

Confined two-dimensional fermions at finite density

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We introduce the chemical potential in a system of two-dimensional massless fermions, confined to a finite region, by imposing twisted boundary conditions in the Euclidean time direction. We explore in this simple model the application of functional techniques which could be used in more complicated situations.

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I. INTRODUCTION

The thermodynamics of hadronic matter has recently received great attention [1], mainly in connection with the possible occurrence of a deconfining phase transition at finite temperature. The difficulty of studying this transition in the framework of QCD comes from the fact that confinement has not been derived in this theoretical context.

Consequently, effective models such as, for example, the bag models [2,3], have been introduced to mimic the confining properties of strong interactions. In these models, fields are confined to bounded regions and subject to adequate boundary conditions. Finite volume effects turn out to be relevant, and they have been studied in the thermodynamical limit, for instance, in [4], and for a nonvanishing chemical potential in [5].

A complete analysis of the free energy for an MIT bag model at $T > 0$, with $\mu = 0$, has been performed in [6]. The corrections introduced by chiral boundary conditions have also been studied [7]. Functional methods have been used in these papers to isolate the finite temperature-dependent pieces from the (divergent) Casimir energy.

The aim of the present paper is to study the possibility of introducing a nonvanishing chemical potential through "twisted" boundary conditions in the Euclidean time direction and treat it following the methods developed in [6,8]. As a first approach to this problem we present here the evaluation of the Gibbs free energy for the simple model of massless $(1+1)$ -dimensional fermions confined to a segment, for T and $\mu \neq 0$.

Even though, for this toy model, the eigenvalue problem can be exactly solved, we will rather follow an alternative approach. We will relate differences of free energies of the system to the Green function satisfying adequate (spatial and temporal) boundary conditions. This approach is expected to be more useful in the realistic four-dimensional case, where eigenvalues cannot be explicitly solved for. In Sec. IV, we will reobtain the re-

sults in Sec. II, making use of the functional method developed in [8], which is based on the evaluation of p determinants.

II. TWO-DIMENSIONAL FERMIONS WITH $\mu \neq 0$

We consider a system of two-dimensional fermions confined in the segment $[0, L]$ and subject to given boundary conditions. As is well known, the chemical potential can be introduced as an imaginary constant temporal gauge field [9]. If the temperature is $T = 1/\beta$ and the chemical potential is μ , the Grand canonical partition function can be expressed as

$$\begin{aligned} \Xi(T, L, \mu) &= e^{-\beta G(T, L, \mu)}, \\ &= \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left(\int_0^1 dt \int_0^1 dx \bar{\psi} [D(\beta, L) - i\mu\gamma^0] \psi \right), \\ &\sim \text{Det} [D(\beta, L) - i\mu\gamma^0]_{\text{BC}}, \end{aligned} \quad (1)$$

where

$$D(\beta, L) = \frac{i}{\beta} \gamma^0 \partial_t + \frac{i}{L} \gamma^1 \partial_x \quad \text{for } 0 \leq t, x \leq 1, \quad (2)$$

with

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\ \gamma^5 &= -i\gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \end{aligned} \quad (3)$$

and "BC" means that we must evaluate the functional determinant of the differential operator defined on func-

tions satisfying the boundary conditions:

$$\begin{aligned} B\psi(t, x) &= 0, \quad \text{for } x = 0, 1, \\ \psi(1, x) &= -\psi(0, x). \end{aligned} \quad (4)$$

Here B denotes an elliptic boundary condition to be satisfied at the spatial edges.

So, we have the following relation for the Gibbs free energy:

$$\begin{aligned} -\beta \frac{\partial G}{\partial \mu}(\beta, L, \mu) &= \text{Tr} \left\{ \frac{\partial}{\partial \mu} \ln [D(\beta, L) - i\mu\gamma^0]_{\text{BC}} \right\} \\ &= \text{Tr} \{-i\gamma^0 K_{\text{BC}}(t, x; t', x')\}, \end{aligned} \quad (5)$$

where $K_{\text{BC}}(t, x; t', x')$ is the Green function of the problem, satisfying

$$\begin{aligned} [D(\beta, L) - i\mu\gamma^0] K_{\text{BC}}(t, x; t', x') &= \delta(x - x')\delta(t - t'), \\ BK_{\text{BC}}(t, x; t', x') &= 0, \quad \text{for } x = 0, 1, \\ K_{\text{BC}}(1, x; t', x') &= -K_{\text{BC}}(0, x; t', x'). \end{aligned} \quad (6)$$

