PONTIFICIA UNIVERSIDAD CATOLICA DE CHILE
INSTITUTO DE ECONOMIA
MAGISTER EN ECONOMIA

TESIS DE GRADO
MAGISTER EN ECONOMIA

Larrain, Langlois, Fernando Antonio

Diciembre, 2014

PONTIFICIA UNIVERSIDAD CATOLICA DE CHILE
INSTITUTO DE ECONOMIA
MAGISTER EN ECONOMIA

DYNAMIC BILATERAL OLIGOPOLY

Fernando Antonio Larrain Langlois

Comisión

Salvador Valdés
Eugenio Bobenrieth

Santiago, diciembre de 2014

Dynamic Bilateral Oligopoly

Pontificia Universidad Católica de Chile

Fernando Larrain Langlois*

January, 2015

Abstract

This thesis reinterprets and adapts the model developed in Besanko and Doraszelski [Besanko, D. & Doraszelski, U. (2004), 'Capacity Dynamics and Endogenous Asymmetries in Firm Size', *The RAND Journal of Economics* **35**(1), pp. 23-49], originally designed for unilaterally oligopolistic markets, to explore and describe the behavior of investment and supply networks in bilaterally oligopolistic ones. In order to do that, it combines axiomatic solutions concepts taken from cooperative game theory with a discrete state dynamic stochastic game framework. In particular, it characterizes a version of the bargaining game proposed by de Fontenay and Gans [de Fontenay, C. C. & Gans, J. S. (2005), 'Vertical Integration in the Presence of Upstream Competition', *The RAND Journal of Economics* **36**(3), pp. 544-572] which considers capacity constraints, showing existence of an equilibrium and uniqueness of payoffs for some particular applications. To illustrate the interaction between the two games, the case of a seller facing two buyers is solved numerically. Preliminary results show that asymmetric industry structures can arise in quantity competition settings as a result of supply network instability. Furthermore, downstream firms tend to over-invest and hold capacity in excess to that of their supplier.

*I would like to thank my thesis advisors, Salvador Valdés and Eugenio Bobenrieth, for their continuous support and patience. I would also like to thank Nicolás Figueroa for his useful comments. All errors are my own. Email: fnlarrai@uc.cl

1 Introduction

This work aims to capture two important aspects of many markets, bilateral market power and dynamics, in a parsimonious way. To that end, I modify the production capacity accumulation game presented in Besanko & Doraszelski (2004) (hereafter BD). First, I adapt it to consider supply-chain relationships, and secondly, I introduce cooperative game-theoretic solution concepts to define the payoffs resulting from product market competition.

Most classic theoretical models of imperfect competition have been concerned with markets in which only one side (i.e. either supply or demand) can exercise market power. However, many markets are better characterized as bilaterally imperfect[1], that is, as markets in which buyers and sellers simultaneously display some degree of market power. For example, in recent years, North American and European retail chains have grown immensely, as documented by Inderst & Wey (2003), with each retailer accounting for a larger share of each manufacturer's sales. The same authors have called this market structure "bilateral oligopoly", extending the concept of bilateral monopoly coined, as far as I know, by Bowley (1928).

The tendency towards modeling unilateral market power can be partially explained by the fact that, when there is bilateral market power, standard economics modeling tools lead to a large number of potential equilibria. The multiplicity or indeterminacy of equilibria is generally due to a poor specification of the circumstances facing the agents. When the process through which agents get to coincide in their expectations (assuming they do) is not part of the model, it leads to multiplicity [2].

One way to solve this problem is to give a more detailed description of the bargaining process followed by the agents, that is, of the timing of their decisions, their alternatives, their preferences, etc. However, this approach requires assumptions that depend too much on the context. A small change in the characteristics of the market under study might render any conclusions from the model useless.

Another way to solve the multiplicity problem is to assume that any outcome of the game should display a series of characteristics that seem reasonable or are intuitively appealing. That is why Rubinstein (1982) named it the "axiomatic approach". It has the advantage of being quite general, because the assumptions it makes are less context-dependent. However, as intuitively appealing as they may be, the axioms might not always be consistent with non-cooperative behavior. Nevertheless, in the following sections I argue that this second approach seems better suited for the objectives of this thesis.

Real-world markets are also characterized by a temporal dimension. Agents interact repeatedly in time, in a possibly uncertain environment, forming and updating beliefs about their payoffs. This involves, among other things, forming expectations of other agents' future behavior. Any firm looking to maximize its value should probably consider the transitional and long-term effects of its own present actions and those of other firms. In particular, their investment or capacity accumulation strategies are a key determinant of their size and, consequently, their bargaining power. These considerations can only be adequately taken into account by using a fully dynamic stochastic game, such as the one in BD.

A preliminary numerical analysis of the model yields interesting results. As opposed to the results in Besanko & Doraszelski (2004), asymmetric industry structures can arise in a quantity competition setting, possibly due to exclusionary behavior by the suppliers. This structures become more likely when investment is more reversible. Also, firms over-invest in production capacity, probably because this can lead to a permanent advantage over their competitors. They tend to hold capacity in excess to that of their suppliers, that is, capacity that can never be used because of the lack of inputs.

The rest of the thesis proceeds as follows. In Section 2, I review the different methods that have been used in the literature to model bilateral market power and dynamics. I also review some of the literature on vertical and horizontal integration, because it is closely related to the subject of this work. In Section 3 I formally define the concepts that will be used to build the model and review some technical results regarding the existence of a solution. In Section 4 the adaptation of the BD model is described and a network-bargaining game is characterized. In Sections 5 and 6 two particular applications are characterized analytically in

[1] Strictly speaking, the idea of market imperfection encompasses many more situations different from the exercise of market power, such as informational asymmetries, externalities of different kinds, etc.

[2] I understand this coincidence of expectations as in the definition of Nash Equilibrium, that is, a situation in which everyone knows how the others will act and responds optimally, all at the same time.

further detail, showing uniqueness of firms' payoffs. In Section 7, preliminary numerical results for one of the applications are presented and compared with the results in Besanko & Doraszelski (2004). Section 8 is for my conclusion.

2 Literature Review

This section reviews the theoretical and applied literature related to the subject of the thesis. To make a clear description, it is subdivided in two subsections. The first one reviews bargaining games and cooperative game theory. The second one focuses on dynamic stochastic games. Most concepts will only be described informally; the important ones will be defined thoroughly in Section 3.

2.1 Bargaining Games and Cooperative Game Theory

A bargaining game is a situation in which several individuals have to jointly decide how to share a "cake" of fixed or variable size. As discussed previously, one strand of literature has solved this kind of problems by posing axioms which determine a unique division of the cake with "reasonable" characteristics, such as symmetry, Pareto optimality, etc. Nash (1950, 1953) was the first one to propose such a result for a two-person bargaining problem, the so called "Nash Bargaining solution", which consisted of dividing the cake in equal parts. Later, it was generalized to games with imperfect information (Harsanyi & Selten (1972)), proportional solutions (Kalai (1977), Roth (1979)), multilateral negotiations (Krishna & Serrano (1996)), etc.

Other authors working in cooperative game theory developed different axiomatic solutions which allowed the formation of coalitions between players. The best known example of this is the "Shapley Value" (Shapley (1953*b*)). It assigns to each individual a payoff equal to its average marginal contribution to the size of the cake. This average is taken over the different possible coalitions in which it can participate. Strictly speaking, coalitional games are different from bargaining games. However, for our purposes, they fall in the same category: they are both games in which a group of individuals must share a cake.

Many other solution concepts were developed later on, similar in essence. Myerson (1977*a*,*b*) extended Shapley's Value to games in partition function form. His results were, in turn, extended by Jackson & Wolinsky (1996), Navarro (2007) and others to general networks. In particular, Navarro (2007) develops a "generalized Myerson-Shapley allocation rule" which admits all forms of externalities between coalitions. This is an important aspect of supply-chain relationships whenever downstream firms compete in the same final good market. It is also relevant when upstream firms compete for the same limited input supply.

As we saw before, another strand of literature has tried to solve bargaining games using a different approach based only non-cooperative game theory. The "strategic approach", as it was referred to by Rubinstein (1982), specifies the game in greater detail and looks for its Nash Equilibria, or for its Subgame Perfect Nash Equilibria. The so called "Nash Program" was carried out by a series of authors as an attempt to connect both approaches and justify the axiomatic solutions through non-cooperative mechanisms. For example, Rubinstein (1982) and Binmore et al. (1986) link the Nash Bargaining solution to the discount factor and the degree of risk aversion of the players in a game of alternating offers. Grossman & Hart (1986), Hart & Moore (1990) and Inderst & Wey (2003), among others, develop bargaining games that implement the Shapley Value. More recent work by de Fontenay & Gans (2005, 2013) and Douven et al. (2011) connects the generalized Myerson-Shapley Value to non-cooperative game theory by means of simultaneous or sequential alternating-offers games like the ones in Binmore et al. (1986).

2.2 Dynamic Stochastic Games

Broadly speaking, a dynamic game is characterized by the existence of a discrete temporal dimension in the interaction of the players. It is stochastic if the players' payoffs depend not only on their actions but on the realization of some random variable. There are several ways to classify them. One of them is according to the length of the time horizon (finite or infinite). A second one is according to the information available to the players on each stage of the game (perfect or imperfect; it is related to whether the players move sequentially or simultaneously). A third one is according to the existence or non existence of a state variable such as production capacity, inventories, etc. It can be discrete or continuous. Typically, the term "dynamic stochastic game" is reserved for a stochastic infinite horizon game with state variables. Other common categories have their own specific name, like "Supergames", which are infinite horizon games with no state variable.

These games are solved by finding an equilibrium. Just as there are many kinds of games, there are several notions of equilibrium. For dynamic games, the most common ones are "Subgame Perfect Equilibrium" and "Markov Perfect Equilibrium" (hereafter SPE and MPE, respectively). The idea of subgame perfection is key to dynamic games, because it rules out equilibria sustained by non-credible threats. However, for many games, the set of SPE's is too large. To reduce the number of equilibria one can focus on a subset of SPE's. A particularly convenient subset is the one constituted of the MPE's. The term "Markov Perfect Equilibrium" was coined by Maskin & Tirole (1988*a*,*b*); they give a series of arguments in favor of this subset of SPE's. The essence of the MPE is that it only admits strategies that depend solely on directly payoff relevant variables (state variables). In other words, it rules out strategies that condition on the whole history of the game.

Supergames are the most common class of infinite horizon games in Industrial Organization. They basically consist of an infinite number of repetitions of a static game called the "stage game". There is no state variable and the stage game is typically simultaneous, leading to an imperfect information supergame. They are the simplest kind of infinite horizon game; they have well known properties and are relatively easy to handle (e.g. there are many theorems that characterize the set of SPE's, the so called Folk Theorems). The key insight they offer is that it is possible to achive better equilibria (in the sense of Pareto) when there is repeated interaction. Hence their main application in IO has been the study of tacit collusion and cartel formation (Green & Porter (1984), Rotemberg & Saloner (1986), Abreu et al. (1986, 1990)). However, they possess limited descriptive power because of the absence of state variables. The only connection between the different periods of time is through the history of the game[3]. There is no "physical" connection, such as the one given by the process of captial stock accumulation. This limits the range of economic problems one can adress using supergames.

Stochastic dynamic games have a higher level of complexity than supergames but allow the study of problems with state variables. The seminal work on this kind of games is Shapley (1953*a*). Several authors have later on developed a series of results regarding the existence of equilibria. They can be classified according to the kind of strategy they consider (pure or behavioral, stationary or non-stationary) and the kind of state space they consider (finite, infinitely countable or a compact subset of a metric space). Some of these results are better suited than others to the analysis of economic problems; in particular, the ones regarding Markov pure strategies. In fact, behavior strategies don't seem to be something that one observes in reality and they may be hard or impossible to compute. Furthermore, Markov strategies are not as restrictive as they may appear and they are computationally tractable. For a recent paper with general results regarding Markovian equilibria in discrete state dynamic stochastic games see Escobar (2013).

The grater complexity of dynamic stochastic games is evident from the fact that it is generally impossible to solve them analytically (as opposed to, for example, supergames). The root of the problem is the same one as in any standard infinite horizon dynamic programming problem, which is that one cannot use backward iteration because the game never ends. That is why several authors have developed numerical solution methods. Pakes & McGuire (1994) and Ericson & Pakes (1995) defined a framework that uses an algorithm for finding MPE's that is analogous to that of dynamic programming (see Doraszelski & Pakes (2007) for a complete review of the framework). However, it not only requires value function conjectures but also policy function conjectures because of the multiple players involved. Based as it is only on an analogy, it does not retain all of the properties of the dynamic programming algorithm. As a matter of fact, convergence to an equilibrium is not guaranteed because it is not based on a contraction. Nevertheless, it has worked well in practice as it is, and is the most widely used algorithm. It is this algorithm that I will use in the following sections.

