



PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE

ESCUELA DE INGENIERÍA

EXTENSIONS OF FUNDAMENTAL HUB LOCATION MODELS

ARMIN MAURICIO LÜER VILLAGRA

Thesis submitted to the Office of Research and Graduate Studies in partial fulfillment of the requirements for the Degree of Doctor in Engineering Sciences

Advisor:

VLADIMIR MARIANOV KLUGE

Santiago de Chile, October, 2015

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To my parents Armin and Nelly.

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EXTENSIONES DE LOS MODELOS DE LOCALIZACIÓN DE HUBS

Tesis enviada a la Dirección de Investigación y Postgrado en cumplimiento parcial de los requisitos para el grado de Doctor en Ciencias de la Ingeniería.

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RESUMEN

Esta investigación está enfocada en la formulación y resolución eficiente de extensiones de los Modelos Fundamentales para Problemas de Localización de Hubs (Fundamental HLPs, en inglés), que se sabe pertenecen a NP-hard para los casos no-triviales.

Los HLPs buscan localizar un tipo de instalaciones conocidas como hubs, donde los flujos desde múltiples pares Origen-Destino (OD pairs, en inglés) son consolidados, ordenados y conmutados, obteniéndose la topología hub-and-spoke, comúnmente utilizada en la aviación comercial, en la entrega postal y de encomiendas, en sistemas de transporte público, etc. Los modelos fundamentales asumen que: (i) todas las rutas OD pasan por uno o dos hubs, (ii) la red entre hubs es completa, (iii) la compañía que localiza sus hubs es monopolista y su demanda es inelástica, y (iv) se aplica un factor de descuento constante sólo a los flujos entre hubs.

El principal objetivo de esta tesis es extender los Modelos Fundamentales. Para esto, se usan tres enfoques diferentes.

Primero, relajando los supuestos (i), (ii) y (iii), se formula un problema competitivo de localización de hubs y fijación de precios, donde una compañía existente opera una red hub-and-spoke que cobra un margen porcentual fijo por sus servicios de transporte y una

nueva compañía debe diseñar su propia red hub-and-spoke para maximizar su beneficio. El comportamiento de los usuarios es modelado mediante un modelo logit simple. Se obtiene una expresión cerrada para los precios óptimos que debe cobrar el entrante, si los diseños de ambas redes están fijos. Se resuelve el problema mediante un algoritmo genético. Se muestra la pertinencia de la maximización del beneficio como un objetivo para localizar competitivamente hubs y la relevancia de considerar simultáneamente la competencia y la fijación de precios en la localización de hubs.

Segundo, relajando los supuestos (i), (ii) y (iv) se desarrolla un esquema de modelamiento para ayudar en la localización de hubs a los tomadores de decisiones. Se formula un modelo de programación matemática que es capaz de representar economías de escala. Se usan indicadores claves de desempeño agregados (KPIs, en inglés) para analizar las soluciones obtenidas, mostrando la pertinencia del enfoque y que las soluciones obtenidas son correctas.

Finalmente, se relajan nuevamente los supuestos (i), (ii) y (iv), para desarrollar un HLP donde debe localizarse un número fijo de hubs, realizando asignación única, y donde el flujo en un arco es descontado sólo si éste excede un umbral predefinido. Se formula como un problema entero mixto (MIP, en inglés), y se resuelve utilizando software típico de Programación Matemática. También se desarrolla un procedimiento heurístico para resolver más rápidamente las instancias de prueba. Se muestra la pertinencia del enfoque, así como el desempeño, tanto del modelo exacto como del procedimiento heurístico.

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ABSTRACT

This research is focused on the formulation and efficient solution of extensions of Fundamental Hub Location Problems (Fundamental HLPs), which belong to NP-hard for the non-trivial cases.

HLPs aim at locating facilities known as hubs, in which the flows from multiple Origin-Destination (OD) pairs are consolidated, sorted and commuted, leading to the hub-and-spoke topology, commonly used, among others, in commercial aviation, parcel and courier delivery, and public transportation systems. Fundamental models assume that: (i) all the OD routes visit one or two hubs, (ii) the inter-hub network is complete, (iii) the company locating hubs is monopolistic and its demand is inelastic, and (iv) a constant discount factor is applied only to the flows between hubs.

The main objective of this thesis is to extend the fundamental HLPs. We use three different approaches.

Firstly, relaxing assumptions (i), (ii) and (iii), we formulate a competitive hub location and pricing problem, where an existing company operates a hub-and-spoke network and applies a fixed percentage of markup to their transportation services, and a newcomer

designs its own hub-and-spoke network in order to maximize its profit. The users' behavior is modeled using a simple logit model. We derive a closed expression for the optimal pricing, when both network topologies are fixed. We solve the problem using a genetic algorithm. Finally, we show the pertinence of profit maximization as a competitive hub location objective, and the relevance of considering simultaneously competition and pricing in hub location.

Secondly, we develop a modeling framework to help decision-makers to locate hubs. We formulate a mathematical model that is able to represent economies of scale, relaxing assumptions (i), (ii) and (iv). We use aggregate Key Performance Indicators (KPIs) to analyze the solutions obtained, showing the pertinence of our approach and the accuracy of the solutions obtained.

Finally, we again relax assumptions (i), (ii) and (iv) and develop a single-allocation p -HLP in which the flow in any arc is discounted if it exceeds a predefined threshold. We formulate it as a Mixed-Integer Problem (MIP), and solve the model using standard mathematical programming software. In order to solve the test instances faster, we also develop a heuristic procedure. We show the pertinence of our assumptions, and the computational tractability of our exact model and heuristic procedure.

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1. INTRODUCTION

This thesis aims at modeling and solving extensions of the Fundamental Hub Location Problems, and more specifically, three problems concerning competition and pricing, more general cost structures, and threshold-based discounts.

Hub Location is a young research field within Location Analysis that began with the seminal papers in which the general problem is stated (O’Kelly, 1986); formulated as a quadratic integer problem (O’Kelly, 1987); and then linearized (Campbell, 1994). Its goal is to locate a special kind of facilities, called hubs, in which flows from multiple origins and destinations are consolidated, sorted and commuted. Hub-and-spoke networks are designed by solving Hub Location Problems (HLP), and are mainly used, among others, in commercial aviation, parcel and courier delivery services, and public transportation systems. Hub Location has attracted much attention lately, as shown by multiple recent literature reviews (Campbell et al., 2002; Alumur & Kara, 2008; Kara & Taner, 2011; Campbell & O’Kelly, 2012; Farahani et al., 2013).

Hub Location Problems (HLPs) can be classified according to their objective function in: median, in which the sum of distances or travel costs are minimized; covering, in which captured demand is maximized, given a capture distance; center, in which the maximum of a set of functions of the involved distances between the hub-and-spoke network and the OD pairs is minimized; and problems with fixed costs, where the number of hubs is not preset, but left to the model to decide (Campbell, 1994).

In order to make the models computationally tractable, the early models in the literature - ‘Fundamental Models’ herein - have the following assumptions:

- The routes between every Origin-Destination (OD) pair go through one or two hubs.
- The inter-hub network is complete.

- The company locating their hubs is monopolistic and the demand is inelastic.
- A constant discount factor is applied to the flows between hubs.

After the initial contributions, the literature has focused on relaxing some of these assumptions, and has led to four main research lines in hub location: competitive models, development of efficient solution approaches to fundamental models, the study of cost structures in hub-and-spoke networks and their extensions, and the use of different inter-hub network topologies.

1.1. Competitive models

Competition has been studied in hub location since Marianov et al. (1999), in which the authors proposed different models for the case in which a newcomer must locate its hubs, aiming at maximizing the traffic capture. An OD pair traffic is captured if the offered cost is lower than the costs of the existing firm. In these models, all the traffic between an origin and a destination (OD pair) is captured by the cheapest option, leading to integer linear models. The direct extension to gradual capture was developed by Eiselt & Marianov (2009), using gravity functions for the utility, and assuming that the users decide what airline to use depending on travel time and cost. The resulting model is integer non-linear, and it is solved using heuristic concentration. Following this line of work, Gelareh et al. (2010) formulated the problem solved by a new liner service provider who wants to maximize its market share, depending on both time and cost. The authors use Lagrangean methods to solve the test instances. Note that this approach does not consider the response of the existing companies.

A natural extension is to use Game Theory to locate hubs. This line of work began with Sasaki & Fukushima (2001), who stated the Stackelberg hub location problem. Under this approach, an existing company (incumbent) competes with a

set of entrants, maximizing its profit, using one hub in every route. The extension to multiple hubs in a route was done by Sasaki et al. (2014), in which two agents locate arcs to maximize their revenues, allowing up to two hubs in every route.

To the best of our knowledge, including pricing in competitive hub location problems has not been studied in the literature. The material in Chapter 2 contributes along this direction. In that chapter, a competitive hub location and pricing problem is formulated and solved, in which an existing company, the incumbent, is operating a hub-and-spoke network using mill pricing, i.e. charging a fixed margin over its costs. A new company, the entrant, has to decide hub locations, network design and strategic pricing to maximize its profit. We used a logit model to address the gradual capture of traffic based on price, and a genetic algorithm to solve the proposed instances (Lüer-Villagra & Marianov, 2013). We also showed that the proposed profit maximization is an appropriate objective for competitive hub location problems, and we derived a closed expression for the optimal pricing if both network designs are fixed.

1.2. Efficient solution approaches to the fundamental models

Besides the development of extensions to the fundamental models, new solution approaches have been devised. Their main goal is to solve larger instances in a reasonable amount of time, or faster compared to a direct implementation. The work along this line has been done through the development of tighter and smaller formulations, and the use of decomposition techniques both on fundamental model and their extensions.

The usual ways to strengthen mathematical programming formulations are reformulation and addition of cutting planes. Reformulation has been used since the beginning of hub location research. For example, a formulation by Campbell

(1994) was strengthened by Skorin-Kapov et al. (1996). Also, it was transformed into flow-based formulations by Ernst & Krishnamoorthy, both for the single allocation (Ernst & Krishnamoorthy, 1996) and the multiple allocation (Ernst & Krishnamoorthy, 1998) cases. Further work has been done in the development of tighter and smaller formulations. For example, Marín et al. (2006) developed new formulations for the multiple allocation p -hub median problem, generalizing the previous ones. Based on its structure, the authors proposed tighter constraints together with a preprocessing procedure, which allowed shorter solution times. Later, García et al. (2012) developed a new formulation for the same problem, together with an ad-hoc branch-and-cut procedure. Their procedures allowed solving larger instances, especially if the number of hubs to be located is large. Following a different approach, Hamacher et al. (2004) compared the uncapacitated single allocation hub location problem polyhedron with the uncapacitated facility location problem, in order to derive a new formulation for the hub location problem only with facet-defining constraints.

Given that most of the hub location problems can be formulated as multi-commodity flow problems with special constraints, the use of decomposition approaches appears as very natural. For example, Benders Decomposition (Benders, 1962) has been applied both to the single-allocation fundamental model (Contreras et al., 2011), and its extensions (de Camargo, Miranda Jr., & Luna, 2008; Rodríguez-Martín & Salazar-González, 2008; de Camargo, Miranda Jr., Ferreira, & Luna, 2009; de Camargo, de Miranda, & Luna, 2009; de Sá, de Camargo, & de Miranda, 2013).

1.3. Cost structures for hub-and-spoke networks

The design and use of hub-and-spoke networks is motivated by economies of scale, i.e. decreasing average unit costs as the amount of flow transported

increases. This is achieved by consolidation of flow from multiple OD pairs. In the fundamental models, economies of scale are modeled in the cost structure by applying a fixed discount to the flow between hubs. However, this discount is independent of the amount of flow. This approach, although computationally appealing, does not represent adequately the economies of scale in hub-of-spoke networks.

Several extensions to fundamental models have been proposed in order to achieve an improved representation of the economies of scale. These extensions consist of changes in the cost structure of the models, and can be classified in: threshold-based discounts, linear-piecewise cost functions, and cost structures with fixed costs.

The inclusion of flow thresholds in HLPs began with Campbell (1994). He used minimum flow thresholds for spoke enabling, adding a fixed cost in the objective function. The relative value of the minimum flow thresholds and the fixed costs allowed him to parameterize its model from single to multiple allocation. Later, Podnar et al. (2002) developed a network design problem, i.e. the model does not locate hubs, and any arc can be discounted if its flow exceeds a fixed threshold.

An alternative approach has been the use of piecewise-linear functions to approximate the usually non-linear nature of economies of scale. Its use in HLPs began with O'Kelly & Bryan (1998) and their FLOWLOC model. The authors stated a non-linear flow discount function for the inter-hub arcs, noting that its use increased the flow consolidation between hubs, compared to the fundamental models. Later, Bryan (1998) extended the FLOWLOC model to allow capacity constraints, require minimum thresholds to enable arcs, relax the fixed number of hubs assumption, and apply flow-dependent discounts everywhere in the hub-and-spoke network. Klincewicz, (2002) proposed a numerical procedure to solve the FLOWLOC model, based on the fact that if the hub locations are fixed, the

resulting model is an instance of the uncapacitated facility location problem. The author proposed a complete enumeration procedure, together with tabu-search and GRASP-based heuristics. Note that the resulting models are quite hard to solve compared to the fundamental models, and tighter formulations need to be solved using more sophisticated techniques, as de Camargo et al. (2009) do, for example.

Finally, fixed costs have been used before in HLPs as proxies to model thresholds or similar structures and not capacitated vehicles utilization. To the best of our knowledge, the first hub location problem with a cost structure with fixed costs was proposed by Kimms (2006). The author modeled the problem as a location and network design problem, where the arcs are traversed by capacitated vehicles having both fixed and variable costs.

In Chapter 3 we propose a modeling framework for hub location that correctly represents the economies of scale present in practice, using a cost structure with fixed costs. As opposed to explicitly solving a hub location problem and assuming an a priori existence of economies of scale as many models do, we formulate a Location-Network Design Model as a Mixed-Integer Problem (MIP). In this model, a company must locate its management and maintenance resources at existing airports (which become hubs), together with defining routes and allocating capacity both on arcs (airplanes), and nodes (airports), minimizing costs, subject to a constraint on the aggregated level of service. The decision process is guided by a set of Key Performance Indicators (KPIs) from the airline industry. We begin analyzing both airline economics and hub location models. Then, we describe the framework and the model to finally show how choosing different operational constraints influences both the network structure and KPIs.

In Chapter 4 we present a single allocation, incomplete inter-hub network, p -hub location problem in which a fixed unit cost discount is applied to the flow in an

arc if it exceeds a fixed threshold. We used standard mathematical programming software to solve to optimality the resulting models for literature instances, and a heuristic procedure to get good feasible solutions. KPIs are used to analyze and compare solutions. Our results show that our model, based on the fundamental model for the p -hub, single allocation problem, is able to represent the existence of economies of scale. It also requires a reasonable computational effort, tending to consolidate flows between hubs, and it can be efficiently solved by the proposed heuristic procedure.

1.4. Use of different inter-hub network topologies

The fundamental models assume that the inter-hub network is complete. This is a reasonable assumption if enabling an arc is relatively inexpensive, or if the network flows are large enough everywhere in the network. Also, it is computationally appealing, since the implied additional constraints provide a stronger lower bound, if integer programming is used to formulate and solve the models.

Some authors have relaxed this assumption. The literature follows three lines: tree-shaped, incomplete, and general inter-hub networks.

To the best of our knowledge, the first authors that considered a tree-shaped inter-hub network were Contreras et al. (2010). They stated the Tree of Hubs Location Problem, which is particularly relevant in telecommunication applications, where the set-up cost of inter-hub arcs is higher. The authors also provide valid inequalities and an exact separation algorithm for them. After that, de Sá et al. (2013b) developed a Benders Decomposition for the problem, devising a new cut selection scheme, leading to a procedure that outperforms

other implementations, both in computational time and maximum instance size solved up to optimality.

There are previous works about logistic systems where the consolidation points are not fully connected. However, to the best of our knowledge, the first HLP-oriented literature review about incomplete inter-hub networks is the work by O’Kelly & Miller (1994), where the authors first reviewed the literature based on the number of hubs (one or multiple); the space in which the problem is stated (planar or discrete); the design objective (min-sum, mini-max); and the problem characteristics. All these works share the following assumptions: complete inter-hub network, single allocation, and forbidden inter-non-hub connections. Secondly, they reviewed the literature in which these assumptions have been relaxed, and proposed a classification system for hub networks, based on the node-hub assignment, the (dis)allowance of inter-non-hub connections and the inter-hub connectivity, concluding the existence of several lines of work.

More recently, Hub-Arc Location Problems (HALPs) were introduced by Campbell et al. (2005a,b). In HALPs, a predefined number of hub arcs must be located. Note that a complete inter-hub network is implied only if the budget is large enough. From a different point of view, Alumur et al. (2009) stated and formulated the single-allocation incomplete hub network design problem, both for cost minimization and covering objectives. Their focus was the efficient solution through tight formulations and the addition of valid inequalities. Along this line, Calik et al. (2009) developed a tabu-search based heuristic for the hub covering problem presented later, showing its efficiency in literature instances.

A relatively unexplored line of work is the use of more general topologies for the inter-hub network. The Fundamental HLPs enforce that all the flow must go through one or at most two hubs. In the passenger transportation case, the inclusion of more hubs in a route implies inconvenience from the user

perspective, because every additional hub in the route implies delays, congestion and additional travel time. In freight transportation, however this is not an issue, if the transfers at the hubs are done efficiently, as Lüer-Villagra et al. (2015) pointed out.

Finally, some authors has been focused in generalize hub location and network design models, as Contreras & Fernández (2012, 2014) did.

In this thesis, all the developed models allow incomplete inter-hub networks, extending the fundamental models. Additionally, the modeling framework developed in Chapter 3, and the model with flow threshold-based discounts presented in Chapter 4, allow the use of more than two hubs in a route.

1.5. Thesis contributions

In synthesis, the main scientific contributions of this thesis are the following:

First, we formulate and solve efficiently a competitive hub location and pricing, for a leader-follower situation, deriving a closed expression for the optimal pricing if the networks of both agents are fixed. It is also the first paper that deals with hub location and pricing simultaneously.

Secondly, we develop a modeling framework for hub location that does represent economies of scale in a general way everywhere in the network. We analyze the solutions using aggregated performance indicators.

Third and finally, we model and solve a hub location problem with flow thresholds in the network, allowing discounting the traffic on any arc, if it is large enough. We show that this model also represents economies of scale, tends to consolidate flows between hubs, and is computational tractable, compared with other extensions of fundamental models.

The remainder of this thesis is organized as follows. Chapter 2 contains the paper “*A Competitive Hub Location and Pricing Problem*”, published in the European Journal of Operational Research. Chapter 3 presents the paper “*A modeling framework for strategic airline network design*”, submitted to Computers & Operations Research. Chapter 4 presents the paper “*On single-allocation p -hub median location problems with flow threshold-based discounts and economies of scale*”, to be submitted.

2. A COMPETITIVE HUB LOCATION AND PRICING PROBLEM

We formulate and solve a new hub location and pricing problem, describing a situation in which an existing transportation company operates a hub-and-spoke network, and a new company wants to enter into the same market, using an incomplete hub-and-spoke network. The entrant maximizes its profit by choosing the best hub locations and network topology and applying optimal pricing, considering that the existing company applies mill pricing. Customers' behavior is modeled using a logit discrete choice model. We solve instances derived from the CAB dataset using a genetic algorithm and a closed expression for the optimal pricing. Our model confirms that, in competitive settings, seeking the largest market share is dominated by profit maximization. We also describe some conditions under which it is not convenient for the entrant to enter the market.

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2.1. Introduction

Most air passenger transportation and package delivery companies have chosen the hub-and-spoke topology for their networks (Gelareh & Pisinger, 2011). This topology makes use of transshipment and flow consolidation facilities called hubs, which significantly reduces the number of routes required to connect all origins and destinations in a region. It also allows taking advantage of any existing economies of scale, by consolidating traffic in inter-hub transportation and on the spokes (arcs that connect hub nodes to non-hub nodes), as compared to a point to point network. Bigger and more efficient vehicles are used on high traffic route segments, and there is higher asset utilization throughout the network.

The first model for the optimal design of hub networks (the Hub Location Problem) was introduced by O’Kelly (1986) and first formulated as an optimization problem by O’Kelly (1987). The literature about hub problems is now extensive. Hub location problems are classified the same way as facility location problems are (Campbell, 1994): median, covering, center and fixed costs problems. Complete reviews of hub location problems can be found in Campbell et al. (2002), Alumur & Kara, (2008), Kara & Taner (2011), Campbell & O’Kelly (2012), Farahani et al. (2013).

Current trends in hub location include the development of new formulations that allow obtaining good or even optimal solutions in less time for larger instances of the problems. The work along this line has explored the use of polyhedral properties of the formulations, as in Hamacher et al. (2004) or the development of tighter and smaller formulations, (Marín et al., 2006; García et al., 2012). From a different viewpoint, Contreras & Fernández (2012) have proposed a unified view, formulations and algorithmic insights of location and network

design problems, including the hub location problems as a special case. Also, solution methods like Benders Decomposition (de Camargo et al., 2008), and Branch and Price (Contreras et al., 2010), have been proposed.

Several extensions of the original problems have been used successfully. Congestion has been considered by constraining queue length at hubs (Marianov & Serra, 2003; Mohammadi, Jolai, & Rostami, 2011), as well as by adding a non-linear term in the objective and solving the problem either using Lagrangean methods (Elhedhli & Wu, 2010), or evolutionary algorithms, as in Köksalan & Soylu (2010).

In regard to economies of scale, particularly interesting and relevant to all the research in hub location is the observation by Campbell (2012, 2013). Through the analysis of a very extensive set of cases, he found that the fundamental hub location models share the following problem: depending on the origin-destination flows, it could happen that the traffic between some hubs is too small for making use of economies of scale, and conversely, the traffic on spokes could be large enough to apply a discounted cost. This shortcoming was also pointed out by Bryan (1998), O'Kelly & Bryan (1998), and de Camargo et al. (2009). The fundamental hub location models apply a fixed, flow independent discount factor to all inter-hub arcs, and they do not apply any discount on high-traffic spokes. Further, the fundamental hub location models have a fully connected network of discounted arcs between all hubs.