This can also be expressed in terms of

$$\begin{aligned} D(\beta, L) - i\mu\gamma^0 &= e^{\mu\beta t} D(\beta, L) e^{-\mu\beta t}, \\ K_{\text{BC}}(t, x; t', x') &= e^{\mu\beta t} k(t, x; t', x') e^{-\mu\beta t}, \end{aligned} \quad (7)$$

where $k(t, x; t', x')$ is the Green function of the problem

$$\begin{aligned} D(\beta, L) k(t, x; t', x') &= \delta(x - x')\delta(t - t'), \\ Be^{\mu\beta t} k(t, x; t', x') &= 0, \quad \text{for } x = 0, 1, \\ k(1, x; t', x') + e^{-\mu\beta t} k(0, x; t', x') &= 0. \end{aligned} \quad (8)$$

This function has the development

$$k(t, x; t', x') = L \sum_{n=-\infty}^{\infty} k_n(x, x') e^{-i\Omega_n\beta(t-t')}, \quad (9)$$

with the frequencies given by

$$\begin{aligned} \Omega_n &= \omega_n - i\mu, \\ \omega_n &= \frac{(2n+1)\pi}{\beta} \quad \text{for } n \in \mathbf{Z}. \end{aligned} \quad (10)$$

The coefficients satisfy

$$\begin{aligned} k_n(x, x') &= e^{\gamma_5 \Omega_n L x} k_n(0, x') \\ &\quad - i\gamma^1 e^{-\gamma_5 \Omega_n L(x-x')} H(x - x'), \end{aligned} \quad (11)$$

with

$$H(x) = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x < 0. \end{cases} \quad (12)$$

Now, we must impose the boundary conditions on the spatial edges. We will adopt a static "baglike" condition

$$(B\psi)(t, x) = (1 + \not{x})\psi(t, x) = 0 \quad \text{for } x = 0, 1, \quad (13)$$

such that, at $x = 0$,

$$(1 - \gamma^1)k_n(0, x') = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} k_n(0, x') = 0. \quad (14)$$

This implies that

$$k_n(0, x') = \begin{pmatrix} 1 \\ i \end{pmatrix} \otimes (f(x') \quad g(x')). \quad (15)$$

On the other hand, at $x = 1$,

$$(1 + \gamma^1)k_n(1, x') = 0, \quad (16)$$

which means that

$$\begin{aligned} (1 - i) \left[e^{\gamma_5 \Omega_n L x} \begin{pmatrix} 1 \\ i \end{pmatrix} \otimes (f(x') \quad g(x')) \right. \\ \left. - i\gamma^1 e^{-\gamma_5 \Omega_n L(1-x')} \right] = 0. \end{aligned} \quad (17)$$

This gives

$$k(t, x; t', x') = iL \sum_{n=-\infty}^{\infty} \left[\frac{e^{\gamma_5 \Omega_n L x}}{2 \cosh(\Omega_n L)} (1 + \gamma^1) e^{-\gamma_5 \Omega_n L(1-x')} - \gamma^1 e^{-\gamma_5 \Omega_n L(x-x')} H(x - x') \right] e^{-i\Omega_n\beta(t-t')}. \quad (18)$$

So we have, for the derivative of the free energy,

$$\begin{aligned} -\beta \frac{\partial G}{\partial \mu}(\beta, L, \mu) &= \text{Tr} \{-i\gamma^0 e^{\mu\beta t} k(t, x; t', x') e^{-\mu\beta t}\} \\ &= L \int_0^1 dx \int_0^1 dt \text{tr} \left\{ \gamma^0 e^{\mu\beta(t-t')} \sum_{n=-\infty}^{\infty} \left[\frac{e^{-\gamma_5 \Omega_n L(1-x-x')}}{2 \cosh(\Omega_n L)} \right. \right. \\ &\quad \left. \left. + \gamma^1 \left(\frac{e^{-\gamma_5 \Omega_n L(1+x-x')}}{2 \cosh(\Omega_n L)} - e^{-\gamma_5 \Omega_n L(x-x')} H(x - x') \right) \right] e^{-i\Omega_n\beta(t-t')} \right\} \Big|_{(t', x')=(x, t)}. \end{aligned} \quad (19)$$

For $0 < x, x' < 1$, the first term in the series converges absolutely and uniformly, even at $(t, x) = (t', x')$. So it can be summed up in any order; in particular, one can first take the trace as a matrix, obtaining a vanishing result. This is not the case for the second term, containing the γ^1 matrix. In fact,

$$\begin{aligned}
 & \sum_{n=-\infty}^{\infty} \left(\frac{e^{-\gamma_5 \Omega_n L(1+x-x')}}{2 \cosh(\Omega_n L)} - e^{-\gamma_5 \Omega_n L(x-x')} H(x-x') \right) e^{-i\Omega_n \beta(t-t')} \\
 &= \sum_{n=0}^{\infty} \left[\frac{e^{-\gamma_5 \Omega_n L(1+x-x')}}{2 \cosh(\Omega_n L)} - e^{-\gamma_5 \Omega_n L(x-x')} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] e^{-i\Omega_n \beta(t-t')} \\
 &+ \sum_{n=-\infty}^{-1} \left[\frac{e^{-\gamma_5 \Omega_n L(1+x-x')}}{2 \cosh(\Omega_n L)} - e^{-\gamma_5 \Omega_n L(x-x')} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] e^{-i\Omega_n \beta(t-t')} \\
 &+ \sum_{n=0}^{\infty} e^{-\gamma_5 \Omega_n L(x-x')} \begin{pmatrix} -H(x-x') & 0 \\ 0 & H(x'-x) \end{pmatrix} e^{-i\Omega_n \beta(t-t')} \\
 &+ \sum_{n=-\infty}^{-1} e^{-\gamma_5 \Omega_n L(x-x')} \begin{pmatrix} H(x'-x) & 0 \\ 0 & -H(x-x') \end{pmatrix} e^{-i\Omega_n \beta(t-t')},
 \end{aligned} \tag{20}$$