Later on, Judd et al. (2003) developed a numerical algorithm for supergames which is based on a set based monotone operator equivalent to the Bellman operator of dynamic programming. It allows one to find

[3]If players were not allowed to condition on its history, the only equilibria left would be the ones from the stage game

the whole set of SPE's of a game, and it guarantees convergence. It is possible to extend it to games with state variables (see working papers cited in the same article). However, as I mentioned earlier, SPE's can be too wide a class to focus on, diminishing the predictive capacity of the model.

The most recent algorithm introduced to this literature was proposed by Besanko et al. (2010) to deal adequately with the problem of multiple equilibria. One of the difficulties with the Pakes & McGuire algorithm is that it provides no systematic approach to finding all MPE's. The method by Besanko et al. overcomes this issue through the use of homotopy methods. These methods can find all the solutions to many difficult polynomial systems of equations by deforming them continuously into simpler ones. As it turns out, in several cases the MPE's of discrete state games can be described as a solution to a system of polynomial equations, so one can employ homotopy methods to find them all. One disadvantage of this algorithm is that it imposes a very heavy computational burden.

3 Preliminary Concepts

In this section I review with more detail three axiomatic solutions for cooperative games. The first one is the generalized Nash Bargaining solution (hereafter NB), the second one is the Shapley Value, and the third one is the generalized Myerson-Shapley allocation rule. All of them can, under certain circumstances, be implemented by a non-cooperative bargaining process [4]. I also discuss the concept of "network stability". Then, I define a general dynamic framework which encompasses the BD model, and review some results related to the existence of MPE's.

Sections 3.1 and 3.2 are based on Thomson (1994) and Winter (2002). Sections 3.3 and 3.4 are based on Navarro (2007), Gilles et al. (2006), and Jackson & Wolinsky (1996). Finally, Sections 3.5, 3.6 and 3.7 are based on Escobar (2013) and Doraszelski & Satterthwaite (2010).

3.1 Nash Bargaining

A *bargaining problem* of $|I|$ players is a pair (B, d), where B is the set of feasible alternatives and $d \in B$. Each player has preferences over B, and can jointly choose a point in it. En case of disagreement, the outcome of the negotiation is d, which is why it is called the "disagreement point". A *solution* is a rule that associates to each pair (B, d) a point in B, that is, it associates an agreement to each possible bargaining game.

Nash (1950, 1953) defined a series of axioms that lead to a solution which assigns a unique point for every pair (B, d). There are, of course, many other possible axiomatizations that lead to different solutions. The *Nash Bargaining solution* is the one that maximizes the following expression over B:

$$\prod_{i \in I} (F_i(b) - F_i(d))$$

where $F_i(b)$ represents the payoff for player i under alternative $b \in B$. This product may be generalized to asymmetric bargaining situations, giving rise to the *generalized Nash Bargaining solution* (Harsanyi & Selten (1972), Roth (1979)):

$$\prod_{i \in I} (F_i(b) - F_i(d))^{\alpha_i}$$

The coefficient α_i can be interpreted as player i's bargaining power.

3.2 The Shapley Value

A *game in coalitional form* over a finite set of players $I = \{1, 2, ..., n\}$ is a function μ mapping the set of all possible coalitions 2^I to the real numbers such that $\mu(\emptyset) = 0$. The total profit that a coalition Γ can obtain in the game μ is $\mu(\Gamma)$. A *value* is am operator F that assigns to every game μ a payoff vector $F(\mu) = (F^1(\mu), F^2(\mu), ..., F^n(\mu)) \in \mathbb{R}^n$. $F_i(\mu)$ is the payoff corresponding to player i in game μ.

[4]See Binmore et al. (1986) for a NB example, Inderst & Wey (2003) for a Shapley Value example and de Fontenay & Gans (2005, 2013) for the Myerson-Shapley Value.

Just like Nash, Shapley (1953*b*) presented a series of axioms which define a value for any coalitional game. The Shapley Value assigns to each player its average marginal contribution to the total profits. The average is taken over the set of all permutations of I. If ξ is one such permutation, $p_{\xi,i}$ the set of players preceding i in it and Ξ the set of all permutations of I, the marginal contribution of i with respect to the order ξ is $\mu(p_{\xi,i} \cup i) - \mu(p_{\xi,i}))$, and the i-th component of the *Shapley Value* is:

$$F_i(\mu) = \frac{1}{n!} \sum_{\xi \in \Xi} [\mu(p_{\xi,i} \cup i) - \mu(p_{\xi,i}))] \tag{1}$$

3.3 The Myerson-Shapley Value

To define the Myerson-Shapley Value, some more groundwork is required. A partition is a set of coalitions $P = \{\Gamma_1, \Gamma_2, ...\Gamma_M\}$ such that (1) $\Gamma_m \neq \emptyset$ for all $m = 1, 2, ..., M$, (2) $\cup_{m=1}^{M} \Gamma_m = I$ and (3) $\Gamma_m \cap \Gamma_l = \emptyset$ if $m \neq l$. Let $\Psi(I)$ be the set of partitions of I. An *embedded coalition* is a pair (Γ, P) where $\Gamma \in P \in \Psi(I)$. Let $C(I)$ be the set of all possible embedded coalitions in I.

A *game in partition function form* (PFF) (Thrall & Lucas (1963), as cited in Navarro (2007)) is a pair $[I, \mu]$, where μ is a function that assigns to every $(\Gamma, P) \in C(I)$ a real number $\mu(\Gamma, P)$. Given the coalitional structure P, the real number $\mu(\Gamma, P)$ is the value which can be perfectly transferred across players in Γ. Notice that the function $\mu(\cdot, \cdot)$ can assign different values to a coalition depending on the partition, that is, depending on the coalitions formed by the players excluded from Γ. For example, if there are two suppliers and two retailers in a market, the profits that a retailer-supplier pair can generate will depend on whether the other supplier and retailer form a competing coalition or exit the market. This possibility is not contemplated by coalitional games (section 3.2).

Myerson (1977*a*,*b*) proposed an allocation rule that generalizes the Shapley Value to games in PFF. A PFF-allocation rule is a function F which assigns to every game in PFF $[I, \mu]$ a payoff vector $F(I, \mu) \in \mathbb{R}^n$. It is equivalent to a value in the context of coalitional games. The *Myerson Value* is defined by the following payoffs for all i:

$$F_i(I, \mu) = \sum_{(\Gamma, P) \in C(I)} (-1)^{|P|-1} (|P| - 1)! (\frac{1}{n} - \sum_{\substack{\tilde{\Gamma} \in P \\ \tilde{\Gamma} \neq \Gamma \\ i \notin \tilde{\Gamma}}} \frac{1}{(|P| - 1)(n - |\tilde{\Gamma}|)}) \tag{2}$$

As it is, the Myerson Value does not allow for general forms of externalities[5]. However, it can be adapted to a general framework which does. General externalities include cases in which the "inner structure" of the coalition could alter its payoff or other coalitions' payoffs. We will need to define precisely what we mean by "inner structure" because, strictly speaking, a coalition can only be distinguished from another if it contains different players.

Let us consider the same finite set of players, I. Following Navarro (2007), assume that there are network relations among the players. These can be formally represented by an undirected graph, which is a set of unordered pairs (i, j), where $i, j \in I$, and $i \neq j$. Each unordered pair will be referred to as a link.

A link (i, j) may not belong to the graph, that is, the effective set of links may be a proper subset of the potential set of links. Starting form a graph g, two players $i, j \in I$ may decide to "break" or "delete" their link, that is, to terminate their network relation. This will lead to the graph $g \setminus (i, j)$, where "$\setminus$" denotes set subtraction. Alternatively, if $(i, j) \notin g$, they might decide to add it to the graph, leading to $g \cup (i, j)$, where $\cup$ denotes set union. In the following section I discuss when this situations are likely to happen.

Continuing with the definitions, let g^I be the set of all unordered pairs in I, that is, the full graph over I. The restriction of g to a coalition Γ is $g|\Gamma = \{(i, j) \in g : i \in \Gamma \wedge j \in \Gamma\}$. A connected component of Γ is a coalition $\tilde{\Gamma} \subseteq \Gamma$ such that (1) for every two players in $\tilde{\Gamma}$, there is a set of consecutive links or path in $g|\Gamma$

[5]For the particular model of this thesis, an oligopoly game, only pecuniary externalities will be considered. That is, they will not take into account non-players such as price-taking consumers or suppliers. Therefore, these externalities will not correspond to the standard use of the word in economic policy

connecting them, and (2) for any players $i \in \tilde{\Gamma}$ and $j \notin \tilde{\Gamma}$, there is no path in $g|\Gamma$ which connects them. The set of all connected components of Γ in g will be denoted by $\Gamma|g$.

Assume, lastly, that for every graph g and connected component $\Gamma \in I|g$ there is a value $w(\Gamma, g)$ which can be perfectly distributed among the agents in Γ. What some authors have called "the generalized Myerson-Shapley Value" is defined in the following theorem, using the specification of $F_i(\cdot, \cdot)$ in equation 2.

Theorem 1 (Myerson Value for Networks) *The* Myerson value for networks *is the allocation rule which assigns to every player i and every graph g the allocation $F_i(I, W_g)$, where $[I, W_g]$ is the game in PFF given, for every $(\Gamma, P) \in C(I)$, by*

$$W_g(\Gamma, P) = \sum_{\tilde{\Gamma} \in \Gamma|g} w(\tilde{\Gamma}, g|P)$$

(Navarro (2007), Theorem 4.2).

$F_i(I, W_g)$ could be called a "network payoff function", as in Gilles et al. (2006), because it assigns a payoff to player i for each network g. I will use the more concise notation $F_i(g)$ when omitting the rest of the information leads to no ambiguities.

It is now clear that the value assigned to a coalition may depend not only on the partition of I but on how the players are linked in the partitioned graph. In other words, a given coalition may have a different values, even for a same partition of I, because the underlying network may change. In fact, it is not necessary that its own links change (that is, its "inner structure"). The value of the coalition may change because of the addition or deletion of links in other coalitions.

3.4 Network Stability

In the presence of externalities among players in the game, a new concern arises: the stability of the network. Cooperative games assume that the network is exogenous, but attempts to understand the network formation process have yielded several insights relating them to non-cooperative game theory. For instance, one could think of a cooperative game as a second stage to a non-cooperative game in which players decide which links to form. This kind of models will typically give rise to several Nash Equilibria. That is why many authors have developed refinements that help reduce the number of equilibria (e.g. Strong Nash Equilibria, Coalition Proof Nash Equilibria, etc.). However, even then, multiple equilibria may persist. In what follows, I describe three simple refinements, based on the principle that a link can be unilaterally dissolved. The first two are probably the earliest notions of stability in the networks literature (Jackson & Wolinsky (1996)). Before stating them, we need to define the *neighborhood* of player i in graph g, which is the set $N_i(g) = \{j \in I | j \neq i \wedge (i, j) \in g\}$.

(i) A network $g \in g^I$ is *link deletion proof* (LDP) if for every $i \in I$ and every $j \in N_i(g)$, it holds that $F_i(g) >= F_i(g \setminus (i, j))$. The operator "\" denotes set subtraction, so the LDP property means that no player wants to break a link with its neighbors, considering them one at a time.

(ii) A network $g \in g^I$ is *link addition proof* (LAP) if for all players $i, jinI$, it holds that $F_i(g \cup (i, j)) > F_i(g) \Longrightarrow F_j(g \cup (i, j)) < F_j(g)$. In other words, LAP means that when a player wants to add a link, its potential neighbor doesn't want to. That is, there are no compatible incentives to form additional links.

(iii) A network $g \in g^I$ is *strong link deletion proof* (SLDP) if for every $i \in g$ and every $\Gamma \subseteq N_i(g)$, it holds that $F_i(g) >= F_i(g \setminus \Gamma)$. SLDP means that no player wants to break any links with its neighbors, considering one or more at a time.

A network that satisfies (i) and (ii) is called *pairwise stable* (Jackson & Wolinsky (1996)). If it satisfies (ii) and (iii), it is called *strongly pairwise stable* (Gilles & Sarangi (2004)). Since (iii) implies (i), strong pairwise stability implies pairwise stability.

3.5 Dynamic Framework

The definitions and theorems in this section and in Sections 3.6 and 3.7 were taken from Escobar (2013) and Doraszelski & Satterthwaite (2010). In what follows I reproduce a formal definition of discrete state dynamic stochastic games by Escobar (2013).

It will be assumed that there exists a finite set I of players. Each period $t \geq 0$ they observe the state of the game, $s_t \in S$, where S is countable. Then, they simultaneously choose actions $a_t = (a_{i,t})_{i \in I}$, where $a_{i,t} \in A_i \subseteq \mathbb{R}^{L_i}$, A_i compact. The evolution of the state of the system is Markovian, that is, the distribution over s_{t+1} is completely determined by (a_t, s_t). The Markovian transition function is $Q(E; a_t, s_t) = \mathbb{P}[s_{t+1} \in E|(a_t, s_t)]$, where $E \subseteq S$. $Q(E; a, s)$ is setwise continuous in a, i.e., for every $E \subseteq S$ and $s \in S$ it is continuous in $a \in A \equiv \prod_{i \in I} A_i$. Each player receives a per period payoff equal to $\pi_i(a_t, s_t)$, where $\pi_i(a, s)$ is bounded. Given realized sequences of actions $(a_t)_{t \geq 0}$ and states $(s_t)_{t \geq 0}$, the total payoff for player i corresponds to the discounted sum of its per period payoffs:

$$\sum_{t=0}^{\infty} (\beta_i)^t \pi_i(a_t, s_t) \tag{3}$$

where $\beta_i \in [0, 1)$. The objective of each player is to maximize the expected value of this sum.

