd class of $\mathcal{T}_Z$ by the following formal sum

$$\operatorname{td}(\mathcal{T}_Z) = 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} + \frac{c_1c_2}{24} - \frac{c_1^4 - 4c_1^2c_2 - 3c_2^2 - c_1c_3 + c_4}{720} + \ldots.$$

As a consequence of the Hirzebruch-Riemann-Roch Theorem, we have the *Noether's identities*, i.e., the analytic Euler characteristic are equal to the d-th summand of $\operatorname{td}(\mathcal{T}_Z)$. For example

$$\chi(\mathcal{O}_Z) = \frac{c_1c_2}{24}, \quad \text{when } d = 3.$$

Example 2.2.1. *If S is a non-singular surface of general type whose Chern numbers equal the BMY-inequality (1.3), then S must be minimal. Indeed, if S is not minimal, take a contraction $S \to S'$ to its minimal model. Since this morphism is a chain of blow-ups at points, for simplicity, let us assume that S is the blow-up of S' at a point p. We have, $c_1^2(S) = c_1^2(S') + 1$ and $c_2(S) = c_2(S') + 1$. So, $c_1^2(S') = 3c_2(S') + 2$ and S' does not satisfy the BMY-inequality, a contradiction.*

Example 2.2.2. *The argument above shows that S is relatively minimal, i.e., any birational morphism $S \to S'$ is an isomorphism. Now let us assume Z a non-singular 3-fold of general type satisfying the equality in the Miyaoka-Yau inequality, i.e., $c_1^3(Z) = \frac{8}{3}c_1c_2(Z)$. Assume that $Z \to Z'$ is a blow-down. If Z is the blow-up at a point $p \in Z'$, we have*

$$c_1^3(Z) = c_1^3(Z') - 8, \quad c_1c_2(Z) = c_1c_2(Z'), \quad c_3(Z) = c_3(Z') + 2$$

If Z is the blow-up along a curve $C \hookrightarrow Z'$, we have

$$c_1^3(Z) = c_1^3(Z') + c_1(\mathcal{N}_{C/Z'}), \quad c_1c_2(Z) = c_1c_2(Z'), \quad c_3(Z) = c_3(Z') + c_1(C),$$

where $\mathcal{N}_{C/Z'}$ is the normal bundle of Z' on C. Thus, c_1^3/c_1c_2 behaves erratically and we cannot apply the argument above if we assume minimality with respect to blow-dows.

Let us construct some examples using just fiber products. Consider a collection $Z_1, \ldots, Z_s$ of non-singular projective varieties. Set $Z = Z_1 \times \ldots \times Z_s$, and $p_i \colon Z \to Z_i$ the corresponding projections. We have $\mathcal{T}_Z = \bigoplus_{i=1}^s p_i^*\mathcal{T}_{Z_i}$, and so in terms of Chern polynomials we can compute

$$c_t(\mathcal{T}_Z) = \prod_{i=1}^{s} p_i^* c_t(\mathcal{T}_{Z_i}).$$

The following proposition is deduced via routine computations.

Proposition 2.2.3. *Consider $Z = Z_1 \times Z_2$ with $\dim Z_i = d_i$. Then for any partition $i_1 + \ldots + i_m = d_1 + d_2$ we have*

$$c_{i_1} \ldots c_{i_m}(Z) = \sum_{\substack{k_j + l_j = i_j \\ k_1 + \ldots + k_m = d_1 \\ l_1 + \ldots + l_m = d_2}} c_{k_1} \ldots c_{k_m}(Z_1) c_{l_1} \ldots c_{l_m}(Z_2).$$

In particular, the topological characteristic satisfy $c_{d+1}(Z) = c_d(Z_1)c_d(Z_2)$.

□

Remark 2.2.4. *The above formula can be extended to products of more varieties in the same way.*

Example 2.2.5. *Let C be a non-singular curve and V any non-singular projective variety of dimension d. Set $Z = V \times C$. From the above formula and for any partition $i_1 + \ldots + i_m = d+1$, we get*

$$c_{i_1} \ldots c_{i_m}(Z) = \left(\sum_{j=1}^{m} c_{i_1} \ldots c_{i_j - 1} \ldots c_{i_m}(V) \right) e(C).$$

In particular, $c_1^{d+1} = d c_1^d(V) e(C)$. If V is a surface, then Z is a 3-fold with

$$c_1^3 = 3K_V^2 e(C), \quad c_1 c_2 = 12\chi(\mathcal{O}_V) e(C), \quad c_3 = e(V) e(C).$$

Then in terms of slopes, we have

$$\left(\frac{c_1^3}{c_1 c_2}, \frac{c_3}{c_1 c_2} \right) = \left(\frac{c_1^2(V)}{4\chi(\mathcal{O}_V)}, \frac{c_2(V)}{12\chi(\mathcal{O}_V)} \right).$$

By Sommese's result [Som84] about the density of slopes c_1^2/c_2 of surfaces of general type in $[1/5, 3]$, the slopes for 3-folds of general type $Z = V \times C$ are dense on the segment in $\mathbb{Q}^2$ connecting the points $(1/2, 5/6)$ and $(9/4, 1/4)$.

Example 2.2.6. *Let Z_1, Z_2 be non-singular surfaces. Then, $Z = Z_1 \times Z_2$ is a 4-fold with*

$$c_1^4(Z) = 6c_1^2(Z_1)c_1^2(Z_2), \quad c_1^2 c_2(Z) = c_1^2(Z_1)c_2(Z_2) + c_1^2(Z_2)c_2(Z_1) + c_1^2(Z_1)c_1^2(Z_2)$$

$$c_2^2(Z) = c_2(Z_1)c_2(Z_2) + c_1^2(Z_1)c_1^2(Z_2), \quad c_1 c_3(Z) = c_1^2(Z_1)c_2(Z_2) + c_1^2(Z_2)c_2(Z_1),$$

$$c_4(Z) = c_2(Z_1)c_2(Z_2).$$

Set $x = c_1^2(Z_1)/c_2(Z_1)$ and $y = c_1^2(Z_2)/c_2(Z_2)$. It is interesting to consider the following slopes

$$\left(\frac{c_1^4}{c_4}, \frac{c_1^2 c_2}{c_4}, \frac{c_2^2}{c_4}, \frac{c_1 c_3}{c_4} \right)(Z) = (6xy, x + y + xy, 1 + xy, x + y).$$

In this case, using again Sommese's density, slopes for 4-folds of general type $Z_1 \times Z_2$ are dense in a non-linear surface in $\mathbb{Q}^4$.

Definition 2.2.7. *A* simple normal crossing *(SNC) divisor $D = \sum_{j=1}^{r} D_j$ is a reduced effective divisor with distinct non-singular components D_j satisfying the following condition: for each $p \in D$ there are local coordinates $x_1, \dots, x_d$ on Z such that the equation defining D on p is $x_1 \dots x_e = 0$, with $e \leq d$.*

From [Iit77] we introduce the following sheaf on Z.

Definition 2.2.8. *For a SNC divisor D, the* sheaf of log-differentials along *D, denoted by $\Omega^1_Z(\log D)$, is the $\mathcal{O}_Z$-submodule of $\Omega^1_Z \otimes \mathcal{O}_Z(D)$ described as follows. Let $p \in Z$ be a point.*

(i) If $p \notin D$, then $(\Omega^1_Z(\log D))_p = \Omega^1_{Z,p}$.

(ii) If $p \in D$, we choose local coordinates $x_1, \dots, x_d$ on Z with $x_1 \dots x_e = 0$ defining D on p. Then, $(\Omega^1_Z(\log D))_p$ is generated as $\mathcal{O}_{Z,p}$-module by

$$\frac{dx_1}{x_1}, \dots, \frac{dx_e}{x_e}, dx_{e+1}, \dots, dx_d.$$

If $D = \sum_{j=1}^{r} \nu_j D_j$ is a divisor on Z, whose associated reduced divisor $D_{red} = \sum_j D_j$ is a SNC divisor, then for simplicity we set

$$\Omega^1_Z(\log D) := \Omega^1_Z(\log D_{red}).$$

In the rest of this section, we assume $D = \sum_{j=1}^{r} \nu_j D_j$ as a divisor with D_{red} a SNC divisor.

Definition 2.2.9. *The* log-Chern classes *of a pair (Z, D) are defined as*

$$\bar{c}_i(Z, D) = c_i(\Omega_Z(\log D)^\vee).$$

The log-Chern numbers *of a pair (Z, D) are defined as the degree of top-dimensional intersections*

$$\bar{c}_{i_1} \dots \bar{c}_{i_m} := \bar{c}_{i_1}(Z, D) \dots \bar{c}_{i_m}(Z, D), \quad i_1 + \dots + i_m = d.$$

We set $\Omega^e_Z(\log D) := \bigwedge^e \Omega^1_Z(\log D)$ for any $1 \leq e \leq d$. We have $\Omega^d_Z(\log D) = \mathcal{O}_Z(K_Z + D_{red})$, i.e., $\bar{c}_1(Z, D) = c_1(Z) - D_{red}$. As an application of the Hirzebruch-Riemann-Roch Theorem, we have (see [Iit78, Prop. 2])

$$\bar{c}_d = e(Z) - e(D_{red}).$$

Lemma 2.2.10. *We have a natural exact sequence*

$$0 \to \Omega_Z^1 \to \Omega_Z^1(\log D) \to \bigoplus_{i=1}^{r} \mathcal{O}_{D_i} \to 0,$$

which is known as the residual exact sequence*.*

Proof. The morphism $\Omega_Z^1 \to \Omega_Z^1(\log D)$ is just the inclusion. The other morphism locally is described as

$$\sum_{j=1}^{e} a_j \frac{dx_j}{x_j} + \sum_{j'=e+1}^{d} b_{j'} dx_{j'} \mapsto \bigoplus_{j=1}^{e} a_j|_{D_j}$$

. See the rest of the proof in [EV92, Proposition 2.3]. □

From the residual exact sequence, we can compute the Chern polynomial through the identity

$$c_t(\Omega_Z^1(\log D)) = c_t(\Omega_Z^1) \prod_{j=1}^{r} \left(\sum_{e=0}^{d} D_j^e t^e \right). \tag{2.1}$$

Let $0 \leq e \leq d$, and let $i_1 + \ldots + i_m = e$ be any partition of positive integers. By convention, for the case $e = 0$ we assume the existence of a unique partition, i.e., $i_1 = 0$. We introduce the following notation,

$$D^{[i_1,\ldots,i_m]} := \sum_{j_1 < \ldots < j_m} D_{j_1}^{i_1} \ldots D_{j_m}^{i_m}, \quad D^{[0]} = 1.$$

Examples of this notation are,

$$D^{[e]} = \sum_{j=1}^{r} D_j^e, \quad D^{[1,1]} = \sum_{j<k} D_j D_k.$$

Lemma 2.2.11. *We have,*

$$\prod_{j=1}^{r} \left(\sum_{e=0}^{d} D^e \right) = \sum_{e=0}^{d} \left(\sum_{i_1+\ldots+i_m=e} D^{[i_1,\ldots,i_m]} \right).$$

Proof. We are taking all combinations $D_{j_1}^{j_1} \ldots D_{j_m}^{i_m}$ without repeated elements. Then the result follows by induction. □

Corollary 2.2.12. *We have the identity,*

$$\bar{c}_d = c_d + \sum_{e=1}^{d} (-1)^e c_{d-e} \left(\sum_{i_1+\ldots+i_m=e} D^{[i_1,\ldots,i_m]} \right).$$

Proof. Using identity (2.1), and Lemma 2.2.11 we can compute $\bar{c}_d$ as the degree d element of the expression

$$\left(\sum_{e=0}^{d} c_e(Z) \right) \left(\sum_{e=0}^{d} \left(\sum_{i_1+\ldots+i_m=e} D^{[i_1,\ldots,i_m]} \right) \right).$$

□

Corollary 2.2.13. *Assume D is non-singular, i.e., its non-singular components are pairwise disjoint. Then for each $1 \leq e \leq d$ we have $\bar{c}_e(Z, D) = c_e(Z) + R_e(D)$ where*

$$R_e(D) = \sum_{\substack{k+l=e \\ k\neq e}} (-1)^l D^{[l]} c_k(Z),$$

for each $e = 1, \ldots, d$.

Proof. Since D is non-singular, then $D_i D_j = 0$ for all $i \neq j$. Thus, from the identity (2.1) , we get

$$\sum_{e=0}^{d} (-1)^e \bar{c}_e(Z, D) t^e = \left(\sum_{e=0}^{d} (-1)^e c_e(Z) t^e \right) \left(\sum_{e=0}^{d} D^{[e]} t^e \right).$$

Using the Cauchy product formula for polynomials we have

$$(-1)^e \bar{c}_e(Z, D) = \sum_{k+l=e} (-1)^k c_k(Z) D^{[l]},$$

and from this, the formula follows. □

Example 2.2.14. *For Z a non-singular curve, set $D = \nu_j P_1 + \ldots + \nu_r P_r$ where $P_i \in Z$ are points. So for the unique* log-*Chern number we have*

$$\bar{c}_1 = c_1 - r = -(2g(X) - 2 + r).$$

Let Z be a non-singular surface, and $D = \sum_{j=1}^{r} \nu_j D_j$ with D_j non-singular curves. Then

$$\bar{c}_1^2 = c_1^2 - 2c_1 D_{red} + D_{red}^2,$$

$$\bar{c}_2 = c_2 + t_2 + 2\sum_{j=1}^{r}(g(D_j) - 2),$$

Where t_2 is the number of nodes of D. For proof, see [Urz09, Prop. 3.1].

Corollary 2.2.15. *Consider a non-singular 3-fold Z, and $D = \sum_j \nu_j D_j$ on Z. We have*

$$\bar{c}_2(Z, D) = c_2(Z) - D_{red}(c_1 - D_{red}) - D^{[1,1]}.$$

Thus, the logarithmic Chern numbers for 3-folds are,

$$\bar{c}_1^3 = (c_1(Z) - D_{red})^3$$

$$\bar{c}_1\bar{c}_2 = c_1c_2 - D_{red}(c_1^2 + c_2) + c_1(2D^{[2]} + 3D^{[1,1]}) - D_{red}(D^{[2]} + D^{[1,1]}).$$

$$\bar{c}_3 = c_3 - c_2D_{red} + c_1(D^{[2]} + D^{[1,1]}) - \left(D^{[3]} + D^{[1,2]} + D^{[2,1]} + D^{[1,1,1]}\right).$$

Proof. Using identity (2.1) for $d = 3$, and looking for the degree 2 terms we obtain $\bar{c}_2(Z, D)$. The other formulas are direct computations using the above lemmas. □

Remark 2.2.16. *As in the case of nodes for surfaces, let us denote the number of triple points of D by t_3. Then we can rewrite*

$$\bar{c}_3 = c_3 - c_2D_{red} + c_1(D^{[2]} + D^{[1,1]}) - \left(D^{[3]} + D^{[1,2]} + D^{[2,1]} + t_3\right).$$

Example 2.2.17. *Let $Z \hookrightarrow \mathbb{P}^m$ be a non-singular projective 3-fold. Let $H_1, \ldots, H_r \sim H$ hyperplane sections defining an SNC arrangement. We have*

$$\bar{c}_1^3 = c_1^3(Z) - r^3\deg(Z) - 3rc_1^2H + 3c_1(Z)H^2$$

$$\bar{c}_1\bar{c}_2 = c_1c_2(Z) - rH(c_1^2+c_2)(Z) + \left(2r + 3\binom{r}{2}\right)c_1(Z)H^2 - \deg(Z)r\left(r + \binom{r}{2}\right),$$

$$\bar{c}_3 = c_3(Z) - rHc_2(Z) + \left(r + \binom{r}{2}\right)c_1(Z)H^2 - \deg(Z)\left(r + 2\binom{r}{2} + \binom{r}{3}\right).$$

Thus as r grows, we have,

$$\lim_{r\to\infty}\frac{\bar{c}_1^3}{\bar{c}_1\bar{c}_2}=\lim_{r\to\infty}\frac{r^3}{r\binom{r}{2}}=2$$

$$\lim_{r\to\infty}\frac{\bar{c}_3}{\bar{c}_1\bar{c}_2}=\lim_{r\to\infty}\frac{\binom{r}{3}}{r\binom{r}{2}}=\frac{1}{3}.$$

Example 2.2.18. *For a non-singular projective variety V of dimension d, the product $Z=V\times\mathbb{P}^1$ has $Pic(Z)=Pic(V)\oplus\mathbb{Z}$. In this case, $\mathbb{Z}$ is generated by a class of a fiber $[V]:=[V\times p]$ for a $p\in\mathbb{P}^1$. This is independent of the chosen point since all of them are linearly equivalent as divisors. For distinct points $\{p_1,\dots,p_r\}$, we have a collection of distinct fibers $V_1,\dots,V_r$ defining a non-singular SNC divisor. In this case, $D_{red}=V_1+\ldots+V_r\sim r[V]$. Since $V_j^e=0$ for $e>1$, from Corollary 2.2.13 we have*

$$\bar{c}_e(Z,D)=c_e(Z)-D_{red}c_{e-1}(Z).$$

For any partition $i_1+\ldots+i_m=d+1$, we can compute

$$\bar{c}_{i_1}\ldots\bar{c}_{i_m}=c_{i_1}\ldots c_{i_m}-r[V]\sum_{j=1}^{m}c_{i_1}\ldots c_{i_j-1}\ldots c_{i_m}(Z).$$

From Example 2.2.5 we have

$$c_{i_1}\ldots c_{i_m}=2\left(\sum_{j=1}^{m}c_{i_1}\ldots c_{i_j-1}\ldots c_{i_m}(V)\right).$$

Since $[V]^e=0$ for $e>1$, we have

$$\bar{c}_{i_1}\ldots\bar{c}_{i_m}=-\left(\sum_{j=1}^{m}c_{i_1}\ldots c_{i_j-1}\ldots c_{i_m}(V)\right)(r-2).$$

Indeed, the same formula is valid if $Z=\mathbb{P}(\mathcal{E})\to\mathbb{P}^1$, replacing V by $\mathbb{P}^{r-1}$. In particular, for V a surface, the 3-fold Z has logarithmic Chern numbers,

$$\bar{c}_1^3=-3c_1^2(V)(r-2),\quad \bar{c}_1\bar{c}_2=-12\chi(\mathcal{O}_V)(r-2),\quad \bar{c}_3=-c_2(V)(r-2).$$

For V a 3-fold, the 4-fold Z has

$$\bar{c}_1^4=-4c_1^3(V)(r-2),\quad \bar{c}_1^2\bar{c}_2=-(c_1^3+2c_1c_2)(V)(r-2),\quad \bar{c}_2^2=-2c_1c_2(V)(r-2)$$

$$\bar{c}_1\bar{c}_3=(c_3+c_1c_2)(V)(2-r),\quad \bar{c}_4=-c_3(V)(r-2).$$

Remark 2.2.19. *For a product $V \times C$, where C is a non-singular curve of genus $g \geq 1$, it is not guaranteed the existence of points $p_1, \ldots, p_r \in C$ having $[V \times p_i] \sim [V \times p_j]$ for arbitrary r. This is due to the fact that $Pic(C)$ could be uncountable infinite. However, if such arbitrarily large collection of points exists, then as above we must have the formula*

$$\bar{c}_{i_1} \ldots \bar{c}_{i_m} = -\left(\sum_{j=1}^{m} c_{i_1} \ldots c_{i_j - 1} \ldots c_{i_m}(V)\right)(r - e(C)).$$

Definition 2.2.20. *Let Z be a non-singular projective variety and D an effective divisor with D_{red} as SNC divisor. A surjective morphism $h \colon Y \to Z$ between non-singular projective varieties is called a* log-morphism*, if $D'_{red} = (h^* D)_{red}$ is a SNC divisor.*

Lemma 2.2.21. *For any* log*-morphism $h \colon (Y, D') \to (Z, D)$, we have an injection*

$$h^* \Omega_Z(\log D) \hookrightarrow \Omega_Y(\log D').$$

Moreover, if h is finite and ramified at D, then we have isomorphism outside the singularities of D.

Proof. See [Vie82, Lemma 1.6]. □

2.3 n-th root covers

In this section, we follow [EV92, Sec. 3].

Consider the following building data $(Z, D, n, \mathcal{L})$ where,

1. Z is a non-singular projective variety of dimension d,
2. $D = \sum_{j=1}^{r} \nu_j D_j$ is an effective divisor on Z, with $D_{red} = \sum_{j=1}^{r} D_j$ a SNC divisor,
3. $n \geq 2$ is a prime number, and
4. $\mathcal{L}$ a line bundle on Z such that $\mathcal{O}_Z(D) \simeq \mathcal{L}^{\otimes n}$.

With this building data, we construct a $\mathcal{O}_Z$-algebra given by

$$\mathcal{F} = \bigoplus_{i=0}^{n-1} \mathcal{L}^{-i},$$

where the structure is defined by the following laws:

a) Fix a section $s \in H^0(Z, \mathcal{L}^n)$ defining D.

b) For any $m \in \mathbb{Z}$, let $\{m\}_n$ its residue modulo n. Then, the line bundle $\mathcal{L}^{-m}$ identifies with $\mathcal{L}^{-\{m\}_n}$ by the $\mathcal{O}_Z$-homomorphism $h \mapsto hs^{\lfloor \frac{m}{n} \rfloor}$.

Using the relative spectrum construction we get an affine morphism

$$f' \colon Y'_n = \mathrm{Spec}_Z\, \mathcal{F} \to Z.$$

The normalization $f\colon Y_n \to Y'_n \to Z$ will be called the *n-th root covering* associated to the building data $(Z, D, n, \mathcal{L})$. Take an affine chart $U = \mathrm{Spec}A \subset Z$ such that $\mathcal{L}^{-1}|_U$ trivialize by a A-module generated by t. So, we have $f'^{-1}(U) = \mathrm{Spec}B'$ where

$$B' = \frac{A[t]}{t^n - s|_U}.$$

The function field of B' is $K = \mathbb{C}(Z)[\alpha]$, where α is a root such that $\alpha^n = s|_U$. By SNC property of D_{red}, we can shrink U and choose local coordinates $z_1, \dots, z_d$ such that $s|_U = z_{j_1}^{\nu_{j_1}} \dots z_{j_e}^{\nu_{j_e}}$. As a consequence, the morphism f is ramified at D, and topologically $f^{-1}(D) \cong D$. Moreover, the singularities of Y'_n and Y_n are over the singularities of D, i.e., over $D_{j_1} \cap \ldots \cap D_{j_e}$ for each $e > 1$. Indeed, since Z is non-singular, the singularities of Y_n over $D_{j_1} \cap \ldots \cap D_{j_e}$ are locally analytically isomorphic to the normalization of

$$\mathrm{Spec}\left(\frac{\mathbb{C}[z_1, \dots, z_d, t]}{t^n - z_{j_1}^{\nu_{j_1}} \dots z_{j_e}^{\nu_{j_e}}}\right).$$

Proposition 2.3.1. *The integral closure B of B' in its fraction field K is freely generated over A by the generators*

$$\alpha^i z_{j_1}^{-\left\lfloor \frac{\nu_{j_1} i}{n} \right\rfloor} \dots z_{j_e}^{-\left\lfloor \frac{\nu_{j_e} i}{n} \right\rfloor}, \quad i = 0, 1, \dots, n-1.$$

Proof. See [Gao11, Th. 3.1]. □

As corollary, let L be a divisor on Z such that $\mathcal{L} = \mathcal{O}_Z(L)$. We have the following decomposition on eigenspaces

$$f_*\mathcal{O}_{Y_n} = \bigoplus_{i=0}^{n-1} \mathcal{O}_Z(-L^{(i)}), \quad L^{(i)} = -iL + \sum_j \left\lfloor \frac{i\nu_j}{n} \right\rfloor D_j.$$

Indeed, $Y_n = \mathrm{Spec} f_*\mathcal{O}_{Y_n}$. Therefore $f : Y_n \to Z$ is an affine finite morphism.