where the first two terms in the right-hand side are absolutely convergent even at $x' = x$. For the last two terms in this equation we have, for $x' \neq x$,

$$\begin{aligned}
 S_+ &= \sum_{n=0}^{\infty} e^{-\gamma_5 \Omega_n L(x-x')} \begin{pmatrix} -H(x-x') & 0 \\ 0 & H(x'-x) \end{pmatrix} e^{-i\Omega_n \beta(t-t')} \\
 &= \begin{pmatrix} -H(x-x') \frac{e^{(i\mu-\pi/\beta)[L(x-x')+i\beta(t-t')]} }{1 - \exp\left(-\frac{2\pi}{\beta} [L(x-x')L + i\beta(t-t')]\right)} & 0 \\ 0 & H(x'-x) \frac{e^{(i\mu-\pi/\beta)[L(x'-x)L+i\beta(t-t')]} }{1 - \exp\left(-\frac{2\pi}{\beta} [L(x'-x)L + i\beta(t-t')]\right)} \end{pmatrix},
 \end{aligned} \tag{21}$$

and

$$\begin{aligned}
 S_- &= \sum_{n=-\infty}^{-1} e^{-\gamma_5 \Omega_n L(x-x')} \begin{pmatrix} H(x'-x) & 0 \\ 0 & -H(x-x') \end{pmatrix} e^{-i\Omega_n \beta(t-t')} \\
 &= \begin{pmatrix} H(x'-x) \frac{e^{-(i\mu+\pi/\beta)[L(x'-x)-i\beta(t-t')]} }{1 - \exp\left(-\frac{2\pi}{\beta} [L(x'-x) - i\beta(t-t')]\right)} & 0 \\ 0 & -H(x-x') \frac{e^{-(i\mu+\pi/\beta)[L(x-x')-i\beta(t-t')]} }{1 - \exp\left(-\frac{2\pi}{\beta} [L(x-x') - i\beta(t-t')]\right)} \end{pmatrix},
 \end{aligned} \tag{22}$$

which are singular at $(t', x') = (t, x)$. Calling $z = L(x-x') + i\beta(t-t')$, $\bar{z} = L(x-x') - i\beta(t-t')$, we have

$$S_+ + S_- = -\frac{\beta}{2\pi} \begin{pmatrix} \frac{1}{z} + i\mu + \pi/\beta [H(x'-x) - H(x-x')] + O(z/\beta^2) & 0 \\ 0 & \frac{1}{\bar{z}} - i\mu + \pi/\beta [H(x'-x) - H(x-x')] + O(\bar{z}/\beta^2) \end{pmatrix}. \tag{23}$$

Now, we can evaluate

$$\begin{aligned}
 L \operatorname{tr} \left\{ \gamma^0 e^{\mu\beta(t-t')} \gamma^1 (S_+ + S_-) \right\} &= -\frac{iL\beta}{2\pi} e^{\mu\beta(t-t')} \left\{ \frac{1}{z} - \frac{1}{\bar{z}} + 2i\mu + O(|z|/\beta^2) \right\} \\
 &\xrightarrow{t' \rightarrow t} \frac{L\beta\mu}{\pi} + O(|x-x'|L^2/\beta).
 \end{aligned} \tag{24}$$

This corresponds to the following (finite) contribution to $-\beta \frac{\partial G}{\partial \mu}(T, L, \mu)$:

$$L \int_0^1 dx \int_0^1 dt \operatorname{tr} \left\{ \gamma^0 e^{\mu\beta(t-t')} \gamma^1 (S_+ + S_-) \right\}_{(t', x')=(t, x)} = \frac{\mu}{\pi} L \beta. \quad (25)$$

For the remaining terms in Eq. (20), we can put $t' = t$ and $x' = x$, and take the matrix trace inside the sum:

$$\begin{aligned} L \int_0^1 dx \int_0^1 dt \operatorname{tr} \left\{ \gamma^0 \gamma^1 \sum_{n=-\infty}^{\infty} \left[\frac{e^{-\gamma_5 \Omega_n L}}{2 \cosh(\Omega_n L)} - \begin{pmatrix} H(-n-1/2) & 0 \\ 0 & H(n+1/2) \end{pmatrix} \right] \right\} \\ = -iL \sum_{n=-\infty}^{\infty} \{ \tanh(\Omega_n L) - \operatorname{sgn}(n+1/2) \}. \end{aligned} \quad (26)$$

