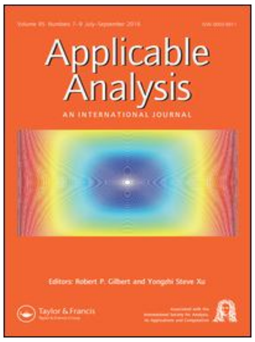

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A posteriori error analysis of semilinear parabolic interface problems using elliptic reconstruction

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ABSTRACT
In this article, *a posteriori* error analysis for space-time discretizations of semilinear parabolic interface problems in a bounded convex domain in $\mathbb{R}^2$ is presented and analyzed. In time discretizations both the backward Euler and the Crank-Nicolson approximations are considered whereas in space we have considered the standard piecewise linear finite elements. *A posteriori* error estimates of optimal order in time and almost optimal order in space are derived in the $L^\infty(L^2)$-norm. The main technical tools used are the energy argument combined with the elliptic reconstruction technique. The forcing term is assumed to satisfy the Lipschitz condition.

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1. Introduction

The study of semilinear interface problems is motivated by models of mass transfer of substances through semipermeable membranes. Such models arise from various applications in biomedical and chemical engineering, e.g. modeling of electrokinetic flows, solute dynamics across arterial walls, and cellular signal transduction, for instance, see [1,2].

The focus of this work is to study $L^\infty(L^2)$-norm *a posteriori* error analysis of semilinear parabolic interface problems of the form

$$u_t(x,t) - \operatorname{div}(\beta(x)\nabla u(x,t)) = f(x,t,u) \quad \text{in} \quad \Omega \times (0,T] \tag{1.1}$$

with the prescribed initial and boundary conditions

$$u(x,0) = u_0(x) \quad \text{in} \quad \Omega; \qquad u = 0 \quad \text{on} \quad \partial\Omega \times [0,T] \tag{1.2}$$

and jump conditions on the interface

$$[u] = 0, \quad \left[\beta \frac{\partial u}{\partial \mathbf{n}}\right] = 0 \quad \text{across} \quad \Gamma \times [0,T], \tag{1.3}$$

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where $0 < T < +\infty$; $u_t = \frac{\partial u}{\partial t}$; Ω is a bounded convex polygonal domain in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$ and Ω_1 be a subdomain of Ω with C^2 boundary $\partial\Omega_1 := \Gamma$. The interface Γ now divides the domain Ω into two subdomains Ω_1 and $\Omega_2 := \Omega \setminus \Omega_1$. Here, $[v]$ denotes the jump of a quantity v across the interface Γ, i.e. $[v](x) = v_1(x) - v_2(x)$, $x \in \Gamma$ with $v_i(x) = v(x)|_{\Omega_i}$, $i = 1, 2$. The symbol $\mathbf{n}$ denotes the unit outward normal to the boundary $\partial\Omega_1 := \Gamma$. The diffusion coefficient $\beta(x)$ is assumed to be positive and piecewise constant on each subdomain, i.e.

$$\beta(x) = \beta_i \quad \text{for} \quad x \in \Omega_i,\ i = 1, 2.$$

The initial function $u_0(x)$ and the forcing term $f(x, t, u)$ are real-valued functions and assumed to be smooth.

Interface problems usually lead to non-smooth solutions across an interface. Due to low global regularity and irregular geometry of the interfaces it is challenging to achieve high order accuracy by the standard finite element methods. Despite of the efforts given to these type of problems in recent years related to *a priori* error analysis, see [3–7] and references therein, the literature seems to lack $L^\infty(L^2)$ *a posteriori* error analysis in energy method. *A priori* error analysis of semilinear parabolic interface problems in H^1-norm has been studied by Sinha et al. [8]. Some *a posteriori* error analysis results for the semilinear parabolic problems can be found in [9,10].

For parabolic problems (in the absence of an interface), it is known that the energy method for *a posteriori* error analysis of finite element discretizations yields suboptimal rates of convergence in the $L^\infty(L^2)$-norm (cf. [11]). An alternative approach for obtaining optimal rates of convergence in the $L^\infty(L^2)$-norm is based on the parabolic duality technique, see [12]. But as energy method is the most fundamental technique in the a priori error analysis, it is therefore natural to follow this method in the corresponding *a posteriori* error analysis to obtain optimal order error estimates in $L^\infty(L^2)$-norm. To restore optimality in the $L^\infty(L^2)$-norm for parabolic problems Makridakis et al. [13] have introduced a novel elliptic reconstruction technique. This elliptic reconstruction so introduced may be regarded as the dual counterpart of Wheeler's elliptic projection method in *a priori* error analysis introduced by Wheeler [14].

The aim of this paper is to follow the reconstruction technique to derive *a posteriori* error estimators for problem (1.1). More precisely, for backward Euler approximation we use piecewise linear space-time reconstruction (cf. [15]) whereas the Crank-Nicolson approximation uses quadratic space-time reconstruction, (see, e.g. [16]) of finite element solution. A key argument of our proof is the appropriate adaption of elliptic reconstruction operator combined with the energy technique. Other worth mentioning technicalities for our analysis are approximation results of the Clément-type interpolation operator [17,18] and the discrete version of Gronwall's lemma. Optimal order estimates in time and almost optimal order estimates in space in the $L^\infty(L^2)$-norm are obtained for both the backward Euler and Crank-Nicolson approximations.

The layout of the paper is as follows. In Section 2, we briefly introduce some notations and preliminaries, present both the backward Euler and Crank-Nicolson approximations and recall some results from the literature. In Section 3 we derive *a posteriori* error estimates for the backward Euler method of semilinear parabolic problems. Section 4 discusses the related *a posteriori* analysis for the Crank-Nicolson approximation. Finally, concluding remarks are presented in Section 5.

2. Preliminaries

This section introduces some standard function spaces, the finite element discretizations of the domain Ω and the fully discrete backward Euler and the Crank-Nicolson finite element Galerkin approximations to the problem (1.1)–(1.3). In addition, we recall some approximation properties of the Clément-type interpolation operator from [17,18].

2.1. Function spaces

Given a Lebesgue measurable set $\mathcal{M} \subset \mathbb{R}^2$ and $1 \leq p \leq \infty$, we denote by $L^p(\mathcal{M})$, the standard Lebesgue spaces with the norm $\|\cdot\|_{L^p(\mathcal{M})}$. For $p = 2$, $L^2(\mathcal{M})$ is a Hilbert space with respect to the norm induced by the inner product $\langle u, v\rangle = \int_{\mathcal{M}} u(x)v(x)\mathrm{d}x$. We denote the norm of $L^2(\mathcal{M})$ by $\|\cdot\|_{\mathcal{M}}$. For an integer $m > 0$, $H^m(\mathcal{M})$ denotes the usual Sobolev space of real functions having their weak derivatives of order up to m in the Lebesgue space $L^2(\mathcal{M})$ with the norm $\|\cdot\|_{m,\mathcal{M}}$. The function space $H_0^1(\mathcal{M})$ is a subspace of $H^1(\mathcal{M})$ whose elements have vanishing traces on the boundary $\partial\mathcal{M}$. For simplicity of notation, we will skip the subscript $\mathcal{M}$ whenever $\mathcal{M} = \Omega$. We will also use the standard space-time function space $L^p(0, T; \mathbf{B})$, $1 \leq p < +\infty$ ($\mathbf{B}$, Banach space) with the standard norms.

In addition, we shall also work on the function space $(X := H_0^1(\Omega) \cap H^2(\Omega_1) \cap H^2(\Omega_2), \|\cdot\|_X)$ with

$$\|v\|_X := \|v\|_{H_0^1(\Omega)} + \|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}.$$

2.2. Space-time discretizations of the domain

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of the time axis and set $I_n := (t_{n-1}, t_n]$ with time steps $k_n := t_n - t_{n-1}$.

In order to describe the triangulation $\mathcal{T}_n = \{K\}(0 \leq n \leq N)$ of the domain $\bar{\Omega}$ at each time level t_n, we first approximate the domain Ω_1 by a polygon P_{Ω_1} with boundary Γ_P such that all the vertices of the polygon lie on the interface Γ. Thus, Γ_P now splits the domain Ω into two subdomains P_{Ω_1} and P_{Ω_2}, where P_{Ω_2} is a polygon approximating the domain Ω_2. Let $h_n := \max\{h_K \mid h_K = \mathrm{diam}(K),\ K \in \mathcal{T}_n\}$. We now make the following assumptions on the triangulation $\mathcal{T}_n$ (cf. [3,15]).

(A1) If $K_1, K_2 \in \mathcal{T}_n$ and $K_1 \neq K_2$, then either $K_1 \cap K_2 = \emptyset$ or $K_1 \cap K_2$ share a common edge or a common vertex. We also assume that each triangle is either in P_{Ω_1} or in P_{Ω_2} or intersects the interface Γ in at most two vertices.

(A2) Two simplicial decompositions $\mathcal{T}_{n-1}$ and $\mathcal{T}_n$ of $\bar{\Omega}$ are said to be compatible if they are derived from the same macro triangulation $\mathcal{T} = \mathcal{T}_0$ by an admissible refinement procedure which preserves shape regularity and assures that for any elements $K \in \mathcal{T}_{n-1}$ and $K' \in \mathcal{T}_n$, either $K \cap K' = \emptyset$, $K \subset K'$, or $K' \subset K$. There is a natural partial ordering on a set of compatible triangulations, namely $\mathcal{T}_{n-1} \leq \mathcal{T}_n$ if $\mathcal{T}_n$ is a refinement of $\mathcal{T}_{n-1}$. Then for a given pair of successive compatible triangulations $\mathcal{T}_{n-1}$ and $\mathcal{T}_n$, we define naturally the finest common coarsening $\hat{\mathcal{T}}_n := \mathcal{T}_n \wedge \mathcal{T}_{n-1}$ with local mesh sizes are given by $\hat{h}_n := \max\{h_{n-1}, h_n\}$. These conditions allow us to bound the elliptic errors which lie in two adjacent finite element spaces, i.e. finite element spaces defined on meshes at adjacent time steps. For a more detailed discussion on compatible triangulations, we refer to [15].

