ANALYSIS OF IRREGULARLY SPACED TIME SERIES

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Abstract

In this thesis, we propose novel stationary time series models that can be used when the observations are taken on irregularly spaced times. First, we present a model with a first-order moving average structure, and then we generalized it to consider an autoregressive component. We called the first model irregularly spaced first-order moving average and the second one irregularly spaced first-order autoregressive moving average. Their definitions and properties are established. We present their state-space representations and their one-step linear predictors. The behavior of the maximum likelihood estimator is studied through Monte Carlo experiments. Illustrations are presented with real and simulated data.

Keywords: Moving average; Autoregressive; Time-dependent data; General backward continued fraction.

Dedication

To my wife, with love.

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Chapter 1

Introduction

In statistics, time series analysis establishes a principal tool when we want to study observations that naturally are dependent. For instance, a researcher might be interested in studying the spatial behavior of air carbon monoxide concentration levels in a city, or he might also be interested in understanding the stock price temporal dynamic of a financial asset in a stock market. In both cases, space and time are important factors that induce dependence between observations. The main objective of this analysis is to understand the nature of such dependence.

Nowadays, in order to study time series, there are many methods assume time series are regularly spaced, that is, the interval between observations is constant over time (see, e.g., Brockwell and Davis, 1991; Hamilton, 1994; Box et al., 2016). However, there are few contributions to treat irregularly spaced time series, which are frequently observed in fields such as climatology, economics, finance, astronomy, medical sciences, geophysics, among others. For example, Mudelsee (2014) mentions that conventional time series analysis largely ignored irregularly spaced structures that climate time series has to take into account.

According to Jones (1985), irregularly spaced time series can occur in two different ways. First, data can be regularly spaced with missing observations. Second, data can be truly irregularly spaced with no underlying sampling interval. Techniques considering time series in the presence of missing data are useful in the first case (see, e.g., Parzen,

1963; Jones, 1980; Dunsmuir, 1983; Reinsel and Wincek, 1987). Nevertheless, these techniques can not be applying if data are really irregularly spaced. This case usually has been treated through two approaches. On the one hand, we can transform irregularly spaced time series in regularly spaced time series through interpolation and to use conventional techniques. Adorf (1995) provides a summary of such transformations, which are frequently used to analyze astronomic data. However, interpolation methods typically produce bias (for instance, over smoothing), changing the dynamic of the process. On the other hand, irregularly spaced time series can be treated as discrete realizations of a continuous stochastic process (see, e.g., Robinson, 1977; Parzen, 1984; Thornton and Chambers, 2013), but continuous time series models tend to be complicated (mostly due to the difficulty of estimating and evaluating them from discretely sampled data). Moreover, we need to handle embedding and aliasing problems[1] that are not easy to solve. The emphasis of this approach has been mainly to model autoregressive moving averages process. However, a stationary condition of these models is that autoregressive order must be strictly greater than moving average order. Thus, a stationary continuous-time first-order moving average process or a stationary continuous-time first-order autoregressive moving average process is not viable, at least for a real-valued process (Chan and Tong, 1987). Consequently, in this thesis, we propose two novel models called irregularly spaced first-order moving average and irregularly spaced first-order autoregressive moving average that allow us to treat either moving averages and autoregressive moving averages structures with irregularly spaced times.

In this chapter, we give an account of some of the main concepts in time series, that will be used to set up notation and basic ideas about the stochastic process. We start the discussion introducing concepts such as irregularly spaced stochastic process, distributional and constructionist viewpoint of stochastic process, Gaussian process, strictly and weakly stationary process, ergodicity, and mixing properties. Later, we introduce two models with first-order autoregressive structures. Finally, we present the properties of these models.

[1]Embedding happens when different continuous-time processes can look the same when observed discretely. Aliasing occurs when the variability due to higher frequencies is mapped into an interval defined by the sampling process (Tómasson, 2015).

1.1 Basic ideas and terminology

1.1.1 Irregularly spaced stochastic process

Let $(\Omega, \Im, P)$ be a probability space, $\mathbb{T}$ be any set, and $\mathbb{R}^{\mathbb{T}}$ be the product space generated by taking a copy of $\mathbb{R}$ for each element of $\mathbb{T}$. Davidson (1994) defines a stochastic process as a measure mapping $x\colon \Omega \mapsto \mathbb{R}^{\mathbb{T}}$, where

$$x(\omega) = \{X_\tau(\omega), \tau \in \mathbb{T}\}.$$

$\mathbb{T}$ is called the index set, and the random variable $X_\tau(\omega)$ is called a coordinate of the process or trajectory. In the case where $\mathbb{T}$ is an interval of $\mathbb{R}$ such as $\mathbb{T} = (0,\infty)$ or $\mathbb{T} = (-\infty,\infty)$, x is called a continuous-index stochastic process. On the other hand, when $\mathbb{T}$ is a countable subset of $\mathbb{R}$, x becomes $\{\cdots, X_{t_{-1}}, X_{t_0}, X_{t_1}, \cdots\}$ and it is called a discrete-index stochastic process. Given the equipotency of $\mathbb{Z}$ and $\mathbb{N}$, it will suffice consider only $\{X_{t_n}\}_{n\geq 1}$.

Now, like Robinson (1977), consider $\mathbb{T}' = \{t_1, t_2, t_3, \ldots\}$ as a set such that its consecutive differences $\Delta_{n+1} = t_{n+1} - t_n$, for $n \geq 1$, are strictly positive, uniformly bounded and bounded away from zero. Thus, there are $\Delta_L > 0$ and $\Delta_U < \infty$ such that $\Delta_L \leq \Delta_{n+1} \leq \Delta_U$ for all n. Without loss of generality, we suppose $\Delta_L = 1$ (otherwise, we can rescale each Δ_{n+1} by $\min_n \Delta_{n+1}$). These conditions are compatibles with any physical recorder and determine $\mathbb{T}'$ as a discrete (and therefore countable) subset of $\mathbb{R}$. We shall call $x' = \{X_\tau, \tau \in \mathbb{T}'\}$ an irregularly spaced stochastic process and we assume that the pattern of irregular spacing is independent to the stochastic properties of the process. When $\mathbb{T}'$ is an arithmetic progression[2], with $\Delta_{n+1} = 1$ for $n \geq 1$, the process x' is called a regularly spaced stochastic process and no relevant information is lost by consider $\mathbb{T}' = \mathbb{N}^+$. An irregularly (or unequally or unevenly) spaced time series is a finite realization of an irregularly spaced stochastic process.

[2] An arithmetic progression is a sequence of numbers such that the difference of any two successive members is a constant.

1.1.2 Two ways to describe a stochastic process

Consider any n element $\tau_1, \tau_2, \ldots, \tau_n$ in $\mathbb{T}$. According to Parzen (1962), one way of describing a stochastic process, x, is to specify the joint distribution of $\{X_{\tau_1}, \ldots, X_{\tau_n}\}$ for all n. On the other hand, another way of describing a stochastic process is to give a formula for the value X_τ of the process at each point τ in terms of a family of random variables whose probability law is known. Viewing a stochastic process via the joint distribution of a finite subset of the components is called the distributional viewpoint. On the other hand, viewing a stochastic process as functions of other processes (often simpler process like a family of iid random variables) is called the constructionist viewpoint (Spanos, 1999).

1.1.3 Gaussian process and their existence

Following to Azencott and Dacunha-Castelle (1986), a stochastic process x is called Gaussian if for every finite subset $\{\tau_1, \tau_2, \ldots, \tau_n\}$ of $\mathbb{T}$ the joint distribution of $\{X_{\tau_1}, X_{\tau_2}, \ldots, X_{\tau_n}\}$ is Gaussian. The following Proposition was taken from Azencott and Dacunha-Castelle (1986, Proposition 2.4.2, pp. 15).

Proposition 1. *Let $m : \mathbb{T} \to \mathbb{R}$ be an arbitrary function, and let $\Gamma : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ be a function satisfying the following two conditions. First, $\Gamma(\tau, \iota) = \Gamma(\iota, \tau)$ for any $\tau, \iota \in \mathbb{T}$. Second, if $\{\tau_1, \tau_2, \ldots, \tau_n\}$ is any finite $\mathbb{T}$ set, the matrix $[\Gamma(\tau_i, \tau_j)]$ is nonnegative definite. Then there is a real-valued Gaussian process $x = \{X_\tau, \tau \in \mathbb{T}\}$ with mean m and covariance Γ. This process is unique up to equivalence.*

1.1.4 Strictly and weakly stationary process

If the probabilistic structure of a process is invariant under a shift of the time origin, then the process is strictly stationary (Priestley, 1981). So, for any set $\{\tau_1, \tau_2, \ldots, \tau_n\}$ of $\mathbb{T}$, the joint distribution of $\{X_{\tau_1}, X_{\tau_2}, \ldots, X_{\tau_n}\}$ must remain unaltered if we shift each element by the same amount. The process x is said to be weakly stationary if, for any subset $\{\tau_1, \tau_2, \ldots, \tau_n\}$ of $\mathbb{T}$, all the joint moments up to order 2 of $\{X_{\tau_1}, X_{\tau_2}, \ldots, X_{\tau_n}\}$ exist and remain invariant under a shift. Thus, a Gaussian process is a strictly stationary process if it is a weakly stationary processs (Cramér and Leadbetter, 1967).

1.1.5 Ergodicity and mixing

Following Spanos (1999), ergodicity refers to the property of a stationary stochastic process, which will enable us to use a single trajectory in order to estimate reliably the moments of the distribution underlying the stochastic process in question. According to Davidson (1994), mixing means that its random variables with indices far apart are almost independent.

The following results were taken from Stout (1974).

Lemma 2 (Lemma 3.5.8, pp. 182)**.** *Let $\{X_{t_n}\}_{n\geq 1}$ be independent identically distributed. Then, $\{X_{t_n}\}_{n\geq 1}$ is stationary ergodic.*

Theorem 3 (Theorem 3.5.8, pp. 182)**.** *Let $\{X_{t_n}\}_{n\geq 1}$ be stationary ergodic and f be a mesurable function $f:\mathbb{R}^{\infty}\to\mathbb{R}$. Let $Y_{t_n}=f(X_{t_n},X_{t_{n+1}},\ldots)$ define $\{Y_{t_n}\}_{n\geq 1}$. Then, $\{Y_{t_n}\}_{n\geq 1}$ is stationary ergodic.*

On the other hand, according to Spanos (1999), the mixing (strong) condition are stronger than ergodicity in the sense that when we impose stationarity on a stochastic process, then the mixing condition implies ergodicity. Following Hannan (1970), a sequence of independent identically distributed random variables with mean zero and finite variance is mixing. Also, we have the following result from the same author.

Theorem 4 (Theorem 3, pp. 204)**.** *If $\{X_{t_n}\}_{n\in\mathbb{Z}}$ is given by*

$$X_{t_n}=\sum_{j=-\infty}^{\infty}a_jY_{t_{n-j}},\qquad \sum_{j=-\infty}^{\infty}a_j^2<\infty,$$

where $\{Y_{t_n}\}_{n\in\mathbb{Z}}$ are independent and identically distributed then $\{X_{t_n}\}_{n\in\mathbb{Z}}$ is mixing and therefore ergodic.

1.2 An irregularly spaced first-order autoregressive process

Robinson (1977) considers the continuous-time first-order autoregressive model sampled at known discrete times. Let $\{\varepsilon_{t_n}\}_{n\in\mathbb{Z}}$ be independent random variables each

$\mathrm{N}(0, \sigma_0^2 \phi(\alpha_0; \Delta_n))$ with $\sigma_0^2 > 0$, $\alpha_0 < 0$, $\Delta_n = t_n - t_{n-1}$ and $\phi(\alpha; \Delta) = {}^{(\mathrm{e}^{2\alpha\Delta}-1)}/_{2\alpha}$. Then, the discrete-time model of Robinson is

$$X_{t_n} = \mathrm{e}^{\alpha_0 \Delta_n} X_{t_{n-1}} + \varepsilon_{t_n}.$$

This model may be thought of as a first-order autoregression with time-varying coefficient and heteroscedastic errors. It is essential to stand out that the only solution of this model for which $\mathrm{E}(X_{t_n}^2) < \infty$, all n, is

$$X_{t_n} = \sum_{j=0}^{\infty} \mathrm{e}^{\alpha_0 (t_n - t_{n-j})} \varepsilon_{t_{n-j}},$$

and the covariance is

$$\mathrm{E}[X_{t_m} X_{t_n}] = -\frac{\sigma_0^2}{2\alpha_0} \mathrm{e}^{\alpha_0 (t_n - t_m)}, \quad m \leq n.$$

Moreover, let X_τ be observed at points $t_0, t_1, \ldots, t_N$. The log-likelihood is

$$-\frac{1}{2} N \log 2\pi\sigma^2 - \frac{1}{2} \sum_{n=1}^{N} \log \phi(\alpha; \Delta_n) - \frac{1}{2} \sum_{n=1}^{N} \frac{(x_{t_n} - \mathrm{e}^{\alpha \Delta_n} x_{t_{n-1}})^2}{\sigma^2 \phi(\alpha; \Delta_n)},$$

where α and σ^2 are any admissible parameter values and the concentrated criterion function is

$$q_N(\alpha) = \log \hat{\sigma}_N^2(\alpha) + \frac{1}{N} \sum_{n=1}^{N} \log \phi(\alpha; \Delta_n),$$

with

$$\hat{\sigma}_N^2(\alpha) = \frac{1}{N} \sum_{n=1}^{N} \frac{(x_{t_n} - \mathrm{e}^{\alpha \Delta_n} x_{t_{n-1}})^2}{\phi(\alpha; \Delta_n)}.$$

The maximun likelihood estimate of α_0, $\hat{\alpha}_N$, is the value minimizing $q_N(\alpha)$ and the estimate of σ_0^2 is $\hat{\sigma}_N^2 = \hat{\sigma}_N^2(\hat{\alpha}_N)$. Robinson proves the strong law of large numbers and central limit theorem for $\hat{\alpha}_N$ and $\hat{\sigma}_N^2$. Consider the following conditions

- $q_N(\alpha)$ is minimized over a closed interval $\mathscr{A} = [\alpha_L, \alpha_U]$, $\alpha_L > -\infty$, $\alpha_U < \infty$ and $\alpha_0 \in \mathscr{A}$.
- $\Delta_L > 0$, $\Delta_U < \infty$, and $\Delta_L \leq \Delta_n \leq \Delta_U$ for all $-\infty < n < \infty$.
- For

$$\tilde{a}_N(\alpha) = \frac{1}{N}\sum_{n=1}^{N} \log \frac{\phi(\alpha_0;\Delta_n)}{\phi(\alpha;\Delta_n)}, \qquad \tilde{b}_N(\alpha) = \frac{\sigma_0^2}{N}\sum_{n=1}^{N} \frac{\phi(\alpha_0;\Delta_n)}{\phi(\alpha;\Delta_n)},$$
$$\tilde{c}_N(\alpha) = \frac{\sigma_0^2}{-2\alpha_0 N}\sum_{n=1}^{N} \frac{\zeta(\alpha;\Delta_n)^2}{\phi(\alpha;\Delta_n)}, \quad \zeta(\alpha;\Delta) = \mathrm{e}^{\alpha_0\Delta} - \mathrm{e}^{\alpha\Delta},$$

  the limits

$$\lim_{N\to\infty} \tilde{a}_N(\alpha) = a(\alpha), \quad \lim_{N\to\infty} \tilde{b}_N(\alpha) = b(\alpha), \quad \lim_{N\to\infty} \tilde{c}_N(\alpha) = c(\alpha), \tag{1.1}$$

  exist, and the convergence is uniform over $\mathscr{A}$.

Under these conditions, Robinson proof that

$$(\hat{\alpha}_N, \hat{\sigma}_N^2) \to (\alpha_0, \sigma_0^2) \text{ a.s.}$$

Next, consider the additional conditions

- $\alpha_0 \neq \alpha_L, \alpha_U$.

- For,

$$\tilde{\omega}_N(\alpha) = \frac{1}{N}\sum_{n=1}^{N}\frac{\partial \log \phi(\alpha;\Delta_n)}{\partial \alpha}, \qquad \tilde{\chi}_N(\alpha) = \frac{1}{N}\sum_{n=1}^{N}\left(\frac{\partial \log \phi(\alpha;\Delta_n)}{\partial \alpha}\right)^2,$$
$$\tilde{\psi}_N(\alpha) = \frac{1}{N}\sum_{n=1}^{N}\frac{\Delta_n^2 e^{2\alpha\Delta_n}}{\phi(\alpha;\Delta_n)},$$

  the limits

$$\lim_{N\to\infty}\tilde{\omega}_N(\alpha) = \omega(\alpha), \quad \lim_{N\to\infty}\tilde{\chi}_N(\alpha) = \chi(\alpha), \quad \lim_{N\to\infty}\tilde{\psi}_N(\alpha) = \psi(\alpha),$$

  exist, and the convergence is uniform in some neighborhood of α_0.