A *Markov strategy* for player i is a function $\bar{a}_i : S \to A_i$ mapping current states to feasible actions. This means that every period t, player i chooses $\bar{a}_i(s_t)$ if the state is s_t. A tuple $\bar{a}_i)_{i \in I}$ of Markov strategies is a *Markov Perfect Equilibria* if it constitutes an SPE of the dynamic game. In other words, a tuple of strategies is an MPE if and only if it is Markov and it constitutes an SPE. Notice that there may be tuples that conform an SPE but are not Markov. Notice too that we have taken the Markov strategies to be pure. It is generally understood that they are. When they're not, the term "behavior strategy" is used. We will denote by a_{-i} the tuple that includes the actions of every player except i. A_{-i} will be defined analogously.

3.6 Some Existence Results

The following result allows one to reduce the problem of existence of an MPE to one of existence of a solution to a system of functional equations:

Theorem 2 *For each $i \in I$, consider a function $\bar{a}_i : S \to A_i$. Suppose that there is a tuple $(J_i)_{i \in I}$, where $J_i : S \to \mathbb{R}$ is bounded, such that for all i and for all $s \in S$*

$$J_i(s) = \max_{x_i \in A_i} \{\pi_i((x_i, \bar{a}_{-i}(s)), s) + \beta_i \sum_{s' \in S} J_i(s') Q(s'; (x_i, \bar{a}_{-i}(s)), s)\}$$

$$\bar{a}_i(s) \in \arg\max_{x_i \in A_i} \{\pi_i((x_i, \bar{a}_{-i}(s)), s) + \beta_i \sum_{s' \in S} J_i(s') Q(s'; (x_i, \bar{a}_{-i}(s)), s)\}$$

Then, $(\bar{a}_i)_{i \in I}$ is an MPE (Stokey et al. (1989), Theorem 9.2, as apparently incorrectly cited by Escobar (2013). However, similar results are used in (Fackler 2002, p. 209) and in several, if not most, articles related to the Ericson and Pakes framework (see Doraszelski & Pakes (2007))).

For $a \in A$, $s \in S$ and bounded functions $J_i : S \to \mathbb{R}$, lets define

$$\Pi_i(a, s; J_i) = \pi_i(a, s) + \beta_i \sum_{s' \in S} J_i(s') Q(s'; a, s)$$

If we fix s and $(J_i)_{i \in I}$, the functions $\Pi(\cdot, s; J_i)$, $i \in I$, define a static game in which the players' action profiles are $a \in A$. It will be called the "reduced game".

Definition 1 (Convex Best Replies) *The dynamic stochastic game in Section 3.5 has* convex best replies *if for all i, all $s \in S$, all $a_{-i} \in A_{-i}$ and all bounded function $J_i : S \to [\frac{\check{\pi}_i}{1-\beta_i}, \frac{\hat{\pi}_i}{1-\beta_i}]$, the best-reply set*

$$\arg\max_{x_i \in A_i} \Pi_i((x_i, a_{-i}), s; J_i)$$

is convex. $\hat{\pi}_i$ and $\check{\pi}_i$ are the supremum and infimum, respectively, of π_i (Escobar (2013)).

The following theorem guarantees the existence of a tuple $(J_i)_{i \in I}$ which satisfies Theorem 2 and which is, consequently, an MPE.

Theorem 3 (Existence of an MPE) *The dynamic stochastic game possesses an MPE if it has convex best replies and for all i, $\pi_i(a, s)$ is upper semi-continuous in $a \in A$ and lower semi-continuous in $a_{-i} \in A_{-i}$* (Escobar (2013), Theorem 1).

3.7 Some Concavity Results

In this section I present sufficient conditions for Theorem 3. More precisely, they are sufficient conditions for the convexity of the players' best replies. This will be true, in particular, if they are unique for each state, which is equivalent to having reduced game objective functions with unique global maximizers. This, in turn, will be true if they are strictly concave. In the remainder of this section, we will assume that $A_i = [0, \hat{x}_i]$.

Definition 2 (Unique Investment Choice (UIC) Admissibility) *The transition probability $Q(s'; a, s)$ is* UIC admissible *if, for all i, a, s and s', it can be written in separable form as*

$$G_i(s'; a_{-i}, s)\Lambda_i(a_i, s) + L_i(s'; a_{-i}, s) \tag{4}$$

where $\Lambda_i(a_i, s)$ is twice differentiable, strictly increasing and strictly concave in a_i, that is:

$$\frac{\partial \Lambda_i(a_i, s)}{\partial a_i} > 0, \qquad \frac{\partial^2 \Lambda_i(a_i, s)}{\partial a_i^2} < 0$$

for all $a_i \in [0, \hat{x}_i]$ (Doraszelski & Satterthwaite (2010), Condition 1).

The name of this property is due to the fact that it was originally proposed for a model in which the action space corresponded to investment choices. It might as well be called "unique action choice admissibility" in a more general context such as this. It is only a matter of interpretation of the action space.[6]

UIC admissibility by itself does not guarantee a unique best reply. In order for it to be sufficient, one needs to impose additional structure to the model. For our purposes, it will be enough to assume that for all $a \in A$ and $s \in S$, $\pi_i(a, s)$ is linear in a_i with $\frac{\partial \pi_i(a,s)}{\partial a_i} = m < 0$. Under this two conditions, it is possible to write the reduced game payoff for player i in the following way:

$$\begin{aligned}
\Pi_i(a, s; J_i) &= \pi_i(a, s) + \beta_i \sum_{s' \in S} J_i(s') Q(s'; a, s) \\
&= \pi_i(a, s) + \beta_i \sum_{s' \in S} J_i(s')(G_i(s'; a_{-i}, s)\Lambda_i(a_i, s) + L_i(s'; a_{-i}, s)) \\
&= \pi_i(a, s) + H_i(s'; a_{-i}, s, J_i)\Lambda_i(a_i, s) + (\beta_i \sum_{s' \in S} J_i(s') L_i(s'; a_{-i}, s))
\end{aligned}$$

and its first order condition (hereafter FOC)

$$\frac{\partial \Pi_i(a, s; J_i)}{\partial a_i} = m + H_i(s'; a_{-i}, s, J_i)\frac{\partial \Lambda_i(a_i, s)}{\partial a_i} = 0$$

where $H_i(s'; a_{-i}, s, J_i) = \beta_i \sum_{s' \in S} J_i(s') G_i(s'; a_{-i}, s)$. If this last term is less than or equal to zero, the only solution to the FOC is $a_i = 0$, because the objective function is strictly decreasing. If it is greater than zero

[6] To avoid confusion, notice that partial derivatives are taken with respect to the action, wich belongs to a compact interval. This is done for any given state. Also, the actions are not equilibrium strategies, but simply an argument of the function. There is no attempt to take a derivative with respect a discrete variable nor is there any attempt to use the Envelope Theorem or anything simmilar.

and a solution to the FOC exists, the latter will be unique by the strict concavity of the objective function. If $H_i(s'; a_{-i}(s), s, J_i)$ is greater than zero and the FOC has no solution, the objective function is either strictly increasing or strictly decreasing in the interval A_i, and the only maximizer is $a_i = 0$ or $a_i = \hat{x}_i$, respectively. It has thus been shown that the best reply of player i is unique for any state (Doraszelski & Satterthwaite (2010),Proposition 3).

4 The Model

The framework developed by Ericson & Pakes (1995) gave origin to a series of dynamic models of imperfect competition. Many of them possess what Doraszelski & Pakes (2007) call a"static-dynamic breakdown": the actions which determine the operational cash flows of the firms do not affect the evolution of the state variables. The operational cash flow corresponds to the payoff obtained through product market competition and is typically affected by pricing and production decisions. The free cash flow, which is what the firm owners ultimately receive and care about, subtracts the additional investment expenditures ($R\&D$, production capacity accumulation, etc.). The static-dynamic breakdown allows the independent specification of the type of market competition, which implies that the operational cash flows can be treated as part of the primitives of the model.

Continuing with the notation in Section 3.5, it will be assumed that there is a set of firms, $I = \{1, 2, ..., n\}$, which can be partitioned in a set of upstream firms (sellers, suppliers or manufacturers), U, and set of downstream firms (buyers or retailers), D, both non-empty. The former provide the latter with an intermediate good which is then processed and sold as a final good in the next stage of the supply chain. In what follows, I will show how to adapt the BD model to this bilateral format.

4.1 Static Competition

Static competition is the one that occurs every period in the product markets. Each seller $i \in U$ has a constant marginal cost of production and a production capacity restriction. Each buyer $i \in D$ has a constant returns to scale technology which uses as input the intermediate good[7], and it also has a capacity restriction. The inverse demand for the final good is $p(q) = \zeta - \eta q$, where ζ and η are two positive constants and q is the total production of the final good. z_i is the amount of the intermediate good bought (sold) by $i \in D$ ($i \in U$), q_i the amount of the final good produced by $i \in D$, $\gamma > 0$ the transformation coefficient from z to q, $c \in [0, \zeta\gamma)$ the marginal cost of production of z and $k_i \in K_i \subset \mathbb{R}_+$ the production capacity of firm i (measured in the same units as its output). I have assumed homogeneous final and intermediate goods to simplify the analysis. This will not impact greatly the solution of the model since it will involve a form of quantity competition.[8]

The symmetry assumptions regarding the transformation coefficient and the marginal cost serve to identify alternative sources of asymmetric market structures. The objective is to isolate endogenous sources of asymmetry that result from market competition from exogenously imposed technological or strategic asymmetries (e.g. Stackelberg leadership).

To simplify the resolution of the model, my approach to the determination of the firms' payoffs in the static game will be mainly axiomatic or cooperative. Each period, depending on the state of the industry, a network configuration g will arise. It will be composed of buyer-seller pairs only, that is, $g \subseteq \hat{g} = \{(i,j)|i \in D \wedge j \in U\} \subset g^I$. In other words, it will exclude the possibility of horizontal links. This could be due to the presence of antitrust regulations, for example.

The restriction to buyer-seller pairs allows me to follow the convention of reserving the first index in the link for downstream firms and the second for upstream firms without loss of generality.

[7] other inputs enter the function in fixed proportion.

[8] As it is well known, price competition is more sensitive regarding the assumption of product differentiation.

4.1.1 The Bargaining Problem

Recall from Section 3.3 that a link (i, j) represents a network relation between players i and j. Each link in network g will negotiate over an output-input-transfer triple denoted by (q_{ij}, z_{ij}, T_{ij}), taking as given the values of other triples. Transfers are defined to be positive when i pays j. The outcome will be determined by NB, considering as the firms' outside options the best payoffs they would obtain by breaking that link (and that link only). The intersection of the results from all bilateral negotiations will determine the static equilibrium values for every output-input pair and the price of the final good. It will also determine the value of each coalition for the graph under consideration.[9]

More specifically, the bargaining game for the link (i, j) will be given by a set of feasible agreements $B = \{(q_{ij}, z_{ij}, T_{ij}) \in \mathbb{R}^2_+ \times \mathbb{R} | q_i \leq k_i \wedge z_j \leq k_j \wedge q_i \leq \gamma z_i\}$, where q_i, z_j and z_i are as defined above[10]. The disagreement point will be $(q_{ij}, z_{ij}, T_{ij}) = (0, 0, 0)$ and it will give players i and j payoffs of $F_i(g \setminus (i, j))$ and $F_j(g \setminus (i, j))$, respectively.

$F_i(g \setminus (i, j))$ and $F_j(g \setminus (i, j))$ represent the payoffs i and j would get in the network $g \setminus (i, j)$, that is, the network composed by the links that remain after the negotiations between i and j break down. In this network, there is no trade between players i and j. A supplier facing two buyers, for example, might prefer to sell inputs to a single buyer to avoid the externalities generated by downstream competition. The link with one of the buyers would cease to exist and the link with the remaining seller would constitute a new network.