Addressing this issue should become a hot research topic, and some better representations of economies of scale have already been proposed by approximating the non-linear inter-hub discount function with a piece-wise linear function (Bryan, 1998; Kimms, 2006; O'Kelly & Bryan, 1998); by using incomplete inter-hub networks (Alumur et al., 2009; Calik et al., 2009; Contreras et al., 2010a); using hub-arc models (Campbell et al., 2005a, 2005b; Sasaki et al.,

2014), and by forcing a minimum flow on inter-hub links (Podnar et al. 2002, Campbell et al., 2005a, 2005b). Currently, however, most of the researchers use the fundamental approach of discounting the flow between hubs, independent of its magnitude, (Campbell & O’Kelly, 2012; Farahani et al., 2013), mainly because of the computational attractiveness of such approach, and the fact that the search for a completely successful model is still open. Among these, we use a model in which a constant (flow-independent) discount between hubs and no discount on spokes are considered, and an incomplete inter-hub network is allowed. Although all these models tend to improve the application of economies of scale, they still do not completely solve the problem. We do not use hub-arc models, because they do not apply economies of scale on spokes with large flows, and they tend to locate a number of hubs that is very large, in times disproportionate for the airline industry (Campbell, 2009). Furthermore, deriving a closed form expression for both piecewise linearization models and models that require a minimum flow on inter-hub arcs would require an additional level of iteration of the procedure in this paper, because the cost and existence of different routes depends on the amount of the predicted flow, making the problem close to intractable. Also, piecewise linearization models are more complicated in terms of number of variables and constraints.

Competition between firms that use hub networks has been studied mainly from a sequential location approach, in which an existing firm, called the incumbent or leader, serves the demand in a region, and a new firm, the entrant or follower, wants to enter the market. In the first article on competitive hub location, Marianov et al., (1999) model a situation in which the entrant captures a flow if its costs are lower than those of the incumbent’s. This approach was extended to gradual capture by Eiselt & Marianov (2009). A related line of research was followed by Gelareh et al. (2010), where the newcoming company is a liner service provider that maximizes its market share, depending both on service time and transportation cost. The formulation is very hard to solve ‘as is’, and a

specialized Lagrangean method is used. Using a game theoretical approach, Sasaki & Fukushima (2001) state the Stackelberg hub location problem, in which the incumbent competes with several entrants to maximize its profit. Only one hub is considered in every origin-destination route. Later, Adler & Smilowitz (2007) introduce a framework to decide the convenience of merging airlines or creating alliances, using a game theory based approach. More recently Sasaki et al. (2014) propose a problem in which two agents locate arcs in order to maximize their respective revenues under the Stackelberg framework, allowing more than one hub in a route.

Dobson & Lederer (1993) propose the problem of maximizing profit of an airline for a network with only one hub, given a discrete consumer density as a function of departure time, duration and price of the route to be travelled. This is an operational problem, not including location decisions. Simultaneous location and pricing problems have been proposed and solved by Serra & ReVelle (1999). To the best of our knowledge, there is no literature on hub location problems explicitly including pricing and location decisions. We study a competitive problem, including discrete choice between routes, using a hub location model with incomplete hub-connectivity.

We propose a novel hub location problem, called the Competitive Hub Location and Pricing Problem (CHLPP). An existing company (or group of companies), called the incumbent, utilizes a transportation network with a hub-and-spoke topology, and charges its costs plus a fixed additional percentage to their customers (mill pricing). A new company, the entrant, wants to offer its services in the same market, using its own hub-and-spoke network and setting prices so to maximize its profit, rather than its market share –a cherry-picking strategy. The profit comes from the revenues because of captured flows, subtracting the fixed and variable costs. Both the incumbent and the newcomer offer several routes. Customers choose which company and route to patronize by price, and their

decision process is modeled using a logit model. The question to be answered is: Can a newcomer obtain profit under these conditions, even with higher operating costs than the incumbent? In order to answer this question, our procedure finds how many hubs to locate, where should they be located, what is the best route network, and the optimal price of the services.

The contributions of this paper are as follows. In the first place, we formulate a hub problem including aspects that were never taken into consideration together, as the optimal pricing decision and a discrete choice by customers. Secondly, we derive a closed form expression for the optimal pricing. Third, we solve the non-linear problem using a genetic algorithm. Finally, we make an extensive analysis of the scenarios that a newcoming company would face, and the best actions it could take, when the objective is profit maximization –as opposed to cost minimization or market share maximization.

Note that hub location decisions are strategic, while pricing decisions are tactical or even operational. Linking these two levels may seem unusual at first sight. However, location or route opening decisions -or even entrance into a market- can be very dependent on the revenues that a company can obtain by operating these locations and routes. Revenues, in turn, depend on the pricing structure and on the competitive context. In other words, without consideration of the feasible range of prices that the entrant can charge, it is difficult to make good location decisions, and we explore here the relationship between both. Once the firm is established, revenue management techniques can be applied to decide on the day to day prices.

The proposed model is applied to the air passenger transportation industry. However, with slight changes in the discrete choice model, it can be applied to mail and freight transportation industries, or any other industry that benefits from a hub-and-spoke network structure.

The remainder of this chapter is organized as follows. Section 2.2 describes the problem and the mathematical model. Section 2.3 describes the genetic algorithm. Section 2.4 presents the computational results using the CAB dataset. Finally, in section 2.5 we provide general conclusions.

2.2. A Competitive Hub Location and Pricing Problem

Air passenger traffic in a region is served by an existing company (or a set of companies already established in the market, collectively), called the incumbent, that utilizes a transportation network with a hub-and-spoke topology. We make the assumption, customary in fundamental hub location models, that there are reduced transportation costs (due to economies of scale) in the traffic between hubs, and not on spokes, and the discount factors are constant. We assume that all the incumbent's hubs are connected, although full interconnection is not required for the entrant's inter-hub network. The incumbent uses mill pricing, i.e., charges its costs plus a fixed profit percentage. The incumbent's hubs are located optimally for cost minimization when serving all the demand, though the incumbent may end up serving less than that after the entrant arrives. A new company, the entrant, intends to enter the same market, using its own hub-and-spoke network and setting prices so to maximize its profit, rather than its market share, i.e., a cherry-picking strategy. The entrant does not share hubs with the incumbent, but could use the same locations (cities) for sitting them. The profit is equal to the revenues from captured flows, once fixed and variable costs are subtracted. Both the incumbent and the newcomer may offer several routes between origins and destinations in the region, i.e. an origin-destination pair may be served by more than one route belonging to the same company. Customers choose which company and route to patronize by price, although the model could trivially accommodate other attributes as travel time or number of legs. Customers' decision process is modeled using a logit model. The logit model is

well validated in the transportation literature (see Ortúzar & Willumsen, 2011). Logit models are currently the most popular models for representing discrete choice, because they provide a closed form expression and because they can accommodate several different attributes of the alternatives as cost, waiting time, travel time, and so on. Logit models serve well in the case of passengers and multiple routes. If mail or package service is to be represented, then, rather than choosing among multiple routes, customers choose among several providers. Again, a situation that can be represented using logit models.

The problem is defined over a graph $G = G(N, A)$, where N is the set of nodes and A is the set of arcs. Each arc has a fixed cost, K_{ij} , and a variable cost c_{ij} per unit of flow. For the formulation we assume that both the incumbent and the entrant have the same arc costs, but this assumption can be trivially relaxed. To model inter-hub discounts, let χ , α and δ be the discount factors due to flow consolidation in collection (origin to hub), transfer (between hubs) and distribution (hub to destination), respectively. Let F_k be the cost of locating a hub at node $k \in N$, and W_{ij} is the given inelastic demand, in terms of the flow to be transported from origin node $i \in N$ to destination node $j \in N$. All demand is served by either the incumbent or the entrant. The percentage over the cost charged by the incumbent is Δ . This percentage could be easily made different for different arcs or competitors. The logit model has a known sensitivity parameter Θ . Higher values of Θ mean that customers are very sensitive to price and they will mostly choose less expensive routes. Smaller values of Θ mean that the customers are less sensitive to price (or price differences), and there will be a higher customers' spread among the different routes. For further details on logit models, see Ortúzar & Willumsen (2011). Finally, P is the set of nodes where the incumbent's hubs are located. The proposed model is the following.

$$Z = \max \sum_{i,j,k,m \in N} (p_{ijkm} - c_{ijkm}) W_{ij} X_{ijkm} - \sum_{(i,j) \in A} K_{ij} H_{ij} - \sum_{k \in N} F_k Y_k \quad (2.1)$$

$$\sum_{k,m \in N} X_{ijkm} + \sum_{k,m \in P} Z_{ijkm} = 1, \forall i, j \in N \quad (2.2)$$

$$X_{ijkm} = \frac{Y_k Y_m H_{ik} H_{km} H_{mj} \exp(-\Theta p_{ijkm})}{\left(\sum_{s,t \in N} Y_s Y_t H_{is} H_{st} H_{tj} \exp(-\Theta p_{ijst}) \right) + \eta_{ij}}, \forall i, j, k, m \in N \quad (2.3)$$

$$Z_{ijkm} = \frac{\exp(-\Theta \bar{P}_{ijkm})}{\left(\sum_{s,t \in N} Y_s Y_t H_{is} H_{st} H_{tj} \exp(-\Theta p_{ijst}) \right) + \eta_{ij}}, \forall i, j, k, m \in N \quad (2.4)$$

$$\bar{P}_{ijst} = (1 + \Delta) c_{ijst}, \forall i, j, s, t \in N \quad (2.5)$$

$$c_{ijkm} = \chi \cdot c_{ik} + \alpha \cdot c_{km} + \delta \cdot c_{mj}, \forall i, j, k, m \in N \quad (2.6)$$

$$\eta_{ij} = \sum_{s,t \in P} \exp(-\Theta \cdot \bar{P}_{ijst}), \forall i, j \in N \quad (2.7)$$

$$Y_k \in \{0, 1\}, \forall k \in N \quad (2.8)$$

$$H_{ij} \in \{0, 1\}, \forall (i, j) \in A \quad (2.9)$$

$$p_{ijkm} \geq 0, \forall i, j, k, m \in N \quad (2.10)$$

Where,

- X_{ijkm} is the fraction of the flow going from $i \in N$ to $j \in N$ through entrants's hubs located at $k, m \in N$.
- Z_{ijkm} is the fraction of the flow going from $i \in N$ to $j \in N$ through incumbent's hubs located at $k, m \in P$.

- $Y_k = 1$, if the entrant locates a hub at node $k \in N$; 0 otherwise.
- $H_{ij} = 1$, if the entrant establishes a direct connection between nodes $i, j \in N : (i, j) \in A$; 0 otherwise.
- c_{ijkm} is the variable cost of the flow between nodes i and $j \in N$, using hubs $k, m \in N$.
- p_{ijkm} is the price charged by the entrant to flows between nodes i and $j \in N$, using intermediate hubs $k, m \in N$.
- $\bar{p}_{ijkm}$ is the price charged by the incumbent to flows between nodes i and $j \in N$, using intermediate hubs $k, m \in N$.

The objective function (2.1) maximizes the entrant's profit, i.e. the net revenue minus the fixed and variable costs. Constraints (2.2) ensure that the flow between nodes $i, j \in N$ is routed through entrant's or incumbent's hubs. Constraints (2.3) and (2.4) assign the flows according to a logit model whose argument are the prices charged by the entrant or the incumbent, respectively. Constraints (2.5) define incumbent's mill pricing strategy, while (2.6) is the definition of the transportation costs over a route $i \rightarrow k \rightarrow m \rightarrow j$. Equations (2.7) define the parameters η_{ij} . Finally, (2.8)-(2.10) state the domain of the decision variables.

2.3. Solution approach

The resulting model is a non-linear mixed integer programming problem. Unfortunately, although the objective might be concave with respect to price, we cannot assure the convexity or concavity of the objective or the constraints with respect to all the variables. For this reason, we cannot guarantee that current

commercial software packages for integer programming would find the optimal solution. Furthermore, the size of real instances of the problem is too large for any exact procedure, because of the 4-index formulation required to make the pricing of every route offered by any agent.

Consequently, we propose using an ad-hoc metaheuristic that, at each step, finds feasible solutions for the location-network design problem and, for each such solution, solves a pricing problem. Given that the location-network design search space includes only binary variables, any metaheuristic able to solve combinatorial problems could be used. However, in this case, any regular metaheuristic would require evaluating the objective at each step and for every solution in the neighborhood of the current solution, which would make the problem computationally intensive and the progress towards finding a solution extremely slow. We chose a genetic algorithm because of several reasons: it does not require local search procedures, as the genetic operators help the algorithm to explore the solution space; solutions can be represented easily; and genetic algorithms have had good success in previous applications involving hub location problems (Topcuoglu et al., 2005; Cunha & Silva, 2007; Kratica et al., 2007). Genetic algorithms have been proven to show an optimizing behavior. See, for example, Rudolph (1994). The proposed approach can be stated as follows: the genetic algorithm explores the space of hub locations and connecting arcs, and finds feasible solutions. From every solution, a valid hub-and-spoke network configuration is derived. Once a valid configuration is found, the pricing problem is solved for this configuration, and the optimal flows and prices are found, for that network configuration. The flows captured and priced by the entrant are used to compute the value of the objective function, after discounting the network costs.

2.3.1. Genetic algorithm

First, a population of n_{pop} random feasible solutions, i.e. valid hub-and-spoke networks, is created and saved in a solution set S . Then, on every iteration, two solutions, called parents, are selected randomly from S . A crossover operator is applied to parents, generating two solutions called offsprings. With probability p_m the algorithm mutates an offspring, favoring population diversity. The objective function is computed and, finally, an offspring is accepted into the set S only if it is better, in terms of the objective function, than the worst solution in S . The algorithm iterates until a stopping condition is met.

The remainder of this section describes the components of the genetic algorithm.

Solution representation

The solution representation is a key issue in the performance of a genetic algorithm. A solution to the location and network design problem can be defined using two elements: a binary vector Y of size $|N|$, called the hub location vector, in which $Y_k = 1$ means that a hub is located at node k ; and a binary matrix H , called the arc utilization matrix, of size $|N|^2$, in which $H_{ij} = 1$ means that the arc $(i, j) \in A$ is used by the entrant's network, for collection, transmission or distribution.

We chose a representation using arcs, as opposed to edges, because it enables the use of classical crossover operators, and it does not bias the search toward edges connecting the low-index nodes. Note that this representation does not preclude infeasible solutions, because it can contain arcs between non-hub nodes. This situation is allowed to keep the diversity of the population; otherwise, there could be a premature convergence. However, these arcs are not considered in the computation of the objective function.

Crossover operator

The crossover operator combines two or more solutions from the population, and results in one or more offsprings. We use the 1-point crossover operator, which starts from two parent solutions and returns two offsprings. An integer number $b \in [1, |N|-1]$ is selected randomly, called the cutting point. The location vectors and arcs utilization matrices of the parents are cut after the b^{th} position in the former, and after the b^{th} column (or row) in the latter. The row and column crossover are applied with equal probability. Then, the resulting pairs of pieces of the hub location vectors and arc utilization matrices of the two parents are exchanged. Figure 2-1a shows two solutions of this location and network design problem, for a 4-node network. Figure 2-1b and Figure 2-1c show the new solutions obtained after applying the crossover operator to the solutions shown in Figure 1a, using $b = 3$ and making column and row exchanges in the arc utilization matrices, respectively.

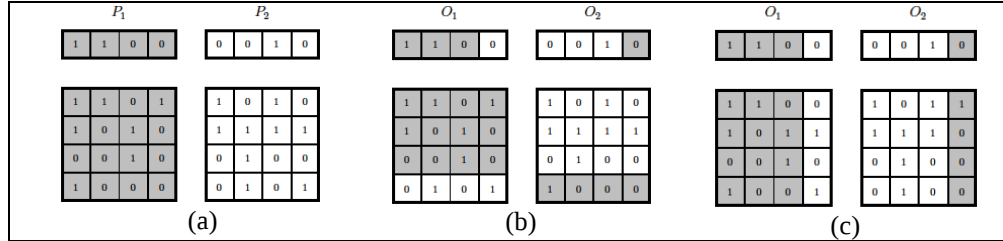


Figure 2-1. (a) Two partial solutions with $|N|=4$, to be used with the proposed genetic algorithm. After applying 1-point: (b) row crossover, (c) column crossover.

Mutation operator

The mutation operator creates a new solution from an old one as follows. A random integer $v \in [1, |N|]$ is selected. In the hub location vector, the value of Y_v is flipped. In the arc utilization matrix, all the elements of either the v -th column

or the v -th row are flipped, choosing at random which it is going to be, with the same probability.

Insertion operator

The insertion operator evaluates every solution generated by crossover and mutation, and includes it in the population if its objective value is better than the worst solution currently in the solution set. If that is the case, the worst solution is replaced by the new one.

2.3.2. Pricing problem

Hub problems have never included the pricing into consideration together with discrete choice models, since deriving a closed expression for optimal pricing is not straightforward in this case. However, in other research fields, e.g. the field of product bundle pricing (Bitran & Ferrer, 2007), pricing has been studied. We adapt a formula from that field to our case, considering that hubs on a route are bundles, as follows. Once a new solution is found by the genetic algorithm, i.e. the values $\{Y_k\}_{k \in N}$ and $\{H_{ij}\}_{(i,j) \in A}$ are known, we define S_{ij} as the set of feasible pairs of hubs (k, m) that can connect the origin-destination (OD) pair (i, j) , that is:

$$S_{ij} = \{(k, m) \in N^2, Y_k = Y_m = H_{ik} = H_{km} = H_{mj} = 1\}, \forall i, j \in N \quad (2.11)$$

Replacing (2.3) in (2.1), and using (2.11), the objective function of the pricing problem is:

$$Z = \max \sum_{i, j \in N} \frac{W_{ij} \sum_{(k, m) \in S_{ij}} (p_{ijkm} - c_{ijkm}) \exp(-\Theta \cdot p_{ijkm})}{\sum_{(k, m) \in S_{ij}} \exp(-\Theta \cdot p_{ijkm}) + \eta_{ij}} - \tau \quad (2.12)$$

with:

$$\tau = \sum_{(i,j) \in A} K_{ij} \hat{H}_{ij} + \sum_{k \in N} F_k \hat{Y}_k \quad (2.13)$$

Optimal prices are derived from the first order conditions, in the next Theorem.

Theorem 1. *The optimal price for every route $i \rightarrow k \rightarrow m \rightarrow j$ is given by the following closed expression.*

$$p_{ijkm}^* = c_{ijkm} + \frac{1}{\Theta} \left\{ 1 + W \left[\frac{1}{\eta_{ij}} \sum_{(s,t) \in S_{ij}} \exp(-\Theta \cdot c_{ijst} - 1) \right] \right\} \quad (2.14)$$

Where $W(\cdot)$ is the W Lambert function, defined as the inverse function of $f(W) = We^W$.

PROOF. Bitran & Ferrer (2007) derive a formula for optimal pricing in the case of a single product bundle. Our formula and proof are a generalization for the case of multiple bundles (multiple hub pairs). It is easy to see that the objective function (2.12) can be decomposed in separate expressions for every OD pair (i, j) . Using the first order conditions $\frac{\partial Z}{\partial p_{ijkm}} = 0, \forall i, j, k, m \in N$, we obtain the

following expression for a particular route $i \rightarrow s \rightarrow t \rightarrow j$:

$$\begin{aligned} & \left[\sum_{(k,m) \in S_{ij}} \exp(-\Theta p_{ijkm}) + \eta_{ij} \right] \left[1 - \Theta (p_{ijst} - c_{ijst}) \right] \\ & + \Theta \left[\sum_{(k,m) \in S_{ij}} (p_{ijkm} - c_{ijkm}) \exp(-\Theta p_{ijkm}) \right] = 0 \end{aligned} \quad (2.15)$$

Consider now the equivalent expression for a route $i \rightarrow u \rightarrow v \rightarrow j$ of the same OD pair. Dividing this expression and (2.15) by Θ , and then subtracting them, we obtain the following equation:

$$\left(p_{ijst} - c_{ijst} - p_{ijuv} + c_{ijuv} \right) \left[\eta_{ij} + \sum_{(k,m) \in S_{ij}} \exp(-\Theta p_{ijk m}) \right] = 0 \quad (2.16)$$

Since the terms in brackets in equation (2.16) are nonnegative, the expression in parenthesis must be zero. In other words, if there are multiple optimal routes for the OD pair (i, j) , the margins $p_{ij\bullet\bullet} - c_{ij\bullet\bullet}$ will be equal. Let $r_{ij} = p_{ijk m} - c_{ijk m}$. Replacing in (2.15), we obtain:

$$\begin{aligned} & (1 - \Theta r_{ij}) \left\{ \eta_{ij} + \sum_{(k,m) \in S_{ij}} \exp[-\Theta(r_{ij} + c_{ijk m})] \right\} \\ & + \Theta \sum_{(k,m) \in S_{ij}} r_{ij} \exp[-\Theta(r_{ij} + c_{ijk m})] = 0 \end{aligned} \quad (2.17)$$

Let $Q_{ij} = \sum_{(k,m) \in S_{ij}} \exp(-\Theta c_{ijk m})$. A reordering of the terms leads to:

$$(-1 + \Theta r_{ij}) \exp(-1 + \Theta r_{ij}) = \frac{Q_{ij} \exp(-1)}{\eta_{ij}} \quad (2.18)$$

The $W(z)$ Lambert function is defined so that $z = W(z) \exp[W(z)]$ holds. Let

$$z_{ij} = \frac{Q_{ij} \exp(-1)}{\eta_{ij}} \text{ and } W(z_{ij}) = -1 + \Theta r_{ij}.$$

Then, $W(z_{ij}) = -1 + \Theta r_{ij} = W\left(\frac{Q_{ij} \exp(-1)}{\eta_{ij}}\right)$, and

$$r_{ij} = \frac{1}{\Theta} \left[1 + W\left(\frac{Q_{ij} \exp(-1)}{\eta_{ij}}\right) \right] \quad (2.19)$$

Replacing back r_{ij} , the closed expression for the optimal prices is:

$$p_{ijk}^* = c_{ijk} + \frac{1}{\Theta} \left\{ 1 + W \left[\frac{1}{\eta_{ij}} \sum_{(s,t) \in S_{ij}} \exp(-\Theta \cdot c_{ijst} - 1) \right] \right\} \quad (2.20)$$

The second order conditions can be used to show that Z is concave on every p_{ijk} ■

Note that in this expression, the price is always greater than the operating cost, because $\Theta > 0$ and $W(z) \in \mathbb{R}^+$ if $z \in \mathbb{R}^+$. Secondly, a lower factor Θ (users' sensitivity to price differences) leads to higher optimal prices. This is intuitively correct, since a lower sensitivity means that there are more customers willing to pay higher prices for the service. These customers can be captured by the entrant.