Example 2.3.2 (Prototypical set-up)**.** *In $\mathbb{P}^d$, let $H_1, , ..., H_{d+1}$ be the hyperplanes given by each coordinate section z_j. For any partition*

$$\nu_1 + \ldots + \nu_{d+1} = n$$

into positive integers, we have $D = \sum_j \nu_j H_j \sim nH$, for a general hyperplane section H. Thus, Y_n is the normalization over $\mathbb{C}$ of the projective variety

$$Y_n' \cong Proj\left(\frac{\mathbb{C}[z_1, ..., z_{d+1}, t]}{(t^n - z_1^{\nu_1} \ldots z_{d+1}^{\nu_{d+1}})}\right).$$

,

Proposition 2.3.3. *Both constructions Y_n' and Y_n are projective and irreducible.*

Proof. Clearly both morphism $f' \colon Y_n' \to Z$ and $f \colon Y_n \to Z$ are finite, so they are projective. Indeed, as the previous example we have Z-isomorphisms $Y_n' \cong \mathrm{Proj}(\mathcal{S}')$ and $Y_n \cong \mathrm{Proj}(\mathcal{S})$, where $\mathcal{S}'$ and $\mathcal{S}$ are $\mathcal{O}_Z$-algebras defined by $\mathcal{S}_0' = \mathcal{S}_0 = \mathcal{O}_Z$, with $\mathcal{S}_i' = \mathcal{F}$ and $\mathcal{S}_i = f_*\mathcal{O}_{Y_n}$ for all $i > 0$ [Vak, 17.3.F]. To prove irreducibility we just have to prove that any open set is irreducible. Let $U = \mathrm{Spec}A \hookrightarrow Z$ any open set. Then, $f'^{-1}(U) = \mathrm{Spec}B'$ with $B' = A[t]/(t^n - s|_U)$. Since n is prime and $s|_U$ is not a constant, we have $t^n - s|_U$ irreducible. Now, Y_n is just the normalization of Y_n', so it is irreducible. □

Remark 2.3.4. *Let $G = \mathbb{Z}/n\mathbb{Z} \cong \langle\zeta\rangle$ for a n-th root $\zeta^n = 1$. We have an induce G-action on $f_*\mathcal{O}_{Y_n}$ given by $h \to \zeta^i h$ for any local section h of $\mathcal{O}_Z(L^{-(i)})$. So in terms of invariants of the G-action, we have ${f_*\mathcal{O}_{Y_n}}^G = \mathcal{O}_Z$ and $Z = Y_n/G$.*

Definition 2.3.5. *A* partial resolution of singularities *of Y_n is a projective, surjective, birational morphism $g : X \to Y_n$ with X a projective normal variety having at most rational singularities. This last means that for any resolution of singularities $g' \colon X' \to X$ we have $R^i g'_* \mathcal{O}_{X'} = 0$ for $i > 0$.*

Theorem 2.3.6. *For the n-th root covering $f\colon Y_n \to Z$ we have*

1. *The morphism f is flat.*

2. *The variety Y_n has rational singularities.*

3. *For any partial resolution of singularities $g\colon X \to Y_n$, the composition $h = f \circ g$ satisfies the following $\mathbb{Q}$-numerical equivalence*

$$K_X \sim_{\mathbb{Q}} h^*\left(K_Z + \frac{n-1}{n} D_{red}\right) + \Delta,$$

where Δ is a divisor supported on the exceptional divisor of g.

Proof. We have a local description of the singularities of Y_n. In Section 2.4 we will see that these are toric singularities, and it is well known that they are (log-terminal) rational singularities [CLS11, Th.11.4.2], so we have (2). Now, Y_n is Cohen-Macaulay and since f is finite, f must be flat [Vak, Th.28.2.11] and we are done with (1). Assertion (3) follows from [Par91, Prop. 3.4].

Lemma 2.3.7. *Let Y a normal variety, and and $g : X \to Y$ a proper, surjective, birational morphism. Assume that X has rational singularities. Then $g_*\mathcal{O}_X = \mathcal{O}_Y$ and $R^i g_*\mathcal{O}_X = 0$ for all $i > 0$ if and only if Y has rational singularities.*

Proof. See [Vie77, Lemma 1.]. □

Corollary 2.3.8. *For any partial resolution of singularities $g : X \to Y_n$, we have $\chi(\mathcal{O}_X) = \sum_{i=0}^{n-1} \chi(\mathcal{O}_Z(-L^{(i)}))$, i.e., the analytic Euler characteristic of X is independent of the chosen partial resolution.*

Proof. Let $g' : X' \to X$ be a resolution of singularities. Since Y_n has rational singularities, by Lemma 2.3.7, we must have $h_*\mathcal{O}_{X'} = g_*\mathcal{O}_X = \mathcal{O}_Y$, and $R^i h_*\mathcal{O}_{X'} = R^i g_*\mathcal{O}_X = 0$ for all $i > 0$. Thus,

$$\chi(\mathcal{O}_{X'}) = \chi(\mathcal{O}_X) = \chi(\mathcal{O}_{Y_n}).$$

Then, we assume that g is just a resolution of singularities, so

$$H^i(X, \mathcal{O}_X) \cong H^i(Y_n, g_*\mathcal{O}_X) \cong H^i(Y_n, \mathcal{O}_{Y_n}), \quad i \geq 0.$$

Since f is an affine morphism, also we have $H^i(Y_n, \mathcal{O}_{Y_n}) \cong H^i(Z, f_*\mathcal{O}_{Y_n})$ for all $i \geq 0$. Thus, we have

$$\chi(\mathcal{O}_X) = \sum_{i=0}^{n-1} \chi(\mathcal{O}_Z(-L^{(i)})).$$

□

Since the degree n of f is a prime number, from Example 2.1.4 we have $f^*D_j = nD'_j$, where $D'_j = (f^*D_j)_{red}$. Thus, for any partial resolution $h\colon X \to Y_n \to Z$ we must have a ramification formula

$$h^*D_j = nD'_j + \Delta_j,$$

where Δ_j is a divisor supported in the exceptional divisors of h over D_j. Finally, we give a state about the connectedness of a partial resolution.

Proposition 2.3.9. *Any partial resolution $g : X \to Y_n$ is irreducible.*

Proof. Since X is normal we reduce the proof to show that X is connected [Sta18, Tag. 0347]. From Corollary 2.3.8 we know that

$$h^0(\mathcal{O}_X) = 1 + \sum_{i=1}^{n-1} h^0(\mathcal{O}_Z(-L^{(i)})).$$

If Y is not connected, then $h^0(\mathcal{O}_X) > 1$, so there exists a $i \geq 1$ such that $h^0(\mathcal{O}_Z(-L^{(i)})) \geq 1$. We have,

$$-nL^{(i)} \sim \sum_{j=1}^{r} \{i\nu_j\}_n D_j.$$

So, we choose curves Γ_j on Z such that $D_j\Gamma_j > 0$. Thus, we get a linear system $A\nu \equiv 0 \mod n$ where $\nu = [\nu_1, ..., \nu_r]^T$ and $A = (D_j\Gamma_k)_{jk}$. Since n is prime, $\nu \equiv 0 \mod n$, and since $0 < \nu_j < n$, we get a contradiction. □

2.4 Toric local picture

In this section, for toric varieties we mainly follow the notation of [CLS11].

Let $n > 0$ be a prime number and $0 \leq \nu_1, \dots, \nu_d < n$ integers. Choose a $\nu_k \neq 0$, and let $0 \leq q_1, \dots, q_d < n$ be integers such that $\nu_j + q_j\nu_k \equiv 0$ modulo n. In particular, $q_k = n-1$. As usual set $N = \mathbb{Z}^d$ and $N_\mathbb{R} \cong \mathbb{R}^d$ with canonical basis $e_1, \dots, e_d$, and $M = N^\vee \cong \mathbb{Z}^d$ with $M_\mathbb{R} = \mathbb{R}^d$. Consider the semigroup

$$S = \left\langle e_1, \dots, e_{k-1}, \sum_{j\neq k} q_j e_j + n e_k, e_{k+1}, \dots, e_d, \sum_{j\neq k} \frac{\nu_j + q_j\nu_k}{n} e_j + \nu_k e_k \right\rangle_\mathbb{N}.$$

Since

$$\sum_{j\neq k} \frac{\nu_j + q_j\nu_k}{n} e_j + \nu_k e_k = \sum_{j\neq k} \frac{\nu_j}{n} e_j + \frac{\nu_k}{n}\left(\sum_{j\neq k} q_j e_j + n e_k\right),$$

we have that the saturation [CLS11, p. 27] of S is $S^{sat} = \sigma^\vee \cap \mathbb{Z}^d$ where

$$\sigma^\vee = C\left(e_1, \dots, e_{k-1}, \sum_{j\neq k} q_j e_j + n e_k, e_{k+1}, \dots, e_{d-1}, e_d\right) \subset M_\mathbb{R}$$

is the simplicial d-cone defined by those elements. It is the dual cone of

$$\sigma = C\left(ne_1 - q_1 e_k, \dots, ne_{k-1} - q_{k-1}e_k, e_k, ne_{k+1} - q_{k+1}e_k, \dots, ne_d - q_d e_k\right) \subset N_\mathbb{R}.$$

Observe that $\mathrm{mult}(\sigma) = n^{d-1}$ and $\mathrm{mult}(\sigma^\vee) = n$. Recall that the *fundamental parallelepiped* of a cone σ are the elements of σ with coordinates in $[0,1)$ respect its generators. We denote it by P_σ.

Proposition 2.4.1. *Every element of $v \in P_\sigma \cap \mathbb{Z}^d$ can be written as*

$$v = \frac{\sum_{j\neq k} v_j(ne_j - q_j e_k) + \left\{\sum_{j\neq k} v_j q_j\right\}_n e_k}{n}, \quad 0 \leq v_i < n.$$

Proof. An element $v \in P_\sigma$ can be written in terms of the canonical basis as

$$v = \sum_{j\neq k} n\alpha_j e_j + \left(\alpha_k - \sum_{j\neq k} \alpha_j q_j\right) e_k, \quad \alpha_j \in [0,1).$$

Since $v \in \mathbb{Z}^d$, we have $\alpha_j \in n\mathbb{Z}$ for $j \neq k$. Let us write $\alpha_j = v_j/n$ with $v_j \in \{0, 1, ..., n-1\}$. This, implies that $n\alpha_k \in \mathbb{Z}$ and satisfies

$$n\alpha_k \equiv_n \sum_{j\neq k} v_j q_j \mod n.$$

Thus, if we write $\alpha_k = v_k/n$ with $v_k \in \{0, ..., n-1\}$, we must have $v_k = \left\{\sum_{j\neq k} v_j q_j\right\}_n$. □

Proposition 2.4.2. *The toric variety associated with the semigroup S is*

$$Spec(\mathbb{C}[S]) = Spec\left(\frac{\mathbb{C}[x_1,\ldots,x_d,t]}{(t^n - x_1^{\nu_1}\ldots x_d^{\nu_d})}\right).$$

Moreover, its normalization corresponds with $Spec(\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^d])$.

Proof. For simplicity, we will prove the result in the case $k = d$. Since $Norm(\mathrm{Spec}(\mathbb{C}[S])) = \mathrm{Spec}(\mathbb{C}[S^{sat}])$ we will prove the first. Take the surjective morphism of semigroups $\phi\colon \mathbb{N}^{d+1} \mapsto S$ such that

$$\phi(e_j) = e_j, \quad \phi(e_d) = \sum_{i=1}^{d-1} q_j e_j + n e_d, \quad \phi(e_{d+1}) = \sum_{j=1}^{d-1} \frac{\nu_j + q_j\nu_d}{n} e_j + \nu_d e_d.$$

It induces a surjective morphism of coordinate rings $f\colon \mathbb{C}[x_1,\ldots,x_d,t] \to \mathbb{C}[S]$, and by [CLS11] in Proposition 1.1.9 it is known that

$$\mathrm{Ker}(f) = (x_1^{a_1}\ldots x_d^{a_d}t^{a_{d+1}} = x_1^{b_1}\ldots x_d^{b_d}t^{b_{d+1}} : \phi(a) = \phi(b), a, b \in \mathbb{N}^{d+1}).$$

If we set $x_j = a_j - b_j$, the condition $\phi(a) = \phi(b)$ gives equations

$$\begin{cases} x_1 + q_1 x_d + \dfrac{\nu_1 + q_1\nu_d}{n} x_{d+1} & = & 0 \\ \cdots & \cdots & \cdots \\ x_{d-1} + q_{d-1} x_d + \dfrac{\nu_{d-1} + q_{d-1}\nu_d}{n} x_{d+1} & = & 0 \\ n x_d + \nu_d x_{d+1} & = & 0 \end{cases}$$

We can assume $x_{d+1} = nc$ with $c > 0$, then $x_d = -\nu_d c$, and the equations reduces to

$$\begin{cases} x_1 & = & -\nu_1 c \\ \cdots & \cdots & \cdots \\ x_{d-1} & = & -\nu_{d-1} c \end{cases} \Rightarrow \begin{cases} b_j & = & a_j + \nu_j c, & 1 \leq j \leq d \\ a_{d+1} & = & b_{d+1} + nc \end{cases}$$

So $\mathrm{Ker}(f)$ is generated by elements of the form

$$x_1^{a_1}\ldots x_d^{a_d} t^{b_{d+1}}((x_1^{\nu_1}\ldots x_d^{\nu_d})^c - (t^n)^c),$$

and the result follows. □

Corollary 2.4.3. *The normalization of the affine varieties* $t^n = x_1^{\nu_1} \dots x_d^{\nu_d}$ *and* $t^n = x_j \prod_{j \neq k} x_i^{n-q_j}$ *are isomorphic.*

Proof. Observe that the cones defining both varieties are the same up to a change of basis, equal to σ. Thus, the semigroups defined by $\sigma^\vee \cap M$ are isomorphic. $\square$

Remark 2.4.4. *We point out the following. Assume that* $j = e$ *and* $\nu_{e+1} = \dots = \nu_d = 0$*. Thus* $q_{e+1} = \dots = q_d = 0$*, and we have a toric description of the normalization of the varieties* $t^n = x_1^{\nu_1} \dots x_e^{\nu_e}$ *embedded in* $\mathbb{A}^d$ *for any* $1 \leq e \leq d$*.*

Remark 2.4.5. *It is known that the cone* $\sigma^\vee$ *defines a toric variety isomorphic to the* d*-dimensional cyclic quotient singularity of type*

$$\frac{(n - q_1, \dots, n - q_{k-1}, 1, n - q_{k+1}, \dots, n - q_d)}{n}.$$

We have $Spec(\sigma \cap \mathbb{Z}^3) \cong \mathbb{C}^d / \langle \phi \rangle$*, where* $\phi \colon \mathbb{C}^d \to \mathbb{C}^d$ *is defined by*

$$\phi \colon (z_1, \dots, z_d) \mapsto (\zeta^{n-q_1} z_1, \dots, \zeta z_k, \dots, \zeta^{n-q_d} z_d),$$

with $\zeta^d = 1$ *a primitive root (Cf. [Ash15]). In this way, cyclic quotient singularities are geometrically dual to the singularities of* n*-th root covers. We will denote a cyclic singularity of this type by* $C_{q_1, \dots, \hat{q}_k, \dots, q_d}$*. In dimension* 2*, it occurs the accident that singularities of* n*-th root covers are also cyclic quotient singularities.*

2.4.1 Toric tools

This subsection is devoted to stating well-known toric results that will be helpful in the rest of this thesis. Let Σ be a fan in $N_\mathbb{R}$, and denote by X_Σ its associated toric variety. Any cone τ in Σ of dimension e is called a e-cone, and the set of e-cones is denoted by Σ^e.

Proposition 2.4.6. *Any* e*-cone* $\tau = C(d_1, ..., d_e)$ *in* Σ *defines a subvariety in* X_Σ *denote by* $V(\tau)$*. Moreover, it is a toric variety whose fan is defined by*

$$Star(\tau) = \{\tilde{\sigma} \subset N(\tau)_\mathbb{R} : \sigma \text{ a cone of } \Sigma \text{ containing } \tau\},$$

where $N(\tau) = N / \langle d_1, ..., d_e \rangle_\mathbb{Z}$*, and* $\tilde{\sigma}$ *is the imagen of* σ *in* $N(\tau)_\mathbb{R}$*.*

Proof. See [CLS11, Prop. 3.2.7]. □

In particular, any 1-cone (or ray) ρ defines a divisor in X_Σ defined by the class of $V(\rho)$. It is denoted by D_ρ.

Theorem 2.4.7. *A divisor $\sum_{\rho\in\Sigma^1} a_\rho D_\rho \in Div(X_\Sigma)$ is a Cartier divisor if and only if for every $\sigma \in \Sigma^d$ there exists a $m_\sigma \in M$ such that $\langle m_\sigma, v_\rho\rangle = -a_\rho$ for every $\rho \in \Sigma^1$ contained in σ.*

Proof. See [CLS11, Ch. 4]. □

The collection $\{m_\sigma\}_{\sigma\in\Sigma}$ is called the *Cartier data* of a Cartier divisor $E = \sum_{\rho\in\Sigma^1} a_\rho D_\rho$. Denote by $|\Sigma|$ the *support of the fan*, i.e., the set-theoretic union of all its cones. In particular, we have a *support function* ϕ_E on $|\Sigma|$ associated to E defined as

$$\phi_E : |\Sigma| \to \mathbb{R}, \quad v \mapsto \langle m_\sigma, v\rangle, \quad \text{if } v \in \sigma.$$

A toric $\mathbb{Q}$-Cartier divisor is defined as a $\mathbb{Q}$-divisor E of index r such that its Cartier data is given by a set $\{m_\sigma\}_{\sigma\in\Sigma} \subset \frac{1}{r}M \subset M_\mathbb{Q}$. The definition of ϕ_E for E extends naturally. Thus

$$E = -\sum_{\rho\in\Sigma^1} \phi_E(v_\rho) D_\rho.$$

Proposition 2.4.8. *Let $f\colon X_{\Sigma'} \to X_\Sigma$ be a toric morphism induced by a refinement Σ' of Σ in $N_\mathbb{R}$. Let E be a $\mathbb{Q}$-Cartier divisor on X_Σ with Cartier data $\{m_\sigma\}_{\sigma\in\Sigma} \subset M_\mathbb{Q}$. Then f^*E has the same Cartier data on $|\Sigma|$, i.e.,*

$$f^*E = -\sum_{\rho\in\Sigma'^1} \phi_E(v_\rho) D_\rho.$$

Proof. This is a particular case of [CLS11, Prop.6.2.7]. □

The canonical divisor of X_Σ is the divisor

$$K_{X_\Sigma} = -\sum_{\rho\in\Sigma^1} D_\rho.$$

By Proposition 2.4.8 K_{X_Σ} is $\mathbb{Q}$-Cartier if and only if for each $\sigma \in \Sigma^d$ there exists a m_σ with $\langle m_\sigma, v_\rho\rangle = 1$ for every $\rho \in \Sigma^1$ contained in σ. Thus, if Σ is simplicial, then the canonical divisor is always a $\mathbb{Q}$-Cartier divisor. Moreover, if X_Σ is smooth, we have the canonical bundle $\omega_{X_\Sigma} = \mathcal{O}_{X_\Sigma}(K_{X_\Sigma})$.

Proposition 2.4.9. *Let X_Σ a toric variety with K_{X_Σ} a $\mathbb{Q}$-Cartier divisor. Let $\phi_{K_{X_\Sigma}}$ be its support function. Then for every toric morphism $f\colon X_{\Sigma'} \to X_\Sigma$ coming from a refinement Σ' of Σ we have*

$$K_{X_{\Sigma'}} = f^*K_{X_\Sigma} + \sum_{\rho\in\Sigma'^1\setminus\Sigma^1} (\phi_{K_{X_\Sigma}}(v_\rho) - 1)D_\rho.$$

Proof. See [CLS11, Lemma.11.4.10]. □

Example 2.4.10. *Consider the cone $\sigma = C(ne_1 - q_1e_d, \ldots, ne_{d-1} - q_{d-1}e_d, e_d)$. The canonical divisor of U_σ has $\mathbb{Q}$-Cartier data given by*

$$m = \frac{(q_1+1)e_1 + \ldots + (q_{d-1}+1)e_{d-1} + ne_d}{n} \in M_\mathbb{Q} \cap \sigma^\vee.$$

Let Σ be a refinement of σ by a lattice point

$$v = \frac{\sum_{j=1}^{d} v_j(ne_j - q_je_d) + \{\sum_{j=1}^{d-1} q_jv_j\}_n e_d}{n}.$$

If we denote by F the divisor associated to the ray defined by v, then we compute

$$\begin{aligned} K_{X_\Sigma} &= f^*K_{U_\sigma} + (\langle m, v\rangle - 1)F \\ &= f^*K_{U_\sigma} + \frac{v_1 + \ldots + v_{d-1} + \{\sum_{j=1}^{d-1} q_jv_j\}_n - n}{n} F \end{aligned}$$

Intersection theory

Here we assume our toric variety X_Σ with Σ simplicial, in this way X_Σ is $\mathbb{Q}$-factorial. The intersection theory on X_Σ is summarized in the following theorem.

Theorem 2.4.11.

1. *The Chow group $A_e(X_\Sigma)$ is generated by $\{[V(\sigma)] : \sigma \in \Sigma^e\}$ as an abelian group.*

2. *There exists a well-defined $\mathbb{Q}$-intersection of cycles in the $\mathbb{Q}$-Chow ring $A_\bullet(X_\Sigma)_\mathbb{Q} = A_\bullet(X_\Sigma)\otimes\mathbb{Q}$: For $\sigma, \sigma', \sigma'' \in \Sigma$, if σ' and σ'' span σ with $\dim\sigma = \dim\sigma' + \dim\sigma''$, then*

$$[V(\sigma')]\,[V(\sigma'')] = \frac{mult(\sigma')mult(\sigma'')}{mult(\sigma)}[V(\sigma)],$$

and $[V(\sigma')].[V(\sigma'')] = 0$ if they does not generate a cone of Σ.

Proof. See [Ful93, Ch. V.]. □

As a first consequence, for any pair of toric divisors D_{ρ_1} and D_{ρ_2}, for two rays ρ_1 and ρ_2 of Σ^1. Let $\tau \in \Sigma^2$ be the face generated by these two rays. Then

$$D_{\rho_1} D_{\rho_2} = \frac{1}{\mathrm{mult}(\tau)}[V(\tau)].$$

Each d-cone σ satisfies $\deg[V(\sigma)] = 1$ since it is represented by its distinguished point. Therefore, if $\rho_1, ..., \rho_d$ generate a d-cone σ, then we have

$$D_{\rho_1} \dots D_{\rho_d} = \frac{1}{\mathrm{mult}(\sigma)}.$$

As a second application, let $\sigma_1, \sigma_2 \in \Sigma^d$ such that $\tau = \sigma_1 \cap \sigma_2 \in \Sigma^{d-1}$. We have $[V(\sigma_i)] = 1$, and $C = [V(\tau)]$ is a closed curve on X_Σ. Assume that $\tau = C(v_1, ..., v_{d-1})$, and

$$\sigma_1 = C(v_0, \tau), \quad \sigma_2 = C(\tau, v_d).$$

Denote by $\rho_i = \mathbb{R}_+ v_i$, and since $\mathrm{mult}(\rho_i) = 1$, we have

$$D_{\rho_0}.C = \frac{\mathrm{mult}(\tau)}{\mathrm{mult}(\sigma_1)}, \quad D_{\rho_d}.C = \frac{\mathrm{mult}(\tau)}{\mathrm{mult}(\sigma_2)}.$$

Moreover, if we consider the unique linear relation up to scalars between $v_0, ..., v_n$ given by

$$a_0 v_0 + a_1 v_1 + ... + a_d v_d = 0,$$

then

$$D_{\rho_i} C = \frac{a_i \mathrm{mult}(\tau)}{a_0 \mathrm{mult}(\sigma_1)} = \frac{a_i \mathrm{mult}(\tau)}{a_d \mathrm{mult}(\sigma_2)}.$$

Corollary 2.4.12.

$$K_{X_\Sigma} C = -\sum_{i=0}^{d} D_{\rho_i} C = -\frac{mult(\tau)}{mult(\sigma_1)} \sum_{i=0}^{d} \frac{a_i}{a_0} = -\frac{mult(\tau)}{mult(\sigma_2)} \sum_{i=0}^{d} \frac{a_i}{a_d}.$$

□

Remark 2.4.13. *Cones defining the same plane has its adjacent walls defining curves with $K_{X_\Sigma} C = 0$. Indeed, there exists a $w \in M_\mathbb{R}$ such that $\langle w, v_i \rangle = \langle w, v_j \rangle > 0$ for all $i \neq j$. This implies that $a_0 + \dots + a_d = 0$, and the assertion follows from the above formula.*

2.5 Planar cones and Hirzebruch-Jung algorithm

Set $N \cong \mathbb{Z}^d$ and $M = N^\vee$. A *planar cone* τ in $N_\mathbb{R} \cong \mathbb{R}^d$ is a cone of dimension 2, i.e., it is generated by two rays defined by primitive generators $v_0, v_{s+1} \in N$ (s will have sense soon). Assume that $\operatorname{mult}(\tau) = n$. It is known that if $n > 1$, then there exists some $v \in \tau \cap N$ such that v_0, v generate $\tau \cap N$, i.e., $|\det(v_0, v)| = 1$. If $v = c_1 v_0 + c_2 v_{s+1}$, with $c_i \in \mathbb{Q}_{\geq 0}$, then

$$\det(v_0, v) = nc_2 = 1 \Leftrightarrow c_2 = 1/n.$$

On the other hand, since $\det(v, v_{s+1}) \in \mathbb{N}$, we have $c_1 \in \frac{1}{n}\mathbb{N}$. If we set $q = nc_1$ with $0 < q < n$, we say that τ *is of type* (n, q) *in direction* v_0 *to* v_{s+1}, or type (n, q') in the opposite direction, where $0 < q' < n$ is the inverse modulo n of q.