Under the above conditions, Robinson proof that

$$N^{1/2}\left(\begin{bmatrix}\hat{\alpha}_N\\ \hat{\sigma}_N^2\end{bmatrix} - \begin{bmatrix}\alpha_0\\ \sigma_0^2\end{bmatrix}\right) \xrightarrow{\mathrm{d}} \mathrm{Z},$$

where

$$\mathrm{Z} \sim \mathrm{N}\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}2\lambda(\alpha_0)^{-1} & 2\sigma_0^2\omega(\alpha_0)\lambda(\alpha_0)^{-1}\\ 2\sigma_0^2\omega(\alpha_0)\lambda(\alpha_0)^{-1} & 2\sigma_0^4(\chi(\alpha_0) - \psi(\alpha_0)/\alpha_0)\lambda(\alpha_0)^{-1}\end{bmatrix}\right)$$

with $\lambda(\alpha) = \chi(\alpha) - \omega(\alpha)^2 - \psi(\alpha)/\alpha$.

Later, Eyheramendy et al. (2018) introduce the so-called irregular autoregressive (IAR) model that allows for Gaussian and non-Gaussian- distributed data, leading to increase flexibility. This model is defined by

$$X_{t_n} = \phi^{t_n - t_{n-1}} X_{t_{n-1}} + \sigma\sqrt{1-\phi^{2(t_n-t_{n-1})}}\,\zeta_{t_n} \quad \text{for} \quad n \in \mathbb{Z},$$

where $0 < \phi < 1$, $\sigma^2 > 0$ and $\{\zeta_{t_n}\}_{n\in\mathbb{Z}}$ is a white noise sequence with zero mean and

unit variance. Note that $\mathrm{E}(X_{t_n}) = 0$ and $\mathrm{Var}(X_{t_n}) = \sigma^2$ for all n. Further, the autocovariance function is $\mathrm{Cov}(X_{t_k}, X_{t_j}) = \sigma^2 \phi^{t_k - t_j}$ for $k \geq j$. Hence, the IAR process correspond to weakly stationary process. Also, if $\{\zeta_{t_n}\}_{n \in \mathbb{Z}}$ is an independent and identically distributed sequence of random variables with mean zero and unit variance and if $t_n - t_{n-j} \geq \mathrm{C} \log j$, as $j \to \infty$, where C is a positive constant that satisfies $\mathrm{C} \log \phi^2 < -1$, then there is a solution to the IAR process and the sequence is strictly stationary and ergodic.

Chapter 2

An irregularly spaced first-order moving average process

A model for irregularly spaced time series can take place in many areas. Often, when we have an irregularly spaced time series, we treat them as a discrete realization of a continuous stochastic process. However, continuous time series models tend to be complicated (mostly due to the difficulty of estimating and evaluating them from discretely sampled data). Here, we need to handle embedding and aliasing problems that are not easy to solve. Additionally, the emphasis of this approach has been mainly to model autoregressive moving averages process. Nevertheless, a stationary condition of these processes is that autoregressive order must be strictly greater than moving average order. Then, a stationary continuous-time first-order moving average process is not viable, at least for a real-valued process (Chan and Tong, 1987). In this chapter, we propose a novel model called an irregularly spaced first-order moving average model that allows us to treat moving averages structures with irregularly spaced times.

The remainder of the Chapter 2 is organized as follows. In Section 2.2, we present a novel class of stochastic processes called irregularly spaced first-order moving average model for the treatment and analysis of unevenly spaced time series. In Section 2.3, we give the state-space representation of the model. In Section 2.4 we provide the one-step linear predictors and the mean squared errors. The maximum likelihood and bootstrap estimation methods are introduced in Sections 2.5 and 2.6, respectively. We study the behavior of the

maximum likelihood and bootstrap estimators via Monte Carlo in Section 2.7. Finally, we present a medical illustration in Section 2.8.

2.1 Introduction

In this section, we would like to build a stationary stochastic process having a moving average structure that allows us to consider irregularly spaced times. We suppose that the pattern of irregular spacing is independent of the stochastic process properties.

Consider the Proposition 1 and let $\mathbb{T}' = \{t_1, t_2, t_3, \ldots\}$ be a set such that its consecutive differences $\Delta_{n+1} = t_{n+1} - t_n$, for $n \geq 1$, are uniformly bounded and bounded away from zero. Now, let $m : \mathbb{T}' \to \mathbb{R}$ be a function such that $m(t_n) = 0$, for any $t_n \in \mathbb{T}'$. Next, let $\Gamma : \mathbb{T}' \times \mathbb{T}' \to \mathbb{R}$ be a function such that, for any $t_n, t_s \in \mathbb{T}'$,

$$\Gamma(t_n, t_s) = \begin{cases} \gamma_0, & |n-s| = 0, \\ \gamma_{1,\Delta_{\max\{n,s\}}}, & |n-s| = 1, \\ 0, & |n-s| \geq 1. \end{cases}$$

Note that Γ can be represented by an infinite real tridiagonal matrix as

$$\Gamma = \begin{bmatrix} \gamma_0 & \gamma_{1,\Delta_2} & 0 & 0 & \cdots \\ \gamma_{1,\Delta_2} & \gamma_0 & \gamma_{1,\Delta_3} & 0 & \\ 0 & \gamma_{1,\Delta_3} & \gamma_0 & \gamma_{1,\Delta_4} & \\ 0 & 0 & \gamma_{1,\Delta_4} & \gamma_0 & \\ \vdots & & & & \ddots \end{bmatrix}. \tag{2.1}$$

Let Γ_n be the $n \times n$ truncation of Γ and assume $\gamma_{1,\Delta_j} \neq 0$, for $j = 2, \ldots, n$. From Appendix A.1, Γ_n is positive definite if $\gamma_0 > 0$ and $(\gamma_{1,\Delta_j}/\gamma_0)^2 \leq 1/4$ for $j = 2, \ldots, n$. Thus, if

$$\gamma_0 > 0 \quad \text{and} \quad \left(\frac{\gamma_{1,\Delta_{n+1}}}{\gamma_0}\right)^2 \leq 1/4, \text{ for } n \geq 1, \text{ with } \gamma_{1,\Delta_{n+1}} \neq 0, \tag{2.2}$$

then there is a stationary Gaussian process $\{X_{t_n}, t_n \in \mathbb{T}'\}$, unique up to equivalence, with mean 0 and covariance Γ. We called this process an irregularly spaced first-order moving average process of general form. In the following section, we are going to give particular expressions to γ_0 and $\gamma_{1,\Delta_{n+1}}$ such that satisfying (2.2). The goal is to obtain a stationary irregularly spaced stochastic process for which we can get the conventional first-order moving average process when $\Delta_{n+1} = 1$ for all $n \geq 1$.

2.2 An Irregular spaced first-order Moving Average model

Next, we define a novel class of stochastic process called Irregularly spaced first-order Moving Average (IMA) process. The definition is made from either a distributional and constructionist viewpoint. Later, we present process properties.

2.2.1 Distributional viewpoint

In (2.1), γ_0 and $\gamma_{1,\Delta_{n+1}}$, for $n \geq 1$, represent variance and first-order covariances, respectively. In order to satisfy (2.2), we define the variance as $\gamma_0 = \sigma^2(1+\theta^2)$ and the first-order covariances as $\gamma_{1,\Delta_{n+1}} = \sigma^2\theta^{\Delta_{n+1}}$, where $\sigma^2 > 0$ and $0 < \theta < 1$. Hence, we obtain a particular stationary irregularly spaced stochastic process with covariance matrix

$$\Gamma = \sigma^2 \begin{bmatrix} 1+\theta^2 & \theta^{\Delta_2} & 0 & 0 & \cdots \\ \theta^{\Delta_2} & 1+\theta^2 & \theta^{\Delta_3} & 0 & \\ 0 & \theta^{\Delta_3} & 1+\theta^2 & \theta^{\Delta_4} & \\ 0 & 0 & \theta^{\Delta_4} & 1+\theta^2 & \\ \vdots & & & & \ddots \end{bmatrix} \tag{2.3}$$

which contains the conventional first-order moving average process as a special case. We call this Gaussian process an irregularly spaced first-order moving average process.

Definition 5 (IMA–distributional viewpoint)**.** Let $\mathbb{T}' = \{t_1, t_2, t_3, \ldots\}$ be a set such that its consecutive differences $\Delta_{n+1} = t_{n+1} - t_n$, for $n \geq 1$, are uniformly bounded and bounded away from zero. The IMA process $\{X_{t_n}, t_n \in \mathbb{T}'\}$ is defined as a Gaussian process with mean 0 and covariance (2.3) with $\sigma^2 > 0$ and $0 < \theta < 1$. We say that $\{X_{t_n}, t_n \in \mathbb{T}'\}$ is an IMA process with mean μ if $\{X_{t_n} - \mu, t_n \in \mathbb{T}'\}$ is an IMA process.

2.2.2 Constructionist viewpoint

Now, as is usual, we would like to specify the IMA process as a function of other (often simpler) stochastic processes. This approach is known as a constructionist viewpoint of the process (Spanos, 1999). Appendix A.1 contains details about how we built this model.

Definition 6 (IMA–constructionist viewpoint)**.** Let $\{\varepsilon_{t_n}\}_{n\geq 1}$ be independent random variables that follow a Gaussian distribution $\mathrm{N}(0, \sigma^2 c_n(\theta))$ with $\sigma^2 > 0$, $0 < \theta < 1$, $c_1(\theta) = 1 + \theta^2$ and

$$c_n(\theta) = 1 + \theta^2 - \frac{\theta^{2\Delta_n}}{c_{n-1}(\theta)} \quad \text{for} \quad n \geq 2,$$

where $\Delta_n = t_n - t_{n-1}$. The process $\{X_{t_n}, t_n \in \mathbb{T}'\}$, with $\mathbb{T}'$ as was defined in Definition 5, is said to be an IMA process if $X_{t_1} = \varepsilon_{t_1}$ and, for $n \geq 2$,

$$X_{t_n} = \varepsilon_{t_n} + \frac{\theta^{\Delta_n}}{c_{n-1}(\theta)} \varepsilon_{t_{n-1}}. \tag{2.4}$$

We say that $\{X_{t_n}, t_n \in \mathbb{T}'\}$ is an IMA process with mean μ if $\{X_{t_n} - \mu, t_n \in \mathbb{T}'\}$ is an IMA process. Additionally, the general backward continued fraction, $c_n(\theta)$, satisfies (proof in Appendix A.2)

$$1 < c_n(\theta) < 2 \quad \text{for all } n.$$

2.2.3 Process properties

If $X_n = [X_{t_1}, \ldots, X_{t_n}]'$ is a random vector from an IMA process, then X_n is a Gaussian random vector with mean $\mathrm{m}_n = 0$ and tridiagonal covariance matrix

$$\Gamma_n = \sigma^2 \begin{bmatrix} 1+\theta^2 & \theta^{\Delta_2} & \cdots & 0 & 0 \\ \theta^{\Delta_2} & 1+\theta^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1+\theta^2 & \theta^{\Delta_n} \\ 0 & 0 & \cdots & \theta^{\Delta_n} & 1+\theta^2 \end{bmatrix}.$$

Thus, the IMA process is a weakly stationary Gaussian process and therefore strictly stationary. Further, since $\{\varepsilon_{t_n}\}_{n\geq 1}$ is ergodic (see Chapter 1, Lemma 2), and $X_{t_n} = f(\varepsilon_{t_n}, \varepsilon_{t_{n-1}})$, with f a measurable function, then $\{X_{t_n}\}_{n\geq 1}$ is also ergodic (see Chapter 1, Theorem 3). In fact, since $\{\varepsilon_{t_n}\}_{n\geq 1}$ is mixing and X_{t_n} is a measurable function of it, then $\{X_{t_n}\}_{n\geq 1}$ is mixing and therefore ergodic (see Chapter 1, Theorem 4).

2.3 State-space representation

Now, using the same notation given in Definition 6, we provide a state-space representation of the model (2.4). This representation has the lowest dimension of the state vector and is given by

$$\begin{aligned} \alpha_{t_{n+1}} &= \frac{\theta^{\Delta_{n+1}}}{c_n(\theta)}\varepsilon_{t_n}, \\ X_{t_n} &= \alpha_{t_n} + \varepsilon_{t_n} \end{aligned}$$

for $n \geq 1$ with $\alpha_{t_1} = 0$. Note that, in this representation, the transition and measurement equation disturbances are correlated. As is suggested by Harvey (1989), to get a new system on which these disturbances are uncorrelated, we use

$$\alpha_{t_{n+1}} = -\frac{\theta^{\Delta_{n+1}}}{c_n(\theta)}\alpha_{t_n} + \frac{\theta^{\Delta_{n+1}}}{c_n(\theta)}X_{t_n}, \tag{2.5}$$

as the transition equation. The inclusion of X_{t_n} in (2.5) does not afect Kalman filter, as X_{t_n} is known at time t_n.

2.4 Prediction

The one-step linear predictors are defined as $\hat{X}_{t_1} = 0$ and

$$\hat{X}_{t_{n+1}} = \phi_{n1}X_{t_1} + \cdots + \phi_{nn}X_{t_n},\ n \geq 1,$$

where $\phi_{n1}, \ldots, \phi_{nn}$ satisfy the prediction equations

$$\Gamma_n \phi_n = \gamma_n. \tag{2.6}$$

In terms of the IMA process we have,

$$\Gamma_n = \sigma^2 \begin{bmatrix} 1+\theta^2 & \theta^{\Delta_2} & 0 & \cdots & 0 \\ \theta^{\Delta_2} & 1+\theta^2 & \theta^{\Delta_3} & & \vdots \\ 0 & \theta^{\Delta_3} & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1+\theta^2 & \theta^{\Delta_n} \\ 0 & \cdots & 0 & \theta^{\Delta_n} & 1+\theta^2 \end{bmatrix}, \phi_n = \begin{bmatrix} \phi_{n1} \\ \phi_{n2} \\ \vdots \\ \phi_{nn} \end{bmatrix} \text{ and } \gamma_n = \sigma^2 \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \theta^{\Delta_{n+1}} \end{bmatrix}.$$

The mean squared errors are $v_{n+1} = \mathrm{E}[(X_{t_{n+1}} - \hat{X}_{t_{n+1}})^2] = \gamma_0 - \gamma_n' \Gamma_n^{-1} \gamma_n$ with $v_1 = \gamma_0$.

From Brockwell and Davis (1991, Proposition 5.1.1, pp.167), if $\gamma_0 > 0$ and $\gamma_{h,\Delta_{n+1}} \to 0$ as $h \to \infty$, for all $n \geq 1$, then the covariance matrix Γ_n is non-singular for every n. Note that, in particular, the covariance structure of the IMA process satisfies these conditions. Hence, there is exactly one solution of 2.6 which is given by

$$\phi_n = \Gamma_n^{-1} \gamma_n.$$

An useful algorithm for solve the prediction equations is known as Innovations algorithm (Brockwell and Davis, 1991). The innovations algorithm gives the coefficients of $X_{t_n} - \hat{X}_{t_n}, \ldots, X_{t_1} - \hat{X}_{t_1}$, in the alternative expansion $\hat{X}_{t_{n+1}} = \sum_{j=1}^{n} \theta_{nj}(X_{t_{n+1-j}} - \hat{X}_{t_{n+1-j}})$. Hence, using this algorithm (see Appendix A.3) we have

$$\begin{aligned} \theta_{n,j} &= 0, \quad 2 \leq j \leq n, \\ \theta_{n,1} &= \frac{\gamma_{1,\Delta_{n+1}}}{v_n} \quad \text{and} \\ v_{n+1} = \gamma_0 - \theta_{n,1}^2 v_n &= \gamma_0 - \frac{\gamma_{1,\Delta_{n+1}}^2}{v_n}, \quad \text{with } v_1 = \gamma_0. \end{aligned}$$

Specifically, for the IMA process, we obtain $\hat{X}_{t_1}(\theta) = 0$ with mean squared error $\sigma^2 c_1(\theta)$ and

$$\hat{X}_{t_{n+1}}(\theta) = \frac{\theta^{\Delta_{n+1}}}{c_n(\theta)}(X_{t_n} - \hat{X}_{t_n}(\theta)),\ n \geq 1,$$

with mean squared errors $\sigma^2 c_{n+1}(\theta)$.

2.5 Maximum likelihood estimation

Let X_t be observed at points $t_1, \ldots, t_{\mathrm{N}}$. The criterion function is

$$C_{\mathrm{N}}(\theta, \sigma^2) = \log \sigma^2 + \frac{1}{\mathrm{N}} \sum_{n=1}^{\mathrm{N}} \log c_n(\theta) + \frac{1}{\mathrm{N}} \sum_{n=1}^{\mathrm{N}} \frac{(X_{t_n} - \hat{X}_{t_n}(\theta))^2}{\sigma^2 c_n(\theta)}.$$

Holding fixed θ, we can optimize $C_{\mathrm{N}}(\theta, \sigma^2)$ with respect to σ^2. The corresponding conditional estimate of σ^2 is

$$\sigma_{\mathrm{N}}^2(\theta) = \frac{1}{\mathrm{N}} \sum_{n=1}^{\mathrm{N}} \frac{(X_{t_n} - \hat{X}_{t_n}(\theta))^2}{c_n(\theta)}.$$

The concentrated criterion function is

$$q_{\mathrm{N}}(\theta) = C_{\mathrm{N}}(\theta, \sigma^2_{\mathrm{N}}(\theta)) = \log \sigma^2_{\mathrm{N}}(\theta) + \frac{1}{\mathrm{N}} \sum_{n=1}^{\mathrm{N}} \log c_n(\theta).$$

Let θ_0 and σ_0^2 be any admissible parameter values. The maximun likelihood estimate of θ_0, $\hat{\theta}_{\mathrm{N}}$, is the value minimizing $q_{\mathrm{N}}(\theta)$ and the estimate of σ_0^2 is $\hat{\sigma}^2_{\mathrm{N}} = \sigma^2_{\mathrm{N}}(\hat{\theta}_{\mathrm{N}})$. Henceforth, we could omit N in the estimates when the notation is overload.

