Pontificia Universidad Católica de Chile

Facultad de Física

Instituto de Física

Loops in Holographic Correlators

Iván Muñoz Sandoval

Thesis submitted for the degree of

Doctor en Física

Advisor:

Máximo Bañados

Comisión: Alberto Faraggi (UNAB)
Benjamin Koch (PUC, TUW)

Santiago, Chile

Enero, 2023

Dedicado a Leonor y a mi abuela Sonia

Abstract

In the context of the Anti de-Sitter (AdS)/Conformal Field Theory (CFT) correspondence, we investigate the computation of holographic correlation functions for quantum fields in the bulk. Unlike the semi-classical approach, quantum computations involve Infra-Red (IR) and Ultra-Violet (UV) divergences. However, consistent with the semi-classical approximation, we find that IR infinities correspond to boundary divergences, while UV divergences correspond to the bulk. We present a systematic procedure for solving the perturbative quantum problem in the bulk.

To illustrate our approach, we consider a Φ^4 scalar field on a fixed AdS background and obtain the boundary correlation function in position and momentum space. In position space, we use two approximations: (i) we assume that the field is composed of the classical solution plus a quantum fluctuation, and we solve the classical part before using the holographic dictionary to obtain the quantum correction to the 2- and 4-point functions, requiring UV and IR renormalizations; (ii) using the quantum effective action, we renormalize the UV divergence from the equation of motions and then use the holographic dictionary to obtain the dual correlation function. Both formulations lead to the same conclusions and demonstrate that the bulk theory is renormalizable up to AdS_7.

Meanwhile, in momentum space, we use the background field method and renormalize the two-point function up to one loop, finding exact agreement with the position space computation. Finally, we provide a general set-up for obtaining the off-shell graviton bulk propagator, which is crucial for obtaining correlation functions for more realistic

models.

Acknowledgements

Quisiera agradecer a mi supervisor, Máximo Bañados por toda la paciencia y siempre empujarme a entender el tema de estudio al nivel más profundo y explicar de forma precisa. He aprendido mucho de él y definitivamente ha marcado mi forma de como enfrentar un tema que aprender y presentar.

Agradecer especialmente a Kostas Skenderis, por su gran apoyo en mi pasantía en la Universidad de Southampton y Ernesto Bianchi, quien también aportó invaluables comentarios en las reuniones de grupo y fue parte fundamental en el desarrollo de este trabajo.

A Ben Koch y Alberto Faraggi por su revisión del trabajo, comentarios y criticas que me han ayudado mucho a mejorar. Agradecer a Beto también los diversos comentarios y conversaciones de física.

A mi familia por su incondicional apoyo en todo momento. En especial a Rebeca por su cariño, en especial en la redacción de este trabajo. A Leonor que aún no llega y ya nos llena de amor.

A mis amigos y compañeros que sirvieron de escape en diversos momentos. No puedo dejar fuera a Negro, Raúl, Cesar, Chirri, Pirigüin, Ivania, Carlos, Manu y Daniel y tantos que han sido parte de este proceso. A Anita Alvarez por sus comentarios, sugerencias y correcciones sobre el ingles del trabajo.
To the guys in Southampton, Aaron, Federico, Elliot, Martina, and Sami, that made

everything much easier and more pleasant.

Agradecer el tremendo trabajo de los administrativos de la facultad de física, Emilio Bravo, Carolina Saavedra, Ximena Martinez, Juan Acuña, Cynthia Castillo y Yasna Rios

Muy probablemente deje a muchas personas importantes fuera, mis disculpas y gratitud hacia ellas.

Este trabajo fue financiado por Agencia Nacional de Investigación (ANID)-PCHA/Doctorado Nacional/2016-21160784 y Fondecyt No. 1201145.

Muchas gracias a todos!

Contents

List of Figures

Chapter 1

Introduction

Among the known four fundamental natural forces, gravity was the first to have a satisfactory description given by Newton's law of universal gravitation [1]. Using Newton's law is possible to describe the motion of planets and explain why we stay stuck to the earth. However, Newton's gravitational law cannot describe Mercury's orbit. To explain this discrepancy, there are two options: another unknown object perturbing Mercury's orbit, or Newton's law is incomplete. Over the coming centuries, no evidence of an extra body may modify Mercury's orbit, suggesting that Newton's law was not the end of the story of describing gravity.

In 1915, Einstein published perhaps the most fantastic theory on physics, the General Theory of Relativity or just General Relativity (GR) [2]. This theory changes our understanding of space and time and describes gravity as an effect of curved space-time. Perhaps the most simple description of GR is Wheeler's quote: *"Space-time tells matter how to move; matter tells space-time how to curve"*.

One of the earliest pieces of evidence in support of GR was the correct description of Mercury's orbit, and not many years later, the observation of the deflection of light when it passes close to a massive object. Even today, there are further and further experimental confirmations of GR predictions. Some examples include the existence of black holes [3, 4, 5, 6, 7], gravitational waves [8, 9]or the frame-dragging effect [10]. Each experiment gives more support to Einstein's theory of gravity. Despite this success, the theory predicts objects we cannot fully explain within the theory. Black holes are the most typical example of this. According to GR, black holes are singularities of space-time where the space-time itself breaks and, therefore, GR breaks. In analogy, due to Mercury's orbit, we knew that Newton's law of gravity was not the last word in the description of gravity. Black holes tell us that GR cannot be the last word in the description of gravity.

Alongside the development of GR, quantum mechanics arose as a successful theory to describe the micro-world. Even more, by considering quantum mechanics with special relativity, the quantum field theory (QFT) was developed and allowed to go into the quest of understanding a fundamental aspect of nature: What compose matter. Among the many possibles QFT, the most successful description is the Standard Model (SM) of particles that accurately describes the fundamental component of matter and how they interact in flat space-time. Despite this, the SM did not predict the mass of the particle. The Higgs mechanism [11, 12, 13] tells us how particle acquires their mass, at least in theory. In 2012 the Higgs particle was discovered by ATLAS [14], and CMS [15] at the LHC, adding the last missing block of the SM. However, as it happens with GR, the SM does not describe the whole microscopic world; it has not been able to describe the neutrino mass or what composes dark matter. Nevertheless, the biggest problem of SM is that it has nothing to say about gravity. Trying to fit gravity, as described by GR, into the quantisation scheme of the SM conduces to infinities that cannot be removed without modifying the original theory, i.e. gravity is non-renormalisable, at least by perturbation theory [16].

So far, our knowledge of how to describe the fundamental forces relies upon two very different theories: On one hand, the GR, a classical theory describing gravity as the curvature of space-time; on the other hand, the SM tells us that a quantum theory can explain all the other forces on a flat space. GR and SM differ in their fundamental aspects, and one explains phenomena with remarkable precision that the other cannot describe and vice versa. Then is natural to ask ourselves how to find a theory that describes gravity in the most fundamental way that is compatible with the quantum field theory.

Finding a theory that makes the quantum theory and GR compatible will lead to a better understanding of black holes or the very early moments of the universe. Nowadays, the most promising candidate for a quantum theory of gravity is string theory.

However, even if we still do not have a complete theory of quantum gravity, we can still learn some features of black holes. For many years, they were believed to be cold, dark objects moving around the universe, absorbing everything and not letting go of anything that falls inside them. This idea changed when Bekenstein [17, 18] proves that black holes have an entropy proportional to the event horizon area; and later on, Hawking [19] found the proportionality constant for the entropy and the temperature for the black hole; proving that black holes are thermodynamic objects. In 1996 Strominger-Vafa [20] worked on counting the black hole microstate for a certain kind of black hole in string theory, which agrees with Bekenstein-Hawking entropy giving more evidence to the area formula for the entropy.

In statistical mechanics, a gas's entropy scales as the volume. So, the fact that black holes' entropy scales as the area gives the intuition that the nature of gravity is encoded in the area enclosing the volume. This is the called holographic principle [21, 22], named after hologram in optics where a two-dimensional image encodes the three-dimensional object we see.

Despite the strong evidence about the holographic nature of gravity, it was not until Maldacena's [23] work that he realised that type IIB superstring theory in certain limits decouples into apparently two different theories. One corresponds to type IIB Super-Gravity (SUGRA) on $AdS_5 \times S^5$, and the other theory is $\mathcal{N} = 4$ SYM living in the boundary of the AdS_5, i.e. in four dimensions. Maldacena conjecture that both theories are the same.

The conjecture, now called *Maldacena's conjecture*, *AdS/CFT*, *Gauge/Gravity duality* or more general as *Holographic correspondance* is systematised after Witten's [24] and Gubser, Klebanov and Polyakov [25] work where they state what is now called the "holographic dictionary" or GPKW rules.

According to the holographic dictionary, for each classical field in the bulk, we demand an arbitrary but fixed boundary condition. From the point of view of the CFT, the boundary condition for the bulks classical field has the interpretation as the source from which we can compute correlation functions for conformal operators. So, to each bulk field correspond a gauge invariant conformal operator, and the boundary condition corresponds to the source of the dual operator. Table 1.1 summarises the correspondence between bulk fields and CFT operators. Other important characteristics are:

Field in AdS_{d+1}	CFT_d Operator	Conformal dimension
Scalar of mass m	$\mathcal{O}$	$\Delta = \frac{d}{2} \pm \frac{1}{2}\sqrt{d^2 + 4m^2}$
Gauge field A_μ	Conserved current J^a	$d-1$
Graviton $g_{\mu\nu}$	Energy momentum tensor T^{ab}	d

Table 1.1: The equivalence between bulk fields and their duals is given by Lorentz invariance. In the table, the $\mu : 0, \dots, d$ are the bulk index and $a = 1, \dots, d$ are the boundary labels. In most cases, the conformal dimension corresponds to the positive solution for the scalar field, but there are cases where we can choose either option [26].

- Strong/weak duality: What is strongly coupled on one side is weakly coupled on the other; this allows us to consider the classical SUGRA solution, which is dual to strongly coupled CFT in the boundary. Because of this property, the AdS/CFT correspondence has application in the study theories that do not allow perturbative expansion.

- UV/IR connection: The UV divergences on one side correspond to IR divergence on the dual theory. This duality will play a substantial role in the discussion of the present work

Following the holographic dictionary, Witten computes the holographic 2-point function for the bulk massless scalar field, finding precisely the expected 2-point function for a CFT. The same happens with the bulk gauge field, giving the known 2-point function for conserved current in CFT. In analogy to the Feynman diagram, the boundary correlation functions correspond to bulk Witten diagrams. They are very similar to the Feynman diagram, but now the external legs are attached to the boundary of AdS.

However, due to the infinite volume of AdS, there are IR divergences that need regularisation and renormalisation. Dealing with these infinities is called holographic renormalisation; it was introduced in [27, 28]. Variational methods [29, 30, 31] have been developed to work the holographic renormalisation problem. In [32] is presented an introductory review. All counterterms needed to cancel the IR divergences are local in the renormalisation process. From the point of view of the CFT, the divergences correspond to the usual UV infinities that arise in standard QFT. The holographic renormalisation mechanism is crucial to obtaining the correct holographic Weyl anomaly of the dual CFT [27].

Although there is no proof of the Maldacena conjecture because we do not know a quantum theory for gravity, no counterexample has been found. Now, it is widely believed that the duality is true. Some reviews on gauge/gravity duality are [33, 34]

The classical bulk computation of holographic correlation functions already gives the expected results for a CFT. Then we may ask ourselves what happens if, in the bulk, we consider not only classical fields but the quantum nature of the field. In particular, what are the consequences of considering quantum fluctuation in the computation of the holographic correlation function. If the conjecture holds, then the quantum effects must be re-organised such that the holographic correlation function respects the known conformal structure.

In analogy with standard QFT, considering quantum fields in the AdS/CFT context is expected to find Ultra-Violet (UV) divergences due to the loops and Infra-Red (IR) divergences due to the infinite volume of AdS. Over the last year, much progress has been reported. Most use the CFT data to obtain the quantum correction of conformal correlation functions; an incomplete list is [35, 36, 37, 38, 39, 40, 41, 42]. While working from the AdS side, most of the work [43, 44, 45, 46, 47, 48, 49, 42, 50, 51] makes use of the spectral decomposition in the embedding space formalism, to directly compute loop Witten diagrams without focusing on the regularisation and renormalisation of the bulks theory. While in [52, 53] worked on the problem of Feynman loop diagram within AdS and then sent the external leg to the boundary.

By working the quantum correction for holographic correlation function, there has been no systematic treatment of the divergent structure on the AdS side. Our objective is to work on this side of the problem.

In this thesis, we study the role of quantum fluctuations in the dual correlation functions. In particular, we focus on the IR and UV divergences. We provide a systematic way to regularise and renormalise the bulk theory to have a well-defined theory and compute the correction to holographic correlation functions.

As usual, in the AdS/CFT correlation function, we use the position space techniques to study quantum correction. Understanding the position space treatment, we re-obtain the quantum holographic correlation function in momentum space.

In contrast with standard QFT, most works on the CFT correlation function are usually done in position space. Mainly because Ward identities in momentum spaces become second-order linear Partial Differential Equations (PDE) which is much more challenging than the position space computation. Even more, if we take the Fourier transformation to the known correlation function in position space, we will find cases in which we cannot compute it. Progress has been made on this topic. For example, in

[54, 55] is calculated, the 3-point function for momentum space finds new anomalies and in [56] is worked the 4-point function. An alternative approach to the problem of the CFT correlation function in momentum space is to use the AdS/CFT correspondence.

We can compute the dual correlation function by considering the bulk fields in momentum space along the coordinates transverse to the boundary and following the holographic dictionary. We work the loop correlation function in momentum space and check that both position and momentum spaces share the same structure, leading to the same conclusions about the renormalisation of the bulk theory.