Let us change the $\mathbb{Z}$-base of N such that $e_1 = v$ and $e_2 = v_0$. So, we have $v_{s+1} = ne_1 - qe_2$. We have,

$$\tau^\vee = C(e_1, pe_1 + ne_2) \oplus \bigoplus_{i=3}^{d-3} \mathbb{R}w_i = C(w_1, w_2) \oplus \mathbb{R}^{d-2},$$

where the $w_i \in M$ are such that $\langle v_0, w_i\rangle = \langle v_{s+1}, w_i\rangle = 0$ and $\langle \cdot, w\rangle \geq 0$ on τ for any $w \in C(w_1, w_2)$. Thus, in terms of toric varieties,

$$X_\tau = C_q \times (\mathbb{C}^\times)^{d-2},$$

where C_q is the cyclic quotient surface singularity of type $\frac{1}{n}(q, 1)$ (Remark 2.4.5).

Assume that n, q are coprime, then consider the *Hirzebruch-Jung algorithm* of division for n/q, i.e., a pair of sequences

$$m_0 = n > m_1 = q > \ldots > m_s = 1 > m_{s+1} = 0,$$

$$n_0 = 0 < n_1 = 1 < \ldots < n_s = q' < n_{s+1} = n,$$

which are related by

$$m_{\alpha+1} = k_\alpha m_\alpha - m_{\alpha-1}$$

$$n_{\alpha+1} = k_\alpha n_\alpha - n_{\alpha-1},$$

where $k_1, \ldots, k_s$ are integers satisfying $k_\alpha \geq 2$. Usually we denote $n/q = [k_1, \ldots, k_s]$. These sequences define the Hirzebruch-Jung continuous fraction as

$$\frac{n}{q} = k_1 - \cfrac{1}{k_2 - \cfrac{1}{\ddots - \frac{1}{k_s}}}.$$

Remark 2.5.1. *Observe that sequence n_α is the sequence m_α for n/q', i.e., if the pair (m'_α, n'_α) is the resolution of n/q' then*

$$(m'_\alpha, n'_\alpha) = (n_\alpha, m_\alpha).$$

Lemma 2.5.2. *For each α, we have the following relations*

1. $m_\alpha n_{\alpha+1} - m_{\alpha+1} n_\alpha = n,$
2. $gcd(m_\alpha, m_{\alpha+1}) = gcd(n_\alpha, n_{\alpha+1}) = 1,$
3. $gcd(m_\alpha, n_\alpha) = 1.$

Proof. The identities in (1) follow by induction using as the main tool the recursion from the $k'_\alpha s$. Since $\gcd(n, q) = \gcd(n, q') = 1$, (2) follows directly. For (3), using (1) and (2) we have

$$\gcd(m_\alpha, n_\alpha) = \gcd(m_\alpha, m_\alpha n_{\alpha+1} - n) = \gcd(m_\alpha, n) = 1.$$

□

The non-singular resolution of the planar cone τ is a refinement by adding the rays defined recursively by

$$v_\alpha = \frac{m_\alpha v_{\alpha-1} + v_{s+1}}{m_{\alpha-1}} = \frac{m_\alpha v_0 + n_\alpha v_{s+1}}{n}, \quad 1 \leq \alpha \leq s.$$

See Figure 2.1. Each cone $C(v_\alpha, v_{\alpha+1})$ is non-singular, since

$$\det(v_\alpha, v_{\alpha+1}) = \frac{1}{n}(m_\alpha n_{\alpha+1} - m_{\alpha+1} n_\alpha) = 1.$$

From Remark 2.5.1 observe that we have a dual non-singular resolution given by the sequence

$$v'_\alpha = \frac{m'_\alpha v'_{\alpha-1} + v_0}{m'_{\alpha-1}} = \frac{m'_\alpha v_{s+1} + n'_\alpha v_0}{n}, \quad 1 \leq \alpha \leq s.$$

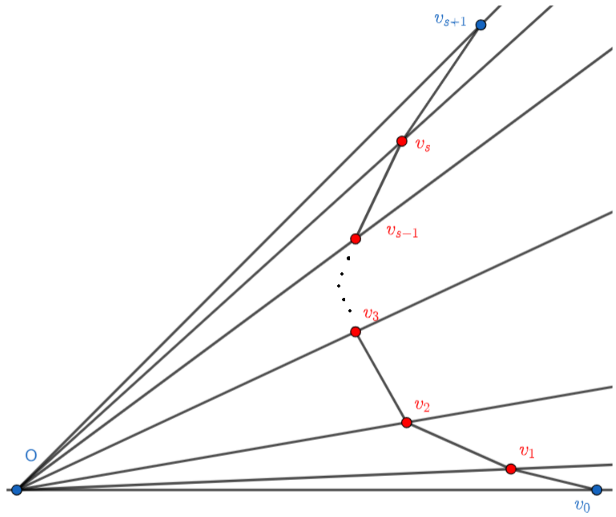


Figure 2.1: Resolved planar cone

from where we have $v_\alpha = v'_{s+1-\alpha}$.

Let us change the $\mathbb{Z}$-base of N such that $e_1 = v$ and $e_2 = v_0$. So, we have $v_{s+1} = ne_1 - qe_2$. We have,

$$\tau^\vee = C(e_1, pe_1 + ne_2) \oplus \bigoplus_{i=3}^{d-3} \mathbb{R}w_i = C(w_1, w_2) \oplus \mathbb{R}^{d-2},$$

where the $w_i \in M$ are such that $\langle v_0, w_i\rangle = \langle v_{s+1}, w_i\rangle = 0$ and $\langle \cdot, w\rangle \geq 0$ on τ for any $w \in C(w_1, w_2)$. Thus, in terms of toric varieties,

$$X_\tau = C_q \times (\mathbb{C}^\times)^{d-2},$$

where $C_q = \mathrm{Spec}(\mathbb{C}[C(w_1, w_2) \cap M])$ is the cyclic quotient surface singularity of type $\frac{1}{n}(q, 1)$ (Remark 2.4.5). Thus, the constructed resolution is a blow-up $h : \mathrm{Bl}_{\mathfrak{a}}(X_\tau) \to X_\tau$, where $\mathfrak{a} = \mathfrak{m} \otimes \mathbb{C}[x_3^{\pm 1}, ..., x_d^{\pm 1}]$ and

$$\mathfrak{m} = \bigoplus_{w \in C(w_1, w_2) \cap (M \setminus 0)} \chi^m.$$

For details see [CLS11, 11.3.6]. In particular, $\mathrm{Bl}_{\mathfrak{a}}(X_\tau) = \mathrm{Bl}_{\mathfrak{m}}(X_\tau) \times (\mathbb{C}^\times)^{d-2}$. We can give a explicit description of $\mathfrak{m}$ noting that the projection $C_q \to \mathbb{A}^2$ is given by the surjection $\mathbb{C}[\chi^{w_1}, \chi^{w_2}] \to \mathbb{C}[C_q]$.

2.6 Dedekind sums and asymptoticity

Consider the *sawtooth function* $((x))\colon \mathbb{R} \to \mathbb{R}$ is defined as

$$((x)) = \begin{cases} x - \lfloor x \rfloor - 1/2 & x \in \mathbb{R} \setminus \mathbb{Z} \\ 0 & x \in \mathbb{Z} \end{cases} .$$

Observe that is an odd periodic function of period 1.

Definition 2.6.1. *For $n \geq 3$ prime and $a_1, \ldots, a_d \in \mathbb{Z}$, we define the* Dedekind sum *of dimension d by*

$$d(a_1, \ldots, a_d, n) = \sum_{i=1}^{n-1} \left(\left(\frac{ia_1}{n}\right)\right) \cdots \left(\left(\frac{ia_d}{n}\right)\right).$$

By periodicity, we can reduce $a_i \in \mathbb{Z}$ to $0 \leq a_i < n$, by

$$d(a_1, \ldots, a_d, n) = d(\{a_1\}_n, \ldots, \{a_d\}_n, n).$$

where $\{a_i\}_n$ is the residue modulo n of a_i. Since $((x))$ is an odd function, we always have

$$d(-a_1, \ldots, -a_d, n) = (-1)^d d(a_1, \ldots, a_d, n),$$

i.e., $d(a_1, ..., a_d, n) = 0$ for any odd dimension d.

We can rewrite this sum as

$$d(a_1, \ldots, a_d, n) = \frac{1}{n^d} \sum_{i=1}^{n-1} \left(\{ia_1\}_n - \frac{n}{2}\right) \cdots \left(\{ia_d\}_n - \frac{n}{2}\right).$$

A well-known result is the Reciprocity Theorem for a Dedekind $d(a_1, \ldots, a_d, n)$ of dimension d even [Zag73, pag.158]. It tells us the existence of a rational function $\phi_n(x_0, x_1, \ldots, x_d)$ with $x_0 \ldots x_d \phi_n$ a polynomial in $d+1$ variables such that

$$d(a_1, \ldots, a_d, n) + \sum_{k=1}^{d} d(a_1, \ldots, a_{k-1}, n, a_{k+1}, \ldots, a_d, a_k) = \phi_n(x_0, x_1, \ldots, x_d),$$

For an explicit construction of ϕ_n see [Zag73, Sec. 3]. Indeed the case $d = 2$ is a classical result given by H. Rademacher in 1956, i.e., we have

$$d(a, b, n) + d(a, n, b) + d(n, b, a) = \frac{1}{4}\left(\frac{a^2 + b^2 + n^2}{3abn} - 1\right).$$

The non-trivial Dedekind sums are those of even dimension. However, by the vanishing of the odd-dimensional Dedekind sums, we can relate sums of powers of residues modulo n with Dedekind sums of even dimension. The following lemma illustrates this.

Lemma 2.6.2. *We have the following relations,*

$$\sum_{i=1}^{n-1}\{ia\}_n\{ib\}_n = n^2 d(a,b,n) + \frac{n^2(n-1)}{4},$$

$$\sum_{i=1}^{n-1}\{ia\}_n^2\{ib\}_n = \sum_{i=1}^{n-1}\{ia\}_n\{ib\}_n^2 = n^3 d(a,b,n) + \frac{n^2(n-1)(2n-1)}{12}$$

$$\sum_{i=1}^{n-1}\{ia\}_n\{ib\}_n\{ic\}_n = \frac{n^3}{2}\left(d(a,b,n)+d(a,c,n)+d(b,c,n)\right) + \frac{n^3(n-1)}{8},$$

Proof. We have an identity

$$\sum_{i=1}^{n-1}\{ia\}_n^k = \sum_{i=1}^{n-1} i^k,$$

for each $k \geq 0$ integer. Indeed, since ia and n are coprimes, the map $i \mapsto \{ia\}_n^k$ is a permutation of the set $\{1,...,n-1\}$. Then, we use repeatedly this identity in the following expressions. The first formula follows from,

$$n^2 d(a,b,n) = \sum_{i=1}^{n-1}\left(\{ia\}_n - \frac{n}{2}\right)\left(\{ib\}_n - \frac{n}{2}\right),$$

and using it in,

$$0 = \sum_{i=1}^{n-1}\left(\{ia\}_n - \frac{n}{2}\right)\left(\{ib\}_n - \frac{n}{2}\right)\left(\{ic\}_n - \frac{n}{2}\right),$$

we get the other two.

□

Recall from Section 2.5 that each pair of coprime numbers n, a have a negative-regular continued fraction

$$\frac{n}{q} = k_1 - \cfrac{1}{k_2 - \cfrac{1}{\ddots - \frac{1}{k_s}}}.$$

A well-known result [Bar77] (also see [Hol88]) is the following formula relating the length s of the continued fraction and Dedekind sums,

Proposition 2.6.3. *Let n be a prime number, and q be an integer such that $0 < q < n$. Let $n/q = [k_1, \ldots, k_s]$. Then*

$$d(q, 1, n) + s = \sum_{\alpha=1}^{s} (k_\alpha - 2) + \frac{q + q'}{n}.$$

□

2.7 Asymptoticity in dimension 2

Let Z be a non-singular projective surface, and let D be an effective divisor with SNC reduced divisor. Assume the necessary hypothesis to construct the normal n-th root cover $Y_n \to Z$ along D (Section 2.3). We have to choose a resolution of singularities $h\colon X_n \to Y_n \to Z$, and the Chern numbers c_1^2, c_2 of X_n will depend on this resolution. The singularities of Y_n over each point of $D_j \cap D_k$ are analytically isomorphic to the normalization of

$$\operatorname{Spec}\left(\frac{\mathbb{C}[x, y, t]}{t^n - x^{n-q_{jk}} y}\right),$$

where $\nu_j + q_{jk}\nu_k \equiv 0$ modulo n. This singularity is a cyclic quotient singularity of type $\frac{1}{n}(q_{jk}, 1)$, and the singular point can be resolved by some weighted blow-ups. The exceptional data will be a chain of non-singular rational curves $\{E_1, \ldots, E_s\}$ with $E_j E_{j+1} = 1$ and $E_j^2 = -k_j$, where the $k_j \geq 2$ are the integers that define the negative regular continued fraction

$$\frac{n}{q_{jk}} = k_1 - \cfrac{1}{k_2 - \cfrac{1}{\ddots - \frac{1}{k_s}}},$$

usually called Hirzebruch-Jung continued fraction. See Section 2.5 for details. The number s is called the *length* of the resolution and we denoted it by $\ell(q_{jk}, n)$. In this way, we resolve all singularities of Y_n obtaining a morphism $g\colon X_n \to Y_n$, with composition $h\colon X_n \to Z$.

In dimension 2, for the chosen resolution X_n Dedekind sums and lengths appear in the following formulas [Urz09],

$$\chi(\mathcal{O}_{X_n}) = n\chi(\mathcal{O}_Z) - \frac{p^2-1}{12n}D^{[2]} - \frac{p-1}{4}e(D) + \sum_{j<k} d(1, q_{jk}, n)D_jD_k$$

$$c_2(X_n) = nc_2(Z) - (n-1)e(D) + \sum_{j<k} \ell(q_{jk}, n)D_jD_k.$$

Then, we can recover a formula for c_1^2 by Noether's identity. In [Gir03] and [Gir06], Girstmair proved that the lengths and the values of Dedekind sums have a particular asymptotical behavior.

Theorem 2.7.1 (Girstmair)**.** *For $n \geq 17$ there exists a set $O_n \subset \{0, \ldots, n\}$ such that for any $q \in O_n$ we have*

$$d(1, q, n) \leq 3\sqrt{n} + 5,$$

$$\ell(q, n) \leq 3\sqrt{n} + 2.$$

Moreover $|\{0, \ldots, n\} \setminus O_n| \leq \sqrt{n}\log(4n)$.

□

Remark 2.7.2. *The set O_n can be constructed as follows. Let $n \geq 2$ be an integer. A* Farey point *is a rational number of the form $n\frac{c}{d}$, with $1 \leq d \leq \sqrt{n}$, $0 \leq c \leq d$, $\gcd(c, d) = 1$. The interval*

$$I_{\frac{c}{d}} = \{x : 0 \leq x \leq n, \left|x - n\frac{c}{d}\right| \leq \frac{\sqrt{n}}{d^2}\},$$

is called a Farey neighborhood *of $n\frac{c}{d}$. Define the* bad set *as*

$$\mathcal{F} = \bigcup_{1 \leq d \leq \sqrt{n}} \bigcup_{c \in \mathcal{C}_d} I_{\frac{c}{d}},$$

where $\mathcal{C}_d = \{c : 0 \leq c \leq d, (c, d) = 1\}$. Thus, O_n is the set of those integers $q \notin \mathcal{F}$ with $0 \leq q < n$.

Remark 2.7.3. *Observe that from the relation of Proposition 2.6.3 combining it with the results of Girstmair, we have an asymptotic behavior for the coefficients of the Hirzebruch-Jung continued fraction in the following sense:*

For a prime number $n \gg 0$, and integers $q \in O_n$ with $n/q = [k_1, \ldots, k_s]$, we have

$$\sum_{\alpha=1}^{s} (k_\alpha - 2) \leq 6\sqrt{n} + 7.$$

Definition 2.7.4. *A collection of prime divisors $\{D_1, \ldots, D_r\}$ on a non-singular d-fold Z is an* asymptotic arrangement *if $D_{red} = D_1 + \ldots + D_r$ is SNC, and for prime numbers $n \gg 0$:*

1. *There are multiplicities $0 < \nu_j < n$, such that for any $j < k$ with $D_j \cap D_k \neq \emptyset$, we have $q_{jk} \in O_n$, the unique integer such that $\nu_j + q_{jk}\nu_k \equiv 0$ modulo n.*

2. *There are line bundles $\mathcal{L}$ such that,*

$$\mathcal{O}_Z\left(\sum_{j=1}^{r} \nu_j D_j\right) \simeq \mathcal{L}^{\otimes n}.$$

Example 2.7.5. *Inside the proof of [Urz09, Th. 6.1], it was proved that for any large prime number n there exist a partition*

$$\nu_1 + \ldots + \nu_r = n,$$

with $q_{jk} \in O_n$ such that $\nu_j + q_{jk}\nu_k \equiv 0$ modulo n. We call it an asymptotic partition *of n. Indeed, the probability of a partition of n to be asymptotic tends to 1 as n grows to infinity. Thus, any collection of hyperplanes $\{H_1, \ldots, H_r\}$ on $\mathbb{P}^d$ defining a SNC divisor, is itself an asymptotic arrangement with*

$$D = \nu_1 H_1 + \ldots + \nu_r H_r = (\nu_1 + \ldots + \nu_r)H = nH,$$

where H is a general hyperplane section on $\mathbb{P}^d$.

Definition 2.7.6. *Let $\mathcal{A} = \{D_1, \ldots, D_r\}$ be an arbitrary collection of hypersurfaces in a non-singular projective d-fold Z. We say that $\mathcal{A}$ has* simple crossing *if each D_j is non-singular, pairwise intersections are transversal, and the components of $D_{j_1} \cap \ldots \cap D_{j_e}$ are non-singular. For $0 \leq e \leq d-2$ and $m > d - e$, a (e,m)*-singularity *in $\mathcal{A}$ is a e-fold through which exactly m of the divisors pass.*

Proposition 2.7.7. *Let Z be a non-singular projective d-fold, and $\mathcal{A} = \{D_1, \ldots, D_r\}$ a collection of hypersurfaces with simple crossings. Assume that $\mathcal{A}$ satisfy the hypothesis (1) and (2) of Definition 2.7.4, then there exists a* log*-morphism $h : Z' \to Z$ such that the components of $(h^*D)_{red}$ define an asymptotic arrangement. The* log*-morphism h is called a* log-resolution *of the pair $(Z, \mathcal{A})$.*

Proof. Let $h\colon Z' \to Z$ be the morphism constructed as follows:

1. Blow-up all $(0, m)$-singularities for $m > d$.

2. Blow-up all $(1, m)$-singularities for $m > d - 1$,

... and inductively,

(d-1) Blow-up all $(d - 2, m)$-singularities for $m > 2$.

The pull-back of the arrangement defines a SNC divisor consisting of the strict transforms $\tilde{D}_j$ of each divisor and the exceptional divisors over each (e, m)-singularity. Thus, for $n \gg 0$ there are multiplicities $0 < \nu_j < n$ such that $D = \sum_j \nu_j D_j \sim nL$ with $q_{jk} \in O_n$. The multiplicity of $\tilde{D}_j$ in h^*D is ν_j. Let E be a exceptional divisor over a (e, n)-singularity defined over a component $D_{j_1} \cap \ldots \cap D_{j_m}$. The multiplicity of E in h^*D is $\nu_{j_1} + \ldots + \nu_{j_m}$. Thus, the multiplicities in h^*D are the same of D together with its combinatorial sums. This setup translates to an arithmetical one, so is a consequence of [Urz09, Th.7.1] that $(h^*D)_{red}$ is again an asymptotic arrangement. ☐

Let us denote by $\bar{c}(Z, D)$ the logarithmic Chern class of the above log-resolution. In [Urz09] the author uses the observations of the above remark to get asymptoticity of invariants for arbitrary arrangements of curves.

Theorem 2.7.8. *[Urzúa] Let Z be a non-singular projective surface, and let $\{D_1, \ldots, D_r\}$ be an asymptotic arrangement of curves. Denote by $X_n \to Y_n \to Z$ the resolution of the n-th root cover Y_n along each $D = \sum_j \nu_j D_j$. Then*

$$\frac{c_1^2(X_n)}{n}, \frac{c_2(X_n)}{n} \xrightarrow[n\to\infty,\, n \text{ prime}]{} c_1^2(Z, D), c_2(Z, D).$$

Moreover, if each K_{Y_n} is nef, then with respect to n, the Chern numbers of the minimal model of X_n are asymptotic to the logarithmic Chern numbers of the base (Z, D).

Proof. For the first assertion see [Urz09, Th.7.1]. See [Urz16, Th.4.3] for the last □

The following example shows a classical family of arrangements of planes. For more examples of this kind we refer to [Hun89].

Example 2.7.9. *Let us consider* Platonic arrangements, *i.e, arrangements of planes* $\{H_1, ..., H_r\}$ *in* $\mathbb{P}^3$ *defined by linear polynomials with real coefficients describing a Platonic solid in* $\mathbb{P}^3_{\mathbb{R}}$*, the projective space over* $\mathbb{R}$*. In this case, no more than 2 planes pass through a line. However, there are points having* $m \geq 3$ *planes passing through them. They are called the* m*-points of the arrangement. Let* t_m *be the number of them. Let* $\sigma\colon Z \to \mathbb{P}^3$ *be the* log-*resolution constructed in Proposition 2.7.7. In this case, from [PSG94, 2.2.14] we have Chern classes*

$$c_1(Z) = -4\sigma^* H - 2\sum_{p\geq 4}\sum_{s=1}^{t_m} E_{m,s},$$

$$c_2(Z) = 6\sigma^* H^2,$$

$$c_3(Z) = 4 + 3\sum_{m\geq 4} t_m,$$

where $E_{m,s}$ *are the exceptional divisors over* m*-points, and they satisfy* $E^3_{m,s} = 1$*. With this in mind, and using formulas from Corollary 2.2.15, we can compute the following tables.*

Name	r	# lines H_{jk}	t_3	t_4	t_5	t_6
Tetrahedron	4	6	4	-	-	-
Hexahedron	6	15	8	3	-	-
Octahedron	8	28	8	12	-	-
Dodecahedron	12	66	40	15	12	-
Icosahedron	20	190	140	90	24	20

Table 2.1: Platonic Solids

They have logarithmic Chern numbers given in the following table.