2.6 Bootstrap estimation

To carry out statistical inference it is necessary to be able to derive the distributions of the statistics used for the estimation of the parameters from the data. If N is small, or if the parameters are close to the boundaries, the asymptotic approximations can be quite poor (Shumway and Stoffer, 2017). Also, in the irregularly spaced time case, the asymptotic approximations need to establish strong conditions such as (1.1) which are difficult to meet. To overcome these difficulties and to can get approximations of the finite sample distributions, we might use the bootstrap method. The idea in time series is to fit a suitable model to the data, to construct residuals from the fitted model, and then to generate new series by incorporating random samples from the residuals into the fitted model. The residuals are typically recentred to have the same mean as the innovations of the model.

Following to Bose (1990), for $n \geq 2$, define the estimated innovations or one-step prediction residuals, as

$$e_{t_n} = \sum_{j=0}^{n-1} (-1)^j \frac{\prod_{k=n-j+1}^{n} \theta^{\Delta_k}}{\prod_{l=n-j}^{n-1} c_l(\theta)} X_{t_{n-j}}$$

and $e_{t_1} = X_{t_1}$. Using the structure of the process and assuming that the fitted model is, in fact, the true model for the data we have

$$e_{t_n} = \varepsilon_{t_n} + (-1)^{n-1} \frac{\prod_{k=1}^{n} \theta^{\Delta_k}}{\prod_{l=0}^{n-1} c_l(\theta)} \varepsilon_{t_0}.$$

Hence e_{t_n} and ε_{t_n} are close for all large n if $0 < \theta < 1$, which shows that resampling is proper in this situation.

Next, we apply the bootstrap method to estimate θ_0 in the IMA process with $\sigma_0^2 = 1$. Let X_t be observed at points $t_1, \ldots, t_{\mathrm{N}}$ and consider $\hat{\theta}_{\mathrm{N}}$ as the respective MLE estimation. The standardized estimated innovations are

$$e^s_{t_n} = \frac{X_{t_n} - \hat{X}_{t_n}(\hat{\theta}_{\mathrm{N}})}{\sqrt{c_n(\hat{\theta}_{\mathrm{N}})}},$$

for $n = 2, \ldots, \mathrm{N}$. The so-called model-based resampling might proceed by equiprobable sampling with replacement from centered residuals $e^s_{t_2} - \bar{e}, \ldots, e^s_{t_{\mathrm{N}}} - \bar{e}$, where $\bar{e} = \sum_{n=2}^{N} e^s_{t_n}/\mathrm{N}-1$, to obtain simulated innovations $\zeta^*_{t_1}, \ldots, \zeta^*_{t_{\mathrm{N}}}$, and then setting

$$X^*_{t_1} = \sqrt{c_1(\hat{\theta}_{\mathrm{N}})}\zeta^*_{t_1},$$
$$X^*_{t_n} = \sqrt{c_n(\hat{\theta}_{\mathrm{N}})}\zeta^*_{t_n} + \frac{\hat{\theta}^{\Delta_n}_{\mathrm{N}}}{c_{n-1}(\hat{\theta}_{\mathrm{N}})}\sqrt{c_{n-1}(\hat{\theta}_{\mathrm{N}})}\zeta^*_{t_{n-1}}, \quad \text{for } n = 2, \ldots, \mathrm{N}.$$

Next, we estimate the parameters through ML assuming the data are $X^*_{t_n}$. Thus, we can repeat this process a large number, B, of times generating a collection of bootstrapped parameter estimates. Then, we can approximate the finite sample distribution of the estimator, $\hat{\theta}_{\mathrm{N}}$, from the bootstrapped parameter values.

2.7 Monte Carlo study

This section provides a Monte Carlo study. Our goal is to study the properties for Maximum Likelihood (ML) and bootstrap estimators. We consider $\sigma_0^2 = 1$, $\theta_0 \in \{0.1, 0.5, 0.9\}$ and $\mathrm{N} \in \{100, 500, 1500\}$, where N represent the length of the serie. We simulated $\mathrm{M} = 1000$ trajectories, $\{\mathrm{S}_m\}_{m=1}^{\mathrm{M}}$, and estimated θ_0. For each set up, we regard regular as well as irregular spaced times $t_1, \ldots, t_{\mathrm{N}}$, where $t_n - t_{n-1} \overset{\text{ind}}{\sim} \exp(\lambda = 1)$, for $n = 2, \ldots, \mathrm{N}$. In Figure

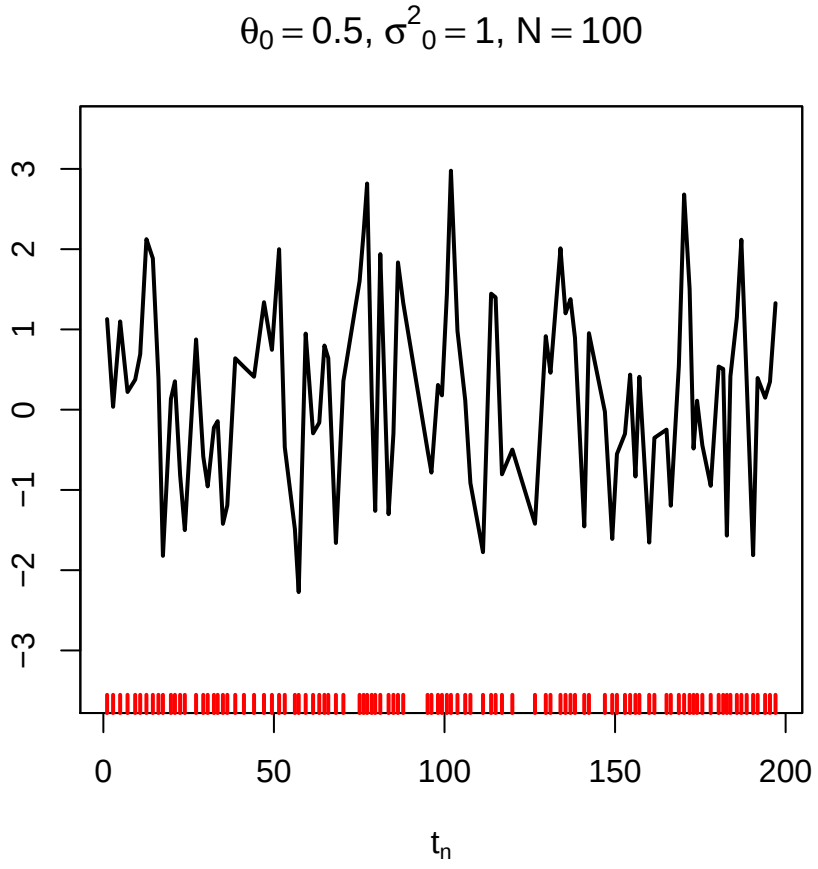

Figure 2.1: IMA trajectory example with $\theta_0 = 0.5$, $\sigma_0^2 = 1$ and N = 100. On the bottom, we found the tick marks of irregularly spaced times.

2.1, we present an IMA trajectory example with the tick marks of irregularly spaced times.

Let $\hat{\theta}_m^{\text{MLE}}$ be the ML estimation and $\widehat{\text{se}}(\hat{\theta}_m^{\text{MLE}})$ be the estimated standar error for the m-th trajectory. The estandar error is estimated by curvature of the likelihood surface at $\hat{\theta}_m^{\text{MLE}}$. We summarised the M maximum likelihood estimations by

$$\hat{\theta}^{\text{MLE}} = \frac{1}{\text{M}} \sum_{m=1}^{\text{M}} \hat{\theta}_m^{\text{MLE}} \quad \text{and} \quad \widehat{\text{se}}(\hat{\theta}^{\text{MLE}}) = \frac{1}{\text{M}} \sum_{m=1}^{\text{M}} \widehat{\text{se}}(\hat{\theta}_m^{\text{MLE}}).$$

On the other hand, for each trayectory, we simulated $\mathrm{B} = 500$ bootstrap trayectories represented by $\{\{\mathrm{S}_{m,b}\}_{b=1}^{\mathrm{B}}\}_{m=1}^{\mathrm{M}}$. Then, we got $\{\{\hat{\theta}_{m,b}^{\mathrm{b}}\}_{b=1}^{\mathrm{B}}\}_{m=1}^{\mathrm{M}}$ (the ML estimations). The bootstrap estimation and their estimated estandar errror are defined as

$$\hat{\theta}_m^{\mathrm{b}} = \frac{1}{\mathrm{B}} \sum_{b=1}^{\mathrm{B}} \hat{\theta}_{m,b}^{\mathrm{b}} \quad \text{and} \quad \widehat{\mathrm{se}}^2(\hat{\theta}_m^{\mathrm{b}}) = \frac{1}{\mathrm{B}-1} \sum_{b=1}^{\mathrm{B}} (\hat{\theta}_{m,b}^{\mathrm{b}} - \hat{\theta}_m^{\mathrm{b}})^2,$$

for $m = 1, \ldots, \mathrm{M}$. Finally, we summarised the M bootstrap estimations by

$$\hat{\theta}^{\mathrm{b}} = \frac{1}{\mathrm{M}} \sum_{m=1}^{\mathrm{M}} \hat{\theta}_m^{\mathrm{b}} \quad \text{and} \quad \widehat{\mathrm{se}}(\hat{\theta}^{\mathrm{b}}) = \frac{1}{\mathrm{M}} \sum_{m=1}^{\mathrm{M}} \widehat{\mathrm{se}}(\hat{\theta}_m^{\mathrm{b}}).$$

Besides, as a measure of the estimator performance, we used the Root Mean Squared Error (RMSE), and the Coefficient of Variation (CV) estimated defined by

$$\widehat{\mathrm{RMSE}}_{\hat{\theta}^{\mathrm{MLE}}} = (\widehat{\mathrm{se}}(\hat{\theta}^{\mathrm{MLE}})^2 + \widehat{\mathrm{bias}}^2_{\hat{\theta}^{\mathrm{MLE}}})^{1/2}, \text{ and}$$
$$\widehat{\mathrm{CV}}_{\hat{\theta}^{\mathrm{MLE}}} = \frac{\widehat{\mathrm{se}}(\hat{\theta}^{\mathrm{MLE}})}{\left|\hat{\theta}^{\mathrm{MLE}}\right|},$$

where $\widehat{\mathrm{bias}}_{\hat{\theta}^{\mathrm{MLE}}} = \hat{\theta}^{\mathrm{MLE}} - \theta_0$. Finally, we estimated the estimator variance by

$$\widetilde{\mathrm{se}}^2(\hat{\theta}^{\mathrm{MLE}}) = \frac{1}{\mathrm{M}-1} \sum_{m=1}^{\mathrm{M}} (\hat{\theta}_m^{\mathrm{MLE}} - \hat{\theta}^{\mathrm{MLE}})^2.$$

Similarly, we defined $\widehat{\mathrm{RMSE}}_{\hat{\theta}^{\mathrm{b}}}$, $\widehat{\mathrm{CV}}_{\hat{\theta}^{\mathrm{b}}}$, $\widehat{\mathrm{bias}}_{\hat{\theta}^{\mathrm{b}}}$ and $\widetilde{\mathrm{se}}^2(\hat{\theta}^{\mathrm{b}})$ for the bootstrap case. Figure 2.2 shows the workflow of the simulation study.

Below, we present the results for the irregularly spaced time case (see Appendix A.4 for the regularly spaced case). First, we show the simulated finite sample distributions for maximum likelihood and bootstrap estimators. Second, for both estimators, we present the measures of performance. Separately, we estimate the Monte Carlo Error (MCE) by

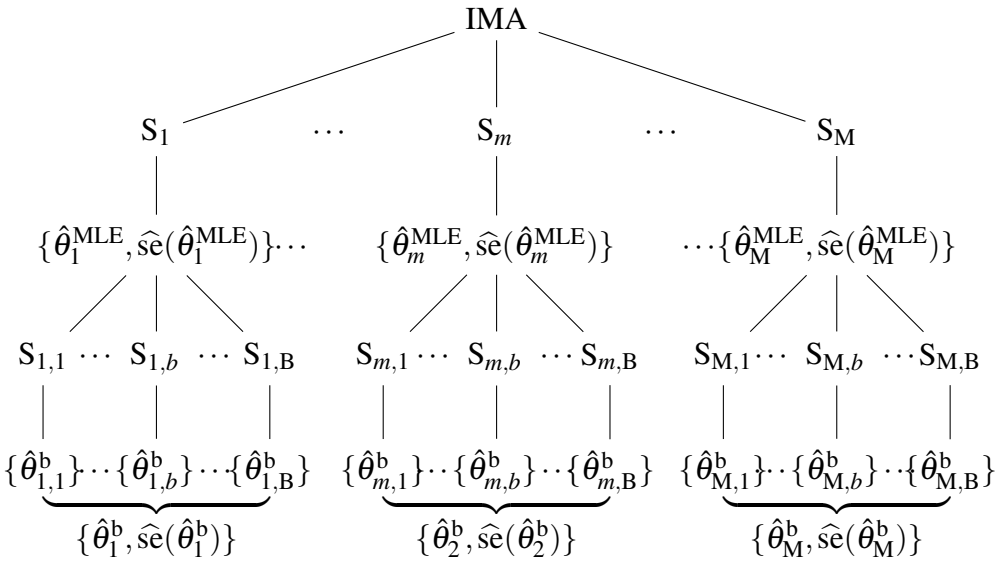

Figure 2.2: General scheme of Monte Carlo study. Here, we show how we got the pairs $\{\hat{\theta}_m^{\text{mle}}, \widehat{\text{se}}(\hat{\theta}_m^{\text{mle}})\}_{m=1}^{\text{M}}$ and $\{\hat{\theta}_m^{\text{b}}, \widehat{\text{se}}(\hat{\theta}_m^{\text{b}})\}_{m=1}^{\text{M}}$ for each trajectory.

asymptotic theory every simulation (see, Koehler et al., 2009), then we present the maximum value.

2.7.1 Finite sample distributions for ML and bootstrap estimators

In Figure 2.3, we present the simulated finite sample distributions for maximum likelihood and bootstrap estimators. As we can see, both estimators have well behavior. They seem unbiased and consistent. However, the smaller θ_0, the more variables are the estimates.

2.7.2 Measures of performance-irregularly spaced case

As we can see in Table 2.1, for either maximum likelihood and bootstrap methods, the estimates have well behaviors. However, the smaller θ_0, the more variables are the estimates. Further, bias and uncertain are reduced when N increases. Both methods suitably estimate the standard error. Also, the irregularly spaced times seem to increase the estimation uncertainty (see the regularly spaced times case in Appendix A.4).