In the current network, g, the payoff for a downstream firm will equal its sales minus any transfers made to upstream firms, that is, $F_i(g) = p(q)q_i - T_i$, where $T_i = \sum_{m \in U} T_{im}$. On the other hand, the payoff for an upstream firm will equal the transfers received from downstream firms minus production costs: $F_j(g) = T_j - cz_j$, with $T_j = \sum_{m \in D} T_{mj}$. NB implies that (q_{ij}, z_{ij}, T_{ij}) will be chosen to maximize over B the product of these payoffs net of outside options:

$$\left(p(q)q_i - T_{ij} - \sum_{m \neq j, m \in U} T_{im} - F_i(g \setminus (i, j))\right) \left(T_{ij} + \sum_{m \neq i, m \in D} T_{mj} - cz_j - F_j(g \setminus (i, j))\right)$$

It is straightforward to show that this is equivalent to maximizing their joint net surplus and choosing T_{ij} to split it evenly [11]. In other words, the optimal final and intermediate good quantities are independent of the way the firms share their joint net surplus. Therefore, we can rewrite link (i, j)'s optimization problem as follows:

$$\begin{aligned} &\max_{(q_{ij}, z_{ij})} p(q)q_i - cz_j \\ &\text{s.t.} \\ &\quad q_i \leq k_i \\ &\quad z_j \leq k_j \\ &\quad q_i \leq \gamma z_i \\ &\quad q_{ij}, z_{ij} \geq 0 \end{aligned}$$

[9]Notice that the assumption of bargaining over short-run variables only is not innocuous. It is quite likely that this will generate externalities among the players, specially when considering the fully dynamic game. Some coordination on investment strategies is likely, but I will assume that for reasons such as informational asymmetries or the existence of non-contractible contingencies, players can only bargain over short-run variables. Of course, the problem of intertemporal commitment can also be an important impediment restriction to long term agreements.

[10]That is, $q_i = \sum_{m \in U} q_{im}$, $z_j = \sum_{m \in D} z_{mj}$ and $z_i = \sum_{m \in U} z_{im}$

[11]This is partly due to the fact that the restrictions in B involving quantities are independent of the one involving the transfer payment, which corresponds to the trivial restriction $T_{ij} \in \mathbb{R}$. Intuitively, this practically unrestricted transfer allow the players to separate the choice of the size of the surplus to be shared from the choice of the distribution of that surplus. Since NB, as shown by Nash (1950, 1953), is (bilaterally) efficient, the joint surplus will be maximized

4.1.2 Existence of a Static Equilibrium

I have not defined formally the notion of equilibrium being employed in this game. Strictly speaking, there is no equilibrium in the non-cooperative sense because there is no non-cooperative behavior. What matters is how to solve simultaneously all negotiation problems. Therefore, I will understand as "static-equilibrium" the quantities that result when all bilateral negotiations are carried out assuming "passive beliefs" as in de Fontenay & Gans (2013). This concept is similar to the idea of rational expectations: all negotiations will be carried out expecting equilibrium values for the outcomes of other links, and this expectation will be self-fulfilling.

To demonstrate that there is such an equilibrium for any network, let us assume that every link (i, j) is restricted to output-input pairs that satisfy $q_{ij} = \gamma z_{ij}$. This reduces the dimensionality of the previous problem. Among other things, it implies that $q_i = \gamma z_i$, $q_j = \gamma z_j$ and the third restriction is trivially satisfied. Writing the inverse demand function explicitly, we get:

$$\max_{q_{ij}} (\zeta - \eta q) q_i - \frac{c}{\gamma} q_j \tag{5}$$

$$\text{s.t.} \tag{6}$$

$$q_i \leq k_i \tag{7}$$

$$q_j \leq \gamma k_j \tag{8}$$

$$q_{ij} \geq 0 \tag{9}$$

Let $\tilde{q}_i = q_i - q_{ij}$, $\tilde{z}_j = z_j - z_{ij}$, $\tilde{z}_i = z_i - z_{ij}$ and $\tilde{q} = q - q_{ij}$. The first and second order conditions (FOC and SOC, respectively) for an interior solution are:

$$\text{FOC:} - \eta q_i + \zeta - \eta q - \frac{c}{\gamma} = 0 \tag{10}$$

$$\text{SOC:} - 2\eta < 0 \tag{11}$$

This conditions lead to a unique unrestricted maximum at:

$$q_{ij} = \frac{1}{2\eta}(\zeta - \frac{c}{\gamma}) - \frac{1}{2}(\tilde{q}_i + \tilde{q})$$

The strictly negative SOC implies that the objective function is strictly concave in q_{ij} (globally). In other words, it is strictly increasing to the left of the unrestricted maximum and strictly decreasing to the right. Consequently, there is a unique restricted maximum determined by:

$$q_{ij} = \max\{0, \min\{\frac{1}{2\eta}(\zeta - \frac{c}{\gamma}) - \frac{1}{2}(\tilde{q}_i + \tilde{q}), k_i - \tilde{q}_i, \gamma k_j - \tilde{q}_j\}$$

Solving for other links, one obtains symmetric expressions. Together, they constitute a system of equations that has at least one solution. To see why this last statement is true, notice that the minimum between two continuous functions is continuous, and so is the maximum. Since every function inside $\min\{\cdot\}$ above is linear and, therefore, continuous, the minimum between them is continuous too. Hence, the two functions inside the maximum are continuous, which leads us to the conclusion that the whole function is continuous.

It is also true that q_{ij} will take values between zero and $\min\{\frac{1}{2\eta}(\zeta - \frac{c}{\gamma}), k_i, \gamma k_j\}$. The interval defined by $[0, \min\{\frac{1}{2\eta}(\zeta - \frac{c}{\gamma}), k_i, k_j\}]$ is non-empty, compact and convex, and so is the Cartesian product $\mathscr{D} = \prod_{j \in U} \prod_{i \in D} [0, \min\{\frac{1}{2\eta}(\zeta - \frac{c}{\gamma}), k_i, k_j\}]$. Since the continuous image of a compact set is compact, the mapping from $\mathscr{D}$ to itself defined by the system of equations satisfies the conditions of the Brouwer Fixed Point Theorem. The system having a fixed point is equivalent to saying that there exists at least one equilibrium defined by the previous optimality conditions.

Remember, however, that I imposed an artificial restriction, namely, that $q_{ij} = \gamma z_{ij}$ for all $(i, j) \in g$. For the previous solution to constitute an equilibrium to the original problem, it is necessary (and sufficient) that no link wants to unilaterally deviate from $q_{ij} = \gamma z_{ij}$. That is, given $q_{ml} = \gamma z_{ml}$ for all $(m, l) \in g$,

$(m, l) \neq (i, j)$, it should be optimal for (i, j) to choose $q_{ij} = \gamma z_{ij}$. This is exactly the case, because $q_{ml} = \gamma z_{ml}$ implies that the third restriction in problem 5 becomes $q_{ij} \leq \gamma z_{ij}$. Since the objective function is strictly decreasing in z_{ij} whenever $c > 0$, and since $q_{ij} \leq \gamma k_j$, the optimal choice for the input will always be $z_{ij} = q_{ij}/\gamma$.[12]

I have shown that the model has at least one equilibrium, but there could be many. This is problematic to the extent that it generates multiple payoff profiles for the same network. Then, a refinement of the equilibrium concept would be required to obtain unique payoffs that can be plugged into the dynamic framework from Section 3.5. This will be unnecessary for the cases examined afterwards. It will be demonstrated that payoffs are uniquely determined in those particular cases. Unfortunately, even then, the equilibrium network might not be unique, as I discuss below, so it will be necessary to use a somewhat arbitrary selection procedure for the payoffs. Still, having unique payoffs for each network represents an improvement from the case in which both the payoffs for each network and the network itself are not uniquely defined.

4.1.3 Payoffs

So far, I have only explained how equilibrium quantities for the final and intermediate goods are determined. After solving for the optimal values of every output-input pair, one way to find the payoff functions and transfers is through the following iterative procedure. Since the value of the outside option is zero for both players in graphs containing a single link, one can begin by defining $F_i((i,j)) = F_j((i,j)) = \frac{1}{2}(p(q^*_{ij})q^*_{ij} - cz^*_{ij})$, where the asterisk denotes optimality (from the link's perspective). In other words, T_{ij} is chosen so that both firms split total industry profits evenly. After finding the payoffs for every single-link graph, the outside options for every double-link graph will be known and it will be possible to find the corresponding payoffs. This procedure continues until finding the payoffs for $\hat{g}$, the largest network.

Alternatively, this hybrid network-bargaining game leads to payoffs that can be described through the generalized Myerson-Shapley allocation rule, as demonstrated by de Fontenay & Gans (2005, 2013). In fact, notice that the iterative procedure described above follows the axioms that define the Myerson-Shapley Value, namely, that the surplus net of outside options is equally shared among players, and that all profits are distributed. Using its explicit formula would require finding the value of every coalition Γ under each possible network structure g, $w(\Gamma, g)$, and applying Theorem 1 afterwards. This second approach seems better suited for solving large scale problems. For the applications in Sections 5 and 6, I used the first one.

Notice that to obtain unique payoffs for each network, it will be necessary and sufficient that the total value of each connected component (defined in Section 3.3) is uniquely determined. For the applications in this thesis, it will suffice to verify that total industry profits are unique, that is, that the amount given by

$$p(q)q - cz \tag{12}$$

is the same for every static equilibrium.

4.1.4 Equilibrium-Network Selection

As mentioned earlier, the model just described was developed by de Fontenay & Gans (2005, 2013). However, in the context of a dynamic investment model, their results are insufficient. Specifically, the authors give feasibility conditions for the complete buyer-seller network to arise in equilibrium, and assume that they are satisfied. Unfortunately, this will rarely be the case when considering endogenous and potentially asymmetric production capacities, which is the situation analyzed in this thesis. Since de Fontenay & Gans do not characterize the equilibria that result when the feasibility conditions are not met, I will use different feasibility criteria. I will assume that every period, a graph is randomly chosen with equal probability from the set of non-empty strongly pairwise stable networks. In other words, the ex-post payoffs of the firms will be given by the generalized Myerson-Shapley Value, but the ex-ante payoffs will be given by its expectation over the set of non-empty strongly pairwise stable networks.

As previously discussed, the equilibrium selection problem is not easily solved. When recurring to non-cooperative link formation games, multiple Nash Equilibira arise. Refinements of the Nash Equilibrium

[12] Notice that when $c = 0$, this choice is also optimal, but it is not the only one.

concept yield a smaller set of equilibria, yet uniqueness is not guaranteed. That is why it seems reasonable and simpler to continue with the cooperative approach, in spite of the arbitrariness involved in my solution. I will employ some filters to avoid unstable networks and obtain payoffs that are fairly consistent with individual rationality, but I will not specify the process through which the players get to one stable network or another. Instead, I will assume that every period "nature" chooses a stable network randomly.

For instance, consider what happens when there is disagreement between a buyer and two sellers and one of these links is broken, but not both of them. In other words, assume that network $g = \{(seller, buyer_1), (seller, buyer_2)\}$ is not link-deletion proof, but networks $g_1 = \{(seller, buyer_1)\}$ and $g_2 = \{(seller, buyer_2)\}$ are strongly pairwise stable. Our assumption is that each buyer will have a 50% chance of remaining in the static game during that period.

This working assumption is designed to avoid exogenous asymmetries between firms and solve the network selection problem in a simple way. It should be analyzed in more detail in future work because it may be inconsistent with profit maximizing behavior.

4.2 Dynamic Competition

Every period, each firm i begins with a production capacity from the finite set $K_i = \{k_{i,0}, k_{i,1}, ..., k_{i,\varkappa_i}\} \subset \mathbb{R}_+$ with $0 = k_{i,0} < k_{i,1} < ... < k_{i,\varkappa_i}$. The state space of the industry is defined as the Cartesian product $\prod_{i \in I} K_i$. For simplicity, when referring to the state of a particular firm i, k_{i,s_i}, the variable $s_i \in \{1, 2, ..., \varkappa_i\}$ will be used. The set S will be defined as the finite set of all vectors $s = (s_1, s_2, ..., s_n)$ corresponding to a vector $k \in \prod_{i \in I} K_i$.

Since the static game is stationary, time subscripts will be eliminated. An apostrophe will indicate next period variables (e.g. s' indicates the state of the game next period) and no apostrophe will indicate current period variables (e.g. s is the state of the game in the current period).

4.2.1 The Capacity Accumulation Process

Firms can invest in the accumulation of production capacity blocks, subject to a risk of non-completion or failure. They can also be subject to depreciation shocks which destroy part of their capacity. The probability that firm i suffers a depreciation shock is δ_i, independent of that of other firms. The probability that an investment project is completed successfully the next period is also independent between firms (and from the depreciation shock) and is defined as:

$$\lambda_i(x_i) = \frac{\theta_i x_i}{1 + \theta_i x_i}$$

where $x_i \geq 0$ is the amount of money invested by the firm and $\theta_i > 0$ measures the effectiveness of investment around $x_i = 0$:

$$\left.\frac{d\lambda_i(x_i)}{dx_i}\right|_{x_i=0} = \theta_i$$

It is possible to interpret $(\lambda_i(x_i))^{-1}$ as the average time it takes for an additional block to come on-line if the firm invests x_i every period. The time-to-build will be geometrically distributed, with variance $(\lambda_i(x_i))^{-1}/(\theta_i x_i)$. The risk measured by this variance translates into stock-outs or idle capacity, making the choice of $theta_i$ an important matter.