Name	r	$\bar{c}_1^3(Z,D)$	$\bar{c}_1\bar{c}_2(Z,D)$	$\bar{c}_3(Z,D)$
Tetrahedron	4	0	0	0
Hexahedron	6	11	27	17
Octahedron	8	76	124	64
Dodecahedron	12	623	705	93
Icosahedron	20	4918	4690	250

Table 2.2: Logarithmic Invariants for Platonic Solids

Chapter 3

Asymptoticity for non-singular branch locus

3.1 Logarithmic asymptoticity

Let $A(Z)_{\mathbb{R}} = A(Z) \otimes_{\mathbb{Z}} \mathbb{R}$ be the extended Chow ring of a non-singular variety Z. For each $n \geq 1$ assume the existence of finite log-morphisms $h_n \colon Y_n \to Z$ between non-singular varieties of the same dimension with $\deg(h_n) = n$ (Definition 2.2.20). We have a morphism of extended Chow rings $h_n^* \colon A(Z)_{\mathbb{R}} \to A(Y_n)_{\mathbb{R}}$ for each n. Since h_n is flat, we have that $h_n^*(A^e(Z)) \subset A^e(Y_n)$ [Har77, III.9.6]. The following definition lets us to reduce the notation in the rest of this section.

Definition 3.1.1. *Let $\{C_n\}_{n\geq 1}$ be a sequence of e-cycles with*

$$C_n \in h_n^*(A^e(Z)_{\mathbb{R}}) \subset A^e(Y_n)_{\mathbb{R}}$$

for each n, and $C \in A^e(Z)_{\mathbb{R}}$. We say that C_n has C as limit *and denoted by $\lim_{n\to\infty} C_n = C$, if for every $b \in A^{d-e}(Z)_{\mathbb{R}}$ we have*

$$\lim_{n\to\infty} \frac{C_n\, h_n^* b}{n} = Cb.$$

This is well-defined since $C_n\, h_n^* b = n h_{n*} C_n\, b$ on Z, i.e., we are dealing with real numbers depending on Z.

Lemma 3.1.2. *For any partition $i_1 + \ldots + i_m = d$, assume that for each $j = 1, \ldots, m$ there are sequences $\{C_n^{i_j}\}_n$ with $\lim_n C_n^{i_j} = C^{i_j} \in A^{i_j}(Z)_{\mathbb{R}}$.*

Then, we have

$$\lim_{n\to\infty} \frac{C_n^{i_1}\dots C_n^{i_m}}{n} = C^{i_1}\dots C^{i_m}.$$

Proof. By definition, there are classes $c_n^i \in A^i(Z)_{\mathbb{R}}$ such that that $C_n^i = h_n^* c_n^i$. Define $c_{n_1,\dots,n_m} = c_{n_1}^{i_1}\dots c_{n_m}^{i_m} \in \mathbb{R}$. Since $\lim_{n\to\infty} C_n^i = C^i$, we have

$$\lim_{n_j\to\infty} c_{n_1,\dots,n_m} = c_{n_1}^{i_1}\dots c_{n_{j-1}}^{i_{j-1}} C^{i_j} c_{n_{j+1}}^{i_{j+1}}\dots c_{n_m}^{i_m}$$

for fixed $n_i \neq n_j$. So, we proceed analogously taking limit for the other n_i, and we get,

$$\lim_{n_1,\dots,n_2\to\infty} c_{n_1,\dots,n_m} = C^{i_1}\dots C^{i_m},$$

and the result follows. □

Theorem 3.1.3. *For $n \geq 1$ assume the existence of finite* log-*morphisms $h_n\colon Y_n \to Z$ ramified at a non-singular divisor D whose reduced form is a SNC divisor. Let $D_1, \dots, D_r$ be the components of D, and D'_j the reduced preimage of each D_j. Assume $h_n^* D_j = nD'_j$, and that D_j^d does not depend on n. Then, we have,*

$$\lim_{n\to\infty} \frac{c_{i_1}\dots c_{i_m}(Y_n)}{n} = \bar{c}_{i_1}\dots \bar{c}_{i_m}(Z, D).$$

Proof. The proof of the theorem is based in proving the following,

$$c_e(Y_n) \in h_n^*(A^e(Z)_{\mathbb{R}}), \quad \text{and} \quad \lim_{n\to\infty} c_e(Y_n) = \bar{c}_e(Z, D),$$

for each e. We proceed by induction on the dimension $1 \leq e \leq d$. Since D is non-singular, by Lemma 2.2.21 we have $\bar{c}_e(Y_n, D) = h_n^*(\bar{c}_e(Z, D))$. Thus, by Corollary 2.2.13 we get

$$c_e(Y_n) = h_n^*(\bar{c}_e(Z, D)) - R_e(D'), \quad R_e(D') = \sum_{\substack{k+l=e\\ k\neq e}} (-1)^l D'^{[l]} c_k(Z).$$

Since $h_n^* D_j = nD'_j$, for the case $e = 1$ we have,

$$c_1(Y_n) = h_n^*(c_1(Z) - D_{red}) + \sum_{j=1}^{r} \frac{h_n^*(D_j)}{n} \in h_n^*(A^1(Z)_{\mathbb{R}}).$$

Then for every $h_n^* b \in h_n^*(A^1(Z))$ we have

$$\lim_{n\to\infty} \frac{c_1(Y_n)\, h_n^* b}{n} = \lim_{n\to\infty} \left[(c_1(Z) - D_{red})b + \sum_{j=1}^{r} \frac{D_j b}{n} \right] = (c_1(Z) - D_{red})b,$$

i.e., $\lim_{n\to\infty} c_1(Y_n) = \bar{c}_1(Z, D)$. We assume as induction hypothesis that the result is true for $k < e$. Thus, for $k = e$ using Corollary 2.2.13 we have,

$$\begin{aligned} R_e(D') &= \sum_{\substack{k+l=e \\ k\neq e}} (-1)^l \sum_{j=1}^{r} (D'_j)^l c_k(X_n) \\ &= \sum_{\substack{k+l=e \\ k\neq e}} (-1)^l \sum_{j=1}^{r} \frac{(h_n^* D_j)^l}{n^l} c_k(X_n). \end{aligned}$$

Since $k < e$, $c_k(Y_n) \in h_n^*(A^k(Z)_{\mathbb{R}})$, also $c_e(Y_n)$ is in $h_n^*(A^e(Z)_{\mathbb{R}})$. We have $\lim_n c_k(Y_n) = c_k(Z)$, thus for $e > 0$ we have

$$\lim_{n\to\infty} \frac{R_e(D')}{n} = 0,$$

from where the result follows. $\square$

3.2 Applications

The main situation to apply Theorem 3.1.3 is the case of n-th root covers. In this case, take a non-singular SNC divisor $D_1 + \ldots + D_r$ on Z, and we restrict our attention for prime numbers $n \geq 2$. Assume that D_j^d does not depend on n. For each n, assume the existence of L and $0 < \nu_j < n$ such that $D = \sum_j \nu_j D_j \sim nL$ (Section 2.3). Construct the non-singular covers $h_n \colon Y_n \to Z$ along each D. Then $h_n^* D_j = nD'_j$, and we get.

Corollary 3.2.1. *Under the above hypothesis, the n-th root covers Y_n satisfy,*

$$\frac{c_{i_1} \ldots c_{i_m}(Y_n)}{n} \approx \bar{c}_{i_1} \ldots \bar{c}_{i_m}(Z, D),$$

for prime numbers $n \gg 0$.

□

For our purposes in geography this result has a disadvantage, the difficulty to find good pairs (Z, D) whose covers Y_n are minimal of general type. For example, from Theorem 2.3.6 would be enough K_Z big and nef, and D ample with many components. However, at least the condition K_Z big seems difficult to assure by the following. In [BPS16] was proved the following: Assume that Z has a collection of disjoint divisors $\{D_j\}_{j\in J}$, if $|J| \gg 0$, then there exists a surjective morphism from Z to a curve such that every D_j is contained in a fiber. For another purposes, these results can be applied to projective bundles.

Example 3.2.2. *Let C be a curve of genus $g \geq 1$, and $\mathcal{L} \neq \mathcal{O}_C$ a line bundle on C of degree 0. Consider the locally free sheaf $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{L}$ of rank 2, and the non-singular ruled surface,*

$$\pi : \mathbb{P}(\mathcal{E}) \to C.$$

Then, $\mathbb{P}(\mathcal{E})$ has two disjoint sections C_1, C_2 with $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(C_i) \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, and $C_i^2 = 0$ [Har77, Ex. V.2.11.2]. Thus, Corollary 3.2.1 can be applied. We construct surfaces $Y_n \to \mathbb{P}(\mathcal{E})$ with

$$\frac{c_1^2(Y_n)}{n} \to \bar{c}_1^2(\mathbb{P}(\mathcal{E}), C_1 + C_2) = 12(1-g),$$

$$\frac{c_2(Y_n)}{n} \to \bar{c}_2(\mathbb{P}(\mathcal{E}), C_1 + C_2) = 2(1-g),$$

for prime numbers $n \gg 0$. In particular, as n grows the slope c_1^2/c_2 of Y_n tends to 6.

By the above discussion, in the rest of this thesis we will study the above results for the case of 3-folds when D has its components with non-empty intersection. Thus Y_n will have singularities.

Remark 3.2.3. *We can extend Corollary 3.2.1 to Abelian covers, i.e., to the case $G_n = \mathbb{Z}/n_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/n_k\mathbb{Z}$ a sequence of Abelian groups of order $n = n_1 \ldots n_k$ with each n_j a prime number with $n_j \neq n_k$. See [Par91] or [Gao11]. In this case, the Abelian covers $Y_n \to X$ depend on a data $D^i \sim n_i L_i$ with D^i_{red} a SNC divisor for $i = 1, \ldots, k$. Then the SNC divisor to take is*

$D_{red} = (D^1 + \ldots + D_k)_{red}$. The ramification numbers for each component D_i^j of D^i are given by

$$h^* D_j^i = \frac{n}{gcd(n_i, \nu_j^i(n))} {D'}_j^i = n{D'}_j^i,$$

where $\nu_j^i(n)$ is the multiplicity of D_j^i in D^i. From here, we leave to the reader the analog asymptotic result. However, we can ask: How do this argument be extended to any Galois cover?

Example 3.2.4. *Since a n-th root cover $Y_n \to Z$ (assume n prime) along a non-singular effective divisor D is non-singular, from Theorem 2.3.6, if K_Z is a nef divisor, then Y_n is minimal. Consider $Z = V \times \mathbb{P}^1$ with projections p_1, p_2, and V a non-singular variety of dimension $d-1$. From Example 2.2.18 consider $V_1, \ldots, V_r \sim [V]$ on Z with $r > 2$. If K_V is nef and $r \geq 3$, then Y_n is minimal. Moreover, if V is of general type, then Y_n is of general type for $n >> 0$. This follows since $K_V^{d-1} > 0$, and from the fact that for $n >> 0$ we have*

$$K_{Y_n}^d \approx nK_V^{d-1}(r-2) > 0.$$

If V is a surface, the asymptotic result Corollary 3.2.1 implies that

$$\left(\frac{\bar{c}_1^3}{\bar{c}_1\bar{c}_2}, \frac{\bar{c}_3}{\bar{c}_1\bar{c}_2}\right)(Y_n) \to \left(\frac{c_1^2(V)}{4\chi(\mathcal{O}_V)}, \frac{c_2(V)}{12\chi(\mathcal{O}_V)}\right),$$

as $n, r \to \infty$. From Sommese's density result, we have a line L in $\mathbb{Q}^2$ connecting $(1/2, 5/6)$ and $(9/4, 1/4)$ whose points are limit points respect to n of sequences of non-singular minimal 3-folds of general type.

This example gives a new point of view of the n-th root construction: In fact, Y_n is isomorphic to $V \times C_n$ where $C_n \to \mathbb{P}^1$ is a n-th root cover of $\mathbb{P}^1$.

Chapter 4

Asymptoticity of invariants

4.1 Asymptoticity of χ for 3-folds

Consider a data $(Z, D, n, \mathcal{L})$ as in Section 2.3, with Z a non-singular projective 3-fold. Let $h\colon X_n \to Y_n \to Z$ be any resolution of singularities of the branched n-th root cover Y_n along the effective divisor $D = \sum_{j=1}^{r} \nu_j D_j \sim nL$ whose reduced form is SNC. We have,

$$L^{(i)} = iL - \sum_{j=1}^{r} \left\lfloor \frac{i\nu_j}{n} \right\rfloor D_j = \frac{1}{n}\left(iD - \sum_{j=1}^{r} n \left\lfloor \frac{i\nu_j}{n} \right\rfloor D_j \right) = \frac{1}{n}\sum_{j=1}^{r} \{i\nu_j\}_n D_j.$$

By Corollary 2.3.8, and Hirzebruch-Riemann-Roch theorem for 3-folds we can compute

$$\begin{aligned}
\chi(\mathcal{O}_{X_n}) &= \sum_{i=0}^{n-1} \chi(\mathcal{O}_Z(-L^{(i)})) \\
&= n\chi(\mathcal{O}_Z) - \frac{1}{12}\sum_{i=1}^{n-1} \left(L^{(i)}(L^{(i)} + K_Z)(2L^{(i)} + K_Z) + c_2(Z).L^{(i)} \right) \\
&= n\chi(\mathcal{O}_Z) - \frac{1}{12}\sum_{i=1}^{n-1} \left(2(L^{(i)})^3 + 3(L^{(i)})^2 K_Z + L^{(i)} K_Z^2 + c_2(Z).L^{(i)} \right) \\
&= n\chi(\mathcal{O}_Z) - \frac{1}{12} R(n, D),
\end{aligned}$$

where $R(n, D)$ is a quantity depending only on n and D. We have the following identities.

$$\sum_{i=1}^{n-1} L^{(i)} = \frac{(n-1)}{2} D_{red}$$

$$\sum_{i=1}^{n-1} (L^{(i)})^2 = \frac{(n-1)(2n-1)}{6n} D^{[2]} + \frac{(n-1)}{2} D^{[1,1]} + 2\sum_{j<k} d(\nu_j, \nu_k, n) D_j D_k.$$

The above is not difficult to deduce from the formulas in Lemma 2.6.2. To illustrate, we compute $\sum_i (L^{(i)})^3$ as follows. The first step, we compute explicitly:

$$\begin{aligned}
(L^{(i)})^3 &= \frac{1}{n^3}\left(iD - \sum_{j=1}^{r} n \left\lfloor \frac{i\nu_j}{n} \right\rfloor D_j\right)^3 \\
&= \frac{1}{n^3}\left(\sum_{j=1}^{r} \{i\nu_j\}_n D_j\right)^3 \\
&= \frac{1}{n^3}\left(\sum_{j=1}^{r} \{i\nu_j\}_n^3 D_j^3 + 3\sum_{j<k} \{i\nu_j\}_n^2 \{i\nu_k\}_n D_j^2 D_k + \{i\nu_j\}_n \{i\nu_k\}_n^2 D_j D_k^2 \right. \\
&\qquad \left. + 6 \sum_{j<k<l} \{i\nu_j\}_n \{i\nu_k\}_n \{i\nu_l\}_n D_j D_k D_l \right)
\end{aligned}$$

Applying directly the formulas in Lemma 2.6.2, we get.

$$\begin{aligned}
\sum_{i=1}^{n-1} (L^{(i)})^3 &= \frac{(n-1)^2}{4n} D^{[3]} + \frac{(n-1)(2n-1)}{4n}(D^{[1,2]} + D^{[2,1]}) + \frac{3(n-1)}{4} D^{[1,1,1]} \\
&\quad + 3\sum_{j<k} d(\nu_j, \nu_k, n)(D_j^2 D_k + D_j D_k^2) \\
&\quad + 3\sum_{j<k<l} (d(\nu_j, \nu_k, n) + d(\nu_j, \nu_l, n) + d(\nu_k, \nu_l, n)) D_j D_k D_l.
\end{aligned}$$

On the other hand,

$$\sum_{i=1}^{n-1} (L^{(i)})^2 K_Z = \frac{(1-n)}{2} c_1(Z) \left(\frac{(2n-1)}{3n} D^{[2]} + D^{[1,1]}\right) + 2\sum_{j<k} d(\nu_j, \nu_k, n) D_j D_k K_Z,$$

$$\sum_{i=1}^{n-1} L^{(i)} K_Z^2 = \frac{(n-1)}{2} K_Z^2 D_{red} = \frac{(n-1)}{2} D_{red} c_1^2(Z),$$

$$\sum_{i=1}^{n-1} L^{(i)} c_2(Z) = \frac{(n-1)}{2} D_{red} c_2(Z).$$

Thus, we have $R(n,D) = R_1(n,D) + R_2(n,D) + R_3(n,D)$, where

$$R_1(n,D) = \frac{(n-1)^2}{2n} D^{[3]} + \frac{(n-1)(2n-1)}{2n}(D^{[1,2]} + D^{[2,1]}) + \frac{3(n-1)}{2} D^{[1,1,1]},$$

$$R_2(n,D) = \frac{(1-n)}{2} c_1(Z) \left(\frac{(2n-1)}{n} D^{[2]} + 3D^{[1,1]} \right) + \frac{(n-1)}{2} D_{red}(c_1^2(Z) + c_2(Z)),$$

$$R_3(n,D) = 6 \left(\sum_{j<k} d(\nu_j, \nu_k, n) D_j D_k (D_j + D_k + K_Z) \right.$$
$$\left. + \sum_{j<k<l} (d(\nu_j,\nu_k,n) + d(\nu_j,\nu_l,n) + d(\nu_k,\nu_l,n)) D_j D_k D_l \right).$$

Theorem 4.1.1. *If $\{D_1, \ldots, D_r\}$ is an asymptotic arrangement, then*

$$\frac{\chi(\mathcal{O}_{X_n})}{n} \approx \frac{\overline{c_1 c_2}(Z,D)}{24},$$

for prime numbers $n \gg 0$.

Proof. First observe the following limits

$$\lim_{n\to\infty} \frac{R_1(n,D)}{n} = \frac{1}{2} D^{[3]} + (D^{[1,2]} + D^{[2,1]}) + \frac{3}{2} D^{[1,1,1]} = \frac{1}{2} D_{red}(D^{[2]} + D^{[1,1]}),$$

$$\lim_{n\to\infty} \frac{R_2(n,D)}{n} = -\frac{1}{2} c_1(Z) \left(2D^{[2]} + 3D^{[1,1]}\right) + \frac{1}{2} D_{red}(c_1^2(Z) + c_2(Z)).$$

From formulas in Corollary 2.2.15, we get the identity

$$\lim_{n\to\infty} \frac{R_1(n,D) + R_2(n,D)}{n} = \frac{1}{2}(c_1 c_2(Z) - \overline{c_1 c_2}(Z,D)).$$

Since the collection of divisors is an asymptotic arrangement, we use Theorem 2.7.1 to get $R_3(n,D)/n \approx 0$ for prime numbers $n \gg 0$. Thus, in this case we have

$$\frac{\chi(\mathcal{O}_{X_n})}{n} \approx \frac{c_1c_2(Z)}{24} - \frac{1}{24}(c_1c_2(Z) - \overline{c_1c_2}(Z,D)) = \frac{\overline{c_1c_2}(Z,D)}{24},$$

for prime numbers $n \gg 0$. □

Remark 4.1.2. *Observe that the formula for $(L^{(i)})^3$ can be extended to higher dimensions. In the same way of Lemma 2.6.2, we can find formulas for $(L^{(i)})^e$ depending only on the combinatorial aspects of D and higher dimensional Dedekind sums. Thus, for asymptoticity of $\chi(\mathcal{O}_{X_n})$ in any dimension, we need asymptoticity of Dedekind sums, i.e., the higher dimensional analogs of Girstmair's results. For dimension $d \geq 4$ this is an open problem.*

4.2 Toric local resolutions

In this section, we study the 3-fold singularity given by the normalization of $t^n = x_1^{\nu_1}x_2^{\nu_2}x_3^{\nu_3}$, with $n \geq 2$ a prime number and $0 < \nu_i < n$. The aim is to achieve good local resolutions of singularities in asymptotic terms with respect to n. Since this singularity is toric, it can be resolved by subdivisions of its associated cone obtaining a refinement fan. To assure asymptotic properties, we have to pay attention to the combinatorial aspects of the refinement. Let $0 < p, q < n$ be integers such that $\nu_1 + p\nu_3 \equiv 0$ and $\nu_1 + q\nu_3 \equiv 0$ modulo n. By Section 2.4, this 3-fold singularity is a toric variety $Y_{p,q} := \mathrm{Spec}(\sigma^\vee \cap \mathbb{Z}^3)$ defined by the cone $\sigma = C(d_1, d_2, d_3) \subset \mathbb{R}^3$, where $d_1 = ne_1 - pe_3, d_2 = ne_2 - qe_3$, and $d_3 = e_3$ are the primitive ray generators. A transversal section of this cone is sketched in Figure 4.1.

The fan defined by the cone σ has 2-dimensional faces (walls) given by $\tau_{jk} = C(d_j, d_k)$ for $j < k$. By Section 2.5, each wall τ_{jk} can be resolved by Hirzebruch-Jung sequences $(m_{jk,\alpha}, n_{jk,\alpha})_{\alpha=0}^{s_{jk}+1}$ in direction d_j to d_k with initial data

$$m_{jk,0} = n, \quad n_{jk,0} = 0, \quad n_{jk,1} = 1, \quad \forall j < k,$$

$$m_{13,1} = p', \quad m_{23,1} = q', \quad m_{12,1} = \{-p'q\}_n,$$

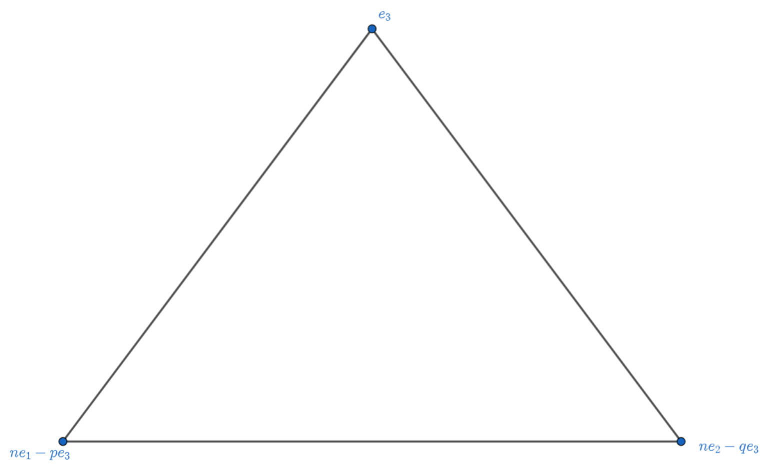


Figure 4.1: Transversal section of σ.

where p' and q' are the inverse modulo n of p and q. Thus, there are integers $k_{jk,\alpha} \geq 2$, such that

$$\frac{n}{m_{jk,1}} = [k_{jk,1}, ..., k_{jk,s_{jk}}].$$

Then, the walls τ_{jk} can be resolved by subdividing them in a sequence of steps by rays with generators $(e_{jk,\alpha})_{\alpha=1}^{s_{jk}}$ defined recursively as

$$e_{jk,\alpha} = \frac{m_{jk,\alpha} d_j + n_{jk,\alpha} d_k}{n}, \quad 1 \leq \alpha \leq s_{jk}.$$

If we fix $j < k$, we denote by $e_{kj,\alpha}$ the exceptional divisors in direction d_k to d_j. We have the relation $e_{jk,\alpha} = e_{kj,s+1-\alpha}$. In Figure 4.2 are illustrated the border generators.