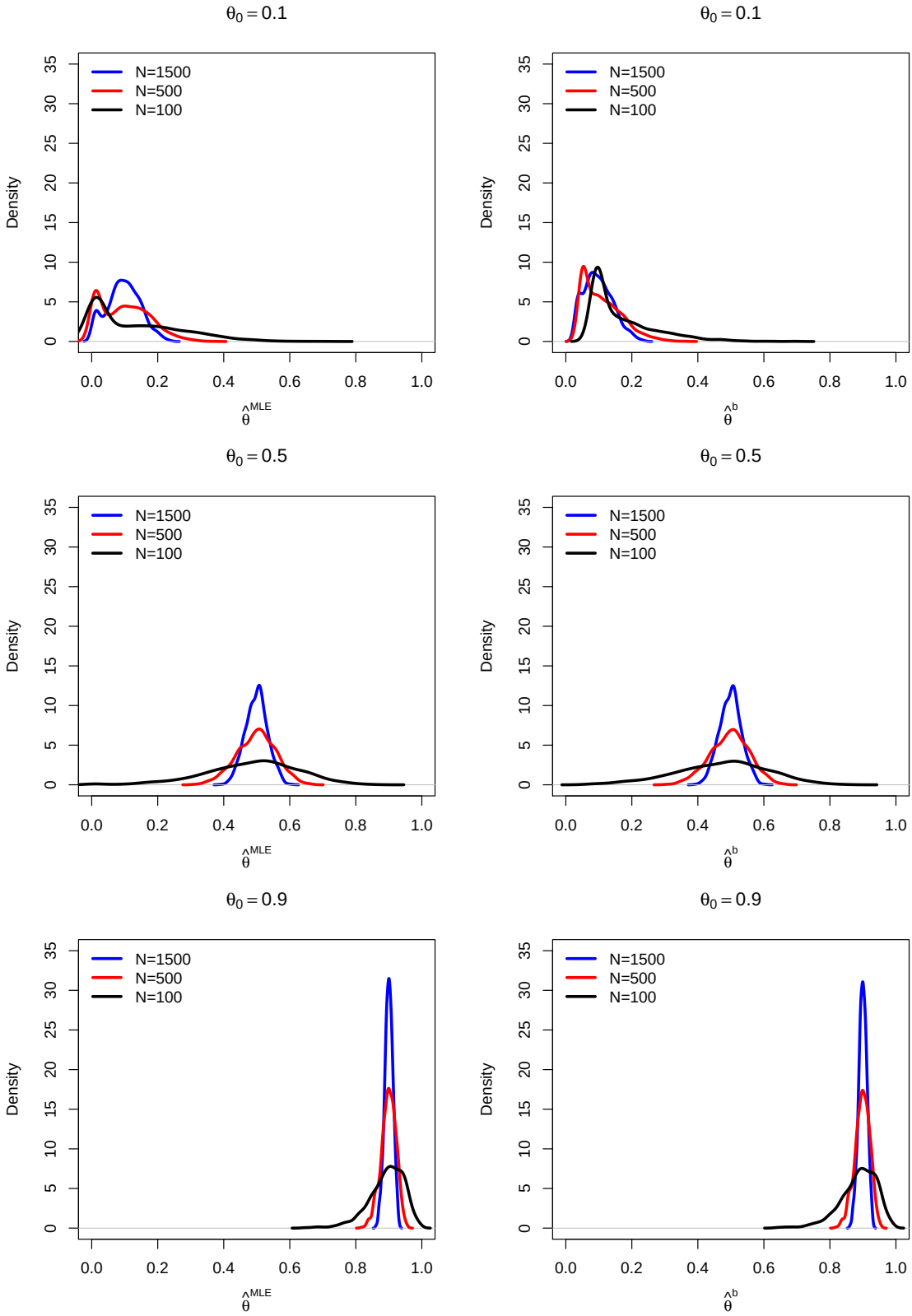

Figure 2.3: Simulated finite sample distributions. On the left, the MLE case. On the right, the bootstrap case.

N	θ_0	$\hat{\theta}^{\text{MLE}}$	$\widehat{\text{se}}(\hat{\theta}^{\text{MLE}})$	$\widetilde{\text{se}}(\hat{\theta}^{\text{MLE}})$	$\widehat{\text{bias}}_{\hat{\theta}^{\text{MLE}}}$	$\widehat{\text{RMSE}}_{\hat{\theta}^{\text{MLE}}}$	$\widehat{\text{CV}}_{\hat{\theta}^{\text{MLE}}}$
100	0.1	0.132	0.184	0.135	0.032	0.186	1.389
	0.5	0.486	0.134	0.144	−0.014	0.135	0.276
	0.9	0.893	0.051	0.055	−0.007	0.052	0.057
500	0.1	0.100	0.093	0.074	0.000	0.093	0.930
	0.5	0.498	0.058	0.059	−0.002	0.058	0.117
	0.9	0.899	0.022	0.022	−0.001	0.022	0.024
1500	0.1	0.097	0.056	0.050	−0.003	0.056	0.574
	0.5	0.499	0.034	0.034	−0.001	0.034	0.068
	0.9	0.900	0.012	0.012	0.000	0.012	0.014
N	θ_0	$\hat{\theta}^{\text{b}}$	$\widehat{\text{se}}(\hat{\theta}^{\text{b}})$	$\widetilde{\text{se}}(\hat{\theta}^{\text{b}})$	$\widehat{\text{bias}}_{\hat{\theta}^{\text{b}}}$	$\widehat{\text{RMSE}}_{\hat{\theta}^{\text{b}}}$	$\widehat{\text{CV}}_{\hat{\theta}^{\text{b}}}$
100	0.1	0.167	0.136	0.098	0.067	0.151	0.811
	0.5	0.473	0.140	0.139	−0.027	0.143	0.296
	0.9	0.887	0.055	0.056	−0.013	0.056	0.062
500	0.1	0.111	0.069	0.059	0.011	0.070	0.626
	0.5	0.496	0.059	0.059	−0.004	0.059	0.118
	0.9	0.898	0.022	0.023	−0.002	0.022	0.024
1500	0.1	0.099	0.047	0.043	−0.001	0.047	0.474
	0.5	0.498	0.034	0.033	−0.002	0.034	0.068
	0.9	0.900	0.012	0.012	0.000	0.012	0.014

Table 2.1: Monte Carlo results for the irregularly spaced time case. The MCE estimated is 0.005.

2.8 Application: lung function of an asthma patient

Belcher et al. (1994) analyzed measurements of the lung function of an asthma patient. Times are measured mostly at 2 hour time intervals but with irregular gaps (see the unequal spaced of tick marks in Figure 2.4). However, as was shown in Wang (2013), the series trend component (obtained by decomposing original time series into trend, seasonal, and irregular components via the Kalman smoother) exhibits structural changes after 100th observation. Thus, bellow, we use the first 100 observations to analyze such a phenomenon. Below, we present the ML and bootstrap estimates with their respective estimated standard errors.

$$\hat{\theta}^{\text{MLE}} = 0.853 \quad \widehat{\text{se}}(\hat{\theta}^{\text{MLE}}) = 0.069 \quad \hat{\sigma}^2_{\text{MLE}} = 258.286 \quad \widehat{\text{se}}(\hat{\sigma}^2_{\text{MLE}}) = 36.537$$

$$\hat{\theta}^{\text{b}} = 0.841 \quad \widehat{\text{se}}(\hat{\theta}^{\text{b}}) = 0.077 \quad \hat{\sigma}^2_{\text{b}} = 259.270 \quad \widehat{\text{se}}(\hat{\sigma}^2_{\text{b}}) = 32.662$$

As we can see in Figure 2.4, the fit looks adequate. Also, the standardized residuals seem to follow a standard normal distribution. Here, we use a nonparametric density estimation of the standardized residuals with normality reference bands (see Bowman and Azzalini, 1997). Further, Figure 2.4 shows the ACF estimated and the results from a Ljung-Box test. As we expected, the series complies with the test at the 5% significance level. The results were obtained through our R package called `istsa` (see Appendix A.5).

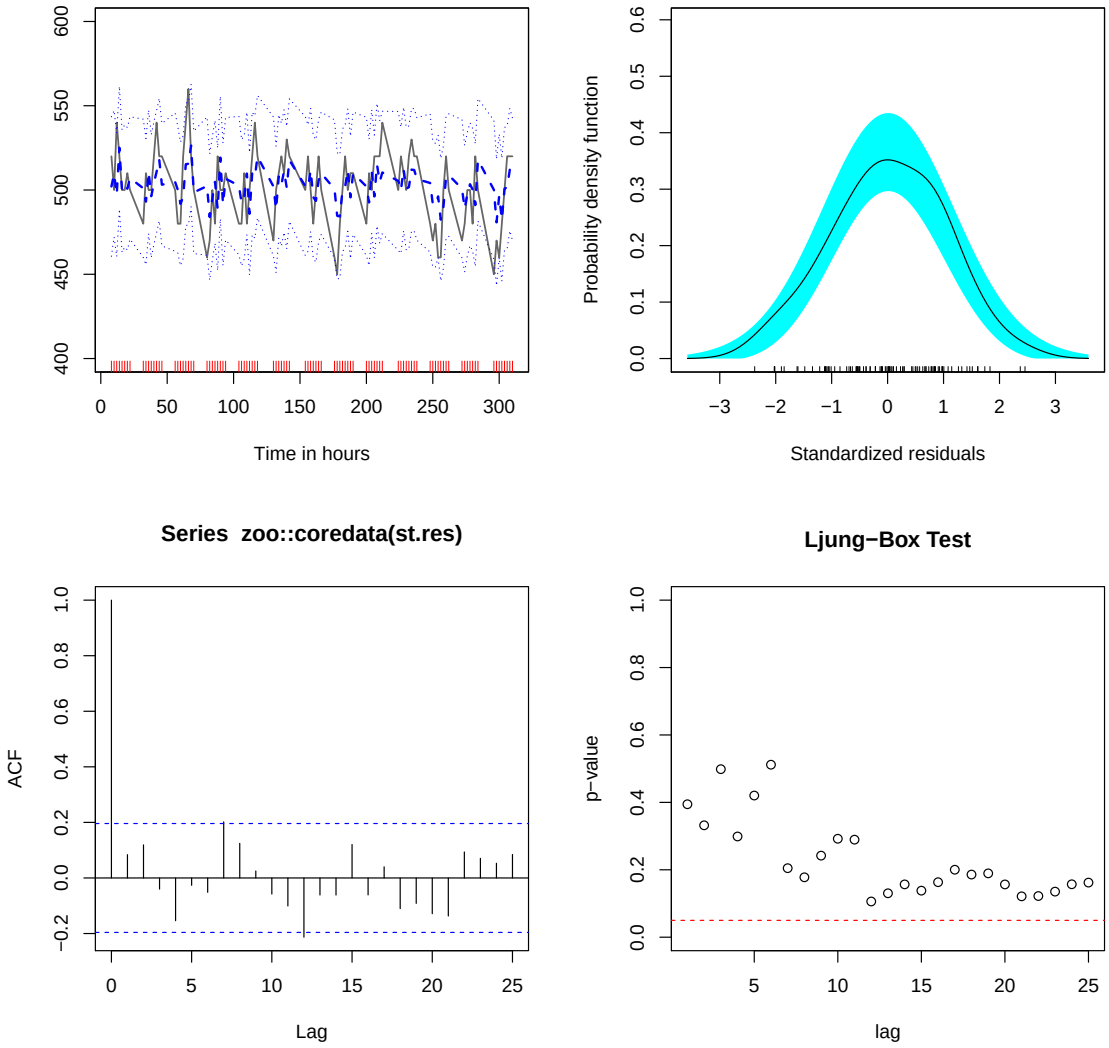

Figure 2.4: On the left-top, the lung function of an asthma patient with the predicted values and their respective variability bands. On the right-top, nonparametric density estimation of the standardized residuals with normality reference bands. In this case, we use $\hat{\theta}^{\text{MLE}}$. On the bottom-left, the autocorrelation function estimated. On the bottom-right, the Ljung-Box test for randomness.

Chapter 3

An irregularly spaced first-order autoregressive moving average process

Often, when we have an irregularly spaced time series, we treat them as a discrete realization of a continuous stochastic process. The emphasis of this approach has been mainly to model autoregressive moving averages process. However, a stationary condition of these processes is that autoregressive order must be strictly greater than moving average order. Then, a stationary continuous-time first-order autoregressive moving average process is not viable, at least for a real-valued process (Chan and Tong, 1987). In this chapter, we propose a novel model called an irregularly spaced first-order autoregressive moving average process that allows us to treat first-order autoregressive moving averages structures with irregularly spaced times.

The remainder of the Chapter 3 is organized as follows. In Section 3.2, we present a novel class of stochastic processes called irregularly spaced first-order autoregressive moving average model for the treatment and analysis of unevenly spaced time series. In Section 3.3, we give the state-space representation of the model. In Section 3.4 we provide the one-step linear predictors and the mean squared errors. The maximum likelihood and bootstrap estimation methods are introduced in Sections 3.5 and 3.6, respectively. We study the behavior of the maximum likelihood and bootstrap estimators via Monte Carlo in Section 3.7. Finally, we present a geophysical illustration in Section 3.8.

3.1 Introduction

In this section, we would like to build a stationary stochastic process having an autoregressive moving average structure that allows us to consider irregularly spaced times. We suppose that the pattern of irregular spacing is independent of the stochastic process properties.

First, let $\{\zeta_\tau, \tau \in \mathbb{T}'\}$ be iid random variables each $\mathrm{N}(0,1)$ and define,

$$\begin{aligned} X_{t_1} &= \upsilon_1^{1/2}\zeta_{t_1}, \\ X_{t_{n+1}} &= \phi^{\Delta_{n+1}} X_{t_n} + \upsilon_{n+1}^{1/2}\zeta_{t_{n+1}} + \varpi_n \upsilon_n^{1/2}\zeta_{t_n}, \quad \text{for } n \geq 1, \end{aligned}$$

where $0 \leq \phi < 1$, $\{\upsilon_n\}_{n\geq 1}$ and $\{\varpi_n\}_{n\geq 1}$ are time-varying sequences that characterize the moments of the process. Thus, for $n \geq 1$, we have $\mathrm{E}(X_{t_n}) = 0$,

$$\begin{aligned} \mathrm{Var}(X_{t_1}) &= \upsilon_1, \\ \mathrm{Var}(X_{t_{n+1}}) &= \phi^{2\Delta_{n+1}}\mathrm{Var}(X_{t_n}) + \upsilon_{n+1} + \varpi_n^2\upsilon_n + 2\phi^{\Delta_{n+1}}\varpi_n\upsilon_n, \text{ and} \\ \mathrm{Cov}(X_{t_n}, X_{t_{n+k}}) &= \begin{cases} \phi^{\Delta_{n+1}}\mathrm{Var}(X_{t_n}) + \varpi_n\upsilon_n, & k = 1, \\ \phi^{\Delta_{n+k}}\mathrm{Cov}(X_{t_n}, X_{t_{n+k-1}}), & k \geq 2. \end{cases} \end{aligned}$$

By successive substitutions we have

$$\mathrm{Cov}(X_{t_n}, X_{t_{n+k}}) = \phi^{t_{n+k}-t_{n+1}}\mathrm{Cov}(X_{t_n}, X_{t_{n+1}}), \text{ for } k \geq 2.$$

In order to obtain a stationary process, we require that, for $n \geq 1$, $\mathrm{Var}(X_{t_{n+1}}) = \mathrm{Var}(X_{t_1}) = \gamma_0$ and $\mathrm{Cov}(X_{t_n}, X_{t_{n+1}}) = \gamma_{1,\Delta_{n+1}}$ with $\gamma_{1,\Delta_{n+1}}$ a function of $\Delta_{n+1} = t_{n+1} - t_n$. Thus,

$$\phi^{2\Delta_{n+1}}\gamma_0 + \upsilon_{n+1} + \varpi_n^2\upsilon_n + 2\phi^{\Delta_{n+1}}\varpi_n\upsilon_n = \upsilon_1 = \gamma_0 \text{ and} \tag{3.1}$$

$$\phi^{\Delta_{n+1}}\gamma_0 + \varpi_n\upsilon_n = \gamma_{1,\Delta_{n+1}} \quad \text{for } n \geq 1. \tag{3.2}$$

From (3.2), we have

$$\varpi_n = \frac{\gamma_{1,\Delta_{n+1}} - \phi^{\Delta_{n+1}}\gamma_0}{\upsilon_n}. \tag{3.3}$$

If we replace (3.3) into (3.1), we obtain

$$\upsilon_{n+1} = \gamma_0 + \phi^{2\Delta_{n+1}}\gamma_0 - 2\phi^{\Delta_{n+1}}\gamma_{1,\Delta_{n+1}} - \frac{(\gamma_{1,\Delta_{n+1}} - \phi^{\Delta_{n+1}}\gamma_0)^2}{\upsilon_n}, \quad \text{with } \upsilon_1 = \gamma_0.$$

Now, let $\gamma_{1,\Delta_{n+1}}$ be $\phi^{\Delta_{n+1}}\gamma_0 + \sigma^2\theta^{\Delta_{n+1}}$, for $n \geq 1$, where $\sigma^2 > 0$ and $0 \leq \theta < 1$. Therefore,

$$\varpi_n = \frac{\sigma^2\theta^{\Delta_{n+1}}}{\upsilon_n} \quad \text{and} \quad \upsilon_{n+1} = \gamma_0(1-\phi^{2\Delta_{n+1}}) - 2\sigma^2\phi^{\Delta_{n+1}}\theta^{\Delta_{n+1}} - \frac{(\sigma^2\theta^{\Delta_{n+1}})^2}{\upsilon_n},$$

with $\upsilon_1 = \gamma_0$. Further, let γ_0 be $\sigma^{2(1+2\phi\theta+\theta^2)}/(1-\phi^2)$. Consequently,

$$\upsilon_{n+1} = \sigma^2\left(\frac{(1+2\theta\phi+\theta^2)}{(1-\phi^2)}(1-\phi^{2\Delta_{n+1}}) - 2\phi^{\Delta_{n+1}}\theta^{\Delta_{n+1}} - \frac{\sigma^2\theta^{2\Delta_{n+1}}}{\upsilon_n}\right)$$

We need to show that, under the above variance and first-order covariances specifications, we have $\upsilon_{n+1} > 0$, for all n. For this goal, let $c_1(\phi,\theta)$ be $(1+2\phi\theta+\theta^2)/(1-\phi^2)$ and

$$c_{n+1}(\phi,\theta) = \frac{(1+2\theta\phi+\theta^2)}{(1-\phi^2)}(1-\phi^{2\Delta_{n+1}}) - 2\phi^{\Delta_{n+1}}\theta^{\Delta_{n+1}} - \frac{\theta^{2\Delta_{n+1}}}{c_n(\phi,\theta)}.$$

Hence, note that $\upsilon_1 = \sigma^2 c_1(\phi,\theta)$ and

$$\upsilon_{n+1} = \sigma^2 c_{n+1}(\phi, \theta).$$

Since $1 \leq \Delta_{n+1} \leq \Delta_U$, for all n, we have

$$c_{n+1}(\phi, \theta) \geq 1 + \theta^2 + 2(\theta\phi - \phi^{\Delta_{n+1}}\theta^{\Delta_{n+1}}) - \frac{\theta^{2\Delta_{n+1}}}{c_n(\phi, \theta)} \geq c_{n+1}(\theta) > 0$$

because $\phi\theta \geq \phi^{\Delta_{n+1}}\theta^{\Delta_{n+1}}$, for all n. Thus, we obtain the desired result. The sequence $c_{n+1}(\theta)$ is defined as in Chapter 2, Definition 6. Below, we define a novel stochastic process having an autoregressive moving average structure that allows us to consider irregularly spaced times.

