Exact fronts for the nonlinear diffusion equation with quintic nonlinearities

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We consider traveling wave solutions of the reaction diffusion equation with quintic nonlinearities $u_t = u_{xx} + \mu u(1-u)(1+\alpha u+\beta u^2+\gamma u^3)$. If the parameters α, β, and γ obey a special relation, then the criterion for the existence of a strong heteroclinic connection can be expressed in terms of two of these parameters. If an additional restriction is imposed, explicit front solutions can be obtained. The approach used can be extended to polynomials whose highest degree is odd.

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I. INTRODUCTION

The nonlinear diffusion equation $u_t = u_{xx} + f(u)$ models phenomena in diverse fields such as population growth, kinetics of phase transitions, chemical reactions, and many others. Of special interest is the case when the function f is such that there exist two steady states, one stable and one unstable. We shall assume that the equation has been scaled so that the unstable state is $u_u = 0$ and the stable state is $u_s = 1$, and we consider functions f which are positive in $(0,1)$. Then sufficiently localized initial conditions evolve into a traveling front which joins the two steady states [1]. The speed at which the front propagates, c^* is equal or greater than the linear marginal stability value $c_L = 2\sqrt{f'(0)}$. In many cases the asymptotic speed of propagation is exactly the linear value $c_L = 2\sqrt{f'(0)}$ obtained by the linear marginal stability criteria [2,3]. There are cases, however, when the front propagates at a speed greater than this value, a case which is referred to as that in which a nonlinear speed selection mechanism [4–6] operates. Explicit expressions for this special nonlinear front or strong heteroclinic connection and its speed have been obtained for particular choices of f. All the known solutions correspond to functions f of the form $f(u) = \mu u + u^n - u^{2n-1}$ which, for μ positive but smaller than a critical value μ_c, are strongly heteroclinic [7]. The purpose of this article is to show, using as an example a quintic polynomial f, that the criterion for the existence of special fronts can be formulated in many cases in a simpler way that enables one to decide whether for a certain f there is a strong heteroclinic connection even if the exact solution for the front is not known. We find exact fronts for this quintic polynomial for f together with a criterion for strong heteroclinicity in terms of the parameters of the polynomial valid even when no explicit solution for the front can be obtained. Similar results can be obtained for polynomials whose highest degree is odd. The knowledge of exact solutions is of interest not only as a curiosity, they are also needed in the framework of the recent proposal of structural stability [8], the knowledge of the speed for a specific form of f enables the calculation of the speed for small perturbations to f using renormalization group techniques.

In Sec. II we state the problem and reformulate already known results, and in Sec. III we give the results for the quintic polynomial.

II. MONOTONIC FRONTS OF THE REACTION DIFFUSION EQUATION

We consider the reaction diffusion equation

$$u_t = u_{xx} + f(u)$$

with $f(0)=0$, $f(1)=0$, $f'(0)>0$, and $f>0$ in $(0,1)$. Given these conditions on f then there exist fronts that connect the unstable fixed point $u=0$ to the stable fixed point $u=1$. Traveling wave fronts $u(x-ct)$ satisfy the ordinary differential equation

$$u_{zz} + cu_z + f(u) = 0, \qquad \lim_{z\to-\infty} u = 1, \quad \lim_{z\to\infty} u = 0, \tag{1}$$

where $z = x - ct$ and we assume that c is positive. A front joining the stable fixed point 1 to the unstable point 0 is monotonic if in addition its derivative du/dz does not change sign. If we search for monotonic fronts it is convenient to consider the dependence of z as a function of u, or rather the dependence of $v(u) = -(dz/du)^{-1}$ as a function of u. For a monotonic solution of Eq. (1), $u(z)$ decreases monotonically as z goes from $-\infty$ to ∞, therefore, the function $v(u)$ is well defined and is positive between 1 and 0 and vanishes at the fixed points. One readily finds that the equation for $v(u)$ is

$$v(u)\frac{dv}{du} - cv(u) + f(u) = 0, \tag{2a}$$

with

$$v(0) = v(1) = 0, \qquad \text{with } v > 0. \tag{2b}$$

Since the endpoints are singular we must determine the behavior near them analytically. If we consider functions f analytic around 0 and with $f'(0) > 0$, then near $u = 0$ we find

$$v(u) = a_1 u + a_{3/2} u^{3/2} + a_2 u^2 + a_{5/2} u^{5/2} + a_3 u^3 + \cdots$$