2.4. Computational experiments and discussion

We tested our model on the CAB data set (O'Kelly, 1987). The fixed cost of opening a hub at node k was set to $F_k = 100, \forall k \in N$. The fixed cost of establishing a link between the pair of nodes i and j was computed using the following expression (Calik et al., 2009).

$$K_{ij} = 100 \frac{c_{ij} / W_{ij}}{\max_{(k,l) \in A} c_{kl} / W_{kl}}, \forall (i, j) \in A \quad (2.21)$$

For the experiments, we used the following setting: the flows in thousands,

$$\delta = \chi = 1, \quad \alpha = \{0.2, 0.4, 0.6, 0.8, 1.0\}, \quad q = |P| = \{1, \dots, 5\},$$

$$\Delta = \{0.05, 0.1, 0.2, 0.3, 0.4, 0.5\}, \quad \text{and} \quad \Theta = \{3.85, 5.78, 7.70, 9.63, 11.55, 15.39\}.$$

These values of Θ correspond to 3σ taking the values $\{1, 0.66, 0.5, 0.4, 0.33, 0.25\}$, where σ is the standard deviation of the users' perception of the price. The 900 resulting instances were run 10 times each, using different random seeds.

We used a PC with a 2.80 GHz Core i7 processor and 6 GB of RAM, and operating system Ubuntu 11.10. The genetic algorithm was programmed in C++ and compiled using GCC 4.6 with the vectorization and code optimization options activated.

As the calculation of the objective function value is separable by origin-destination pairs, we parallelize it using the library GOMP (GNU-OpenMP).

The genetic algorithm was run up to a maximum of 10,000 iterations, with 100 solutions in the set S , and a mutation probability of 1%. However, the preliminary tests shown that after 5,000 iterations there was no improvement in the quality of the solutions, and we used this last value in the reported numerical experiments. As we mention before, optimality is not necessarily achieved.

2.4.1. The role of inter-hub economies of scale on the entrant's profit

We first study the case in which inter-hub transportation is cheaper, and analyses the effects of these discounted costs or economies of scale on the entrant's profit.

From the entrant's point of view, there are three basic situations.

1. The incumbent has only one hub located. In this case, the larger the inter-hub discount, the higher the benefit of the entrant.
2. Both the incumbent and the entrant have two or more hubs. Inter-hub economies of scale are less relevant to the entrant, because both competitors can take advantage of them.
3. The incumbent operates a large hub-and-spoke network. The entrant will obtain benefit only if there are low economies of scale or none at all. In this case, the only advantage of the entrant is the *a priori* knowledge of the incumbent's network.

Figure 2-2 shows the results for these three scenarios. The profit earned by the entrant is shown on the left vertical axis of each graph, the income perceived by the incumbent on the right vertical axis, and the inter-hub discount factor (α) on the horizontal axis.

We display the incumbent's income (and not the profit) because the incumbent is supposed to have been in the market for a while, so its investment costs are sunk. Figure 2-2a shows the case in which the incumbent has only one hub located ($q=1$) and charges a low margin ($\Delta=0.05$) over his costs, with customers having an intermediate sensitivity factor ($\Theta=5.78$). In this case, for lower inter-hub costs (lower values of α), the entrant can increase its customer capture and profit by opening more than one hub, taking advantage of the reduced inter-hub costs, which the incumbent, with only one hub, cannot.

Figure 2-2b and Figure 2-2c show what happens when $q=2$ and $q=3$ (the incumbent has two and three open hubs, respectively). Please note the different scale for the entrant profit on these Figures, since now the entrant's profit is significantly smaller than when $q=1$, because the incumbent can take advantage of the inter-hub discounts, achieving a better competitive position and reducing the entrant's capability of obtaining a higher profit. Figure 2-2c shows how, if the incumbent has a more extensive network, with more than two hubs, it is not convenient for the entrant to start operations in the same market, unless there are no inter-hub economies of scale at all. Our tests show that this situation does not change for different values of Θ .

If the leader's margin Δ increases, the entrant's profit potentially grows and becomes less dependent on α , even if the incumbent has a larger network with several hubs. Naturally, the incumbent can easily change its margins, making the entrant's option of competing in this market very risky. The effect of the margin charged by the incumbent on the entrant's profit is shown in Figure 2-3, for

$\alpha = 0.6$ and $\Theta = 15.39$. The entrant's profit Z is shown on the vertical axis, while the margin Δ is shown on the horizontal axis. Each series is associated with a different number of incumbent's hubs, q . Note that the entrant's profit increases almost linearly on Δ , especially for low values; but not on q . As before, there are situations in which it is not possible for a new competitor to enter the market ($q = 5$ and small margins, for example).

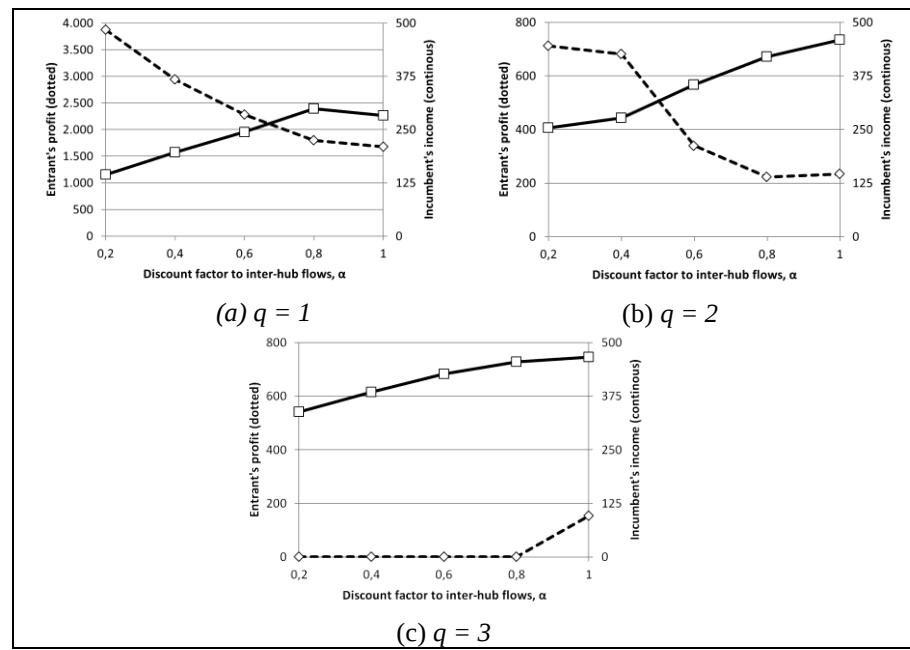


Figure 2-2. Entrant's objective function value as a function of α , with $\Delta=0.05$ and $\Theta = 5.78$.

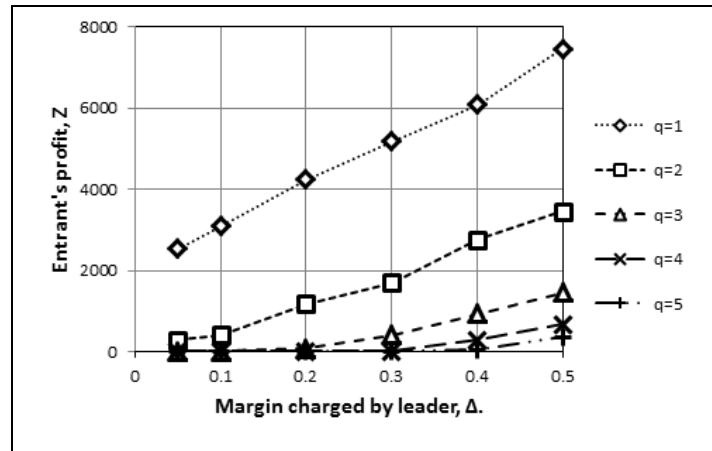


Figure 2-3. The effect of sensitivity to price differences on the entrant's profit.

Figure 2-4 shows how the entrant's profit varies as a function of the users' sensitivity to price differences, Θ . We focus on the case in which the incumbent charges a small margin over its costs ($\Delta = 0.05$), for different values of α .

When $q = 1$ (Figure 2-4a), for high values of Θ , i.e. most of the customers choose the least expensive routes. In other words, there is little spread of customers among the different routes. Since the entrant can locate two or more hubs –taking advantage of inter-hub economies of scale– it can offer routes that are cheaper than those offered by the incumbent, obtaining a reasonable profit. As the value of Θ decreases, more customers are willing to pay higher prices, and the entrant's advantage due to inter-hub economies of scale, as well as its profit, decreases. However, when customers' sensitivity to price further decreases, the entrant can increase its prices, obtaining a higher profit.

If $q = 2$, the market is more competitive, because the incumbent is already taking advantage of the inter-hub economies of scale. This is shown in Figure 2-4b. High and intermediate values of Θ put the entrant in a disadvantageous situation, particularly if the incumbent has optimized its hub locations and network. As Θ

continues decreasing, customers are less sensitive and the entrant can increase its prices and profit.

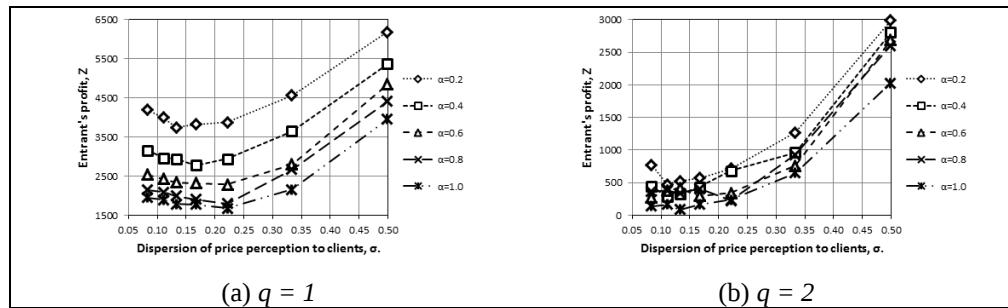


Figure 2-4. Entrant's objective function value as a function of σ , with $\Delta=0.05$, for different values of α .

As intuitively expected, the larger the margin charged by the incumbent, the greater the entrant's potential profit.

Finally, note that curves are not monotonic, and in occasions they intersect each other. This is due to the fact that the genetic algorithm does not guarantee optimality of the solutions obtained.

2.4.2. Optimal pricing decisions

The pricing problem is decomposable by OD pairs, each pair being an individual market. Every feasible route is a separate product in this market. In this subsection we will not consider the fixed costs of using arcs of the network, to do a fair comparison between both agents. From the entrant's point of view, there are two possible scenarios: with and without (variable) cost advantage over the incumbent.

The entrant has a cost advantage over the incumbent.

If the location and network design decisions allow the entrant to open a route with lower operating costs than the incumbent for a specific OD pair, then it has a competitive advantage in this particular market. An intuitive decision would be to price that particular route just below the incumbent's cheapest price. However, this is not always the optimal decision. Consider, for example, the situation depicted in Table 2-1, that shows a solution in which the incumbent has two hubs located at nodes 2 and 5, and the entrant has also two hubs, at nodes 10 and 25, with $\alpha = 0.2$ (strong inter-hub discount), $\Delta = 0.05$ (low incumbent margin), and $\Theta = 15.39$ (customers are very sensitive to price differences). The Table shows all routes and optimal costs for the (8, 3) OD pair. The entrant has lower costs than the incumbent, leading to a competitive advantage. The first column shows the possible routes for both the entrant and the incumbent. The remaining columns are the cost, price, likelihood of usage, market share and profit of each route, respectively. The least expensive routes of both the entrant and the incumbent are highlighted. Note that although the entrant has the cheapest cost for this (O,D) pair (through route 8, 10, 25, 3), and in spite of the high price sensitivity of the customers, the optimal price of this entrant's route (which implies charging a margin of about 0.112 units) is higher than the incumbent's lowest price (route 8,5,2,3). As counterintuitive as it seems, this is the optimal decision when the objective of the entrant is profit maximization. Note that this pricing policy does not lead to a maximum market share. Decreasing this price would most likely increase the entrant's market share to over 50%, but it would decrease its profit. By making this decision, the entrant captures less customers, but the customers that it captures are those willing to pay higher prices (cherry-picking). This behavior is similar to what Sasaki et al. (2014) found for heterogeneous customers.

Table 2-1. Optimal pricing by the entrant, with cost advantage, $\Theta = 15.39$, $\Delta = 0.05$, $\alpha = 0.2$, for the (8,3) OD pair. Entrant's hubs on nodes 10 and 25, and incumbent's hubs on nodes 2 and 5.

	Route	Cost	Price	$\exp(-\Theta \cdot \text{Price})$	MS (%)	Profit
Entrant's routes	8 $\rightarrow$ 10 $\rightarrow$ 3	2.478	2.590	4.927E-18	0.00%	0.000
	8 $\rightarrow$ 10 $\rightarrow$ 25 $\rightarrow$ 3	1.521	1.633	1.213E-11	41.71%	0.269
	8 $\rightarrow$ 25 $\rightarrow$ 10 $\rightarrow$ 3	3.320	3.432	1.159E-23	0.00%	0.000
	8 $\rightarrow$ 25 $\rightarrow$ 3	1.881	1.993	4.773E-14	0.16%	0.001
incumbents's routes	8 $\rightarrow$ 2 $\rightarrow$ 3	1.872	1.966	7.322E-14	0.25%	0.001
	8 $\rightarrow$ 2 $\rightarrow$ 5 $\rightarrow$ 3	2.338	2.454	3.936E-17	0.00%	0.000
	8 $\rightarrow$ 5 $\rightarrow$ 2 $\rightarrow$ 3	1.536	1.613	1.668E-11	57.38%	0.254
	8 $\rightarrow$ 5 $\rightarrow$ 3	1.830	1.921	1.436E-13	0.49%	0.003
		Sum		2.908E-11	100.00%	0.528

Using the same parameter values as in Table 2-1, Figure 2-5 shows both competitors' market share and profit for different values of the margin over cost charged by the entrant. The incumbent's profit and market share increase as the entrant increases its margin. However, although the entrant's market share decreases as its margin increases, its profit is not monotonic, achieving a maximum at the point predicted by expression (2.14), charging a margin of about 0.112 units over its costs.

The entrant does not have a cost advantage over the incumbent.

Consider now the situation in which it is the incumbent who has the least cost for an OD pair, on one of its routes, as shown in Table 2-2. Still the entrant has some room for capturing the customers that are willing to pay a higher price, provided it charges a low margin over its costs.

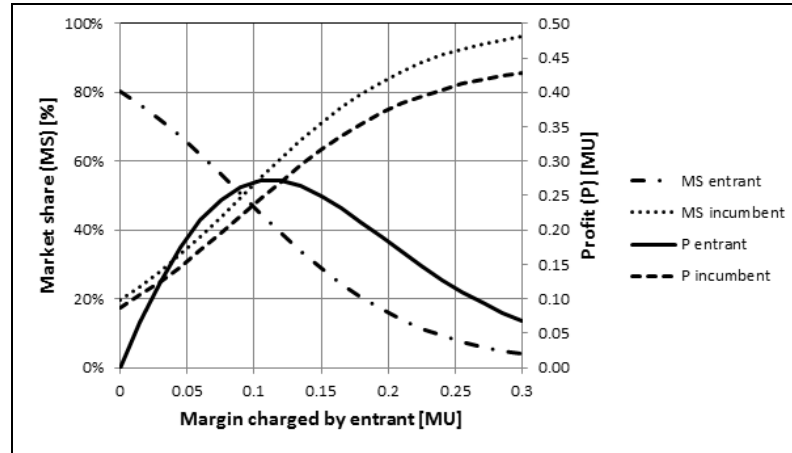


Figure 2-5. Incumbent's and entrant's market share and profit, for different values of entrant's margin over cost. Entrant has lower costs than the incumbent. Entrant's hubs on nodes 10 and 25; incumbent's hubs on nodes 2 and 5; it is the (8,3) OD pair, with $\alpha = 0$.

We focus on the (4,6) OD pair, with $\alpha = 0.2$, and $\Delta = 0.05$, as before. For this example, however, we use $\Theta = 3.85$ i.e., customers are less sensitive to price differences. We use this value to illustrate more explicitly the rationale behind the pricing decisions that our model suggests.

As Table 2-2 shows, in this situation, the entrant, taking advantage of the low sensitivity to price differences (Θ), charges a margin of about 0.065 units, that enables the capture of the customers willing to pay more for the service, achieving some profit, as the rightmost column shows. For higher values of Θ , the margin charged and the total profit are smaller.

2.4.1. Entrant's network structure

We studied three scenarios to understand the resulting entrant's network structure: (1) intermediate inter-hub discount, high sensitivity to price differences, and a moderate margin charged by the incumbent; (2) intermediate

inter-hub discount, intermediate sensitivity to price differences, and a moderate margin charged by the incumbent; and (3) no inter-hub discount, intermediate sensitivity to price differences, and low incumbent margin. Figure 2-6 and Figure 2-7 show the entrant's network structure for scenarios 1 and 3, respectively. The narrower arcs connect hubs and non-hub nodes. The thicker arcs connect entrant's hubs, shown as black circles. The incumbent's hubs are shown as stars, but for the sake of clarity, the network is not drawn. A grey star shows co-location of entrant's and incumbent's hubs.

Table 2-2. Optimal pricing by the entrant, without cost advantage, $\Theta=3.85$, $\Delta=0.05$, $\alpha=0.2$, for the (4,6) OD pair. Entrant's hubs on nodes 10 and 25, and incumbent's hubs on nodes 2 and 5.

	Route	Cost	Price	$\exp(-\Theta \cdot \text{Price})$	MS. (%)	Profit
Entrant's routes	4 $\rightarrow$ 10 $\rightarrow$ 6	2.037	2.102	3.060E-04	0.11%	0.002
	4 $\rightarrow$ 10 $\rightarrow$ 25 $\rightarrow$ 6	1.472	1.537	2.689E-03	0.93%	0.021
	4 $\rightarrow$ 25 $\rightarrow$ 10 $\rightarrow$ 6	1.938	2.003	4.480E-04	0.15%	0.004
	4 $\rightarrow$ 25 $\rightarrow$ 6	0.891	0.956	2.522E-02	8.68%	0.198
incumbents's routes	4 $\rightarrow$ 2 $\rightarrow$ 6	0.926	0.972	2.368E-02	8.15%	0.133
	4 $\rightarrow$ 2 $\rightarrow$ 5 $\rightarrow$ 6	0.925	0.971	2.379E-02	8.19%	0.133
	4 $\rightarrow$ 5 $\rightarrow$ 2 $\rightarrow$ 6	0.654	0.686	7.117E-02	24.51%	0.281
	4 $\rightarrow$ 5 $\rightarrow$ 6	0.481	0.505	1.431E-01	49.28%	0.416
			Sum	2.904E-01	100.00%	1.188

Scenario 1

In this case, $\alpha = 0.6$, $\Theta = 15.39$, and $\Delta = 0.3$, i.e. there are moderate inter-hub economies of scale; the users are very sensitive to price differences; and the incumbent charges a moderate margin over his costs. Figure 2-6 shows the networks for different values of q .

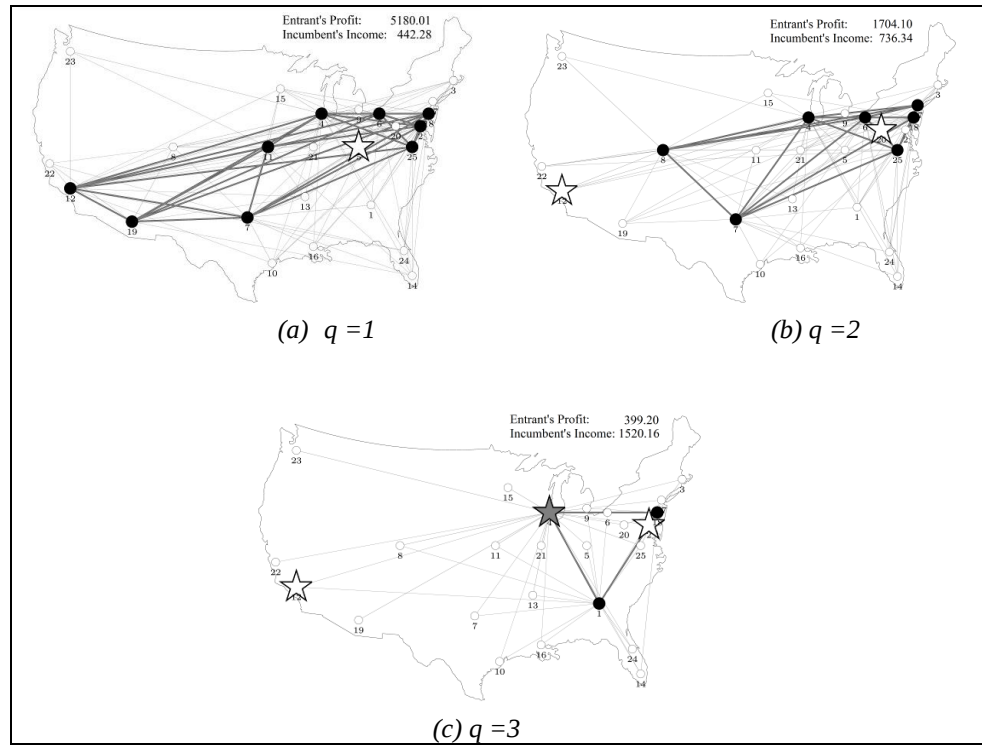


Figure 2-6. Solutions for $\alpha=0.6$, $\Theta=15.39$, and $\Delta=0.3$, and different values of q . White circles are cities; black circles are locations of entrant's hubs; white stars are locations of incumbent's hubs. Gray stars indicate colocation of both incumbent's and entrant's.

If $q = 1$ (Figure 2-6a), the incumbent cannot use the inter-hub economies of scale, so the entrant has the incentive to locate several hubs, separated from each other by long arcs.

When $q = 2$ (Figure 2-6b), both agents can use the inter-hub economies of scale, and the scenario is more competitive. The entrant's hub interconnection network is less extended; i.e., the number and distance between hubs decreases.

Finally, then $q = 3$ (Figure 2-6c), the incumbent is even stronger, allowing the entrant to locate only three hubs and an even smaller network.