3.2 An Irregular spaced first-order Autoregressive Moving Average model

We define a novel class of stochastic process called Irregularly spaced first-order Autoregressive Moving Average (IARMA) process. The definition is made from a constructionist viewpoint. Later, we present process properties.

Definition 7 (IARMA–constructionist viewpoint)**.** Let $\{\varepsilon_{t_n}\}_{n\geq 1}$ be independent random variables each $\mathrm{N}(0, \sigma^2 c_n(\phi, \theta))$ with $\sigma^2 > 0$, $0 \leq \phi, \theta < 1$, $c_1(\phi, \theta) = \frac{1+2\theta\phi+\theta^2}{1-\phi^2}$ and

$$c_n(\phi, \theta) = c_1(\phi, \theta)(1 - \phi^{2\Delta_n}) - 2\phi^{\Delta_n}\theta^{\Delta_n} - \frac{\theta^{2\Delta_n}}{c_{n-1}(\phi, \theta)} \quad \text{for} \quad n \geq 2,$$

where $\Delta_n = t_n - t_{n-1}$. The process $\{X_{t_n}, t_n \in \mathbb{T}'\}$, with $\mathbb{T}'$ as was defined in Chapter 2, Definition 5, is said to be an IARMA process if $X_{t_1} = \varepsilon_{t_1}$ and, for $n \geq 2$,

$$X_{t_n} = \phi^{\Delta_n} X_{t_{n-1}} + \varepsilon_{t_n} + \frac{\theta^{\Delta_n}}{c_{n-1}(\phi, \theta)}\varepsilon_{t_{n-1}}. \tag{3.4}$$

We say that $\{X_{t_n}, t_n \in \mathbb{T}'\}$ is an IARMA process with mean μ if $\{X_{t_n} - \mu, t_n \in \mathbb{T}'\}$ is an IARMA process.

Note that, when $\phi = 0$, we obtain an IMA process while when $\theta = 0$, we get an IAR process. Next, we present some properties of this process.

3.2.1 Process properties

If $X_n = [X_{t_1}, \ldots, X_{t_n}]'$ is a random vector from an IARMA process, then X_n is a Gaussian random vector with mean $\mathrm{m}_n = 0$ and covariance matrix

$$\Gamma_n = \begin{bmatrix} \gamma_0 & & & & & \\ \gamma_{1,\Delta_2} & \gamma_0 & & & & \\ \phi^{t_3-t_2}\gamma_{1,\Delta_2} & \gamma_{1,\Delta_3} & \ddots & & & \\ \vdots & \vdots & & \gamma_0 & & \\ \phi^{t_{n-1}-t_2}\gamma_{1,\Delta_2} & \phi^{t_{n-1}-t_3}\gamma_{1,\Delta_3} & \cdots & \gamma_{1,\Delta_{n-1}} & \gamma_0 & \\ \phi^{t_n-t_2}\gamma_{1,\Delta_2} & \phi^{t_n-t_3}\gamma_{1,\Delta_3} & \cdots & \phi^{t_n-t_{n-1}}\gamma_{1,\Delta_{n-1}} & \gamma_{1,\Delta_n} & \gamma_0 \end{bmatrix},$$

where $\gamma_0 = \sigma^2(1+2\phi\theta+\theta^2)/(1-\phi^2)$ and $\gamma_{1,\Delta_n} = \phi^{\Delta_n}\gamma_0 + \sigma^2\theta^{\Delta_n}$, for $n \geq 2$. Thus, the IARMA process is a weakly stationary Gaussian process and therefore strictly stationary. Further, note that when $\Delta_n = 1$, for $n \geq 2$, we get the conventional ARMA process. On the other hand, since we can express (3.4) as a function of $\{\varepsilon_{t_j}\}_{j=1}^n$, for each n, then X_{t_n} is mixing and therefore ergodic (see Chapter 1, Theorem 4). Also, from (3.4), consider $Y_{t_n} = \varepsilon_{t_n} + \theta^{\Delta_n}/c_{n-1}(\phi,\theta)\varepsilon_{t_{n-1}}$, for $n \geq 2$, with $\mathrm{Var}(Y_{t_n}) = \sigma^2(c_1(\phi,\theta)(1-\phi^{2\Delta_n}) - 2\phi^{\Delta_n}\theta^{\Delta_n})$. Hence, we have $X_{t_1} = \varepsilon_{t_1}$ and, for $n \geq 2$,

$$X_{t_n} = \phi^{\Delta_n} X_{t_{n-1}} + Y_{t_n},$$

with $\mathrm{Cov}(X_{t_{n-1}}, Y_{t_n}) = \phi^{\Delta_n}\sigma^2$. By successive substitutions, we obtain

$$X_{t_n} = \phi^{t_n - t_1}\varepsilon_{t_1} + \sum_{j=2}^{n} \phi^{t_n - t_j} Y_{t_j}, \quad \text{for } n \geq 2.$$

Consequently, the larger n, the initial condition effect vanishes. Then, the process "forget" its initial starting value.

3.3 State-space representation

Now, using the same notation given in Chapter 2, Definition 7, we provide a state-space representation of the model (3.4). This representation has the lowest dimension of the state vector and is given by

$$\begin{aligned} \alpha_{t_{n+1}} &= \phi^{\Delta_{n+1}}\alpha_{t_n} + \left(\phi^{\Delta_{n+1}} + \frac{\theta^{\Delta_{n+1}}}{c_n(\phi,\theta)}\right)\varepsilon_{t_n}, \\ X_{t_n} &= \alpha_{t_n} + \varepsilon_{t_n}, \end{aligned}$$

for $n \geq 1$, with $\alpha_{t_1} = 0$. Note that, in this representation, the transition and measurement equation disturbances are correlated. As is suggested by Harvey (1989), to get a new system on which these disturbances are uncorrelated, we use

$$\alpha_{t_{n+1}} = \left(\phi^{\Delta_{n+1}} + \frac{\theta^{\Delta_{n+1}}}{c_n(\phi,\theta)}\right)X_{t_n} - \frac{\theta^{\Delta_{n+1}}}{c_n(\phi,\theta)}\alpha_{t_n} \tag{3.5}$$

as the transition equation. The inclusion of X_{t_n} in (3.5) does not afect Kalman filter, as X_{t_n} is known at time t_n.

3.4 Prediction

In this Section, we provide the one-step linear predictors to the IARMA model. Using the innovations algorithm (see Appendix A.3), the one-step linear predictors are $\hat{X}_{t_1}(\phi,\theta)=0$, with mean squared error $\upsilon_1=\sigma^2 c_1(\phi,\theta)$, and

$$\hat{X}_{t_{n+1}}(\phi,\theta)=\phi^{\Delta_{n+1}}X_{t_n}+\frac{\theta^{\Delta_{n+1}}}{c_n(\phi,\theta)}(X_{t_n}-\hat{X}_{t_n}(\phi,\theta)),\ n\geq 1,$$

with mean squared errors $\upsilon_{n+1}=\sigma^2 c_{n+1}(\phi,\theta)$.

3.5 Maximum likelihood estimation

Let X_t be observed at points $t_1,\ldots,t_{\mathrm{N}}$. The criterion function is

$$C_{\mathrm{N}}(\phi,\theta,\sigma^2)=\log\sigma^2+\frac{1}{\mathrm{N}}\sum_{n=1}^{\mathrm{N}}\log c_n(\phi,\theta)+\frac{1}{\mathrm{N}}\sum_{n=1}^{\mathrm{N}}\frac{(X_{t_n}-\hat{X}_{t_n}(\phi,\theta))^2}{\sigma^2 c_n(\phi,\theta)}.$$

Holding fixed ϕ and θ, we can optimize $C_{\mathrm{N}}(\phi,\theta,\sigma^2)$ with respect to σ^2. The corresponding conditional estimate of σ^2 is

$$\sigma^2_{\mathrm{N}}(\phi,\theta)=\frac{1}{\mathrm{N}}\sum_{n=1}^{\mathrm{N}}\frac{(X_{t_n}-\hat{X}_{t_n}(\phi,\theta))^2}{c_n(\phi,\theta)}.$$

The concentrated criterion function is

$$q_{\mathrm{N}}(\phi,\theta)=C_{\mathrm{N}}(\phi,\theta,\sigma^2_{\mathrm{N}}(\phi,\theta))=\log\sigma^2_{\mathrm{N}}(\phi,\theta)+\frac{1}{\mathrm{N}}\sum_{n=1}^{\mathrm{N}}\log c_n(\phi,\theta).$$

Let ϕ_0, θ_0 and σ_0^2 be any admissible parameter values. The maximum likelihood estimates of ϕ_0 and θ_0, denoted $\hat{\phi}_{\mathrm{N}}$ and $\hat{\theta}_{\mathrm{N}}$, respectively, are the values minimizing $q_{\mathrm{N}}(\phi,\theta)$ and the estimate of σ_0^2 is $\hat{\sigma}^2_{\mathrm{N}}=\sigma^2_{\mathrm{N}}(\hat{\phi}_{\mathrm{N}},\hat{\theta}_{\mathrm{N}})$. Henceforth, we could omit N in the estimates when the notation is overload.

3.6 Bootstrap estimation

In this section, we apply the bootstrap method to estimate ϕ_0 and θ_0 in the IARMA process with $\sigma_0^2 = 1$. Let X_t be observed at points $t_1, \ldots, t_{\mathrm{N}}$ and consider $\hat{\phi}_{\mathrm{N}}$ and $\hat{\theta}_{\mathrm{N}}$ as the respective MLE estimations. The standardized estimated innovations are

$$e^s_{t_n} = \frac{X_{t_n} - \hat{X}_{t_n}(\hat{\phi}_{\mathrm{N}}, \hat{\theta}_{\mathrm{N}})}{\sqrt{c_n(\hat{\phi}_{\mathrm{N}}, \hat{\theta}_{\mathrm{N}})}},$$

for $n = 2, \ldots, \mathrm{N}$. The so-called model-based resampling might proceed by equi-probable sampling with replacement from centered residuals $e^s_{t_2} - \bar{e}, \ldots, e^s_{t_{\mathrm{N}}} - \bar{e}$, where $\bar{e} = \sum_{n=2}^{N} e^s_{t_n}/\mathrm{N}-1$, to obtain simulated innovations $\zeta^*_{t_1}, \ldots, \zeta^*_{t_{\mathrm{N}}}$, and then setting

$$X^*_{t_1} = \sqrt{c_1(\hat{\phi}_{\mathrm{N}}, \hat{\theta}_{\mathrm{N}})}\zeta^*_{t_1},$$
$$X^*_{t_n} = \hat{\phi}_{\mathrm{N}}^{\Delta_n} X^*_{t_{n-1}} + \sqrt{c_n(\hat{\phi}_{\mathrm{N}}, \hat{\theta}_{\mathrm{N}})}\zeta^*_{t_n} + \frac{\hat{\theta}_{\mathrm{N}}^{\Delta_n}}{c_{n-1}(\hat{\phi}_{\mathrm{N}}, \hat{\theta}_{\mathrm{N}})}\sqrt{c_{n-1}(\hat{\phi}_{\mathrm{N}}, \hat{\theta}_{\mathrm{N}})}\zeta^*_{t_{n-1}},$$

for $n = 2, \ldots, \mathrm{N}$. Next, we estimate the parameters through ML, assuming the data are $X^*_{t_n}$. Thus, we can repeat this process a large number, B, of times generating a collection of bootstrapped parameter estimates. Then, we can approximate the finite sample distribution of the estimators, $\hat{\phi}_{\mathrm{N}}$ and $\hat{\theta}_{\mathrm{N}}$, from the bootstrapped parameters values.

3.7 Monte Carlo study

This Section provides a Monte Carlo study. Our goal is to study the properties for Maximum Likelihood (ML) and bootstrap estimators. We consider $\sigma_0^2 = 1$, $\phi_0 \in \{0.5\}$, $\theta_0 \in \{0.1, 0.5, 0.9\}$ and $\mathrm{N} \in \{100, 500, 1500\}$ where N represent the length of the serie. We simulated $\mathrm{M} = 1000$ trajectories, and estimated ϕ_0 and θ_0. For each set up, we regard regular (see Appendix B.1) as well as irregular spaced times, $t_1, \ldots, t_{\mathrm{N}}$, where $t_n - t_{n-1} \overset{\text{ind}}{\sim} \exp(\lambda = 1)$, for $n = 2, \ldots, \mathrm{N}$. In Figure 3.1, we present an IARMA trajectory

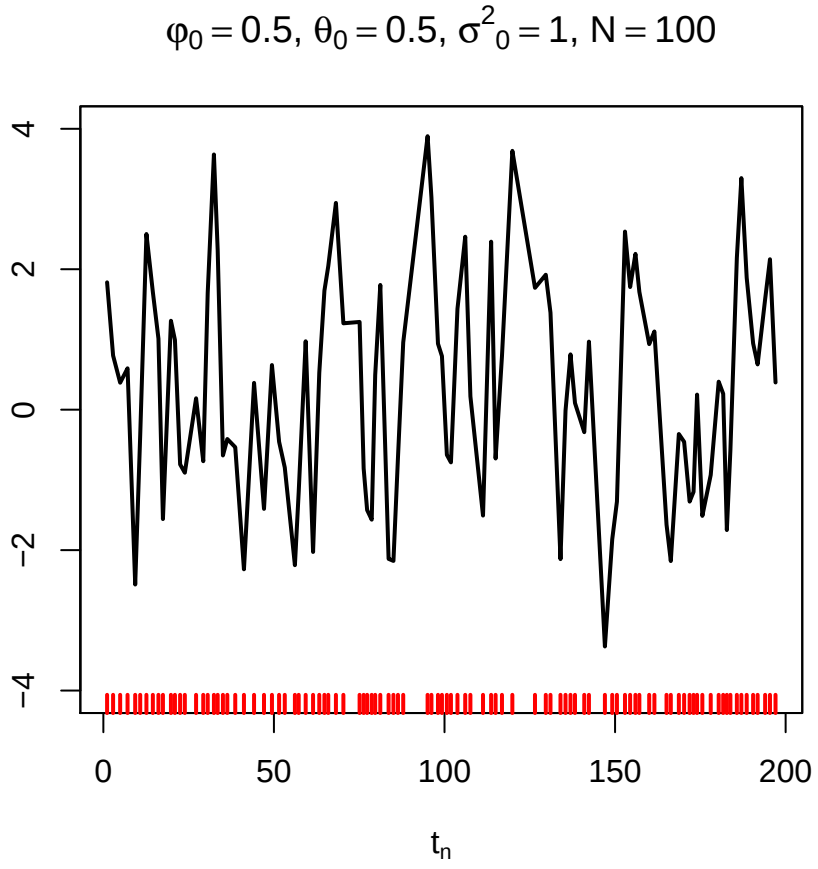

Figure 3.1: IARMA trajectory example with $\phi_0 = 0.5$, $\theta_0 = 0.5$, $\sigma_0^2 = 1$ and N = 100. On the bottom, we found the tick marks of irregularly spaced times.

example with the tick marks of irregularly spaced times.

The workflow of the simulation study is the same used in the IMA model (see Figure 2.2). Next, we present the results for the irregularly spaced time case (see Appendix B.1 for the regularly spaced case). First, we show the simulated finite sample distributions for maximum likelihood and bootstrap estimators. Second, for both estimators, we present the measures of performance. Separately, we estimate the Monte Carlo Error (MCE) by

asymptotic theory every simulation (see, Koehler et al., 2009), then we present the maximum value.

3.7.1 Finite sample distributions for ML and bootstrap estimators

In Figures 3.2 and 3.3, we provide the simulated finite sample distribution for ML and bootstrap estimators. As we can see, both estimators have well behavior. They seem unbiased and consistent. However, the smaller θ_0, the more variables are the estimates. The autoregressive estimator seems to have the same pattern.

3.7.2 Measures of performance-irregularly spaced case

As we can see in Table 3.1, for either maximum likelihood and bootstrap methods, the estimates have well behaviors. However, the smaller θ_0, the more variables are the estimates. Further, bias and uncertain are reduced when N increases. The standard error is suitably estimated by either ML and bootstrap methods. Also, the irregularly spaced times seem to increase the estimation uncertainty (see Appendix B.1). The autoregressive parameter shows similar results as moving average cases, at least for values considered.