We shall also need the following notations for future use. For $0 \leq n \leq N$, $\mathcal{E}_n = \{E\}$ be the set of all edges of the triangles $K \in \mathcal{T}_n$ which do not lie on $\partial\Omega$, and $\Sigma_n := \cup_{E \in \mathcal{E}_n} E$. Furthermore, we will also use the sets $\hat{\Sigma}_n := \Sigma_n \cap \Sigma_{n-1}$ and $\check{\Sigma}_n := \Sigma_n \cup \Sigma_{n-1}$.

2.3. The fully discrete finite element approximations

For the purpose of the finite element approximation of the interface problem (1.1)–(1.3), we begin by writing the problem in weak form: Find $u \in L^\infty(0, T; H_0^1(\Omega))$ such that

$$\langle u_t(t), \varphi\rangle + a(u(t), \varphi) = \langle f(x, t, u), \varphi\rangle \quad \forall \varphi \in H_0^1(\Omega), \quad \text{a.e. } t \in (0, T], \tag{2.1}$$

where $a(\cdot, \cdot)$ is a bilinear form on $H_0^1(\Omega)$ defined by

$$a(v, w) = \langle \beta(x)\nabla v, \nabla w\rangle \quad \forall v, w \in H_0^1(\Omega).$$

Note that the bilinear form $a(\cdot,\cdot)$ is bounded and coercive on $H_0^1(\Omega)$, i.e. $\exists \alpha_0, \gamma_0 > 0$ such that

$$|a(v,w)| \leq \alpha_0 \; \|v\|_1 \|w\|_1 \qquad \text{and} \qquad a(v,v) \geq \; \gamma_0 \|v\|_1^2 \quad \forall v,w \in H_0^1(\Omega). \tag{2.2}$$

We assume that $f : \bar{\Omega} \times [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the Lipschitz condition in the third argument, i.e. there exists a constant $\mathcal{C}_L > 0$ such that

$$|f(x,t,v) - f(x,t,w)| \leq \mathcal{C}_L |v-w| \quad \forall v,w \in \mathbb{R}. \tag{2.3}$$

For the existence, uniqueness and regularity of the solution of the semilinear parabolic interface problems, one may refer to Feng and Shen [19].

For each $n = 0, \ldots, N$, we consider the finite element space $\mathbb{S}^n$ corresponding to the triangulation $\mathcal{T}_n$ as follows:

$$\mathbb{S}^n := \left\{ \chi \in H_0^1(\Omega) \;\;|\;\; \chi|_K \in \mathbb{P}_1(K) \;\; \text{for all} \;\; K \in \mathcal{T}_n \right\},$$

where $\mathbb{P}_1(K)$ is the space of polynomials of degree less than or equal to 1 on K. For $v \in \mathbb{S}^n$, let $f^n(v) := f(x,t_n,v)$. Henceforth, we shall use the following shorthand notations: For $1 \leq n \leq N$,

$$f^{n-\frac{1}{2}}(v^{n-\frac{1}{2}}) := \frac{f^n(v^n) + f^{n-1}(v^{n-1})}{2}, \quad v^{n-\frac{1}{2}} := \frac{v^n + v^{n-1}}{2} \quad \text{and} \quad \partial v^n := \frac{v^n - v^{n-1}}{k_n}.$$

Since both the backward Euler and the Crank-Nicolson approximations will be analyzed, we first state these two methods below.

The fully discrete backward Euler approximation:

The standard backward Euler approximation for problem (1.1)–(1.3) may be stated as follows: Given $U^0 = I_h^0 u(0)$, seek $U^n \in \mathbb{S}^n$ ($1 \leq n \leq N$) such that

$$\left\langle \frac{U^n - U^{n-1}}{k_n}, \chi_n \right\rangle + a(U^n, \chi_n) = \langle f^n(U^n), \chi_n \rangle \quad \forall \chi_n \in \mathbb{S}^n. \tag{2.4}$$

Here, the operator I_h^0 is a suitable projection from $H_0^1(\Omega)$ into the finite-dimensional subspace $\mathbb{S}^0$.

The fully discrete Crank-Nicolson approximation:

The fully discrete Crank-Nicolson approximation of the problem (1.1)–(1.3) is stated as follows: Let $U^0 = I_h^0 u(0)$, where I_h^0 is a suitable projection operator from $H_0^1(\Omega)$ into the finite-dimensional subspace $\mathbb{S}^0$. Then, for $1 \leq n \leq N$, find $U^n \in \mathbb{S}^n$ such that

$$\left\langle \frac{U^n - U^{n-1}}{k_n}, \chi_n \right\rangle + a\left(U^{n-\frac{1}{2}}, \chi_n\right) = \left\langle f^{n-\frac{1}{2}}(U^{n-\frac{1}{2}}), \chi_n \right\rangle \qquad \forall \chi_n \in \mathbb{S}^n. \tag{2.5}$$

We now recall the following projection operators for our subsequent use.

Discrete elliptic operator: The discrete elliptic operator associated with the bilinear form $a(\cdot,\cdot)$ and the finite element space $\mathbb{S}^n$ is the operator $\mathcal{A}_h^n : H_0^1(\Omega) \longrightarrow \mathbb{S}^n$ such that for $v \in H_0^1(\Omega)$ and $0 \leq n \leq N$,

$$\langle \mathcal{A}_h^n v, \; \chi_n \rangle \; = \; a(v, \chi_n) \qquad \forall \chi_n \in \mathbb{S}^n. \tag{2.6}$$

L^2-projection operator: The L^2-projection operator is a map $\Pi_0^n : L^2(\Omega) \longrightarrow \mathbb{S}^n$ such that for $v \in L^2(\Omega)$ and $0 \le n \le N$,

$$\langle \Pi_0^n v, \chi_n \rangle = \langle v, \chi_n \rangle \quad \forall \chi_n \in \mathbb{S}^n. \tag{2.7}$$

Using the above projections, (2.5) can be expressed in distributional form as

$$\frac{U^n - \Pi_0^n U^{n-1}}{k_n} + \frac{1}{2}(\mathcal{A}_h^n)U^n + \frac{1}{2}(\mathcal{A}_h^n)U^{n-1} = \Pi_0^n f^{n-\frac{1}{2}}(U^{n-\frac{1}{2}}).$$

For parabolic problems, Bänsch et al. [16] has observed that the discrete elliptic operator $\mathcal{A}_h^n$ on the finer mesh when applied to the coarse grid function U^{n-1} in the above may lead to oscillation during refinement. The same behavior is naturally expected for the parabolic interface problems as well. Therefore, following the discussion of [16], we consider the following modified Crank-Nicolson approximation.

Modified Crank-Nicolson approximation: Given $U^0 = I_h^0 u(0)$, for $1 \le n \le N$ seek $U^n \in \mathbb{S}^n$ such that

$$\frac{1}{k_n}(U^n - P_1^n U^{n-1}) + \frac{1}{2}(\mathcal{A}_h^n U^n) + \frac{1}{2}P_2^n(\mathcal{A}_h^{n-1}U^{n-1}) = \Pi_0^n f^{n-\frac{1}{2}}(U^{n-\frac{1}{2}}),\ 1 \le n \le N, \tag{2.8}$$

where $P_1^n, P_2^n : \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^n$ be any suitable projection operators.

Representation of elliptic operator: Let $v \in \mathbb{S}^n (0 \le n \le N)$. Then, the bilinear form $a(\cdot,\cdot)$ can be rewritten using the Green's formula as

$$a(v,\varphi) = \sum_{K\in\mathcal{T}_n} \langle -\mathrm{div}(\beta(x)\nabla v), \varphi\rangle_K + \sum_{E\in\mathcal{E}_n} \langle j[\beta v], \varphi\rangle_E \tag{2.9}$$

$$= \langle (v)_{\mathrm{el}}, \varphi\rangle + \langle j[\beta v], \varphi\rangle_{\Sigma_n} \quad \forall \varphi \in H_0^1(\Omega), \tag{2.10}$$

where $(v)_{\mathrm{el}}$ in (2.10) denotes the regular part of the distribution $-\mathrm{div}(\beta\nabla v)$, and is defined as a piecewise continuous function such that

$$\langle (v)_{\mathrm{el}}, \varphi\rangle = \sum_{K\in\mathcal{T}_n} \langle -\mathrm{div}(\beta\nabla v), \varphi\rangle_K \quad \forall \varphi \in H_0^1(\Omega).$$

The quantity $j[\beta v]$ denotes the spatial jump of $\beta\nabla v$ across an element side $E \in \mathcal{E}_n$ and is defined as

$$j[\beta v]|_E(x) = \lim_{\epsilon\to 0} \big(\beta\nabla v(x + \epsilon\eta_E) - \beta\nabla v(x - \epsilon\eta_E)\big)\cdot \eta_E,$$

where η_E is an arbitrary unit normal vector to E at the point x.

2.4. Clément-type interpolation estimates

A residual-based *a posteriori* error estimates mainly uses the approximation properties of the Clément-type interpolation operator introduced by Scott and Zhang [17]. The approximation properties for such type of operator are established in [17, Theorem 4.1] under needed regularity assumptions on functions. However, in the present case, due to the discontinuity of the diffusion coefficient β along the interface Γ, the solution has a lower regularity in the entire domain Ω, usually one has only $u \in X$. Thus, the existing approximation results do not apply directly. Therefore, new approximation results obtain in [18] yield nearly optimal order convergence up to $|\log h_n|$ factor with $u \in X$. We now recall the following approximation properties of the Clément-type interpolation operator from [17,18].