where the first terms are given by

$$a_1^2 - c a_1 + f'(0) = 0, \tag{3a}$$

$$a_{3/2}(\tfrac{5}{2} a_1 - c) = 0, \tag{3b}$$

$$a_2(3a_1 - c) + \tfrac{1}{2} f''(0) = 0, \tag{3c}$$

$$a_{5/2}(c - \tfrac{7}{2} a_1) - \tfrac{7}{2} a_{3/2} a_2 = 0, \tag{3d}$$

and so on. That the leading term in the expansion of v near zero is linear in u is due to the fact that the front in the original coordinates $u(z)$ approaches the fixed points exponentially. Since v must be positive between 0 and 1, a_1 must be real and positive. The two roots for a_1 are given by $a_{1P} = [c + \sqrt{c^2 - 4f'(0)}\,]/2$ and $a_{1M} = [c - \sqrt{c^2 - 4f'(0)}\,]/2$. The minimum speed c for which there may be a monotonic front is the linear marginal speed $c_L = 2\sqrt{f'(0)}$ at which the roots coincide $a_{1P} = a_{1M} \equiv a_{1L}$. For speeds greater than this value $a_{1M} < a_{1L} < a_{1P}$. Strong heteroclinic solutions or special nonlinear front profiles are those associated with a_{1P}. From the expansion at the origin it follows from (3b) that either $c = 5a_1/2$ or $a_{3/2} = 0$. In the first case we find that $c = 5\sqrt{f'(0)/6} \approx 2.041\sqrt{f'(0)}$ and $a_1 = \sqrt{2f'(0)/3} = a_{1M}$. The solution that arises in this case is not a preferred asymptotic state. In fact, it is known [1] that a small positive initial condition will evolve into the monotonic front of lower speed. This condition has been shown to be equivalent to the statement that the selected front is that with the steepest decay to zero [6], i.e., that with larger a_1 for any given c. For any $c > c_L$ the larger of the two values of a_1 is a_{1P} hence $c = 5a_1/2$ is not a possible asymptotic state and we shall not consider it any further. Strong heteroclinic connections can be achieved only if $a_{3/2} = 0$, all half integer coefficients vanish then and $v(u) = a_1 u + a_2 u^2 + \cdots$.

Near $u = 1$, assuming $f'(1) < 0$,

$$v(1-u) = b_1(1-u) + b_2(1-u)^2 + b_3(1-u)^3 + \cdots,$$

where b_1 is the positive solution of

$$b_1^2 + c b_1 + f'(1) = 0.$$

There is only one positive solution for b_1, the rest of the coefficients follow easily.

It is convenient to introduce a new parameter λ defined by $c = \lambda a_1$. It is not difficult to realize that whenever $1 < \lambda < 2$ then the solution for v is strongly heteroclinic, that is, associated with a_{1P} and when $\lambda > 2$ it becomes associated with a_{1M}; hence for $\lambda > 2$ the linear marginal speed is selected. If $c = \lambda a_1$ then

$$c = \lambda \sqrt{\frac{f'(0)}{\lambda - 1}}, \qquad a_1 = \sqrt{\frac{f'(0)}{\lambda - 1}}. \tag{4}$$

At $\lambda = 2$, the speed attains its linear value $c_L = 2\sqrt{f'(0)}$. The problem then is to determine the value of λ. This transition value $\lambda = 2$ is not associated with any specific nonlinearity, it is valid for any f that satisfies the conditions given above.

An extensive classification of exact solutions, not restricted to the search for fronts, has been given for cubic f [9]. All exact front solutions given in the literature, [4,6,7,10,11] correspond to functions f for which an exact solution for v is of the form

$$v_n(u) = a_1 u (1 - u^{n-1})$$

which is an exact solution of Eq. (2) for

$$f_n(u) = f'(0) \left(u + \frac{(1 + n - \lambda)}{\lambda - 1} u^n - \frac{n}{\lambda - 1} u^{2n-1} \right). \tag{5}$$

We observe that for $\lambda = n + 1$ we recover the solutions of Kaliappan [10]; since $n > 1$, λ is greater than two, so none of them are strongly heteroclinic. The front corresponding to v_n is given implicitly by

$$z = -\int \frac{du}{v_n(u)} \tag{6}$$

and explicitly by [7,12]

$$u_n(z) = \frac{e^{-z a_1}}{(1 + e^{-(n-1) z a_1})^{\frac{1}{n-1}}}.$$

The criterion for the existence of strongly heteroclinic fronts together with their exact expression has been given [7] for functions f of the form $f(u) = \mu u + u^n - u^{2n-1}$. The critical value for μ given in Ref. [7] for the transition from a strong heteroclinic connection to a simple nongeneric connection (a solution associated with a_{1M}) is equivalent to the value $\lambda = 2$ after suitable rescaling. It is perhaps convenient to see it in the example given by van Saarloos [6]