3.8 Application: relative abundance of an oxygen isotope in an ocean core

Belcher et al. (1994) analyzed 164 measurements of relative abundance of an oxygen isotope (δ^{18}O) in an ocean core. The data correspond to unequally spaced time points in the past, with an average separation of 2000 years. Unevenly spaced tick marks indicate the corresponding irregularly spaced sampled times in Figure 3.4. Below, we present the ML and bootstrap estimates with their respective estimated standard errors.

$$\hat{\phi}^{\mathrm{MLE}} = 0.954 \quad \widehat{\mathrm{se}}(\hat{\phi}^{\mathrm{MLE}}) = 0.010 \quad \hat{\sigma}^2_{\mathrm{MLE}} = 0.014 \quad \widehat{\mathrm{se}}(\hat{\sigma}^2_{\mathrm{MLE}}) = 0.002$$

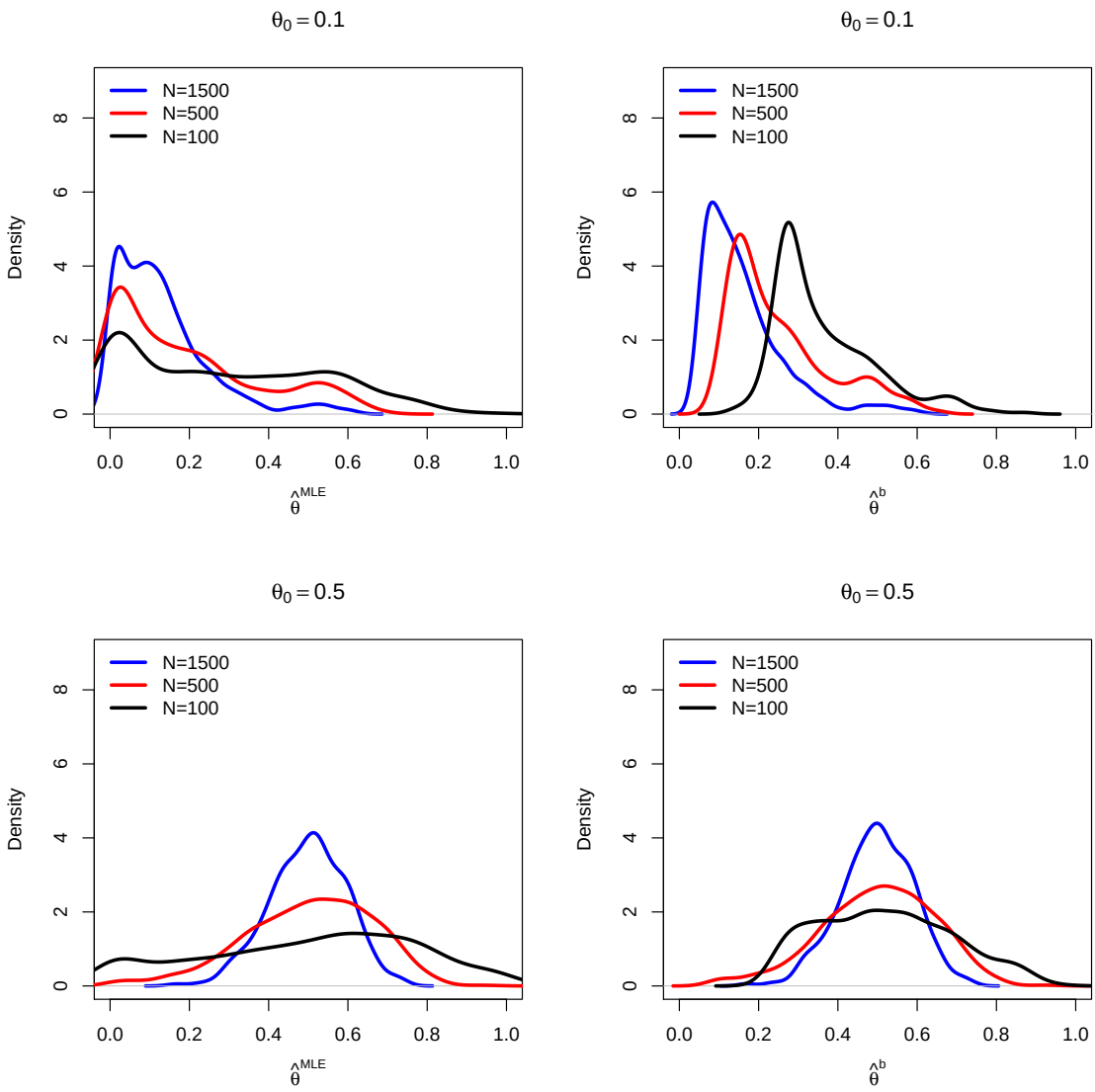

Figure 3.2: Simulated finite sample distributions. On the left, the MLE case. On the right, the bootstrap case.

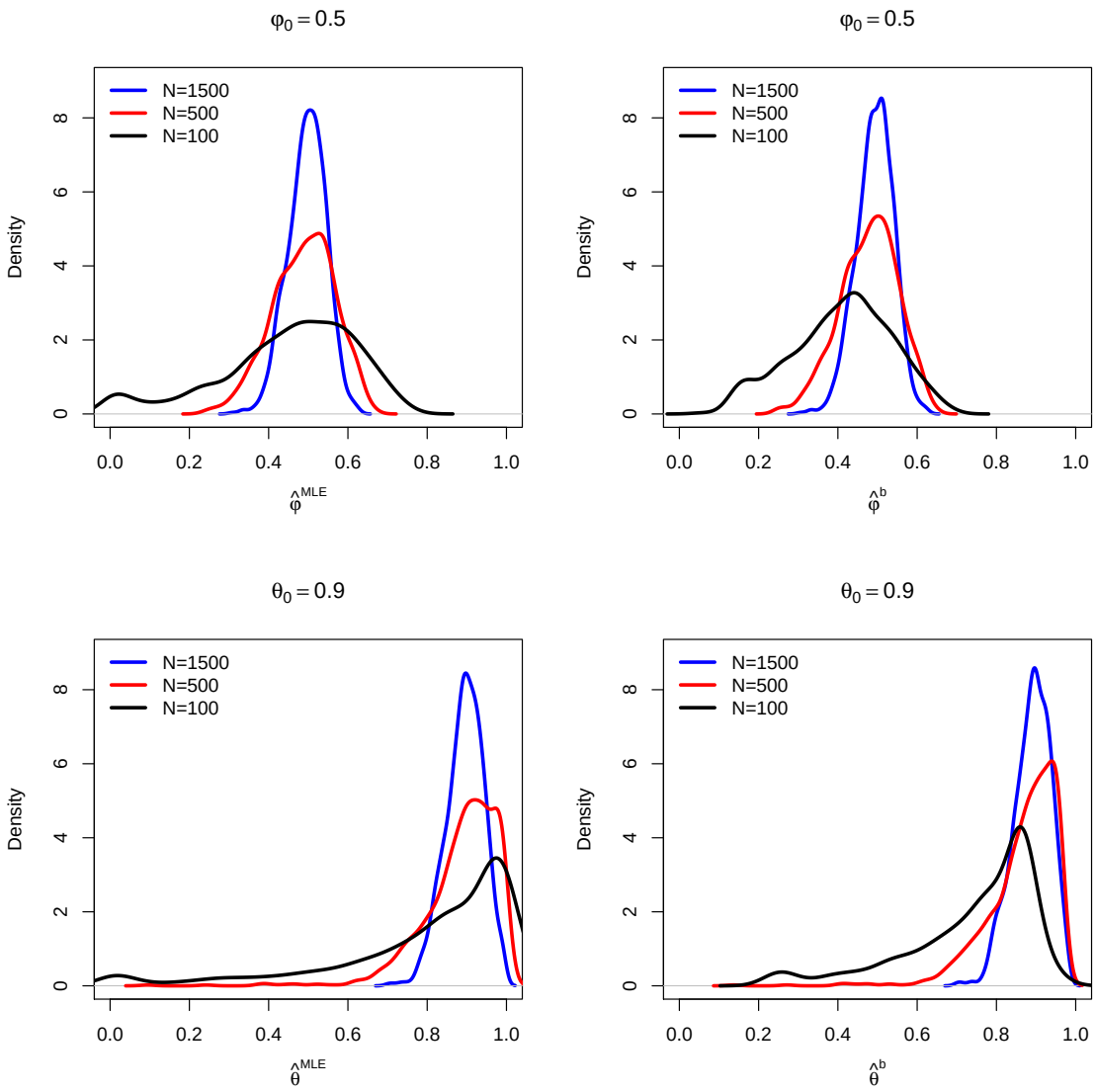

Figure 3.3: Simulated finite sample distributions. On the left, the MLE case. On the right, the bootstrap case. When $\phi_0 = 0.5$, we use $\theta_0 = 0.5$.

N	θ_0	$\hat{\theta}^{\text{MLE}}$	$\widehat{\text{se}}(\hat{\theta}^{\text{MLE}})$	$\widetilde{\text{se}}(\hat{\theta}^{\text{MLE}})$	$\widehat{\text{bias}}_{\hat{\theta}^{\text{MLE}}}$	$\widehat{\text{RMSE}}_{\hat{\theta}^{\text{MLE}}}$	$\widehat{\text{CV}}_{\hat{\theta}^{\text{MLE}}}$
100	0.1	0.294	0.245	0.245	0.194	0.312	0.835
	0.5	0.500	0.252	0.263	0.000	0.252	0.505
	0.9	0.796	0.232	0.228	−0.104	0.255	0.292
500	0.1	0.192	0.158	0.179	0.092	0.183	0.827
	0.5	0.501	0.149	0.160	0.001	0.149	0.298
	0.9	0.885	0.090	0.094	−0.015	0.091	0.102
1500	0.1	0.131	0.102	0.116	0.031	0.106	0.780
	0.5	0.499	0.094	0.098	−0.001	0.094	0.188
	0.9	0.895	0.050	0.049	−0.005	0.050	0.056
N	ϕ_0	$\hat{\phi}^{\text{MLE}}$	$\widehat{\text{se}}(\hat{\phi}^{\text{MLE}})$	$\widetilde{\text{se}}(\hat{\phi}^{\text{MLE}})$	$\widehat{\text{bias}}_{\hat{\phi}^{\text{MLE}}}$	$\widehat{\text{RMSE}}_{\hat{\phi}^{\text{MLE}}}$	$\widehat{\text{CV}}_{\hat{\phi}^{\text{MLE}}}$
100	0.5	0.448	0.155	0.167	−0.052	0.163	0.346
500		0.488	0.076	0.079	−0.012	0.077	0.156
1500		0.497	0.046	0.048	−0.003	0.046	0.092
N	θ_0	$\hat{\theta}^{\text{b}}$	$\widehat{\text{se}}(\hat{\theta}^{\text{b}})$	$\widetilde{\text{se}}(\hat{\theta}^{\text{b}})$	$\widehat{\text{bias}}_{\hat{\theta}^{\text{b}}}$	$\widehat{\text{RMSE}}_{\hat{\theta}^{\text{b}}}$	$\widehat{\text{CV}}_{\hat{\theta}^{\text{b}}}$
100	0.1	0.365	0.235	0.129	0.265	0.354	0.643
	0.5	0.520	0.252	0.169	0.020	0.253	0.485
	0.9	0.739	0.228	0.165	−0.161	0.280	0.309
500	0.1	0.253	0.169	0.129	0.153	0.227	0.667
	0.5	0.501	0.151	0.143	0.001	0.151	0.301
	0.9	0.867	0.090	0.089	−0.033	0.096	0.104
1500	0.1	0.162	0.112	0.103	0.062	0.128	0.689
	0.5	0.498	0.095	0.092	−0.002	0.095	0.190
	0.9	0.889	0.049	0.047	−0.011	0.051	0.056
N	ϕ_0	$\hat{\phi}^{\text{b}}$	$\widehat{\text{se}}(\hat{\phi}^{\text{b}})$	$\widetilde{\text{se}}(\hat{\phi}^{\text{b}})$	$\widehat{\text{bias}}_{\hat{\phi}^{\text{b}}}$	$\widehat{\text{RMSE}}_{\hat{\phi}^{\text{b}}}$	$\widehat{\text{CV}}_{\hat{\phi}^{\text{b}}}$
100	0.5	0.409	0.172	0.126	−0.091	0.194	0.420
500		0.478	0.081	0.073	−0.022	0.084	0.169
1500		0.493	0.047	0.047	−0.007	0.047	0.094

Table 3.1: Monte Carlo results for the irregularly spaced time case. The MCE estimated is 0.008. When $\phi_0 = 0.5$, we use $\theta_0 = 0.5$.

$$\hat{\phi}^{\mathrm{b}} = 0.950 \quad \widehat{\mathrm{se}}(\hat{\phi}^{\mathrm{b}}) = 0.011 \quad \hat{\sigma}^2_{\mathrm{b}} = 0.014 \quad \widehat{\mathrm{se}}(\hat{\sigma}^2_{\mathrm{b}}) = 0.002$$

As we can see in Figure 3.4, the fit looks adequate. Also, the standardized residuals seem to follow a standard normal distribution. Here, we use a nonparametric density estimation of the standardized residuals with normality reference bands (see Bowman and Azzalini, 1997). Further, Figure 3.4 exhibits the ACF estimated and the results from a Ljung-Box test. As we expected, the series complies with the test at the 5% significance level. On the other hand, Belcher et al. (1994) and Wang (2013) analyzed the same data set through continuous-time autoregressive models via the Kalman filter. They fitted a model of order 7 to the data. However, their estimates have considerable variation. When we compare their fit with ours, we observed similar results (the relative Mean Absolute Error (rMAE) is 1.143, see Davydenko and Fildes, 2013). Nevertheless, our model is more parsimonious. The results were obtained through our R package called `istsa` (see Appendix B.2).

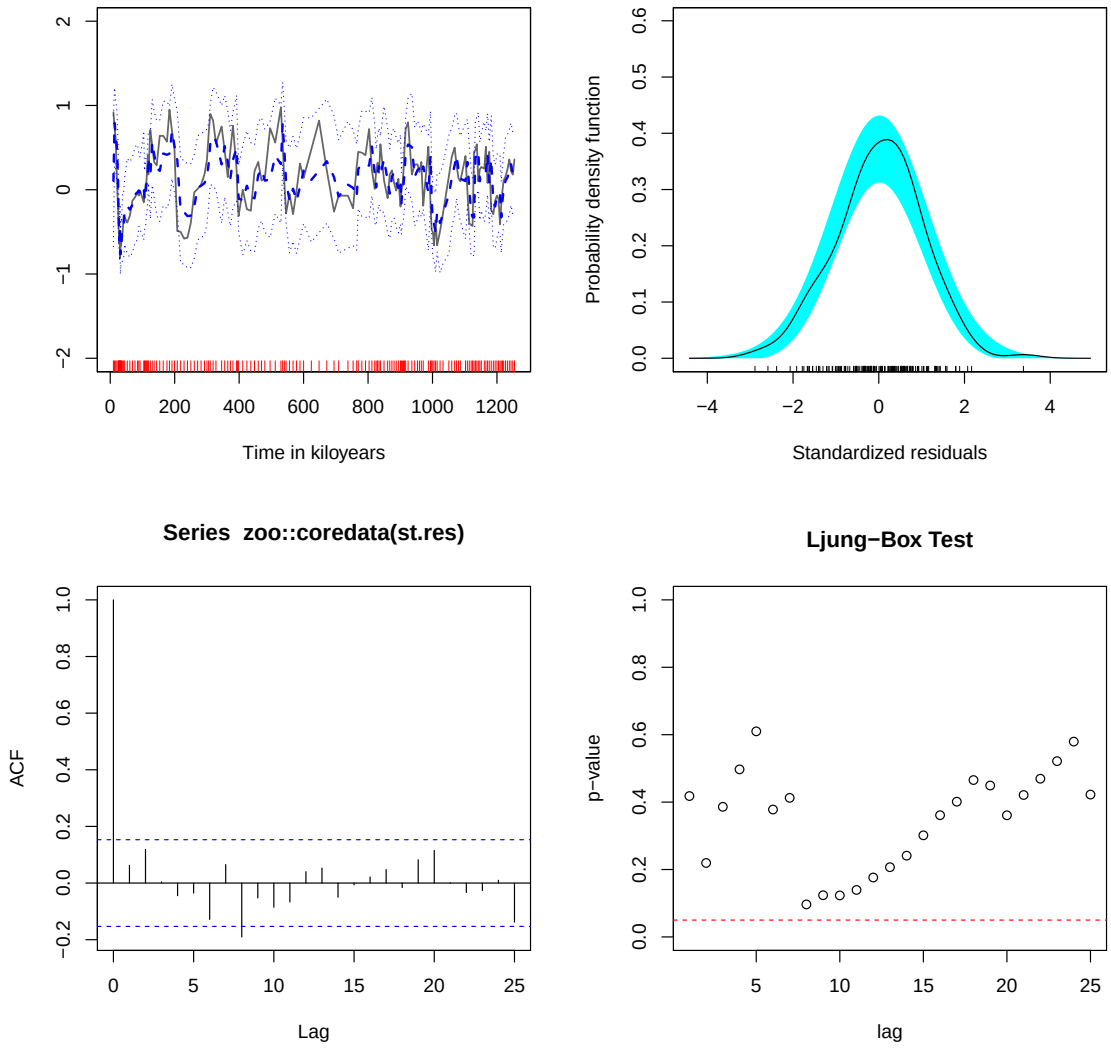

Figure 3.4: On the left-top, the relative abundance of an oxygen isotope (δ^{18}O) in an ocean core with the predicted values and their respective variability bands. On the right-top, non-parametric density estimation of the standardized residuals with normality reference bands. In this case, we use $\hat{\phi}^{\text{MLE}}$. On the bottom-left, the autocorrelation function estimated. On the bottom-right, the Ljung-Box test for randomness.