In order to choose a good asymptotic local resolution of σ, imitating the 2-dimensional case, we can ask for a minimal local resolution, i.e., with nef canonical bundles. However, minimal varieties in higher dimensions may have terminal singularities, minimal singular models are not necessarily unique, and there are no efficient algorithms in the toric case. In this last, at least there exists a kind of optimal method. Minimal resolutions can be obtained by the canonical resolution of σ which is obtained by the canonical refinement of the cone [CLS11, Prop. 11.4.15]. However, the canonical refinement

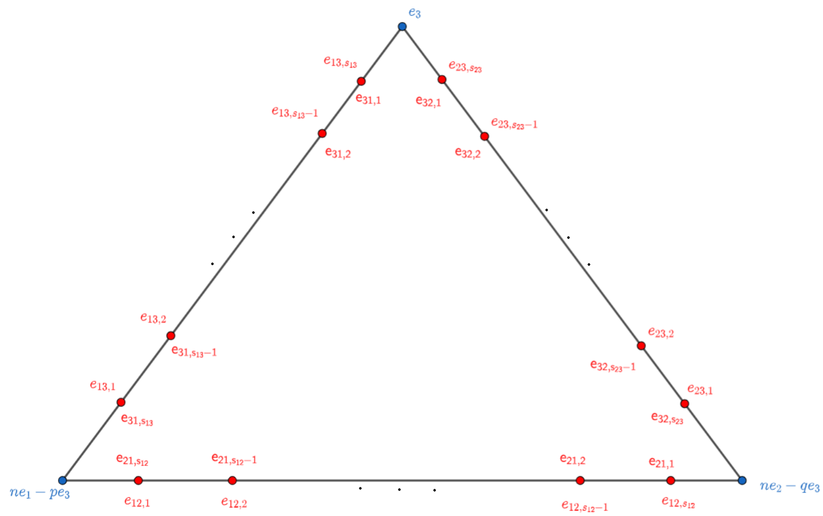


Figure 4.2: Lattice points that resolve each wall

appears not to have a regular pattern for any p, q.

In the following, we will see some examples of minimal resolutions. With the irregular forms that they take, in Section 4.2.2, we propose a cyclic resolution that imitates the resolution for the case $\{p+q\}_n = 1$. Then, in Section 4.3 we observe that this resolution has the asymptotic properties that we have been looking for.

4.2.1 Minimal resolutions

4.2.1.1 Case $(p, q) = (n-1, q)$

Following the notation of Section 2.5 for each τ_{jk} we have

$$\tau_{12} \text{ and } \tau_{23} \text{ of type } (n, q'),$$

$$\tau_{13} \text{ of type } (n, n-1),$$

thus τ_{12} and τ_{23} have the same Hirzebruch-Jung resolution. We construct a *minimal resolution* in the following steps.

Step 1 : Since τ_{12} and τ_{23} are of the same type, their resolutions have the same length, so let us denote $s = s_{12} = s_{23}$. In particular, we will denote

$$m_\alpha := m_{12,\alpha} = m_{23,s-1+\alpha}, \quad n_\alpha := n_{12,\alpha} = n_{23,s-1+\alpha}.$$

Then, we do toric blow-ups in both respective walls of σ in the following order

$$e_{12,1} \to e_{23,s} \to \dots \to e_{12,\alpha} \to e_{23,s-1+\alpha} \to \dots \to e_{12,s} \to e_{23,1}.$$

We get a fan σ^* as a refinement of σ. This fan can be illustrated in Figure 4.3.

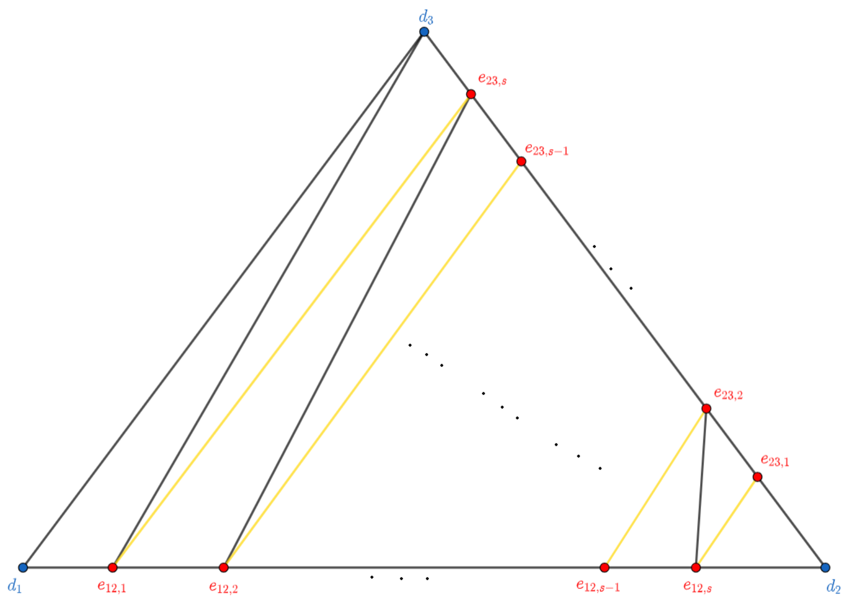


Figure 4.3: The fan σ^*. Black and yellow walls will be identified below.

Denote by $X^* \to Y_{p,q}$ the corresponding projective morphism induced by the refinement. Denote

$$\sigma_{12,\alpha} = C(e_{12,\alpha}, e_{12,\alpha+1}, e_{23,s+1-\alpha}), \quad 0 \le \alpha \le s,$$

$$\sigma_{23,\alpha} = C(e_{12,\alpha+1}, e_{23,s+1-\alpha}, e_{23,s+2-\alpha}), \quad 0 \le \alpha \le s,$$

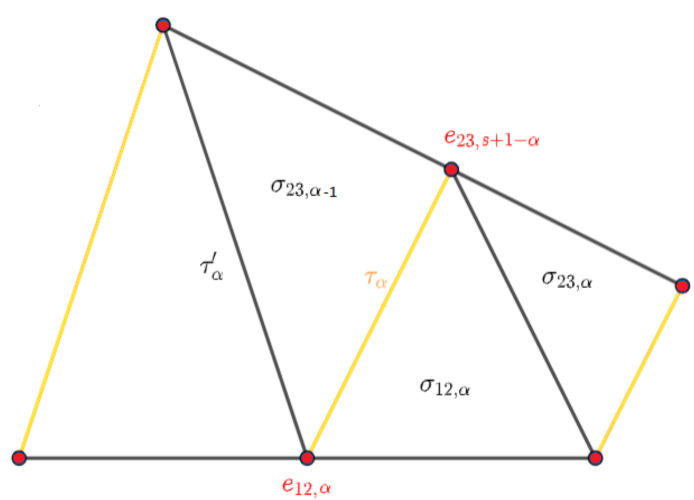
$$\tau_\alpha = C(e_{12,\alpha}, e_{23,s+1-\alpha}), \quad \tau'_\alpha = C(e_{12,\alpha}, e_{23,s+2-\alpha}), \quad 1 \le \alpha \le s.$$

Observe that τ_α are the yellow walls in Figure 4.3, while τ'_α are the black ones. We will denote by Σ_α the non-simplicial cone conformed by generator of $\sigma_{12,\alpha}$ and $\sigma_{23,\alpha}$.

Proposition 4.2.1. *For the refinement σ^* we have,*

1. *mult*$(\sigma_{12,0}) = n$.
2. *mult*$(\sigma_{12,\alpha}) =$ *mult*$(\sigma_{23,\alpha-1}) = m_\alpha, \quad 1 \leq \alpha \leq s$.
3. *mult*$(\tau_\alpha) = m_\alpha$, *and mult*$(\tau'_\alpha) = 1$.
4. *The canonical divisor of X^* has the explicit Cartier data given by:*
$$w_\alpha = \left(n_{\alpha+1} - n_\alpha, \frac{(m_\alpha - m_{\alpha+1}) + q(n_{\alpha+1} - n_\alpha)}{n}, n_{\alpha+1} - n_\alpha \right) \in M,$$
on each Σ_α. In particular K_{X^*} is Cartier.
5. *X^* has canonical singularities.*

Proof. (1) and (2) are direct calculations using determinants. (3) follows from the fact that n is prime and $(m_\alpha, n_\alpha) = 1$. (4) is a direct computation since $w_{\cdot,\alpha}$ must satisfy $\langle w_\alpha, \cdot \rangle = 1$ on $\sigma_{12,\alpha}$ and $\sigma_{23,\alpha}$ with the symmetry of the context. Since $m_\alpha \equiv qn_\alpha \mod n$, we have $w_\alpha \in M$, this implies K_{X^*} Cartier, and (5). □



Each primitive generator of σ^* defines a divisor $E_{jk,\alpha} = V(C(e_{jk,\alpha}))$ Section 2.4.1. Also, we have closed curves $C_\alpha = V(\tau_\alpha)$ and $C'_\alpha = V(\tau'_\alpha)$. In terms of intersection theory, we have

$$E_{12,\alpha}.E_{23,s+1-\alpha} = C_\alpha,$$
$$E_{12,\alpha}.E_{23,s+2-\alpha} = C'_\alpha.$$

Proposition 4.2.2. *We have,*

$$K_{X^*}\, C'_\alpha = 0$$
$$K_{X^*}\, C_\alpha = k_\alpha - 2 \geq 0.$$

Proof. We have the following relation between primitive generators

$$e_{12,\alpha} = e_{23,s+1-\alpha} + m_\alpha(1,0,-1).$$
$$e_{12,\alpha-1} + k_\alpha e_{12,\alpha} + e_{12,\alpha+1} = 0,$$
$$e_{23,s+2-\alpha} + k_\alpha e_{23,s+1-\alpha} + e_{23,s-\alpha} = 0.$$

By combining them we have

$$2e_{12,\alpha-1} + k_{12,\alpha}e_{12,\alpha} - 2e_{12,s+2-\alpha} - k_{23,\alpha}e_{23,\alpha} = 0,$$
$$e_{23,s+2-\alpha} + (k_{12,\alpha} - 1)e_{12,\alpha} + e_{23,s+1-\alpha} + e_{12,\alpha+1} = 0,$$

and the result follows using Corollary 2.4.12. □

Step 2 : Since $\mathrm{mult}(\tau_\alpha) = m_\alpha$, observe that

$$\frac{1}{m_\alpha}\left((m_\alpha - 1)e_{23,s+1-\alpha} + e_{12,\alpha}\right) = e_{23,s+1-\alpha} + (1,0,-1) \in N,$$

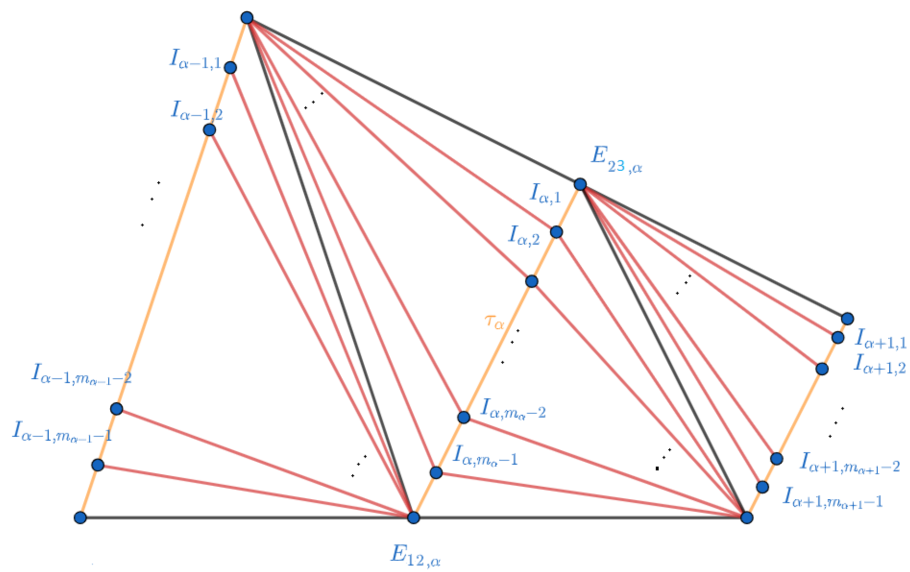
thus, this singularity is of type $(m_\alpha, m_\alpha - 1)$, i.e., a canonical singularity. Then τ_α can be resolved by the following ray generators $I_{\alpha,\ell}$ defined recursively by

$$I_{\alpha,\ell+1} = 2I_{\alpha,\ell} - I_{\alpha,\ell-1}, \quad I_{\alpha,1} = e_{23,s+1-\alpha} + (1,0,-1), I_{\alpha,0} = e_{23,s+1-\alpha},$$

or explicitly $I_{\alpha,\ell} = e_{12,s+1-\alpha} + \ell(1,0,-1)$. In the next we will use interchanged the same notations for divisors defined by this ray generators. So we define a refinement σ^t of of σ^* doing toric blow-ups through the generators $I_{\alpha,\ell}$ in the following order

$$I_{1,1} \to I_{1,2} \to \ldots \to I_{1,m_1-1} \to I_{2,1} \to \ldots \to I_{2,m_2-1} \to \ldots \to I_{s,m_s-1}.$$

Denote by X the toric variety defined by σ^t, and by $g\colon X \to X^* \to Y_{p,q}$ the corresponding projective morphism. In Section 4.2.1.1 is illustrated the refinement σ^t at the level of blocks Σ_α.



Proposition 4.2.3. *$h\colon X \to Y_{p,q}$ is a minimal resolution of singularities, i.e., K_X is nef.*

Proof. Since τ_α defines a canonical singularity, each one of $I_{\alpha,\cdot}$ are in the same plane. So by [CLS11, Prop 11.4.17] the toric blow-ups along each $I_{\alpha,\cdot}$ defines a projective crepant birational morphism, and the resulting toric variety will have at worst terminal singularities. The singularities of X are determined precisely by its cones of multiplicity strictly greater than 1. Now each one of these cones is of the form

$$\sigma_{23,\alpha,\ell} = C(E_{23,\alpha-1}, I_{\alpha,\ell}, I_{\alpha,\ell+1}) \text{ or } \sigma_{12,\alpha,\ell} = C(E_{12,\alpha+1}, I_{\alpha,\ell}, I_{\alpha,\ell+1}).$$

So the multiplicity is computed by

$$\begin{aligned}
\operatorname{mult}(\sigma_{23,\alpha,\ell}) &= |\det(E_{23,s+2-\alpha}, I_{\alpha,\ell}, I_{\alpha,\ell+1})| \\
&= |\det(E_{23,s+2-\alpha}, E_{23,s+1-\alpha} + \ell(1,0,-1), E_{23,s+1-\alpha} + (\ell+1)(1,0,-1))| \\
&= |\det(E_{23,s+2-\alpha}, E_{23,s+1-\alpha} + \ell(1,0,-1), (1,0,-1))| \\
&= |\det(E_{23,s+2-\alpha}, E_{23,s+1-\alpha}, (1,0,-1))| \\
&= \left| n_{\alpha-1}\left(\frac{m_\alpha - qn_\alpha}{n}\right) - n_\alpha\left(\frac{m_{\alpha-1} - qn_{\alpha-1}}{n}\right)\right| \\
&= \frac{1}{n}|n_{\alpha-1}m_\alpha - n_\alpha m_{\alpha-1}| = 1,
\end{aligned}$$

by Lemma 2.5.2. And the result is analogous for $\mathrm{mult}(\sigma_{12,\alpha,\ell}) = 1$. Moreover the Cartier data of K_X is the same on the blocks Σ_α in σ^t as before, this is due to the collection $\{I_{\alpha,\ell}\}_\ell$ lies on the same plane of the block Σ_α. So each curve given by a face $C(E_{23,s+2-\alpha}, I_{\alpha,\ell})$ or $C(E_{12,s+1-\alpha}, I_{\alpha,\ell})$ is K-trivial, therefore K_X is a nef divisor. □

So if we denote by $h\colon X \to Y_{p,q} \to \mathbb{A}^3$ the composition of $g\colon X \to Y_{p,q}$ given by the refinement, and $f\colon Y_{p,q} \to \mathbb{A}^3$. Let us denote by $\tilde{D}_j$ the strict transform of the coordinate divisor H_j under h. Abusing notation, we also denote $I_{\alpha,\ell}$ to the divisor defined by its correspondent ray generator. Using Proposition 2.4.8 we get explicitly

$$h^*H_1 = n\tilde{D}_1 + \sum_{\alpha=1}^{s}\left(m_\alpha E_{12,\alpha} + \sum_{\ell=1}^{m_\alpha-1} \ell I_{\alpha,\ell}\right) + \sum_{\beta=1}^{n-1} \beta E_{13\beta},$$

$$h^*H_2 = n\tilde{D}_2 + \sum_{\alpha=1}^{s} n_\alpha(E_{12,\alpha} + I_{\alpha,1} + \dots + I_{\alpha,m_\alpha-1} + E_{23,\alpha}),$$

$$h^*H_3 = n\tilde{D}_3 + \sum_{\alpha=1}^{s}\left(n_\alpha E_{23,\alpha} + \sum_{\ell=1}^{m_\alpha-1} (n_\alpha-\ell) I_{\alpha,\ell}\right) + \sum_{\beta=1}^{n-1} (n-\beta) E_{13,\beta}.$$

Moreover, since the resolution $X \to X^*$ is crepant we have

$$K_X = h^*K_{Y_{p,q}} + \sum_{\alpha=1}^{s}\left(\frac{m_\alpha + n_\alpha}{n} - 1\right) E_\alpha,$$

where

$$E_\alpha = E_{12,\alpha} + I_{\alpha,1} + \dots + I_{\alpha,m_{\alpha-1}} + E_{23,s+1-\alpha}.$$

Since the Cartier data of K_X is unchanged, we have

$$K_X\, E_{23,s+1-\alpha}\, I_{\alpha,1} = K_X\, I_{\alpha,1}\, I_{\alpha,2} = \dots = K_X\, I_{\alpha,m_\alpha-1}\, E_{12,\alpha} = k_\alpha - 2,$$

$$K_X\, E_{23,s+2-\alpha}\, I_{\alpha,\ell} = K_X\, E_{12,\alpha+1}\, I_{\alpha,\ell} = 0.$$

4.2.1.2 Case $p + q = n + 1$

Consider for $0 < p, q < n$ the toric variety $Y_{p,q}$ isomorphic to the normalization of $t^n = x^{n-p} y^{n-q} z$ with $(n-p) + (n-q) + 1 = n$, i.e., $p + q = n + 1$. To construct the *minimal smooth resolution* of σ we refine the fan associated in two steps:

Step 1: Since $p+q=n+1$, it is not difficult to check that $v=e_1+e_2-e_3\in\sigma\cap N$ always maximizes the distance from the lattice points of $\mathrm{Conv}(\sigma(1)\cup\{0\})$ to the plane generated by $\sigma(1)$. So as first step, we do a toric blow up through the ray generated by v, obtaining a refinement σ^* of σ.

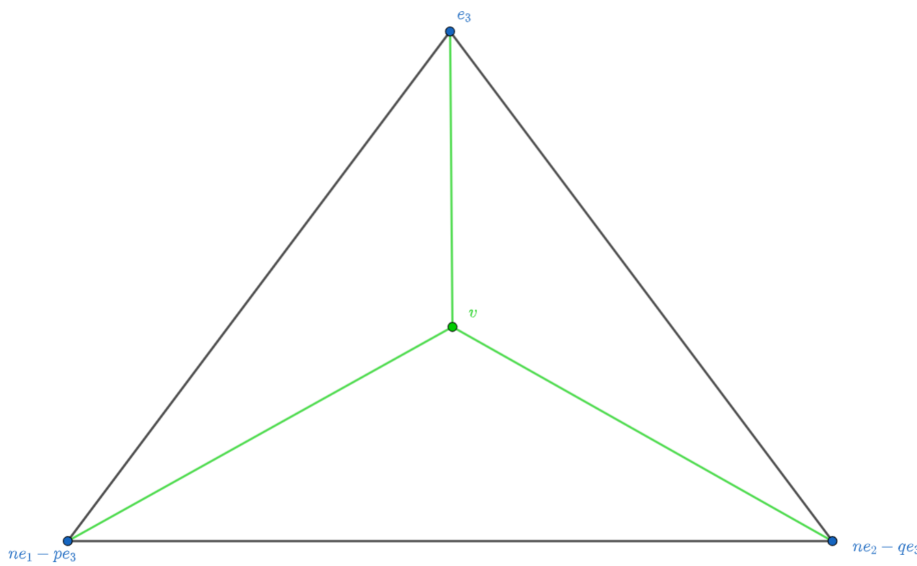


Figure 4.4: The fan σ^*

Step 2: Now do toric blow-ups through each wall following the non singular resolution associated in directions d_j to d_k where $j<k$. So we obtain a refinement σ^t, and denote X the toric variety associated to this fan. This refinement give us a birational projective morphism $g_{loc}: X\to Y_{p,q}$ which is the *minimal resolution* of $Y_{p,q}$. In the following, we prove that X is a quasi-projective minimal smooth 3-fold over Y_{loc}. Denote each cone of $(\sigma^t)^3$ by

$$\sigma_{jk,\alpha}=C(v,e_{jk,\alpha},e_{jk,\alpha+1}),\quad 0\le\alpha\le s_{jk}.$$

We will use the same notation to speak about the pair $(m_{jk,\alpha},n_{jk,\alpha})$ of a Hirzebruch-Jung resolution for a 2-face of σ. We also denote each inner wall by $\tau_{jk,\alpha}=C(v,e_{jk,\alpha})$.

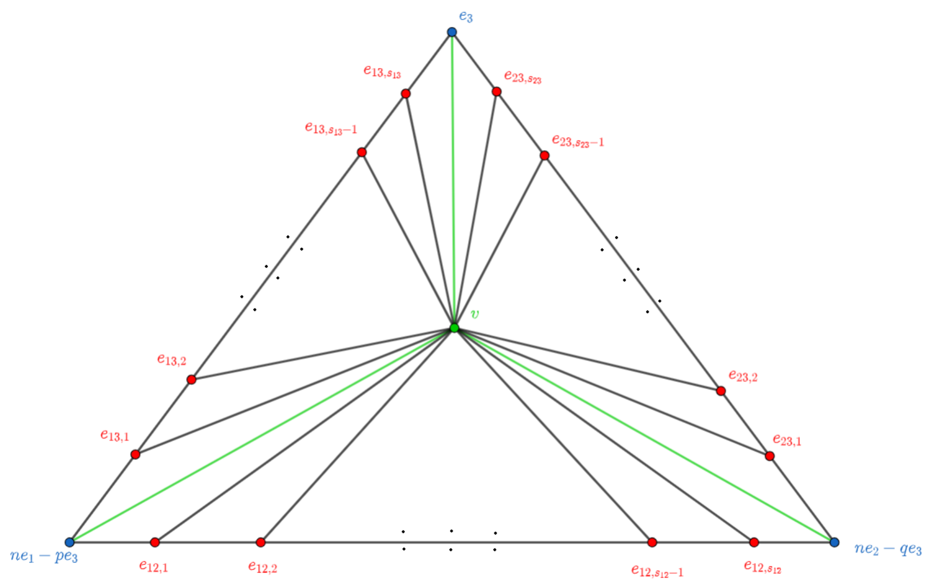


Figure 4.5: The fan σ^t

Lemma 4.2.4. *We have mult*$(\sigma_{jk,\alpha}) = 1$*, and mult*$(\tau_{jk,\alpha}) = 1$.

Proof. The first assertion is a direct compute using determinants and properties of 2.5. □

Each ray generated by some $e_{jk,\alpha}$ defines a toric divisor on X given by $E_{jk,\alpha} = V(C(e_{jk,\alpha}))$, at the say time we fix notation for $j < k$ by $E_{kj,\alpha} := E_{jk,s+1-\alpha}$. The inner ray generated by v also defines a toric divisor which we denote by F. Each inner wall $\tau_{jk,\alpha}$ defines a closed curve on X given by $C_{jk,\alpha} = V(C(\tau_{jk,\alpha}))$. Since the refinement σ^t is simplicial with cones of multiplicity one, X is smooth, and the canonical divisor K_X is a Cartier divisor. Then we have defined an intersection theory on X.