We finally get

$$-\beta \frac{\partial G}{\partial \mu}(\beta, L, \mu) = \frac{\mu}{\pi} L \beta - iL \sum_{n=-\infty}^{\infty} \{ \tanh[(\omega_n - i\mu)L] - \operatorname{sgn}(n+1/2) \}. \quad (27)$$

Notice that, while the first term in the right-hand side (RHS) of (27), coming from the singular behavior of the Green function, is linear in μ , the second one, a π -periodic function of μL , contains the finite size effects (vanishing for $L \rightarrow \infty$). It is easy to see from this expression that, in the $\beta \rightarrow \infty$ limit, the mean value of the particle number is

$$\lim_{\beta \rightarrow \infty} \bar{N} = \left[\frac{\mu L}{\pi} \right], \quad (28)$$

where $[x]$ means the integer part of x .

Now, (27) can be integrated to obtain

$$-\beta \{ G(\beta, L, \mu) - G(\beta, L, 0) \} = \frac{\mu^2}{2\pi} L \beta + \sum_{n=-\infty}^{\infty} \ln \left\{ \frac{\cosh[(\omega_n - i\mu)L]}{\cosh[\omega_n L]} + i\mu L \operatorname{sgn}(n+1/2) \right\}. \quad (29)$$

Taking into account that $\omega_{-n} = -\omega_{(n-1)}$ we have, for the piece of the free energy which depends on μ , the (finite) result

$$\begin{aligned} -\beta \{ G(\beta, L, \mu) - G(\beta, L, 0) \} &= \frac{\mu^2}{2\pi} L \beta + \ln \left\{ \prod_0^{\infty} \frac{(1 + 2 \cos(2\mu L) e^{-2\omega_n L} + e^{-4\omega_n L})}{(1 + 2e^{-2\omega_n L} + e^{-4\omega_n L})} \right\} \\ &= \frac{\mu^2}{2\pi} L \beta + \ln \left\{ \frac{\theta_3(\mu L, e^{-2\pi L/\beta})}{\theta_3(0, e^{-2\pi L/\beta})} \right\}, \end{aligned} \quad (30)$$

where $\theta_3(u, q)$ is the Jacobi theta function [10].

III. THE FREE ENERGY FOR $\mu = 0$

In order to have the complete Gibbs function, we will evaluate in this section the free energy of our system of two-dimensional fermions as a function of temperature, for $\mu = 0$. We take

$$\frac{\partial}{\partial \beta} \ln \operatorname{Det} [D(\beta, L)]_{\text{BC}} = \operatorname{Tr} \left\{ \frac{-i}{\beta^2} \gamma^0 \partial_t k(t, x; t', x') \right\}, \quad (31)$$

where the Green function of the operator is given by (18) with $\mu = 0$:

$$k(t, x; t', x') = iL \sum_{n=-\infty}^{\infty} \left[\frac{e^{\gamma_5 \omega_n L x}}{2 \cosh(\omega_n L)} (1 + \gamma^1) e^{-\gamma_5 \omega_n L(1-x')} - \gamma^1 e^{-\gamma_5 \omega_n L(x-x')} H(x-x') \right] e^{-i\omega_n \beta(t-t')}. \quad (32)$$

As in Sec. I, the term in the RHS of Eq. (31) which does not contain γ^1 gives a vanishing contribution. For the remaining, we get two contributions to the trace: the first coming from the absolutely convergent series (for $0 < x, x' < 1$)

$$\begin{aligned} & \frac{L}{\beta} \text{Tr} \left\{ \gamma_5 \sum_{n=-\infty}^{\infty} \omega_n e^{-i\omega_n \beta(t-t')} e^{-\gamma_5 \omega_n L(x-x')} \left[\frac{e^{-\gamma_5 \omega_n L}}{2 \cosh(\omega_n L)} - \begin{pmatrix} H(-n-1/2) & 0 \\ 0 & H(n+1/2) \end{pmatrix} \right] \right\} \\ &= -\frac{L}{\beta} \sum_{n=-\infty}^{\infty} \omega_n \{ \tanh(\omega_n L) - \text{sgn}(n+1/2) \} = 2 \frac{\partial}{\partial \beta} \sum_{n=0}^{\infty} \ln \{ 1 + e^{-2\omega_n L} \}; \end{aligned} \quad (33)$$

the second one turns out to be divergent, and therefore a regularization procedure is required. We will use a ‘‘point-splitting’’ prescription to get