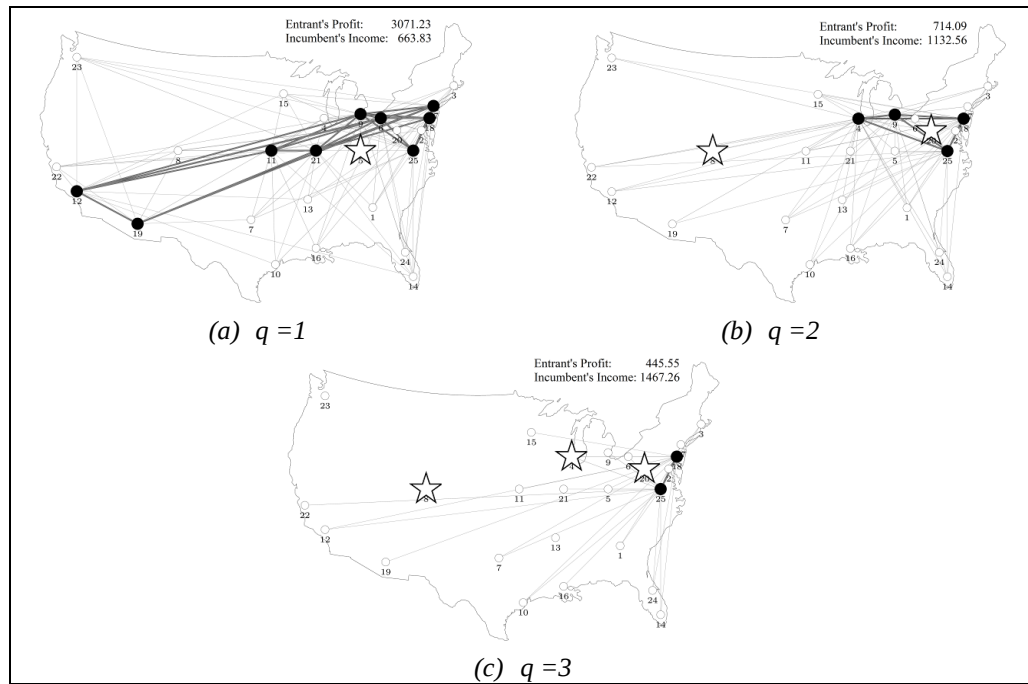


Figure 2-7. Solutions for $\alpha=1$, $\Theta=7.7$, and $\Delta=0.2$, and different values of q . White circles are cities; black circles are locations of entrant's hubs; white stars are locations of incumbent's hubs.

Scenario 2

Now, we consider the case in which the users are less sensitive to price differences. Let $\Theta = 7.7$, with all the other parameters as in the previous scenario. The results for this scenario are very similar to Scenario 1, but with a higher income for the incumbent.

Scenario 3

If there are no inter-hub economies of scale ($\alpha = 1$), the incumbent charges a lower margin ($\Delta = 0.2$) and there is intermediate sensitivity to price differences, the resulting networks for different values of q , are shown in Figure 2-7.

Without inter-hub economies of scale, and if the incumbent's margin is low, it is harder for the entrant to capture customers in any scenario. If $q = 1$, the resulting entrant's network is smaller (Figure 2-7a). When $q = 2$ (Figure 2-7b), the entrant creates a more sparse network, and locates four hubs. Finally, when $q = 3$ (Figure 2-7c), the incumbent is even stronger, and the entrant's network is even smaller. We remark that even though there are no economies of scale, some passengers will choose using routes including inter-hub arcs. This is due to the fact that, as opposed to hub location models with no user choice, there is dispersion in the preferences of the customers.

Note that the networks in Figure 2-7 have most (or all) the hubs very close together. This is due to the following possible reasons: Each origin-destination pair uses all possible routes going through one or two hubs (there is multiple-assignment of demands to hubs). Also, the model requires every flow to go through at least one hub. These two conditions, together with the fact that the largest flows and the highest density of cities are on the east coast (on Figure 2-7b, half of the flows are either originating or having as a destination the hub nodes), and the low opening cost of hubs, make the east side of the country a good location for several hubs. It is also important to note that the cost structure influences the resulting entrant's network. For example, with larger fixed hub costs, there would likely be fewer hubs; and using the fixed costs structure in Calik et al. (2009) for arcs could give an incentive to use short links between cities with large flows. Similarly, Table 2-3 summarizes the results of all scenarios. In general, the stronger the incumbent's position is, i.e. lower margin and larger network, the harder it is for the entrant to obtain any profit. Stronger incumbent's positions result in a decrease in both profit and the number of hubs open by the entrant. Also, when the incumbent is strong, in terms of the number of located hubs, the entrant's best option tend to be to concentrate in a small area (if possible) and obtain the highest profit there. Finally, we remark that the objectives of both firms are different: while the incumbent minimizes cost to

serve the entire demand, which resembles the profit maximization of a monopolist, the entrant's objective is profit maximization given the incumbent's network and prices.

Table 2-3. Number of open hubs (# hubs) and arcs (# arcs), running time (Time), Entrant's Profit and Incumbent's Income, for all scenarios and values of q .

Scenario	Parameters	q	# hubs	# arcs	Time (s)	Entrant's Profit	Incumbent's Income
1	$\alpha=0.6$	1	9	102	7.67	5180.01	442.28
	$\Theta=15.39$	2	7	84	7.61	1704.10	736.34
	$\Delta=0.3$	3	3	36	7.24	399.20	1520.16
2	$\alpha=0.6$	1	12	152	15.23	4823.74	587.47
	$\Theta=7.7$	2	7	93	6.89	1799.74	964.43
	$\Delta=0.3$	3	3	31	6.30	315.23	1771.64
3	$\alpha=1$	1	9	107	6.32	3071.23	663.83
	$\Theta=7.7$	2	4	64	4.47	714.09	1132.56
	$\Delta=0.2$	3	2	29	6.43	445.55	1467.26

2.5. Conclusions

We present a new approach to analyze a situation in which two firms compete in a transportation market. An existing firm operates a hub-and-spoke network, and applies mill pricing. A new firm wants to enter the same market, maximizing its profit by building a possibly incomplete hub network, and by making optimal pricing decisions. Customers' choice of provider and route depends solely on price, as would predict a simple logit model, although including other factors would be very easy.

We formulate a non-linear mixed integer programming model. We derive a closed form expression for the optimal pricing policy, and solve the problem as a location-network design problem (combinatorial) with an embedded pricing problem. We use a genetic algorithm for the location and network design problem. We thoroughly analyze the results of the model using the CAB dataset.

The computational experience shows that considering optimal pricing decisions when solving the hub location-network design problem leads to a better estimation of the maximum profit that the entrant will be able to obtain. Without solving the pricing problem, it is not possible to estimate the demand captured, its behavior and the final profit.

As opposed to Eiselt & Marianov (2009), who studied the hub location problem considering that users choose according to a gravity model, we use a logit model, which enables introducing customers' sensitivity to prices. The consideration of sensitivity provides new insights about the competitive hub location problem. We show how this sensitivity plays either in favour of the entrant or the incumbent, depending on the incumbent's margin and network configuration. We show that, if sensitivity to price differences is considered, the optimal pricing policy for the entrant does not necessarily always consist in charging a price that is below the incumbent's cheapest price for a given OD pair. In fact, for low sensitivities to price, customers will spread among the routes more evenly, so all routes will capture some traffic. This is a conceptual difference with the work by Marianov *et al.* (1999), who consider that the firm with the cheapest route captures all customers.

Also, we show that, under competition, inter-hub discounted costs strongly influence the decision of entering a market. Inter-hub economies of scale, together with low prices, can be used by a strong operator to block a new agent from entering the market. On the other hand, inter-hub discounted costs can be the key to success for a new agent, who can take advantage of his knowledge of the incumbent's network and prices, whenever incumbent's location and network design are not the best, or its prices are high. We remark, though, that the modelling of the economies of scale is still an open question in the hub location literature, as Campbell (2012, 2013) pointed out recently, and using inter-hub

discount factors of the incurred costs is just an approximation of the actual dynamics in a hub-and-spoke network.

A scenario analysis like the one presented here can be a valuable tool for a firm that is evaluating its entrance to a market. Furthermore, using this analysis, some situations can be foreseen in which entering the market is not convenient even if customer sensitivity is low.

We also show that, for competitive situations, a formulation that maximizes profit provides different results and insights than a model that seeks market share maximization, while more adapted to situations in which a competitor aims at a higher profit.

Naturally, there are some factors that we did not take into account, which could be relevant in a competitive situation. One of these is the fact that the incumbent could react to the newcomer's entry to the market, using for example smaller planes in some spokes to increase the service frequency; or decreasing the prices charged on certain routes; improving the benefits offered within frequent-flyer programs, and so on. However, taking all these factors into account is left as a future challenge.

Further analysis could be performed to explore situations in which profit is required to exceed a certain bound and the number of routes opened by the competing firms is limited. Extensions to this work include the analysis of the same scenarios using multinomial logit models, since in the airline hub problem customers choose based on prices, flight time, the number of stops (hubs), and other factors.

Also of interest is the effect of sharing hubs by different companies, as it reduces fixed location costs, but potentially increases congestion. Finally, we leave for future research the analysis of von Stackelberg-type games.

3. A MODELLING FRAMEWORK FOR STRATEGIC AIRLINE NETWORK DESIGN

We develop a modelling framework to assist airline managers evaluating their strategic network design decisions. As opposed to explicitly solving a hub location problem and assuming an a priori existence of economies of scale, as many models do, we formulate a Location-Network Design Model as a Mixed-Integer Problem (MIP), in which a company must locate their management and maintenance resources at existing airports, together with defining routes and allocating capacity both on arcs (airplanes), and nodes (airports), minimizing costs, subject to a constraint on the aggregated level of service. The decision process is guided by a set of Key Performance Indicators (KPIs) from the airline industry. We begin analyzing both airline economics and hub location models, follow by describing the framework and the model and, finally, we show how choosing different operational constraints influences both the network structure and KPIs.

This chapter was formatted as a manuscript titled “A modeling framework for strategic airline network design”, and submitted for review to Computer & Operations Research in December 29, 2014.

3.1. Introduction

After deregulation, the airline industry has changed significantly, mainly in the USA (Berechman & de Wit, 2014). Firstly, the market of air passengers has increased consistently (Oum & Zhang, 1997). This increase in the demand has led to unpleasant externalities to users, as congestion and delays. Secondly, part of the industry is now concentrated on firms that operate large networks, in terms of the number of OD (origin-destination) pairs served. These firms (“trunk”, “network carriers,” or “hub-and-spoke” airlines) were created through expansion and mergers between companies, in order to achieve economies of scale, network and density, through consolidation of flows, using hub-and-spoke network topology. The remainder of the market is served by low-cost carriers, which fiercely compete with each other, cherry-picking on the demand of hub-and-spoke airlines through the provision of direct services between important OD pairs, and offering routes that use secondary airports, as opposed to congested hub airports. Finally, the flying material (airplane) technology has evolved following two trends: providing small and medium-sized airplanes with greater autonomy, and very efficient wide-body and long range airplanes.

All of these changes have decreased the market share of hub-and-spoke airlines, increasing the competition in the markets, and making possible the existence of direct services between more OD pairs. These changes have stressed the need to develop tools to design more efficient airline networks for the hub-and-spoke companies.

The strategic design of airline networks has been addressed mainly using hub location models, which have been under discussion recently (Campbell, 2013). The main concern relates to the appropriate modelling of economies of scale in transport networks, and the relative amount of spoke and inter-hub flows.

The contribution of this paper lies along those lines: we aim at an improved modelling of economies of scale when designing an airline network. The contribution is three-fold: we provide an extensive review on literature referring to economies of scale in the airline industry, hub location models and network design, unifying and integrating concepts from these areas. Then, we develop a general framework to design airline networks, which we believe approximates better the economies of a hub-and-spoke airline network, and we measure its performance proposing and using key performance indicators which, to the best of our knowledge, has never been done to analyze solutions of a hub network design problem. Finally, we formulate a general location and network design problem for airlines, providing computational experience and managerial insights.

The remainder of the article is organized as follows: subsection 3.2 reviews the literature on Economies of Scale (EoS) in the airline industry, hub location models, and airline network design. Subsection 3.3 states our framework for airline network design, composed of parameters, a set of possible policies, a location and network design problem, and KPIs. Subsection 3.4 is devoted to computational experiments and managerial insights. Subsection 3.5 discusses the results and states the concluding remarks.

3.2. Literature review

3.2.1. Economies of scale in the airline industry

The existence of hub-and-spoke airlines suggests that, in practice, there are economies of scale. However, there is no agreement in the literature about this issue. Finding whether there are in fact economies of scale is important, because these are a source of cost advantages for larger companies.

Formally, a company/industry has EoS if, as the production increases, the average costs decrease, i.e. the increase in total costs is less than proportional to the increase in production. The classical way to estimate the amount of economies of scale is to state a production function, and calculate its elasticity with respect to the production factors. Recent research on the subject began with the work by Caves, Christensen, & Tretheway (1984), who, by means of a translog (transcendental logarithm) model adjusted using panel data on large and small airlines, studied the differences between trunk and local service airline costs, showing that density of traffic is the primary factor that explains EoS. This result explains why local airlines can compete with trunk airlines in dense traffic markets. Their paper has also been the seminal reference for further research in terms of methodology.

Relevant papers on airline costs from the economical perspective can be classified in those dealing with estimation of returns of scale, density and/or traffic, together with their respective economies; and those determining the factors that generate savings in the airline cost function.

Estimation of return of scale, density and their economies

After the work of Caves et al. (1984), the economic analysis of airlines has been a controversial topic, because researchers have not used a common terminology and methodology for their calculations. Also, a correct formal definition of EoS in the transport context was only stated about 20 years later (Basso & Jara-Díaz, 2006). During the 20 years between Caves et al. (1984) and this definition, the theoretical analyses suggested that there were no strong economies of scale, contradicting the practical fact that the average firm size and industry concentration had increased significantly in the USA after deregulation.

One of the first efforts to improve the calculation of EoS in the airline industry was attempted by Oum & Zhang (1997), noting the scarce evidence in the

literature on the existence of such economies. The divergence between theory and practice was caused by the traditional way returns of scale were measured, on which EoS are based, using cost elasticities of outputs and network size, and ignoring the effects of the operating characteristics, as load factor, leg length (flying distance), network size (the number of OD pairs), etc. The authors show empirically that all these operating characteristics are relevant and affect the computation of economies of scale, allowing a better approximation of what happens in practice.

Later, and from a theoretical point of view, Basso & Jara-Díaz (2006) pointed out that the cause of the practice-theory divergence was the definition of EoS used in the transport industry. Previous literature assumes that the route structure remains constant after an increase in traffic, which is not so. They distinguish two concepts: the economies of density, i.e. the decrease in the unitary transport cost when the traffic increases but the route structure is fixed; and the economies of scale, which appear when the number of origin-destination pairs remains constant, but the route structure changes. Based on this analysis, for the US airline data set covering period 1980 – 1989, they estimate that while the transport firms get closer to exhaust economies of density, there are still economies of scale to exploit.

More recently, Johnston & Ozment (2013) pointed out another reason for the lack of strong evidence in the literature on the existence of EoS in the airline industry. They noted that all the previous studies were done using data prior to 1989, when deregulation did not have yet a full effect on industry. The authors' goal was to “reconcile the disparity between the traditional economic definition of economies of scale and those that have evolved in the transportation literature, while assessing the potential effects of multicollinearity”. The data used ranged from 1985 to 2009, two models were adjusted to the data, and economies of scale

were calculated using the economic and transport definitions. They detected economies of scale on the USA airline industry in all the cases.

It is important to note that EoS are not only related to the flying material operation, as Pels, Nijkamp, & Rietveld (2003) stated in a study on European airports using DEA (Data Envelopment Analysis). They found that, in average, the European airports have constant returns to scale in air transport movements, but increasing return to scale in passenger movements, which makes relevant the inclusion of the operation of airports in the analysis of EoS.

In summary, the literature has been drifting towards an agreement in that economies of scale do exist, at least in the USA airline industry, so an accurate modelling, both considering airplane and airport operation is pertinent and important when designing airline networks.

Factors that determine changes in airline cost functions

In order to properly model airlines' costs, it is important to detect the factors that generate savings. After Caves et al. (1984), most of the studies refer to the USA and European airline industry, with the interesting exception of Kirby's (1986) work, who discusses some air transport policies for the Australian industry, given the fear of monopoly development on those years. The study reveals the existence of large potential cost savings, and economies of operation on load factors, airplane size and stage length.

Later, and focused on the USA airline industry, Baltagi, Griffin, & Rich (1995) studied the effects of deregulation using a translog variable cost function. They found that the cost savings had four main sources: improvements in load factor, reduction on union wage rates, lower fares (especially in local airlines), and changes in route structure based mostly on greater leg length and hubbing (the percentage of routes with two or more legs, with connection in at least one hub). This result is consistent with that of Wei & Hansen (2003) when only the

operating cost of airplanes is computed. The authors found economies of airplane size and leg length, with an optimal airplane size increasing with leg length. It implies that an appropriate model for airline operation must include decisions on the type of airplane used.

The load factor appears consistently as a proper performance indicator for airlines, both for passenger and cargo transportation. On the latter case, Mayer & Scholz (2012), using a log linear specification of the average cost function of every airline, noting that both the fuel price and the flight related labor have the highest impact on airplane operating costs. Also, the load factor is a proper indicator of the performance on the flight level. The authors also noticed the existence of economies of density on the USA cargo transporting airlines studied.

More recently, the effects on airfares of the interaction between low-cost carriers and network carriers has been studied. Brueckner, Lee, & Singer (2013) analyze two cases: non-stop and connecting flights. Their results suggest “*that most forms of legacy-carrier competition (competition between network airlines) have weak effects on average (fare of) flights. Low-cost carrier competition, on the other hand, has dramatic fare impacts*”.

It follows that important savings can be achieved by using appropriately the flying material (airplanes) capacity, avoiding short flights and using progressively more efficient airplanes. It motivates the development of better models for the (existing) economies of scale and to the use of the airlines’ operational assets.

3.2.2. Hub location and network design models

Hub Location Problems (HLPs) have been used since deregulation for the strategic design of airline networks. Current research focuses on extending in

different directions and relaxing assumptions of the fundamental model proposed by O’Kelly (1986, 1987) and linearized by Campbell (1994).

The fundamental model assumes the existence of a fixed discount factor for inter-hub flows, due to economies of scale. The use of this fixed discount in the fundamental model leads to networks in which an economy of scale discount is applied even to legs where the traffic is very low. The literature afterwards has followed one of the following approaches: continuing its use, because of its simplicity, or proposing better models for representing economies of scale in HLPs.

To the best of our knowledge, most of the research follows the first approach, as Campbell & O’Kelly (2012) and Farahani et al. (2013) have pointed out. In Kimms' (2006) words: “Surprisingly enough, although the economies of scale phenomenon is one of the main motivations for installing hub-and-spoke systems, the way costs are modelled has not really been questioned by many authors in this area.”

Nevertheless, better models to represent the economies of scale in HLPs have been indeed proposed. These can be classified in models that use flow dependent inter-hub discount factors; models using incomplete inter-hub networks; hub-arc models; and models considering thresholds to enable links with discounted cost in the network.

Flow dependent discount factors

Given that the fundamental model uses a fixed factor to discount the inter-hub flows, a natural extension is to use non-linear or linear piecewise (and flow-dependent) discount factors.

O’Kelly & Bryan (1998) were the first to use this approach. They used a linear piecewise flow-dependent discount factor for a multiple assignment HLP to

approximate a continuous non-linear objective cost function. The resulting model for location and hub network design, called FLOWLOC, tends to consolidate most of the inter-hub flows in a particular inter-hub arc, if the discount is stronger, *ceteris paribus*. It suggests that a better way to model the economies of scale, rather than a fixed discount factor, is some function of the flow.

On the same year, Bryan (1998) further extended the latter article, including models considering capacity, minimum thresholds on inter-hub arcs, non-fixed number of hubs to locate, and flow-dependent cost functions for every arc in the network. The author stated a relationship between the FLOWLOC model and the user equilibrium, which is quite interesting, because it finds in the HLP, the conditions under which the flows distributes (spontaneously) in the network according to a closest demand-hub assignment. An interesting fact was highlighted later by Klineciewicz (2002), who noted that the FLOWLOC model can be reduced to the Uncapacitated Facility Location Problem, if the hub locations are known. Also the author proposed enumeration and search procedures, extending the O'Kelly & Bryan's (1998) approach to larger instances.

Later, O'Kelly & Bryan (2002) studied in more detail the fundamental and FLOWLOC models, both with single and multiple allocations. They studied the concept of 'fractional facilities' to analyze the behavior of the models when ties exist on route costs. Their work showed that the FLOWLOC model tends to concentrate the flows in the inter-hub arcs, as it should be.

Extending the FLOWLOC, Horner & O'Kelly (2001) propose a network design problem where a non-linear cost function exists on each arc, rewarding the appearance of economies of scale everywhere in the network, with conditions taken from the equilibrium traffic assignment. The location of hubs is not considered explicitly. The contribution of their work is the use of the notion of

equilibrium, which leads to new patterns of flows, with more consolidation of them, together with the appearance of gateway nodes in the network. Also, they noted that the consolidation of flows are more likely if there are considerable costs discounts and the interactions occurs between distant points of the network.

HLPs are not only devoted to the passenger airline industry, as Racunica & Wynter (2005) pointed out, proposing a model to design a hub-and-spoke network in which passenger and freight rail transports are combined. Concave linear piece-wise costs functions are defined for every arc in the network, with stronger discounts on inter-hub arcs. Also, the discount depends on the absolute amount of flow that traverses the arc, and not the proportion of the total flow that goes through that arc, as in O’Kelly & Bryan (1998). The authors noted that the solution is sensitive both to the fixed costs of locating a hub, and the number of pieces used to approximate the costs functions.

A few years later, instead of trying to extend the FLOWLOC model, Kimms (2006) focused on modelling the economies of scale as accurately as possible. The author argued that “In a hub-and-spoke network design setting, however, economies of scale due to quantity discounts may appear only if the transportation is done by a third party.” He proposed to include both fixed and variable costs on every arc of the network. Also, his base model is extended to include multiple traveling modes, capacitated facilities, limits in the size of the fleet of vehicles, etc. In some of these cases, the author noted that models including multiple modes and vehicles must include conservation constraints of vehicles. His contribution is a model with a mixed cost structure, having both fixed and variable costs of using the vehicles/airplanes, because it allows modelling EoS on HLPs.