The thesis structure is: In chapter 2, we briefly introduce conformal field theory and AdS space-time. In the end, we present more details of the holographic dictionary. Having introduced the holographic dictionary. In Chapter 3, we go in deep with the holographic renormalisation, compute the correlation function in the semi-classical approach and introduce Witten diagrams. In chapter 4[1], we work on the bulk quantum computation following two approaches:

1. Based on [57], we use the background field method, which consists in splitting the field into a classical part with non-trivial boundary conditions plus a quantum fluctuation. Following the holographic dictionary, we obtain the corresponding loop Witten diagram and compute the quantum corrections by regularising and renormalising the infinities.

2. We face the same problem but following the quantum effective action. We solve the UV problem by studying the equation of motion that comes from the effective quantum action. Then we solve the IR divergence.

In chapter 5, we review the computation of tree-level and quantum-level holographic correlation functions in momentum space. This is done considering the Fourier transformation

[1]This chapter is thanks to the excellent collaboration with Máximo Bañados, Ernesto Bianchi and Kostas Skenderis.

along the transverse coordinates and computing the 1-loop correction to the 2-point function. Last, in chapter 6, we discuss how to compute the off-shell AdS graviton propagator. This is a work in progress

Chapter 2

AdS/CFT basic

To introduce the AdS/CFT correspondence, we must present what we understand as CFT and AdS space-time. This chapter aims to introduce both topics. At the end of the chapter, we connect both theories via the so-called holographic dictionary.

The first part, 2.1, consists of an introduction to conformal symmetry and its consequence at the classical and quantum level. In particular, how conformal invariance restricts the structure of correlation function between conformal invariant operators.

In the second part 2.2, we will introduce AdS space-time, the different coordinates, isometries and the euclidean version of AdS.

In section 2.3, we will introduce the so-called AdS/CFT correspondence in which both, AdS and CFT are conjectured to be equals. The connection between both theories is understood via the holographic dictionary, allowing us to compute holographic correlation functions.

2.1 Conformal Symmetry

By definition, [58] a conformal transformation of the coordinates is an invertible map $x \to x'$ such that the metric tensor is invariant up to a scale

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x). \tag{2.1}$$

This transformation has Poincare transformation as a particular case when $\Lambda(x) = 1$, so it can be seen as a generalisation of Poincare transformation.

Considering an infinitesimal transformation, this is $x'^{\mu} = x^{\mu} + \epsilon^{\mu}$ where $\epsilon << 1$, the metric at first order will change as

$$g'_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta} \tag{2.2}$$

$$= g_{\mu\nu} - \left(\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu}\right). \tag{2.3}$$

Using $\Lambda(x) = 1 - f(x)$ we have

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = f(x) g_{\mu\nu}. \tag{2.4}$$

Taking the trace we find $f(x) = \frac{2}{d}\partial_{\lambda}\epsilon^{\lambda}$. For simplicity, let us consider $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(1, \ldots, 1)$ for simplicity. Applying a derivative on (2.4) and doing some algebra, we find

$$2\partial_{\mu}\partial_{\nu}\epsilon_{\rho} = \eta_{\mu\rho}\partial_{\nu} f + \eta_{\nu\rho}\partial_{\mu} f - \eta_{\mu\nu}\partial_{\rho} f, \tag{2.5}$$

contracting with $\eta^{\mu\nu}$, this becomes

$$2\partial^2\epsilon_{\mu} = (2-d)\partial_{\mu} f. \tag{2.6}$$

Applying ∂_ν on the former equation and ∂^2 on (2.4),

$$(2-d)\partial_\mu \partial_\nu f = \eta_{\mu\nu}\partial^2 f, \tag{2.7}$$

and then contracting with $\eta^{\mu\nu}$ we have

$$(d-1)\partial^2 f = 0. \tag{2.8}$$

For different dimensions, the equation demands different treatments.

For $d=1$, the f function is arbitrary, and this can be seen as the absence of angles in $d=1$. For $d=2$ corresponds to the holomorphic transformation $z \to f(z)$. CFT in $d=2$ plays a crucial role in string theory and is a whole topic on its own. We refer to [59, 60, 58, 61] for further information.

For $d \geq 3$ the eq. (2.8) and eq. (2.7) implies that f has to be at most linear in the coordinates. From $f = \frac{2}{d}\partial_\lambda \epsilon^\lambda$ we see that ϵ_μ is, at most, quadratic on the coordinates. So, we propose,

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\lambda}x^\nu x^\lambda. \tag{2.9}$$

We have to find what are $a_\mu, b_{\mu\nu}$ and $c_{\mu\nu\lambda}$ for general ϵ. We will plug each term on the equations above, finding a_μ to be unconstrained and interpreted as translation. Substituting the linear term on (2.4) gives,

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d} b\eta_{\mu\nu},\ b = b^\lambda_{\ \lambda}, \tag{2.10}$$

this means that $b_{\mu\nu}$ is given by an anti-symmetric part and a pure trace part $b_{\mu\nu} = m_{\mu\nu} + \alpha\eta_{\mu\nu}$, where $m_{\mu\nu} = -m_{\nu\mu}$. The interpretation of α is as dilatation, and $m_{\mu\nu}$

corresponds to a rigid rotation. The quadratic term yields

$$c_{\mu\nu\lambda} = \eta_{\mu\lambda} b_\nu + \eta_{\mu\nu} b_\lambda - \eta_{\nu\rho} b_\mu, \text{ where } b_\mu = \frac{1}{d} c^\sigma{}_{\sigma\mu}. \tag{2.11}$$

From here we can check that the infinitesimal transformation is

$$x'^\mu = x^\mu + 2x^\lambda b_\lambda x^\mu - x^2 b^\mu, \tag{2.12}$$

which is called special conformal transformation (SCT). Note that SCT are non-linear. The finite form of SCT is

$$x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b_\lambda x^\lambda + b^2 x^2}. \tag{2.13}$$

A different way to obtain the SCT is by applying a sequence of inversion-translation-inversion,

$$\frac{x'^\mu}{x'^2} = \frac{x^\mu}{x^2} - b^\mu. \tag{2.14}$$

So we have the following finite transformation

- Translation $x'^\mu = x^\mu + a^\mu$,
- Dilatation $x'^\mu = \alpha x^\mu$,
- rotation $x'^\mu = M^\mu{}_\nu x^\nu$,
- SCT $x'^\mu = \dfrac{x^\mu - b^\mu x^2}{1 - 2b_\mu x^\mu + b^2 x^2}$.

The corresponding generator of each transformation is,

- Translation $P_\mu = -i\partial_\mu$,
- Dilatation $D = -ix^\mu \partial_\mu$,
- rotation $L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial \mu)$,

- SCT $K_\mu = -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu)$.

These generators satisfy the following commutation rules,

$$[D, P_\mu] = iP_\mu, \tag{2.15}$$

$$[D, K_\mu] = -iK_\mu, \tag{2.16}$$

$$[K_\mu, P_\nu] = 2i(\eta_{\mu\nu} D - L_{\mu\nu}), \tag{2.17}$$

$$[K_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu), \tag{2.18}$$

$$[P_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu), \tag{2.19}$$

$$[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho} L_{\mu\sigma} + \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho}), \tag{2.20}$$

which defines the $d-$dimensional conformal algebra for $d \geq 3$.

By defining the following generators,

$$J_{\mu\nu} = L_{\mu\nu},\ J_{d+1,\mu} = \frac{1}{2}(P_\mu - K_\mu),\ J_{d+1,0} = D,\ J_{0,\mu} = \frac{1}{2}(P_\mu + K_\mu), \tag{2.21}$$

where $J_{ab} = -J_{ab}$ with $a, b = 0, 1, \ldots, d, d+1$. The new generators satisfies de $SO(d+1, 1)$ algebra,

$$[J_{ab}, J_{cd}] = i(\eta_{ad} J_{bc} + \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac}), \tag{2.22}$$

where the $\eta_{ab} = \text{diag}(1, \ldots, 1, -1)$.

For $d = 2$ conformal symmetry, the algebra is infinite-dimensional and is known as Virasoro algebra.

2.1.1 Conformal invariance in quantum field theory

We want to build a QFT with conformal symmetry. Consider a conformal transformation, this is an invertible map $x \to x'(x) = x'$ such that equation (2.1) holds. Then, for example, a scalar field transforms as

$$\mathcal{O}'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta} \mathcal{O}(x), \tag{2.23}$$

with Δ is some real number called conformal dimension. Notice that in the language of a general change of coordinate, the field $\mathcal{O}$ transforms as a tensor of Δ indices. This is characteristic of conformal invariant theory, where even the scalar field may acquire a Jacobian under a change of coordinates.

If the field satisfies (2.23), we call it a primary field of conformal dimension Δ. Considering the quantum theory, we will call $\mathcal{O}$ indistinctly field or operator.

In QFT, and CFT is not the exception, our main object of interest is the correlation function. If we have the correlation function and use the Lehmann, Symanzik and Zimmermann (LSZ) reduction formula, we have solved the scattering process and found the S-matrix.

The $n-$point correlation function for conformal invariant operators is

$$\langle \mathcal{O}_{\Delta_1}(x_1)\dots\mathcal{O}_{\Delta_n}(x_n)\rangle = N\int D\mathcal{O}\ \left(\mathcal{O}_{\Delta_1}(x_1)\dots\mathcal{O}_{\Delta_n}(x_n)\right) e^{-S}, \tag{2.24}$$

with S a conformal invariant action and N a normalisation factor. By demanding conformal invariance of the correlation function then, we find that under re-scaling, the correlation function must satisfy,

$$\langle \mathcal{O}_{\Delta_1}(x_1)\dots\mathcal{O}_{\Delta_n}(x_n)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{-\frac{\Delta_1}{d}}\dots\left|\frac{\partial x'}{\partial x}\right|_{x=x_n}^{-\frac{\Delta_n}{d}} \langle \mathcal{O}_{\Delta_1}(x')\dots\mathcal{O}_{\Delta_n}(x')\rangle, \tag{2.25}$$

where d is the space-time dimension. We will sketch an exciting feature of conformal symmetry: it fixes the structure of the 2-point function and constrains the 3-point function up to a constant factor.

For the 2-point function, by translation invariance, it can only depend on $\vec{x}_{12} \equiv \vec{x}_1 - \vec{x}_2$ and rotation invariance forces us to depend on $|\vec{x}_{12}|$. Invariance under dilatation $x \to x' = \lambda x$ implies

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\rangle = \frac{C_{\Delta_1\Delta_2}}{|\vec{x}_{12}|^{\Delta_1+\Delta_2}}, \tag{2.26}$$

where $C_{\Delta_1\Delta_2}$ is a normalisation constant. Finally by using SCT we find $C_{\Delta_1\Delta_2} = c\delta_{\Delta_1,\Delta_2}$ so

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\rangle = \frac{C\delta_{\Delta_1,\Delta_2}}{|\vec{x}_{12}|^{2\Delta_1}}. \tag{2.27}$$

We may absorb the constant C by redefining the field $\mathcal{O}$.

By similar reasoning, we may find that the 3-point function is

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\rangle = \frac{C_{123}}{|\vec{x}_{12}|^{\Delta_1+\Delta_2-\Delta_3}|\vec{x}_{23}|^{\Delta_2+\Delta_3-\Delta_1}|\vec{x}_{13}|^{\Delta_1+\Delta_3-\Delta_2}}. \tag{2.28}$$

It is tempting to keep pushing forward with higher correlation functions. However, for $n \geq 4$, the conformal symmetry does not fully determine the correlation function. For example, by using the conformal symmetry for the 4-point function, we will find

$$\langle \mathcal{O}_{\Delta_1}(x_1)\ldots\mathcal{O}_{\Delta_4}(x_4)\rangle = f(u,v)\prod_{i<j}|\vec{x}_{ij}|^{-\Delta_i+\Delta_j+\frac{\Delta}{3}} \tag{2.29}$$

where

$$u = \frac{|\vec{x}_{12}||\vec{x}_{34}|}{|\vec{x}_{13}||\vec{x}_{24}|}, \quad v = \frac{|\vec{x}_{12}||\vec{x}_{34}|}{|\vec{x}_{23}||\vec{x}_{41}|}, \tag{2.30}$$

are known as crossing ratios, $f(u,v)$ is an arbitrary function not determined by the conformal symmetry and $\Delta = \sum_i \Delta_i$. In general, the n-point function will have $\frac{n(n-3)}{2}$ crossing ratios.

We can obtain the correlation function via the conformal Ward identity. But we will not follow this path.

Conformal symmetry notably fixes the 2-point function and, up to a factor, the 3-point function. However, to obtain the correlation function for some conformal invariant theory, we use the path integral (2.24), which is hard and, in most cases, impossible to compute. Thus, we use the functional generator

$$Z[J] = \int D\mathcal{O} e^{-S+\int dx J^a \mathcal{O}_a}, \tag{2.31}$$

where $a = \Delta_1, \ldots, \Delta_n$ is a label.

As we have added a new object to the path integral, we must demand to be conformal invariant. Then, under conformal transformation, the source must transform as

$$J'(\vec{x}') = \left|\frac{\partial x'}{\partial x}\right|^{-(d-\Delta)} J(x). \tag{2.32}$$

The correlation function (2.24) is obtained by taking derivatives with respect to the source J and then setting it to zero.

2.2 Anti-de Sitter Space

For the sake of coherence with the following sections, we will consider a $d+1$-dimensional theory of gravity.

The best theory that describes gravitation interaction is given by general relativity (GR). According to GR, gravity is nothing but the curvature of space-time, and the dynamical field is the metric. The Einstein equations are the equation of motion that govern the dynamics of the metric,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = \kappa^2 T_{\mu\nu}, \tag{2.33}$$

where $R_{\mu\nu}$ is the Ricci tensor, R the Ricci scalar, $g_{\mu\nu}$ is the metric, Λ the cosmological constant, $T_{\mu\nu}$ is the energy-momentum tensor and $\kappa = 8\pi G_N$ is a constant.