According to the previous definitions, the transition function for firm i should therefore be:

$$Q_i(s_i'|x_i, s_i) = \mathbb{P}[s_i'|x_i, s_i] = \begin{cases} \lambda_i(x_i)(1 - \delta_i) & \text{if } s_i' = s_i + 1 \\ (1 - \lambda_i(x_i))(1 - \delta_i) + \lambda_i(x_i)\delta_i & \text{if } s_i' = s_i \\ (1 - \lambda_i(x_i))\delta_i & \text{if } s_i' = s_i - 1 \\ 0 & \text{in other case} \end{cases}$$

when starting from interior values of the production capacity set, i.e., for s_i different from 1 or $\varkappa_i$. For $s_i = 1$,

$$Q_i(s_i'|x_i, s_i) = \mathbb{P}[s_i'|x_i, s_i] = \begin{cases} \lambda_i(x_i)(1-\delta_i) & \text{if } s_i' = s_i + 1 \\ 1 - \lambda_i(x_i)(1-\delta_i) & \text{if } s_i' = s_i \\ 0 & \text{in other case} \end{cases}$$

Similarly, for $s_i = \varkappa_i$,

$$Q_i(s_i'|x_i, s_i) = \mathbb{P}[s_i'|x_i, s_i] = \begin{cases} 1 - (1-\lambda_i(x_i))\delta_i & \text{if } s_i' = s_i \\ (1-\lambda_i(x_i))\delta_i & \text{if } s_i' = s_i - 1 \\ 0 & \text{in other case} \end{cases}$$

The independence between firms implies that the transition function of s, the state of the whole market or industry, is $Q(s'; x, s) = \prod_{i\in I} \mathbb{P}[s_i'|x_i, s_i] = \prod_{i\in I} Q_i(s_i'; x_i, s_i)$ for s corresponding to $(s_1, s_2, ..., s_n)$ and s' to $(s_1', s_2', ..., s_n')$, with $x = (x_1, x_2, ..., x_n)$. $Q(s'; x, s)$ is setwise continuous because, for any given initial and final state pair, it is an n degree polynomial in the success probabilities of the players $(\lambda_i(x_i))$, which are themselves continuous in the investment decisions.

It may be too restrictive to assume that it is only possible to transition to adjacent states, and that the firms' transition probabilities are independent of each other. However, both restrictions can be easily relaxed (see Doraszelski & Satterthwaite (2010)). For example, it is possible to introduce shocks that simultaneously affect all firms, that is, industry-wide shocks. For the sake of simplicity, I will only consider the restricted transitions. They will generate sufficiently rich dynamics, as we will see.

Entry and exit are only implicitly considered through the investment decision. If a firm wants to exit the market it must stop investing and, in a finite number of periods, depreciation will take care of it. If it wants to enter, it must invest to accumulate capacity, which will also grow in a finite number of periods. A firm exiting the market does not receive a scrap value for its assets, and a firm entering it does not pay setup costs (beyond investing in capacity). It is possible to extend the model to consider entry and exit decisions explicitly the same way Doraszelski & Satterthwaite (2010) extend the BD model, following the tradition of the Ericson & Pakes (1995) framework.

4.2.2 Existence of an MPE

The free cash flow of firm i in a given period corresponds to the operational cash flow minus the amount invested in capital accumulation. Considering explicitly in the notation that the operational cash flow depends on the state of the industry, we obtain the following expression for the free cash flow of i:

$$\pi_i(x, s) = F_i(s) - x_i$$

where $F_i(s)$ is defined according to the cooperative game described in 4.1.

The space of available actions for each player is the interval $A_i = [0, \hat{x}_i]$, where $\hat{x}_i > 0$. It will always be possible to choose its upper bound in way that it is not binding for the firm. Similarly, the state space of the game, S, has been defined as a finite set. It can be demonstrated that there will be no loss of generality in doing so because, in equilibrium, only a finite subset of the "true", infinitely countable state space, will be visited with positive probability (see Ericson & Pakes (1995) for a rigorous demonstration in a similar context). With $A = \prod_{i\in I} A_i$ and S defined in such a way, it can be shown that the function $\pi_i(x, s)$ is bounded (that is, for the relevant pairs $(x, s) \in A \times S$). Furthermore, it is possible to define the objective function of each player as the discounted sum of its future cash flows for some nonnegative discount factor $\beta_i < 1$, just like in the general framework defined previously (see equation 3).

For $x \in A$, $s \in S$ and bounded functions $J_i : S \to \mathbb{R}$, the payoffs from the reduced game implicit in this model are defined as (Section 3.6):

$$\Pi_i(x, s; J_i) = F_i(s) - x_i + \beta_i \sum_{s'\in S} J_i(s')Q(s'; x, s)$$

To prove that there is at least one MPE we need to show first that $Q(s'; x, s)$ is UIC admissible. This has already been demonstrated by Doraszelski & Satterthwaite (2010), but, for the sake of completeness, I reproduce their result. First, notice that $Q(s'_i; x_i, s_i)$ can always be written in the form $h(s'_i; s_i)\lambda_i(x_i)+l(s'_i; s_i)$, i.e., it is linear in the success probability of i. Let us define

$$
\begin{aligned}
G_i(s'; x_{-i}, s) &= \left(\prod_{j\in I, j\neq i} Q_j(s'_j; x_j, s_j)\right) h(s'_i; s_i) \\
L_i(s'; x_{-i}, s) &= \left(\prod_{j\in I, j\neq i} Q_j(s'_j; x_j, s_j)\right) l(s'_i; s_i) \\
\Lambda_i(x_i, s) &= \lambda_i(x_i)
\end{aligned}
$$

Now it is straightforward to show that $Q(s'; x, s)$ can always be written in the separable form of equation 4:

$$
\begin{aligned}
Q(s'; x, s) &= \prod_{j\in I} Q_j(s'_j; x_j, s_j) \\
&= \left(\prod_{j\in I, j\neq i} Q_j(s'_j; x_j, s_j)\right) Q_i(s'_i; x_i, s_i) \\
&= \left(\prod_{j\in I, j\neq i} Q_j(s'_j; x_j, s_j)\right) (h(s'_i; s_i)\lambda_i(x_i) + l(s'_i; s_i)) \\
&= G_i(s'; x_{-i}, s)\Lambda_i(x_i, s) + L_i(s'; x_{-i}, s)
\end{aligned}
$$

Notice also that the first derivative of $\lambda_i(x_i)$ is strictly positive on A_i, and its second derivative is strictly negative:

$$
\begin{aligned}
\frac{d\lambda_i(x_i)}{dx_i} &= \frac{\theta_i}{(1+\theta_i x_i)^2} \\
\frac{d^2\lambda_i(x_i)}{dx_i^2} &= \frac{-2\theta_i^2}{(1+\theta_i x_i)^3}
\end{aligned}
$$

Hence, $Q(s'; x, s)$ is UIC admissible, which concludes the first part of the proof.[13]

Since the transition function is UIC admissible and the cash flow $\pi_i(x, s) = F_i(s) - x_i$ is continuous in x for all i, there exists a unique $x_i \in A_i$ which maximizes the payoff $\Pi_i(x, s; J_i)$ (i.e. the reduced game has convex best replies) and the conditions for Theorem 3 are satisfied. Moreover, the conditions for Theorem 2 are also satisfied, that is, there exists a tuple $(J_i)_{i\in I}$, where $J_i : S \to \mathbb{R}$ is bounded, such that for all i and all $s \in S$

$$
J_i(s) = \max_{x_i\in A_i}\{F_i(s) - x_i + \beta_i \sum_{s'\in S} J_i(s')Q(s'; (x_i, \bar{x}_{-i}(s)), s)\} \tag{13}
$$

$$
\bar{x}_i(s) = \arg\max_{x_i\in A_i}\{F_i(s) - x_i + \beta_i \sum_{s'\in S} J_i(s')Q(s'; (x_i, \bar{x}_{-i}(s)), s)\} \tag{14}
$$

and the functions $\bar{x}_i : S \to A_i$ form an MPE.

[13] The decreasing returns to investment implied by the concavity of λ generate incentives for firms to develop multiple projects at the same time instead of one. However, this is not allowed by the model. Nevertheless, when considering mergers, for example, it is relevant to determine if the merged firm will be able to make multiple investment decisions or not. This is also an important question when determining a benchmark for this oligopoly.

4.2.3 Best-Reply Functions

The best-reply functions of the reduced game can be obtained from the FOC that characterize the maximum of $\Pi_i(x, s; J_i)$:

$$-1 + \beta_i \sum_{s' \in S} J_i(s') \frac{\partial Q(s'; x, s)}{\partial x_i} = 0 \tag{15}$$

The partial derivative of the individual transition function is:

$$\frac{\partial Q_i(s'_i | x_i, s_i)}{\partial x_i} = \begin{cases} \frac{(1-\delta_i)\theta_i}{(1+\theta_i x_i)^2} & \text{if } s'_i = s_i + 1 \\ \frac{(2\delta_i - 1)\theta_i}{(1+\theta_i x_i)^2} & \text{if } s'_i = s_i \\ \frac{-\delta_i \theta_i}{(1+\theta_i x_i)^2} & \text{if } s'_i = s_i - 1 \\ 0 & \text{in other case} \end{cases}$$

for interior s_i, i.e. different from 1 or $\varkappa_i$. For $s_i = 1$,

$$\frac{\partial Q_i(s'_i | x_i, s_i)}{\partial x_i} = \begin{cases} \frac{(1-\delta_i)\theta_i}{(1+\theta_i x_i)^2} & \text{if } s'_i = s_i + 1 \\ -\frac{(1-\delta_i)\theta_i}{(1+\theta_i x_i)^2} & \text{if } s'_i = s_i \\ 0 & \text{in other case} \end{cases}$$

Similarly, for $s_i = \varkappa_i$,

$$\frac{\partial Q_i(s'_i | x_i, s_i)}{\partial x_i} = \begin{cases} \frac{\delta_i \theta_i}{(1+\theta_i x_i)^2} & \text{if } s'_i = s_i \\ \frac{-\delta_i \theta_i}{(1+\theta_i x_i)^2} & \text{if } s'_i = s_i - 1 \\ 0 & \text{in other case} \end{cases}$$

If the equation 15 has a nonnegative solution, we can solve for x_i by multiplying by $(1 + \theta_i x_i)^2$ and taking square root. If it has a negative solution or no solution at all, x_i can only be equal to zero (if it approaches ∞, the objective function approaches $-\infty$). Therefore, the best reply from i is:

$$x_i = \begin{cases} \max\left\{0, \frac{\sqrt{\Delta_i(x_{-i}, s)} - 1}{\theta_i}\right\} & \text{if } \Delta_i(x_{-i}, s) \geq 0 \\ 0 & \text{in other case} \end{cases} \tag{16}$$

where

$$\Delta_i(x_{-i}, s) = \beta_i \sum_{s' \in S} \left(J_i(s') \prod_{j \in I, j \neq i} Q_j(s'_j; x_j, s_j) \frac{\partial Q_i(s'_i; x_i, s_i)}{\partial x_i} (1 + \theta_i x_i)^2 \right) \tag{17}$$

4.3 Computation

The solution algorithm is similar to the dynamic programming algorithm, but as previously mentioned, it does not guarantee convergence. The existence of an MPE is demonstrated through the use of Kakutani's Fixed Point Theorem, not in the Contraction Mapping Theorem.

What it does, basically, is to start from initial guesses for the value functions ($J_i(s)$) and investment policies ($x_i(s)$) to obtain the reduced game best replies and payoffs ($\Pi_i(x, s; J_i)$) for every player. If the payoffs differ greatly from the initial value function guesses, the process is repeated substituting the initial guesses by the functions obtained from the reduced game. Otherwise, the new functions are accepted as a good approximation to the solution of the dynamic game.

More specifically, the solution algorithm is as follows:

1. Initial guesses: $x_i(s) = 0$ y $J_i(s) = \frac{\pi_i(0,s)}{1-\beta_i}$, $\forall s \in S, i \in I$.
2. For each i, update its optimal policy to $x_i'(s)$ using equation 16 evaluated in the initial guesses.
3. Update its value function:

$$J_i'(s) = \pi_i(x_i'(s), s) + \beta_i \sum_{s' \in S} J_i(s') Q(s'|x'(s), s)$$

4. Convergence criterion:

$$\left\| \frac{J_i'(s) - J_i(s)}{1 + |J_i'(s)|} \right\|_\infty < \epsilon$$

for all i, where $\| \cdot \|_\infty$ denotes the sup norm and $\epsilon > 0$ is the tolerance level. The optimal policies can be used as an additional stopping criterion. If the inequation is satisfied, the last policies and value functions are accepted as a solution to the system of functional equations in 13. If not, the process is repeated from step 2 onwards using them as initial guesses.

The convergence criterion is practically free of unit of account. This is convenient because it is hard to get a notion of the magnitudes involved and determine an appropriate level of precision for the algorithm. If it is too demanding, the program may take too long to converge, and if not, the result may be inaccurate. This is especially the case when the model is not being calibrated according to real data.