$$\begin{aligned} & \frac{iL}{\beta^2} \text{Tr} \left\{ \gamma_5 \partial_t \sum_{n=-\infty}^{\infty} e^{-i\omega_n \beta(t-t')} e^{-\gamma_5 \omega_n L(x-x')} \right. \\ & \quad \times \left. \begin{pmatrix} H(x'-x)H(-n-1/2) & & & 0 \\ & H(x-x')H(n+1/2) & & \\ & & H(x'-x)H(n+1/2) & \\ 0 & & & -H(x-x')H(-n-1/2) \end{pmatrix} \right\} \\ &= 2 \frac{L}{\beta^2} \int_0^1 dx \int_0^1 dt \text{Im} \partial_t \left\{ H(x-x') \sum_{n=0}^{\infty} e^{-(2n+1)\frac{\pi}{\beta} [L(x-x') + i\beta(t-t')]} \right. \\ & \quad \left. + H(x'-x) \sum_{n=0}^{\infty} e^{-(2n+1)\frac{\pi}{\beta} [L(x'-x) + i\beta(t-t')]} \right\} \Big|_{(t',x')=(t,x)} \quad (34) \\ &= \frac{L}{\beta^2} \int_0^1 dx \int_0^1 dt \text{Im} \partial_t \left\{ \frac{1}{\sinh \left(\frac{\pi}{\beta} [L |x-x'| + i\beta(t-t')] \right)} \right\} \Big|_{(t',x')=(t,x)} \\ & \xrightarrow{t' \rightarrow t} -\frac{\pi L}{\beta^2} \frac{\cosh(\pi\epsilon/\beta)}{[\sinh(\pi\epsilon/\beta)]^2} = \frac{\partial}{\partial \beta} \left\{ -\frac{L\beta}{\pi\epsilon^2} - \frac{\pi L}{3\beta} + O(\epsilon^2) \right\}, \end{aligned}$$

for $\epsilon = L |x' - x| \ll \beta$, which shows a singular piece proportional to L .

So, we have, for the free energy at $\mu = 0$,

$$\begin{aligned} F(\beta, L) &= -\frac{2}{\beta} \sum_{n=0}^{\infty} \ln \{ 1 + e^{-2\omega_n L} \} + \frac{L}{\pi\epsilon^2} - \frac{\pi L}{3\beta^2} + \frac{C}{\beta} \\ &= -\frac{1}{\beta} \ln \left\{ \theta_3 \left(0, e^{-\frac{2\pi L}{\beta}} \right) \right\} + \frac{1}{\beta} \sum_{n=1}^{\infty} \ln \left(1 - e^{-\frac{4\pi L n}{\beta}} \right) + \frac{L}{\pi\epsilon^2} - \frac{\pi L}{3\beta^2} + \frac{C}{\beta}, \end{aligned} \quad (35)$$

with C an integration constant.

The Casimir energy is obtained through the limit of this expression for vanishing temperature:

$$\begin{aligned} \lim_{\beta \rightarrow \infty} F(\beta, L) &= E_{\text{Cas}}(L) \\ &= -\frac{1}{\pi} \int_0^{\infty} d\omega \ln \{ 1 + e^{-2\omega L} \} + \frac{L}{\pi\epsilon^2} \quad (36) \\ &= -\frac{\pi}{24L} + \frac{L}{\pi\epsilon^2}, \end{aligned}$$

where one can see that the singular $O(\epsilon^{-2})$ part can be eliminated through a renormalization of the vacuum energy density. The finite part of $E_{\text{Cas}}(L)$, $-\frac{\pi}{24L}$, coincides with the result obtained in Ref. [11], and gives rise to an attractive force between the edges of the segment where the fermions are confined.

Then, we get the finite result

$$\begin{aligned} F(\beta, L) - E_{\text{Cas}}(L) &= -\frac{1}{\beta} \ln \left\{ \theta_3 \left(0, e^{-\frac{2\pi L}{\beta}} \right) \right\} \\ & \quad + \frac{1}{\beta} \sum_{n=1}^{\infty} \ln \left(1 - e^{-\frac{4\pi L n}{\beta}} \right) \\ & \quad - \frac{\pi L}{3\beta^2} + \frac{C}{\beta} + \frac{\pi}{24L}. \end{aligned} \quad (37)$$

The undetermined constant C can be evaluated by imposing the vanishing of entropy at zero temperature. Since

$$S = \beta^2 \frac{\partial}{\partial \beta} F(\beta, L), \quad (38)$$

for $\beta \rightarrow \infty$ the Poisson formula for the previous series

allows us to write

$$S = \beta^2 \frac{\partial}{\partial \beta} \left\{ \frac{C}{\beta} + O(\beta^{-2}) \right\} = -C + O(\beta^{-1}), \quad (39)$$

implying that $C = 0$.