Chapter 4

Concluding remarks and future directions

In this thesis, we have presented two novel models to analyze irregularly spaced time series with either moving averages and autoregressive moving averages structures.

In Chapter 2, we proposed an irregularly spaced first-order moving average model that allows us to treat moving averages structures with irregularly spaced times. Its formal definition and properties were established. Also, we showed the state space representation along with one-step linear predictors. Further, we studied the estimation procedures for either maximum likelihood and bootstrap methods through Monte Carlo. For both methods, the estimates had well behaviors, and the standard error was suitably estimated. Also, bias and uncertain were reduced when N increased. The medical illustration showed a good fit.

In Chapter 3, we proposed an irregularly spaced first-order autoregressive moving average model that allows us to treat first-order autoregressive moving averages structures with irregularly spaced times. We have presented the definition and properties. The state-space representation, along with one-step linear predictors, was given. We studied the estimation procedures for either maximum likelihood and bootstrap methods through Monte Carlo. For both methods, the estimates (either autoregressive and moving average) had well behaviors, and the standard error was suitably estimated. The geophysical application gave well behavior. In spite of other authors have already analyzed the same data through a

continuous-time autoregressive model, their estimates have considerable variation, and the model obtained was less parsimonious that our estimated model.

In future directions, it would be desirably generalized the first-order IARMA process to higher orders to consider more complex structures. Also, it would be interesting to find stochastic times that allow us to improve the phenomena modeling process. Furthermore, since we have established our models with positive covariance structures, it would be attractive to generalize such structures in order to consider negative covariances. Finally, the IMA and IARMA models can be used in longitudinal data problems since there we usually have irregularly spaced times for each observation (see, e.g., Jones and Boadi-Boateng, 1991; Muñoz et al., 1992).

Appendix A

Appendix Chapter 2

A.1 Constructionist viewpoint

Next, we give some details about how we built the IMA model from the constructionist viewpoint. The main idea is to specify the IMA process as a function of other (often simpler) stochastic processes.

Let $\{\zeta_{t_n}\}_{n\geq 1}$ be iid random variables each $\mathrm{N}(0,1)$ and consider,

$$\begin{aligned} X_{t_1} &= \nu_1^{1/2}\zeta_{t_1}, \\ X_{t_{n+1}} &= \nu_{n+1}^{1/2}\zeta_{t_{n+1}} + \omega_n \nu_n^{1/2}\zeta_{t_n}, \quad \text{for } n \geq 1, \end{aligned}$$

where $\{\nu_n\}_{n\geq 1}$ and $\{\omega_n\}_{n\geq 1}$ are time-varying sequences that characterize the moments of the process. Thus, for $n \geq 1$, we have $\mathrm{E}(X_{t_n}) = 0$,

$$\begin{aligned} &\mathrm{Var}(X_{t_1}) = \nu_1,\ \mathrm{Var}(X_{t_{n+1}}) = \nu_{n+1} + \omega_n^2 \nu_n, \text{ and} \\ &\mathrm{Cov}(X_{t_n}, X_{t_{n+k}}) = \begin{cases} \omega_n \nu_n, & k = 1, \\ 0, & k \geq 2. \end{cases} \end{aligned}$$

Our main is to find $\{\nu_n\}_{n\geq 1}$ and $\{\omega_n\}_{n\geq 1}$ so that $\{X_{t_n}\}_{n\geq 1}$ be a stationary process. For this, we need that, for $n \geq 1$, $\text{Var}(X_{t_{n+1}}) = \text{Var}(X_{t_1}) = \gamma_0$ and $\text{Cov}(X_{t_n}, X_{t_{n+1}}) = \gamma_{1,\Delta_{n+1}}$ with $\gamma_{1,\Delta_{n+1}}$ a function of $\Delta_{n+1} = t_{n+1} - t_n$. Thus,

$$\begin{aligned} &\nu_{n+1} + \omega_n^2 \nu_n = \nu_1 = \gamma_0 \text{ and} \\ &\omega_n \nu_n = \gamma_{1,\Delta_{n+1}} \quad \text{for } n \geq 1. \end{aligned}$$

From these equations, we obtain

$$\omega_n = \frac{\gamma_{1,\Delta_{n+1}}}{\nu_n} \quad \text{and} \quad \nu_{n+1} = \gamma_0 - \frac{\gamma_{1,\Delta_{n+1}}^2}{\nu_n}, \quad \text{with } \nu_1 = \gamma_0.$$

Therefore, we can set a real-valued stationary process defining γ_0 and $\{\gamma_{1,\Delta_{n+1}}\}_{n\geq 1}$ suitably, that is, in such a way that $\nu_n > 0$, for all n.

According to Kiliç (2008), the sequence $\{\nu_n\}_{n\geq 1}$ is known as a general backward continued fraction and if $\gamma_0 > 0$ and, for $n \geq 1$, $\left(\frac{\gamma_{1,\Delta_{n+1}}}{\gamma_0}\right)^2 \leq 1/4$ with $\gamma_{1,\Delta_{n+1}} \neq 0$, then $\{\nu_n\}_{n\geq 1}$ is a strictly positive sequence. Hence, under these conditions, $\{X_{t_n}\}_{n\geq 1}$ is a well defined real-valued stationary stochastic process. Specifically, in the IMA process, the variance and the first-order covariance functions were defined as

$$\gamma_0 = \sigma^2(1+\theta^2) \qquad \text{and} \qquad \gamma_{1,\Delta_{n+1}} = \sigma^2\theta^{\Delta_{n+1}} \quad \text{for } n \geq 1.$$

In this specification, we achieve the conventional MA process when $\Delta_{n+1} = 1$ for $n \geq 1$.

Next, we present several results that allow us to obtain the conditions under which a general backward continued fraction is a strictly positive sequence. The sequence $\{\nu_n\}_{n\geq 1}$ is known as a general backward continued fraction and its $n+1$-th convergent is

$$\nu_{n+1} = \left[\gamma_0 + \frac{-\gamma_{1,\Delta_2}^2}{\gamma_0+} \frac{-\gamma_{1,\Delta_3}^2}{\gamma_0+} \cdots \frac{-\gamma_{1,\Delta_{n+1}}^2}{\gamma_0}\right]_b = \gamma_0 + \cfrac{-\gamma_{1,\Delta_{n+1}}^2}{\gamma_0 + \cfrac{-\gamma_{1,\Delta_n}^2}{\ddots \cfrac{}{\gamma_0 + \cfrac{-\gamma_{1,\Delta_2}^2}{\gamma_0}}}}.$$

Thus, few convergent are

$$\nu_2 = \left[\gamma_0 + \frac{-\gamma_{1,\Delta_2}^2}{\gamma_0+}\right]_b = \frac{\gamma_0^2 - \gamma_{1,\Delta_2}^2}{\gamma_0},$$

$$\nu_3 = \left[\gamma_0 + \frac{-\gamma_{1,\Delta_2}^2}{\gamma_0+} \frac{-\gamma_{1,\Delta_3}^2}{\gamma_0+}\right]_b = \frac{\gamma_0^3 - \gamma_0\gamma_{1,\Delta_2}^2 - \gamma_0\gamma_{1,\Delta_3}^2}{\gamma_0^2 - \gamma_{1,\Delta_2}^2},$$

$$\nu_4 = \left[\gamma_0 + \frac{-\gamma_{1,\Delta_2}^2}{\gamma_0+} \frac{-\gamma_{1,\Delta_3}^2}{\gamma_0+} \frac{-\gamma_{1,\Delta_4}^2}{\gamma_0+}\right]_b = \frac{\gamma_0^4 - \gamma_0^2\gamma_{1,\Delta_2}^2 - \gamma_0^2\gamma_{1,\Delta_3}^2 - \gamma_0^2\gamma_{1,\Delta_4}^2 + \gamma_{1,\Delta_2}^2\gamma_{1,\Delta_4}^2}{\gamma_0^3 - \gamma_0\gamma_{1,\Delta_2}^2 - \gamma_0\gamma_{1,\Delta_3}^2}.$$

The ratio for this general backward continued fraction is

$$\nu_1 = \frac{P_1}{P_0} \text{ and } \nu_{n+1} = \frac{P_{n+1}}{P_n},$$

for $n \geq 1$, where the sequence $\{P_n\}_{n\geq 0}$ is

$$P_{n+1} = \gamma_0 P_n - \gamma_{1,\Delta_{n+1}}^2 P_{n-1}$$

with $P_1 = \gamma_0$ and $P_0 = 1$. This sequence is obtained by what is known as the Wallis-Euler recurrence relations (Loya, 2017). El-Mikkawy and Karawia (2006) proves that $P_k > 0$, for $k = 1, \ldots, n+1$, if and only if the matrix

$$\Gamma_{n+1} = \begin{bmatrix} \gamma_0 & \gamma_{1,\Delta_2} & \cdots & 0 & 0 \\ \gamma_{1,\Delta_2} & \gamma_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \gamma_0 & \gamma_{1,\Delta_{n+1}} \\ 0 & 0 & \cdots & \gamma_{1,\Delta_{n+1}} & \gamma_0 \end{bmatrix}$$

is positive definite with $\Gamma_1 = [\gamma_0]$. Further, they proved that $\det\Gamma_1 = P_1$ and $\det\Gamma_{n+1} = P_{n+1}$.

Andelić and da Fonseca (Theorem 3.3, pp. 158, 2011) present the so-called Wall-Wetzel Theorem (Wall and Wetzel, 1944), which relates positive definiteness of a tridiagonal matrix (taking $\gamma_{1,\Delta_{n+1}} \neq 0$ for all n) with a certain related sequence. Indeed, the real tridiagonal matrix Γ_{n+1} is positive definite, for all n, if and only if $\gamma_0 > 0$ and

$$\left\{ \left(\frac{\gamma_{1,\Delta_{k+1}}}{\gamma_0} \right)^2 \right\}_{k=1}^{n}$$

is a chain sequence for all n (see, Theorem 3.2, pp. 70, Ismail and Muldoon, 1991).

Wall (1948) worked the theory of chain sequences studying orthogonal polynomials. A finite or infinite sequence $\{\alpha_n\}_{n\geq 1}$ is called a (positive) chain sequence if there is a parameter sequence $\{g_n\}_{n\geq 0}$, where $0 \leq g_0 < 1$ is called an initial parameter, such that

$$\alpha_n = g_n\left(1 - g_{n-1}\right), n \geq 1, \text{ with } 0 < g_n < 1, \text{ for } n \geq 1.$$

It is important to consider the following result due to Wall (Theorem 20.1, pp. 86, 1948). This result was called by Chihara (Theorem 5.7, pp. 97, 1978) the comparison test because this says that any positive sequence bounded by a chain sequence is also a chain sequence. Thus, if $\{\alpha_n\}_{n\geq 1}$ is a chain sequence and

$$0 < \beta_n \leq \alpha_n, \text{ for all } n \geq 1,$$

then $\{\beta_n\}_{n\geq 1}$ is also a chain sequence. Further, Wall (Theorem 19.1, pp. 79, 1948) characterize all constant term chain sequences. A constant term sequence $\{\alpha\}_{n\geq 1}$ is a chain

sequence if and only if $0 < \alpha \leq 1/4$. Thus, $\{\beta_n\}_{n\geq 1}$ is a chain sequence if

$$0 < \beta_n \leq 1/4, \text{ for all } k \geq 1.$$

Thus, if $\gamma_0 > 0$ and, for $n \geq 1$, $\left(\frac{\gamma_{1,\Delta_{n+1}}}{\gamma_0}\right)^2 \leq 1/4$ with $\gamma_{1,\Delta_{n+1}} \neq 0$, then $\{\nu_n\}_{n\geq 1}$ is a strictly positive sequence.

A.2 On the general backward continued fraction in the IMA process

Given $0 < \theta < 1$, $c_n(\theta) > 0$ and $\Delta_{n+1} \geq 1$, for all $n \geq 1$, we have

$$c_{n+1}(\theta) = 1 + \theta^2 - \frac{\theta^{2\Delta_{n+1}}}{c_n(\theta)} < 1 + \theta^2 < 2 \quad \text{for } n \geq 1.$$

Now, we know that $c_1(\theta) = 1 + \theta^2 > 1$. Hence,

$$c_2(\theta) = 1 + \theta^2 - \frac{\theta^{2\Delta_2}}{c_1(\theta)} > 1 + \theta^2 - \theta^{2\Delta_2} > 1 + \theta^2 - \theta^2 = 1.$$

Hence,

$$c_2(\theta) = 1 + \theta^2 - \frac{\theta^{2\Delta_2}}{c_1(\theta)} > 1 + \theta^2 - \theta^{2\Delta_2} > 1 + \theta^2 - \theta^2 = 1.$$

In general, suppose that $c_n(\theta) > 1$. Then,

$$c_{n+1}(\theta) = 1 + \theta^2 - \frac{\theta^{2\Delta_{n+1}}}{c_n(\theta)} > 1 + \theta^2 - \theta^{2\Delta_{n+1}} > 1.$$

Thus,

$$1 < c_n(\theta) < 2 \quad \text{for all } n.$$

A.3 Innovations algorithm

Following to Brockwell and Davis (1991, Proposition 5.2.2, pp.172), we obtain the coefficients of $\hat{X}_{t_{n+1}} = \sum_{j=1}^{n} \theta_{nj}(X_{t_{n+1-j}} - \hat{X}_{t_{n+1-j}})$ through the next proposition.

Proposition 8. *If* $\{X_{t_n}\}_{n\geq 1}$ *has zero mean and* $E(X_{t_i}X_{t_j}) = \gamma(t_i,t_j)$*, where the matrix* $\left[\gamma(t_i,t_j)\right]_{i,j=1}^{n}$ *is non-singular for each* $n = 1,2,\ldots$*, then the one-step predictors* $\hat{X}_{t_{n+1}}$*,* $n \geq 0$*, and their mean squared errors* υ_n*,* $n \geq 1$*, are given by*

$$\hat{X}_{t_{n+1}} = \begin{cases} 0 & \text{if} \quad n = 0, \\ \sum_{j=1}^{n} \theta_{nj}\left(X_{t_{n+1-j}} - \hat{X}_{t_{n+1-j}}\right) & \text{if} \quad n \geq 1, \end{cases}$$

and

$$\begin{cases} \upsilon_1 = \gamma(t_1,t_1), \\ \theta_{n,n-k} = \upsilon_{k+1}^{-1}\left(\gamma(t_{n+1},t_{k+1}) - \sum_{j=0}^{k-1}\theta_{k,k-j}\theta_{n,n-j}\upsilon_{j+1}\right), \quad k = 0,1,\ldots,n-1, \\ \upsilon_{n+1} = \gamma(t_{n+1},t_{n+1}) - \sum_{j=0}^{n-1}\theta_{n,n-j}^2\upsilon_{j+1}. \end{cases}$$

For the irregularly spaced first-order moving average process of general form we have,

$$\theta_{n,j} = 0, \quad 2 \leq j \leq n,$$
$$\theta_{n,1} = \upsilon_n^{-1}\gamma_{1,\Delta_{n+1}},$$
$$\upsilon_{n+1} = \gamma_0 - \theta_{n,1}^2\upsilon_n = \gamma_0 - \frac{\gamma_{1,\Delta_{n+1}}^2}{\upsilon_n}, \text{ and } \upsilon_1 = \gamma_0.$$

A.4 Measures of performance-regularly spaced case

In Table A.1, we present the Monte Carlo results when we consider regularly spaced times to compare with the irregularly spaced case. As we can see, the Monte Carlo results for the regularly spaced time case has the same behavior that the irregularly spaced time case. Nevertheless, the regularly spaced time case has less uncertain.