One of the practical applications of HLPs using linear-piecewise costs is given by Cunha & Silva (2007). They solved a HLP with fixed costs and non-linear

discount factors in the inter-hub arcs, using a genetic algorithm with a local search procedure. The heuristic achieves good results in the models, and in the case of study. They recognized the existence of EoS in the Less-Than-Truck (LTL) Brazilian freight industry. The authors highlight the importance of studying other inter-hub cost structures, including explicitly the quality of service of the industry.

Most of research on HLPs using linear piecewise costs has been done on p -median and fixed costs HLPs. An exception is the work of Wagner (2007), who proposed new formulations for hub covering problems, using both fixed and flow-dependent discount factor on the inter-hub flows. The author focused on modelling, rather than on the interpretation of economies of scale. His contribution is the formulation of the problem, and exploiting its structure by means of pre-processing.

A promising line of work on this area is the use of decomposition approaches to solve HLPs with more general cost structures. The work by de Camargo, de Miranda, & Luna (2009) proposed a Benders Decomposition for a variation of the FLOWLOC model. The authors first developed a tighter formulation, and then solved it using Benders Decomposition, solving the sub-problems by inspection, speeding up the method as compared to the proposed model ‘as is’, being able to solve larger instances, also.

In summary, most of the research aimed at improving the representation of economies of scale has been done on p -median and fixed costs HLPs, using linear piece-wise cost functions for the inter-hub links. The added complexity of the resulting models is a computational challenge, so the structure must be exploited, perhaps through decomposition approaches.

Incomplete inter-hub networks

In occasions, the optimal solutions of the fundamental model carry very few units of flow between some inter-hub arcs, which are discounted by assumption. This is partially due to the fact that the inter-hub network is assumed completely connected. This complete connection can be relaxed, however.

In the fundamental models for HLPs, the scarce resources are the hubs, which need to be located efficiently, together with complete inter-hub networks. Campbell, Ernst, & Krishnamoorthy (2005a,b) were the first authors to formulate and study Hub Arc Location Problems, which locate a fixed number of inter-hub arcs, where the hubs must be located on both ends of every located inter-hub arc. As in this case ‘the scarce resource’ is the connectivity between hubs, the flows tend to concentrate more in inter-hub arcs. On the other hand, these models tend to locate more hubs than desired, so the solution could be unrealistic for some industries.

From another point of view, Alumur, Kara, & Karasan (2009) studied the HLPs with single allocation and incomplete inter-hub networks, where both the number of hubs and inter-hub arcs could be parameters. The authors focused on the covering and center HLPs. The main insights are that if the incomplete inter-hub networks are designed properly, the same level of service can be achieved doing a lower investment.

More recently, Lüer-Villagra & Marianov (2013) stated a competitive pricing and hub location problem, with an incomplete hub network, and solved it using a genetic algorithm. They showed that the structure of the inter-hub network affects the way an agent competes in a hub-and-spoke network. They noted also that the stronger the other agent’s position is, the more incomplete the inter-hub network is.

Using incomplete inter-hub networks looks appropriate to model hub-and-spoke networks, if the number of arcs is not set exogenously, because it could consolidate the flows artificially, leading to the appearance of flow concentration where it is not reasonable. Furthermore, incomplete inter-hub networks can achieve almost the same quality of service of complete networks, with a lower investment.

Flow thresholds on arcs

Another way to avoid the appearance of inter-hub discounted arcs with little flows is to incorporate minimum thresholds. It also makes sense in some industries, where minimum utilization rates are required.

To the best of our knowledge, the models proposed by Campbell (1994) are the first ones including thresholds. The author models spoke arcs having both fixed cost and minimum flow threshold to be enabled. This setting should lead to fewer connections between every non-hub node and hubs, tending to consolidate flows between them.

One of the approaches by Bryan (1998) was to develop extensions of the FLOWLOC model including minimum thresholds to enable inter-hub arcs. She found that, in terms of their optimal objective function value, the solutions are not sensitive to the threshold value and tend to concentrate more the flows on inter-hub arcs, compared to the FLOWLOC model.

Instead of locating hubs, Podnar, Skorin-Kapov & Skorin-Kapov (2002) developed a network design problem with fixed discount factors. If the flow in a certain arc exceeds a minimum threshold, the cost incurred by the flow is discounted by a fixed factor, as in the fundamental model. The key difference between the fundamental HLP and their model is that the economies of scale are allowed to appear everywhere in the network. This approach could be accurate when the industry uses only two kinds of vehicles, for large and small shipments,

for example. Their main contribution is recognizing that the economies of scale could appear not only between hubs, but also in spokes.

More recently, Skorin-Kapov & Skorin-Kapov (2005) used a game theoretical approach. They developed a cooperative game, based on the model of Podnar et al. (2002). The problem consists in dividing evenly the costs of operating the network, obtained by Podnar et al.'s model, between the network users, according to their contribution to the appearance of the economies of scale. This insightful work provides further directions of research, bonding together network design with economies of scale and game theory.

In other words, flow-dependent discounts factors, or fixed costs of enabling certain arcs in the network are a reasonable approach to the location or network design models that seek to represent appropriately the economies of scale. The goal is to make average costs dependent on the amount of flow.

Airline Network Design

Jaillet, Song, & Yu, 1996 used a different approach to the design of airline networks. They described and formulated problems based on a single airline with a fixed share of the market designing its network, following three different service policies, based on the maximum number of stops in a route: One-stop, Two-stop and All-stop. Then, solution approaches are presented and discussed, followed by heuristic procedures. Rather than locating hubs, the authors determine hub candidates, based on six indicators including the outflow of each node. They found strong or dominant connecting cities, depending more on their position than on the demand amount, and that the network structure is very different to a hub-and-spoke topology. Also, they suggested that on an efficient network, considering two kinds of airplanes does not decrease the total cost by a relevant amount, as compared to the case of a single type of airplane. This conclusion is partially due to the fact that only the fixed cost of arc capacities is

considered in the objective function, but not the fixed costs associated with the nodes.

As opposed to previous approaches, the framework and model we propose locate neither hubs nor hub-arcs explicitly, but allocates resources to airports, some of which become naturally hubs, because they end up performing hub functions given their size. Instead of using non-linear discount functions, we implicitly consider all factors that have an influence on economies of scale, including fixed and variable costs both at airports and flying material. The aim is to let the model decide on the network that uses optimally all the possible economies, for hub-and-spoke airlines.

3.3. Modelling framework for strategic airline network design

An appropriate modelling framework must include consideration of the following issues:

- It must adequately represent the economies of scale, whenever they exist;
- Given the relevance of airport performance for airlines (Pels, Nijkamp, & Rietveld, 2003), it should allocate the adequate capacity to the airports. Naturally, if an airport exceeds a certain capacity, it will require a greater investment, leading to an airport that could be called a main airport or a hub of the company.
- The framework should consider the capacities of the flying material.
- The framework has to be flexible enough to include operational constraints, which are airline-dependent, and can affect the operation of airports or flying material. For example, the transoceanic flights using wide-haul airplanes must be operated between airports with maintenance

facilities, or the longest legs must be symmetrical, in order to ease the crew scheduling, maintenance, catering, etc.

- It needs to consider the current airline network
- It must include consideration of the service quality, in some aggregate form, since a decrease of the service quality (e.g., total flight time, number of legs in a route, etc.) can make the airline lose capture or revenues.
- It must be flexible enough to accommodate different types of topologies. Looking at actual airlines, it is clear that most of them do not operate pure network topologies, namely point-to-point or hub-and-spoke. So, only general assumptions on the network structure must be done a-priori.

We then propose a framework that addresses these issues, providing also the managers with KPIs, and the possibility of feedback, based on a MIP formulation of the airline network design problem.

3.3.1. Framework structure

Figure 3-1 shows a block diagram of our proposed framework. The inputs for the framework are the operational constraints, as described before, the specifications and number of the airplanes available, the demand matrix, and all the relevant costs.

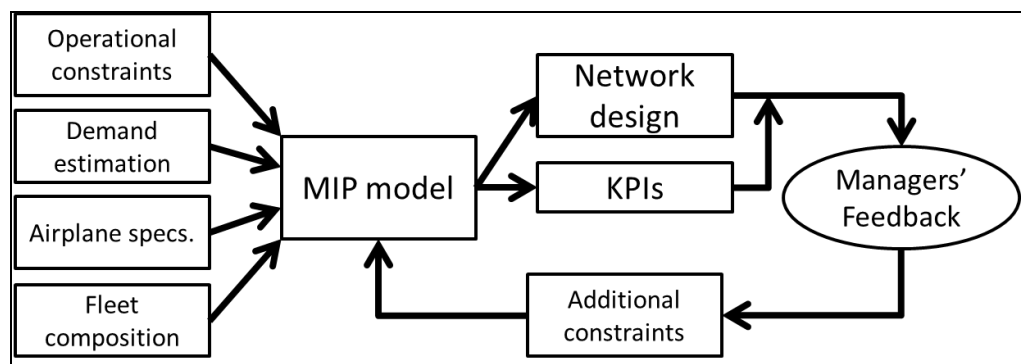


Figure 3-1. Block diagram of our proposed framework.

First, the MIP model is solved (ideally up to optimality), leading to a first network design. It includes the legs operated; the capacity allocated to arcs (number and type of airplanes) and nodes (capacity units on airports); the airports that would be central in the operation of the airline, i.e. hubs; together with the paths followed by flows.

The solution obtained by the MIP model can be assessed by the use of different KPIs, in order to analyze the solutions achieved in an aggregated way. Some possible KPIs are averages, as route length, number of legs in a route, and airplane utilization; or ratios, like total cost per unit of available capacity, total

distance travelled per unit of available capacity; total cost per unit of total demand (unitary cost); etc.

The network design and the KPIs can be used by the managers to analyze the solution, and to generate additional constraints to be added to the MIP model, for example, to set an upper bound on the number of legs for some routes or OD pairs, force the appearance of a direct flight between certain nodes, etc. Then, the model is solved again, leading to a new network design and KPIs, until the managers reach a suitable solution.

3.3.2. MIP model for the airline network design problem

Parameters

The problem is formulated as a discrete network design problem over a graph $G = (N, A)$, where N is the set of nodes and A is set of arcs, where the length of arc $(i, j) \in A$ is denoted d_{ij} . The transport demand between every $o \in N$ and $d \in N$ is considered static, inelastic, and denoted by w^{od} . The fleet of the airline is composed of a set T of different airplane models. The airplane type $t \in T$ has limited autonomy Au^t , capacity Cap^t , and at most $\overline{M}^t$ airplanes can fly the same leg. The set of flights than a type of airplane can cover is defined as $A^t = \{(i, j) \in A : d_{ij} \leq Au^t\}$. The airline has P^t airplanes of type $t \in T$. Note that the airplanes allocated to a certain arc do not represent actual flights, since the proposed model is strategic.

The airports, which can receive at most $\overline{Q}$ airplanes simultaneously, are classified in two kinds: small and large, in terms of the capacity allocated to them

by the company. A large airport is a central point for flights, maintenance or catering operations. A small airport can receive at most $\bar{R}$ airplanes.

We remark that airports are usually owned by third parties. An airline can allocate capacity to any airport in the network, and transform it into either a hub (in which legs of different routes are connected) or a technical base (in which there is concentration, maintenance, operative or administrative facilities, large flows, etc.) or both (Martín & Voltes-Dorta, 2007). An existing airport can be large itself, but if the airline does not allocate a large capacity to it for its operations, is considered a small airport for the effects of our model.

Four kinds of costs are considered in our strategic airline network design problem. First, fixed costs of using a node $k \in N$ as a large airport, denoted by CF_k , and incurred if more than $\bar{R}$ airplanes will be at that node simultaneously. Second, fixed costs of operating a direct flight from $i \in N$ to $j \in N$, using an airplane of type $t \in T$, denoted by CA_{ij}^t , $(i, j) \in A^t$. Third, variable costs incurred in a flight, associated with the number of passengers flying, denoted by c_{ij}^t , $(i, j) \in A^t$. Finally, fixed costs of allocating the s – th unit of capacity to a node $k \in N$, i.e. the costs incurred to receive the s – th airplane at node $k \in N$, denoted by CFC_{sk} , where $s \in 1, \dots, \bar{Q}$.

Finally, there is an upper bound on the total distance travelled by the passengers, that can be seen as the proxy of an ‘aggregated level of service’, denoted by χ .

Variables

Four groups of variables are used, defined as follows.

- Z_{ik} : 1, if the node $i \in N$ is assigned (connected by at least one flight) to the large airport located at node $k \in N$; 0, otherwise.

- $Y_{sk} : 1$, if airport at node $k \in N$ has at least $s \in 1, \dots, \bar{Q}$ units of capacity allocated; 0, otherwise.
- $W_{ij}^{mt} : 1$, if at least m airplanes of type $t \in T$ are operating a direct flight on $(i, j) \in A^t$; 0, otherwise.
- $F_{ij}^{ot} : \text{flow (number of passengers) flying on an airplane of type } t \in T \text{ on } (i, j) \in A^t, \text{ coming from node } o \in N.$

Base model

$$\min \sum_{k \in N} CF_k Z_{kk} + \sum_{s=1}^{\bar{Q}} \sum_{k \in N} CFC_{sk} Y_{sk} + \sum_{t \in T} \sum_{m=1}^{\bar{M}^t} CA_{ij}^t W_{ij}^{mt} + \sum_{o \in N} \sum_{t \in T} \sum_{(i,j) \in A^t} c_{ij}^t F_{ij}^{ot} \quad (3.1)$$

$$Z_{ik} \leq Z_{kk}, \quad \forall i, k \in N \quad (3.2)$$

$$\sum_{k \in N} Z_{ik} \geq 1, \quad \forall i \in N \quad (3.3)$$

$$\sum_{t \in T: (i,j) \in A^t} W_{ij}^{1t} \geq Z_{ij}, \quad \forall (i, j) \in \bigcup_{t \in T} A^t \quad (3.4)$$

$$Y_{\bar{R}+1,k} \leq Z_{kk}, \quad \forall k \in N \quad (3.5)$$

$$W_{ij}^{mt} \geq W_{ij}^{m+1,t}, \quad \forall t \in T, (i, j) \in A^t, m \in 1, \dots, \bar{M}^t - 1 \quad (3.6)$$

$$Y_{sk} \geq Y_{s+1,k}, \quad \forall k \in N, s \in 1, \dots, \bar{Q} - 1 \quad (3.7)$$

$$\sum_{t \in T} \sum_{(i,j) \in A^t} \sum_{m=1}^{\bar{M}^t} W_{ij}^{mt} \leq \sum_{s=1}^{\bar{Q}} Y_{sj}, \quad \forall j \in N \quad (3.8)$$

$$\sum_{o \in N} F_{ij}^{ot} \leq Cap^t \left(\sum_{m=1}^{\bar{M}^t} W_{ij}^{mt} \right), \quad \forall t \in T, (i, j) \in A^t \quad (3.9)$$

$$\sum_{(i,j) \in A^t} \sum_{m=1}^{\bar{M}^t} W_{ij}^{mt} \leq P^t, \quad \forall t \in T \quad (3.10)$$

$$\sum_{o \in N} \sum_{t \in T} \sum_{(i,j) \in A^t} d_{ij} F_{ij}^{ot} \leq \chi \quad (3.11)$$

$$\sum_{t \in T} \left(\sum_{(i,j) \in A^t} F_{ij}^{ot} - \sum_{(j,i) \in A^t} F_{ji}^{ot} \right) = \begin{cases} \sum_{\substack{d \in N: \\ o \neq d}} w^{od}, & i = o \\ -w^{oi}, & i \neq o \end{cases}, \quad \forall i, o \in N \quad (3.12)$$

$$\sum_{(i,j) \in A^t} \sum_{m=1}^{\bar{M}^t} W_{ij}^{mt} = \sum_{(j,i) \in A^t} \sum_{m=1}^{\bar{M}^t} W_{ji}^{mt}, \quad \forall j \in N, t \in T \quad (3.13)$$

$$\sum_{\substack{t \in T: \\ (i,j) \in A^t}} W_{ij}^{1t} \leq 1, \quad \forall (i,j) \in \bigcup_{t \in T} A^t \quad (3.14)$$

$$Z_{ik} \in \{0,1\}, \quad \forall i, k \in N \quad (3.15)$$

$$Y_{sk} \in \{0,1\}, \quad \forall k \in N, s \in 1, \dots, \bar{Q} \quad (3.16)$$

$$W_{ij}^{mt} \in \{0,1\}, \quad \forall t \in T, m \in 1, \dots, \bar{M}^t, (i,j) \in A^t \quad (3.17)$$

$$F_{ij}^{ot} \geq 0, \quad \forall t \in T, o \in N, (i,j) \in A^t \quad (3.18)$$

Objective function (3.1) minimizes the total cost incurred by the airline on locating a large capacity at a node (large airport or hub), the fixed capacity allocation costs, fixed cost of operating airplanes, and the variable costs per passengers transported. Constraints (3.2) allow allocating a demand node only to a located large airport. On the other hand, constraints (3.3) assure the assignment of every small airport to at least one large airport, while constraints (3.4) force the existence of at least one airplane flying the implied leg $i \rightarrow j$. Constraints (3.5) assure that a node is a large airport only if it has more than $\bar{R}$ units of

capacity allocated. Also, (3.6) and (3.7) are used to sequentially add capacity on nodes and arcs, respectively. Constraints (3.8) enforce the allocation of enough capacity at the nodes (airports) to meet the capacity on incoming arcs (airplanes). Moreover, constraints (3.9) assure that the flow in a certain type of airplane over an arc is at most the capacity implied by the airplanes allocated to it. The allocable arc capacity (fleet) is limited by (3.10), while (3.11) is an aggregated quality of service constraint, setting an upper bound on the total distance travelled in the network. (3.12) are flow balance constraints, and (3.13) are airplane conservation constraints. Constraints (3.14) assures that only one kind of airplane operates every leg, as usually happens on airlines. Finally (3.15)-(3.18) state the domain of decision variables.

Examples of operational constraints

Depending of the company, more constraints can be added. For example,

- a. If flights operated using certain type $\hat{t} \in T$ of airplanes must be only between large airports, constraints (3.19) are required.

$$\begin{aligned} W_{ij}^{\hat{t}} &\leq Z_{ii}, \\ W_{ij}^{\hat{t}} &\leq Z_{jj}, \end{aligned} \quad \forall (i, j) \in A^{\hat{t}} \quad (3.19)$$

- b. If certain type $\hat{t} \in T$ of airplanes is only allowed to make ‘back-and-forth’ routes, constraints (3.20) must be added to the formulation.

$$W_{ij}^{m\hat{t}} = \begin{cases} W_{ji}^{m\hat{t}}, & \text{if } (j, i) \in A^{\hat{t}}, \\ 0, & \text{o.c.} \end{cases} \quad \forall (i, j) \in A^{\hat{t}}, m = 1, \dots, \overline{M}^{\hat{t}} \quad (3.20)$$

Comparison with previous literature

It is important to note that this work does not assume any particular topology of the network to be designed, considering only technical and operational constraints, designing the network in terms of capacity allocation. In this sense,

the most similar work is (Jaillet et al., 1996), with the All-stop policy, because they design an airline network without having in mind a particular topology. However, there are differences. A first difference is that their focus is on determining candidate nodes for hubs, not to actually locate them. Secondly, in our MIP model, the objective includes the total operating costs, and not only the fixed costs of operating flights. Together with the network design, our model allocates capacity both on nodes and arcs, and explicitly locates facilities (large airports). In the third place, our aim is to correctly represent and use the economies of scale to design the network. This is achieved by using the right airplanes for the right legs (e.g., use low per-seat cost airplanes for long distance flights), optimally assigning the capacity, etc.

The MIP model can be seen as a generalization of previous models, which can be obtained deleting the node capacity allocation, setting parameters or adding additional constraints. For instance, the models of (Jaillet et al., 1996) can be obtained omitting the large airport location variables, considering unlimited supply of airplanes, and adding constraints on the maximum number of stops in a route (for the one and two-stop models).

The p -Hub Median Problem (Campbell, 1994; Ernst & Krishnamoorthy, 1996, 1998; O’Kelly, 1986, 1987) can be obtained fixing the number of large airports to p , dropping the capacity constraints and the fixed costs, defining two types of airplanes, the first ones operating only between large airports (hubs), and the second ones on the spokes, also requiring complete connectivity between hubs. These models can be extended to those using piecewise linear costs (Bryan, 1998; Ricardo Saraiva de Camargo et al., 2009; O’Kelly & Bryan, 1998), by the use of more than one type of airplane between hubs (every discount range), with decreasing unit variables costs, plus additional constraints for the sequential use of every discount factor.

The models in (Kimms, 2006) can be derived adding constraints on the maximum number of arcs (3) on every route, and dropping the capacity constraints on the nodes.

Hub Arc Location Models (Campbell et al., 2005a, 2005b) can also be obtained using the same changes as in the p-Hub Median Models, but bounding the number of (uncapacitated) airplanes flying between hubs, instead of the number of hubs.

3.4. Computational experiments

We tested our framework using the CAB10 instance from the literature. As there are no previous studies on the proposed airline network problem and hence, no full set of required parameters, we developed a base instance. We scaled the original demand $(w^{od})_0$ and distance $(d_{ij})_0$ matrices in order to get appropriate values for the other parameters. We considered two kinds of airplanes, denoted l (large) and s (small). The rest of the parameters are set as follows.

$$\begin{aligned} w^{od} &= 0.001w_0^{od}; \quad d_{ij} = 0.01d_{ij0}; \quad \overline{M}^l = 6; \quad \overline{M}^s = 6; \quad \overline{Q} = 10; \quad \overline{R} = 5; \quad CA_{ij}^l = 20d_{ij}; \\ CA_{ij}^s &= 10d_{ij}; \quad c_{ij}^l = 0.01d_{ij}; \quad c_{ij}^s = 0.02d_{ij}; \quad Au^l = 18; \quad Au^s = 6; \quad Cap^l = 200; \\ Cap^s &= 100; \quad CF_k = 1000, \forall k \in 1, \dots, 10; \\ CFC_{sk} &= 10 - 0.5(s-1), \forall k \in 1, \dots, 10, s \in 1, \dots, \overline{Q} \end{aligned}$$

In what follows, this set of parameters will be called the base instance for our experiments.