Proposition 2.1: *Let $\mathscr{I}_n : X \longrightarrow \mathbb{S}^n$ be the standard Clément-type interpolation operator as introduced in* [17]. *Then, for the finite element polynomial space of degree ≤ 1, the following interpolation estimates hold: For $v \in H_0^1(\Omega)$, we have*

$$\begin{cases} \|v - \mathscr{I}_n v\| \ \leq \mathcal{C}_{I,1}\, h_n\, \|v\|_1, \\ \|v - \mathscr{I}_n v\|_{\Sigma_n} \ \leq \mathcal{C}_{I,2}\, h_n^{\frac{1}{2}}\, \|v\|_1, \end{cases} \tag{2.11}$$

and for $v \in X$,

$$\begin{cases} \|v - \mathscr{I}_n v\| \ \leq \mathcal{C}_{I,3}\, h_n^2\, |\log h_n|^{\frac{1}{2}}\, \|v\|_X, \\ \|v - \mathscr{I}_n v\|_{\Sigma_n} \ \leq \mathcal{C}_{I,4}\, h_n^{\frac{3}{2}}\, |\log h_n|^{\frac{1}{2}}\, \|v\|_X, \end{cases} \tag{2.12}$$

where the constants $\mathcal{C}_{I,k}$, $k \in \{1,2,3,4\}$ depend only on the shape-regularity of the family of triangulations.

Next, we state the Clément-type interpolation inequalities relative to the finest common coarsening of $\mathcal{T}_n$ and $\mathcal{T}_{n-1}$ which reflects the mesh change behavior.

Proposition 2.2: *Let $\hat{\mathscr{I}}_n : X \longrightarrow \mathbb{S}^n \cap \mathbb{S}^{n-1}$ be the Clément-type interpolation operator with respect to the finest common coarsening of $\mathcal{T}_n$ and $\mathcal{T}_{n-1}$, i.e. $\hat{\mathcal{T}}_n := \mathcal{T}_n \wedge \mathcal{T}_{n-1}$ corresponding to the finite element space $\mathbb{S}^n \cap \mathbb{S}^{n-1}$ with mesh size $\hat{h}_n := max\{h_n, h_{n-1}\}$. Then, for the finite element polynomial space of degree ≤ 1, the following is true for $v \in X$:*

$$\|v - \hat{\mathscr{I}}_n v\|_{\check{\Sigma}_n \setminus \hat{\Sigma}_n} \leq \mathcal{C}_{I,5}\, \hat{h}_n^{\frac{3}{2}}\, |\log \hat{h}_n|^{\frac{1}{2}}\, \|v\|_X,$$

where the constant $\mathcal{C}_{I,5}$ depends on the shape regularity of the family of triangulations and on the number of steps required to move from $\mathcal{T}_{n-1}$ to $\mathcal{T}_n$.

Further, the approximation properties (2.11) *and* (2.12) *hold true in the finite element space $\mathbb{S}^n \cap \mathbb{S}^{n-1}$ with $\hat{h}_n$ replacing h_n.*

3. Abstract backward Euler error analysis

In this section, we first introduce the elliptic reconstruction operator and then discuss the related *a posteriori* error analysis for the backward Euler approximation.

Definition 3.1 (Elliptic reconstruction): For a fully discrete finite element solution $U^n \in \mathbb{S}^n$ obtained from (2.4), we define the elliptic reconstruction $\mathcal{R}_b^n U^n \in H_0^1(\Omega)$ of $U^n \in \mathbb{S}^n$ as the solution of the following elliptic problem

$$a(\mathcal{R}_b^n U^n, \varphi) = \langle \tilde{f}^n, \varphi \rangle \ \ \forall \varphi \in H_0^1(\Omega), \tag{3.1}$$

where

$$\tilde{f}^n := \begin{cases} \mathcal{A}_h^0 U^0, & n = 0, \\ f^n(U^n) - k_n^{-1}(U^n - U^{n-1}), & 1 \leq n \leq N. \end{cases}$$

Note that the operator $\mathcal{R}_b^n$ satisfies the Galerkin orthogonality property. We now state the following elliptic reconstruction error bound in the L^2-norm. For a proof, we refer the reader to [18, Lemma 5.1] for details.

Lemma 3.1: *For a finite element approximation $U^n \in \mathbb{S}^n$ of the elliptic Equation* (3.1), *the following is true for $0 \leq n \leq N$:*

$$\|(\mathcal{R}_b^n - I)U^n\| \leq \mathcal{C}_{I,6}\, h_n^2\, |\log h_n|^{\frac{1}{2}} \|\tilde{f}^n - (U^n)_{\mathrm{el}}\| + \mathcal{C}_{I,7}\, h_n^{\frac{3}{2}}\, |\log h_n|^{\frac{1}{2}}\, \|j[\beta U^n]\|_{\Sigma_n},$$

where $\mathcal{C}_{I,6} := \mathcal{C}_{I,3}\mathcal{C}_R$ *and* $\mathcal{C}_{I,7} := \mathcal{C}_{I,4}\mathcal{C}_R$.

In order to derive the *a posteriori* error bound, we split the total error $e(t) := u(t) - U(t)$ by considering the reconstruction $\tilde{\Theta}(t)$ as an intermediate object as follows:

$$e(t) := \tilde{\rho}(t) + \tilde{\varepsilon}(t), \quad t \in I_n \quad \text{where} \quad \tilde{\rho}(t) := u(t) - \tilde{\Theta}(t), \; \tilde{\varepsilon}(t) := \tilde{\Theta}(t) - U(t), \tag{3.2}$$

where $U(t)$ and $\tilde{\Theta}(t)$, $t \in I_n$ (piecewise linear interpolant of the solution and reconstruction operator) are defined by

$$U(t) := l_{n-1}(t)U^{n-1} + l_n(t)U^n, \quad t \in I_n \; (n = 1\ldots, N), \tag{3.3}$$

$$\tilde{\Theta}(t) := l_{n-1}(t)\, \mathcal{R}_b^{n-1}\, U^{n-1} + l_n(t)\, \mathcal{R}_b^n\, U^n, \quad t \in I_n \; (n = 1\ldots, N), \tag{3.4}$$

with

$$l_{n-1}(t) := \frac{t_n - t}{k_n} \quad \text{and} \quad l_n(t) := \frac{t - t_{n-1}}{k_n} \quad \text{for } t \in I_n. \tag{3.5}$$

With $\tilde{\rho}(t)$ (parabolic error) and $\tilde{\varepsilon}(t)$ (elliptic reconstruction error) as above, for $1 \le n \le N$ and for each $\varphi \in H_0^1(\Omega)$, $t \in I_n$, we have by simple calculation the following parabolic error equation:

$$\langle \tilde{\rho}_t(t), \varphi\rangle + a(\tilde{\rho}(t), \varphi) = -\langle \tilde{\varepsilon}_t(t), \varphi\rangle - a(\tilde{\Theta}(t) - \tilde{\Theta}^n, \varphi) + \big\langle f(t,u) - f^n(U^n), \varphi\big\rangle. \tag{3.6}$$

Now we define the following residual-based error estimators which will be used in the subsequent analysis of the fully discrete backward Euler approximation.

The elliptic reconstruction error estimator:

For $0 \le n \le N$,

$$\mathscr{O}_{\mathrm{BE},n} := \mathcal{C}_{I,6}\, h_n^2\, |\log h_n|^{\frac{1}{2}} \|\tilde{f}^n - (U^n)_{\mathrm{el}}\| + \mathcal{C}_{I,7}\, h_n^{\frac{3}{2}}\, |\log h_n|^{\frac{1}{2}}\, \|j[\beta U^n]\|_{\Sigma_n}. \tag{3.7}$$

The space-mesh error estimator:

For $1 \le n \le N$,

$$\begin{aligned}\mathscr{M}_{\mathrm{BE},n} := {}& \mathcal{C}_{I,6}\, \hat{h}_n^2\, |\log \hat{h}_n|^{\frac{1}{2}}\, \|\partial \hat{\mathbf{R}}_n\| + \mathcal{C}_{I,7}\, \hat{h}_n^{\frac{3}{2}}\, |\log \hat{h}_n|^{\frac{1}{2}} \|\partial \hat{\mathbf{J}}_n\|_{\hat{\Sigma}_n} \\ & + \mathcal{C}_{I,8}\, \hat{h}_n^{\frac{3}{2}}\, |\log \hat{h}_n|^{\frac{1}{2}}\, \|\partial \hat{\mathbf{J}}_n\|_{\check{\Sigma}_n \setminus \hat{\Sigma}_n}, \quad \mathcal{C}_{I,8} = \mathcal{C}_{I,7}\mathcal{C}_R,\end{aligned} \tag{3.8}$$

where

$$\hat{\mathbf{R}}_0 := (U^0)_{\mathrm{el}} - (\mathcal{A}_h^0 U^0) \quad \text{and} \quad \hat{\mathbf{R}}_n := k_n^{-1}(U^n - U^{n-1}) + (U^n)_{\mathrm{el}} - \tilde{f}^n, \quad \text{for } 1 \le n \le N$$

denote the *element residuals* and $\hat{\mathbf{J}}_n := j[\beta U^n]$, for $0 \le n \le N$ refers to the *jump residual.*

The temporal error estimator:

For $1 \le n \le N$,

$$\mathscr{T}_{\mathrm{e,BE},n} := \begin{cases} \frac{1}{2}\, \|\mathcal{A}_h^0 U^0 - f^1(U^1) + \partial U^1\|, & n = 1, \\ \frac{1}{2}\, k_n\, \|\partial\left(f^n(U^n) - \partial U^n\right)\|, & 2 \le n \le N. \end{cases} \tag{3.9}$$

The data approximation error estimators:

For $1 \le n \le N$,

$$\begin{cases} \mathscr{D}_{\mathrm{BE},n,1} := \sqrt{C_L}\max\{\|\tilde{\varepsilon}^n\|, \|\tilde{\varepsilon}^{(n-1)}\|\}, \\ \mathscr{D}_{\mathrm{BE},n,2} := \frac{1}{k_n}\int_{t_{n-1}}^{t_n} \|f(t,U) - f^n(U^n)\| \mathrm{d}t. \end{cases} \tag{3.10}$$

A posteriori error bound for the parabolic error $\tilde{\rho}(t)$ relies on a sequence of auxiliary lemmas. Below, we shall state Lemmas 3.2–3.3 without proofs. The proof of the Lemma 3.2 is immediate from [18, Lemma 5.4]. Following the idea of [15], the proof of the Lemma 3.3 follows. We therefore, refrain from giving the details.