The l.h.s of Einstein's equations can be obtained from the Einstein-Hilbert action

$$S_{EH} = \frac{1}{2\kappa}\int d^{d+1}x\sqrt{-g}\,(R - 2\Lambda)\,. \tag{2.34}$$

While the r.h.s is obtained from the matter action, S_M,

$$T_{\mu\nu} = \frac{2}{\sqrt{g}}\frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{2.35}$$

From now on, we will consider the vacuum Einstein equation, which is $T_{\mu\nu} = 0$ and find a solution. In particular, we will find a maximally symmetric space-time solution.

Maximally symmetric space-time has the same number of symmetries as Minkowski space, which has $d+1$-translations and $(d+1)d/2$ rotations. So maximally symmetric $(d+1)$ dimensional space-times has $(d+1)(d+2)/2$ linearly independent Killing vectors.

Thinking in translation and rotation symmetry, the curvature must be constant on every point of the manifold. Because of this, the Riemann tensor has to be proportional to the Ricci scalar and using the symmetries of the Riemann tensor, it is

$$R_{\mu\nu\rho\sigma} = \frac{R}{d(d+1)}\left(g_{\nu\sigma}g_{\mu\rho} - g_{\nu\rho}g_{\mu\sigma}\right). \tag{2.36}$$

Using this in vacuum Einstein equation, we find $R = 2(d+1)/(d-1)\Lambda$ so the cosmological constant can be negative, positive or zero. If the cosmological constant is positive, we have a *De Sitter* space-time; if the cosmological constant is zero, then we have a flat space and if it is negative *Anti-De Sitter*(AdS_{d+1}) space. From now on, we will focus on AdS_{d+1} dimensions.

To describe the AdS_{d+1} space we will consider a $(d+2)-$dimensional Minkowski space with coordinates $\left(X^0, \ldots, X^d, X^{d+1}\right)$ and metric $\eta_{AB} = \mathrm{diag}(-,+,\ldots,+,-)$. The line element of the $d+2$-dimensional Minkowski space is

$$ds^2 = -\left(dX^0\right)^2 + \left(dX^1\right)^2 + \ldots \left(dX^d\right)^2 - \left(dX^{d+1}\right)^2 = \eta_{AB} dX^A dX^B, \tag{2.37}$$

where $A, B \in \{0, 1, \ldots, d+1\}$. The AdS_{d+1} can be seen as the hyper-surface embedded in the Minkowski space given by

$$-\left(X^0\right)^2 + \sum_{i=1}^{d} \left(X^i\right)^2 - \left(X^{d+1}\right)^2 = -l^2, \tag{2.38}$$

where l is the radius of curvature of AdS. The hyper-surface has $SO(d,2)$ invariance.

Anti-de-Sitter space has a conformal boundary given by X^A large. In this case, the hyperboloid approach, the lightcone given by

$$-\left(X^0\right)^2 + \sum_{i=1}^{d} \left(X^i\right)^2 - \left(X^{d+1}\right)^2 = 0, \tag{2.39}$$

which is invariant under $X \to \lambda X$, $\lambda \in \mathbb{R}$. The AdS_{d+1} boundary will be denoted by ∂AdS_{d+1} and corresponds to a compactification of $d-$dimensional Minkowski space. To see this, we must recall that the conformal group $SO(d,2)$ does not act on the Minkowski space because conformal transformation maps point to infinity, so we must add this point to the Minkowski space. To see this [24], let's introduce the lightcone coordinates, $u = X^d + X^{d+1}$ and $v = X^d - X^{d+1}$ so the boundary (2.39) is

$$uv - \eta_{\mu\nu} X^\mu X^\nu = 0, \quad \mu, \nu = 0, \ldots, d-1, \tag{2.40}$$

which is invariant under re-scaling. For $v \neq 0$ using the scaling invariance, we can set $v = 1$ and then solve for u; therefore, we have a $d-$dimensional Minkowski space. For $v = 0$, we have to add infinity to the Minkowski space. Having determined that the boundary of AdS_{d+1} is scaling invariant and corresponds to a compactified Minkowski space (from now on, we will call it just Minkowski space), we will now look for coordinates that may represent AdS_{d+1} space-time.

We will study different coordinate systems for AdS_{d+1} defined through (2.38). For example, we may use the parametrization

$$X^0 = l \cosh \rho \cos \tau, \tag{2.41}$$

$$X^{d+1} = l \cosh \rho \sin \tau, \tag{2.42}$$

$$X^i = l \Omega_i \sinh \rho, \;\; i = 1, \ldots, d. \tag{2.43}$$

Here Ω_i is the angular coordinates, while the range of the other coordinates is $0 < \rho < \infty$ and $0 \leq \tau \leq 2\pi$. These are known as global coordinates of AdS_{d+1}. Following this parametrization, we can find the AdS_{d+1} line element which is given by

$$ds^2 = l^2 \left(- \cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\Omega_{d-1}^2 \right). \tag{2.44}$$

Notice that the coordinate τ range is $[0, 2\pi]$ giving closed time-like curves. To solve this, we unwrap the τ coordinate such that $-\infty < \tau < \infty$.

We can write the metric more suggestively. Doing $\tan \theta = \sinh \rho$, we find for the global AdS metric

$$ds^2 = \frac{l^2}{\cos^2 \theta} \left(-d\tau^2 + d\theta^2 + \sin^2 \theta d\Omega_{d-1} \right) \tag{2.45}$$

with τ arbitrary and $\theta \in [0, \frac{\pi}{2}]$ for $d \neq 1$; and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ for $d = 1$.

Another useful parametrization that solves (2.38) is

$$X^0 = \frac{z}{2}\left(1 + \frac{1}{z^2}\left(\vec{x}^2 - t^2 + l^2\right)\right), \tag{2.46}$$

$$X^i = l\frac{x^i}{z}, \quad i = 1, \dots, d-1, \tag{2.47}$$

$$X^d = \frac{z}{2}\left(1 + \frac{1}{z^2}\left(\vec{x}^2 - t^2 - l^2\right)\right), \tag{2.48}$$

$$X^{d+1} = l\frac{t}{z}, \tag{2.49}$$

where, $z > 0$, $t \in \mathbb{R}$ and $\vec{x}$ is a $d-$dimensional real vector. These are the Poincare patch coordinates because $z > 0$ only covers one-half of the AdS_{d+1} space. The metric is

$$ds^2 = \frac{l^2}{z^2}\left(dz^2 - dt^2 + d\vec{x}^2\right). \tag{2.50}$$

If we change coordinates such that $z = le^{-y}$, then

$$ds^2 = e^{2y}\left(-dt^2 + d\vec{x}^2\right) + l^2 dy^2. \tag{2.51}$$

In these coordinates, if we send a light ray to infinity in y we have $ds^2 = 0$ along $\vec{x}$ constant we obtain

$$t = \int dt = l\int^{\infty} e^{-y} dy < \infty. \tag{2.52}$$

Thus, the light ray travels to the boundary in $y \to \infty$ in a finite time. This is due to the AdS hyperbolic structure.

In these coordinates, the Poincare invariance on the transverse coordinates $(t, \vec{x})$ and the dilatation invariance $(z, t, \vec{x}) \to (\lambda z, \lambda t, \lambda \vec{x})$ are very explicit.

2.2.1 Euclidean AdS

For this work, we will work on the Euclidean version of AdS_{d+1} which is obtained by performing a Wick rotation on the X^0, then (2.38) becomes,

$$\sum_{i=0}^{d}(X^i)^2 - \left(X^{d+1}\right)^2 = -l^2. \tag{2.53}$$

The isometry group now is $SO(d+1,1)$.

In parametrization (2.41), under the Wick rotation, $\tau \to i\tau$ means that $\cos\tau \to \cosh\tau$ and $\sin\tau \to \sinh\tau$ and the global euclidean AdS metric is

$$ds_G^2 = \frac{l^2}{\cos^2\theta}\left(d\tau^2 + d\theta^2 + \sin^2\theta d\Omega_{d-1}\right), \tag{2.54}$$

so the boundary metric in these coordinates is,

$$ds_G^2|_{boundary} = d\tau^2 + d\Omega_{d-1}^2. \tag{2.55}$$

On the other hand, by doing the Wick rotation to the Poincare coordinates we have,

$$ds^2 = \frac{1}{z^2}\left(dz^2 + \delta_{ij}dx^i dx^j\right), \tag{2.56}$$

with δ_{ij} is the flat space metric, i.e $\mathbb{R}^d$ space.

The boundary of Poincare coordinates is,

$$ds^2|_{boundary} = dt^2 + d\vec{x}^2 = d\rho^2 + \rho^2 d\Omega_{d-1}^2 = e^{2\tau}\left(d\tau^2 + d\Omega_{d-1},^2\right) \tag{2.57}$$

We wrote the flat space in spherical coordinates and then defined $\rho = e^\tau$. Comparing with (2.55), we notice that the boundary of global Euclidean AdS coordinates and the boundary of Poincare coordinates are, up to a possible conformal factor, the same.

The line element (2.56) is invariant under $SO(d+1,1)$. But is also invariant under inversion

$$z = \frac{z'}{z'^2 + \vec{x}'^2}, \quad \vec{x} = \frac{\vec{x}'}{z'^2 + \vec{x}'^2}. \tag{2.58}$$

They will play a significant role in understanding the structure of some relevant integrals that we will see in the following chapters.

From now on, we will work on euclidean AdS space-time in Poincare coordinates. For simplicity, we will call it AdS.

2.3 The AdS/CFT Correspondence

In the previous section, we notice that the boundary of AdS$_{d+1}$ has conformal symmetry and is connected causally with the interior of AdS. But, having the same symmetries does not enough implies that both theories are equivalent

More evidence on the possible connection between AdS and CFT is provided by Brown-Henneaux [62] where they find, by studying the global charges, that the asymptotic symmetry of AdS$_3$ develops a central charge that corresponds to the central charge of CFT$_2$.

So far, there are only hints on a possible relation between AdS and CFT, but no concrete example has been shown. The first concrete example of how is related AdS and CFT was provided by Maldacena in a very celebrated paper [23]. Maldacena gives the first well-defined example of a holographic correspondence between type IIB super-gravity on a $AdS_5 \times S^5$ background and $\mathcal{N} = 4$ Super Yang-Mills (SYM) with conformal symmetry. In Maldacena's seminal work, he works on both sides of the duality and conjectures that quantum gravity on AdS times a compact surface is equivalent to a CFT in the boundary of the AdS space. However, to prove the conjecture, we need to know the full theory of quantum gravity.

One of the main features of the AdS/CFT correspondence is the strong/weak duality. This duality states that the strong coupling regime on one side of the duality corresponds to the weak coupling on the other, so considering weakly coupled string theory we find SUGRA and should correspond to strong coupling CFT. This aspect of the duality has been exploited to understand the non-perturbative properties of CFT. Another crucial characteristic of the AdS/CFT is the IR/UV connection, in which an IR divergence on one side corresponds to an UV divergence on the other.

Despite all the properties that the AdS/CFT correspondence possesses, we still need a systematic way to compute the object of one side using only the other side. The first step for this objective was formulated by Gubser, Klebanov and Poliakov [25]; and independently by Witten [24], where they introduce the holographic dictionary. The holographic dictionary tells us how to relate a theory on AdS to some CFT on the boundary. The procedure, also known as the GKPW dictionary, gives us the recipe to compute the CFT correlation function using information from the AdS side.

The correspondence says that an AdS$_{d+1}$ space-time is dual to a CFT$_d$ in the conformal boundary. This can be seen as the partition function of each are the same:

$$Z_{AdS} = Z_{CFT}. \tag{2.59}$$

However, to truly relate both theories, we need further information.

We will focus on the AdS side to obtain information on the boundary theory. In particular, we will enunciate the dictionary that translates a theory in AdS such that we may get the correlation function of the dual CFT.

To find correlation functions of QFT, particularly in CFT, we modify the partition function by adding an arbitrary function that we call source, so the partition function is now a functional of the source. The correlation function is obtained by taking derivatives of the partition function with respect to the source. As the CFT partition function depends on the arbitrary source, the AdS partition function must also depend on an arbitrary function.

According to the GKPW dictionary, every field on AdS, let us say $\Phi(z, \vec{x})$, corresponds to a gauge-invariant CFT operator, namely $\mathcal{O}(\vec{x})$. In the boundary, the source of the gauge invariant operator is obtained by demanding that the bulk field have an arbitrary but fixed boundary condition. In other words, demanding that the bulk field in the boundary behave as $\Phi|_{\text{boundary}} \sim \phi_0(\vec{x})$, where $\phi_0(\vec{x})$ is identified as the source of the dual operator. Then the AdS/CFT correspondence can be stated as,

$$Z_{AdS}[\phi_0] \equiv \int_{\Phi\sim\phi_0} D\Phi e^{-S[\Phi]} = Z_{CFT}[J] \equiv \left\langle \exp\left(-\int dx J\mathcal{O}\right)\right\rangle_{CFT}, \tag{2.60}$$

where $J(\vec{x})$ is the source of the gauge invariant CFT operator and $\phi_0(\vec{x}) = J(\vec{x})$ is understood.

In (2.60), the partition function for AdS means the partition function of the complete theory of quantum gravity, which is unknown. While on the CFT side, in principle, we do not know what theory we are working on, but we know that by symmetries, it must have a determined structure such as (2.27).