The MIP model was solved using the CPLEX 12.6 C++ library, on a PC with Intel(R) Core(TM) i7-2600 CPU @ 3.40GHz, 16 GB of RAM, Ubuntu 14.04.1

LTS Operating System. For the tests, CPLEX is allowed to run only 1 processing thread. The code was compiled using GCC 4.8.2.

In our experiments, we computed the following KPIs: Average Route Length, ARL; Average Leg Number, ALN; Average Airplane Utilization, AAU; Average Unit Cost, AUC; Average Leg Length, ALL; Total Distance Travelled, TDT; and (Optimal) Objective Function Value, OFV. The CPU time required to solve the instances is also reported.

We studied the solutions in terms of network structure and the values of the KPIs, from five different perspectives. First, we determine if the MIP model reproduces the existing economies of scale. Next, we studied the effect of changing the upper bound of the total distance travelled, as one proxy of the quality of service. Also, different fleet composition and operational constraints were tested. Finally, we analyze the effect of changes on flying material specifications.

3.4.1. Economies of scale

Recall that EoS exists when, keeping the network size constant, the average transportation cost is decreasing in the amount of demand.

In order to test whether our MIP model reproduces the economies of scale existing in practice, we computed iteratively all KPI's while successively multiplying all the demands w^{od} by a factor $(1+\nu)$, where the parameter ν represents the percentage of increase in demand, with respect to the base case. The complete results obtained are displayed in Table A.1, Appendix A.

Figure 3-2 shows the values of AUC (Average Unit Cost) and ARL (Average Route Length) for different values of ν . Note that AUC is decreasing (at a

decreasing rate) in ν , i.e. our MIP model indeed reproduces the economies of scale (EoS). Also, rather than assuming EoS *a priori*, as it is done in the fundamental model, these appear by correctly modelling the cost structure. On the other hand, the increase in ARL with ν is not strict, i.e., the function “jumps” at places where a marginal increase in demand enables cost minimization through changes in the network topology. However, in average, the larger the demand is, the longer is the route average, consistently with what is seen in practice. The stability, under certain circumstances, of network design to demand changes is also consistent with practice.

Also, Figure 3-3 shows the values of the increasing behavior both on of Average Leg Number (ALN) and Average Airplane Utilization (AAU) with respect to ν , consistent with reality. Note that an increase of ALN means that more complex networks are required and, in average, more unpleasant trips for the passengers. Meanwhile, the increase of AAU implies airplanes with fewer empty seats.

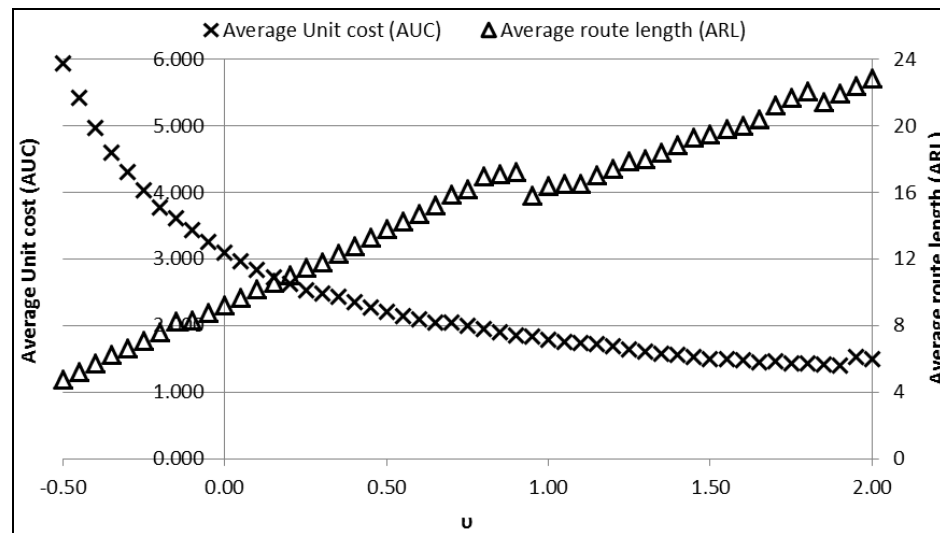


Figure 3-2. Average Unit cost (AUC) and Average route length (ARL) obtained varying ν in the base instance.

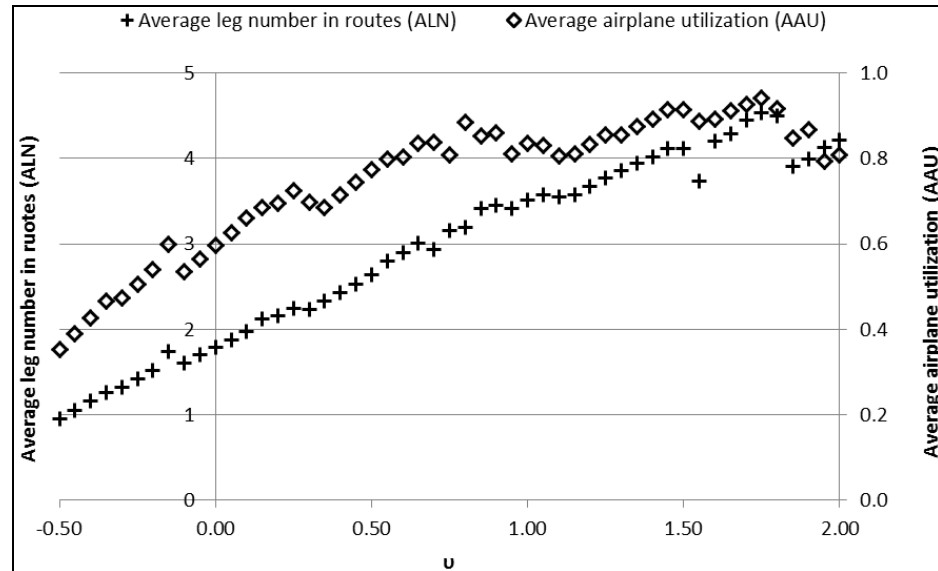


Figure 3-3. Average leg number in routes (ALN) and Average airplane utilization (AAU) obtained varying v in the base instance.

3.4.2. Upper bound on the total distance travelled

Following, we tested if setting an upper bound (χ) on the total distance travelled by passengers allows us to improve the service quality provided to them. First, we solved the unconstrained problem, and then we decreased χ , until the problem became infeasible. The complete results obtained are shown in Table A.2, Appendix A.

Figure 3-4 shows the values of the ARL and ALN, as a function of χ . Firstly, ARL is increasing in χ , denoting that the routes are, in average, shorter when χ is reduced, i.e. the average level of service is increased. Interestingly, ALN is non-monotonically increasing in χ , see for example the region around $\chi = 7000$

and $\chi = 7800$. At these points, in the solution, some small airplanes are replaced by large airplanes, and the average number of legs per route decreases.

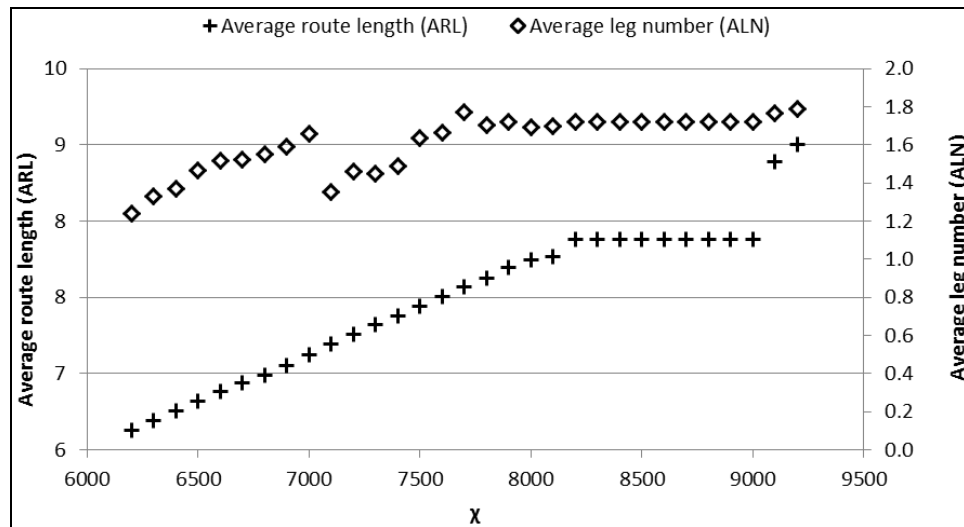


Figure 3-4. Average route length (ARL) and Average number of legs per route (ALN) obtained varying χ in the base instance.

This sudden change is also seen in the values of AAU in Figure 3-5, showing that around these regions the utilization of the airplanes decreases, caused by the higher number of large airplanes prescribed by the MIP model. In this figure, it can be also noted that AUC is decreasing in χ , i.e. the lower the average service level is, the cheaper is the average trip for the airline.

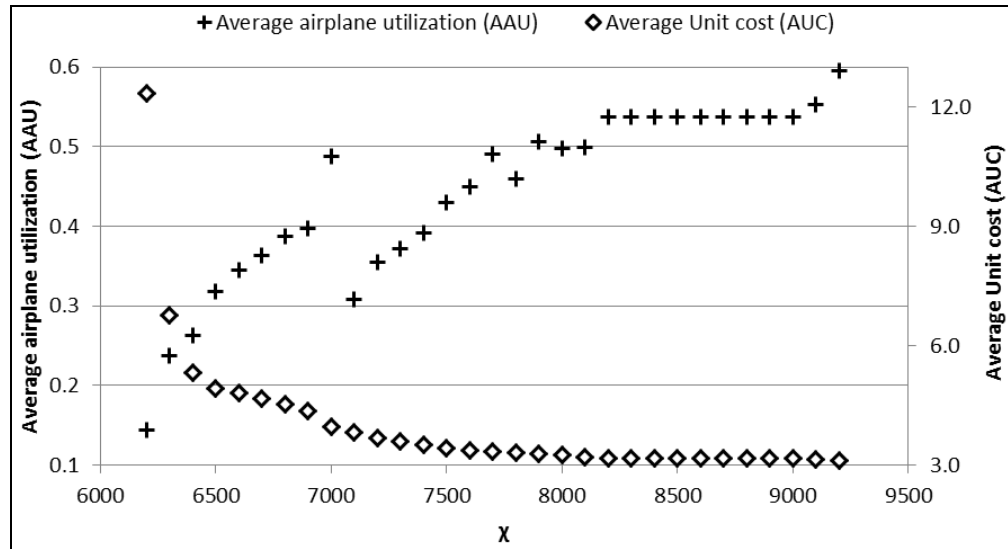


Figure 3-5. Average airplane utilization (AAU) and Average Unit cost (AUC) obtained varying χ in the base instance.

3.4.3. Fleet composition

It is unrealistic to consider an airline with an unlimited number of airplanes. Consequently, we tested the effect both on KPIs and network structure of changing the available fleet. In order to get useful insights, we modify the instance progressively.

First, we solved the base instance constraining only the number of large airplanes. Table 3-1 shows, for every feasible number of large airplanes (P^l), the number of small airplanes to be used (N^s), and the resulting KPIs. Note that AAU, AUC and OFV are decreasing in P^l , because in the base instance the MIP model, unconstrained on the number of airplanes, tries to use as many large airplanes as it is cost-efficient. Also, ALN is empirically decreasing in P^l as expected, caused by the progressively greater fleet flexibility.

Table 3-1. KPIs obtained by limiting the maximum number of large planes (P^l) available in the base instance.

P^l	N^s	ALN	AAU	AUC	OFV	Time (s)
3	18	2.001	0.833	3.523	3519.98	17.84
4	16	2.019	0.841	3.464	3460.38	106.11
5	16	2.019	0.841	3.464	3460.38	78.83
6	16	1.842	0.657	3.236	3233.20	172.98
7	16	1.842	0.657	3.236	3233.20	88.25
8	13	1.885	0.649	3.108	3105.44	53.02
9	13	1.885	0.649	3.108	3105.44	70.64
10	10	1.789	0.596	3.101	3097.88	90.20

Then, we constrained only the small airplanes. Table 3-2 shows the KPIs for every tested value of maximum numbers of small airplanes (P^s), together with the number of large airplanes prescribed by the MIP model (N^l) to cover all the demand. Note than in this case the fleet suggested by the model has 20 airplanes, varying the mix between the two kinds. On the other hand, ARL, ALN, AUC, TDT and OFV are decreasing in P^s , because of the greater flexibility that the MIP model has to design the network. Also, AAU and ALL are slightly increasing in P^s , as expected.

Table 3-2. KPIs obtained by limiting the maximum number of available small planes (P^s) in the base instance.

P^s	N^l	ARL	ALN	AAU	AUC	ALL	TDT	OFV	Time (s)
0	20	9.467	1.892	0.472	3.387	5.004	9457.38	3384.17	7.89
2	18	9.467	1.892	0.497	3.307	5.004	9457.38	3303.51	40.08
4	16	9.467	1.892	0.525	3.238	5.004	9457.38	3234.36	30.98
6	14	9.203	1.789	0.526	3.187	5.144	9193.69	3183.57	30.99
8	12	9.203	1.789	0.558	3.142	5.144	9193.69	3138.96	21.66
10	10	9.203	1.789	0.596	3.101	5.144	9193.69	3097.88	45.86

Finally, we constrained both large and small airplanes. In this case the KPIs do not show a clear trend, because the change on the maximum number of more than one kind of airplanes affects the way the MIP model allocates capacity both to nodes and arcs. Of course, smaller fleets lead to higher values of ARL, ALN, ALL, AAU and TDT. Also, the tighter the capacity is, the sparser the network topology is.

3.4.4. Operational constraints

Airlines operate under different operational constraints. We studied the effect on both the network structure and KPIs of adding constraints (3.19) or (3.20). To illustrate the effect, take the base instance with $P^l = 4$ and $P^s = 18$.

Table 3-3 shows the results obtained, in terms of KPIs, while Figure 3-6 show the network structures obtained. Taking Case 1 as the base of the analysis, note that the inclusion of constraints (3.19) for large airplanes, i.e. forcing them to operate only between large airports (Case 2), greatly increases OFV (and AUC), while keeping ARL, AAL relatively stable. In other words, this policy increases the costs but does not improve the performance, from the users' point-of-view.

Table 3-3. Results on base instance with $P^l = 4$ and $P^s = 18$, including constraints (3.19) or (3.20).

Case	Using (3.19)	Using (3.20)	ARL	ALN	AAU	AUC	ALL	TDT	OFV	Time (s)
1	No	No	9.469	2.019	0.841	3.464	4.689	9459.37	3460.38	137.08
2	Yes	No	9.737	2.001	0.833	4.524	4.867	9727.13	4519.98	25.44
3	No	Yes	8.323	1.980	0.761	3.517	4.203	8314.95	3513.14	55.39
4	Yes	Yes	8.807	1.932	0.742	4.616	4.558	8798.46	4611.03	12.79

Including constraints (3.20) for large airplanes, i.e. forcing symmetrical legs for large airplanes (Case 3), increases marginally the costs, i.e. AUC and OFV, but improves the user-related KPIs (ARL and ALL).

Finally, including constraints (3.19) and (3.20) together, i.e. Case 4, produce an intermediate result. It is important to highlight that ALN is quite stable on all the cases, been more dependent on fleet composition.

From a graphical point of view, Figure 3-6a shows that on Case 1, it is optimal to locate large airports at Dallas (node 7) and Cleveland (node 6), with large airplanes operating between these cities, Denver (node 8) and Chicago (node 4). Inclusion of constraints (3.19), namely Case 2, in Figure 3-6b, does not affect significantly the network topology, but the increase in costs is explained by the location of a large airport at Denver (node 8), only by operational purposes.

On the other hand, adding constraints (3.20), i.e. Case 3, in Figure 3-6c, affects the network topology, but keeping the location and number of the large airports. It is important to note that in this case, the large airport at node 7 (Dallas) is used mainly for maintenance of large airplanes, while the large airport located at node 6 (Cleveland) is used for connection between flights operated with small airplanes.

Finally, Figure 3-6d, obtained by adding constraints (3.19) and (3.20), shows a mixed result, locating more large airports and using symmetrically large airplanes.

This experiment shows both the large impact of operational constraints on network design and KPIs, and the way our framework can be successfully used as a decision support tool.

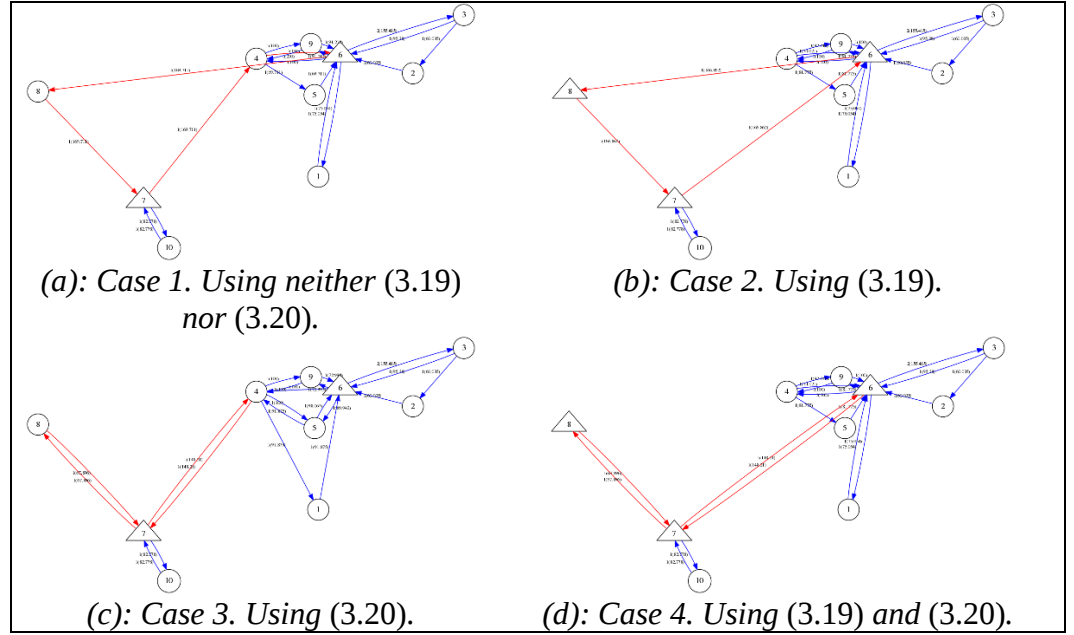


Figure 3-6. Network designs of base instance with $P^l = 4$ and $P^s = 18$, using (3.19) or (3.20).

3.4.5. Flying material specifications

Our last experiment studies the effect on KPIs and network structure of changing the specifications of the airplanes.

First we increased progressively the autonomy of small airplanes (Au^s) from 6 to 18 units of distance. Table 4 shows the value of previously used KPIs for the values of Au^s tested on the base instance.

It is clear that the AAU increases with Au^s , because more legs can be operated using small airplanes, which is desirable in markets with small demands and distant origin and destination nodes. Also the AUC is decreasing in Au^s , mainly

caused by the higher fleet flexibility. Note that the solution is sensitive to Au^s only for small values of Au^s , staying the same for $Au^s \geq 12$.

Table 3-4. KPIs obtained by changing the autonomy of small airplanes (Au^s) in the base instance.

Au^s	ALN	AAU	AUC	ALL	TDT	OFV	Time (s)
6	1.789	0.596	3.101	5.144	9193.69	3097.88	56.65
8	1.860	0.664	2.936	4.649	8640.87	2932.97	85.07
10	1.579	0.751	2.540	5.618	8861.15	2537.54	31.32
12	1.890	0.821	2.525	4.806	9075.03	2522.98	46.10
14	1.890	0.821	2.525	4.806	9075.03	2522.98	71.11
16	1.890	0.821	2.525	4.806	9075.03	2522.98	99.92
18	1.890	0.821	2.525	4.806	9075.03	2522.98	66.91

Note the impact on KPIs of increasing the autonomy of small airplanes, mainly on AUC. It could be considered as consistent with practice, particularly considering the current state of the airplane manufacturing industry, with both major manufacturers providing new long-ranged and mid-sized airplanes, both for network and low-cost carriers.

Later, we change the autonomy of large airplanes (Au^l), ranging from the original autonomy (18 units to distance), to 8 units of distance, where Table 3-5 shows the results. Solution is stable to changes in the autonomy of large airplanes for $Au^l \geq 12$. Also, ALN is decreasing in Au^l , mainly because larger airplanes with greater autonomy allow decreasing the average number of connections. Note that the KPIs are quite stable to changes in autonomy of airplanes, for the instances tested.

Table 3-5. KPIs obtained by changing the autonomy of large airplanes (Au^l) in the base instance.

Au^l	ARL	ALN	AAU	AUC	ALL	TDT	OFV	Time (s)
8	10.575	2.068	0.712	2.944	5.113	10565.00	2941.53	39.06
10	10.205	1.988	0.735	2.851	5.134	10194.70	2848.58	159.04
12	9.203	1.789	0.596	3.101	5.144	9193.69	3097.88	110.83
14	9.203	1.789	0.596	3.101	5.144	9193.69	3097.88	64.54
16	9.203	1.789	0.596	3.101	5.144	9193.69	3097.88	43.31
18	9.203	1.789	0.596	3.101	5.144	9193.69	3097.88	56.65

3.5. Concluding remarks

We present a framework and a model for the design of airline networks, including allocation of resources to airports, some of which become hubs. The model is able to represent economies of scale, if the cost structure and situation are such that these economies do exist. These economies are caused by the existence of fixed costs of using the airplanes, which are shared by the passengers flying a specific leg.

The framework can be easily extended to use different models to analyze more general problems. For example, if a competitive situation is addressed, a game-theoretical oriented model can be used instead. The same would occur if the demand is elastic to the transport offer. We leave these topics as future research.

Although the difficulty of solving the MIP model is high, it is important to highlight that even the small instances tested (based on CAB10) are useful to get insights on strategic airline network design, by testing different policies, either oriented to reductions of cost or towards a higher quality of service, i.e., towards the customers. For example, an upper bound on the total distance travelled can be used as a proxy for quality of service. In this case, a better way of improving quality would be to force the compliance of a strict standard of quality on every OD pair, but this can be added later, as additional constraints. Also, the effect on

KPIs of the available fleet, the operational constraints, and airplane autonomy can be derived from the instance of interest. The latter is important, because the appearance of short-haul airplanes with extended range is a tendency on the industry, adding flexibility to current and future fleets. On the other hand, in certain scenarios, the entire network can be aggregated in just a few ‘super nodes’, making location, allocation and flow routing decisions easier.