Lemma 4.2.5. *We have*

$$E_{jk,\alpha}E_{jk,\alpha\pm 1}F = 1, \quad E^2_{jk,\alpha}F = -k_{jk,\alpha}, \quad E_{jk,\alpha}F^2 = 0,$$

$$\tilde{D}_jF^2 = \tilde{D}_j^{\,2}F = -1, \quad F^3 = n, \quad K_XF^2 = -(n-3),$$

$$K_X\,C_{jk,\alpha} = \begin{cases} k_{jk,\alpha} - 2 & 1 \leq \alpha \leq s_{jk} \\ 0 & \alpha = 0, s_{jk+1} \end{cases}$$

where $k_{jk,\alpha} \geq 2$ are the coefficients of the resolution $(m_{jk,\alpha}, n_{jk,\alpha})$ defined in 2.5. Then for the inner divisor F we have

$$K_X^2\, F = -\sum K_X\, C_{jk,\alpha} = -\sum (k_{jk,\alpha} - 2),$$

$$K_X\, F^2 = 3 - n.$$

Proof. Using 2.4.11, and noting that each $C_{jk,\alpha}$ is defined by a wall between two cones in $(\sigma^t)^3$. Since we are over a point, then $h^* D_j F^2 = 0$ by projection formula. So we have

$$n\tilde{D}_j F^2 + \sum_k \sum_{\alpha=1}^{s_{jk}} m_{1k,\alpha} E_{jk,\alpha} F^2 + F^3 = -n + F^3 = 0.$$

□

Therefore X is a smooth quasi-projective variety with nef canonical divisor, i.e., X is minimal. Denote by $h\colon X \to Y_{p,q} \to \mathbb{A}^3$ the composition morphism $g \circ f$, from 2.4.8 we can compute

$$h^* H_j = n\tilde{D}_j + \sum_k \sum_{\alpha=1}^{s_{jk}} m_{jk,\alpha} E_{jk,\alpha} + F,$$

$$K_X = g^* K_{U_\sigma} + \sum_{j<k} \sum_{\alpha=1}^{s_{jk}} N_{jk,\alpha} E_{jk,\alpha} - \frac{n-3}{n} F,$$

where

$$N_{jk,\alpha} := \frac{m_{jk,\alpha} + n_{jk,\alpha}}{n} - 1.$$

4.2.1.3 Case $p + q = n$

Since $p + q \equiv_n 0$, we have τ_{12} of type $\frac{n}{1}$, i.e., that wall can be resolved with just one blow-up. We construct the *minimal resolution* in the following steps.

Step 1: We blow-up the wall τ_{12} at the ray generator

$$e_{12,1} = \frac{1}{n}(d_1 + d_2) = \frac{1}{n}(n, n, -(p+q)) = (1, 1, -1).$$

Indeed this generator is that one minimizes the distance of lattice cone points with the origin. Thus we do the corresponding toric blow-up at that generator obtaining a fan σ^* illustrated Section 4.2.1.3.

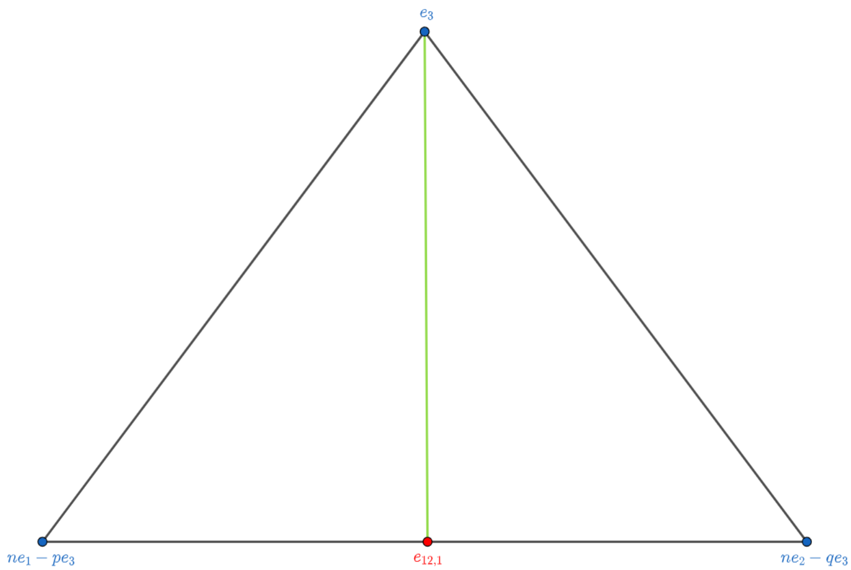


Figure 4.6: The fan σ^*

The fan σ^* has two 3-cones of multiplicity n, and since the remainder walls are of multiplicity n, these 3-cones do have not lattice points inside their parallelepiped associated.

Step 2: Now we blow-up completely the other walls following the Hirzebruch-Jung process. We get a fan σ^t illustrated in Section 4.2.1.3.

Denote by X the toric variety defined by σ^t, and $h : X \to Y_{p,q}$ the projective birational morphism induce by this refinement of σ.

Proposition 4.2.6. *$h : X \to Y_{p,q}$ is a resolution of singularities.*

Proof. For a pair $j, k \neq 1, 2$ a cone $\sigma_{jk,\alpha} = C(e_{jk,\alpha}, jk, \alpha + 1, e_{12,1})$ have

$$\begin{aligned}\operatorname{mult}(\sigma_{jk,\alpha}) &= \frac{1}{n^2}|\det(m_{jk,\alpha}d_j + n_{jk,\alpha}d_k, m_{jk,\alpha+1}d_j + n_{jk,\alpha+1}d_k, e_{12,1})| \\ &= \frac{|m_{jk,\alpha}n_{jk,\alpha+1} - m_{jk,\alpha+1}n_{jk,\alpha}|}{n^2}|\det(d_j, d_k, e_{12,1})| \\ &= \frac{|\det(d_j, d_k, e_{12,1})|}{n} = 1,\end{aligned}$$

by Lemma 2.5.2. □

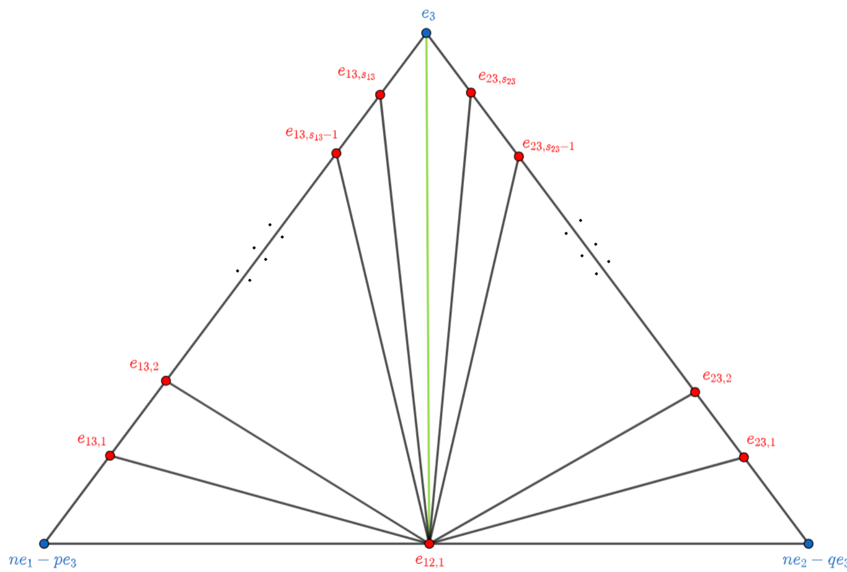


Figure 4.7: The fan σ^t

We denote the inner curves on X by

$$C = V(C(e_3, e_{12,1})), \quad C_{jk,\alpha} = V(C(e_{jk,\alpha}), e_{12,1}).$$

Proposition 4.2.7. *We have*

$$K_X\, C = 0,$$

$$K_X\, C_{jk,\alpha} = k_{jk,\alpha} - 2,$$

where $k_{jk,\alpha} \geq 2$ are the coefficients of the Hirzebruch-Jung resolution of the wall τ_{jk}.

Proof. The proof is direct using Corollary 2.4.12, and the relations between ray generators

$$e_1 + e_2 - e_3 - e_{12,1} = 0,$$

$$e_{jk,\alpha} - k_{jk,\alpha} e_{jk,\alpha} + e_{jk,\alpha} = 0.$$

□

Thus as before, for the composed resolution $h\colon X \to Y_{p,q} \to \mathbb{A}^3$ we have the following pull-backs

$$h^* H_j = n\tilde{D}_j + \sum_k \sum_{\alpha=1}^{s_{jk}} m_{jk,\alpha} E_{jk,\alpha} + F,$$

$$K_X = g^* K_{U_\sigma} + \sum_{j<k} \sum_{\alpha=1}^{s_{jk}} N_{jk,\alpha} E_{jk,\alpha} - \frac{n-3}{n} F,$$

where

$$N_{jk,\alpha} := \frac{m_{jk,\alpha} + n_{jk,\alpha}}{n} - 1.$$

4.2.2 Cyclic resolution

From Proposition 2.4.1 we know that every $v \in P_\sigma \cap N$ can be written as

$$v = \frac{v_1 d_1 + v_2 d_2 + v_3 d_3}{n}, \quad v_3 = \{pv_1 + qv_2\}_n \quad 0 \le v_i < n.$$

Fix a v, with $v_1 + v_2 + v_3 \le n$ and $v_i > 0$.

Step 1: We refine by a star subdivision along v obtaining a fan as illustrate Section 4.2.2.

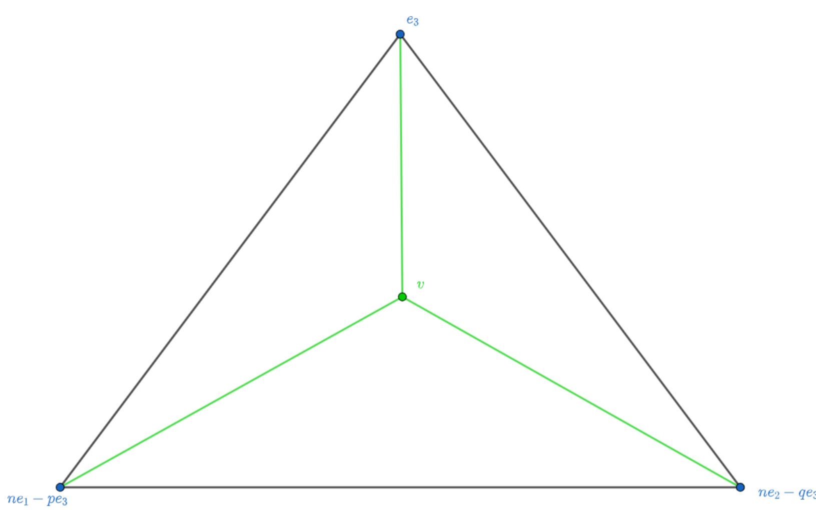


Figure 4.8: Star subdivision along minimizer v

Step 2: Now we refine each wall by doing toric blow-ups following the Hirzebruch-Jung algorithm. So we obtain a refinement σ^* and denote by X

the toric variety associated. This refinement gives us a birational projective morphism $g\colon X \to Y_{p,q}$ [CLS11, 11.1.6]. This refinement is sketched in Figure 4.9. Denote each 3-cone of σ^* by

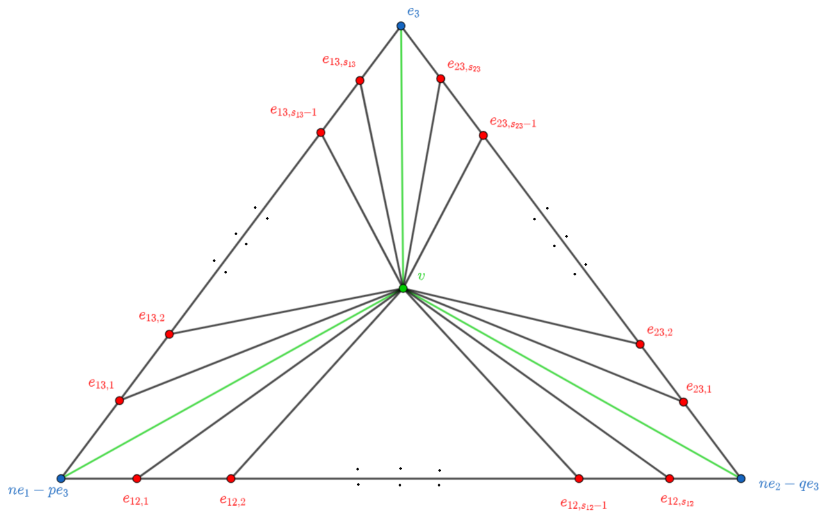


Figure 4.9: Cyclic local resolution

$$\sigma_{jk,\alpha} = C(v, e_{jk,\alpha}, e_{jk,\alpha+1}), \quad 0 \leq \alpha \leq s_{jk}.$$

The 2-cones of σ^* are given in two types. The *exterior walls* $\tau_{jk,\alpha} = C(e_{jk,\alpha}, e_{jk,\alpha+1})$, and the *inner walls* $\rho_{jk,\alpha} = C(v, e_{jk,\alpha})$. For any permutation (v_j, v_k, v_l) with $j < k$, using determinants and properties of Section 2.5, we have

$$\operatorname{mult}(\sigma_{jk,\alpha}) = v_l, \quad \operatorname{mult}(\rho_{jk,\alpha}) = \gcd(v_j n_{jk,\alpha} - v_k m_{jk,\alpha}, v_l), \quad \operatorname{mult}(\tau_{jk,\alpha}) = 1.$$

Lemma 4.2.8. *Each cone $\sigma_{jk,\alpha}$ is a cyclic singularity of type*

$$\frac{(a_{jk,\alpha}, b_{jk,\alpha}, 1)}{v_l}, \quad a_{jk,\alpha} = \{m_{jk,\alpha+1}v_k - n_{jk,\alpha+1}v_j\}_{v_l}, \quad b_{jk,\alpha} = \{m_{jk,\alpha}v_k - n_{jk,\alpha}v_j\}_{v_l},$$

where $\{\cdot\}_{v_l}$ is the residue modulo v_l.

Proof. Since $\tau_{jk,\alpha}$ is non-singular, then $\sigma_{jk,\alpha}$ is semi-unimodular respect to v. By [Ash15, Prop. 1.2.3] if there is positive integer x, y such that

$$\frac{xe_{jk,\alpha} + ye_{jk,\alpha+1} + v}{v_l} \in \mathbb{Z}^3,$$

then x, y define the type of the cyclic singularity. Since $\gcd(n, v_l) = 1$, we can solve the equations modulo v_l and the result follows.

□

Denote by $h\colon X \to Y_{p,q} \to \mathbb{A}^3$ the composition with the natural projection to $\mathbb{A}^3$. Denote by D_j the divisor in $\mathbb{A}^3$ defined by the coordinate x_j. Each ray on σ^* generated by $d_j, e_{jk,\alpha}$, or v defines a toric divisor on X given by

$$\tilde{D}_j = V(C(d_j)), \quad E_{jk,\alpha} = V(C(e_{jk,\alpha})), \quad F = V(C(v)),$$

where $V(\cdot)$ denotes the orbit closure of a cone [CLS11, p. 121]. At the same time we fix notation for $j < k$ by $E_{kj,\alpha} := E_{jk,s+1-\alpha}$. Using Proposition 2.4.8 we can compute

$$h^* D_j = n\tilde{D}_j + \sum_k \sum_{\alpha=1}^{s_{jk}} m_{jk,\alpha} E_{jk,\alpha} + v_j F,$$

$$K_X = g^* K_{Y_{p,q}} + \sum_{j<k} \sum_{\alpha=1}^{s_{jk}} N_{jk,\alpha} E_{jk,\alpha} + \frac{v_1 + v_2 + v_3 - n}{n} F, \tag{4.1}$$

where

$$N_{jk,\alpha} := \frac{m_{jk,\alpha} + n_{jk,\alpha}}{n} - 1.$$

It is satisfied the relation $k_{jk,\alpha} N_{jk,\alpha} - N_{jk,\alpha+1} = N_{jk,\alpha-1} - (k_{jk,\alpha} - 2)$, which gives

$$\sum_{\alpha=1}^{s_{jk}} N_{jk,\alpha}(k_{jk,\alpha} - 2) = -(N_{jk,1} + N_{kj,1}) - \sum_{\alpha=1}^{s_{jk}} (k_{jk,\alpha} - 2).$$

Proposition 4.2.9. *The $\mathbb{Q}$-divisor*

$$h^* \left(\frac{n-1}{n} (D_1 + D_2 + D_3) \right) + \sum_{j<k} \sum_{\alpha} N_{jk,\alpha} E_{jk,\alpha} + \frac{v_1 + v_2 + v_3 - n}{n} F,$$

is an effective $\mathbb{Z}$-divisor.

Proof. The local pullback $h^*(D_1 + D_2 + D_3)$ equals

$$n(\tilde{D}_1 + \tilde{D}_2 + \tilde{D}_3) + \sum_{j<k,\alpha} (m_{jk,\alpha} + n_{jk,\alpha}) E_{jk,\alpha} + (v_1 + v_2 + v_3) F_{jkl,p}.$$

Thus, $h^*\left(\frac{n-1}{n}(D_j+D_k+D_l)\right)+\Delta$ equals to

$$(n-1)(\tilde{D}_1+\tilde{D}_2+\tilde{D}_3)+\sum_{j<k,\alpha}(m_{jk,\alpha}+n_{jk,\alpha}-1)\,E_{jk,\alpha}+(v_1+v_2+v_3-1)F,$$

i.e., an effective $\mathbb{Z}$-divisor. $\square$

Each inner wall defines a closed curve on X given by

$$C_j=V(C(d_j,v)),\quad C_{jk,\alpha}=V(C(\rho_{jk,\alpha})).$$

The refinement σ^* is simplicial with cones of multiplicity one, and the canonical divisor K_X is a $\mathbb{Q}$-Cartier divisor. For any pair v_j,v_k, $j<k$, let v_l be the another coordinate. The following relation among lattices generators,

$$e_{jk,\alpha-1}+(-k_{jk,\alpha})e_{jk,\alpha}+0\cdot v+e_{jk,\alpha+1}=0,$$

$$v_je_{lj,1}+\left(\frac{v_l-m_{lj,1}v_j-m_{lk,1}v_k}{n}\right)d_l+(-1)v+v_ke_{lk,1}=0,$$

describe the intersection theory on X. Using Theorem 2.4.11 we have,

$$E_{jk,\alpha}C_{jk,\alpha\pm1}=\frac{\mathrm{mult}(\rho_{jk,\alpha})}{v_l},\quad E_{jk,\alpha}C_{jk,\alpha}=-\frac{k_{jk,\alpha}\mathrm{mult}(\rho_{jk,\alpha})}{v_l},\quad FC_{jk,\alpha}=0,$$

$$\tilde{D}_l\,C_l=\frac{\gcd(v_j,v_k)(v_l-m_{lj,1}v_j-m_{lk,1}v_k))}{nv_jv_k},\quad F\,C_l=-\frac{\gcd(v_j,v_k)}{v_jv_k},$$

$$K_X\,C_l=-\frac{\gcd(v_j,v_k)}{v_jv_k}\left(v_j+v_k-1+\frac{v_l-m_{lj,1}v_j-m_{lk,1}v_k}{n}\right),$$

$$K_X\,C_{jk,\alpha}=\frac{\mathrm{mult}(\rho_{jk,\alpha})}{v_l}(k_{jk,\alpha}-2),\quad 1\le\alpha\le s_{jk}. \tag{4.2}$$

Lemma 4.2.10. *We have*

$$F^3=\frac{n}{v_1v_2v_3}.$$

Proof. The divisor v_jF is Cartier for any j, then from the pullback identities above we have

$$v_1v_2v_3F^3=\prod_{j=1}^{3}\left(h^*D_j-nD_j-\sum_k\sum_\alpha m_{jk,\alpha}E_{jk,\alpha}\right)=h^*D_1\,h^*D_2\,h^*D_3=n,$$

where the last is by projection formula. $\square$

As a consequence, using Corollary 2.4.12 we can compute,

$$\begin{aligned} K_X F^2 &= -F^2 \left(D_1 + D_2 + D_3 + F\right) \\ &= \frac{v_1 + v_2 + v_3 - n}{v_1 v_2 v_3}. \end{aligned}$$

From the last one, we get

$$K_X^2\, F = -\sum_l \frac{K_X\, C_l}{\gcd(v_j, v_k)} - \sum_{j<k,\alpha} \frac{K_X\, C_{jk,\alpha}}{\operatorname{mult}(\rho_{jk,\alpha})} - K_X F^2.$$

The following arithmetic lemma will be useful in Section 4.3.

Lemma 4.2.11. *There exists $v \in P_\sigma \cap N$ such that $v_1 + v_2 + v_3 = n$, so K_X has multiplicity zero at F. Moreover, for $n \gg 0$ we can choose v such that the slopes $v_j/v_k \leq 3$.*

Proof. By Proposition 2.4.1 we have $v_3 = \{v_1 p + v_2 q\}_n$. So $v_1 + v_2 + v_3 = n$, implies that (v_1, v_2) are solutions (x, y) of the Diophantine equation

$$y \equiv cx \mod n, \quad c = \{-(p+1)(q+1)'\}_n.$$

Moreover, for any of those solutions with $x + y < n$, we have $v_3 = n - x - y$. A degenerate case is $p + q = n - 2$, equivalently $c = 1$, thus the solution to the equation is the diagonal. Thus, we can choose $x = y = \lfloor \frac{n}{3} \rfloor$, and the result follows for this case. Let us assume that $c < n/2$, otherwise we do $(x, y) \mapsto (-x, y)$. The integer points in the square $[1, n-1]^2$ solving the equation distribute in $\mathbb{R}^2$ along the lines $L_\beta : y = cx - \beta n$ for $0 \leq \beta \leq c - 1$. Thus, over each L_β the integer solutions over the line are defined by those integers in the interval

$$I_\beta = \left(\left\lfloor \frac{\beta n}{c} \right\rfloor, \left\lfloor \frac{(\beta+1)n}{c} \right\rfloor \right].$$

For each $1 \leq k \leq \lfloor \frac{n}{c} \rfloor$, we have

$$y_k = y\left(\left\lfloor \frac{\beta n}{c} \right\rfloor + k \right) = ck - r,$$

where $0 \leq r < c$ is the residue of βn modulo c. So, $c(k-1) \leq y_k \leq ck$. Let us choose $\beta = \lfloor \frac{c-1}{3} \rfloor$, and $k = \lfloor \frac{n}{3c} \rfloor$. In particular $\lfloor \frac{n}{3} \rfloor \in I_\beta$. So, as $n \gg 0$ we have $y_k \approx \frac{n}{3}$. By construction, we have $x_k = \left\lfloor \frac{\beta n}{c} \right\rfloor + k$. Let $0 \leq r^* \leq 2$ the residue of $c-1$ modulo 3, then $x_k \approx \frac{n(c-r^*)}{3c}$. Since $\frac{1}{2} \leq \frac{c-r^*}{c} \leq 1$, the result follows chosing $v_1 = x_k$ and $v_2 = y_k$ as $n \gg 0$. $\square$

Example 4.2.12. Case $\{p+q\}_n = 2$: *In this case, again $v_1 = v_2 = 1$ defines a interior lattice point v minimizing $v_1 + v_2 + v_3$. In this case, we have $\sigma_{13,\alpha}$ and $\sigma_{23,\alpha}$ as non-singular cones. On the other hand, each $\sigma_{12,\alpha}$ is cyclic singularity of order 2. Thus, they define canonical and terminal singularities. The first ones achieve a terminal resolution by one blow-up. Moreover,*

$$K_X C_3 = 0, \quad K_X C_j \in \left\{0, -\frac{1}{2}\right\}, \quad j = 1, 2$$

so there are p, q with canonical divisor K_X nef. In the worst case, i.e., $K_X Cj < 0$ for $j = 1, 2$, we can do toric flips a to get a nef toric variety given whose fan is sketched in Figure 4.10.

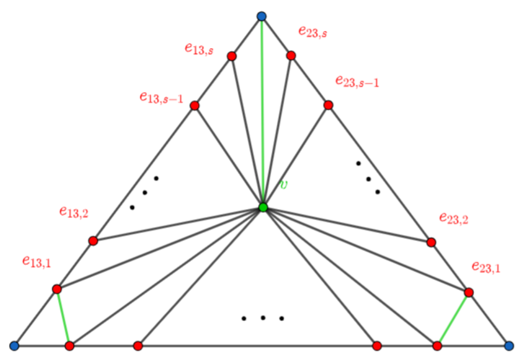


Figure 4.10: Flipped fan for $\{p+q\}_n = 2$.