Through the LSZ reduction formula, if we can determine the correlation function of the CFT, we have solved the theory. The CFT correlation functions are obtained by taking derivatives of (2.60) with respect to the source ϕ_0 and setting the source to zero. So, according to the AdS/CFT conjecture, we can compute the CFT correlation function from the AdS partition function,

$$\begin{aligned}
\langle \mathcal{O}(x) \rangle &= \frac{\delta Z_{AdS}}{\delta \phi_0(\vec{x})}\Big|_{\phi_0=0}, \\
\langle \mathcal{O}(x)\mathcal{O}(y) \rangle &= -\frac{\delta^2 Z_{AdS}}{\delta \phi_0(\vec{x})\delta \phi_0(\vec{y})}\Big|_{\phi_0=0}, \\
&\vdots \\
\langle \mathcal{O}(\vec{x}_1)\ldots\mathcal{O}(\vec{x}_n) \rangle &= (-1)^{n+1}\frac{\delta^n Z_{AdS}}{\delta \phi_0(\vec{x})\ldots\delta \phi_0(\vec{x}_n)}\Big|_{\phi_0=0}.
\end{aligned} \tag{2.61}$$

However, we do not know the full theory of gravity. Nevertheless, using the strong/weak duality, we approximate the AdS partition function by considering the on-shell action. To obtain the on-shell action, we have to solve the equation of motion with a given arbitrary boundary condition such that now the bulk partition function is

$$Z_{AdS} \sim e^{-S[\phi_0]_{os}} \Rightarrow W_{CFT} \sim -S[\phi_0]_{os}, \tag{2.62}$$

where *os* stands for on-shell.

As it is well known, quantum field theories have UV divergences, and we expect them on the CFT side. From the point of view of the bulk, these divergences appear as IR divergences due to the infinite volume of AdS. So if we want to use the AdS space to compute holographic correlators, we need to regulate and renormalise the bulk theory before computing the correlation function. In particular, the IR divergences play a central role in obtaining the correct correlation function and deriving the holographic Weyl anomaly [28, 27].

Chapter 3

Tree level Correlation Function: Position space

In contrast with QFT, in CFT, most of the computation of the correlation function is done in position space, mainly due to the difficulty of analysing Ward identities in momentum space. Thus, most of the known structures of conformal correlation function are in position space. In this chapter, we will compute them using the AdS/CFT correspondence in the saddle point approximation in the bulk.

In this chapter, we will apply the holographic dictionary and carefully treat the divergences that will arise along the computation. We will focus on a scalar field over an AdS background with quartic interaction and obtain the dual CFT correlation function. From the point of view of the bulk, computing the holographic correlation function corresponds to computing Witten diagrams.

We will work at the semi-classical level. To deal with the IR divergence we will introduce the holographic renormalisation method. The method does not demand knowing the full solution of the equation of motion but understanding the asymptotic behaviour of the bulks field. Having the renormalised bulk theory, we will compute the holographic correlation function. To do this, we must solve the equation of motion by introducing propagators allowing us to draw the corresponding Witten diagram.

3.1 Holographic renormalisation

One of the main aspects of QFT is that they suffer UV divergences which demand regularisation and renormalisation. The renormalisation requires adding local counterterms such that the theory is finite. In the holographic conjecture, through the UV/IR connection, the field theory's high energy divergences correspond to the bulk's low energy divergences. The IR divergence of the bulk is due to the infinite volume of AdS.

If the correspondence holds, we should be able to renormalise the bulk theory only using the information within the AdS (and its boundary). To have a finite bulk action, all counterterms must be local. The way to achieve this is by the so-called holographic renormalisation. The procedure successfully computed the Weyl anomaly in [28, 27] and was promoted to a systematic way in [63]. However, to renormalise the on-shell theory, we will follow a simpler way given by [31], which relies on variational methods to find the corresponding counterterms.

For the fields in the bulk, we will demand Dirichlet boundary condition. This means that the field must have an arbitrary but fixed boundary condition, ϕ_0, such that the asymptotic expansion of the field has the generic expansion

$$\Phi(z, \vec{x}) \to z^{\alpha_0}\phi_0 + z^{\alpha_1}\phi_1 + \dots, \quad z \to 0, \tag{3.1}$$

where $\alpha_0 < \alpha_1 < \alpha_2 < \dots$ and the AdS boundary is at $z \to 0$.

We will guarantee that the classical field dominates the path integral by demanding well-posed variational problem. This is, in order to hold

$$W_{CFT} \simeq S[\phi_0]_{os}, \tag{3.2}$$

we must have

$$\delta S[\Phi] = \int_{AdS} d^{d+1}x \, \mathcal{E}\delta\Phi + \int_{\partial AdS} d^d x \, [A\delta\phi_0] \,, \tag{3.3}$$

where $\mathcal{E}$ are the equation of Euler-Lagrange and A is some finite coefficient.

Due to the infinite volume of AdS, we need to regularise the volume. In practice, this means we must regularise the action. The regularised action is

$$S_{reg} = \int_{z=\epsilon} dz \int d\vec{x} L(\Phi, \partial\Phi) + \int d\vec{x} B\Big|_{z=\epsilon} \tag{3.4}$$

where the regulator isolates the divergence at $z \to 0$.

Given the Dirichlet boundary condition for the field Φ, the on-shell variation of the action will be

$$\delta S = \lim_{\epsilon \to 0} \int d\vec{x} \left(\frac{\partial L}{\partial \Phi'} \delta\Phi + \delta B \right) \Big|_{z=\epsilon} = \int d\vec{x} A \delta\phi_0 \tag{3.5}$$

where Φ' is the derivative with respect to the radial coordinate z. From here, we can see that the role of the boundary term B is twofold. It guarantees that the Dirichlet problem is well defined, and as A is finite, the boundary term B must remove any possible divergence such that the limit is finite. If the second equality holds, then the problem is solved, and from (2.61) we can read that the 1-point function corresponds to,

$$\langle \mathcal{O} \rangle = A. \tag{3.6}$$

3.1.1 The Renormalisation problem

To work on the renormalisation problem, we consider the asymptotic expansion of the field given by the Frobenius series and the variation of the field

$$\Phi = z^a \left(\phi_0 + z^2\phi_2 + \ldots\right), \tag{3.7}$$

$$\delta\Phi = z^a \left(\delta\phi_0 + z^2\delta\phi_2 + \ldots\right). \tag{3.8}$$

Then the action will have the following shape

$$\int_{z=\epsilon} d^d x \frac{\partial L}{\partial(\partial_z \Phi)}\delta\Phi = \int_{z=\epsilon} d^d x \sum_{n=-K}^{\infty} z^{2n} C_n(\phi_i, \delta\phi_j), \tag{3.9}$$

where the terms ϕ_i and $\delta\phi_j$ come from the expansion, K is a positive number and represents the fact that usually, the asymptotic expansion has terms with negative power in z.

It can be proven [31] that the coefficient $C_n(\phi_i, \delta\phi_j) = \delta D_n(\phi_i)$ for all $n \neq 0$. Apart from the zero mode, any term is either zero or divergent in the $z = \epsilon \to 0$ limit. If it is divergent, it can be absorbed in a boundary term δB that plays the role of counterterm.

3.1.2 The variational problem

Now that we know that every divergent term can be renormalised away by a proper choice of the boundary term, we still have to find the B_0 boundary term such that

$$\delta S = \int d^d x \left(\frac{\partial L}{\partial(\partial_z \Phi)}\delta\Phi\Big|_{\text{zero mode}} + \delta B_0 \right) \tag{3.10}$$

$$= \int d^d x A \delta\phi_0. \tag{3.11}$$

However, this problem is not always possible to solve. It may happen that B_0 does not exist. If we are able to solve it, then A is interpreted as the vacuum expectation value (vev).

3.1.3 The Free Scalar field

We will apply the variational method for the scalar field.

Consider a scalar field on an AdS background with regularised action

$$S_{reg} = \frac{1}{2}\int_{z=\epsilon} d^{d+1}x\sqrt{g}\left(g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + m^2\phi^2\right) + B, \tag{3.12}$$

where B is a suitable boundary to be determined. The first step consists in solving the equation of motion

$$(-\Box + m^2)\Phi = 0, \quad \text{where } \Box\Phi = \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu\phi\right). \tag{3.13}$$

Working in Poincare coordinates (2.56) with $l^2 = 1$, the equation of motion is

$$-z^2\partial_z^2\Phi + (d-1)z\partial_z\Phi - z^2\delta^{ij}\partial_i\partial_j\Phi + m\Phi^2 = 0 \tag{3.14}$$

subject to the Dirichlet boundary condition $\Phi(z\to 0,\vec{x}) \sim \phi_0(\vec{x})$. To solve the asymptotic problem, we propose a Frobenius series such that

$$\Phi(z,\vec{x}) = z^a\sum_{n=0}^{\infty} z^n\phi_n(\vec{x}) \tag{3.15}$$

with a to be determined. From demanding $\phi_0 \neq 0$ we find

$$\Delta(\Delta - d) = m^2 \Rightarrow \Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2}. \tag{3.16}$$

Notice that $\Delta_- = d - \Delta_+$. From now, we will call $\Delta = \Delta_+$. This leads us to find $a_1 = d - \Delta$ and $a_2 = \Delta$.

In the Frobenius method, if both solutions of the indicial equation, a_1 and a_2, are different and $a_2 - a_1 \notin \mathbb{N}$, then we have two linearly independent solutions to the differential equation. In practice, this means that the asymptotic behaviour of the field is

$$\Phi = z^{d-\Delta}\left(\phi_0 + z\phi_1 + \ldots\right) + z^{\Delta}\left(\phi_{2\Delta-d} + \ldots\right), \quad \Delta = \frac{d}{2} + f, \quad f \notin \mathbb{N}. \tag{3.17}$$

If the solutions differ by an integer, i.e $a_2 - a_1 = 2\Delta - d = N \in \mathbb{N}$, then the solution is given by,

$$\Phi = z^{d-\Delta}\left(\phi_0 + z\phi_1 + \ldots\right) + z^{\Delta}\left(\phi_{2\Delta-d} + \psi_{2\Delta-d}\log(z) + \ldots\right), \quad \Delta = \frac{d}{2} + N,\ N \in \mathbb{N}. \tag{3.18}$$

For $\Delta > \frac{d}{2}$, we call ϕ_0 as the non-normalisable mode and $\phi_{2\Delta-d}$ as the normalisable mode.[1] This will be crucial when considering quantum fluctuation to the classical field.

Demanding (3.16) on the eom gives us $\phi_1 = 0$. The recursive relations tell us that all $\phi_{2n+1},\ n = 0, 1, 2, \ldots$ vanish. So in the series, we only consider terms with even power in the z coordinate.

For simplicity, we will consider $\Delta - \frac{d}{2} \notin \mathbb{N}$. Then the expansion is

$$\Phi = z^{d-\Delta}\sum_n z^{2n}\phi_{2n}(\vec{x}) + z^{\Delta}\sum_n z^{2n}\phi_{2\Delta-d+2n}. \tag{3.19}$$

Evaluating this expansion in the equation of motion (3.14) we find that each ϕ_{2n} term can be solved algebraically in terms of local terms of ϕ_0. The generic form of the ϕ_{2n}

[1] It is worth mentioning that for the case $\frac{d-1}{2} < \Delta < \frac{d}{2}$ then the normalisable mode and non-normalisable mode may swap or both to be normalisable [26]. These cases will not be discussed in this work.

coefficient is,

$$\phi_{2n} = \frac{1}{2n(2\Delta - d - 2n)} \Box \phi_{2n-2}, \quad n = 1, 2, 3 \ldots. \tag{3.20}$$

Notice that the denominator es never zero, so we can determine all the ϕ_{2n} terms as local functions of ϕ_0. However, we can not determine $\phi_{2\Delta-d+2n}$ in terms of ϕ_0.

In the case of $\Delta - \frac{d}{2} = k,\ k = 1, 2, \ldots$, we have to add a logarithmic term at order z^{Δ} to the series expansion. Under this case, the expansion is

$$\Phi = z^{d-\Delta}\phi_0 + \ldots + z^{\Delta}\left(\phi_{2\Delta-d} + \psi_{2\Delta-d}\log z^2\right) + \ldots. \tag{3.21}$$

From the equation of motion, we can determine each term on the first ellipsis to be (3.20), i.e., local functions of ϕ_0. Similarly, $\psi_{2\Delta-d}$ is also given as a local function of ϕ_0. However, we cannot write $\phi_{2\Delta-d}$ as a local function of ϕ_0.

We cannot write $\phi_{2\Delta-d}$ as a local function of the source because we have solved a second-order differential equation, i.e. we will find two solutions, but we gave only the boundary condition. To find $\phi_{2\Delta-d}$, we will impose regularity at the interior of AdS, and the entire scalar field will be written in terms of the boundary condition.

Having determined the scalar field's asymptotic behaviour, we now look at the variational problem.

An arbitrary variation of the action yields

$$\delta S_{reg} = \int_{z=\epsilon} d^{d+1}x\sqrt{g}\,\delta\Phi(-\Box + m^2)\Phi + \int_{z=\epsilon} d^{d}x\sqrt{g}g^{zz}\delta\Phi\partial_z\Phi + \delta B, \tag{3.22}$$

here B is in $z=\epsilon$. Plugging the solution (3.19), with (3.20), provided we will have

$$
\begin{aligned}
\delta S_{reg} &= \int_{z=\epsilon} d^d x \frac{1}{z^{d-1}} \delta\Phi \partial_z \Phi + \delta B \\
&= \int_{z=\epsilon} d^d x \left[\frac{1}{z^{d-1}} \left(z^{d-\Delta} \sum_n z^{2n} \delta\phi_{2n}(\vec{x}) + z^{\Delta} \sum_n z^{2n} \delta\phi_{2\Delta-d+2n} \right) \right. \\
&\left. \left(z^{d-\Delta} \sum_m (d-\Delta+2m) z^{2m-1} \phi_{2m}(\vec{x}) + z^{\Delta} \sum_n (\Delta+2m) z^{2m-1} \phi_{2\Delta-d+2m} \right) \right] + \delta B.
\end{aligned}
$$

As we are interested in taking the limit of $\epsilon \to 0$, we will keep only the terms that are either divergent or constant under the limit. This means we will have

$$
\delta S_{reg} = \int d\vec{x} \left[\sum_{nm} (d-\Delta+2m) \epsilon^{2n+2m+d-2\Delta} \phi_{2m} \delta\phi_{2n} + (d-\Delta)\phi_0 \delta\phi_{2\Delta-d} + \Delta\phi_{2\Delta-d}\delta\phi_0 \right] + \delta B. \tag{3.23}
$$

From (3.20) we can prove that

$$
\phi_{2m}\delta\phi_{2n} = C_n C_m \left(\Box_0^{n+m} \phi_0 \right) \delta\phi_0 = \frac{1}{2} C_n C_m \delta \left(\Box_0^{n+m} \phi_0^2 \right) \tag{3.24}
$$

for $(n+m) < d-2\Delta$. This range is exactly the region in which we will find divergences, so all the divergent terms can be written as a variation and, therefore, can be absorbed in the boundary term.