Regarding future work, extensions include the efficient solution of the MIP model for larger instances, because it is a capacitated (both on arcs and nodes) network design model, with special nodes to be located, and modular capacities, make it hard to solve for larger sizes. Then, heuristic or exact decomposition approaches must be studied.

4. ON SINGLE-ALLOCATION P-HUB MEDIAN LOCATION PROBLEMS WITH FLOW THRESHOLD-BASED DISCOUNTS AND ECONOMIES OF SCALE

The design and improvement of hub-and-spoke systems is a hot topic in air passenger transportation, postal and parcel service industries. However, the most extended class of models for hub networks, which we call the class of “fundamental models”, does not adequately represent economies of scale in these networks. An improved form of representation of the economies is an active research trend. We present a single allocation, incomplete inter-hub network, p-hub location problem in which a fixed unit cost discount is applied to the flow in an arc if it exceeds a fixed threshold. We use standard mathematical programming software to solve to optimality the resulting model for literature instances, and a heuristic procedure to get good feasible solutions. Aggregated performance indicators are used to analyze and compare solutions. Our results show that our model, based on the fundamental model for the p-hub, single allocation problem, is able to represent the existence of economies of scale, requires a reasonable computational effort, tends to consolidate flows between hubs, and can be efficiently solved by the proposed heuristic procedure.

4.1. Introduction

Hub Location Problems (HLPs) address the location of hubs and the allocation of demand to them. Hubs are a special kind of facility used mainly in air passenger, freight, and courier transportation for flow consolidation, sorting and commuting. The resulting network topology is called hub-and-spoke, where the spokes are the arcs connecting non-hub nodes with hub nodes.

Hub Location is a relatively young and active research area that began with the pioneering works of O’Kelly (1986, 1987), and the linearization proposed by Campbell (1994). The models were developed under a few assumptions: supply and offer are inelastic; the company is monopolistic; the inter-hub network is complete; and the flows between hubs are discounted by a fixed factor, i.e. independent of its volume. In the following, we will denote the models that use these assumptions as ‘fundamental models’.

Literature reviews by (Campbell et al., 2002; Alumur & Kara, 2008; Kara & Taner, 2011; Campbell & O’Kelly, 2012; Farahani et al., 2013) show the uninterrupted interest in hub location within the field of Location Analysis. Two research trends can be observed: improvements in the solution methods of the fundamental models, and the development of extensions of them.

The improvements in the solution methods of HLPs have been achieved by model reformulation (Ernst & Krishnamoorthy, 1996, 1998; O’Kelly, Bryan, Skorin-Kapov, & Skorin-Kapov, 1996; Boland, Krishnamoorthy, Ernst, & Ebery, 2004), the addition of cuts (Labbé, Yaman, & Gourdin, 2005; Rodríguez-Martín & Salazar-González, 2008; García, Landete, & Marín, 2012), and the use of decomposition schemes (de Camargo, de Miranda, & Luna, 2008, 2009; Rodríguez-Martín & Salazar-González, 2008; Contreras, Cordeau, & Laporte,

2011). Up to now, instances of the fundamental models with hundreds of nodes can be solved to optimality, while the extensions of the fundamental models can be solved only for small or medium-size instances.

To extend and adapt the fundamental models to different contexts, some of the assumptions have been changed. One such extension includes competition (Marianov et al., 1999; M. Sasaki & Fukushima, 2001; Adler & Smilowitz, 2007; Eiselt & Marianov, 2009; Lüer-Villagra & Marianov, 2013; Mihiro Sasaki et al., 2014). On another front, the completeness of the inter-hub network has been replaced by the use of other topologies, such as incomplete networks or trees (Contreras, Fernández, & Marín, 2010; de Sá, de Camargo, & de Miranda, 2013). Another strand of research are hub arc location problems where, as opposed to locating a number of hubs, a limited number of inter-hub arcs must be located, probably leading to incomplete inter-hub networks (Campbell, Ernst, & Krishnamoorthy, 2005a, 2005b). Finally, different cost structures have been proposed for HLPs to relax the fixed discount factor assumption, in order to represent the economies of scale. These can be classified in three categories: the use of thresholds, piecewise linear cost functions, and cost structures with fixed costs.

In the fundamental model, Campbell (1994) used both thresholds and fixed costs in the spokes of the network. A spoke would not be used unless the flow reaches the threshold. Fixed costs on spokes were used to control the structure of the network: if there were no fixed costs, every non-hub node would be connected through spoke edges to every hub, as this structure is optimal (multiple-allocation). As the fixed cost of enabling spokes increases, the number of spoke edges connecting non-hub nodes to hubs decreases, becoming eventually a single-allocation network. Modifying the threshold also changes the shape of the network, as an increasing threshold reduces the number of open spokes. Following similar lines, Podnar, Skorin-Kapov, & Skorin-Kapov (2002)

proposed a network design problem (rather than a hub location problem), where all the arcs can be discounted if a certain flow threshold is achieved.

The use of piecewise linear cost functions for HLPs began with (O’Kelly & Bryan, 1998) and the FLOWLOC model. Instead of considering a discontinuous cost structure, i.e. using thresholds, they linearize a non-linear objective function, where the discount factor between hubs is a flow-dependent function. In a follow up, Bryan (1998) extends the FLOWLOC model to include capacity constraints, minimum thresholds, non-fixed number of hubs and the use of the flow-dependent discount function everywhere in the network. Later, Klincewicz (2002) develops an optimal enumeration procedure for the FLOWLOC model, noting that if the hub locations are fixed, the model reduces to an uncapacitated facility location problem. Also, tabu-search and GRASP-based heuristic are provided. Given their structure, all these models tend to be hard to solve, or require more sophisticated solution techniques to solve tighter formulations, see de Camargo et al. (2009).

The fixed costs in Campbell (1994), rather than representing actual costs, were required for an appropriate modelling of the threshold-enabled spokes. Without the fixed costs, too many spokes would be enabled. To the best of our knowledge, the first author in recognizing that fixed costs do exist in practice, and that they should be estimated and included in the modelling, was Kimms (2006). He modelled an HLP where capacitated vehicles are used, having both fixed and variable costs. More recently, Lüer-Villagra, Marianov, & Latorre-Núñez (2015) proposed a modelling framework for airline network design, in which the underlying model extends the previous approaches by allocating capacity both to arcs and nodes. The solutions are analyzed using aggregated indicators of the network performance, such as the average number of legs in a route, vehicle utilization, etc. In industry, these indicators are commonly called *Key Performance Indicators* (KPI). It is important to note that, although previous

models with piecewise linear cost functions and fixed costs do represent economies of scale properly, they are hard to solve for practical sizes, i.e. 25 or more nodes, without further algorithmic enhancement.

Figure 4-1 shows the different cost structures that appear in the models in the literature. It is important to highlight that, in the fundamental models (Figure 4-1a), there are two linear cost functions, representing respectively the costs of travel over spoke arcs (upper line) and inter-hub arcs (lower line). It is assumed a priori that there will be a larger flow between hubs and hence, the unitary cost will be lower. Should the flow volume change on any arc (spoke or inter-hub), the unitary cost remains the same and the model cannot adequately represent economies of scale. The remaining curves in Figure 4-1 consider a threshold, marking a flow volume at which costs change. If piecewise linear costs are used, Figure 4-1b, the total cost function is continuous in the amount of flow, but the slope decreases once the threshold is reached, so representing decreasing costs when flow is consolidated in the network. If fixed cost structures are used, Figure 4-1c would represent the case in which vehicles with limited capacity are considered. The increase in the total cost after the threshold, without any change in the slope, occurs at the flow at which all vehicles travelling on an arc are full, and another vehicle with the same fixed and variable cost must be allocated to the arc. Finally, if threshold-based discounts are used, Figure 4-1d, after the threshold, both the slope and the total cost decrease. We assume the last type of structure, which somehow keeps the features of the fundamental model, but corrects its main flaw: the lack of adequate representation of the actual economies of scale.

Note that the existence of different cost structures has led to interest in the study of economies of scale in hub-and-spoke networks. According to Basso & Jara-Díaz (2006), economies of scale are defined as the less than proportional increase in total costs caused by a proportional increase in transportation demand, i.e. the

average unit cost is decreasing in the demand, keeping the network size fixed but allowing the network structure to change. Although the existence of economies of scale has a central role in the use of hub-and-spoke systems, the issue of their appropriate modeling has been raised only recently, see for example (Kimms, 2006; Campbell & O’Kelly, 2012; Campbell, 2013). From an empirical point of view, recent evidence shows that these economies do exist in the USA airline industry (Johnston & Ozment, 2013).

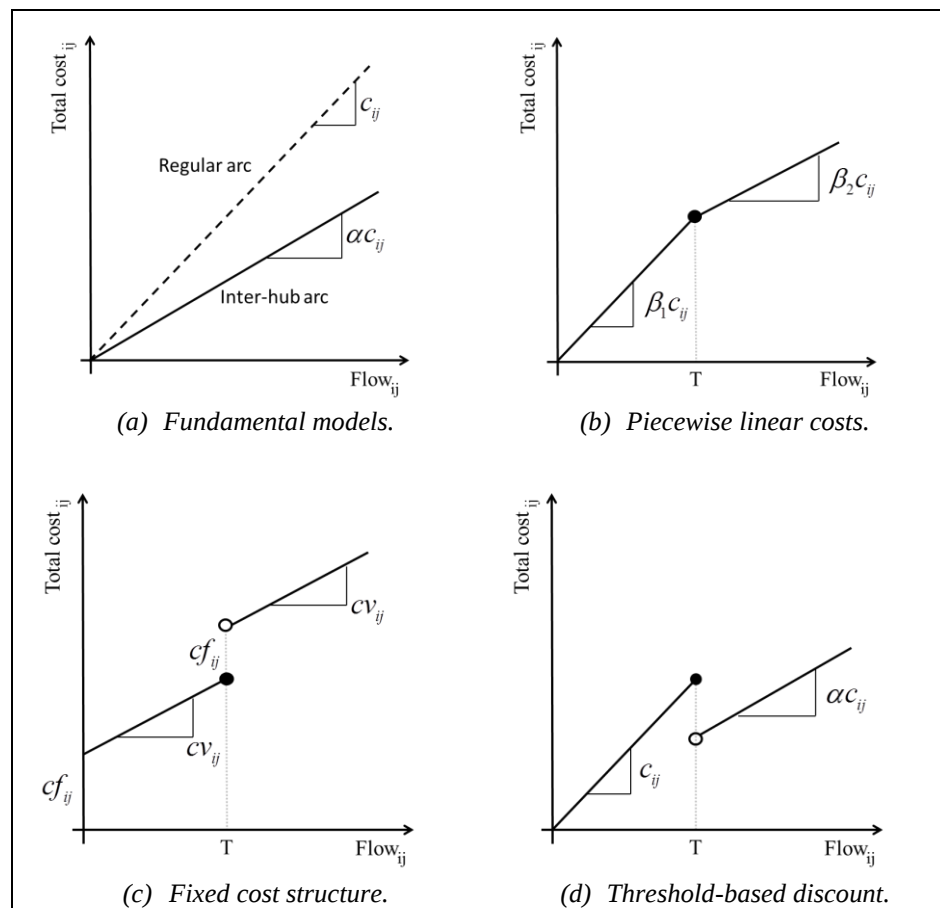


Figure 4-1. Comparison of different costs structures used in HLPs, for a generic arc $(i, j) \in A$.

A review of the literature shows that the threshold-based discount applied to every arc of a hub-and-spoke network has not been explicitly used before, and that the study of the cost structure is relevant in the representation of economies of scale in hub-and-spoke networks, in order to extend the fundamental models.

Our contribution is multiple. First, we formulate a single-allocation p -HLP based on the fundamental model, in which the flow in all the arcs of the network, both spokes and inter hub arcs, is discounted if and only if it is larger than a fixed threshold. Secondly, we show that our model is able to represent economies of scale and flow consolidation between hubs. Thirdly, we compare our approach with the fundamental model, both in terms of solution characteristics (Key performance indicators) and the computational effort required. Finally, we propose a fast heuristic procedure.

The remainder of this chapter is organized as follows. In subsection 4.2, the problem is described, and the model stated. Following, subsection 4.3 contains the experiments and discussion. Finally, subsection 4.4 provides concluding remarks and future research.

4.2. The problem

A non-competing company wants to locate p hubs in a network, represented by a graph $G(N, A)$, where N is the set of nodes, and A the set of arcs. For every pair of nodes $o, d \in N$ the inelastic and deterministic transportation demand is known and denoted w_{od} . For every arc $(i, j) \in A$, the cost per unit of flow is undiscounted if the flow in that arc is less than T units of flow, and denoted by c_{ij} ; otherwise, a discount factor $0 < \alpha < 1$ is applied. Also, the total flow that originates from node $o \in N$ is denoted by $O_o = \sum_{j \in N} w_{oj}$. Similarly, the total flow

that ends in node $d \in N$ is denoted by $D_d = \sum_{i \in N} w_{id}$. Finally, every non-hub node is allocated to a single hub node, i.e. we consider industries where the cost of enabling a spoke is high. First we present the exact model for the problem, followed by an ad-hoc heuristic procedure.

4.2.1. Exact model

Consistent with the notation in (Ernst & Krishnamoorthy, 1996), we define our model as follows.

Variables

- Z_{ik} : 1, if node $i \in N$ is allocated to a hub located at node $k \in N$; 0, otherwise.
- Y_{kl}^i : Flow that originates from node $i \in N$ that goes through hubs located in $k, l \in N$.
- f_{ij} : Regular flow through arc $(i, j) \in A$.
- g_{ij} : Discounted flow through arc $(i, j) \in A$.

Objective function

$$\min \sum_{(i,j) \in A} c_{ij} (f_{ij} + \alpha g_{ij}) \quad (4.1)$$

Constraints

$$\sum_{k \in N} Z_{ik} = 1, \forall i \in N \quad (4.2)$$

$$Z_{ik} \leq Z_{kk}, \forall i, k \in N \quad (4.3)$$

$$\sum_{k \in N} Z_{kk} = p \quad (4.4)$$

$$\sum_{l \in N} Y_{kl}^i - \sum_{l \in N} Y_{lk}^i = O_i Z_{ik} - \sum_{j \in N} w_{ij} Z_{jk}, \forall i, k \in N \quad (4.5)$$

$$\sum_{l \in N} Y_{kl}^i \leq O_i Z_{kk}, \forall i, k \in N \quad (4.6)$$

$$\sum_{k \in N} Y_{kl}^i \leq O_i Z_{ll}, \forall i, l \in N \quad (4.7)$$

$$O_i Z_{ij} + \sum_{m \in N} Y_{ij}^m + D_j Z_{ji} = f_{ij} + g_{ij}, \forall (i, j) \in A \quad (4.8)$$

$$\{f_{ij}, g_{ij}\} \in \text{SOS-1}_{ij}, \forall (i, j) \in A \quad (4.9)$$

$$f_{ij} \in [0, T], \forall (i, j) \in A \quad (4.10)$$

$$g_{ij} \in \{0\} \cup [T, +\infty), \forall (i, j) \in A \quad (4.11)$$

$$Z_{ik} \in \{0, 1\}, \forall i, k \in N \quad (4.12)$$

$$Y_{kl}^i \geq 0, \forall i, k, l \in N \quad (4.13)$$

Objective function (4.1) minimizes the total costs. Constraints (4.2) assure that every non-hub node is allocated to a single hub node, not allowing connections between non-hub nodes. Constraints (4.3) allocate non-hub nodes only to located hub nodes, while constraint (4.4) ensures that p hubs are located, and (4.5) are flow conservation constraints. Only feasible inter-hub flows are enforced by (4.6)-(4.7). Constraints (4.8) compute the total flow in every arc. The threshold-based discount is modelled through (4.9)-(4.11), while (4.12)-(4.13) state the domain of the remaining decision variables.

Note that for every arc, f_{ij} and g_{ij} belong to a Special Ordered Set Type 1 (SOS-1), i.e. at most one variable in the set is non-zero in a feasible solution. Also, is important to highlight that variables g_{ij} are semi-continuous. We use these

modelling structures to avoid the use of Big-M constraints, and to let the solver exploit the problem structure in a more efficient way.

4.2.2. Heuristic procedure

Preliminary computational experiments showed that although the time required to solve our model is shorter than solving other piecewise linear/fixed costs hub models, it is not competitive with the standard computational implementation of the fundamental models. Due to this fact, we developed an iterative heuristic that starts by solving the formulation by Ernst & Krishnamoorthy (1996). In the solution, it checks whether the flow through discounted arcs exceeds the threshold and the amount of flow through non-discounted arcs is below the threshold. If this is so, the solution is optimal. Otherwise, the heuristic progressively updates the discounts applied in every arc, so they approach the right discounts corresponding to the amount of flow in it.

Let $AD^{(t)}$ be the following sub-problem, in iteration t of the heuristic execution.

$$\min \sum_{(i,j) \in A} \gamma_{ij}^{(t)} c_{ij} h_{ij} \quad (4.14)$$

$$O_i Z_{ij} + \sum_{m \in N} Y_{ij}^m + D_j Z_{ji} = h_{ij}, \forall (i, j) \in A \quad (4.15)$$

$$h_{ij} \geq 0, \forall (i, j) \in A \quad (4.16)$$

Together with expressions (4.2)-(4.7), (4.12)-(4.13). Note that $AD^{(t)}$ is essentially a single-allocation p -hub location problem in which the flows are discounted by the parameters $\gamma_{ij}^{(t)}$.

Let EK be the formulation of the fundamental model given by Ernst & Krishnamoorthy (1996) and P_{LR} the linear relaxation of problem P , and ε , $\beta_{\uparrow}$ and $\beta_{\downarrow}$ execution parameters. The pseudo-code of the heuristic procedure is presented in Figure 4-2.

```

1      Solve  $EK$ 
2       $t \leftarrow 0$ 
3      For  $(i, j) \in A$ 
4          If  $Z_{ii} = 1$  and  $Z_{jj} = 1$  then
5               $\gamma_{ij}^{(0)} \leftarrow \alpha$ 
6          Else
7               $\gamma_{ij}^{(0)} \leftarrow 1$ 
8      stop  $\leftarrow 0$ 
9      Do
10         stop  $\leftarrow 1$ 
11         Solve  $AD_{LR}^{(t)}$ 
12         For  $(i, j) \in A$ 
13             If  $h_{ij}^* \geq T$  then
14                  $\gamma_{ij}^{(t+1)} = \max \{ \alpha, \gamma_{ij}^{(t)} - \beta_{\downarrow} \}$ 
15             Else
16                  $\gamma_{ij}^{(t+1)} = \min \{ 1, \gamma_{ij}^{(t)} + \beta_{\uparrow} \}$ 
17             If  $|\gamma_{ij}^{(t)} - \gamma_{ij}^{(t+1)}| < \varepsilon$  then
18                 stop  $\leftarrow 0$ 
19         If stop = 0 then
20              $t \leftarrow t + 1$ 
21     While stop = 0
22     Solve  $AD^{(t)}$ 
23     Return solution obtained

```

Figure 4-2. Pseudo-code of proposed heuristic procedure.

The heuristic procedure starts by solving the fundamental model (line 1), and initializing the iteration counter (line 2). Then, using the solution of problem EK, initial values for parameters $\gamma_{ij}^{(0)}$ are set (lines 3-7), as well as the stopping condition (line 8). After that, until the stopping condition is met (lines 9-21), successive linear relaxations of problem AD are solved (line 11), and if the ‘fractional flow’ in an arc exceeds threshold T , its discount factor is decreased in $\beta_{\downarrow}$ at most (lines 13-14), otherwise it is increased in $\beta_{\uparrow}$ at most (lines 15-16), and if the discounts do not change between iterations, the loop stops (lines 17-18), or the iteration counter is updated (lines 19-20). Finally, AD is integer-solved (line 22) and the solution obtained is returned (line 23).

4.3. Computational Experiments and Discussion

To test our model, we used the 25 node instance of the CAB dataset (CAB25), dividing all demands by 1000. To implement and solve our model we used AMPL and CPLEX 12.6.1, on a PC with Intel(R) Core(TM) i7-4790 CPU @ 3.60GHz, 32 GB of RAM, Ubuntu 14.04 LTS Operating System. For the tests, CPLEX is allowed to use up to 4 threads.

Together with the solutions obtained in our experiments, and as a way to analyze their quality, we computed some aggregated performance indicators. These indicators are related to the objectives of the network operator, and the quality expectations of the users of the network. The Average Route Length indicates how long is the average trip for a passenger; the Average Number of Legs per Route counts the number of transfers a passenger would require and, from the point of view of the operator, the number of legs (or planes) required to cover a route; the Average Unit Cost is an indicator of the cost per passenger or per unit

of load; the Average Leg Length; the Fraction of Flows that are Discounted gives the operator an indication of the percentage of flow that is carried at a discounted cost in that solution; and the (Optimal) Objective Function Value, Z^* . The CPU time required to solve the instances is also reported.

In our experiments, we have multiple goals. The first goal is to show that our model actually does correctly represent economies of scale. With this goal in mind, we progressively increase the total demand, registering the behavior of the average unit cost. Secondly, we aim at characterizing the solutions obtained by changing the flow threshold T and the discount factor α , in terms of the aggregated performance indicators. Thirdly, we seek to compare our approach with the corresponding fundamental model (using the performance indicators), and measure how far off is this last one by fixing the discount structure. And finally, we want to assess the performance of our heuristic.

4.3.1. Existence of economies of scale

We first progressively increase the demand in every origin-destination (OD) pair, solving the model for the new demands and computing the AvgUnitCost. If AvgUnitCost is decreasing, economies of scale are indeed reproduced by the model. Let ν be the percentage of increase in transport demand on every OD pair. Figure 4-3 shows the values of AvgUnitCost and FracDiscFlow obtained by our model for $p=3$, $T=400$ and $\alpha=0.5$. It clearly shows that our model is able to represent economies of scale, because AvgUnitCost is decreasing when demand increases. Note that FracDiscFlow is approximately increasing in ν , because the model is using more intensively the discounted arcs in the network, without changing in average what could be taken as a proxy of quality of service, i.e. the values of AvgLegLen, AvgRouteLen y AvgLegNum, which are insensitive to ν for this case.