$$\phi_t = \phi_{xx} + \phi + d\phi^3 - \phi^5$$

which has a strongly heteroclinic connection for $d > 2/\sqrt{3}$, of speed

$$v^\dagger = \frac{-2 + 2\sqrt{4 + d^2}}{\sqrt{3}}.$$

To identify the value of λ from Eq. (5) we must scale the equation for ϕ so that the stable point is at 1. To do so we let $\phi = K u$, u satisfies $u_t = u_{xx} + u + K^2 d u^3 - K^4 u^5$ and the stable state is $u = 1$ if

$$K^2 = \frac{d + \sqrt{4 + d^2}}{2},$$

where the positive sign is chosen to obtain a real K. Now we compare with Eq. (5) with $n = 3$. We see that $f'(0) = 1$ and that

$$\lambda = 4 - \frac{3d}{K^2}$$

or, in terms of d,

$$\lambda = \frac{4\sqrt{4+d^2} - 2d}{d + \sqrt{4+d^2}}.$$

It is straightforward to see that the critical value $d = 2/\sqrt{3}$ is exactly $\lambda = 2$ and that the speed

$$c = \frac{\lambda}{\sqrt{\lambda - 1}} = v^{\dagger}.$$

Although in the developments above we have imposed the restriction $f'(0) > 0$ so that f is positive in $(0,1)$, monotonic fronts joining $u = 1$ to $u = 0$ also exist in the subcritical case $f'(0) < 0$ which is of interest in many physical applications [6]. We consider now this subcritical or heterozygote inferior [1] case. Consider f such that $f(0) = f(1) = 0$ with $f'(0) < 0$ and with an additional fixed point $0 < u_0 < 1$. In this case, monotonic fronts joining $u = 1$ to $u = 0$ exist provided that [1]

$$\int_0^1 f(u)\, du > 0. \tag{7}$$

As we shall see below, this case corresponds to the extension of the parameter λ to the region $0 < \lambda < 1$, a regime where all solutions are associated with a_{1P}. Since the condition imposed by Eq. (7) guarantees the existence of a monotonic front, Eq. (2) is valid. When $f'(0) < 0$ the only real positive solution of Eq. (3a) is a_{1P}. Letting again $c = \lambda a_1$ we find that $a_1^2 = -f'(0)/(1-\lambda)$. Therefore, when $f'(0) < 0$, we must impose $0 < \lambda < 1$ and Eq. (4) is still valid. This condition on λ insures condition (7). Again it is useful to show this for the exactly solvable case (5). Integrating (5) we find,

$$\int_0^1 f_n(u)\, du = \frac{f'(0)\lambda(n-1)}{2(\lambda-1)(n+1)}.$$

Since $n > 1$ and we are considering the case $f'(0) < 0$, the integral is positive when $0 < \lambda < 1$ and the solution $v_n(u)$ remains valid in this range as well. For the example discussed by van Saarloos [6] $f(u) = \epsilon u + c_1 u^3 - (c_1 + \epsilon)u^5$, condition (7) implies $\epsilon/c_1 > -(1/4)$. In terms of λ comparing $f(u)$ with f_3 given by (5) we identify $\epsilon = f'(0)$ and $\lambda = (4\epsilon + c_1)/(\epsilon + c_1)$. The requirement $0 < \lambda < 1$ is $\epsilon/c_1 > -(1/4)$ [which is equivalent to van Saarloos' condition $\epsilon c_2/c_1^2 > -3/16$ when one writes the function f as $f(\phi) = \epsilon\phi + c_1\phi^3 - c_2\phi^5$]. To sum up, the subcritical case is the continuous extension to the regime $0 < \lambda < 1$.

III. SOLUTIONS

Now consider fronts for f being a quintic polynomial in u. This problem was considered in Ref. [11] but no explicit solutions were found and no attempt to examine the conditions for the transition from the linear to the nonlinear regime were made. Here we show under which conditions a closed form can be obtained, together with some examples and the condition for strong heteroclinicity $\lambda < 2$ in terms of the parameters of the function f. Evidently, the value of the parameter λ can be determined analytically only if an exact solution for v is known. The most general form of a quintic polynomial that vanishes at 0 and 1 is

$$f(x) = \mu x(1-x)(1 + \alpha x + \beta x^2 + \gamma x^3), \tag{8}$$

where μ, α, β, and γ are four arbitrary parameters whose only restriction is given by the requirement $f'(0) > 0$ and $f > 0$ in $(0,1)$. On the other hand, the most general closed form solution for v given a quintic f is given by

$$v(u) = a_1 u(1-u)(1+bu), \tag{9}$$

where $b > -1$. Introducing again the parameter λ given above, so that a_1 and c are given by Eq. (4), Eq. (9) is the exact solution of Eq. (2) with

$$f(u) = f'(0)u(1-u)\left(1 + \frac{(2+\lambda b - 3b)}{\lambda - 1}u + \frac{b(5-2b)}{\lambda-1}u^2 + \frac{3b^2}{\lambda-1}u^3\right). \tag{10}$$