*** A small comparison** Following the standard method of holographic renormalisation [32] we shall use the asymptotic solution and plug it into the regularised action, finding

$$S_{reg} = \int d\vec{x} \left[\epsilon^{d-2\Delta} a_{(0)} + \epsilon^{d-2\Delta+2} a_{(2)} + \ldots - a_{(2\Delta-d)} \log(\epsilon)\right] \tag{3.25}$$

where $a_{(2n)}$ $n = 0, 1, \ldots, \Delta - \frac{d}{2}$ are local functions of ϕ_0. In particular,

$$a_{(0)} = -\frac{1}{2}(d-\Delta)\phi_0^2, \tag{3.26}$$

This is the non-variational analogous to (3.23).

Having renormalised the boundary divergence, we are left with the zero mode and the renormalised action

$$\delta S_{ren} = (2\Delta - d) \int d\vec{x} \phi_{2\Delta-d} \delta\phi_0. \tag{3.27}$$

So, from the holographic dictionary, we find

$$\langle \mathcal{O} \rangle_s = -(2\Delta - d)\phi_{2\Delta-d} \tag{3.28}$$

the subscript s denotes that, in general, $\phi_{2\Delta-d}$ depends on the source. This dependence is obtained by fully solving the equation of motion. If we find $\phi_{2\Delta-d}(\phi_0)$, then using (2.61) we may find higher point correlation functions.

If we have considered

$$\Delta = \frac{d}{2} + N, \quad N = 0, 1, 2, \ldots \tag{3.29}$$

then, the procedure is equivalent, but in the asymptotic series, we have to include a logarithmic term which will give divergent terms (local in ϕ_0) the boundary term can remove that or zero in the limit of $\epsilon \to 0$. Then, we will have

$$\langle \mathcal{O} \rangle_s = -(2\Delta - d)\phi_{2\Delta-d} + X[\phi_0], \tag{3.30}$$

where $X[\phi_0]$ are local terms.

3.2 Correlation function: Classical computation

So far, our analysis has only treated the field's boundary behaviour, which is enough to renormalise the theory. Despite this, to obtain the dual theory's correlation function, we need to find the solution to the equation of motion for the field. Finding a solution is possible if the field's equation of motion is linear. If the equation of motion is non-linear, then we may solve it perturbatively.

We start by solving the free equation of motion with the Dirichlet boundary condition and compute the 2-point function

Free scalar field and the 2-point function

From the holographic dictionary, the 2-point function corresponds to

$$\langle \mathcal{O}_\Delta(\vec{x})\mathcal{O}(\vec{y}) \rangle = -\frac{\delta S_{ren}}{\delta\phi_0(\vec{x})\delta\phi_0(\vec{y})}\Big|_{\phi_0=0} \tag{3.31}$$

$$= -(2\Delta - d)\frac{\delta\phi_{2\Delta-d}(\vec{x})}{\delta\phi_0(\vec{y})}\Big|_{\phi_0=0}. \tag{3.32}$$

So we have to find $\phi_{2\Delta-d}$ as a function of the boundary source.

Consider the equation of motion

$$(-\Box + m^2)\Phi = 0, \quad \Phi(z \to 0, \vec{x}) = z^{d-\Delta}\phi_0(\vec{x}). \tag{3.33}$$

This equation was solved by Witten [24]. The solution is

$$\Phi(z,\vec{x}) = \int d\vec{y} K_{\Delta}(z,\vec{x}-\vec{y})\phi_0(\vec{y}) \tag{3.34}$$

where

$$K_{\Delta}(z,\vec{x}-\vec{y}) = c_{\Delta}\left(\frac{z}{z^2+(\vec{x}-\vec{y})^2}\right)^{\Delta}, \quad c_{\Delta} = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma\left(\Delta-\frac{d}{2}\right)} \tag{3.35}$$

is the bulk-to-boundary propagator, a deduction in momentum space is found in Appendix B.

When the radial coordinate z goes to the boundary, the bulk-to-boundary propagator is proportional to a Dirac delta over the transverse coordinates

$$\lim_{z\to 0} K_{\Delta}(z,\vec{x}-\vec{y}) = \frac{1}{c_{\Delta}} z^{d-\Delta}\delta\left(\vec{x}-\vec{y}\right) + \dots. \tag{3.36}$$

Expanding the field $z \to 0$, we will find

$$\Phi(z\to 0,\vec{x}) = \int d\vec{y}\left(z^{d-\Delta}\delta(\vec{x}-\vec{y}) + \ldots + z^{\Delta}\frac{1}{|\vec{x}-\vec{y}|^{2\Delta}}\right)\phi_0(\vec{y}), \tag{3.37}$$

from here, we can read

$$\phi_{2\Delta-d}(\vec{x}) = \int d\vec{y}\frac{1}{|\vec{x}-\vec{y}|^{2\Delta}}\phi_0(\vec{y}). \tag{3.38}$$

Then, the holographic two-point function is

$$\langle \mathcal{O}_{\Delta}(\vec{x})\mathcal{O}_{\Delta}(\vec{y})\rangle = \frac{2\Delta-d}{|\vec{x}-\vec{y}|^{2\Delta}}, \tag{3.39}$$

which corresponds to the two-point function of two conformal invariant operators if dimension Δ.

Interactions and n-point function

Interacting CFT admits higher point function. To obtain the correlation function from the AdS, we have to consider an interacting bulk system. Another reason to study interacting fields in AdS is that they will be the main object to work at the quantum level, so it is compulsory to get the holographic correlation function at the classical level first and then move on to quantum computation.

In particular, let us take the Φ^4 theory on AdS. The action is

$$S = \int d^{d+1}x\sqrt{g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + \frac{1}{2}m^2\Phi^2 + \frac{\lambda}{4!}\Phi^4\right] + B. \tag{3.40}$$

We look for the coefficient $\phi_{2\Delta-d}$ of the classical field to obtain the holographic correlation function. Of course, there can be local terms, but we will not work them.

The first step is to solve the equation of motion

$$(-\Box + m^2)\Phi = -\frac{\lambda}{3!}\Phi^3. \tag{3.41}$$

As non-linear, we solve this perturbatively in the coupling λ

$$\Phi = \Phi_0 + \lambda\Phi_1 + \dots, \tag{3.42}$$

where the subscript denotes the power of the coupling. Then, the equation of motion becomes

$$(-\Box + m^2)\Phi_0 = 0, \quad \Phi(z \to 0, \vec{x}) = z^{d-\Delta}\phi_0(\vec{x}), \tag{3.43}$$

$$(-\Box + m^2)\Phi_1 = -\frac{1}{3}\Phi_0^3. \tag{3.44}$$

The equation (3.43) was solved in the previous subsection,

$$\Phi_0 = \int d\vec{y} K_\Delta(z, \vec{x} - \vec{y}) \phi_0(\vec{y}). \tag{3.45}$$

Having solved the zeroth order equation, we are left with the non-homogeneous equation (3.44) which is solved by

$$\Phi_1(x) = \int d^{d+1}x' G_\Delta(x, x') \left(\Phi_0(x')\right)^3. \tag{3.46}$$

Where $G_\Delta(x, x')$ is the bulk-to-bulk propagator

$$(-\Box + m^2) G_\Delta(x, x') = \delta(x, x'), \quad \delta(x, x') = \frac{1}{\sqrt{g}} \delta(x - x') \tag{3.47}$$

The explicit form of the bulk-to-bulk propagator is [64]

$$G_\Delta(x, x') = \frac{2^{-\Delta} c_\Delta}{2\Delta - d} \xi^\Delta \,{}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta + 1}{2}, \Delta - \frac{d}{2} + 1, \xi^2\right), \tag{3.48}$$

$$\xi = \frac{2zz'}{z^2 + (z')^2 + (\vec{x} - \vec{x}')^2}. \tag{3.49}$$

Where ${}_2F_1$ is the hypergeometric function (the definition and some properties are shown in Appendix A) and ξ is the chordal distance which is AdS invariant. The deduction of the bulk-to-bulk propagator in momentum space is given in Appendix B.

Having the full solution up to order λ, we now look for the contribution of Φ_1 to the term $\phi_{2\Delta - d}$ in the asymptotic expansion. This can be done by using

$$G_\Delta(x, x')\Big|_{z \to 0} = z^\Delta \frac{1}{2\Delta - d} K_\Delta(z', \vec{x}' - \vec{x}) + \mathcal{O}(z^{\Delta + 2}). \tag{3.50}$$

So,

$$\Phi_1(z \to 0, \vec{x}) = z^{\Delta} \frac{1}{2\Delta - d} \int d^{d+1}x' K_\Delta(z', \vec{x}' - \vec{x}) \left(\Phi_0(x')\right)^3 + \mathcal{O}(z^{\Delta+2}). \tag{3.51}$$

We notice that the contribution to $\phi_{2\Delta-d} \sim \phi_0^3$, so we can compute the 4-point function [65]

$$\begin{aligned}\langle \prod_{i=1}^{4} \mathcal{O}(\vec{y}_i) \rangle &= 2 \frac{\delta^3 \phi_{2\Delta-d}(\vec{y}_1)}{\delta\phi_0(\vec{y}_2)\delta\phi_0(\vec{y}_3)\delta\phi_0(\vec{y}_4)} \\ &= \lambda \int d^{d+1}x \sqrt{g} \prod_{j=1}^{4} K_\Delta(z, \vec{y}_j - \vec{x}).\end{aligned} \tag{3.52}$$

We are left with the product of four bulk-to-bulk propagators. In the literature, this product is called $D-$function and is defined as

$$D_{\Delta\Delta\Delta\Delta}\left(\vec{y}_1, \vec{y}_2, \vec{y}_3, \vec{y}_4\right) = \int d^{d+1}x \sqrt{g} \prod_{j=1}^{4} K_\Delta(z, \vec{y}_j - \vec{x}) \tag{3.53}$$

which has been extensively treated [66, 67, 68]. In [69], the $D-$ function was obtained for generic conformal dimension on each bulk-to-boundary propagator, just quoting the result we have

$$D_{\Delta\Delta\Delta\Delta}\left(\vec{y}_1, \vec{y}_2, \vec{y}_3, \vec{y}_4\right) = \frac{\Gamma\left(2\Delta - \frac{d}{2}\right)}{2\left(\Gamma(\Delta)\right)^4} \frac{1}{|\vec{y}_{13}|^{\Delta} |\vec{y_{24}}|^{\Delta}} \times \bar{D}_{\Delta\Delta\Delta\Delta}(u, v) \tag{3.54}$$

where u, and v are the crossing ratios given in (2.30). The D-function has exactly the expected form of a CFT 4-point function (2.29) and $\bar{D}$ depends on the

3.2.1 Witten Diagram

Computing the holographic correlation function has a diagrammatic representation in terms of Witten diagrams. The building block for the diagrams is the bulk-to-boundary propagator (3.35) and the bulk-to-bulk propagator (3.48). In contrast with Feynman diagrams, Witten diagrams have an arbitrary fixed boundary condition, so the external points are attached to the boundary of AdS.

The rules are:

- The AdS space is represented by a circle, and the CFT is the boundary of the circle
- Each boundary point connected with the bulk is given by a bulk-to-boundary propagator
- Bulk points are connected with a bulk-to-bulk propagator.
- An interaction vertex in the point x is

$$N\lambda \int d^{d+1}x\sqrt{g}, \tag{3.55}$$

  where N is the corresponding symmetry factor that is the same that is found in standard Feynman diagrams.

The Witten diagram of the holographic 4-point function (3.52) is shown in Figure 3.1.

$$\left\langle \prod_{i=1}^{4} \mathcal{O}(\vec{y}_i) \right\rangle =$$

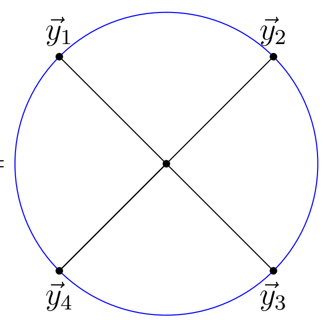


Figure 3.1: tree level 4-point function for Φ^4 bulk interaction. All four bulk-to-boundary propagator collides in a single bulk point.

For different bulk interactions, we will find further Witten diagrams. For example, if we consider a Φ^3 interaction, we will find the Witten diagram shown in Figure 3.2

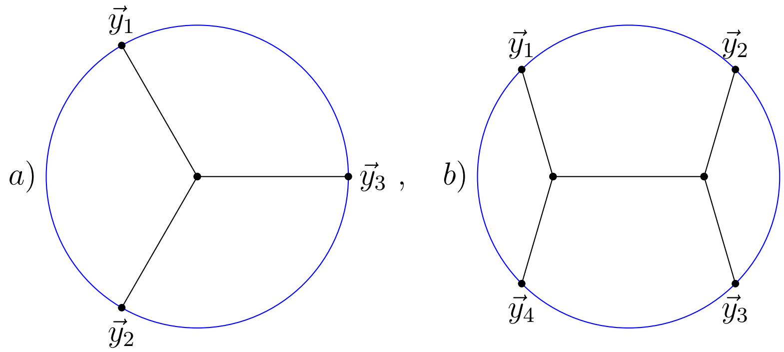


Figure 3.2: a) Witten diagram for 3-point function, b) Witten diagram for 4-point function

Witten diagram will be central in studying the quantum holographic correlation function. They will help us to have a graphic intuition of what kind of loop correction we are computing.