We finally get, for the free energy as a function of β, L, μ ,

$$\begin{aligned} G(\beta, L, \mu) - E_{\text{Cas}}(L) \\ = -\frac{\mu^2 L}{2\pi} - \frac{1}{\beta} \ln \left\{ \theta_3 \left(\mu L, e^{-\frac{2\pi L}{\beta}} \right) \right\} \\ + \frac{1}{\beta} \sum_{n=1}^{\infty} \ln \left(1 - e^{-\frac{4\pi L n}{\beta}} \right) - \frac{\pi L}{3\beta^2} + \frac{\pi}{24L}. \end{aligned} \quad (40)$$

IV. THE CHEMICAL POTENTIAL THROUGH A BOUNDARY VALUE PROBLEM

In this section, we will show how the result in Sec. I can alternatively be obtained with the methods developed in Ref. [8], by introducing the chemical potential through suitable boundary conditions.

As before, we will consider the differential operator

$$D(\beta, L) = \frac{i}{\beta} \gamma^0 \partial_t + \frac{i}{L} \gamma^1 \partial_x, \quad (41)$$

with the dimensionless variables t, x taking values in the segment $[0, 1]$. We will take $D(\beta, L)$ to act on differentiable functions satisfying the boundary conditions

$$\begin{aligned} (A(\eta)\chi)(x) &\equiv \chi(t=1, x) + e^{i\eta}\chi(t=0, x) = 0, \\ (B\chi)(t, x) &\equiv [1 + \not{n}]\chi(t, x) = 0, \quad \text{for } x = 0, 1, \end{aligned} \quad (42)$$

and call this operator $[D(\beta, L)]_{A(\eta), B}$.

The method requires a basis in the kernel of the differential operator, $\ker D(\beta, L)$. The calculation can be greatly simplified by choosing a complete system of functions satisfying the boundary conditions at the spatial edges. Such basis can be constructed from the eigenfunctions of the Hermitian operator $(-i\gamma_5 \frac{d}{dx})_B$ in $[0, 1]$:

$$\begin{aligned} -i\gamma_5 \frac{d}{dx} \chi_n(x) &= \lambda_n \chi_n(x), \\ (B\chi_n)(0) &= 0 = (B\chi_n)(1), \end{aligned} \quad (43)$$

which are given by

$$\chi_n(x) = e^{i\pi(n+1/2)x\gamma_5} \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad (44)$$

with $\lambda_n = \pi(n+1/2)$, $n \in \mathbf{Z}$.

Now, a basis in $\ker D(\beta, L)$ is

$$\psi_n(t, x) = e^{(n+1/2)\frac{\pi}{L}[\beta t + iLx\gamma_5]} \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad \text{with } n \in \mathbf{Z}. \quad (45)$$

The following step is to get the projected boundary values of $\psi_n(t, x)$ through the boundary operators $A(\eta)$ and B :

$$H_n(t, x; \eta) = \begin{pmatrix} (A(\eta)\psi_n)(x) \\ (B\psi_n)(t, 0) \\ (B\psi_n)(t, 1) \end{pmatrix} = \begin{pmatrix} h_n(x; \eta) \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{aligned} h_n(x; \eta) &= \psi_n(t=1, x) + e^{i\eta}\psi_n(t=0, x) \\ &= \left[e^{(n+1/2)\pi\beta/L} + e^{i\eta} \right] e^{i\pi(n+1/2)x\gamma_5} \begin{pmatrix} 1 \\ i \end{pmatrix}. \end{aligned} \quad (46)$$

Forman's operator, $\tilde{\Phi}(\eta', \eta)$, is defined as

$$H_n(t, x; \eta') = \tilde{\Phi}(\eta', \eta) H_n(t, x; \eta), \quad (47)$$

for any basis in $\ker D(\beta, L)$. Since the operator B does not depend on the parameter η , $\tilde{\Phi}(\eta', \eta)$ has the form

$$\tilde{\Phi}(\eta', \eta) = \begin{pmatrix} \Phi(\eta', \eta) & \Phi'(\eta', \eta) \\ 0 & Id_{2 \times 2} \end{pmatrix}, \quad (48)$$

and our election of basis allows us to determine $\Phi(\eta', \eta)$ from

$$\begin{aligned} h_n(x; \eta') &= \Phi(\eta', \eta) h_n(x; \eta) \\ &= \left(\frac{e^{(n+1/2)\pi\beta/L} + e^{i\eta'}}{e^{(n+1/2)\pi\beta/L} + e^{i\eta}} \right) h_n(x; \eta). \end{aligned} \quad (49)$$

Note that the $h_n(x; \eta) \sim \chi_n(x)$, for $n \in \mathbf{Z}$, constitute a complete system in $L^2(0, 1)$, and $\Phi(\eta', \eta)$ is diagonal in this basis:

$$[\Phi(\eta', \eta)]_{n,m} = \delta_{n,m} \left(\frac{1 + e^{i\eta' - (n+1/2)\pi\beta/L}}{1 + e^{i\eta - (n+1/2)\pi\beta/L}} \right). \quad (50)$$