Example 4.2.13. *If we drag the lattice point v to one of the generators of the cone σ, we get a degenerated fan as in Figure 4.11. In this case, the singularities are of order n, and as an advantage, we do not have a divisor F.*

As we see, having $v_j \leq 2$ gives us good singularities to work. Indeed, if the $v_j's$ are small enough, the singularities are also good in asymptotic terms.

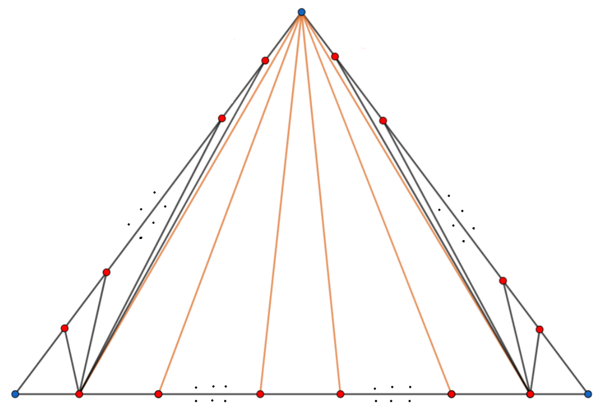


Figure 4.11: Degenerated cyclic resolution

Step 3 (Optional): Now we can desingularize each non-terminal cyclic singularity using the Fujiki-Oka algorithm. See [Ash19] for a modern treatment. Denote the complete resolution as $\tilde{X} \to X \to Y_{p,q}$. Since the cone of toric cyclic singularities of any type $\frac{a,b}{c}$ has multiplicity c, then there is a resolution of singularities with length at most c. In this case, if we assume v_j bounded by $n^{1/C}$ with $C \geq 1$, then the lengths on σ^* can be bounded by

$$\sum_{j<k} s_{jk} n^{1/C} \leq 3C' n^{2/C},$$

for another constant $C' > 0$ using the result of Girstmair (Theorem 2.7.1). For $C > 2$, it is guaranteed that $\tilde{X}$ is a resolution of singularities with good asymptotical behavior.

4.3 Global resolution

Let $\{D_1, ..., D_r\}$ be an asymptotic arrangement on Z. Thus, for prime numbers $n \gg 0$ we have multiplicities $0 < \nu_j < n$ depending on n, with its respective $q_{jk} \in O_n$. We have n-th root covers $f_n \colon Y_n \to Z$ branched at each $D = \sum_j \nu_j D_j$. Let us fix a $n \gg 0$, so we drop the subscript n of the morphisms, i.e., we are fixing a $f \colon Y_n \to Z$.

For $j < k$, the singularities of Y_n over curves in $D_{jk} := D_j \cap D_k$ are locally analytically isomorphic to $C_{q_{jk},n} \times \mathbb{C}$ where $C_{q_{jk},n}$ is the surface cyclic quotient singularity of type $\frac{1}{n}(q_{jk}, 1)$. For a triple $j < k < l$, the singularity of Y_n over a point in $D_{jkl} := D_j \cap D_k \cap D_l$ is locally analytically isomorphic to the normalization of $\mathrm{Spec}(\sigma_{jkl}^\vee \cap M)$ where σ_{jkl} is a cone with walls of types $\frac{1}{n}(q_{jk}, 1), \frac{1}{n}(q_{jl}, 1), \frac{1}{n}(q_{kl}, 1)$ as we see in Section 4.2. We will get the *cyclic resolution* $X_n \to Y_n$ via weighted blow-ups in two steps.

Step 1: Since singularities over D_{jkl} are isolated, for each σ_{jkl}, we do a weighted blow-up at a convenient interior lattice point v^{jkl}. So, this refinement locally gives a projective morphism which is a blow-up over an isolated point [CLS11, 11.1.6]. In this case, we get a projective birational morphism $h' : X'_n \to Y_n$. We have exceptional divisors F_{jkl} whose components are over the points of D_{jkl} and they are isomorphic to weighted fake projective planes [Buc08]. For future computations, we fix the notation of v^{jkl} and F_{jkl} independent of the order of the triple j, k, l. For example, $v^{jkl} = v^{kjl}$.

Step 2: Since the centers of the above blow-ups are points, the singularity type over intersections D_{jk} was not affected. For curves in D_{jk}, locally by SNC property, we can assume that they are supported on a local equation $xy = 0$ for local coordinates x, y. Then over such curves, the singularities on X'_n are locally analytically isomorphic to $C_{q_{jk},n} \times \mathbb{C}$, thus we use the Hirzebruch-Jung algorithm which is a weighted blow-up to resolve $C_{q_{jk},n}$. The Hirzebruch-Jung resolution can be realized by a single blow-up $Bl_{\mathfrak{m}}(C_{q_{jk,n}}) \to C_{q_{jk,}}$ where $\mathfrak{m}$ is a maximal ideal determined explicitly in coordinates x, y as we see at the end of Section 2.5 (also see [KM92, 10.5]). In terms of local resolutions, we need to follow an order compatible with the resolution, i.e., if we locally blow-up a curve in D_{jk} then this operation must be reflected on the other local toric pictures following the centers to blowing-up. See Figure 4.12. Thus, this construction extends and we have resolved the curves D_{jk}. Consequently, we get a projective morphism denoted by $g\colon X_n \to X'_n \to Y_n$, and denote by $h\colon X_n \to Z$ the composition. Since X_n was constructed by a sequence of weighted blow-ups with cyclic singularities, then, X_n is an embedded $\mathbb{Q}$-resolution of Y_n [ABMMOG12, 2.1]. As we see in Proposition 2.3.9, the varieties X_n are irreducible. We summarize this in the following.

Proposition 4.3.1 (and Definition). *There exists a* cyclic partial resolution $g : X_n \to Y_n$, *i.e. a projective, surjective, birational morphism such that X_n is irreducible and, it has at most isolated cyclic quotient singularities of order* *lower than n.*

□

Over each $D_{jk} = D_j D_k$, we get exceptional divisors $E_{jk,\alpha}$, $0 \leq \alpha \leq s_{jk}$, where $s_{jk} = \ell(n, q_{jk})$ and whose components are over those of D_{jk}. From the local computations of the section above, we have

$$h^* D_j = n\tilde{D}_j + \sum_{D_k D_j \neq \emptyset} \sum_{\alpha=1}^{s_{jk}} m_{jk,\alpha} E_{jk,\alpha} + \sum_{D_{kl} D_j \neq \emptyset} F_{kl,j}, \tag{4.3}$$

where $F_{kl,j}$ is a divisor whose components are the exceptional divisors over points in D_{jkl}.

Explicitly, for any triple of positive integers numbers j, k, l, let $\rho_{kl}(j) \in \{1, 2, 3\}$ the position of j if we order the triple. For example $\rho_{23}(1) = 1$, $\rho_{57}(6) = 2$ or $\rho_{54}(8) = 3$. Thus, if $F_{jkl,p}$ is the exceptional divisor over a point $p \in D_j D_k D_l$, then

$$F_{jkl} = \sum_{p \in D_{jkl}} F_{jkl,p}$$

$$F_{kl,j} = \sum_{p \in D_{jkl}} \rho_{kl}(j) F_{jkl,p}. \tag{4.4}$$

In terms of intersection theory, we have

$$F_{jkl}^3 = \frac{n}{v_1^{jkl} v_2^{jkl} v_3^{jkl}} D_{jkl}, \quad \tilde{D}_j \tilde{D}_k = 0.$$

From Theorem 2.3.6 we have $K_{X_n} \sim_{\mathbb{Q}} h^*(K_Z + \frac{n-1}{n} \sum_j D_j) + \Delta$ where

$$\Delta = \sum_{j<k} E_{jk} + \sum_{j<k<l} V_{jkl} F_{jkl},$$

$$V_{jkl} = \frac{v_1^{jkl} + v_2^{jkl} + v_3^{jkl} - n}{n},$$

$$E_{jk} = \sum_{\alpha=1}^{s_{jk}} N_{jk,\alpha} E_{jk,\alpha}.$$

The following proposition will be useful in Chapter 5. From Remark 2.1.1, recall that on a normal variety X, a curve C is a m-curve if $K_X C = m$. A m-curve C is *K-negative, K-positive or K-trivial* if $m < 0$, $m > 0$ or $m = 0$ respectively.

Proposition 4.3.2. *The $\mathbb{Q}$-divisor $h^*\left(\frac{n-1}{n}\sum_{i=1}^r D_i\right)+\Delta$ is an effective $\mathbb{Z}$-divisor. Thus, if K_Z is nef, then the K-negative curves of X_n are contained in the support of h^*D.*

Proof. The first assertion is a direct consequence of Proposition 4.2.9. Now assume that K_Z is nef, and let C be a curve in X. If C is not contained in the support of h^*D, then

$$K_{X_n}C = \left(h^*\left(K_Z + \frac{n-1}{n}\sum_{i=1}^r D_i\right) + \Delta\right).C > 0,$$

since h^*K_Z is nef (projection formula), and effectiveness of $h^*\left(\frac{n-1}{n}\sum_{i=1}^r D_i\right)+\Delta$. Thus, if C is negative must lie in the support of h^*D.

□

Now we describe how the intersection theory on X_n behaves under pull-backs of divisors of Z. In what follows, we set $E_{jk,0} = \tilde{D}_j$ and $E_{jk,s_{jk}+1} = \tilde{D}_k$.

Proposition 4.3.3. *Let G, G' any divisors on Z, then $h^*GF_{kl,j} = 0$ as 2-cycle, for any $1 \le \alpha \le s_{jk}$,*

$$h^*Gh^*G'E_{jk,\alpha} = 0,$$

$$h^*GE_{jk,\alpha}^2 = -k_{jk,\alpha}GD_{jk}, \quad h^*GE_{jk,\alpha}E_{jk,\alpha\pm1} = GD_{jk}.$$

Proof. We use the projection formula repeatedly. The first one is given by the fact that $h_*F_{jk,l}$ has codimension 3. Now for a $1 \le \alpha \le s_{jk}$ we have $h_*E_{jk,\alpha}$ supported in codimension 2, thus

$$h^*Gh^*G'E_{jk,\alpha} = GG'h_*E_{jk,\alpha} = 0.$$

Finally, for any α we have

$$h^*D_jh^*GE_{jk,\alpha} = 0 = h^*G(m_{jk,\alpha-1}E_{jk,\alpha-1}E_{jk,\alpha} + m_{jk,\alpha}E_{jk,\alpha}^2 + m_{jk,\alpha+1}E_{jk,\alpha+1}E_{jk,\alpha}),$$

$$h^*D_kh^*GE_{jk,\alpha} = 0 = h^*G(n_{jk,\alpha-1}E_{jk,\alpha-1}E_{jk,\alpha} + n_{jk,\alpha}E_{jk,\alpha}^2 + n_{jk,\alpha+1}E_{jk,\alpha+1}E_{jk,\alpha}).$$

The recursive relations with $k_{jk,\alpha}$ give

$$\begin{bmatrix} m_{jk,\alpha} & m_{jk,\alpha+1} \\ n_{jk,\alpha} & n_{jk,\alpha+1} \end{bmatrix} \begin{bmatrix} h^*G(E_{jk,\alpha}^2 + k_{jk,\alpha}E_{jk,\alpha-1}E_{jk,\alpha}) \\ h^*G(E_{jk,\alpha+1}E_{jk,\alpha} - E_{jk,\alpha-1}E_{jk,\alpha}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

From Lemma 2.5.2 we have the determinant $m_{jk,\alpha}n_{jk\alpha+1} - m_{jk,\alpha+1}n_{jk,\alpha} = n$, thus

$$h^*G(E_{jk,\alpha}^2 + k_{jk,\alpha}E_{jk,\alpha-1}E_{jk,\alpha}) = 0$$
$$h^*G(E_{jk,\alpha+1}E_{jk,\alpha} - E_{jk,\alpha-1}E_{jk,\alpha}) = 0.$$

In particular, we have

$$h^*GE_{jk,\alpha+1}E_{jk,\alpha} = h^*G\tilde{D}_jE_{jk,1} = GD_{jk},$$

and the result follows. □

Corollary 4.3.4. *For any divisor G on Z we have*

$$h^*GE_{jk}K_X = -D_{jk}G\left((N_{jk,1} + N_{kj,1}) + \sum_{\alpha=1}^{s_{jk}}(k_{jk,\alpha} - 2)\right).$$

Proof. Since $h^*G\,F$ vanishes at top-dimensional intersections, and $E_{jk,\alpha_1}E_{jk,\alpha_2} = 0$ for $|\alpha_1 - \alpha_2| > 1$, we have

$$\begin{aligned}
h^*GE_{jk}K_X = h^*G(E_{jk})^2 &= h^*G\sum_{\alpha=1}^{s_{jk}}N_{jk,\alpha}^2E_{jk,\alpha}^2 + 2N_{jk,\alpha}N_{jk,\alpha+1}E_{jk,\alpha}E_{jk,\alpha+1}\\
&= D_{jk}G\sum_{\alpha=1}^{s_{jk}}-k_{jk,\alpha}N_{jk,\alpha}^2 + 2N_{jk,\alpha}N_{jk,\alpha+1}\\
&= D_{jk}G\sum_{\alpha=1}^{s_{jk}}N_{jk,\alpha}N_{jk,\alpha+1} - N_{jk,\alpha}N_{jk,\alpha-1} + N_{jk,\alpha}(k_{jk,\alpha} - 2)\\
&= D_{jk}G\sum_{\alpha=1}^{s_{jk}}N_{jk,\alpha}(k_{jk,\alpha} - 2)\\
&= -D_{jk}G\left((N_{jk,1} + N_{kj,1}) + \sum_{\alpha=1}^{s_{jk}}(k_{jk,\alpha} - 2)\right).
\end{aligned}$$

where the last identity is by telescoping sum argument. □

Corollary 4.3.5. *If C is a curve on X_n contained in $\tilde{H}_j$ and disjoint to any exceptional divisor F_{jkl}, then*

$$\tilde{D}_jC = \frac{h_*C}{n}\left(D_j - \sum_{j\neq k}m_{jk,1}D_k\right).$$

Proof. From (4.3) we have

$$h^*D_jC = D_jh_*C = n\tilde{D}_j + \sum_k m_{jk,1}E_{jk,1}C,$$

$$h^*D_kC = D_kh_*C = E_{jk,1}C, \quad \forall j \neq k,$$

and the result follows. $\square$

4.3.1 Asymptoticity of $K^3_{X_n}$.

For simplicity, let us denote $K_X = K_{X_n}$. Let us introduce the following notation

$$|D|_{jk} := |D_j|_k + |D_k|_j,$$

where $|D_j|_k$ satisfy

$$\sum_{D_{ll'}D_j \neq 0} F_{ll',j}E_{jk,\alpha}K_X = |D_j|_k(k_{jk,\alpha} - 2).$$

Remark 4.3.6. *Using equations from (4.2), observe that $|D_j|_k$ depends on slopes of weights $v^{jkl}_1, v^{jkl}_2, v^{jkl}_3$ of the lattice points v^{jkl}. Explicitly, using (4.4), we have*

$$|D_j|_k = \sum_l \frac{v^{jkl}_{\rho_{kl}(j)}}{v^{jkl}_{\rho_{kj}(l)}} D_{jkl}.$$

We need this to compute the intersection of K_X with the external walls of the local toric resolution. Recursively we denote,

$$x_{jk,\alpha} = K_X E_{jk,\alpha-1}E_{jk,\alpha}, \quad 1 \leq \alpha \leq s_{jk} + 1$$

$$y_{jk,\alpha} = K_X E^2_{jk,\alpha}, \quad 1 \leq \alpha \leq s_{jk}.$$

Thus, we can write

$$h^*D_jE_{jk,\alpha}K_X = m_{jk,\alpha-1}x_{jk,\alpha} + m_{jk,\alpha}y_{jk,\alpha} + m_{jk,\alpha+1}x_{jk,\alpha+1} + |D_j|_k(k_{jk,\alpha} - 2) \tag{4.5}$$

Using the $\mathbb{Q}$-numerical equivalence of K_X in 2.3.6, we compute

$$x_{jk,1} = h^*D_k\tilde{D}_jK_X = D_{jk}\left(K + \sum_{l\neq j} N_{jl,1}D_l\right),$$

$$x_{jk,s+1} = h^*D_j\tilde{D}_kK_X = D_{jk}\left(K + \sum_{l\neq k} N_{kl,1}D_l\right).$$

Proposition 4.3.7. *We have*

$$x_{jk,\alpha} = x_{jk,1} + \frac{1}{n}(m^*_{jk,\alpha}(D_{jk}D_k - |D_k|_j) - n^*_{jk,\alpha}(D_{jk}D_j - |D_j|_k)),$$

$$y_{jk,\alpha} = -k_{jk,\alpha}x_{jk,\alpha} + \frac{(k_{jk,\alpha}-2)}{n}(n_{jk,\alpha+1}(D_{jk}D_j - |D_j|_k) - m_{jk,\alpha+1}(D_{jk}D_k - |D_k|_j)).$$

Where $m^*_{jk,\alpha} = m_{jk,\alpha} - m_{jk,\alpha-1} - m_{jk,1} + m_{jk,0}$ *and analogous for* $n^*_{jk,\alpha}$.

Proof. Using the recursion given by the $k'_{jk,\alpha}s$, and formulas for $h^*D_j\, E_{jk,\alpha}K_X$ and $h^*D_k\, E_{jk,\alpha}K_X$ of (4.5), we have

$$\begin{bmatrix} (D_j^2D_k - |D_j|_k)(k_{jk,\alpha}-2) \\ (D_jD_k^2 - |D_k|_j)(k_{jk,\alpha}-2) \end{bmatrix} = \begin{bmatrix} m_{jk,\alpha} & m_{jk,\alpha+1} \\ n_{jk,\alpha} & n_{jk,\alpha+1} \end{bmatrix} \begin{bmatrix} k_{jk,\alpha}x_{jk,\alpha} + y_{jk,\alpha} \\ x_{jk,\alpha+1} - x_{jk,\alpha} \end{bmatrix}$$

The determinant $m_{jk,\alpha}n_{jk,\alpha+1} - m_{jk,\alpha+1}n_{jk,\alpha} = n$, implies second relation for $y_{jk,\alpha}$, and

$$x_{jk,\alpha+1} = x_{jk,\alpha} + \frac{(k_{jk,\alpha}-2)}{n}(m_{jk,\alpha}(D_{jk}D_k - |D_k|_j) - n_{jk,\alpha}(D_{jk}D_j - |D_j|_k))$$

The recurrence for $x_{jk,\alpha}$ with a telescopic sum arguments give the result. □

Theorem 4.3.8. *If* $\{D_1, \ldots, D_r\}$ *is an asymptotic arrangement, then*

$$\frac{K^3_{X_n}}{n} \approx -\bar{c}_1^3(Z, D),$$

for prime numbers $n \gg 0$.

Proof. We will compute K_X^3 using the above numerical equivalence, squaring we get

$$K_X^2 \sim_{\mathbb{Q}} h^*K^2 + \left(\sum_{j<k} E_{jk}\right)^2 + \sum_{j<k<l} V_{jkl}^2F_{jkl}^2 + 2\sum_{j<k} E_{jk}\left(h^*K + \sum_{j<k<l} V_{jkl}F_{jkl}\right).$$

We have explicitly

$$(h^*K)^2K_X = (h^*K)^3 = nK^3.$$

$$E_{jk}F_{jkl}K_X = D_{jkl}\sum_{\alpha=1}^{s_{jk}} N_{jk,\alpha}(k_{jk,\alpha}-2) = -D_{jkl}\left((N_{jk,1} + N_{kj,1}) + \sum_{\alpha=1}^{s_{jk}}(k_{jk,\alpha}-2)\right).$$

In the rest, we will denote

$$(k-2)_{jk} = \sum_{\alpha=1}^{s_{jk}} (k_{jk,\alpha} - 2).$$

Using Corollary 4.3.4 for $G = K$, we get

$$\begin{aligned} K_X^3 =& nK^3 - 2\sum_{j<k}\left(D_{jk}K + \sum_{j<k<l} V_{jkl}D_{jkl}\right)(N_{jk,1} + N_{kj,1} + (k-2)_{jk}) \\ &+ \sum_{j<k<l} \frac{nV_{jkl}^3}{v_1^{jkl}v_2^{jkl}v_3^{jkl}} D_{jkl} + K_X\left(\sum_{j<k} E_{jk}\right)^2. \end{aligned}$$

Just rest to compute

$$K_X\left(\sum_{j<k} E_{jk}\right)^2 = \sum_{j<k}\sum_{\alpha=1}^{s_{jk}} N_{jk,\alpha}(N_{jk,\alpha-1}x_{jk,\alpha} + N_{jk,\alpha}y_{jk,\alpha} + N_{jk,\alpha+1}x_{jk,\alpha+1}).$$

From Corollary 4.3.4, we have

$$\begin{aligned} \frac{D_{jk}(D_j+D_k)(k_{jk,\alpha}-2)}{n} &= \frac{h^*(D_j+D_k)E_{jk,\alpha}K_X}{n} \\ &= N_{jk,\alpha-1}x_{jk,\alpha} + N_{jk,\alpha}y_{jk,\alpha} + N_{jk,\alpha+1}x_{jk,\alpha+1} \\ &+ (x_{jk,\alpha} + y_{jk,\alpha} + x_{jk,\alpha+1}) + \frac{(k_{jk,\alpha}-2)}{n}|D|_{jk}, \end{aligned}$$

So, we have explicitly

$$\begin{aligned} K_X\left(\sum_{j<k} E_{jk}\right)^2 &= \sum_{j<k}\sum_{\alpha=1}^{s_{jk}} \frac{D_{jk}(D_j+D_k) - |D|_{jk}}{n} N_{jk,\alpha}(k_{jk,\alpha}-2) \\ &- \sum_{j<k}\sum_{\alpha=1}^{s_{jk}} N_{jk,\alpha}(x_{jk,\alpha} + y_{jk,\alpha} + x_{jk,\alpha+1}). \end{aligned}$$

The first term of the sum contains $\sum_\alpha N_{jk,\alpha}(k_{jk}-2)$, which is asymptotic respect to n by previous discussion (Section 2.7). Thus, we just have to prove asymptoticity for

$$\sum_{j<k}\sum_{\alpha=1}^{s_{jk}} N_{jk,\alpha}(x_{jk,\alpha} + y_{jk,\alpha} + x_{jk,\alpha+1}) \tag{4.6}$$

Proceeding as above, it is not difficult to show the following identity,

$$\sum_{\alpha=1}^{s_{jk}} \frac{D_{jk}(D_j + D_k) - |D|_{jk}}{n}(k_{jk,\alpha} - 2) = \sum_{\alpha=1}^{s_{jk}} (N_{jk,\alpha} + 1)(x_{jk,\alpha} + y_{jk,\alpha} + x_{jk,\alpha+1}) - N_{jk,1}x_{jk,1} - N_{jk,s}x_{jk,s+1}.$$

So, the asymptoticity of (4.6) depends only on the asymptoticity of

$$\sum_{j<k} \sum_{\alpha=1}^{s_{jk}} (x_{jk,\alpha} + y_{jk,\alpha} + x_{jk,\alpha+1}). \tag{4.7}$$

By Proposition 4.3.7, $x_{jk,\alpha} + y_{jk,\alpha} + x_{jk,\alpha+1}$ equals to

$$\frac{k_{jk,\alpha} - 2}{n}(m^{**}_{jk,\alpha}(D_{jk}D_k - |D_k|_j) - n^{**}_{jk,\alpha}(D_{jk}D_j - |D_j|_k) - nx_{jk,1}),$$

where $m^{**}_{jk,\alpha} = m_{jk,\alpha-1} - m_{jk,\alpha+1} - m_{jk,0} + m_{jk,1}$, and analogous for $n^{**}_{jk,\alpha}$. Observe that these terms are bounded by Cn for some constant $C > 0$. On the other hand, the terms $(D_{jk}D_j - |D_j|_k)$ and $(D_{jk}D_k - |D_k|_j)$ asymptotically depend only on the slopes of coordinates of the chosen lattice points v^{jkl} on each intersection D_{jkl}. By Lemma 4.2.11, we can choose lattice points with slopes asymptotically bounded by 3 as n grows, with K_X having $V_{jkl} = 0$ for all $j < k < l$. So, we have

$$|D|_{jk} \leq 6 \sum_l D_{jkl}.$$

Thus, as n grows, all the terms in K_X^3/n vanish except $nK^3 \approx -n\bar{c}_1^3(Z, D)$. □

4.3.2 Asymptoticity of $e(X_n)$.

The topological characteristic can be computed from the topology of (Z, D) and the exceptional divisors $E_{jk,\alpha}, F_{jkl}$. The divisors $F_{jkl} = \sum_{p \in D_{jkl}} F_{jkl,p}$ where $F_{jkl,p}$ is the corresponding exceptional divisor over a $p \in D_{jkl}$. Thus, $e(F_{jkl}) = D_{jkl}e(F_{jkl,p})$. In the toric picture (Section 4.2) of X_n over p, let $v = v^{jkl}$ the ray generator defining $F_{jkl,p}$.