Chapter 4

Loop Correlation Function: Position space

Using the classical fields in the bulk, we obtained the holographic correlation function, and they give the expected result for the correlation function of a CFT. However, the AdS/CFT correspondence states that the equivalences should hold beyond the saddle point approximation. So, we may ask ourselves what is the effect of working beyond the semi-classical approach in the holographic correlation function.

We will work the problem for the quantum scalar field with a Φ^4 interaction on a fixed AdS background in two ways: Using the background field method and following the quantum effective action.

The background field method consists in decomposing the scalar field into a classical part plus a quantum fluctuation. The classical field was worked in 3, so we will only focus on the quantum fluctuation and recall the already obtained result whenever necessary. We will compute the path integral for the quantum fluctuation up to the second order in the coupling and follow the GKPW dictionary to calculate the corresponding correlation function, finding loops in Witten diagrams.

As it has been well established in standard quantum field theory, the quantum treatment of the theory leads to UV divergence (in the bulk) that will demand renormalisation and, thus, redefine the parameters, making them finite. We will explicitly work on this finding a new conformal dimension, so we have to re-adapt the holographic renormalisation to the new parameters of the theory.

At the same time, following the quantum effective action, we will start from the partition function and compute the quantum effective action up to the second order in the coupling. By demanding a finite equation of motion, we will renormalise the UV divergences. Having renormalised the UV divergence, we can perform the holographic renormalisation with the renormalised parameters.

In this chapter, we will use the notation

$$\int d^{d+1}x\sqrt{g} = \int dx. \tag{4.1}$$

The organisation of the chapter is as follows: In 4.1, we will generally refer to the quantum fluctuation and holographic correlation function. In subsection 4.2, we will compute some useful integrals relevant to the following subsections. As some integrals are UV divergent, we present the UV regularisation in section 4.3. In subsection 4.4, we will study the background field method and the renormalisation problem. We will follow the quantum effective action and renormalisation in subsection 4.5.

4.1 General Statement

The AdS/CFT conjecture states that quantum gravity on AdS is equivalent to CFT in the boundary of AdS. At low energy, the relation corresponds to a duality between AdS SUGRA and strongly coupled CFT. The CFTs admitted in the AdS/CFT usually allow a large N 't Hooft limit, where the bulks dual correspond to tree-level computations. Going to loop in the bulk should correspond to $\frac{1}{N^2}$ corrections in the CFT.

We will focus on having a well-posed problem for the bulk theory. That is, we mean to have a finite bulk theory. However, as we are dealing with quantum corrections, it is natural that UV divergence will arise altogether with the known IR divergence. Because of the IR/UV connection, UV divergence in the bulk corresponds to IR divergences in the field theory. As is expected on QFT, the IR divergence should cancel on its own and shall not interfere with the UV structure. So the bulk's UV divergence shall be regularised with an AdS invariant regulator so we do not add new anomalies to the CFT.

We regulate the UV divergence with a regulator κ that does not break the AdS isometries. The IR is regulated with the standard cut-off ϵ.

So the general statement is that the renormalised bulk theory is

$$Z_{AdS}^{\text{ren}}[\phi_0^{\text{ren}}, \lambda_i^{\text{ren}}] = \lim_{\epsilon\to 0}\lim_{\kappa\to 0} Z_{AdS}^{\text{reg}}[\phi_0, \lambda_i, \epsilon, \kappa], \tag{4.2}$$

where ϕ_0^{ren} and λ_i^{ren} are the renormalized source and coupling, while ϕ_0 and λ_i the bare source and bare coupling. So to compute the renormalised correlation function, we need to take derivatives with respect to the renormalised source. Although in the background field method, we take the derivative with respect to the bare source and renormalise the theory such that we find the renormalise source.

4.2 Relevant Integrals

From the point of view of the bulk, the holographic correlation function corresponds to Witten diagrams. They are built with the bulk-to-boundary propagator and bulk-to-bulk propagator. Therefore, to work at loop order within the bulk, we need to know how to compute some relevant integral that will arise from the corresponding Witten diagrams. This subsection aims to work on some of the essential integrals.

For some integral, using AdS isometries, we can give the general structure isolating the UV divergent part, but not the complete solution.

Product of two bulk-to-boundary propagators

The product of two bulk-to-boundary propagators also called the mass gap [53, 52], falls into a particular case of a $D-$function. It is given by

$$D_{\Delta_1,\Delta_2}(\vec{y}_1,\vec{y}_2) = \int dx K_{\Delta_1}(z,\vec{x}-\vec{y}_1)K_{\Delta_2}(z,\vec{x}-\vec{y}_2). \tag{4.3}$$

The diagrammatic expression of this integral is shown in Figure 4.1.

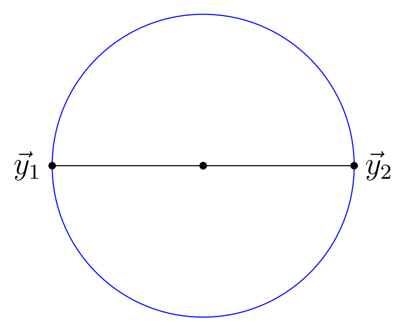
$D_{\Delta\Delta} =$ 

Figure 4.1: Witten diagram for the $D_{\Delta,\Delta}$ function

The asymptotic behaviour of the integrand is $z^{d-\Delta_1-\Delta_2-1}$. So the D_{Δ_1,Δ_2} is IR divergent when $\Delta_1+\Delta_2 > d$. As we are considering that any conformal dimension $\Delta_i > \frac{d}{2}$, then D_{Δ_1,Δ_2} is always IR divergent so it demands regularisation. We will regulate the divergence considering a cut-off in the lower limit of the radial direction.

Then, the regulated integral is

$$D_{\Delta_1,\Delta_2}(\epsilon,\vec{y}_1,\vec{y}_2) = \int_\epsilon^\infty \frac{dz}{z^{d+1}} \int d\vec{x} K_{\Delta_1}(z,\vec{x}-\vec{y}_1)K_{\Delta_2}(z,\vec{x}-\vec{y}_2) \tag{4.4}$$

where we made explicit the $\sqrt{g}$ term. We can translate the transverse coordinate

$$D_{\Delta_1,\Delta_2}(\epsilon,\vec{y}_1,\vec{y}_2) = c_{\Delta_1}c_{\Delta_2}\frac{1}{|\vec{y}_{12}|^{\Delta_1+\Delta_2}}\int_\sigma^\infty dz\int d\vec{x}\frac{z^{\Delta_1+\Delta_2-d-1}}{\left[z^2+(\vec{x}-\hat{n})^2\right]^{\Delta_1}\left[z^2+\vec{x}^2\right]^{\Delta_2}} \tag{4.5}$$

with $\sigma = \frac{\epsilon}{|\vec{y}_{12}|}$ and $\hat{n}$ is defined through $\vec{y}_{12} = |\vec{y}_{12}|\hat{n}$. Notice that the complete dependence of the boundary points is in the prefactor and the lower limit of the radial integral. So, we need to find how the integral depends on $\vec{y}_{12}$. To find this, we will call the integral

$$\tilde{D}_{\Delta_1,\Delta_2} = \int_\sigma^\infty dz\int d\vec{x}\frac{z^{\Delta_1+\Delta_2-d-1}}{\left[z^2+(\vec{x}-\hat{n})^2\right]^{\Delta_1}\left[z^2+\vec{x}^2\right]^{\Delta_2}} \tag{4.6}$$

and take a derivative with respect to σ,

$$\frac{d}{d\sigma}\tilde{D}_{\Delta_1,\Delta_2} = -\int d\vec{x}\frac{\sigma^{\Delta_1+\Delta_2-d-1}}{\left[\sigma^2+(\vec{x}-\hat{n})^2\right]^{\Delta_1}\left[\sigma^2+\vec{x}^2\right]^{\Delta_2}}. \tag{4.7}$$

It is possible to solve the integral directly by using spherical coordinates. Instead, we will follow a different approach. By using the following Dirac delta representation,

$$c_\Delta \lim_{z\to 0}\frac{z^{2\Delta-d}}{(z^2+\vec{x}^2)^\Delta} = \delta(\vec{x}) \tag{4.8}$$

we will find,

$$\lim_{\sigma\to 0}\frac{d}{d\sigma}\tilde{D}_{\Delta_1,\Delta_2} = -2\delta_{\Delta_1,\Delta_2}\frac{1}{c_{\Delta_1}}\frac{1}{\sigma} \Rightarrow \tilde{D}_{\Delta_1,\Delta_2} = -2\frac{\delta_{\Delta_1,\Delta_2}}{c_{\Delta_1}}\log\left(\frac{\epsilon}{|\vec{y}_1-\vec{y}_2|}\right)+C+\mathcal{O}(\epsilon), \tag{4.9}$$

where C is an integration constant. Then, we will have,

$$D_{\Delta,\Delta}(\epsilon,\vec{y}_1,\vec{y}_2) = -\frac{2c_\Delta}{|\vec{y}_1-\vec{2}|^{2\Delta}}\left(\log\left(\frac{\epsilon}{|\vec{y}_1-\vec{y}_2|}\right)+C\right)+\mathcal{O}(\epsilon). \tag{4.10}$$

Notice that we have obtained the dependence on the external point (modulo a constant) without doing any integral.

In this treatment there is a caveat, by defining $\sigma = \frac{\epsilon}{|\vec{y}_{12}|}$ we are assuming that $\vec{y}_1 \neq \vec{y}_2$, thereby can be more divergence.

To obtain the full solution integral for any boundary points and to find the constant C, we use the momentum representation of the bulk-boundary-propagator (5.7) to compute $D_{\Delta,\Delta}$. Quoting the result from Appendix C.4 taking the limit of $\epsilon \to 0$ and using the inverse Fourier transformation (C.38) we find

$$
\begin{aligned}
D_{\Delta,\Delta}(\vec{y}_1,\vec{y}_2) = & \frac{1}{\epsilon^{2\Delta-d}}\frac{\delta(\vec{y}_1-\vec{y}_2)}{2\Delta-d} + \dots \\
& -\frac{2c_\Delta}{|\vec{y}_1-\vec{y}_2|^{2\Delta}}\left(\log\left(\frac{\epsilon}{|\vec{y}_1-\vec{y}_2|}\right) + \frac{\psi\left(\Delta\right)-\psi\left(\Delta-\frac{d}{2}\right)}{2}\right) + \mathcal{O}(\epsilon).
\end{aligned}
\tag{4.11}
$$

Regulating the IR divergence is crucial and will play an essential role in computing the quantum correction to the holographic correlation function.

The GK integral

The integral of the product between a bulk-to-bulk propagator and the bulk-to-boundary propagator is,

$$I_1(x,\vec{y}) = \int dx' G_\Delta(x,x')K_\Delta(z',\vec{x}'-\vec{y}), \tag{4.12}$$

using the asymptotic behaviour of each propagator, we find that the integrand behaves as z^{-1}, then we have to regulate the integral by introducing a cut-off in the lower limit,

$$I_1(\epsilon,x,\vec{y}) = \int_\epsilon dz' \int d\vec{x}' \sqrt{g} G_\Delta(x,x')K_\Delta(z',\vec{x}'-\vec{y}). \tag{4.13}$$

In Figure. 4.2 is shown the corresponding Witten diagram for the I_1 integral.

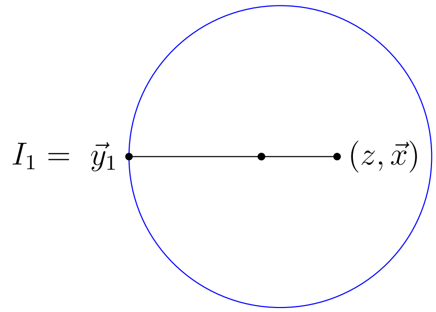


Figure 4.2: Witten diagram for the GK integral

This integral can be computed using the series representation of the bulk-to-bulk propagator, integrating using (C.1) and re-sum into a single expression. This procedure is cumbersome and too long.

It is easier to follow a similar procedure as is done in [70]. We will consider the following equation

$$(-\Box + m^2) I_1 = K_\Delta(z, \vec{x} - \vec{y}), \tag{4.14}$$

where we used (3.47). The boundary condition for the differential equation corresponds to the limit of $z \to \epsilon$, so using (3.50), we have

$$I_1(\epsilon, \vec{x}, \vec{y}) = \frac{\epsilon^\Delta}{2\Delta - d} D_{\Delta,\Delta}(\vec{x}, \vec{y}). \tag{4.15}$$

By direct computation, it is possible to prove that the solution to this differential equation is,

$$I_{1,p} = -\frac{1}{2\Delta - d} K_\Delta(z, \vec{x} - \vec{y}) \log\left(\frac{z}{z^2 + (\vec{x} - \vec{y})^2}\right) + C_1 K_\Delta(z, \vec{x} - \vec{y}) + C_2 K_{d-\Delta}(z, \vec{x} - \vec{y}) \tag{4.16}$$

Demanding regularity in the interior, we determine $C_2 = 0$ and imposing the boundary condition we find

$$C_1 = -\frac{1}{2\Delta - d}\log\left(\epsilon\right) - \frac{1}{2\Delta - d}\left[\psi\left(\Delta\right) - \psi\left(\Delta - \frac{d}{2}\right)\right] \tag{4.17}$$

where $\psi(x)$ the digamma function.

Then, we have

$$I_1(\epsilon, x, \vec{y}) = -\frac{1}{2\Delta - d}K_\Delta(z, \vec{x} - \vec{y})\left[\log\left(\frac{\epsilon z}{z^2 + (\vec{x} - \vec{y})^2}\right) + \psi\left(\Delta\right) - \psi\left(\Delta - \frac{d}{2}\right)\right]. \tag{4.18}$$

An alternative proof can be found in Appendix C.2

The $G^n K$ integral for $n \geq 2$

The generalisation of the GK integral to an arbitrary power on the bulk-to-bulk propagator is

$$I_{2,n}(x, \vec{y}) = \int dx' G_\Delta(x, x')^n K_\Delta(z', \vec{x}' - \vec{y}), \quad n = 2, 3, \ldots. \tag{4.19}$$

Having one bulk-to-boundary propagator and adding more bulk-to-bulk propagators, the integral becomes IR convergent, so we can infer that I_2 is IR finite. Indeed, by power counting, we can check that the IR behaviour of the integrand is $z^{n\Delta - \Delta - 1}$; thus, for $n \geq 2$, the I_2 integral in IR is finite.