In the solution for v we have three adjustable parameters, $\lambda, b, f'(0)$ whereas in the most general form for f, four adjustable parameters exist. Hence, an exact solution for v can be found chosing three parameters of f arbitrarily and the fourth one in terms of them. Choosing μ, β, and γ arbitrarily, we identify

$$f'(0) = \mu,$$

$$\lambda = 1 + \frac{75\gamma}{(3\beta + 2\gamma)^2},$$

and

$$b = \frac{5\gamma}{3\beta + 2\gamma}$$

and the exact solution exists if

$$\alpha = \frac{(2 + \lambda b - 3b)}{\lambda - 1}.$$

For any other value of α a closed form solution does not exist and we cannot determine the value of λ. The criterion for the solution to be strongly heteroclinic $1 < \lambda < 2$ is expressed now in terms of the free parameters β and γ.

Now we show that an explicit solution for the front in the original coordinates exists only if an additional condition on b, hence a relation between the free parameters β and γ is satisfied. Proceeding as above in Eq. (6) we find that $u(z)$ is the solution of

$$e^{-(b+1)\,a_1\,z} = \frac{u^{1+b}}{(1-u)(1+bu)^b}. \tag{11}$$

Writing $b = n/p$ the equation for u is

$$e^{-(n+p)\,a_1\,z} = \frac{u^{n+p}}{(1-u)^p(1+bu)^n}. \tag{12}$$

This can be inverted to obtain the explicit solution for $u(z)$ if $n + p = 2, 3, 4$. The detailed inversion of all the solvable cases is not instructive, here we give one example. Choose $n = 2, p = 1$, then $b = 2$, and the front is a solution of the cubic equation

$$u^3(1 + 4e^{-3a_1 z}) - 3ue^{-3a_1 z} - e^{-3a_1 z} = 0. \tag{13}$$

This cubic has two complex roots and a single real positive root which is the desired front, given by

$$u(z) = \frac{2^{\frac{1}{3}}}{\sqrt{4 + e^{3a_1 z}}\left(e^{\frac{3a_1 z}{2}} + \sqrt{4 + e^{3a_1 z}}\right)^{\frac{1}{3}}} + \frac{\left(e^{\frac{3a_1 z}{2}} + \sqrt{4 + e^{3a_1 z}}\right)^{\frac{1}{3}}}{2^{\frac{1}{3}}\sqrt{4 + e^{3a_1 z}}}. \tag{14}$$

Again this is an exact front for f of the form given by Eq. (8). It corresponds to a strongly heteroclinic connection for $\lambda < 2$. If one chooses the case $n + p = 4$ the quartic equation that arises has a pair of complex conjugate solutions, a negative solution and a positive solution which is the desired front. For values of b which do not allow the explicit computation of the front $u(z)$ we still have the speed selection criteria in terms of the free parameters of the polynomial. The results above are also valid in the subcritical case $f'(0) < 0$ if $0 < \lambda < 1$. The integral of Eq. (10) between 0 and 1 is $(b+2)\lambda f'(0)/12(\lambda - 1)$ which is positive when $f'(0) < 0$ for $0 < \lambda < 1$ and, therefore, all results hold in this range.

Closed form solutions $v(u)$ for polynomial $f's$ can be obtained only if f is an odd polynomial. In general, if f is a polynomial of degree $2k + 1$ that vanishes at 0 and 1, there are $2k$ free parameters (restricted only by the requirement of positivity of f), whereas the corresponding closed form solution for v has $k+1$ parameters, which implies that a closed form for v, and an explicit expression for λ is possible if $k - 1$ parameters of f are chosen adequately in terms of the $k + 1$ remaining free parameters.

IV. CONCLUSION

We have studied the existence of exact strongly heteroclinic fronts for the reaction diffusion with quintic nonlinearities. We find that the use of phase space enables one to characterize the transition from strongly heteroclinic to simple nongeneric fronts in terms of a single parameter λ which is the ratio between the speed and the rate of decay at infinity. The introduction of this parameter gives a unified way in which to describe the type of solution that is independent of the nature of the nonlinearities. The exact value of this parameter cannot be determined analytically when the highest nonlinearity is even, if the highest derivatives are odd it can be determined for special choices of parameters. In the case studied here, quintic nonlinearities, the value of λ can be determined exactly if a special relation betwen the parameters of the equation is satisfied. It is not necessary to know the exact solution $u(x - ct)$ in order to determine whether a strong heteroclinic connection exists. If an additional restriction on the parameters is imposed, exact solutions can be found. We have illustrated this situation for one particular choice, a whole family of exact solutions can be constructed. The use of phase space is not only useful as an aid to find exact solutions, it can be used to obtain a lower bound on the speed, valid for all f, which allows one to determine the range of parameters for which strongly heteroclinic connections exist [13].

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