Lemma 3.2 (Space-mesh error estimate): *With $\mathscr{M}_{\mathrm{BE},n}$ as in (3.8), let $\mathfrak{I}_{n,1}$ represent the space-mesh error term and be given by*

$$\mathfrak{I}_{n,1} := \int_{t_{n-1}}^{t_n} |\langle \tilde{\varepsilon}_t(t), \tilde{\rho}(t)\rangle| \, \mathrm{d}t.$$

Then we have

$$\mathfrak{I}_{n,1} \le k_n \, \mathscr{M}_{\mathrm{BE},n} \max_{t\in \bar{I}_n} \|\tilde{\rho}(t)\|.$$

Lemma 3.3 (Temporal error estimate): *With $\mathscr{T}_{\mathrm{e,BE},n}$ as in (3.9), let $\mathfrak{I}_{n,2}$ denote the temporal error term and be defined by*

$$\mathfrak{I}_{n,2} := \int_{t_{n-1}}^{t_n} \left| a\left(\tilde{\Theta}(t) - \tilde{\Theta}^n, \tilde{\rho}(t)\right)\right| \mathrm{d}t.$$

Then we have

$$\mathfrak{I}_{n,2} \le k_n \, \mathscr{T}_{\mathrm{e,BE},n} \max_{t\in \bar{I}_n} \|\tilde{\rho}(t)\|.$$

Lemma 3.4 (Data approximation error estimate): *With $\mathscr{D}_{\mathrm{BE},n,1}$ and $\mathscr{D}_{\mathrm{BE},n,2}$ as in (3.10), let the data approximation error term be represented by $\mathfrak{I}_{n,3}$ and be defined as*

$$\mathfrak{I}_{n,3} := \int_{t_{n-1}}^{t_n} \left|\left\langle f(t,u) - f^n(U^n), \tilde{\rho}(t)\right\rangle\right| \mathrm{d}t.$$

Then we have

$$\begin{aligned}\mathfrak{I}_{n,3} \le \frac{\sqrt{C_L}}{2\epsilon} k_n \max_{t\in \bar{I}_n} \|\tilde{\rho}(t)\|^2 + \frac{\epsilon\sqrt{C_L}}{2} \int_{t_{n-1}}^{t_n} \|\tilde{\rho}(t)\|_1^2 \mathrm{d}t + k_n \, \mathscr{D}_{\mathrm{BE},n,1} \max_{t\in \bar{I}_n} \|\tilde{\rho}(t)\| \\ + k_n \mathscr{D}_{\mathrm{BE},n,2} \max_{t\in \bar{I}_n} \|\tilde{\rho}(t)\|.\end{aligned}$$

Proof: We rewrite $\mathfrak{I}_{n,3}$ as

$$\begin{aligned}\mathfrak{I}_{n,3} \le & \int_{t_{n-1}}^{t_n} \left|\left\langle f(t,u) - f(t,\tilde{\Theta}(t)), \tilde{\rho}(t)\right\rangle\right| \mathrm{d}t + \int_{t_{n-1}}^{t_n} \left|\left\langle f(t,\tilde{\Theta}(t)) - f(t,U(t)), \tilde{\rho}(t)\right\rangle\right| \mathrm{d}t \\ & + \int_{t_{n-1}}^{t_n} \left|\left\langle f(t,U(t)) - f^n(U^n), \tilde{\rho}(t)\right\rangle\right| \mathrm{d}t \\ := & \; T_{n,1} + T_{n,2} + T_{n,3}.\end{aligned} \tag{3.11}$$

Using the Cauchy–Schwarz inequality, (2.3) and the Young's inequality (with $\epsilon > 0$), we obtain

$$\begin{aligned}\left|\left\langle f(t,u) - f(t,\tilde{\Theta}(t)), \tilde{\rho}(t)\right\rangle\right| &\le \|f(t,u) - f(t,\tilde{\Theta}(t))\| \|\tilde{\rho}(t)\| \\ &\le \sqrt{C_L}\left\{\frac{\|\tilde{\rho}(t)\|^2}{2\epsilon} + \frac{\epsilon}{2}\|\tilde{\rho}(t)\|_1^2\right\}.\end{aligned}$$

Thus,

$$T_{n,1} \leq \frac{\sqrt{\mathcal{C}_L}}{2\epsilon}\, k_n \max_{t\in\bar{I}_n} \|\tilde{\rho}(t)\|^2 + \frac{\epsilon\sqrt{\mathcal{C}_L}}{2} \int_{t_{n-1}}^{t_n} \|\tilde{\rho}(t)\|_1^2 \mathrm{d}t. \tag{3.12}$$

To estimate the second term in (3.11), we use the Cauchy–Schwarz inequality and (2.3) to obtain

$$\left|\left\langle f(t,\tilde{\Theta}(t)) - f(t,U(t)), \tilde{\rho}(t)\right\rangle\right| \leq \sqrt{\mathcal{C}_L}\, \|\tilde{\Theta}(t) - U(t)\|\, \|\tilde{\rho}(t)\|.$$

In view of (3.4), it follows that

$$\left|\left\langle f(t,\tilde{\Theta}(t)) - f(t,U(t)), \tilde{\rho}(t)\right\rangle\right| \leq \sqrt{\mathcal{C}_L}\, \left\{\left|\frac{t_n - t}{k_n}\right|\, \|\tilde{\varepsilon}^{n-1}\| + \left|\frac{t - t_{n-1}}{k_n}\right|\, \|\tilde{\varepsilon}^{n}\|\right\}\, \|\tilde{\rho}(t)\|.$$

Therefore,

$$T_{n,2} \leq \frac{\sqrt{\mathcal{C}_L}}{2}\, k_n\, \left\{\|\tilde{\varepsilon}^{n-1}\| + \|\tilde{\varepsilon}^{n}\|\right\} \max_{t\in\bar{I}_n} \|\tilde{\rho}(t)\| = k_n\, \mathscr{D}_{\mathrm{BE},n,1} \max_{t\in\bar{I}_n} \|\tilde{\rho}(t)\|. \tag{3.13}$$

Finally to estimate $T_{n,3}$, we use the Cauchy–Schwarz inequality to have

$$T_{n,3} \leq \max_{t\in\bar{I}_n} \|\tilde{\rho}(t)\| \int_{t_{n-1}}^{t_n} \|f(t,U(t)) - f^n(U^n)\|\, \mathrm{d}t = k_n\, \mathscr{D}_{\mathrm{BE},n,2} \max_{t\in\bar{I}_n} \|\tilde{\rho}(t)\|, \tag{3.14}$$

which in conjunction with (3.12) and (3.13) complete the desired proof. □

Now, we apply the above lemmas to derive the *a posteriori* error bound for the parabolic error $\tilde{\rho}(t)$ in the $L^\infty(L^2)$-norm.

Theorem 3.5: *Let u be the exact solution of (1.1)–(1.3) and let U^n be its finite element approximation obtained by the backward Euler approximation (2.4). Then, for $1 \leq m \leq N$, the following a posteriori error bound holds:*

$$\left\{\max_{t\in[0,t_m]} \|\tilde{\rho}(t)\|^2 + \int_0^{t_m} \|\tilde{\rho}(t)\|_1^2 \mathrm{d}t\right\}^{\frac{1}{2}} \leq \left\{2\,\mathcal{C}_G(m)\, \|\tilde{\rho}(0)\|^2\right\}^{\frac{1}{2}} + 4\,\mathcal{C}_G(m) \sum_{n=1}^{m} k_n \left\{\mathscr{M}_{\mathrm{BE},n} + \mathscr{T}_{\mathrm{e,BE},n} + \mathscr{D}_{\mathrm{BE},n,1} + \mathscr{D}_{\mathrm{BE},n,2}\right\},$$

where $\mathcal{C}_G(m)$ is a positive constant due to the Gronwall's lemma and the quantities $\mathscr{M}_{\mathrm{BE},n}$, $\mathscr{T}_{\mathrm{e,BE},n}$, $\mathscr{D}_{\mathrm{BE},n,i} (i = 1, 2)$ are given in (3.8)–(3.10), respectively.

***Proof*:** Setting $\varphi = \tilde{\rho}(t)$ in (3.6) and using (2.2), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{\rho}(t)\|^2 + \frac{\gamma_0}{2}\|\tilde{\rho}(t)\|_1^2 \leq |\langle\tilde{\varepsilon}_t(t), \tilde{\rho}(t)\rangle| + |a(\tilde{\Theta}(t) - \tilde{\Theta}^n, \tilde{\rho}(t))| + |\langle f(t,u) - f^n(U^n), \tilde{\rho}(t)\rangle|.$$

Integrate the above from t_{n-1} to t_n to have

$$\frac{1}{2}||\tilde{\rho}(t_n)||^2 - \frac{1}{2}\|\tilde{\rho}(t_{n-1})\|^2 + \frac{\gamma_0}{2}\int_{t_{n-1}}^{t_n} \|\tilde{\rho}(t)\|_1^2 \mathrm{d}t \leq \mathfrak{I}_{n,1} + \mathfrak{I}_{n,2} + \mathfrak{I}_{n,3},$$

where $\mathfrak{I}_{n,i} (i = 1, 2, 3)$ are defined in Lemmas 3.2–3.4, respectively. Summing up over $n = 1 : m$ we have

$$\|\tilde{\rho}(t_m)\|^2 + \gamma_0 \int_0^{t_m} \|\tilde{\rho}(t)\|_1^2 \mathrm{d}t \le \|\tilde{\rho}(0)\|^2 + 2 \sum_{n=1}^{m} \left\{\mathfrak{I}_{n,1} + \mathfrak{I}_{n,2} + \mathfrak{I}_{n,3}\right\}. \tag{3.15}$$