If the parameterization of the model is symmetric, it is possible to partially reduce the computational burden by reducing the number of optimization problems to be solved and the size of the state space. For example, if all buyers have equal payoff functions, production technologies and transition functions and the same is true for the sellers, it suffices to solve the optimization problem of one buyer and one seller. Furthermore, it is possible to iterate over a smaller subset of the state space. As a matter of fact, the only relevant information about the sellers' state from a buyer's point of view is the combination of their capacity levels. A redistribution of these among the sellers does not alter the buyer's optimization problem and best reply. The same thing happens when considering a seller's the point of view and the buyers' capacity levels.

To verify that the definition of the state space is not restrictive, it is possible to obtain the investment strategies of the firms for a given S and check that they are zero near the boundaries of S. If they're not, S should be amplified or the parameters of the game rescaled until they are (for example, it is possible to move the demand curve towards the origin). It must not be the case that a firm wants to accumulate more capacity but cannot do so because of the artificial restriction of the state space. This could distort the solution of the game.

5 Two Buyers and One Supplier

In this section I study the case of two downstream firms, $D = \{1, 2\}$, and one upstream firm, $U = \{3\}$. The largest network they can form corresponds to the graph $\hat{g} = \{(1, 3), (2, 3)\}$. Then come networks with only one link, which are equivalent to a bilateral monopoly. Finally, there is the empty network, in which no firm produces and all get a payoff equal to zero. In what follows I characterize the production levels for every possible $g \subseteq \hat{g}$.

5.1 Two links

The optimization problem for link $(1,3)$ is the following:

$$\begin{aligned}
&\max_{q_{13},z_{13}} (\zeta - \eta(q_{13}+q_{23}))q_{13} - c(z_{13}+z_{23}) \\
&\text{s.t.} \\
&\quad q_{13} \le k_1 \\
&\quad z_{13}+z_{23} \le k_3 \\
&\quad q_{13} \le \gamma z_{13} \\
&\quad q_{13}, z_{13} \ge 0
\end{aligned}$$

Since $c > 0$, for every feasible level of q_{13} the third restriction will be active, so we can safely assume that $q_{13} = \gamma z_{13}$. This reduces the dimensionality of the problem to one. Replacing the expression in the objective function, we get:

$$(\zeta - \eta(q_{13}+q_{23}))q_{13} - c(\frac{q_{13}}{\gamma} + z_{23})$$

Notice that the SOC is the same as in the general case (see equation 10). Consequently, solving the FOC for q_{13} and taking into account the remaining restrictions yields the following expression for the optimum production level:

$$q_{13} = \max\{0, \min\{\frac{1}{2\eta}(\zeta - \frac{c}{\gamma}) - \frac{1}{2}q_{23}, k_1, \gamma(k_3 - z_{23})\}\}$$

Since the problem for link $(2,3)$ is symmetric, we can simply exchange the subscripts 1 and 2 to obtain the other optimum production level:

$$q_{23} = \max\{0, \min\{\frac{1}{2\eta}(\zeta - \frac{c}{\gamma}) - \frac{1}{2}q_{13}, k_2, \gamma(k_3 - z_{13})\}\}$$

For the second link it will also be true that its third restriction is always active in the relevant range, that is, $q_{23} = \gamma z_{23}$. Therefore, we can eliminate all intermediate good quantities from the equations to obtain a system in the variables q_{13} and q_{23}:

$$\begin{aligned}
q_{13} &= \max\{0, \min\{\frac{1}{2\eta}(\zeta - \frac{c}{\gamma}) - \frac{1}{2}q_{23}, k_1, \gamma k_3 - q_{23}\}\} \\
q_{23} &= \max\{0, \min\{\frac{1}{2\eta}(\zeta - \frac{c}{\gamma}) - \frac{1}{2}q_{13}, k_2, \gamma k_3 - q_{13}\}\}
\end{aligned}$$

Even though this system of equations can have multiple solutions, I will demonstrate next that the total profits generated by the network are always unique.

5.1.1 Uniqueness of Total Industry Profits

Notice that the first equation represents a continuous piecewise linear function of q_{23} that has a slope between -1 and 0. The second equation, when inverted, represents a correspondence, and not a function, of q_{23}. However, this is only due to the non-negativity constraint $q_{23} \ge 0$ and the capacity constraint $q_{23} \le k_2$. In between, it is a continuous piecewise linear function which has a slope between -1 and -2. Considering the constraints, the slope would be between -1 and $-\infty$.

It is easy to see that a line of slope -1 would only intersect the first function in one point unless it coincided with the line $q_{13} = \gamma k_3 - q_{23}$. If it did, there would be an interval of intersections where the function takes that form. Let this interval be $[\check{q}^1_{23}, \hat{q}^1_{23}]$, $\check{q}^1_{23} \le \hat{q}^1_{23}$. For any $q_{23} \le \check{q}^1_{23}$, the line will be higher than the function, because it decreases at a higher rate over that range. The same reason implies that for any $q_{23} \ge \hat{q}^1_{23}$, the line will be lower than the function.

The same argument can be used to show that the opposite is true for the correspondence. Let the interval over which the correspondence takes the form $q_{13} = \gamma k_3 - q_{23}$ be $[\check{q}^2_{23}, \hat{q}^2_{23}]$, $\check{q}^2_{23} \leq \hat{q}^2_{23}$. For any $q_{23} \leq \check{q}^2_{23}$, the line will be lower than the function, because it decreases at a lower rate over that range. The same reason implies that for any $q_{23} \geq \hat{q}^2_{23}$, the line will be higher than the function.

Using Brouwer's Fixed Point Theorem as in Section 4.1.2, it can be shown that the function and the correspondence intersect at least once. Since the value of the slope of the correspondence is less or equal than that of the function, it can only be the case that they intersect once or that there is an overlap of the intervals in which both are equal to $\gamma k_3 - q_{23}$. To see that, notice that the line of slope -1 going through any intersection point q^*_{23} is lower than the function and higher than the correspondence for $q_{23} \geq q^*_{23}$ and is higher than the function and lower than the correspondence for $q_{23} \leq q^*_{23}$. That is, unless $q^*_{23} \in [\check{q}_{23}, \hat{q}_{23}] = [\check{q}^1_{23}, \hat{q}^1_{23}] \cap [\check{q}^2_{23}, \hat{q}^2_{23}]$, which would require the intersection to be non-empty. If this were the case, then all points in $\check{q}_{23}, \hat{q}_{23}]$ would be intersections. More over, no $q_{23} \leq \check{q}_{23}$ is an intersection point because the function is lower than the line and the correspondence is higher, and no $q_{23} \geq \hat{q}_{23}$ is an intersection point because the function is higher than the line and the correspondence is lower. Since all equilibria, if more than one, satisfy $q_{13} + q_{23} = \gamma k_3$, the total profits generated by the network, given by $(p(q_{13} + q_{23}) - c)(q_{13} + q_{23})$, will be unique.

5.2 One Link: Bilateral Monopoly

Assume that the graph is $g = \{(1, 3)\}$. The other one-link graph can be treated symmetrically, so there is no loss of generality. The problem for one link is simpler because it only involves an optimization problem. It does not require the solution of an equilibrium. It is equivalent to assuming that $k_2 = 0$ in the previous problem. This would imply that $q_{23} = z_{23} = 0$. Consequently, the final good production level will be:

$$q_{13} = \max\{0, \min\{\frac{1}{2\eta}(\zeta - \frac{c}{\gamma}), k_1, \gamma k_3\}\}$$

By symmetry, the solution for link $(2, 3)$ would be:

$$q_{23} = \max\{0, \min\{\frac{1}{2\eta}(\zeta - \frac{c}{\gamma}), k_2, \gamma k_3\}\}$$

5.3 Equilibrium Production Levels

In this section I characterize all possible equilibria for network $\hat{g}$. The purpose of this is to obtain analytical expressions for the equilibrium production levels to build an algorithm that finds the firms' payoffs. The following cases are mutually exclusive unless one of them yields the same industry profits as another. Then, both can be equilibria.

- $\boldsymbol{[q_{13} = 0]}$ **:** When the non-negative constraint for $(1, 3)$ is active, the quantity set by link $(2, 4)$ is:

$$q_{23} = \max\{0, \min\{\frac{1}{2\eta}(\zeta - \frac{c}{\gamma}), k_2, \gamma k_3\}\}$$

- $\boldsymbol{[q_{13} = \frac{1}{2\eta}(\zeta - \frac{c}{\gamma}) - \frac{1}{2}q_{23};\ q_{23} = \frac{1}{2\eta}(\zeta - \frac{c}{\gamma}) - \frac{1}{2}q_{13}]}$ **:** Notice that this case is equivalent to a traditional Cournot game, resulting in the following quantities:

$$q_{13} = q_{23} = \frac{1}{3\eta}(\zeta - \frac{c}{\gamma})$$

- $\boldsymbol{[q_{13} = \frac{1}{2\eta}(\zeta - \frac{c}{\gamma}) - \frac{1}{2}q_{23};\ q_{23} = k_2]}$ **:** In this case, link $(2, 3)$ is restricted by firm 2's capacity. This leads link $(1, 3)$ to an increase in output, whenever its capacity is not restrictive:

$$q_{13} = \frac{1}{2\eta}(\zeta - \frac{c}{\gamma}) - \frac{1}{2}k_2$$

- $[\boldsymbol{q_{13} = \frac{1}{2\eta}(\zeta - \frac{c}{\gamma}) - \frac{1}{2}q_{23}};\ \boldsymbol{q_{23} = \gamma k_3 - q_{13}}]$ **:** Instead of any downstream firm's capacity being restrictive, it is now the upstream firm that is incapable of providing Cournot quantities of input:

$$q_{13} = \frac{1}{\eta}(\zeta - \frac{c}{\gamma}) - \gamma k_3 q_{23} \qquad\qquad = 2\gamma k_3 - \frac{1}{\eta}(\zeta - \frac{c}{\gamma})$$

- $[\boldsymbol{q_{13} = k_1};\ \boldsymbol{q_{23} = k_2}]$ **:** In this trivial case, both downstream firms are constrained by their own capacity levels.

- $[\boldsymbol{q_{13} = k_1};\ \boldsymbol{q_{23} = \gamma k_3 - q_{13}}]$ **:** A downstream and an upstream restriction are activated leading to:

$$q_{23} = \gamma k_3 - k_1$$

- $[\boldsymbol{q_{13} + q_{23} = \gamma k_3}]$ **:** As discussed above, there may be multiple equilibria. However, if there are, they must always satisfy this equation, yielding constant industry profits.

Ignoring symmetric cases, these seven types of equilibrium summarize all possibilities. A computer algorithm for finding equilibrium industry profits would verify the consistency of each case by replacing the quantities in the optimality conditions until obtaining a coincidence. Since industry payoffs are uniquely determined, no further search is required. Because the last case has undetermined production levels, it could be harder to verify. However, knowing that there is always an equilibrium, it is possible to check the consistency of all other cases first. If none of them work, it has to be true that the last case is the equilibrium. Then, one simply replaces $q_{13} + q_{23}$ by k_3 in the industry profits definition (see equation 12).

6 One Buyer and Two Suppliers

In this section I explore the inverse industry structure. There will be one downstream firm, $D = \{1\}$, and two upstream firms, $U = \{3, 4\}$. The largest network they can form corresponds to the graph $\hat{g} = \{(1, 3), (1, 4)\}$. Breaking a link would lead to a bilateral monopoly, and breaking both, to the empty network. I will only discuss the case of two links; the bilateral monopoly can be treated the same way as in Section 5.2.