N	θ_0	$\hat{\theta}^{\text{MLE}}$	$\widehat{\text{se}}(\hat{\theta}^{\text{MLE}})$	$\widetilde{\text{se}}(\hat{\theta}^{\text{MLE}})$	$\widehat{\text{bias}}_{\hat{\theta}^{\text{MLE}}}$	$\widehat{\text{RMSE}}_{\hat{\theta}^{\text{MLE}}}$	$\widehat{\text{CV}}_{\hat{\theta}^{\text{MLE}}}$
100	0.1	0.108	0.103	0.087	0.008	0.103	0.952
	0.5	0.502	0.089	0.096	0.002	0.089	0.178
	0.9	0.910	0.051	0.055	0.010	0.052	0.056
500	0.1	0.097	0.045	0.044	−0.003	0.045	0.461
	0.5	0.499	0.039	0.039	−0.001	0.039	0.078
	0.9	0.901	0.020	0.021	0.001	0.020	0.022
1500	0.1	0.099	0.026	0.026	−0.001	0.026	0.261
	0.5	0.500	0.022	0.022	0.000	0.022	0.045
	0.9	0.901	0.011	0.011	0.001	0.011	0.013
N	θ_0	$\hat{\theta}^{\text{b}}$	$\widehat{\text{se}}(\hat{\theta}^{\text{b}})$	$\widetilde{\text{se}}(\hat{\theta}^{\text{b}})$	$\widehat{\text{bias}}_{\hat{\theta}^{\text{b}}}$	$\widehat{\text{RMSE}}_{\hat{\theta}^{\text{b}}}$	$\widehat{\text{CV}}_{\hat{\theta}^{\text{b}}}$
100	0.1	0.120	0.082	0.073	0.020	0.084	0.685
	0.5	0.502	0.093	0.099	0.002	0.093	0.186
	0.9	0.915	0.049	0.050	0.015	0.051	0.054
500	0.1	0.098	0.041	0.042	−0.002	0.042	0.422
	0.5	0.499	0.039	0.041	−0.001	0.039	0.079
	0.9	0.902	0.020	0.021	0.002	0.021	0.023
1500	0.1	0.098	0.026	0.026	−0.002	0.026	0.260
	0.5	0.500	0.022	0.023	0.000	0.022	0.045
	0.9	0.902	0.011	0.012	0.002	0.011	0.013

Table A.1: Monte Carlo results for the regularly spaced time case. The MCE estimated is 0.003.

A.5 R package: `istsa`

We present our package called `istsa`, which means irregularly spaced time series analysis. The functions to simulate irregularly spaced times are:

timeSimExp Exponential increments. This function simulate irregularly spaced times $t_1, t_2, \ldots, t_N$ with independent exponential increments. Let $\Delta_1, \ldots, \Delta_N$ be time increments with $\Delta_1 = t_1$. The function define $\Delta_n = \texttt{rexp}(\texttt{rate} = \texttt{rate1}) + 1$ with probability `w`, or $\Delta_n = \texttt{rexp}(\texttt{rate} = \texttt{rate2}) + 1$ with probability $1 - \texttt{w}$ for $n = 1, \ldots, N$. The assignation is made randomly. We sum one to ensure $\Delta_n \geq 1$ for all n.

```
time <- timeSimExp(N = 100, rate1 = 1, rate2 = 1, w = 0.50)
Delta <- diff(time)
summary(Delta)
```

timeSimPois Poisson increments. This function simulate irregularly spaced times $t_1, t_2, \ldots, t_N$ with independent Poisson increments. Let $\Delta_1, \ldots, \Delta_N$ be time increments with $\Delta_1 = t_1$. The function define $\Delta_n = \texttt{rpois}(\texttt{lambda} = \texttt{lambda1}) + 1$ with probability `w`, or $\Delta_n = \texttt{rpois}(\texttt{lambda} = \texttt{lambda2}) + 1$ with probability $1 - \texttt{w}$ for $n = 1, \ldots, N$. The assignation is made randomly. We sum one to ensure $\Delta_n \geq 1$ for all n.

```
time <- timeSimPois(N = 100, lambda1 = 1, lambda2 = 1, w = 0.50)
Delta <- diff(time)
table(Delta)
```

perSamp Periodic sampling, see Robinson (1977). Let $\tau_1, \ldots, \tau_M$ be a set of times such that $\tau_n - \tau_{n-1} \geq 1$ for $n = 2, \ldots, M$. This function create a periodic pattern as follow:

$$\tau_1, \ldots, \tau_M, \tau_1 + \pi, \ldots, \tau_M + \pi, \ldots, \tau_1 + (k-1)\pi, \ldots, \tau_M + (k-1)\pi.$$

```
time <- perSamp(tau = 1:5, k = 3, pi = 24)
Delta <- diff(time)
table(Delta)
```

The functions of the IMA model are:

`imaSim` Simulation. Given parameters and a time vector of length N, this function simulates M trajectories from an IMA process. Trajectories are simulated as random samples from a multivariate Gaussian distribution with mean 0 and covariance matrix Γ (the IMA covariance matrix). Times vector can be simulated by `timeSimExp(N)`.

```
set.seed(1234)
imaSeries <- imaSim(theta = 0.4, var = 1,
        time = timeSimExp(N = 250, rate1 = 0.1),
        M = 4)

plot(imaSeries, nc = 2, panel = function(...){
        lines(...)
        rug(x = zoo::index(imaSeries), col = 2)})
```

imaMinusLogLik Minus log-likelihood. This function finds minus log-likelihood of IMA process with likelihood $= \text{MN}(0, \Gamma)$, where Γ is the IMA covariance matrix.

```
set.seed(1234)
imaSeries <- imaSim(theta = 0.4,
        time = timeSimExp(N = 250, rate1 = 0.1))
```

```
imaMinusLogLik(par = c(theta = 0.2, var = 2),
        serie = zoo::coredata(imaSeries),
        time = zoo::index(imaSeries))
```

imaGbcf General backward continued fraction. This function calculates the general backward continued fraction $c_1(\theta), \ldots, c_N(\theta)$ for the IMA process.

```
imaGbcf(theta = 0.8,
        time = timeSimExp(N = 250, rate1 = 0.1))
```

`imaPredictOneStep` Finite one-step linear prediction and mean squared prediction errors. This function calculates the finite one-step linear prediction and the mean squared prediction errors for the IMA process.

```
data(asth, package = 'cts')

mleEst <- imaMLE(serie = asth[1:100, 2], time = asth[1:100, 1])

predV <- imaPredictOneStep(theta = unname(mleEst$par['theta']),
        var   = unname(mleEst$par['var']),
        serie = asth[1:100, 2], time  = asth[1:100, 1])

plot(predV$serie, xlab = "Time in hours", ylab = "",
        lwd = 1.5, col = "gray40")
rug(asth[, 1], col = "red")

lines(predV$PredictOneStep, col = 4, lty = 2, lwd = 2)
lines(predV$PredictOneStep - (1.96*sqrt(predV$mspeOneStep)),
        col = 4, lty = 3)
```

```
lines(predV$PredictOneStep + (1.96*sqrt(predV$mspeOneStep)),
        col = 4, lty = 3)
```

`imaMLE` Maximum likelihood estimation. This function estimates the parameters of IMA process by maximum likelihood. It uses the L-BFGS-B method with bounds `lower = c(l.theta = 0.01, l.var = 0.01)` and `upper = c(u.theta = 0.99, u.var = Inf)`. We can to pass arguments to `optim`. For instance, `hessian`, a logical: should a numerically differentiated Hessian matrix be returned?

```
set.seed(1234)
imaSeries <- imaSim(theta = 0.4,
        time = timeSimExp(N = 250, rate1 = 0.1))

mleEst <- imaMLE(iPar = c(theta = 0.5, var = 1),
        serie = zoo::coredata(imaSeries),
        time = zoo::index(imaSeries),
        hessian = TRUE)

# ml estimation
mleEst$par

# estimated standard errors
sqrt(diag(solve(mleEst$hessian)))
```

`imaBootSamples` Bootstrap sample generation. This function generates bootstrap samples from a trajectory of the IMA process.

```
data(asth, package = 'cts')

mleEst <- imaMLE(serie = asth[1:100, 2], time = asth[1:100, 1])

bootSam <- imaBootSamples(theta=unname(mleEst$par['theta']),
var = unname(mleEst$par['var']),
serie = asth[1:100, 2], time = asth[1:100, 1], B = 6)

plot(bootSam)

bootEst <- apply(X = zoo::coredata(bootSam), MARGIN = 2,
FUN = function(s){imaMLE(serie = s,
time = asth[1:100, 1])$par})

# bootstrap estimation
rowMeans(bootEst)

# estimated standard errors
apply(X = bootEst, MARGIN = 1, FUN = sd))
```

Appendix B

Appendix Chapter 3

B.1 Measures of performance-regularly spaced case

In Table B.1, we present the Monte Carlo results when we consider regularly spaced times to compare with the irregularly spaced case. As we can see, the Monte Carlo results for the regularly spaced time case has the same behavior that the irregularly spaced time case. Nevertheless, the regularly spaced time case has less uncertain. The autoregressive parameter shows similar results as moving average cases, at least for values considered.

B.2 R package: `istsa`

Below, we present additional functions of the istsa package. The following functions are related to the IARMA model:

`iarmaSim` Simulation. Given parameters and a time vector of length N, this function simulates M trajectories from an IARMA process. Trajectories are simulated through the transition equation from the state-space representation of the model. Times vector can be simulated by `timeSimExp(N)`.

N	θ_0	$\hat{\theta}^{\text{MLE}}$	$\widehat{\text{se}}(\hat{\theta}^{\text{MLE}})$	$\widetilde{\text{se}}(\hat{\theta}^{\text{MLE}})$	$\widehat{\text{bias}}_{\hat{\theta}^{\text{MLE}}}$	$\widehat{\text{RMSE}}_{\hat{\theta}^{\text{MLE}}}$	$\widehat{\text{CV}}_{\hat{\theta}^{\text{MLE}}}$
100	0.1	0.165	0.183	0.146	0.065	0.195	1.111
	0.5	0.521	0.110	0.117	0.021	0.112	0.211
	0.9	0.914	0.052	0.056	0.014	0.054	0.057
500	0.1	0.109	0.079	0.071	0.009	0.079	0.724
	0.5	0.503	0.049	0.049	0.003	0.049	0.097
	0.9	0.901	0.021	0.021	0.001	0.021	0.023
1500	0.1	0.103	0.045	0.044	0.003	0.045	0.438
	0.5	0.502	0.028	0.028	0.002	0.028	0.056
	0.9	0.901	0.012	0.012	0.001	0.012	0.013
N	ϕ_0	$\hat{\phi}^{\text{MLE}}$	$\widehat{\text{se}}(\hat{\phi}^{\text{MLE}})$	$\widetilde{\text{se}}(\hat{\phi}^{\text{MLE}})$	$\widehat{\text{bias}}_{\hat{\phi}^{\text{MLE}}}$	$\widehat{\text{RMSE}}_{\hat{\phi}^{\text{MLE}}}$	$\widehat{\text{CV}}_{\hat{\phi}^{\text{MLE}}}$
100	0.5	0.466	0.112	0.114	-0.034	0.117	0.240
500		0.493	0.049	0.049	-0.007	0.049	0.099
1500		0.497	0.028	0.028	-0.003	0.028	0.057
N	θ_0	$\hat{\theta}^{\text{b}}$	$\widehat{\text{se}}(\hat{\theta}^{\text{b}})$	$\widetilde{\text{se}}(\hat{\theta}^{\text{b}})$	$\widehat{\text{bias}}_{\hat{\theta}^{\text{b}}}$	$\widehat{\text{RMSE}}_{\hat{\theta}^{\text{b}}}$	$\widehat{\text{CV}}_{\hat{\theta}^{\text{b}}}$
100	0.1	0.218	0.144	0.112	0.118	0.186	0.658
	0.5	0.538	0.114	0.111	0.038	0.120	0.212
	0.9	0.921	0.049	0.048	0.021	0.053	0.053
500	0.1	0.124	0.068	0.064	0.024	0.072	0.550
	0.5	0.506	0.049	0.049	0.006	0.049	0.097
	0.9	0.904	0.021	0.022	0.004	0.021	0.023
1500	0.1	0.107	0.043	0.042	0.007	0.043	0.397
	0.5	0.503	0.028	0.029	0.003	0.028	0.056
	0.9	0.902	0.012	0.012	0.002	0.012	0.013
N	ϕ_0	$\hat{\phi}^{\text{b}}$	$\widehat{\text{se}}(\hat{\phi}^{\text{b}})$	$\widetilde{\text{se}}(\hat{\phi}^{\text{b}})$	$\widehat{\text{bias}}_{\hat{\phi}^{\text{b}}}$	$\widehat{\text{RMSE}}_{\hat{\phi}^{\text{b}}}$	$\widehat{\text{CV}}_{\hat{\phi}^{\text{b}}}$
100	0.5	0.435	0.115	0.108	-0.065	0.132	0.263
500		0.486	0.049	0.049	-0.014	0.051	0.101
1500		0.495	0.028	0.028	-0.005	0.029	0.057

Table B.1: Monte Carlo results for the regularly spaced time case. The MCE estimated is 0.005. When $\phi_0 = 0.5$, we use $\theta_0 = 0.5$.

```
set.seed(1234)
iarmaSeries <- iarmaSim(phi = 0.8, theta = 0.3, var = 15,
        time = timeSimExp(N = 250), M = 4)

plot(iarmaSeries, nc = 2, panel = function(...){
        lines(...)
        rug(x = zoo::index(iarmaSeries), col = 2)})
```

iarmaSS Space-state representation. This function create a state-space representation out of the IARMA model (see Luethi et al. (2018)).

```
sim <- iarmaSim(phi = 0.8, theta = 0.3, var = 15,
        time = timeSimExp(N = 250), M = 4)

serie <- zoo::coredata(sim[,1])
time <- zoo::index(sim)
par <- c(0.2, 0.6, 30)

iarmaSS(phi = par[1], theta = par[2], var = par[3],
        serie, time)
```

iarmaMinusLogLik Minus log-likelihood. This function finds minus log-likelihood of IARMA process through the space-state representation of the model.

```
sim <- iarmaSim(phi = 0.8, theta = 0.3,
        var = 15, time = timeSimExp(N = 100),
        M = 4)

iarmaMinusLogLik(par = c(0.8, 0.3, 15),
```

```
        serie = zoo::coredata(sim[,1]),
        time = zoo::index(sim))
```

iarmaGbcf General backward continued fraction. This function calculates the general backward continued fraction $c_1(\phi,\theta),\ldots,c_N(\phi,\theta)$ for the IARMA process.

```
iarmaGbcf(phi = 0.2, theta = 0.6,
        time = timeSimExp(N = 100))
```

`iarmaPredictOneStep` Finite one-step linear prediction and mean squared prediction errors. This function calculates the finite one-step linear prediction and the mean squared prediction errors for the IARMA process.

```
sim <- iarmaSim(phi = 0.8, theta = 0.3, var = 15,
        time = timeSimExp(N = 1000), M = 4)

fit <- iarmaMLE(serie = zoo::coredata(sim[,1]),
        time = zoo::index(sim), hessian = TRUE)

predV <- iarmaPredictOneStep(phi = fit$par["phi"],
        theta = fit$par["theta"], var = fit$par["var"],
        serie = zoo::coredata(sim[,1]),
        time = zoo::index(sim))

plot(predV$serie, xlab = "Time in hours", ylab = "",
        lwd = 1.5, col = "gray40")
rug(zoo::index(predV), col = "red")

lines(predV$PredictOneStep, col = 4, lty = 2, lwd = 2)
```

```
lines(predV$PredictOneStep - (1.96*sqrt(predV$mspeOneStep)),
        col = 4, lty = 3)
lines(predV$PredictOneStep + (1.96*sqrt(predV$mspeOneStep)),
        col = 4, lty = 3)
```

`iarmaMLE` Maximum likelihood estimation. This function estimates the parameters of IARMA process by maximum likelihood. It uses the L-BFGS-B method with bounds `lower = c(l.phi = 0.01, l.theta = 0.01, l.var = 0.01)` and `upper = c(u.phi = 0.99, u.theta = 0.99, u.var = Inf)`. We can to pass arguments to `optim`. For instance, `hessian`, a logical: should a numerically differentiated Hessian matrix be returned?

```
sim <- iarmaSim(phi = 0.8, theta = 0.3, var = 15,
        time = timeSimExp(N = 1000), M = 4)

fit <- iarmaMLE(iPar = c(phi = 0.5, theta = 0.5, var = 1),
        serie = zoo::coredata(sim[,1]),
        time = zoo::index(sim), hessian = TRUE)

# ml estimation
fit$par

# estimated standard errors
sqrt(diag(solve(fit$hessian)))
```

`iarmaBootSamples` Bootstrap sample generation. This function generates bootstrap samples from a trajectory of the IARMA process.

```
sim <- iarmaSim(phi = 0.8, theta = 0.3, var = 15,
time = timeSimExp(N = 1000), M = 4)

fit <- iarmaMLE(serie = zoo::coredata(sim[,1]),
time = zoo::index(sim), hessian = TRUE)

bootSam <- iarmaBootSamples(phi = fit$par["phi"],
theta = fit$par["theta"],
var = fit$par["var"],
serie = zoo::coredata(sim[,1]),
time = zoo::index(sim), B = 5)

bootEst <- apply(X = zoo::coredata(bootSam),
MARGIN = 2, FUN = function(s){
iarmaMLE(serie = s,
time = zoo::index(sim))$par
})

# bootstrap estimation
rowMeans(bootEst)

# estimated standard errors
apply(X = bootEst, MARGIN = 1, FUN = sd))
```

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