Since $\tilde{\rho}(t)$ is continuous in $[0, t_m]$, there exists $t_{0,m} \in [0, t_m]$ such that

$$\|\tilde{\rho}_{0,m}\| := \|\tilde{\rho}(t_{0,m})\| = \max_{t\in[0,t_m]} \|\tilde{\rho}(t)\|,$$

and therefore,

$$\|\tilde{\rho}(t_{0,m})\|^2 + \gamma_0 \int_0^{t_m} \|\tilde{\rho}(t)\|_1^2 \mathrm{d}t \le 2\|\tilde{\rho}(0)\|^2 + 4 \sum_{n=1}^{m} \left\{\mathfrak{I}_{n,1} + \mathfrak{I}_{n,2} + \mathfrak{I}_{n,3}\right\}.$$

Now, using Lemmas 3.2–3.4, we obtain

$$\begin{aligned}\max_{t\in[0,t_m]} \|\tilde{\rho}(t)\|^2 \;&\le\; 2\|\tilde{\rho}(0)\|^2 + (2\,\epsilon\,\sqrt{\mathcal{C}_L} - \gamma_0) \int_0^{t_m} \|\tilde{\rho}(t)\|_1^2 \mathrm{d}t \\ &+ 4 \max_{t\in[0,t_m]} \|\tilde{\rho}(t)\| \sum_{n=1}^{m} k_n \left\{\mathscr{M}_{\mathrm{BE},n} + \mathscr{T}_{\mathrm{e,BE},n} + \mathscr{D}_{\mathrm{BE},n,1} + \mathscr{D}_{\mathrm{BE},n,2}\right\} + \frac{2\sqrt{\mathcal{C}_L}}{\epsilon} \sum_{n=1}^{m} k_n \max_{t\in[0,t_n]} \|\tilde{\rho}(t)\|^2.\end{aligned}$$

Choose $\epsilon > 0$ be such that $(2\,\epsilon\,\sqrt{\mathcal{C}_L} - \gamma_0) > 0$ and a use of the discrete Gronwall's Lemma imply

$$\begin{aligned}\max_{t\in[0,t_m]} \|\tilde{\rho}(t)\|^2 + \mathcal{C}_G(m) \int_0^{t_m} \|\tilde{\rho}(t)\|_1^2 \mathrm{d}t \;&\le\; 2\,\mathcal{C}_G(m)\,\|\tilde{\rho}(0)\|^2 \\ &+ 4\,\mathcal{C}_G(m) \max_{t\in[0,t_m]} \|\tilde{\rho}(t)\| \sum_{n=1}^{m} k_n \left\{\mathscr{M}_{\mathrm{BE},n} + \mathscr{T}_{\mathrm{e,BE},n} + \mathscr{D}_{\mathrm{BE},n,1} + \mathscr{D}_{\mathrm{BE},n,2}\right\},\end{aligned}$$

where $\mathcal{C}_G(m) := 2\ \max\left\{1,\ \sum_{n=1}^{m} \frac{2\sqrt{\mathcal{C}_L}}{\epsilon}\, k_n\ \exp\left\{\frac{2\sqrt{\mathcal{C}_L}}{\epsilon}\ \left(\sum_{n<j<m} k_j\right)\right\}\right\}$.

Finally, we take

$$a_0 := \max_{t\in[0,t_m]} \|\tilde{\rho}(t)\|, \quad a_n := \left\{\mathcal{C}_G(m) \int_{t_{n-1}}^{t_n} \|\tilde{\rho}(t)\|_1^2\, \mathrm{d}t\right\}^{\frac{1}{2}} \ (1 \le n \le m), \quad c := \left\{2\,\mathcal{C}_G(m)\,\|\tilde{\rho}(0)\|^2\right\}^{\frac{1}{2}},$$

$$b_0 := 4\,\mathcal{C}_G(m) \sum_{n=1}^{m} k_n \left\{\mathscr{M}_{\mathrm{BE},n} + \mathscr{T}_{\mathrm{e,BE},n} + \mathscr{D}_{\mathrm{BE},n,1} + \mathscr{D}_{\mathrm{BE},n,2}\right\}, \quad \text{and} \quad b_n := 0, (1 \le n \le m),$$

and use standard inequality [15, (80)] to complete the proof. □

The following theorem presents the fully discrete backward Euler *a posteriori* error estimate in the $L^\infty(L^2)$-norm for the semilinear parabolic interface problem (1.1)–(1.3).

Theorem 3.6: *Let u be the exact solution of (1.1)–(1.3) and let U^n be its finite element approximation obtained by the backward Euler approximation (2.4). Then, for each $1 \le m \le N$, the following a posteriori error estimate holds:*

$$\begin{aligned}\max_{t\in[0,t_m]} \|u(t) - U(t)\| \le \{2\,\mathcal{C}_G(m)\}^{\frac{1}{2}}\, \|\mathcal{R}^0 U^0 - u(0)\| + 2 \max_{0\le n\le m} \mathscr{O}_{\mathrm{BE},n} \\ + 4\,\mathcal{C}_G(m) \sum_{n=1}^{m} k_n \left\{\mathscr{M}_{\mathrm{BE},n} + \mathscr{T}_{\mathrm{e,BE},n} + \mathscr{D}_{\mathrm{BE},n,1} + \mathscr{D}_{\mathrm{BE},n,2}\right\}.\end{aligned}$$

The estimators $\mathscr{O}_{\mathrm{BE},n}$, $\mathscr{M}_{\mathrm{BE},n}$, $\mathscr{T}_{\mathrm{e,BE},n}$, and $\mathscr{D}_{\mathrm{BE},n,i}(i = 1, 2)$ are given by (3.7)–(3.10), respectively.

***Proof*:** By the triangle inequality, we have

$$\|e(t)\| = \|u(t) - U(t)\| \leq \|\tilde{\rho}(t)\| + \|\tilde{\varepsilon}(t)\|, \quad t \in I_n. \tag{3.16}$$

Now,

$$\|\tilde{\varepsilon}(t)\| = \|l_{n-1}(t)\tilde{\varepsilon}^{(n-1)} + l_n(t)\tilde{\varepsilon}^n\| \leq 2\ \max\{\|\tilde{\varepsilon}^n\|, \|\tilde{\varepsilon}^{(n-1)}\|\}, \quad t \in I_n.$$

Again, for $t \in [0, t_m]$, using Lemma 3.1 we obtain

$$\|\tilde{\varepsilon}(t)\| \leq\ 2 \max_{0\leq n\leq m} \mathscr{O}_{\mathrm{BE},n},$$

which combine with Theorem 3.5 proves the result. □

4. Abstract Crank-Nicolson error analysis

For the purpose of the fully discrete Crank-Nicolson error analysis, we now define the space-time quadratic reconstruction for the Crank-Nicolson approximation (2.8). For this, we recall the definition of elliptic reconstruction from [15,16].

Definition 4.1 (elliptic reconstruction): For $v \in \mathbb{S}^n$, we define the elliptic reconstruction $\mathcal{R}_c^n v$ of v as the solution of the following elliptic problem

$$a(\mathcal{R}_c^n v, \varphi) = \langle \mathcal{A}_h^n v, \varphi\rangle \quad \forall \varphi \in H_0^1(\Omega),\ 0 \leq n \leq N. \tag{4.1}$$

Now, we shall introduce some notations for further use.

Let $\Psi : [0, T] \longrightarrow H_0^1(\Omega)$ be continuous piecewise linear function in time defined by

$$\Psi(t) := l_{n-1}(t)\,(P_2^n(\mathcal{A}_h^{n-1}))\,U^{n-1} + l_n(t)\,(\mathcal{A}_h^n)\,U^n, \quad t \in I_n\ (n = 1\ldots, N), \tag{4.2}$$

where $l_{n-1}(t)$ and $l_n(t)$ are given by (3.5).

Also, let $\breve{\Phi} : [0, T] \longrightarrow H_0^1(\Omega)$ be a continuous piecewise linear interpolant of $f(t)$ defined by

$$\breve{\Phi}(t) := l_{n-1}(t)f^{n-1}(U^{n-1}) + l_n(t)f^n(U^n), \quad t \in I_n\ (n = 1\ldots, N). \tag{4.3}$$

Next, to define space-time reconstruction we rewrite the fully discrete Crank-Nicolson approximation (2.8) in the compact form as:

$$\frac{U^n - P_1^n U^{n-1}}{k_n} = H(t_{n-\frac{1}{2}}), \quad n \geq 1, \tag{4.4}$$

where

$$H(t_{n-\frac{1}{2}}) := \Pi_0^n f^{n-\frac{1}{2}}(U^{n-\frac{1}{2}}) - \Psi(t_{n-\frac{1}{2}}), \quad n \geq 1. \tag{4.5}$$

We also define $\breve{H} : [0, T] \longrightarrow H_0^1(\Omega)$ be a piecewise linear function in time defined as

$$\breve{H}(t) := \Pi_0^n\breve{\Phi}(t) - \Psi(t), \quad t \in I_n\ (n = 1, \ldots, N), \tag{4.6}$$

and $\breve{H}(t_{n-\frac{1}{2}}) = H(t_{n-\frac{1}{2}})$.

Inspired by the idea of [16,20], we now define below the space-time Crank-Nicolson reconstruction $\breve{U}$ of the Crank-Nicolson finite element solution U.