As we do not add an IR cut-off, we are not breaking the AdS symmetry, so we can use the whole set of AdS isometries to work the integral. In particular, by translating both transverse coordinates, $\vec{x} \to \vec{x} - \vec{y}$ and $\vec{x}' \to \vec{x}' - \vec{y}$, we can prove that the integral is independent of the boundary coordinate $\vec{y}$. So we have

$$I_{2,n}(z,\vec{x}) = \int dx' G_\Delta(x,x')^n K_\Delta(z',\vec{x}'). \tag{4.20}$$

Now, we do an inversion given by equation (2.58) to x and x', as is an isometry, the volume element and the bulk-to-bulk propagator do not change

$$I_{2,n}(z,\vec{x}) = c_\Delta \int dx' G_\Delta(x,x')^n z'^{\,\Delta}. \tag{4.21}$$

By translating in $\vec{x}' \to \vec{x}' - \vec{x}$, then

$$I_{2,n}(z) = c_\Delta \int dx' G(z,0;z',\vec{x}')^n z'^{\,\Delta}. \tag{4.22}$$

Now we do a dilatation such that $x' \to zx'$, so we will have

$$I_{2,n}(z) = c_\Delta z^\Delta \int dx' G(1,0;z',\vec{x}')^n z'^\Delta. \tag{4.23}$$

Notice that the r.h.s integral is an (eventually UV divergent) constant. Despite this, the only dependence is the z term. Inverting back the z variable, we are left with

$$I_{2,n}(z,\vec{x}) = g_n K_\Delta\left(z,\vec{x}\right), \quad g_n = \int dx' G_\Delta(1,0;z',\vec{x}')^n z'^{\,\Delta}. \tag{4.24}$$

The g_n factor is UV divergent, so computing it demands regularisation. However, we have found a general structure that the $G^n K$ integral has without computing it explicitly. In [48], this integral was worked using the spectral representation finding the same structure.

The $G^n K$ integral has a clear interpretation in terms of Witten diagram shown in Figure 4.3

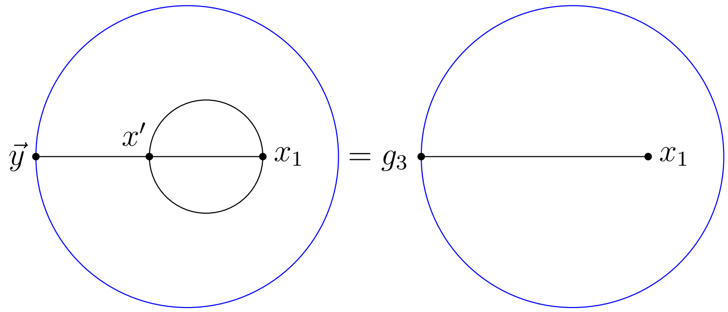


Figure 4.3: The $n = 3$ case. On the l.h.s the integration is in x' and in the r.h.s g_3 is a constant .

The G^2 integral

Now we turn our attention to the integral of a square of two bulk-to-bulk propagators

$$I_3(x) = \int dx' G(x, x')G(x', x). \tag{4.25}$$

The IR convergence is guaranteed because the asymptotic expansion of the integrand has the form $z^{2\Delta-d-1}$, which is convergent for $\Delta > \frac{d}{2}$. Therefore there is no need to add a lower cut-off regulator, and we can use AdS isometries to get the structure of the integral.

Doing a translation $\vec{x}' \to \vec{x}' - \vec{x}$ and dilatation $x' \to zx'$ we will have

$$I_3 = \int dx' G(1, 0; z', \vec{x}')^2 \tag{4.26}$$

which does not depends on x, so it is constant.

It turns out to be that this integral will be UV divergent, so regularisation is needed to compute it.

In Appendix C.3 is proven that

$$-\frac{1}{2\Delta - d}\frac{d}{d\Delta}G_\Delta(1) = \int dx G_\Delta(x, x')G(x', x) \tag{4.27}$$

where we understand $G_\Delta(1)$ to be regulated. This formula makes it much easier to compute the r.h.s integral.

The $KK\log$ integral

Now we turn our attention to the integral

$$I_4(\vec{y}_1,\vec{y}_2)=\int dxK_\Delta(z,\vec{x}-\vec{y}_1)\int dx'G_\Delta(x,x')K_\Delta(z',\vec{x}'-\vec{y}_1). \tag{4.28}$$

The integral over the primed variable corresponds to the one studied in the GK integral subsection and is IR divergent. Then, using (4.18), we will have

$$I_4(\epsilon,\vec{y}_{12})=-\frac{1}{2\Delta-d}\int dxK_\Delta(z,\vec{x}-\vec{y}_{12})K_\Delta(z,\vec{x})\left[\log\left(\frac{z\epsilon}{z^2+\vec{x}^2}\right)+\psi(\Delta)-\psi\left(\Delta-\frac{d}{2}\right)\right]. \tag{4.29}$$

the integral is IR-divergent as well. So, the I_4 has two IR divergences. Regularising it, we have

$$I_4(\epsilon,\vec{y}_{12})=-\frac{1}{2\Delta-d}\int_{z=\epsilon}dxK_\Delta(z,\vec{x}-\vec{y}_{12})K_\Delta(z,\vec{x})\left[\log\left(\frac{z\epsilon}{z^2+\vec{x}^2}\right)+\psi(\Delta)-\psi\left(\Delta-\frac{d}{2}\right)\right]. \tag{4.30}$$

We may notice that a part of the integral is proportional to the $D_{\Delta,\Delta}$ function (4.11)

$$\begin{aligned}
I_4&=-\frac{1}{2\Delta-d}\left(\log(\epsilon)+\psi(\Delta)-\psi\left(\Delta-\frac{d}{2}\right)\right)D_{\Delta,\Delta}+\tilde{I}_4 \qquad (4.31)\\
\tilde{I}_4&=-\frac{c_\Delta^2}{2\Delta-d}\int_{z=\epsilon}dx\left(\frac{z}{z^2+(\vec{x}-\vec{y}_{12})^2}\right)^\Delta\left(\frac{z}{z^2+\vec{x}^2}\right)^\Delta\log\left(\frac{z}{z^2+\vec{x}^2}\right).
\end{aligned}$$

To study the $\tilde{I}_4$, we may notice

$$\tilde{I}_4=-\frac{c_\Delta^2}{2\Delta-d}\int_{z=\epsilon}dx\left(\frac{z}{z^2+(\vec{x}-\vec{y}_{12})^2}\right)^\Delta\frac{d}{d\Delta}\left(\frac{z}{z^2+\vec{x}^2}\right)^\Delta. \tag{4.32}$$

Doing integration by parts, a translation on the transverse coordinate and using that $\vec{y}_{12}$

is invariant under rotation we will have,

$$\tilde{I}_4 = -\frac{1}{c_\Delta(2\Delta - d)} D_{\Delta,\Delta} \frac{dc_\Delta}{d\Delta} + \frac{1}{2(2\Delta - d)} \frac{d}{d\Delta} D_{\Delta,\Delta} \tag{4.33}$$

which is direct to compute.

So the integral is

$$I_4(\epsilon, \vec{y}_1, \vec{y}_2) = \frac{\epsilon^{d-2\Delta}\delta(\vec{y}_1 - \vec{y}_2)}{(2\Delta - d)^3} + \ldots + \frac{2c_\Delta}{2\Delta - d} \frac{1}{|\vec{y}_1 - \vec{y}_2|^{2\Delta}} \left[\log\left(\frac{\epsilon}{|\vec{y}_1 - \vec{y}_2|}\right)^2 + \right. \tag{4.34}$$
$$\log\left(\frac{\epsilon}{|\vec{y}_1 - \vec{y}_2|}\right) \left[\psi(\Delta) - \psi\left(\Delta - \frac{d}{2}\right)\right] + \frac{1}{4}\left[\psi^{(1)}(\Delta - \psi^{(1)}\left(\Delta - \frac{d}{2}\right)\right]$$
$$\left. +\frac{1}{4}\left[\psi(\Delta - \psi\left(\Delta\right) - \frac{d}{2}\right)\right]^2 \Bigg] .$$

The KKG^2 integral

[1] The integral to be studied is

$$I_5(x, \vec{y}_1, \vec{y}_2) = \int dx_1 K_\Delta(z_1, \vec{x}_1 - \vec{y}_1) K_\Delta(z_1, \vec{x}_1 - \vec{y}_2) G_\Delta(x_1, x)^2. \tag{4.35}$$

This integral is more challenging and demands much more work to understand it.

It is not hard to check that this integral is IR convergent for $d > 1$. Then, there is no need for IR regularisation.

By translation invariance, we find that the integral depends on $\vec{y}_{12} = \vec{y}_1 - \vec{y}_2$. Then we invert each coordinate $z \to z'$, $\vec{x} \to \vec{x}'$ given by (2.58) and $\vec{y} \to \vec{y} = \frac{\vec{y}'}{|\vec{y}'|^2}$, as inversion is an AdS isometry, then $G(x', x)$ remains unchanged, but the bulk-to-bulk propagator transform as

$$K_\Delta\left(z, \vec{x} - \vec{y}\right) = \frac{1}{|\vec{y}|^{2\Delta}} K_\Delta\left(z', \vec{x}' - \vec{y}'\right). \tag{4.36}$$

[1] We thank Ernesto Bianchi for very useful comments on this part of the work.

On the r.h.s, the prefactor of K is given in terms of the uninverted vector $\vec{y}$. With this, the integral becomes

$$I_5(x', \vec{y}_1, \vec{y}_2) = \frac{1}{|\vec{y}|^{2\Delta}} \int dx' K_\Delta(z', \vec{x}' - \vec{y}'_{12}) z'^\Delta G_\Delta(x'_1, x')^2. \tag{4.37}$$

The bulk-to-bulk propagator is given by a hypergeometric function (3.48). Using the explicit form of the bulk-to-boundary propagator and using the series representation of the hypergeometric function, we will have,

$$\begin{aligned} I_5(x', \vec{y}_1, \vec{y}_2) = &-\frac{c_\Delta^4}{(2\Delta - d)^2 y_{12}^{2\Delta}} \sum_{n_1 n_2} \alpha_{n_1} \alpha_{n_2} z'^{2\Delta + 2(n_1+n_2)} \\ &\times \int_0^\infty dz'_2 \int d\vec{x}'_2 \frac{z_2'^{4\Delta+2(n_1+n_2)-d-1}}{\left[z_2'^2 + (\vec{x}'_2 - \vec{y}'_{12})^2\right]^\Delta \left[z'^2 + z_2'^2 + (\vec{x}' - \vec{x}'_2)^2\right]^{2\Delta+2(n_1+n_2)}} \end{aligned} \tag{4.38}$$

where

$$\alpha_n = \frac{\left(\frac{\Delta}{2}\right)_n \left(\frac{\Delta+1}{2}\right)_n}{\left(\Delta - \frac{d}{2} + 1\right)_n} = \frac{4^{-n}\Gamma\left(-\frac{d}{2} + \Delta + 1\right)\Gamma(2n + \Delta)}{\Gamma(\Delta)\Gamma\left(-\frac{d}{2} + n + \Delta + 1\right)}. \tag{4.39}$$

The last integral is solved in Appendix C. By using the integral result and re-arranging the terms, we have

$$\begin{aligned} I_5(x, \vec{y}_1, \vec{y}_2) = \pi^{\frac{d+1}{2}} \left(\frac{2^{-2\Delta} c_\Delta}{2\Delta - d}\right)^2 K_\Delta(z, \vec{x} - \vec{y}_1) K_\Delta(z, \vec{x} - \vec{y}_2) \sum_{i=0}^{\infty} \Big[\alpha_i + \beta_i \\ \sigma_i \log\left[\frac{z}{z^2 + (\vec{x} - \vec{y}_1)^2} \frac{z}{z^2 + (\vec{x} - \vec{y}_2)^2} |\vec{y}_{12}|^2\right] \left(\frac{z}{z^2 + (\vec{x} - \vec{y}_1)^2} \frac{z}{z^2 + (\vec{x}_1 - \vec{y}_2)^2} |\vec{y}_{12}|^2\right)^i\Big]. \end{aligned} \tag{4.40}$$

For the renormalisation, the relevant term is

$$\begin{aligned} \alpha_i = (-1)^i (d-1) \frac{(\Delta)_i (\Delta)_i (d)_i}{(2)_i i!} \frac{\Gamma\left(2\Delta - \frac{d}{2} + 1 + i\right)}{\Gamma\left(\Delta + \frac{3}{2} + i\right)\Gamma(\Delta + 1 + i)} \\ {}_4F_3\left(2\Delta - \frac{d}{2} + 1 + i, 1, 1, d + i; \Delta + \frac{3}{2} + i, \Delta + i + 1, 2 + i, 1\right) \end{aligned} \tag{4.41}$$

In general, for any a and b positive, the ${}_{q+1}F_q(a_1, \ldots a_{q+1}; b_1, \ldots, b_q, 1)$, is convergent for

$\sum a_i - \sum b_i > 0$. In our case, this is translated into

$$\frac{3-d}{2} + i > 0. \tag{4.42}$$

So, the α_i term is finite for $d < 3$. For $d = 3, 4$, the coefficient $\alpha_{i=0}$ is divergent. For $d = 5, 6$, the $\alpha_{i=0}$ and $\alpha_{i=1}$ are divergent. For $d = 7, 8$ the terms $\alpha_{i=0}, \alpha_{i=1}$ and $\alpha_{i=2}$ are divergent. The coefficients β_i and σ_i are finite.