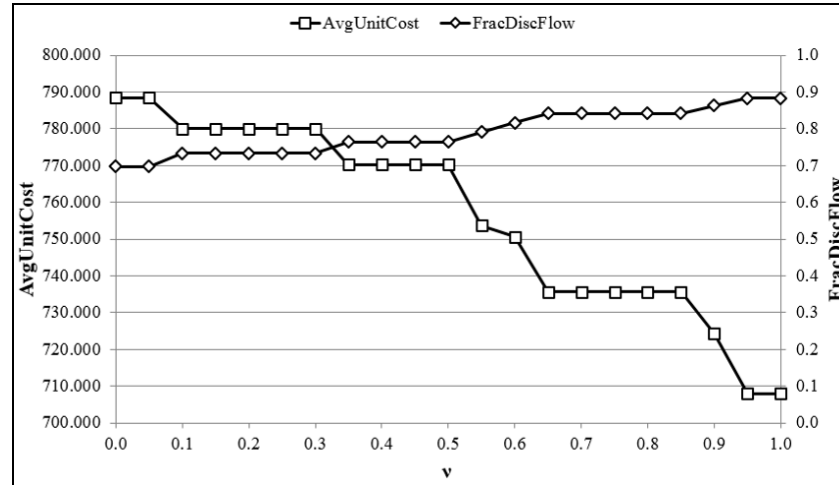


Figure 4-3. AvgUnitCost and FracDiscFlow obtained for $p=3$, $T=400$ and $\alpha=0.5$

4.3.2. Sensitivity in the flow threshold T

As a parameter of our model, the flow discount threshold T has a key role in the solutions obtained. Figure 4-4 shows the values of AvgUnitCost obtained when $\alpha = 0.5$ for different values of T and p . As the demand is fixed for this experiment, the change in AvgUnitCost is proportional to Z^* . As expected, AvgUnitCost is increasing in T and decreasing in p . If the threshold value is higher, fewer arcs are discounted, and it is more expensive to satisfy the demand in average. Also, as p increases, AvgUnitCost decreases because the constraint on the number of hubs is less tight. The non-concavity of AvgUnitCost is caused by the discontinuous nature of our problem and the changes in the network structure caused by the different values of T .

This fact is confirmed by Figure 4-5, where FracDiscFlow is appears quasi-decreasing in T , implying that fewer units of flow are transported through

discounted arcs if the discount threshold is increased. The non-monotonicity can be explained analogously to the previous figure.

4.3.3. Effect of the discount factor

As it happens also with the fundamental model, our approach depends on the appropriate estimation of the value of the discount factor α , which is mostly industry-dependent. Note that lower values of α imply bigger incentives to consolidate flows in certain arcs. Figure 4-6 shows the values of AvgRouteLen and FracDiscFlow obtained for different values of α when for $p=4$ and $T=500$. Note that FracDiscFlow is approximately decreasing in α , implying that a lower portion of the flow is transported through discounted arcs if there is less incentive to use them. Also, AvgRouteLen is decreasing in α , as expected, meaning that – conversely – the routes tend to be longer when strong discounts are applied to arcs where the flow exceeds the threshold, because the hubs tend to spread apart, taking advantage of the low cost of inter-hub travel. In other words, longer routes with consolidated and discounted flows are preferred in average.

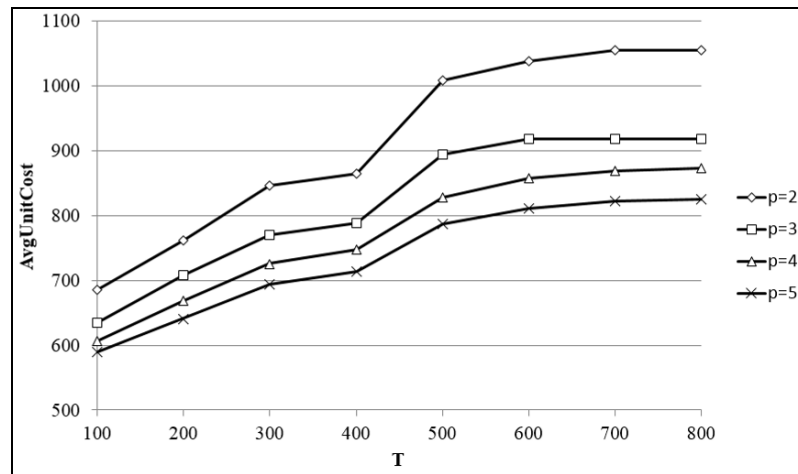


Figure 4-4. Values of AvgUnitCost obtained for different values of T, for $\alpha=0.5$.

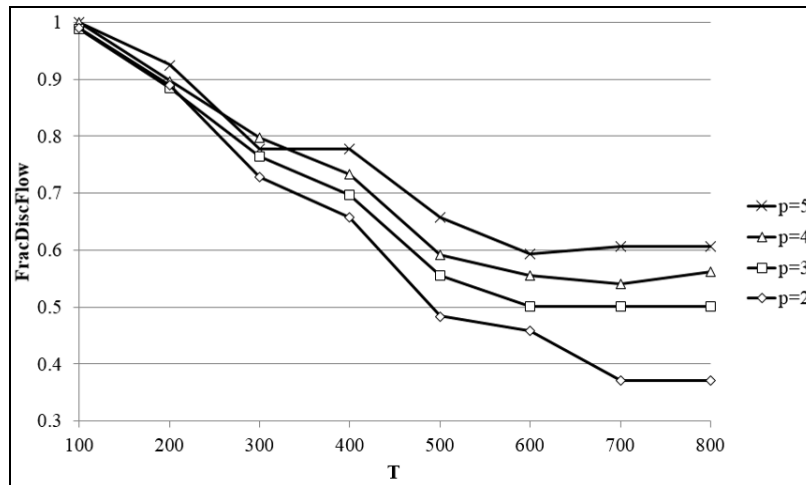


Figure 4-5. Values of FracDiscFlow obtained for different values of T, for $\alpha=0.5$.

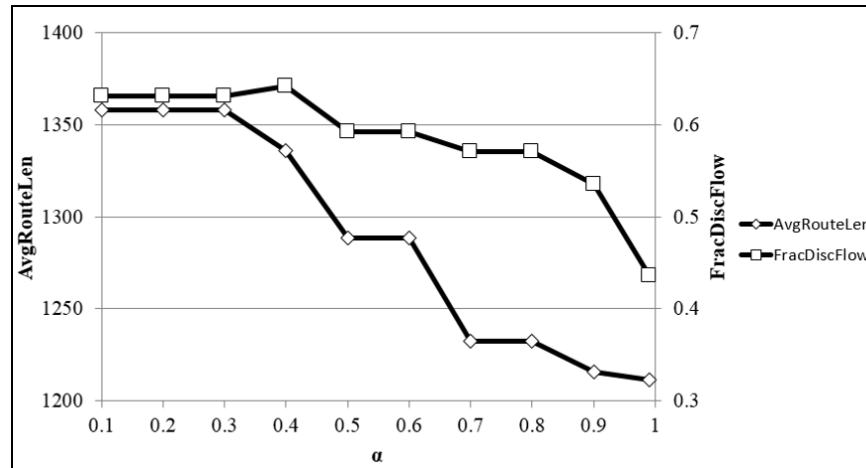


Figure 4-6. Values of AvgRouteLen and FracDiscFlow obtained for different values of α , for $p=4$, $T=500$.

The effects in flow consolidation (FracDiscFlow) and hub dispersion (AvgRouteLen) are confirmed in Figure 4-7, that shows the solutions obtained when $p=4$ and $T=500$, for $\alpha = 0.1$, $\alpha = 0.5$ and $\alpha = 0.9$. The circles are regular nodes; triangles are the nodes with located hubs; and the thicker the arc, the larger is the flow through it.

In Figure 4-7a, when weak discounts are applied ($\alpha = 0.9$) the hubs are located not far from each other, because long arcs do not provide a significant cost difference if they are discounted. Also, some spokes carry large flows, and the inter-hub flows are not significantly larger than spokes' flows. If the discount is stronger, for example $\alpha = 0.5$ in Figure 4-7b, the network topology changes, and long discounted arcs appear, consistent with the higher value of AvgRouteLen. Note also that the inter-hub network is sparser, and less spokes carry large flows. In other words, the flows tend to concentrate between hubs. Finally, if the discount is even stronger ($\alpha = 0.1$ in Figure 4-7c), the network topology,

although it stretches a little more, is essentially the same, confirming the stability of AvgRouteLen and FracDiscFlow for low values of α shown in Figure 4-6.

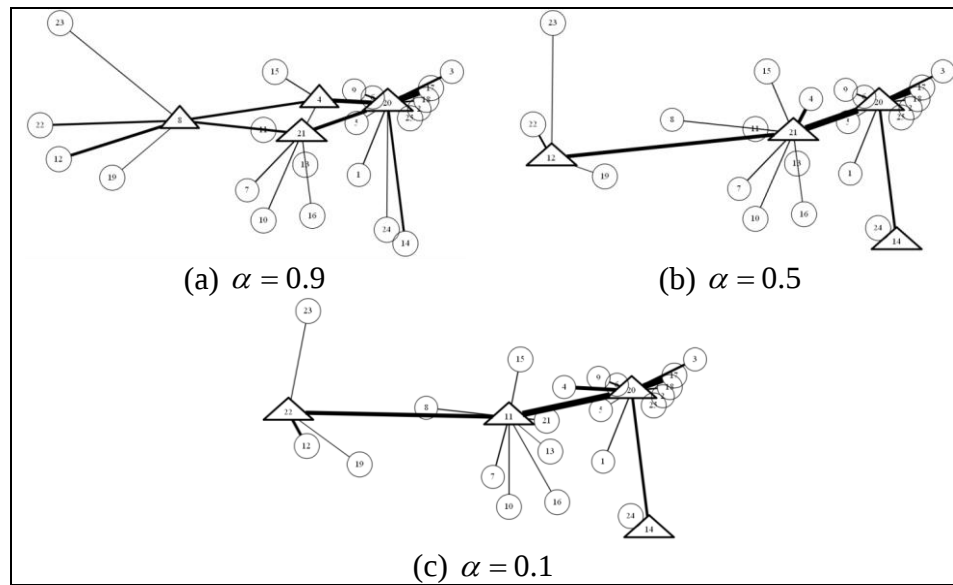


Figure 4-7. Solutions obtained for $p=4$, $T=500$ and different values of α .

4.3.4. Comparison of our exact model with the fundamental p -hub model

In the following, we compare our approach with the fundamental single-allocation p -hub median model. A graphical comparison could be appealing, but ambiguous. We prefer to use KPIs. Table 4-1 shows the results obtained by comparing our model for different values of the threshold T , with the fundamental model, if $\alpha = 0.5$. The KPIs were computed assuming travel costs proportional to the distances, and they are in comparable units. The first row of each part of the table (*FM*) shows the values of the KPIs obtained by solving the fundamental model. The remaining rows show the figures for different values of the threshold T .

First of all, the fundamental model tends to use longer legs in average (AvgLegLen), but not always longer routes (AvgRouteLen), because the average leg number (AvgLegNum) tends to be smaller. This is important from the perspective of users, e.g. in passenger flights, but when there is no user-comfort involved, as in parcel delivery or courier services, this indicator has no relevance.

From this point of view, the fundamental models show results with a better quality of service (comfort) at the expense of efficiency. Recall, though, that the results of the fundamental models do not necessarily describe what happens in practice, because some legs that in the fundamental model appear as discounted, in practice are not, and vice versa. In second place, comparatively, our model tends to build networks that use more intensively the discounted arcs, obtaining higher values of FracDiscFlow, even when the flow threshold T is large. From a cost-perspective, the lower value of AvgLegNum and FracDiscFlow for the fundamental model implies higher costs. On the other hand, from a computational perspective, our model is harder to solve than the fundamental model, but easier than other models with piecewise linear costs that require the use of decomposition approaches and more sophisticated algorithms to be solved for this instance size.

4.3.5. Heuristic performance

We tested the performance of our heuristic with the following experimental settings: $\nu = \{0.00, 0.05, \dots, 1.00\}$, $p = \{2, 3, 4, 5\}$, $T = \{100, 200, \dots, 800\}$, and $\alpha = 0.5$.

For these settings, the heuristic procedure provides in average solutions that are 8.81% away from optimum, in 0.02% of the CPU time required by the exact method. Figure 4-8 and Figure 4-9 show the histograms for the proposed

measures, noting that both have positive kurtosis and reasonable spread around the mode, especially for the CPU time ratio.

Table 4-1. Comparison between our exact model and the fundamental model, for $\alpha=0.5$.

p	T	AvgLegLen	AvgRouteLen	AvgLegNum	FracDiscFlow	AvgUnitCost	Z*	Time
2	FM	690.212	1403.796	2.034	0.116	1151.990	9838004	1.24
	100	600.877	1365.175	2.272	0.990	685.524	5854376	171
	200	633.339	1366.235	2.157	0.889	761.417	6502508	233
	300	607.223	1366.606	2.251	0.728	845.924	7224198	269
	400	607.223	1366.606	2.251	0.658	864.210	7380362	280
	500	633.339	1366.235	2.157	0.484	1007.715	8605895	680
	600	607.223	1366.606	2.251	0.459	1037.680	8861791	626
	700	678.060	1384.081	2.041	0.371	1054.023	9001363	386
3	800	678.060	1384.081	2.041	0.371	1054.023	9001363	637
	FM	612.365	1298.597	2.121	0.254	967.849	8265432	2.02
	100	576.463	1256.630	2.180	0.989	633.906	5413558	162
	200	548.631	1274.257	2.323	0.884	708.040	6046667	718
	300	548.631	1274.257	2.323	0.765	770.242	6577873	570
	400	548.631	1274.257	2.323	0.697	788.528	6734037	446
	500	563.399	1292.682	2.294	0.555	893.652	7631789	1004
	600	563.399	1292.682	2.294	0.502	918.215	7841561	1282
4	700	563.399	1292.682	2.294	0.502	918.215	7841561	993
	800	563.399	1292.682	2.294	0.502	918.215	7841561	946
	FM	650.643	1242.587	1.910	0.342	863.360	7373102	2.82
	100	554.849	1212.774	2.186	1.000	606.387	5178547	328
	200	528.533	1216.933	2.302	0.897	668.305	5707332	2692
	300	513.477	1238.630	2.412	0.797	725.492	6195708	481
	400	513.070	1246.857	2.430	0.734	747.892	6387002	2272
	500	543.256	1288.514	2.372	0.592	827.270	7064887	2376
5	600	558.155	1297.289	2.324	0.556	857.958	7326970	3344
	700	507.243	1283.903	2.531	0.540	868.763	7419241	4432
	800	521.589	1290.265	2.474	0.562	872.534	7451444	5873
	FM	669.797	1214.380	1.813	0.370	792.137	6764854	2.79
	100	509.888	1178.466	2.311	1.000	589.233	5032053	1211
	200	503.801	1187.905	2.358	0.925	640.929	5473535	2606
	300	524.262	1196.236	2.282	0.777	694.050	5927193	3601
	400	470.512	1210.737	2.573	0.778	713.043	6089389	14498
	500	538.048	1285.861	2.390	0.658	786.518	6716872	54647
	600	505.491	1261.530	2.496	0.594	810.980	6925775	65504
	700	461.033	1248.123	2.707	0.606	821.823	7018371	76618
	800	463.621	1255.131	2.707	0.606	825.327	7048297	86326

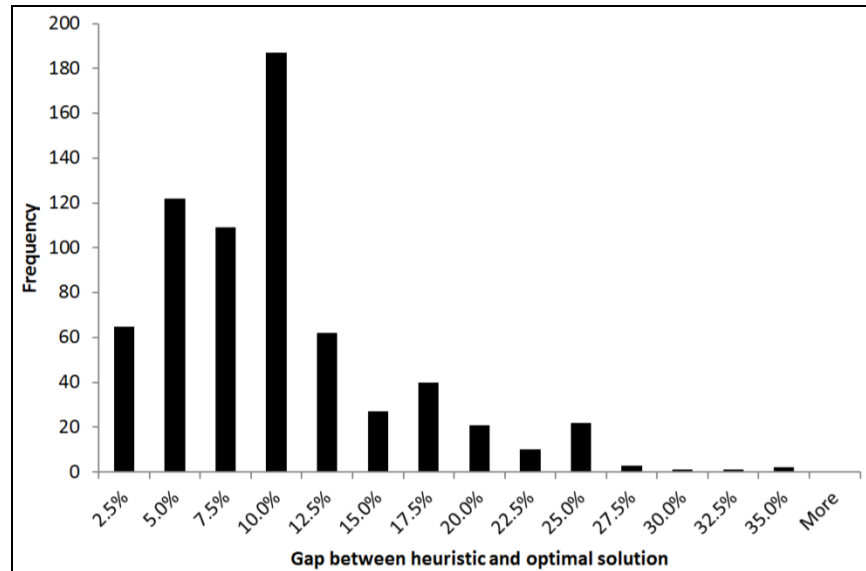


Figure 4-8. Percentage gap between heuristic and optimal solutions, for the tested instances based on CAB25.

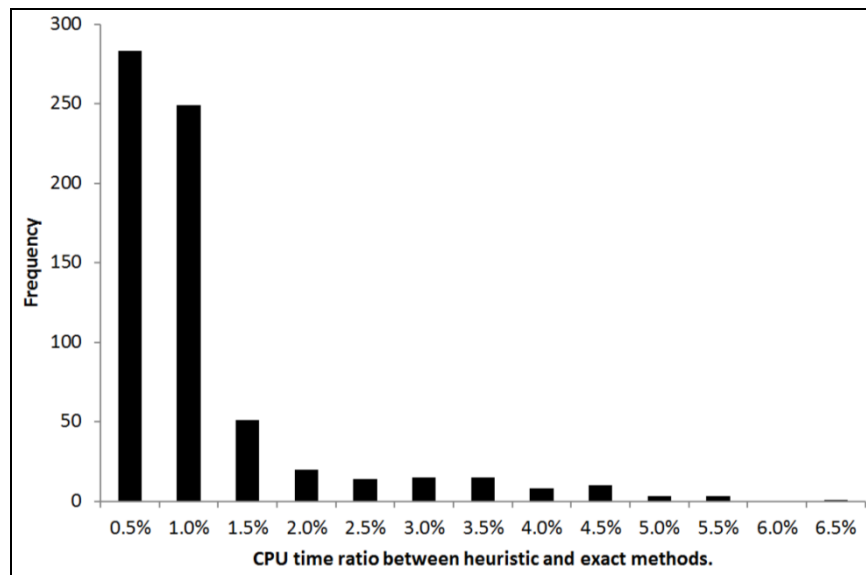


Figure 4-9. Percentage ratio of CPU times of heuristic and optimal methods, for the tested instances based on CAB25.

4.3.6. Summary

We showed that, as opposed to the fundamental model, our model prescribes network designs that do represent economies of scale, behaves properly to changes in the main parameters, and produces solutions coherent with a flow-dependent discount assumption. In other words, the solutions obtained by our model are a better representation of reality than the fundamental model for the same problem. Also, we tested our heuristic procedure, noting that it is fast and efficient in computing good quality feasible solutions.

4.4. Concluding remarks

We formulated a single-allocation p -HLP with threshold-based discounts anywhere in the network. We showed that our model for the single-allocation p -hub median, based on the fundamental model for the same problem, is able to represent economies of scale, tends to consolidate flows between hubs, obtaining similar or better (for some industries) values of the KPIs used, in exchange for a higher computational effort, which, in any case, is comparatively smaller than existing more complex models. We also developed a fast heuristic procedure, in order to quickly get high quality feasible solutions.

Note that a good heuristic that obtains solutions reproducing the real economies of scale is a powerful tool that will allow a substantial leap in hub research, as it makes possible to analyze complex extensions of hub problems, without the flaw of the fundamental models. In fact, future research includes the use of these heuristic feasible solutions as starting points for more sophisticated fast heuristics, its extension to other hub location problems, as for example the hub location problem with fixed costs, considering covering and center objectives, including competition, different service quality constraints, etc.

5. CONCLUSION

This thesis successfully develops extensions to fundamental hub location models. The contributions are both theoretical and practical.

In Chapter 2 we state, formulate and heuristically solve a competitive hub location problem that includes pricing decisions. Two firms compete in a transportation market. An incumbent firm operates a hub-and-spoke network and applies mill pricing. The entrant firm wants to enter in order to maximize its profit by designing and operating a hub-and-spoke network, and by making optimal pricing decisions. The customers' choice between the routes of the incumbent and the entrant is based only on price, following a simple logit model.

The resulting non-linear mixed integer programming model is solved using a genetic algorithm, and given that if the networks of both companies are fixed, a closed form expression for the optimal entrant's policy can be derived. We tested our model using the CAB dataset.

Our computational experiments show that both pricing and users' sensitivity parameter are relevant in competitive hub location problems. Their role for both companies depends mainly on the incumbent's margin and network configuration. We also show that if sensitivity is considered, it is not optimal to the entrant to price its routes below the incumbent's cheapest, for a given OD pair.

Finally, we show that if competition is considered, the inter-hub costs strongly influence entrant's decision of entering a market. We also note that a profit-maximizing model provides different solutions and insights than a market share-maximizing model.

Future work includes incorporating the reaction of the incumbent after entrant's appearance, the extension of our users' behavior model to include elastic demand and other factors like travel time, number of stops in a route, etc. Finally, we think that the study of sharing hubs between companies to reduce congestion and costs is a very promising research line.

In Chapter 3 we develop a framework and a model for airline network design. Our model represents economies of scale, if these are available by the cost structure. Our framework provides a flexible decision support tool, which can be easily extended to other situations, like competitive environments, elastic demand, etc.

We show that although our model is hard to solve, we are able to get insights on airline network design using small instances. For example, setting and upper bound to the total distance traveled can be used to increase the average service quality to customers.

The use of KPIs (Key Performance Indicators) is also relevant, and provides an appropriate way to analyze quantitatively network design and location problems.

Future research includes the efficient (heuristic or exact) solution of our model for larger instances, perhaps through decomposition, model reformulation or formulation strengthening.

In Chapter 4 we formulate and solve a single-allocation p -HLP, where threshold-based discounts can be applied anywhere in the network.

Our model, based on the fundamental single-allocation p -hub median model, can represent economies of scale, consolidate flows between hubs if there are incentives to do it, and obtains similar or better KPI values. In the other hand, solving our model implies a computational effort between the fundamental model

and other existing complex models. Finally, we develop a fast heuristic procedure, in order to quickly get high quality feasible solutions.

Future work along this line includes using the solutions obtained by our heuristic procedure in more sophisticated heuristics and meta-heuristics, and the use of this flow threshold-based discount structure in other hub location problems.

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