Definition 4.2 (space-time reconstruction): The quadratic space-time reconstruction $\breve{U} : [0, T] \longrightarrow H_0^1(\Omega)$ of U is defined by

$$\begin{aligned}\breve{U}(t) &:= \mathcal{R}_c^{n-1}U^{n-1} + k_n{}^{-1}(t-t_{n-1})\left\{(\mathcal{R}_c^n P_1^n)U^{n-1} - \mathcal{R}_c^{n-1}U^{n-1}\right\}\\ &\quad + \int_{t_{n-1}}^{t} \mathcal{R}_c^n \hat{H}_3(s)\mathrm{d}s, \quad t \in I_n \ (n = 1, \dots, N).\end{aligned}$$

Observe that $\breve{U}$ is a continuous function in time and satisfies the relation

$$\breve{U}_t(t) = k_n{}^{-1}\left\{(\mathcal{R}_c^n P_1^n)U^{n-1} - \mathcal{R}_c^{n-1}U^{n-1}\right\} + \mathcal{R}_c^n \breve{H}(t), \quad t \in I_n \ (n = 1, \dots, N). \tag{4.7}$$

To derive the *a posteriori* estimates we decompose the total error $e(t) := u(t) - U(t)$ as

$$e(t) := \breve{\rho}(t) + \breve{\sigma}(t) + \breve{\varepsilon}(t), \quad t \in I_n, \tag{4.8}$$

where $\breve{\rho}(t) := u(t) - \breve{U}(t)$ denotes the parabolic error, $\breve{\sigma}(t) := \breve{U}(t) - \breve{\Theta}(t)$ refers to the time reconstruction error and $\breve{\varepsilon}(t) := \breve{\Theta}(t) - U(t)$ denotes the elliptic reconstruction error. Here, $\breve{\Theta}(t)$ is the continuous piecewise linear function in time defined by

$$\breve{\Theta}(t) := l_{n-1}(t)\, \mathcal{R}_c^{n-1}\, U^{n-1} + l_n(t)\, \mathcal{R}_c^n\, U^n, \quad t \in I_n \ (n = 1 \dots, N). \tag{4.9}$$

The following lemma yields *a posteriori* error bounds for the elliptic reconstruction error $\breve{\varepsilon}(t)$. The proof follows from the backward Euler case (cf. Lemma 3.1).

Lemma 4.1: *For a finite element approximation $U^n \in \mathbb{S}^n$ of the elliptic Equation 4.1, the following is true for $0 \le n \le N$:*

$$\|(\mathcal{R}_c^n - I)v\| \le C_{I,6}\, h_n^2\, |\log h_n|^{\frac{1}{2}} \|\mathcal{A}_h^n v - (v)_{\mathrm{el}}\| + C_{I,7}\, h_n^{\frac{3}{2}}\, |\log h_n|^{\frac{1}{2}}\, \|j[\beta v]\|_{\Sigma_n},$$

where $C_{I,6} := C_{I,3}C_R$ and $C_{I,7} := C_{I,4}C_R$.

Now we define the various residual-based estimators for our subsequent use.

The elliptic reconstruction error estimator:

For $0 \le n \le N$,

$$\mathscr{O}_{\mathrm{CN},n} := C_{I,6}\, h_n^2\, |\log h_n|^{\frac{1}{2}}\, \|(\mathcal{A}_h^n)U^n - (U^n)_{\mathrm{el}}\| + C_{I,7}\, h_n^{\frac{3}{2}}\, |\log h_n|^{\frac{1}{2}}\, \|j[\beta U^n]\|_{\Sigma_n}. \tag{4.10}$$

The space-mesh error estimator:

For $1 \le n \le N$,

$$\begin{aligned}\mathscr{M}_{\mathrm{CN},n} &:= C_{I,6}\, \hat{h}_n^2\, |\log \hat{h}_n|^{\frac{1}{2}}\, \|k_n^{-1}\left\{(\mathcal{A}_h^n)U^n - (\mathcal{A}_h^{n-1})U^{n-1} - (U^n)_{\mathrm{el}} + (U^{n-1})_{\mathrm{el}}\right\}\|\\ &\quad + C_{I,7}\, \hat{h}_n^{\frac{3}{2}}\, |\log \hat{h}_n|^{\frac{1}{2}}\, \|k_n^{-1}\left\{j[\beta U^n] - j[\beta U^{n-1}]\right\}\|_{\check{\Sigma}_n}\\ &\quad + C_{I,8}\, \hat{h}_n^{\frac{3}{2}}\, |\log \hat{h}_n|^{\frac{1}{2}}\, \|k_n^{-1}\left\{j[\beta U^n] - j[\beta U^{n-1}]\right\}\|_{\hat{\Sigma}_n \setminus \check{\Sigma}_n}\end{aligned} \tag{4.11}$$

with $C_{I,8} := C_{I,5}C_R$.

The temporal reconstruction error estimator:

$$\mathscr{T}_{\mathrm{re,CN},n} := \frac{k_n^2}{8}\left\{\mathcal{C}_{I,6}h_n^2|\log h_n|^{\frac{1}{2}}\|(\mathcal{A}_h^n)\mathcal{Z}_n - (\mathcal{Z}_n)_{\mathrm{el}}\| + \mathcal{C}_{I,7}h_n^{\frac{3}{2}}|\log h_n|^{\frac{1}{2}}\|j[\beta\mathcal{Z}_n]\|_{\Sigma_n} + \|\mathcal{Z}_n\|\right\} \tag{4.12}$$

with $\mathcal{Z}_n := -\Pi_0^n\breve{\Phi}_t(t) + \Psi_t(t)$.

The space-error estimator:

$$\mathscr{S}_{\mathrm{CN},n} := \frac{k_n}{4}\left\{\mathcal{C}_{I,6}\; h_n^2\;|\log h_n|^{\frac{1}{2}}\;\|(\mathcal{A}_h^n)\mathcal{Z}_n - (\mathcal{Z}_n)_{\mathrm{el}}\| + \mathcal{C}_{I,7}\; h_n^{\frac{3}{2}}\;|\log h_n|^{\frac{1}{2}}\;\|j[\beta\mathcal{Z}_n]\|_{\Sigma_n}\right\}. \tag{4.13}$$

The temporal error estimator:

$$\mathscr{T}_{\mathrm{e,CN},n} := \frac{1}{\gamma_0}\sqrt{\frac{\alpha_0}{120}}\,k_n^2\left\{\mathcal{C}_{I,1}\; h_n\;\|(\mathcal{A}_h^n)\mathcal{Z}_n - (\mathcal{Z}_n)_{\mathrm{el}}\| \;+\mathcal{C}_{I,2}\; h_n^{\frac{1}{2}}\;\|j[\beta\mathcal{Z}_n]\|_{\Sigma_n} + \alpha_0\;\|\mathcal{Z}_n\|_1\right\}. \tag{4.14}$$

The coarsening error estimator:

$$\mathscr{C}_{\mathrm{CN},n} := k_n^{-1}\|(I-\Pi_0^n)U^{n-1}\| + \frac{1}{2}\|(P_2^n - I)(-\Delta_h^{n-1})U^{n-1}\|. \tag{4.15}$$

The data approximation error estimators:

$$\left\{\begin{aligned}
\mathscr{D}_{\mathrm{CN},n,1} &:= \tfrac{1}{12}\,\sqrt{\mathcal{C}_L}\;k_n^2\Big\{\mathcal{C}_{I,6}\;h_n^2\;|\log h_n|^{\frac{1}{2}}\;\|(\mathcal{A}_h^n)\mathcal{Z}_n - (\mathcal{Z}_n)_{\mathrm{el}}\|\\
&\qquad + \mathcal{C}_{I,7}\;h_n^{\frac{3}{2}}\;|\log h_n|^{\frac{1}{2}}\;\|j[\beta\mathcal{Z}_n]\|_{\Sigma_n} + \|\mathcal{Z}_n\|\Big\},\\
\mathscr{D}_{\mathrm{CN},n,2} &:= \sqrt{\mathcal{C}_L}\;\max\{\|\breve{\varepsilon}^{n-1}\|,\|\breve{\varepsilon}^n\|\},\\
\mathscr{D}_{\mathrm{CN},n,3} &:= \tfrac{1}{k_n}\textstyle\int_{t_{n-1}}^{t_n}\;\|f(x,t,U) - \breve{\Phi}(t)\|\mathrm{d}t,\\
\mathscr{D}_{\mathrm{CN},n,4} &:= \tfrac{1}{\sqrt{\gamma_0}}\;\mathcal{C}_{I,1}\;h_n\;\left\{\left\|(\Pi_0^n - I)\tfrac{\{f^n(U^n) - f^{n-1}(U^{n-1})\}}{2}\right\|\right\}.
\end{aligned}\right. \tag{4.16}$$

Next, we state a series of lemmas to drive *a posteriori* error bounds for the parabolic error $\breve{\rho}(t)$. The proofs can be treated in a similar manner as [21, Lemmas 4.4–4.6 and 4.8] and hence, the details are omitted.

Lemma 4.2 (Temporal error estimate): *With $\mathscr{T}_{\mathrm{e,CN},n}$ as in (4.14), let $\mathcal{I}_{n,1}$ refer to the error term due to time discretization and is defined by*

$$\mathcal{I}_{n,1} := \alpha_0\int_{t_{n-1}}^{t_n}\;\|\breve{\sigma}(t)\|_1^2\mathrm{d}t.$$

Then

$$\mathfrak{I}_{n,1} \leq k_n\;\mathscr{T}^2_{\mathrm{e,CN},n}.$$

Lemma 4.3 (Space-mesh error estimate): *With $\mathscr{M}_{\mathrm{CN},n}$ as in (4.11), let $\mathcal{I}_{n,2}$ represent the space-mesh error term and be given by*

$$\mathcal{I}_{n,2} := \int_{t_{n-1}}^{t_n} |\langle \breve{\varepsilon}_t(t), \breve{\rho}(t)\rangle| \, \mathrm{d}t.$$

Then the following is true:

$$\mathcal{I}_{n,2} \leq k_n \, \mathscr{M}_{\mathrm{CN},n} \max_{t\in \bar{I}_n} \|\breve{\rho}(t)\|.$$

Lemma 4.4 (Space error estimate): *With $\mathscr{S}_{\mathrm{CN},n}$ as in (4.13), let $\mathcal{I}_{n,3}$ denote the space error term and be defined as*

$$\mathcal{I}_{n,3} := \int_{t_{n-1}}^{t_n} \left|\left\langle (\mathcal{R}_c^n - I)\left(\breve{H}(t) - H(t_{n-\frac{1}{2}})\right), \breve{\rho}(t)\right\rangle\right| \mathrm{d}t.$$

Then

$$\mathcal{I}_{n,3} \leq \; k_n \mathscr{S}_{\mathrm{CN},n} \max_{t\in \bar{I}_n} \|\breve{\rho}(t)\|.$$