In what follows, we will need the quotient of Φ 's for two sets of parameters (β, L) and (β_0, L_0) respectively;

$$[\Phi(\eta', \eta) \Phi_0^{-1}(\eta', \eta)]_{n,m} = \delta_{n,m} \left(\frac{1 + e^{i\eta' - (n+1/2)\pi\beta/L}}{1 + e^{i\eta - (n+1/2)\pi\beta/L}} \right) \left(\frac{1 + e^{i\eta - (n+1/2)\pi\beta_0/L_0}}{1 + e^{i\eta' - (n+1/2)\pi\beta_0/L_0}} \right). \quad (51)$$

It is easy to see that $[\Phi(\eta', \eta) \Phi_0^{-1}(\eta', \eta) - 1]$ is a trace class operator, so the determinant $\det_1 [\Phi(\eta', \eta) \Phi_0^{-1}(\eta', \eta)]$ exists:

$$\begin{aligned} \det_1 [\Phi(\eta', \eta) \Phi_0^{-1}(\eta', \eta)] &= \prod_0 \left| \frac{1 + e^{i\eta' - (n+1/2)\pi\beta/L}}{1 + e^{i\eta - (n+1/2)\pi\beta/L}} \right|^2 \left| \frac{1 + e^{i\eta - (n+1/2)\pi\beta_0/L_0}}{1 + e^{i\eta' - (n+1/2)\pi\beta_0/L_0}} \right|^2 \\ &= \frac{\theta_3 \left(\frac{\eta'}{2}, e^{-\pi\beta/2L} \right)}{\theta_3 \left(\frac{\eta}{2}, e^{-\pi\beta/2L} \right)} \frac{\theta_3 \left(\frac{\eta}{2}, e^{-\pi\beta_0/2L_0} \right)}{\theta_3 \left(\frac{\eta'}{2}, e^{-\pi\beta_0/2L_0} \right)}. \end{aligned} \quad (52)$$

The result established in Ref. [8] is

$$\det_1 \left\{ (D(\beta, L))_{A(\eta'), B} (D(\beta_0, L_0))_{A(\eta'), B}^{-1} \right. \\ \left. \times (D(\beta_0, L_0))_{A(\eta), B} (D(\beta, L))_{A(\eta), B}^{-1} \right\} \quad (53) \\ = \det_1 (\Phi(\eta', \eta) \Phi_0^{-1}(\eta', \eta)).$$

Now, we choose the parameter $\eta' = i\mu'\beta$, $\eta = i\mu\beta$ and noticing that

$$(D(\beta, L))_{A(\eta), B} = (e^{-i\eta t} D(\beta, L) e^{i\eta t})_{A(0), B} \quad (54) \\ = \left(\frac{i}{\beta} \gamma^0 \partial_t + \frac{i}{L} \gamma^1 \partial_x - \frac{\eta}{\beta} \gamma^0 \right)_{A(0), B},$$

we can write

$$-\beta G(\beta, L, \mu') + \beta_0 G(\beta_0, L_0, \mu' \beta / \beta_0) \\ -\beta_0 G(\beta_0, L_0, \mu \beta / \beta_0) + \beta G(\beta, L, \mu) \\ = \ln \left\{ \frac{\theta_3 \left(\frac{i\mu'\beta}{2}, e^{-\pi\beta/2L} \right) \theta_3 \left(\frac{i\mu\beta}{2}, e^{-\pi\beta_0/2L_0} \right)}{\theta_3 \left(\frac{i\mu\beta}{2}, e^{-\pi\beta/2L} \right) \theta_3 \left(\frac{i\mu'\beta}{2}, e^{-\pi\beta_0/2L_0} \right)} \right\}. \quad (55)$$

For $\mu' = 0$ this reduces to

$$\beta [G(\beta, L, \mu) - G(\beta, L, 0)] \\ -\beta_0 [G(\beta_0, L_0, \mu \beta / \beta_0) - G(\beta_0, L_0, 0)] \\ = \ln \left\{ \frac{\theta_3(0, e^{-\pi\beta/2L})}{\theta_3 \left(\frac{i\mu\beta}{2}, e^{-\pi\beta/2L} \right)} \frac{\theta_3 \left(\frac{i\mu\beta}{2}, e^{-\pi\beta_0/2L_0} \right)}{\theta_3(0, e^{-\pi\beta_0/2L_0})} \right\}, \quad (56)$$

and we are still free to choose the parameters β_0 and L_0 conveniently.