6.1 Two Links

The optimization problem for link $(1, 3)$ is the following:

$$\begin{aligned}
&\max_{q_{13}, z_{13}} (\zeta - \eta(q_{13} + q_{14}))(q_{13} + q_{14}) - cz_{13} \\
&\quad \text{s.t.} \\
&\qquad q_{13} + q_{14} \leq k_1 \\
&\qquad z_{13} \leq k_3 \\
&\qquad q_{13} + q_{14} \leq \gamma(z_{13} + z_{14}) \\
&\qquad q_{13}, z_{13} \geq 0
\end{aligned}$$

Since $c > 0$, for any feasible levels of q_{13}, q_{14} and z_{14}, the third restriction will be active. Therefore, we can replace z_{13} in the objective function to obtain the following expression:

$$(\zeta - \eta(q_{13} + q_{14}))(q_{13} + q_{14}) - c(\frac{q_{13} + q_{14}}{\gamma} - z_{14})$$

The SOC is the same as in the general case (see equation 10), and the FOC yields:

$$q_{13} = \frac{1}{2\eta}(\zeta - \frac{c}{\gamma}) - q_{14}$$

Notice that the equation corresponds to the static monopoly production level, and not the Cournot level. The externalities between downstream firms are internalized. Using the third restriction again, one gets:

$$z_{13} = \frac{1}{2\eta\gamma}(\zeta - \frac{c}{\gamma}) - z_{14}$$

Rewriting the capacity restrictions in terms of input quantities, we get:

$$z_{13} = \max\{0, \min\{\frac{1}{2\eta\gamma}(\zeta - \frac{c}{\gamma}) - z_{14}, \frac{k_1}{\gamma} - z_{14}, k_3\}\}$$

By symmetry,

$$z_{14} = \max\{0, \min\{\frac{1}{2\eta\gamma}(\zeta - \frac{c}{\gamma}) - z_{13}, \frac{k_1}{\gamma} - z_{14}, k_4\}\}$$

These two equations form a system. It is possible to show that it has either one solution or an interval of solutions satisfying $z_{13} + z_{14} = \min\{\frac{1}{2\eta\gamma}(\zeta - \frac{c}{\gamma}), \frac{k_1}{\gamma}\}$. In fact, the first equation is a function of z_{14} with slope between 0 and -1. The second equation, when inverted, has slopes between -1 and $-\infty$, that is, slopes less or equal than the first equation. Therefore, the conditions for applying the same argument from Section 5.1.1 are met. Recalling that $z_{13} + z_{14} = (q_{13} + q_{14})/\gamma$, it is easy to see that total industry payoffs will be uniquely determined. The only additional restriction to be kept in mind is $q_{13}, q_{14} \geq 0$. Since the system simultaneously guarantees that $z_{13} + z_{14} \leq \frac{k_1}{\gamma}$ and $z_{13}, z_{14} \geq 0$, it will always be possible to take $q_{13}, q_{14} \geq 0$ such that $z_{13} + z_{14} = (q_{13} + q_{14})/\gamma$.

6.2 Equilibrium Production Levels

The system of equations can be expressed in the following way:

$$z_{13} = \max\{0, \min\{\Theta - z_{14}, k_3\}\}$$
$$z_{14} = \max\{0, \min\{\Theta - z_{14}, k_4\}\}$$

where $\Theta = \min\{\frac{1}{2\eta\gamma}(\zeta - \frac{c}{\gamma}), \frac{k_1}{\gamma}\}$. The different possibilities for equilibrium production levels are considered next, ignoring the symmetric cases.

- $[\boldsymbol{z_{13} = 0}]$ **:** In this case, $z_{14} = \min\{\Theta, k_4\}$. Notice that the $\max\{\cdot\}$ can be omitted because all expression inside the $\min\{\cdot\}$ are non-negative.
- $[\boldsymbol{z_{13} + z_{14} = \Theta}]$ **:** This is the case of multiple equilibria. They always add up to the same amount, so industry profits remain constant.
- $[\boldsymbol{z_{13} = \Theta - z_{14}; z_{14} = k_4}]$ **:** This leads to $z_{13} = \Theta - k_4$.
- $[\boldsymbol{z_{13} = k_3; z_{14} = k_4}]$ **:** Both upstream firms are constrained by their capacities.

In other words, the downstream firms produces the monopoly level unless it is restricted by its own capacity or that of its suppliers.

7 Preliminary Numerical Results

In this section I numerically solve the model for the case of two buyers and one seller and compare the results to those obtained for a quantity duopoly by Besanko & Doraszelski. I choose the same specification for the parameters, which is the following:

- $\beta_i = 1/1.05$, for all $i \in I$. This discount factor can be interpreted as equivalent to an interest rate of 5%.

- $c = 0$, that is, I normalized marginal input production costs to zero.[14]
- $\gamma = 1$, which means that one unit of the intermediate good results in one unit of the final good.
- $K_i = \{0, 5, 10, ..., 45\}$ and $\varkappa_i = 10$, for all $i \in I$. That is, the state space corresponds to 9 equally sized production capacity blocks equivalent to five units of final or intermediate good, depending on the case. Since $\gamma = 1$, the unit of account is the same.
- $\zeta = 4$ and $\eta = 0.1$. This, together with the assumption of zero marginal costs, implies that the monopoly production level corresponds to $q = 20$ (4 blocks) and the total Cournot level to $q = 26.\bar{6}$ (between 2 and 3 blocks each buyer). Notice that a single firm is potentially able to serve the market, since $k_{10} = 45 > 40 = \zeta/\eta$, the demand level for $p = 0$.
- $\delta_i = \delta$, for all $i \in I$. I will assume the same depreciation rate for all firms, and I will solve the MPE for different values of δ.
- $\theta_i = \theta$, for all $i \in I$. I will assume the same effectiveness of investment for all firms. To partially neutralize the effect of changes in δ on the probability of accumulating a capacity block, thereby isolating the incentive for maintenance investment, I will follow the same procedure as Besanko & Doraszelski. It consists of choosing θ so that the probability of accumulating an additional block given an investment of $x = 20$ remains constant at a level of 0.5:

$$\theta = \frac{0.5}{(1 - \delta - 0.5)20}$$

For all the cases examined next, I verified that the state space was not restrictive by checking that $x_i(k_i, k_{-i}) = 0$ for all $i \in I$ and for all $k_i = 9$. In all cases it was true. It was also true that the MPE's obtained had a unique invariant distribution. In fact, the transition matrices had only one eigenvalue equal to 1. When that is the case, there is only one probability distribution which, when post-multiplied by the corresponding transition matrix, yields the same distribution over next-period states. It can be interpreted as a steady state distribution, because it represents the percentage of time the supply-chain will spend on a given state when looking at an infinite sample of periods.

To make a graphic analysis of the invariant distribution, I obtained the marginal distribution of the seller's capacity level by summing over the buyers' capacity levels. I also obtained the marginal joint distribution of the buyers' capacity levels by summing over the seller's capacity level. The results are displayed in Figure 1. Summary statistics are provided in Table 1, including the mean, median and inter-quartile range (IQR) of relevant variables. I also included a measure co-movement between the capacity levels of both downstream firms, k_1 and k_2.

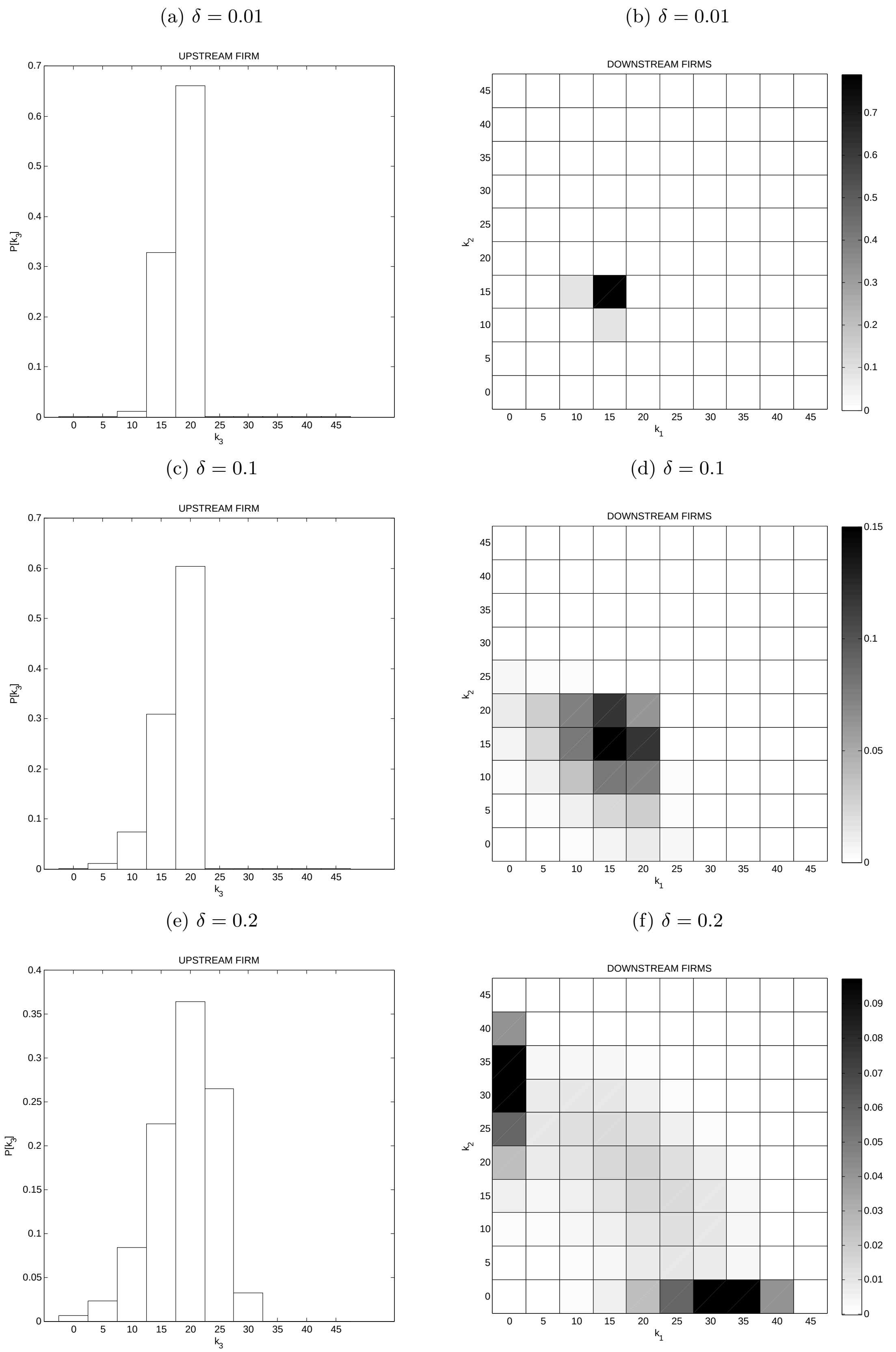
Before discussing the results, I state the main conclusions from Besanko & Doraszelski (2004):

- when firms compete in quantities, the size-distribution of firms is symmetric no matter what the value of δ is. If it takes the value of zero, firms invest until they accumulate Cournot levels of capacity (state $(4, 4)$, precisely). If it is positive, they begin investing for maintenance or precautionary motives, leading to idle capacity in equilibrium.
- when firms compete in prices, the size-distribution of firms is asymmetric, and increasingly so when investment is more reversible, as measured by a higher δ.

The distributions found in this paper (see Figure 1) show the interesting results that asymmetries can arise in a quantity competition setting, and they behave in a similar way as in the price competition case, meaning that they are exacerbated when δ increases. Moreover, the mechanism behind this results are quite different. The asymmetries arise when one of the links between buyer and seller becomes unstable. In fact, for 73% of the states, the complete network (that is, the one that includes both sellers) is the only stable network;

[14] This normalization can be used assuming that the behavior of the supplier is as if $c > 0$, meaning that it prefers to produce less than more, everything else constant.

Figure 1: Marginal Invariant Distributions of Capacity Levels



(a) $\delta = 0.01$ (b) $\delta = 0.01$

(c) $\delta = 0.1$ (d) $\delta = 0.1$

(e) $\delta = 0.2$ (f) $\delta = 0.2$

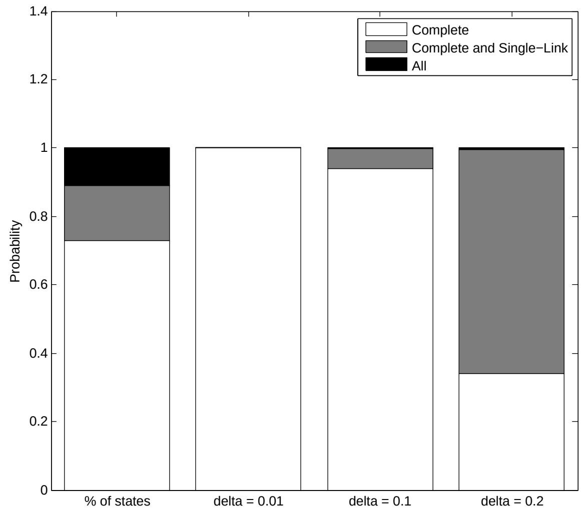
Table 1: Summary Statistics

		$\delta = 0.01$	
Variable	$\mathbb{E}[\cdot]$	Median	IQR
q, z	18.2422	20.0000	[15.0000,20.0000]
k_1	14.4293	15.0000	[15.0000,15.0000]
k_3	18.2451	20.0000	[15.0000,20.0000]
$k_1 \cdot k_2$	208.0825	-	-
p^*	2.1758	3.0000	[3.0000,3.0000]
T/z^{**}	1.4067	1.2917	[1.2917,1.6667]
π_1	6.7126	7.0833	[6.2500,7.0833]
π_3	24.9740	25.8333	[24.6016,25.8333]
x_1	0.2181	0.0000	[0.0000,0.0000]
x_3	0.2249	0.0000	[0.0000,0.3984]
		$\delta = 0.1$	
Variable	$\mathbb{E}[\cdot]$	Median	IQR
q, z	17.3459	20.0000	[15.0000,20.0000]
k_1	14.4462	15.0000	[10.0000,20.0000]
k_3	17.5172	20.0000	[15.0000,20.0000]
$k_1 \cdot k_2$	201.4364	-	-
p^*	2.2654	3.0000	[2.5000,3.0000]
T/z^{**}	1.4160	1.3125	[1.2500,1.5833]
π_1	5.2243	5.4618	[1.9794,7.5000]
π_3	21.5572	22.9167	[19.4927,25.8333]
x_1	1.9027	1.6082	[0.0000,3.4005]
x_3	2.2261	0.0000	[0.0000,4.4498]
		$\delta = 0.2$	
Variable	$\mathbb{E}[\cdot]$	Median	IQR
q, z	17.8641	20.0000	[15.0000,20.0000]
k_1	16.4611	20.0000	[0.0000,30.0000]
k_3	19.2197	20.0000	[15.0000,25.0000]
$k_1 \cdot k_2$	108.7674	-	-
p^*	2.2136	2.5000	[2.0000,4.0000]
T/z^{**}	1.2023	1.0333	[1.0000,1.3125]
π_1	6.3339	3.8462	[-0.0104,14.4533]
π_3	17.0085	17.4806	[12.9009,20.0000]
x_1	2.1759	1.5280	[0.2032,3.4502]
x_3	3.5561	2.6012	[0.8125,5.8703]

* Marginal cost $c = 0$.