Lemma 4.5 (Coarsening error estimate): *With $\mathscr{C}_{\mathrm{CN},n}$ as in (4.15), let $\mathcal{I}_{n,5}$ denote the coarsening error term and be defined as*

$$\mathcal{I}_{n,5} := \int_{t_{n-1}}^{t_n} \left|\left\langle k_n^{-1}(I - P_1^n)U^{n-1} + l_{n-1}(t)(P_2^n - I)(\mathcal{A}_h^{n-1})U^{n-1}, \breve{\rho}(t)\right\rangle\right| \mathrm{d}t.$$

Then

$$\mathcal{I}_{n,5} \leq k_n \, \mathscr{C}_{\mathrm{CN},n} \max_{t\in \bar{I}_n} \|\breve{\rho}(t)\|.$$

Lemma 4.6 (Data approximation error estimate): *With $\mathscr{D}_{\mathrm{CN},n,i} (i = 1, \ldots, 4)$ as in (4.16), let $\mathcal{I}_{n,4}$ denote the data approximation error term and be defined as*

$$\mathcal{I}_{n,4} := \int_{t_{n-1}}^{t_n} \left|\left\langle f(t,u) - \breve{H}(t) - \Psi(t), \breve{\rho}(t)\right\rangle\right| \mathrm{d}t.$$

Then we have

$$\begin{aligned}\mathcal{I}_{n,4} \leq{}& \frac{\sqrt{\mathcal{C}_L}}{2\,\epsilon}\, k_n \max_{t\in \bar{I}_n} \|\breve{\rho}(t)\|^2 + \frac{\epsilon\sqrt{\mathcal{C}_L}}{2} \int_{t_{n-1}}^{t_n} \|\breve{\rho}(t)\|_1^2 \mathrm{d}t \\ &+ k_n \left\{\mathscr{D}_{\mathrm{CN},n,1} + \mathscr{D}_{\mathrm{CN},n,2} + \mathscr{D}_{\mathrm{CN},n,3}\right\} \max_{t\in \bar{I}_n} \|\breve{\rho}(t)\| + k_n^{\frac{1}{2}} \, \mathscr{D}_{\mathrm{CN},n,4} \left(\gamma_0 \int_{t_{n-1}}^{t_n} \|\breve{\rho}(t)\|_1^2 \mathrm{d}t\right)^{\frac{1}{2}}.\end{aligned}$$

Proof: With an aid of (4.6), we first rewrite the integrand in $\mathcal{I}_{n,4}$ as

$$f(t,u) - \breve{H}(t) - \Psi(t) = f(t,u) - \Pi_0^n \breve{\Phi}(t) = \left(f(t,u) - \breve{\Phi}(t)\right) - (\Pi_0^n - I)\breve{\Phi}(t). \tag{4.17}$$

Further, we split the first term of (4.17) to obtain

$$\begin{aligned}\mathcal{I}_{n,4} \leq{}& \int_{t_{n-1}}^{t_n} \left|\left\langle f(t,u) - f(t,\breve{U}), \breve{\rho}(t)\right\rangle\right| \mathrm{d}t + \int_{t_{n-1}}^{t_n} \left|\left\langle f(t,\breve{U}) - f(t,\breve{\Theta}(t), \breve{\rho}(t)\right\rangle\right| \mathrm{d}t \\ &+ \int_{t_{n-1}}^{t_n} \left|\left\langle f(t,\breve{\Theta}(t)) - f(t,U(t)), \breve{\rho}(t)\right\rangle\right| \mathrm{d}t + \int_{t_{n-1}}^{t_n} \left|\left\langle f(t,U(t)) - \breve{\Phi}(t), \breve{\rho}(t)\right\rangle\right| \mathrm{d}t \\ &+ \int_{t_{n-1}}^{t_n} \left|\left\langle (\Pi_0^n - I)\breve{\Phi}(t), \breve{\rho}(t)\right\rangle\right| \mathrm{d}t \\ :={}& T_1^n + T_2^n + T_3^n + T_4^n + T_5^n.\end{aligned}$$

Then using the Cauchy–Schwarz inequality and the Young's inequality, we have

$$\begin{aligned}\left|\langle f(t,u)-f(t,\breve{U}),\breve{\rho}(t)\rangle\right| &\le \sqrt{\mathcal{C}_L}\ \|\breve{\rho}(t)\|\ \|\breve{\rho}(t)\|_1\\ &\le \sqrt{\mathcal{C}_L}\left(\frac{1}{2\epsilon}\|\breve{\rho}(t)\|^2+\frac{\epsilon}{2}\|\breve{\rho}(t)\|_1^2\right),\end{aligned}$$

and hence,

$$T_1^n \le \frac{\sqrt{\mathcal{C}_L}}{2\epsilon}\,k_n\max_{t\in\bar{I}_n}\|\breve{\rho}(t)\|^2+\frac{\epsilon\sqrt{\mathcal{C}_L}}{2}\int_{t_{n-1}}^{t_n}\|\breve{\rho}(t)\|_1^2\mathrm{d}t.$$

Again, using the Cauchy–Schwarz inequality once more we obtain

$$\begin{aligned}\left|\left\langle f(t,\breve{U})-f(t,\breve{\Theta}(t),\breve{\rho}(t)\right\rangle\right| &\le \|f(t,\breve{U})-f(t,\breve{\Theta}(t)\|\ \|\breve{\rho}(t)\|\\ &\le \frac{1}{2}\sqrt{\mathcal{C}_L}\ |t-t_{n-1}|\ |t_n-t|\left\{\|(\mathcal{R}_c^n-I)\mathcal{Z}_n\|+\|\mathcal{Z}_n\|\right\}\ \|\breve{\rho}(t)\|,\end{aligned}$$

where in the last step we have used the fact that $\breve{\sigma}(t)=\breve{U}(t)-\breve{\Theta}(t)=\frac{1}{2}(t-t_{n-1})(t_n-t)\mathcal{R}^n\mathcal{Z}_n$. Therefore,

$$\begin{aligned}T_2^n &\le \frac{k_n^3}{12}\sqrt{\mathcal{C}_L}\ \left\{\|(\mathcal{R}_c^n-I)\mathcal{Z}_n\|+\|\mathcal{Z}_n\|\right\}\ \max_{t\in\bar{I}_n}\|\breve{\rho}(t)\|\\ &= k_n\,\mathscr{D}_{\mathrm{CN},n,1}\ \max_{t\in\bar{I}_n}\|\breve{\rho}(t)\|.\end{aligned}$$

Exactly following the same argument of (3.13), we have

$$T_3^n \le k_n\,\mathscr{D}_{\mathrm{CN},n,2}\ \max_{t\in\bar{I}_n}\|\breve{\rho}(t)\|.$$

Again, invoking the Cauchy–Schwarz inequality it follows that

$$T_4^n \le \max_{t\in\bar{I}_n}\|\breve{\rho}(t)\|\int_{t_{n-1}}^{t_n}\|f(t,U(t))-\breve{\Phi}(t)\| = k_n\,\mathscr{D}_{\mathrm{CN},n,3}\ \max_{t\in\bar{I}_n}\|\breve{\rho}(t)\|.$$

Finally to estimate T_5^n, we exploit the orthogonality property of Π_0^n and Proposition 2.1 to have

$$\begin{aligned}\left\langle(\Pi_0^n-I)\breve{\Phi}(t),\breve{\rho}(t)\right\rangle &= \left\langle(\Pi_0^n-I)\breve{\Phi}(t),\breve{\rho}(t)-\mathscr{I}_n\breve{\rho}(t)\right\rangle\\ &\le \|(\Pi_0^n-I)\breve{\Phi}(t)\|\ \|\breve{\rho}(t)-\mathscr{I}_n\breve{\rho}(t)\|\\ &\le \mathcal{C}_{I,1}\,h_n\,\|(\Pi_0^n-I)\breve{\Phi}(t)\|\ \|\breve{\rho}(t)\|_1.\end{aligned}$$

As $\max_{t\in\bar{I}_n}|l_n(t)|=1$ and $\max_{t\in\bar{I}_n}|l_{n-1}(t)|=1$, it follows from (4.3) that

$$\begin{aligned}\|(\Pi_0^n-I)\breve{\Phi}(t)\| &\le \max_{t\in\bar{I}_n}|l_{n-1}(t)|\ \|(\Pi_0^n-I)f^{n-1}\|+\max_{t\in\bar{I}_n}|l_n(t)|\ \|(\Pi_0^n-I)f^n\|\\ &= \|(\Pi_0^n-I)f^{n-1}\|+\|(\Pi_0^n-I)f^n\|,\end{aligned}$$

and hence,

$$T_5^n \le k_n^{\frac{1}{2}}\ \mathscr{D}_{\mathrm{CN},n,4}\ \left(\gamma_0\int_{t_{n-1}}^{t_n}\|\breve{\rho}(t)\|_1^2\mathrm{d}t\right)^{\frac{1}{2}},$$

and this completes the proof of the lemma. □

As a consequence of the above lemmas we derive the *a posteriori* error bound for the parabolic error $\breve{\rho}(t)$ in the following theorem.