Lemma 4.3.9. *We have $e(F_{jkl,p}) = s_{jk} + s_{jl} + s_{kl} + 3$.*

Proof. It is well-known that $F_{jkl,p}$ is the toric variety associated to the star-fan $\mathrm{Star}(C(v)) \subset (N_v)_{\mathbb{R}}$, i.e., the induced fan by the lattice quotient $N_v = N/vN$ (Proposition 2.4.6). In this case, the 2-cones of $\mathrm{Star}(C(v))$ are the induced by each $C(e_{jk,\alpha}, e_{jk,\alpha+1})$. Since $e(F_{jkl,p})$ is the sum of its top-dimensional cones [CLS11, Thm. 12.3.9], we have the result. □

The components of divisors $E_{jk,\alpha}$ are determined locally as exterior divisor of the toric picture of X_n. They intersect F at the rational curves $C_{jk,\alpha}$. Locally each component of $E_{jk,\alpha}$ is isomorphic to $\mathbb{A}^1 \times \mathbb{P}^1$, this follows for the star-fan construction. Thus, their closure in X_n are birationally ruled surfaces over it associated component of D_{jk} [Har77, Rmk. 2.2.1].

Lemma 4.3.10. *If $E_{jk,\alpha} = \sum_{C \in D_{jk}} E_{jk,\alpha,C}$ is the decomposition in componenets, then we compute*

$$e(E_{jk,\alpha}) = 4 \sum_{C \in D_{jk}} (1 - p_g(C)).$$

Proof. We have $e(E_{jk,\alpha}) = \sum_C e(E_{jk,\alpha,C})$. Since $E_{jk,\alpha,C}$ is a fibration over C with fiber $F = \mathbb{P}^1$, it is known that $e(E_{jk,\alpha,C}) = e(\mathbb{P}^1)e(C) = 4(1 - p_g(C))$. □

Lemma 4.3.11. *If X is a complex algebraic variety, and $A \subset X$ is a subvariety such that $X \setminus A$ is smooth, then $e(X) = e(X \setminus A) + e(A)$.*

Proof. See [Ful93, p. 141]. □

Remark 4.3.12. *The above lemma implies the* exclusion-inclusion principle*, i.e., for subvarieties $V, W \hookrightarrow X$ we have $e(V \cup W) + e(V \cap W) = e(V) + e(W)$.*

Theorem 4.3.13. *If $\{D_1, \ldots, D_r\}$ is an asymptotic arrangement, then*

$$\frac{e(X_n)}{n} \approx \bar{c}_3(Z, D),$$

for prime numbers $n \gg 0$.

Proof. Denote by R the ramification divisor of $h \colon X_n \to Z$ is a $n:1$ morphism which is an isomorphism outside R we have

$$e(X \setminus R) = ne(Y \setminus D) = n(e(Y) - e(D)).$$

On the other hand, $R = \bigcup_j \tilde{D}_j \cup \mathrm{Exc}(h)$, where $\mathrm{Exc}(h)$ is the exceptional data of h. Topologically is given by

$$\mathrm{Exc}(h) = \bigcup_{j<k<l} F_{jkl} \cup \bigcup_{j<k} \bigcup_{\alpha} E_{kj,\alpha}.$$

By the exclusion-inclusion observe that

$$e\left(\bigcup_j \tilde{D}_j\right) - e\left(\bigcup_j \tilde{D}_j \cap \mathrm{Exc}(h)\right) = e(D) - e(\mathrm{Sing}(D)).$$

So, we get

$$e(R) = e(D) - e(\mathrm{Sing}(D)) + e(\mathrm{Exc}(h)).$$

On the other hand, the components of $E_{jk,\alpha}E_{jk,\alpha}$ are curves over D_{jk} isomorphic to their respective components. Also, each component of $E_{jk,\alpha}F_{jkl}$ is a rational curve over its corresponding point in D_{jkl}. Thus, we have identities,

$$e(E_{jk,\alpha}E_{jk,\alpha+1}) = e(D_{jk})$$

$$e(E_{jk,\alpha}F_{jkl}) = 2D_{jkl}.$$

Using repeatedly the exclusion-inclusion principle we we compute

$$e(\mathrm{Exp}(h)) = \sum_{j<k} \sum_{C\in D_{jk}} [s_{jk}(3 - 4p_g(C)) - 1] - \sum_{j<k<l} (s_{jk} + s_{jl} + s_{kl} - 3)D_{jkl}.$$

By the previous discussion in Section 2.7, the lengths s_{jk}/n are asymptotically zero as n grows. Thus, $e(X_n) \approx n(e(Z) - e(D))$ as n grows. $\square$

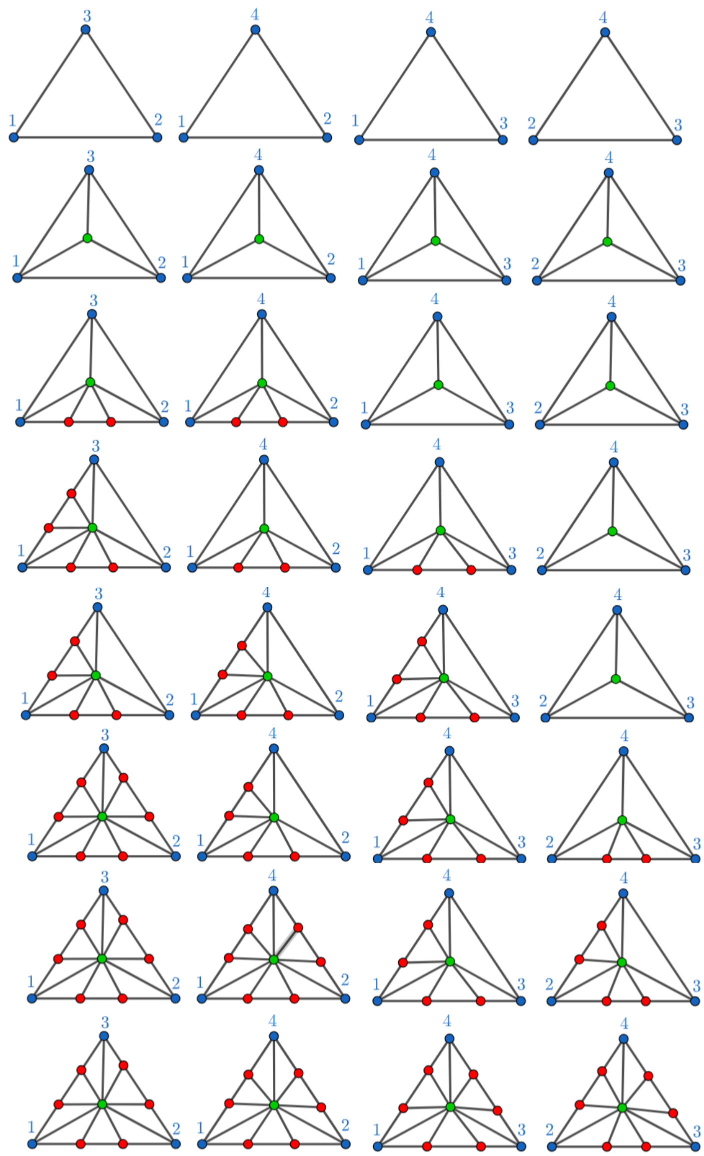


Figure 4.12: Assume $r = 4$ with $D_{jkl} = 1$, then on Y the singularities over each D_{jkl} can be sketched as in the figure. So the resolution process is in the following order: First the internals blow-ups, and then the walls in the following order $D_{12}, D_{13}, D_{14}, D_{23}, D_{24}, D_{34}$.

Chapter 5

Applications to geography of 3-folds

5.1 Hyperplane sections arrangements.

The above partial resolution can be seen as a resolution of pairs

$$h\colon (X_n, \tilde{D}_{red}) \to (Z, D_{red}),$$

where $\tilde{D}$ is the inverse direct image of D. The reduced divisor of D' is an SNC divisor. Indeed, in terms of log-resolutions [KM92, p. 5], we can see that our partial resolution has a good behavior in logarithmic terms, i.e., they preserves the log-structure of the variety n-th root cover Y_n. The following result illustrate this ideas.

Theorem 5.1.1. *Let Z be a minimal non-singular projective 3-fold, and let $\{H_1, \ldots, H_r\}$ be a collection of hyperplane sections in general position. Then, for prime numbers $n \gg 0$ there are* log*-morphisms $(X_n, \tilde{D}_{red}) \to (Z, D_{red})$ of degree n such that:*

1. *X_n is of* log*-general type, i.e., $K_{X_n} + \tilde{D}_{red}$ is big and nef,*
2. *X_n has cyclic quotient singularities, and so* log*-terminal of order lower than n, and*
3. *the slopes $(-K^3/24\chi, e/24\chi)$ of X_n are arbitrarily near to $(2, 1/3)$.*

Proof. We take $D = \sum_{j=1}^{r} \nu_j H_j$, where H_j are hyperplane sections on Z and $\sum_{j=1}^{r} \nu_j = n$ an asymptotic partition. Recall that $H_j H_k H_l = H_j^2 H_k = \deg(Z)$ for any $j < k < l$. Take $h\colon X_n \to Y_n \to Z$ the asymptotic cyclic resolution constructed in Theorem 4.3.13. Again for simplicity let us denote $K_X = K_{X_n}$. From, the explicit description given in Proposition 4.2.9, we have

$$K_X + D'_{red} = K_X + \sum_j \tilde{D}_j + \sum_{j<k,\alpha} E_{jk,\alpha} + \sum_{j<k<l} F_{jkl} = h^*(K_Z + D_{red}).$$

First observe that for any curve C outside the exceptional data of h, we have $(K_X + D'_{red})C \geq 0$, by projection formula and since $K_Z + D_{red}$ is ample. For every closed curve $C = C_{jk,\alpha}$ of $C = C_l$ of the local toric picture (Section 4.2) of the resolution, we have $(K_X + D'_{red})\, C = 0$. For the remainder curves, we just need to concern about the positivity of its intersection with K_X. Since K_Z is a nef divisor, by Proposition 4.3.2 we must have any K_X-negative curve contained in the support of $h^*(D)$. Thus, the rest of rational curves in $\mathrm{Supp}(h^*D)$ are of the following types:

1. Curves defined by the closure of a wall $E_{jk,\alpha-1}E_{jk,\alpha}$ for $1 \leq \alpha \leq s_{jk}$.
2. A curve contained in $E_{jk,\alpha}$ but not in $E_{jk,\alpha\pm1}$ for $1 \leq \alpha \leq s_{jk}$.
3. A curve contained in $\tilde{H}_j$.

If C is of type (1), from Proposition 4.3.3 we have,

$$(K_X + D'_{red})E_{jk,\alpha}E_{jk,\alpha+1} = h^*(K_Z + D)E_{jk,\alpha}E_{jk,\alpha+1} = (K_Z + D_{red})H_{jk} > 0,$$

for any α. If C is of type (2), then C must be a fiber of the ruled surface $E_{jk,\alpha}$,i.e., is in the class of $C_{jk,\alpha}$. But, by (4.1) we have $K_X C > 0$. Finally, if C is of type (3), we assume that it does not intersect interior divisors F_{jkl}. If does it, then by the toric local description C must be of the form $E_{jk,1}\tilde{H}_j$ for some k. Again by projection formula, we have $(K_X + D'_{red})C = (K_Z + D)h_*C \geq 0$. Then, $K_X + D'_{red}$ is a nef divisor, and moreover $(K_X + D'_{red})^3 = (K_Z + D)^3 > 0$.Thus, by [Laz04, Th. 2.2.16.], the divisor $K_X + D'_{red}$ is big. Now, from Theorem 4.3.13 we know that for $n \gg 0$,

$$\begin{aligned} \frac{K_X^3}{n} &\approx -c_1^3(Z, D) = (K_Z + rH)^3 \\ &= K_Z^3 + r^3 \deg(Z) + 3rK_Z H^2 + 3K_Z^2 H, \end{aligned}$$

where H is a generic hyperplane section on Z. Thus, if we choose r depending on n with $r(n)/n \to 0$ as n grows, then the numbers $|D|_{jk}$ goes to zero respect with n. Then, we have $K_X^3 > 0$, so X_n is of general type. Moreover, from Example 2.2.17 we have $(-K^3/24\chi, e/24\chi)(X)$ arbitrarily near to $(2, 1/3)$. $\square$

5.2 A degenerated situation.

Consider $Z \hookrightarrow \mathbb{P}^4$ of degree $d = \deg(Z)$. In this case, we have explicitly

$$K_Z = (d-5)H|_Z, \quad c_2(Z) = (10 + d(d-5))H^2|_Z,$$

$$c_3(Z) = -d\left(d^2(d-5) + 10d - 10\right),$$

for a generic hyperplane section H. Take 3 hyperplane sections $\{H_1, H_2, H_3\}$ in general position, and asymptotic partitions $\nu_1 + \nu_2 + \nu_3 = n$. Along $D = \sum_j \nu_j H_j \sim nH$ consider the respective n-th root cover $Y_n \to Z$. Its singularities are over d points in $H_1H_2H_3$. As we see in Section 4.2.1.2, these singularities admit a locally nef non-singular resolution. Unfortunately, in this resolution the lattice point v does not satisfy the condition of Lemma 4.2.11, i.e., the volume K_X^3 will not be completely asymptotic to the logarithmic Chern number $\bar{c}_1^3(Z, D)$. However, since the chosen v satisfy $v_j = 1$. So, following the methods of Theorem 4.3.13 to compute K_X^3, we get

$$x_{jk,\alpha} = E_{jk,\alpha}E_{jk,\alpha+1}K_X = d(d-3), \quad 1 \le \alpha \le s_{jk}.$$

Now we compute

$$K_X^3 = d(d-3)\left(nd^2 - 3nd + 3n - 9d + 18 - 3\sum_{j<k}(k-2)_{jk}\right),$$

since

$$\sum_{j<k}(N_{jk,1} + N_{kj,1}) = -3\frac{n-3}{n},$$

$$K_X\left(\sum_{j<k} E_{jk}\right)^2 = -d(d-3)\sum_{j<k}(N_{jk,1} + N_{jk,s} + (k-2)_{jk}).$$

In particular for prime numbers $n \gg 0$,

$$\frac{K_X^3}{n} \approx (K_Z + H_1 + H_2 + H_3)^3 - d = d(d-2)^3 - d.$$

On the other hand, from Section 4.1 we have

$$\chi(\mathcal{O}_X) = n\chi(Z, \mathcal{O}_Z) - \frac{1}{12}(R_1(n) + R_2(n) + R_3(n)),$$

where

$$\chi(Z, \mathcal{O}_Z) = -\frac{d(d-5)(10 + d(d-5))}{24}$$

$$R_1(n) = \frac{9d(n-1)(2n-1)}{2n},$$

$$R_2(n) = \frac{3d(d-5)(n-1)(5n-1)}{2n} + \frac{3d((d-5)^2 + d(d-5) + 10)(n-1)}{2}$$

$$R_3(n) = 6d(d-2)(d(\nu_1, \nu_2, n) + d(\nu_1, \nu_3, n) + d(\nu_2, \nu_3, n)).$$

Since, the partition is asymptotic, for $n \gg 0$ we have

$$\frac{\chi(\mathcal{O}_X)}{n} \approx -\frac{d(d-2)(d-1)^2}{24}.$$

For $n \gg 0$, the topological characteristic behaves as

$$\frac{e(X)}{n} \approx c_3(\mathbb{P}^3, D_{red}) = -d(d-5)(d^2 + 2d + 6).$$

Following the proof of Theorem 5.1.1, we get K_X nef for $n \gg 0$. As a consequence of the above computations, we have.

Theorem 5.2.1. *For $d \geq 5$ and $n \gg 0$ there are minimal non-singular 3-folds X of general type having degree n over Z with slopes*

$$\frac{c_1^3}{c_1 c_2} \approx \frac{(d-2)^3 - 1}{(d-2)(d-1)^2}, \quad \frac{c_3}{c_1 c_2} \approx \frac{(d-5)(d^2 + 2d + 6)}{(d-2)(d-1)^2}.$$

In particular, as the degree of Z grows, the slopes have limit point $(1, 1)$.

□

Chapter 6

Discussion & Future Work

In this section, we will see the possible future paths in order to extend this work.

6.1 Asymptoticity through minimal models

One of the main horizons of this research is to achieve the asymptoticity of invariants through minimal models. This means that, as we see in Theorem 2.7.8, the invariants of X_n, with respect to n, could be asymptotically equal to the respective invariants of its minimal model. Thus, we will be in a very nice position to do geography, i.e., the study of arrangements of hypersurfaces is identified through the slopes of Chern numbers with a "region" of minimal projective varieties. As we see in Theorem 5.1.1 and Theorem 5.2.1, if the basis pair (Z, D) has Z minimal of general type and D composed by ample divisors, then our constructions preserves important features in terms of minimal models. However, this in general it is not something easy to work. For the future of this work 3 aspects are important.

1. Asymptotic study of (partial) desingularization of cyclic quotient singularities of dimension ≥ 3.

2. Hirzebruch-Riemann-Roch for singular varieties with terminal and log-terminal singularities with their asymptotic analogs.

3. The behavior of the invariants after applying the MMP to our constructed varieties.

In the next section, we discuss (1). If we achieve our goal we will be, able to construct *good partial resolutions* $X_n \to Y_n$, i.e., the Chern numbers, with respect to n, are asymptotically equal to the logarithmic Chern numbers of the basis (Z, D). We expect that we can improve the singularities to the terminal ones, so we will be able to run the MMP, i.e. we want to construct a *terminal good partial resolution.* For (2), we have results of [Rei87] and [BS05] which are a kind of starting point for future work. These contain versions of the Hirzebruch-Riemann-Roch theorem for varieties with canonical and cyclic quotient singularities. For (3), we think that the answer could be hidden in all the massive previous work done around the minimal model program [BCHM10], [KM92]. We expect, that the involved invariants do not suffer dramatic changes after flipping or contractions operations as occur in the case of surfaces. Then, asymptotically with respect to n, the invariants remain unchanged. We state the above discussion as conjecture.

Conjecture 6.1.1. *Let $X_n \to Y_n \to (Z, D)$ be a terminal good partial resolution of singularities of the n-th root cover construction. Assume that K_{Y_n} is nef, and let X'_n a minimal model of X_n. Then, for any partition $i_1 + \ldots + i_m = d$ we have*

$$\frac{c_{i_1} \ldots c_{i_m}(X'_n)}{n} \approx \bar{c}_{i_1} \ldots \bar{c}_{i_m}(Z, D),$$

for prime numbers $n \gg 0$.

6.2 What about the length of resolution of 3-fold c.q.s

Cyclic quotient singularities of dimension 3 can be desingularized using a generalization of the Hirzebruch-Jung algorithm, this is the Fujiki-Oka algorithm. See [Ash19] for a modern treatment. After the local cyclic toric resolution of Section 4.2.2, instinctively we want to desingularize each one with the Fujiki-Oka process. However, since we want asymptoticy of invariants in our resolutions, so we ask for the *topological length* and the *intersection number behavior* of such an algorithm. For the first, we mean the amount of new topological data, i.e., how the Betti numbers grow for the chosen resolutions. For last, we mean how the new curves and divisors on the exceptional data affect the volume K_X^3. As we discussed in Section 2.7, the algorithm in dimension 2 has both aspects behaving as $\sim \sqrt{n}$ for a suitable class of integer numbers.

Let us assume that we choose a partial resolution for the local cyclic resolution, so the amount of new topological data will behave approximately as

$$\sim 3\sqrt{n} \sum_{j<k,\alpha} \ell(v_l, a_{jk,\alpha}, b_{jk,\alpha}),$$

where $\ell(v_l, a_{jk,\alpha}, b_{jk,\alpha})$ is a length number depending on each cyclic singularities given in Lemma 4.2.8. Thus, asymptotically respect with n, we require that $\ell(v_l, a_{jk,\alpha}, b_{jk,\alpha}) \sim n^{1/c}$ for $c < 1/2$. In particular, Fujiki-Oka algorithm for a cyclic quotient singularity of type $\frac{1}{n}(a, b, 1)$ contains the processes for those of dimension two $\frac{1}{n}(a, 1)$ and $\frac{1}{n}(b, 1)$. Thus, in the best case, we will have $\ell(v_l, a_{jk,\alpha}, b_{jk,\alpha}) \sim \sqrt{v_l}$. To assure asymptoticity in Theorem 4.3.13 we must have $v_l \sim n/3$, thus after resolve we lose the asymptoticity on the topological side. On the other hand, if we admit all $v_l's$ small as we see in Theorem 5.2.1, then after resolve we lose the asymptoticity of the volume. These observations lead us to a well-known question: the existence of a *terminal resolution* for cyclic quotient singularities, i.e., having only terminal singularities.

The terminalization of a toric singularity is a well-known process [CLS11, Sec. 11.4]. Indeed, assume that our toric singularity has associated cone $\sigma \subset \mathbb{R}^d$. First, we have to compute the convex hull of $\sigma \cap \mathbb{Z}^d - \{0\}$. This will give us a refinement of σ, which is a canonical resolution, i.e. having at most canonical singularities with ample canonical bundle. Finally, each canonical toric singularity defined by a cone can be terminalizated by blowing-up each lattice point on the plane generated by the primitive generators of the cone. However, we do not know the growing behavior of this algorithm. In fact, it is known that the best convex-hull algorithm behaves as $\sim n \log n$ when the number of lattice points is n [Gre90]. This not seems like a good algorithm to choose.

Question 6.2.1. *How can we construct a terminal algorithm for cyclic quotient singularities with the desired asymptotic properties? Is it possible?*

As we see in Section 4.2.1.1, 4.2.1.2 and 4.2.1.3, to achieve a well-behaved resolution it is probable that we will have to impose different conditions on the integer a and b.

6.3 Geography questions

Another of the objectives of this work was to achieve minimal non-singular 3-folds of general type in the *unknown zone* of the map in Figure 1.2. Explicitly, this is the zone over the lines connecting $(1/2, 5/6)$ and $(1/16, 43/8)$ with $(2, 1/3)$, i.e., the region

$$R = \left\{ (x, y) \in \mathbb{Q}^2 : y + \frac{1}{3}x - 1 > 0, \quad y + \frac{242}{93}x - \frac{515}{93} > 0 \right\}.$$

However, with all our constructions we were unable to establish a point on that zone. So, we repeat the question asked by Hunt in [Hun89, Ch. 10].

Question 6.3.1. *Are there minimal non-singular* 3*-folds with slopes on the unknown zone R?*

In Theorem 5.1.1 we see that there are 3-folds with cyclic quotient singularities accumulating in the well-known point of the map $(2, 1/3)$. We are curious if after applying the process proposed in Section 6.1, the minimal 3-folds expected will preserve the accumulating point or they move out.

Finally, the principal motivation for all this work is the profound connection between arrangements of hypersurfaces and the geography of invariants of minimal varieties. For us will be interesting to explore the geography through arbitrary arrangements of planes on $\mathbb{P}^3$. We are curious about the regions that minimal models of n-th root covers could cover in the map of Figure 1.2 through arrangements. As we see in Proposition 2.7.7, for us the most important tool is the log-resolution.

Question 6.3.2. *What is the region covered by minimal models of n-th root cover Y_n along arrangements of planes in $\mathbb{P}^3$?.*

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Index