Notice that, for $\beta/\beta_0 \ll 1$, we can write

$$\beta_0 [G(\beta_0, L_0, \mu \beta / \beta_0) - G(\beta_0, L_0, 0)] \\ = \beta_0 \frac{\mu \beta}{\beta_0} \frac{\partial G}{\partial \mu_0}(\beta_0, L_0, \mu_0) = -\mu \beta \bar{N}(\beta_0, L_0, \mu_0), \quad (57)$$

for $0 < \mu_0 < \mu \beta / \beta_0$, and $\bar{N}(\beta_0, L_0, \mu_0)$ the mean value of the particle number. So

$$\beta_0 [G(\beta_0, L_0, \mu \beta / \beta_0) - G(\beta_0, L_0, 0)] \xrightarrow{\beta_0 \rightarrow \infty} 0. \quad (58)$$

On the other hand

$$\lim_{\beta_0 \rightarrow \infty} \frac{\theta_3 \left(\frac{i\mu\beta}{2}, e^{-\pi\beta_0/2L_0} \right)}{\theta_3(0, e^{-\pi\beta_0/2L_0})} = 1, \quad (59)$$

for any value of $\frac{\mu\beta}{2}$. Then, we have

$$\beta [G(\beta, L, \mu) - G(\beta, L, 0)] = \ln \left\{ \frac{\theta_3(0, e^{-\pi\beta/2L})}{\theta_3 \left(\frac{i\mu\beta}{2}, e^{-\pi\beta/2L} \right)} \right\} = \frac{-\mu^2 \beta L}{2\pi} + \ln \left\{ \frac{\theta_3(0, e^{-2\pi L/\beta})}{\theta_3(\mu L, e^{-2\pi L/\beta})} \right\}, \quad (60)$$

where we have used the inversion formula for the Jacobi θ_3 function:

$$\theta_3 \left(\frac{i\mu\beta}{2}, e^{-\pi\beta/2L} \right) = e^{\frac{\mu^2 \beta L}{2\pi}} \theta_3(\mu L, e^{-2\pi L/\beta}). \quad (61)$$

This result coincides with the one obtained in Eq. (30).

We have seen that the chemical potential can be introduced through an analytic extension in the parameter defining the twisted boundary conditions in the Euclidean time direction. From this point of view, the presence of Jacobi theta functions should not be surprising since these functions appear in the study of conformally invariant Dirac operators (see, for example, [12]).

In fact, in Ref. [13] the problem of massless two-dimensional fermions at $T > 0$, satisfying twisted boundary conditions in the spatial direction, has been considered using the techniques employed in the present paper. Taking into account that, for this simple model, baglike (local) boundary conditions imply the antiperiodicity of the Hamiltonian eigenfunctions in the segment $x \in [0, 2]$

[see Eqs. (43), (44)], it is not difficult to reobtain from Eq. (60) the results in [13], by changing $\beta \rightarrow L$, $L \rightarrow \beta/2$.

V. CONCLUSIONS

In this paper, we studied the possibility of simulating the presence of a chemical potential by imposing “twisted” boundary conditions in the Euclidean time direction. This mechanism allowed us to relate the free energy of a system of massless confined fermions to a functional determinant of the Dirac operator.

To study such determinant, we made use of two approaches. The first one is based on the knowledge of the Green function satisfying adequate boundary conditions. The other relies on functional techniques [8] which allow to perform the calculation starting from boundary values of functions in the kernel of the Dirac operator.

In Sec. II, the difference between Gibbs free energies with and without chemical potential was written in terms of the μ integral of a trace involving the Green function.

A careful treatment of this trace (which is not absolutely convergent), through point-splitting regularization, put in evidence the presence of a μ^2 -dependent term.

For completeness, in Sec. III, the Helmholtz free energy was computed by a similar method, employing this time the β derivative. The constant of integration was fixed from the vanishing of entropy at zero temperature. Moreover, the $\beta \rightarrow \infty$ limit allowed us to identify the Casimir energy of the system.

Although the results of Sec. II and III, i.e., the Gibbs free energy of this simple model, could also have been obtained directly from the eigenvalues of the Dirac Hamiltonian (which, in this case, can be exactly evaluated), the method developed in those sections can be applied to more complicated and realistic situations, e.g., the (3+1)-dimensional bag model. In fact, as mentioned in Sec. I, in Refs. [6,7] a hybrid chiral bag model with $\mu = 0$ and $T \neq 0$ in 3 + 1 dimensions has been studied. The Green function (for a spherical bag) can be determined through the extension to the $T \neq 0$ case of the results in [14], where one must take into account that the symmetry of the boundary conditions corresponds to the diagonal subgroup of $SU(2)_{\text{rotational}} \otimes SU(2)_{\text{isospin}}$. It should be

noticed that, in this model, the eigenvalue problem of the Dirac Hamiltonian leads to transcendental equations. So, the evaluation of the μ -dependent contribution to the Gibbs free energy could be performed following the Green function approach developed in the present paper, task which is in progress.

In Sec. IV, we made an alternative calculation following the techniques introduced in [8]. Starting from an adequately selected basis in the kernel of the Dirac operator, the Fredholm determinant of quotients of Forman's operators could be easily constructed. Through an analytic extension in the parameter defining the twisted temporal boundary condition, we reobtained the μ -dependent piece of the Gibbs free energy of the system.

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