Theorem 4.7: *Let u be the exact solution of (1.1)–(1.3) and let U^n be its finite element approximation obtained by the Crank-Nicolson approximation (2.8). Then, for $1 \le m \le N$, the following is true:*

$$\left\{\max_{t\in[0,t_m]} \|\breve{\rho}(t)\|^2 + \int_0^{t_m} \|\breve{\rho}(t)\|_1^2 \mathrm{d}t\right\}^{\frac{1}{2}} \le \left\{2\,\mathcal{C}_G(m)\left(\|\breve{\rho}(0)\|^2 + \sum_{n=1}^m k_n \mathscr{T}_{\mathrm{e,CN},n}^2\right)\right\}^{\frac{1}{2}} + \left(\Upsilon_{m,1}^2 + \Upsilon_{m,2}^2\right)^{\frac{1}{2}},$$

where

$$\Upsilon_{m,1} := 4\mathcal{C}_G(m) \sum_{n=1}^m k_n \left(\mathscr{M}_{\mathrm{CN},n} + \mathscr{S}_{\mathrm{CN},n} + \mathscr{D}_{\mathrm{CN},n,1} + \mathscr{D}_{\mathrm{CN},n,2} + \mathscr{D}_{\mathrm{CN},n,3} + \mathscr{C}_{\mathrm{CN},n}\right), \tag{4.18}$$

and

$$\Upsilon_{m,2} := 4\mathcal{C}_G(m) \sum_{n=1}^m k_n^{\frac{1}{2}} \mathscr{D}_{\mathrm{CN},n,4}. \tag{4.19}$$

$\mathscr{M}_{\mathrm{CN},n}$, $\mathscr{S}_{\mathrm{CN},n}$, $\mathscr{T}_{\mathrm{e,CN},n}$, $\mathscr{C}_{\mathrm{CN},n}$, $\mathscr{D}_{\mathrm{CN},n,i}(i = 1,\dots,4)$ are defined in (4.11) and (4.13)–(4.16), respectively.

Proof: For each $\varphi \in H_0^1(\Omega)$ and for $1 \le n \le N$, using (4.7), (2.1) and rearranging the terms we have the following error equation for $\breve{\rho}(t)$

$$\langle \breve{\rho}_t(t), \varphi\rangle + a(\rho(t), \varphi) = \langle \mathscr{R}(t), \varphi\rangle, \quad t \in I_n, \tag{4.20}$$

where

$$\mathscr{R}(t) := -\breve{\varepsilon}_t(t) - (\mathcal{R}_c^n - I)\left(\breve{H}(t) - H(t_{n-\frac{1}{2}})\right) + \left(f(t,u) - \breve{H}(t) - \Psi(t)\right) + l_{n-1}(t)(P_2^n - I)(-\Delta_h^{n-1})U^{n-1} - k_n^{-1}(P_1^n - I)U^{n-1}.$$

Setting $\varphi = \breve{\rho}(t)$ in (4.20), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\breve{\rho}(t)\|^2 + a\left(\rho(t), \breve{\rho}(t)\right) = \langle \mathscr{R}(t), \breve{\rho}(t)\rangle.$$

Using the identity $2a(v,w) = a(v,v) + a(w,w) - a(v-w, v-w) \quad \forall v, w \in H_0^1(\Omega)$ and using (2.2), we arrive at

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\breve{\rho}(t)\|^2 + \frac{\gamma_0}{2}\left(\|\rho(t)\|_1^2 + \|\breve{\rho}(t)\|_1^2\right) \le \frac{\alpha_0}{2}\|\breve{\sigma}(t)\|_1^2 + |\langle \mathscr{R}(t), \breve{\rho}(t)\rangle|. \tag{4.21}$$

Integrating the above from t_{n-1} to t_n and summing up over $n = 1 : m$, we obtain

$$\|\breve{\rho}(t_m)\|^2 + \gamma_0 \int_0^{t_m} \left(\|\rho(t)\|_1^2 + \|\breve{\rho}(t)\|_1^2\right) \mathrm{d}t \le \|\breve{\rho}(0)\|^2 + \sum_{n=1}^m \left\{\mathcal{I}_{n,1} + 2\sum_{i=2}^5 \mathcal{I}_{n,i}\right\}, \tag{4.22}$$

where $\mathcal{I}_{n,i}$ $(i = 1, \ldots, 5)$ are defined in Lemmas 4.3–4.6. Proceeding as in Theorem 3.5, we lead to

$$
\begin{aligned}
\max_{t\in[0,t_m]} \|\breve{\rho}(t)\|^2 + \mathcal{C}_G(m)\int_0^{t_m} \|\breve{\rho}(t)\|_1^2 \mathrm{d}t \;\le\; & 2\,\mathcal{C}_G(m)\left\{\|\breve{\rho}(0)\|^2 + \sum_{n=1}^{m} k_n \mathscr{T}^2_{\mathrm{e,CN},n}\right\} \\
& + \; 4\,\mathcal{C}_G(m) \max_{t\in[0,t_m]} \|\breve{\rho}(t)\| \sum_{n=1}^{m} k_n \left\{\mathscr{M}_{\mathrm{CN},n} + \mathscr{S}_{\mathrm{CN},n} + \mathscr{D}_{\mathrm{CN},n,1} + \mathscr{D}_{\mathrm{CN},n,2} + \mathscr{D}_{\mathrm{CN},n,3} + \mathscr{C}_{\mathrm{CN},n}\right\} \\
& + 4\mathcal{C}_G(m)\, k_n^{\frac{1}{2}}\, \mathscr{D}_{\mathrm{CN},n,4} \left(\gamma_0 \int_{t_{n-1}}^{t_n} \|\breve{\rho}(t)\|_1^2 \mathrm{d}t\right)^{\frac{1}{2}},
\end{aligned}
$$

where $\mathcal{C}_G(m) := 2\ \max\left\{1,\ \sum_{n=1}^{m} \frac{2\sqrt{\mathcal{C}_L}}{\epsilon}\, k_n\ \exp\{\frac{2\sqrt{\mathcal{C}_L}}{\epsilon}\ \left(\sum_{n<j<m} k_j\right)\}\right\}$.

Finally, we take

$$
\begin{aligned}
a_0 &:= \max_{t\in[0,t_m]} \|\breve{\rho}(t)\|, \quad a_n := \left\{\mathcal{C}_G(m)\int_{t_{n-1}}^{t_n} \|0\breve{\rho}(t)\|_1^2\ \mathrm{d}t\right\}^{\frac{1}{2}} \ (1 \le n \le m), \\
c &:= \left\{2\,\mathcal{C}_G(m)\left(\|\breve{\rho}(0)\|^2 + \sum_{n=1}^{m} k_n \mathscr{T}^2_{\mathrm{e,CN},n}\right)\right\}^{\frac{1}{2}}, \\
b_0 &:= 4\,\mathcal{C}_G(m)\sum_{n=1}^{m} k_n \left\{\mathscr{M}_{\mathrm{CN},n} + \mathscr{S}_{\mathrm{CN},n} + \mathscr{D}_{\mathrm{CN},n,1} + \mathscr{D}_{\mathrm{CN},n,2} + \mathscr{D}_{\mathrm{CN},n,3} + \mathscr{C}_{\mathrm{CN},n}\right\}, \\
b_n &:= 4\,\mathcal{C}_G(m)\, k_n^{\frac{1}{2}}\ \mathscr{D}_{\mathrm{CN},n,2} \ (1 \le n \le m),
\end{aligned}
$$

and invoke the inequality [15, (80)] to complete the rest of the proof. □

We are now prepared to state the fully discrete Crank-Nicolson *a posteriori* error estimate in the $L^\infty(L^2)$ norm for the semilinear parabolic interface problem (1.1)–(1.3).

Theorem 4.8: *Let u be the exact solution of (1.1)–(1.3) and let U^n be its finite element approximation obtained by the Crank-Nicolson approximation (2.8). Then, for each $1 \le m \le N$, the following a posteriori error estimate holds:*

$$
\begin{aligned}
\max_{t\in[0,t_m]} \|u(t) - U(t)\| \le & \left\{2\,\mathcal{C}_G(m)\left(\|\mathcal{R}^0 U^0 - u(0)\|^2 + \sum_{n=1}^{m} k_n \mathscr{T}^2_{\mathrm{e,CN},n}\right)\right\}^{\frac{1}{2}} \\
& + \left(\Upsilon^2_{m,1} + \Upsilon^2_{m,2}\right)^{\frac{1}{2}} + 2 \max_{0\le n\le m} \mathscr{O}_{\mathrm{CN},n} + \max_{0\le n\le m} \mathscr{T}_{\mathrm{re,CN},n},
\end{aligned}
$$

where the estimators are given in Theorem 4.7.

***Proof*:** By the triangle inequality, we have for $t \in I_n$

$$
\|e(t)\| \le \max_{t\in[0,t_m]} \|\breve{\rho}(t)\| + \max_{t\in[0,t_m]} \|\breve{\sigma}(t)\| + \max_{t\in[0,t_m]} \|\breve{\epsilon}(t)\|.
$$

Now,

$$
\|\breve{\varepsilon}(t)\| = \|l_{n-1}(t)\breve{\varepsilon}^{n-1} + l_n(t)\breve{\varepsilon}^{n}\| \le 2\ \max\{\|\breve{\varepsilon}^{n}\|, \|\breve{\varepsilon}^{n-1}\|\}, \quad t \in I_n.
$$

Again, for $t \in [0, t_m]$, using Lemma 4.1, we obtain

$$\|\breve{\varepsilon}(t)\| \leq 2 \max_{0\leq n\leq m} \|\breve{\varepsilon}^n\| \leq 2 \max_{0\leq n\leq m} \mathscr{O}_{\mathrm{CN},n}. \tag{4.23}$$

Further,

$$\begin{aligned}\|\breve{\sigma}(t)\| &= \|\frac{1}{2}(t-t_{n-1})(t_n-t)\mathcal{R}_c^n \mathcal{Z}_n\| \\ &\leq \frac{1}{2}\max_{t\in \bar{I}_n}\{|(t-t_{n-1})(t_n-t)|\}\left(\|(\mathcal{R}_c^n - I)\mathcal{Z}_n\| + \|\mathcal{Z}_n\|\right) \\ &\leq \mathscr{T}_{\mathrm{re,CN},n},\end{aligned} \tag{4.24}$$

which in combination with (4.23) and Theorem 4.7 completes the proof. □

5. Conclusion and extension

This paper investigates a residual-based $L^\infty(L^2)$-norm *a posteriori* error estimates for semilinear parabolic interface problem in a bounded convex domain in $\mathbb{R}^2$. An appropriate adaption of elliptic reconstruction technique and the energy method play a crucial in deriving *a posteriori* error bounds. It is interesting to extend these results to problems in $\mathbb{R}^3$ and many computational issues which need to be addressed in future. We remark that such an extension is not straightforward. However, the authors feel that the idea of universal extension results for Sobolev spaces [22] could be useful.

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