** Distribution conditional on the variable being defined.

Figure 2: Distribution of Stable Networks



for 16%, both the complete and some single-link network are stable, and for the remaining percentage, all networks are stable. However, when considering the distribution over states, one can see that for $\delta = 0.01$, the complete network has a probability close to 100%, but when $\delta = 0.2$, it has only a probability of 34%, shifting almost all the probability mass towards the cases where single-link networks are stable.

Figure 2 illustrates the interaction between the investment game and the network formation game, which is what ultimately gives rise to the asymmetries. The first bar shows the percentage of states that fall under each of the three categories. These percentages can be interpreted as the probability of each category under a uniform distribution over states. When this distribution is endogenized through the dynamic investment process, one obtains the three remaining bars, depending on the depreciation rate.

Examining the marginal distributions in more detail shows that downstream firms have an incentive to over-invest, whether the depreciation rate is high or low. Consider first the unimodal or symmetric distributions. Both of them have modal capacities of 3 blocks, which is similar to what Besanko & Doraszelski found for low δ under quantity competition. In their case, this represented no over-investment compared to the static equilibrium output levels, which were Cournot. However, in our case, it does represent over-investment, because the supplier never has enough capacity to produce Cournot levels of inputs. In fact, it appears as if the supplier would be restricting its own production capacity, thereby reducing the output of downstream firms, in order to achieve the monopoly production level.

In the case of the bimodal or asymmetric distribution, the same pattern can be observed: upstream modal capacity is equivalent to the monopoly level and downstream modal capacity is equivalent to the Cournot level. This seems counterintuitive, because in the modal states only one downstream firm has non-zero production capacity. It appears that the possibility of loosing the advantage over the excluded firm prevents the remaining firm from reducing its capacity to the monopoly level.

8 Conclusions

I have characterized in detail a network-bargaining game based on de Fontenay & Gans (2005, 2013) that can be plugged into the dynamic framework of Besanko & Doraszelski (2004). I have also developed two particular applications and demonstrated that they yield unique payoffs. The preliminary results obtained from the numerical solutions are partially different from the ones obtained by Besanko & Doraszelski, and they are the outcome of a different mechanism that involves the formation of supply relationships. In particular, firm-size distribution can be asymmetric under quantity competition, meaning that aggressive investment policies can lead to a permanent advantage over the competitor, excluding it from the supply-network. Furthermore, firms tend to over-invest and hold capacity in excess to that of their supplier, a behavior which is probably motivated by the possibility of achieving that advantage. I also find that increases in investment reversibility lead to greater asymmetries, as in the price competition case in Besanko & Doraszelski (2004).

References

Abreu, D., Pearce, D. & Stacchetti, E. (1986), 'Optimal Cartel Equilibria with Imperfect Monitoring', *Journal of Economic Theory* **39**(1), 251 – 269.
URL: *http://www.sciencedirect.com/science/article/pii/0022053186900281*

Abreu, D., Pearce, D. & Stacchetti, E. (1990), 'Toward a Theory of Discounted Repeated Games with Imperfect Monitoring', *Econometrica* **58**(5), pp. 1041–1063.
URL: *http://www.jstor.org/stable/2938299*

Besanko, D. & Doraszelski, U. (2004), 'Capacity Dynamics and Endogenous Asymmetries in Firm Size', *The RAND Journal of Economics* **35**(1), pp. 23–49.
URL: *http://www.jstor.org/stable/1593728*

Besanko, D., Doraszelski, U., Kryukov, Y. & Satterthwaite, M. (2010), 'Learning-by-Doing, Organizational Forgetting, and Industry Dynamics', *Econometrica* **78**(2), pp. 453–508.
URL: *http://www.jstor.org/stable/40664481*

Binmore, K., Rubinstein, A. & Wolinsky, A. (1986), 'The Nash Bargaining Solution in Economic Modelling', *The RAND Journal of Economics* **17**(2), pp. 176–188.
URL: *http://www.jstor.org/stable/2555382*

Bowley, A. L. (1928), 'Bilateral Monopoly', *The Economic Journal* **38**(152), pp. 651–659.
URL: *http://www.jstor.org/stable/2224123*

de Fontenay, C. C. & Gans, J. S. (2005), 'Vertical Integration in the Presence of Upstream Competition', *The RAND Journal of Economics* **36**(3), pp. 544–572.
URL: *http://www.jstor.org/stable/4135229*

de Fontenay, C. C. & Gans, J. S. (2013), 'Bilateral Bargaining with Externalities', *Melbourne Business School Discussion Paper No. 2004-32.* .
URL: *http://ssrn.com/abstract=59168, http://dx.doi.org/10.2139/ssrn.591688*

Doraszelski, U. & Pakes, A. (2007), 'A Framework for Applied Dynamic Analysis in IO', *Handbook of Industrial Organization* **3**, 1887–1966.

Doraszelski, U. & Satterthwaite, M. (2010), 'Computable Markov-Perfect Industry Dynamics', *The RAND Journal of Economics* **41**(2), pp. 215–243.
URL: *http://www.jstor.org/stable/40649298*

Douven, R., Halbersma, R., Katona, K. & Shestalova, V. (2011), 'Vertical Integration and Exclusive Vertical Restraints between Insurers and Hospitals', *TILEC Discussion Paper No. 2011-016.* .
URL: *http://ssrn.com/abstract=1781685, http://dx.doi.org/10.2139/ssrn.1781685*

Ericson, R. & Pakes, A. (1995), 'Markov-Perfect Industry Dynamics: A Framework for Empirical Work', *The Review of Economic Studies* **62**(1), 53–82.

Escobar, J. F. (2013), 'Equilibrium Analysis of Dynamic Models of Imperfect Competition', *International Journal of Industrial Organization* **31**(1), 92 – 101.
URL: *http://www.sciencedirect.com/science/article/pii/S0167718712001154*

Fackler, P. L. (2002), *Applied Computational Economics and Finance*, MIT press.

Gilles, R., Chakrabarti, S. & Sarangi, S. (2006), Social Network Formation with Consent: Nash Equilibrium and Pairwise Refinements, Technical report, Working paper, Department of Economics, Virginia Tech, Blacksburg, VA.

Gilles, R. P. & Sarangi, S. (2004), 'Social Network Formation with Consent'.
URL: *http://ssrn.com/abstract=603341, http://dx.doi.org/10.2139/ssrn.603341*

Green, E. J. & Porter, R. H. (1984), 'Noncooperative collusion under imperfect price information', *Econometrica: Journal of the Econometric Society* **52**(1), 87–100.

Grossman, S. J. & Hart, O. D. (1986), 'The costs and benefits of ownership: A theory of vertical and lateral integration', *Journal of Political Economy* **94**(4), pp. 691–719.
URL: *http://www.jstor.org/stable/1833199*

Harsanyi, J. C. & Selten, R. (1972), 'A Generalized Nash Solution for Two-Person Bargaining Games with Incomplete Information', *Management Science* **18**(5), pp. P80–P106.
URL: *http://www.jstor.org/stable/2661446*

Hart, O. & Moore, J. (1990), 'Property Rights and the Nature of the Firm', *Journal of Political Economy* **98**(6), pp. 1119–1158.
URL: *http://www.jstor.org/stable/2937753*

Inderst, R. & Wey, C. (2003), 'Bargaining, Mergers, and Technology Choice in Bilaterally Oligopolistic Industries', *The RAND Journal of Economics* **34**(1), pp. 1–19.
URL: *http://www.jstor.org/stable/3087440*

Jackson, M. O. & Wolinsky, A. (1996), 'A Strategic Model of Social and Economic Networks', *Journal of Economic Theory* **71**(1), 44 – 74.
URL: *http://www.sciencedirect.com/science/article/pii/S0022053196901088*

Judd, K. L., Yeltekin, S. & Conklin, J. (2003), 'Computing Supergame Equilibria', *Econometrica* **71**(4), pp. 1239–1254.
URL: *http://www.jstor.org/stable/1555496*

Kalai, E. (1977), 'Proportional Solutions to Bargaining Situations: Interpersonal Utility Comparisons', *Econometrica* **45**(7), pp. 1623–1630.
URL: *http://www.jstor.org/stable/1913954*

Krishna, V. & Serrano, R. (1996), 'Multilateral Bargaining', *The Review of Economic Studies* **63**(1), pp. 61–80.
URL: *http://www.jstor.org/stable/2298115*

Maskin, E. & Tirole, J. (1988*a*), 'A Theory of Dynamic Oligopoly, I: Overview and Quantity Competition with Large Fixed Costs', *Econometrica* **56**(3), pp. 549–569.
URL: *http://www.jstor.org/stable/1911700*

Maskin, E. & Tirole, J. (1988*b*), 'A Theory of Dynamic Oligopoly, II: Price Competition, Kinked Demand Curves, and Edgeworth Cycles', *Econometrica* **56**(3), pp. 571–599.
URL: *http://www.jstor.org/stable/1911701*

Myerson, R. B. (1977*a*), 'Graphs and Cooperation in Games', *Mathematics of Operations Research* **2**(3), pp. 225–229.
URL: *http://www.jstor.org/stable/3689511*

Myerson, R. B. (1977*b*), 'Values of Games in Partition Function Form', *International Journal of Game Theory* **6**(1), 23–31.
URL: *http://dx.doi.org/10.1007/BF01770871*

Nash, J. (1953), 'Two-Person Cooperative Games', *Econometrica* **21**(1), pp. 128–140.
URL: *http://www.jstor.org/stable/1906951*

Nash, J. F. (1950), 'The Bargaining Problem', *Econometrica* **18**(2), pp. 155–162.
URL: *http://www.jstor.org/stable/1907266*

Navarro, N. (2007), 'Fair Allocation in Networks with Externalities', *Games and Economic Behavior* **58**(2), 354 – 364.
URL: *http://www.sciencedirect.com/science/article/pii/S0899825606000443*

Pakes, A. & McGuire, P. (1994), 'Computing Markov-Perfect Nash Equilibria: Numerical Implications of a Dynamic Differentiated Product Model', *The RAND Journal of Economics* **25**(4), pp. 555–589.
URL: *http://www.jstor.org/stable/2555975*

Rotemberg, J. J. & Saloner, G. (1986), 'A supergame-theoretic model of price wars during booms', *The American Economic Review* **76**(3), 390–407.

Roth, A. E. (1979), 'Proportional Solutions to the Bargaining Problem', *Econometrica* **47**(3), pp. 775–778.
URL: *http://www.jstor.org/stable/1910423*

Rubinstein, A. (1982), 'Perfect Equilibrium in a Bargaining Model', *Econometrica* **50**(1), pp. 97–109.
URL: *http://www.jstor.org/stable/1912531*

Shapley, L. S. (1953*a*), 'Stochastic Games', *Proceedings of the National Academy of Sciences of the United States of America* **39**(10), 1095.

Shapley, L. S. (1953*b*), 'A Value for N-Person Games', *Annals of Mathematical Studies* **28**, pp. 307–317.

Stokey, N., Lucas, R. & Prescott, E. (1989), 'Recursive Methods in Economic Dynamics', *Cambridge MA* .

Thomson, W. (1994), Cooperative Models of Bargaining, *in* R. Aumann & S. Hart, eds, 'Handbook of Game Theory with Economic Applications', 1 edn, Vol. 2, Elsevier, chapter 35, pp. 1237–1284.
URL: *http://EconPapers.repec.org/RePEc:eee:gamchp:2-35*

Thrall, R. M. & Lucas, W. F. (1963), 'N-Person Games in Partition Function Form', *Naval Research Logistics Quarterly* **10**(1), 281–298.
URL: *http://dx.doi.org/10.1002/nav.3800100126*

Winter, E. (2002), The Shapley Value, *in* R. Aumann & S. Hart, eds, 'Handbook of Game Theory with Economic Applications', 1 edn, Vol. 3, Elsevier, chapter 53, pp. 2025–2054.
URL: *http://EconPapers.repec.org/RePEc:eee:gamchp:3-53*