4.3 UV regulator

In the previous section, we studied relevant integrals, finding some of them to be IR convergent, e.g. the $G^n K$ integral. Despite being IR convergent, the finiteness of the integral is not guaranteed because there can be UV divergence. Indeed, the coefficient c_n in (4.24), or the integral I_3 in (4.26) are UV divergent.

The first case is the bulk-to-bulk propagator at the same point. From the definition (3.48), when we evaluate in $x = x'$ we have $\xi = 1$ then

$$G_\Delta(x,x) \equiv G_\Delta(1) = \frac{2^{-\Delta} c_\Delta}{2\Delta - d} \, {}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta+1}{2}, \Delta - \frac{d}{2} + 1, 1\right). \tag{4.43}$$

In general, the hypergeometric function ${}_2F_1(a,b,c,1)$ is convergent for $c > a+b$, and is

$${}_2F_1(a,b,c,1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{4.44}$$

In this case, the condition $c > a+b$ corresponds to $d < 1$. Then, we can write

$$G_\Delta(1) = \frac{2^{-\Delta} c_\Delta}{2\Delta - d} \frac{\Gamma\left(\Delta - \frac{d}{2} + 1\right)\Gamma\left(\frac{1-d}{2}\right)}{\Gamma\left(\frac{\Delta-d}{2} + 1\right)\Gamma\left(\frac{\Delta-d+1}{2}\right)}. \tag{4.45}$$

We can consider the former result's analytical continuation, allowing us to have $d > 1$.

For d even, (4.45) is finite and well-defined. For d odd, it may be ill-defined and demand regularisation. Using dimensional regularisation, $d \to d - \kappa$, the bulk-to-bulk propagator at the same point is finite. The divergent piece is obtained by taking the limit of $\kappa \to 0$.

We will denote the regulated bulk-to-bulk propagator as $G_{\Delta,\kappa}(1)$.

However, for the coming discussion, another regulator will be used. This is because there are integrals that we still do not know how to use in dimensional regularisation.

Inspired in [52, 53] we regulate the bulk-to-bulk propagator,

$$G_\Delta(\xi(x, x')) \to G_\Delta\left(\frac{\xi}{1+\kappa}\right) \tag{4.46}$$

where κ is the geodesic distance among the points. In this regularisation scheme, we keep the dimension fixed but consider the two-point x and x' apart and expand the bulk-to-bulk propagator in powers of the geodesic distance [71]. The regulated bulk-to-bulk propagator inherently is AdS invariant because chordal distance ξ is AdS invariant, and we keep κ fixed.

In the case $x = x'$, then we will have

$$G_{\Delta,\kappa}(1) \equiv G_\Delta\left(\frac{1}{1+\kappa}\right) \tag{4.47}$$

which is finite for $d \geq 1$.

4.4 Background field method

We will now move on to work on one of our main problems: the role of quantum correction in the holographic correlation function, considering an interacting scalar field over an AdS background. Despite being an effective theory and not including backreaction from space-time, the model is enough to understand how to work the quantum correction.

Consider the scalar field with quartic interaction with action given by

$$S = \int dx \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{2} m^2 \Phi^2 + \frac{\lambda}{4!} \Phi^4 \right) + B. \tag{4.48}$$

As we saw in section 3.1, the boundary term plays a crucial role in the renormalisation of IR divergences.

We will consider the field to be composed by

$$\Phi = \Phi_{cl} + h \tag{4.49}$$

where Φ_{cl} solves the equation of motion while h is the quantum fluctuation. As the classical field solves the equation of motion, we will impose a non-normalisable Dirichlet boundary condition, and the quantum fluctuation will have a normalisable boundary condition

$$\Phi_{cl}(z \to 0, \vec{x}) \sim z^{d-\Delta} \phi_0 + \ldots, \quad h(z \to 0, \vec{x}) \sim z^{\Delta} h_0 + \ldots \tag{4.50}$$

where Δ is the positive solution to $\Delta(\Delta - d) = m^2$. Notice that Δ depends on the mass in the Lagrangian. As we may suppose from standard QFT, by considering quantum correction, we will have to renormalise the mass; therefore, we expect that the conformal dimension Δ will also be renormalised.

Plugging the decomposition (4.49) into the action we will have

$$S[\Phi] = S_{cl} + S_q, \tag{4.51}$$

where we called $S_{cl} = S[\Phi_{cl}] + B$ and $S_q = S[\Phi_{cl}, h]$ which is at least quadratic in h. The linear term in h vanishes because of the equation of motion.

Then, the partition function will be

$$
\begin{aligned}
Z &= Z_{cl} Z_q \\
&= e^{-S_{cl}} \int Dh e^{-S_q}
\end{aligned}
$$

where we called $Z_{cl} = e^{-S_{cl}-B}$ and Z_q stands for the path integral over h. In this section, we will focus on the path integral over the quantum fluctuation.

Notice that the S_q action has a non-trivial dependence on the classical field Φ_{cl}, which depends on the "source" of the conformal theory through the boundary condition. Thus, the quantum part of the partition function depends on the source ϕ_0

$$
Z_q = Z_q[\phi_0] = \int Dh e^{-S_q}. \tag{4.52}
$$

If we have a non-quadratic potential, the path integral cannot be fully solved. However, it can be computed perturbatively.

In exact agreement with standard QFT computations, the functional $W_q = \ln Z_q$ generates the connected correlation function. In this section, we will work with W_q to see the effect of quantum correction on the holographic correlation function.

As we have seen before, the correlation functions are obtained by taking functional derivatives with respect to the boundary condition of the classical field. The quantum correction to the tree-level correlation function is given by

$$
\langle \mathcal{O}_{\Delta_1}(\vec{x}) \rangle_q = \frac{\delta}{\delta \phi_0(\vec{x})} W_q \Big|_{\phi_0=0}, \tag{4.53}
$$

$$
\langle \mathcal{O}_{\Delta_1}(\vec{x}) \mathcal{O}_{\Delta_2}(\vec{x}_2) \rangle_q = -\frac{\delta^2}{\delta \phi_0(\vec{x}_1) \delta \phi_0(\vec{x}_2)} W_q \Big|_{\phi_0=0}, \tag{4.54}
$$

$$
\vdots
$$

where the subscript q denotes that it corresponds to the quantum part of the correlation function.

Computing the quantum corrections to the correlation function, we will find the already-known IR divergences and UV divergences. The high energy divergences (in the bulk) do not arise in the tree-level computation, and we must know how to deal with them.

The $S_q = S[\Phi_{cl}, h]$ action for the bulk (4.48) is

$$S_q[\phi_0] = \int dxh \left(-\Box + m^2 + \frac{\lambda}{4}\Phi_{cl}^2 + \frac{\lambda}{3!}\Phi_{cl} h + \frac{\lambda}{4!}h^2 \right) h. \tag{4.55}$$

Where $\Phi_{cl} = \Phi_{cl}(\phi_0)$ is given by the perturbative solution to the equation of motion.

From here, we see that the partition function will be a path integral over a non-Gaussian integral. To work the non-quadratic exponential in the path integral, we proceed by adding a bulk source J. Then, we define the double source partition function as

$$Z[\phi_0, J] = \int Dh \; e^{-S_q + \int dx Jh}. \tag{4.56}$$

We must remember that the source we use to obtain the correlation function is ϕ_0, and the source J is just an intermediate device that allows us to work the non-Gaussian integral.

Following the same tricks of standard QFT, we can take out the non-quadratic part of the action as derivatives with respect to the source J and integrate it out of the partition function

$$Z[\phi_0, J] = e^{-\lambda \int dx \left(\frac{1}{4!}\frac{\delta^4}{\delta J^4} + \frac{1}{4}\Phi_{cl}^2 \frac{\delta^2}{\delta J^2} + \frac{1}{3}\Phi_{cl}\frac{\delta^3}{\delta J^3} \right)} \int Dh e^{-\int dx \left[h(-\Box + m^2)h + Jh \right]} \tag{4.57}$$

the path integral is now Gaussian. Computing the path integral

$$Z[\phi_0, J] = e^{-\lambda \int dx \left(\frac{1}{4!}\frac{\delta^4}{\delta J^4} + \frac{1}{4}\Phi_{cl}^2 \frac{\delta^2}{\delta J^2} + \frac{1}{3}\Phi_{cl}\frac{\delta^3}{\delta J^3} \right)} e^{\frac{1}{2}\int d^{d+1}x_1 d^{d+1}x_2 J(x_1) G_\Delta(x_1, x_2) J(x_2)}, \tag{4.58}$$

where the function $G(x_1, x_2)$ satisfies,

$$(-\Box + m^2)G_\Delta(x_1, x_2) = \frac{\delta(x_1 - x_2)}{\sqrt{g}} \tag{4.59}$$

which is the same equation (3.47) found when we solved the non-linear equation of motion. So $G_\Delta(x_1, x_2)$ corresponds to the bulk-to-bulk propagator.

In equation (4.58), the classical fields are on the left exponential; these terms are the ones that will have the bulk-to-boundary propagator. Therefore, any external leg is contained in this part of the partition function.

We will be interested in the connected correlation functions that are generated by $W[\phi_0] = -\log Z[\phi_0]$. Along the following lines, we will keep up to $\mathcal{O}(\lambda^2)$.

To obtain $W[\phi_0]$ we work term by term in (4.58) and then take the logarithm. The result up to the second order in the coupling is,

$$\begin{aligned}
W_q[\phi_0] = &-\frac{\lambda}{4}\int dx\Phi_{cl}^2(x)G_\Delta(x,x) + \frac{\lambda^2}{8}\int dx_1dx_2\Phi_{cl}^2(x_1)G_\Delta(x_1,x_2)^2G_\Delta(x_2,x_2) \\
&+ \frac{\lambda^2}{6}\int dx_1dx_2\Phi_{cl}(x_1)\Phi_{cl}(x_2)G_\Delta(x_1,x_2)^3 \\
&+ \frac{\lambda^2}{8}\int dx_1dx_2\Phi_{cl}(x_1)G_\Delta(x_1,x_1)G_\Delta(x_1,x_2)G_\Delta(x_2,x_2)\Phi_{cl}(x_2) \\
&+ \frac{\lambda^2}{32}\int dx_1dx_2\Phi_{cl}^2(x_1)G_\Delta(x_1,x_1)\Phi_{cl}^2(x_2)G_\Delta(x_2,x_2) \\
&+ \frac{\lambda^2}{16}\int dx_1dx_2\Phi_{cl}^2(x_1)G_\Delta(x_1,x_2)^2\Phi_{cl}^2(x).
\end{aligned} \tag{4.60}$$

We must remember that Φ_{cl} has its own series in λ given by (3.42). Therefore the order λ term in W_q will have a λ^2 correction.

The loop correction to the two and 4-point functions are given by,

$$\langle \mathcal{O}_\Delta(\vec{y}_1)\mathcal{O}_\Delta(\vec{y}_2)\rangle_q = \frac{\delta^2 W_q[\phi_0]}{\delta\phi_0(\vec{x}_1)\delta\phi_0(\vec{x}_2)}\Big|_{\phi_0=0}, \tag{4.61}$$

$$\langle \mathcal{O}_\Delta(\vec{y}_1)\mathcal{O}_\Delta(\vec{y}_2)\mathcal{O}_\Delta(\vec{y}_3)\mathcal{O}_\Delta(\vec{y}_4)\rangle_q = \frac{\delta^4 W_q[\phi_0]}{\delta\phi_0(\vec{x}_1)\delta\phi_0(\vec{x}_2)\delta\phi_0(\vec{x}_3)\delta\phi_0(\vec{x}_4)}\Big|_{\phi_0=0}. \tag{4.62}$$

By symmetry argument, the 3-point function vanishes.

Performing the derivatives we find for the 2-point function

$$\langle \mathcal{O}_\Delta(\vec{y}_1)\mathcal{O}_\Delta(\vec{y}_2)\rangle_q = -\frac{\lambda}{2}T_1 + \frac{\lambda^2}{4}E + \frac{\lambda^2}{6}S + \frac{\lambda^2}{4}T_2 \tag{4.63}$$

where we called

$$T_1 = \int dx_1 K_\Delta(z_1, \vec{x}_1 - \vec{y}_1)K_\Delta(z_1, \vec{x}_1 - \vec{y}_2)G_\Delta(x_1, x_1), \tag{4.64}$$

$$E = \int dx_1 K_\Delta(z_1, \vec{x}_1 - \vec{y}_1)K_\Delta(z_1, \vec{x}_1 - \vec{y}_2)G_\Delta(x_1, x_2)^2 G_\Delta(x_1, x_1), \tag{4.65}$$

$$S = \int dx_1 K_\Delta(z_1, \vec{x}_1 - \vec{y}_1)G_\Delta(x_1, x_2)^3 K_\Delta(z_2, \vec{x}_2 - \vec{y}_2), \tag{4.66}$$

$$T_2 = \int dx_1 K_\Delta(z_1, \vec{x}_1 - \vec{y}_1)G_\Delta(x_1, x_1)G_\Delta(x_1, x_2)G_\Delta(x_2, x_2)K_\Delta(z_2, \vec{x}_2 - \vec{y}_2). \tag{4.67}$$

Each of these integrals has an explicit diagrammatic expression given in Figure 4.4, where we see that the names $T_1,\ E,\ S,\ T_2$ stands for $1-T$tadpole, Eight, Sunset and $2-T$adpole.

Similarly, we can compute the 4-point function (4.62), finding

$$\langle \mathcal{O}_\Delta(\vec{y}_1)\mathcal{O}_\Delta(\vec{y}_2)\mathcal{O}_\Delta(\vec{y}_3)\mathcal{O}_\Delta(\vec{y}_4)\rangle_q = \frac{\lambda^2}{2}B_1 + \frac{\lambda^2}{2}B_2 + \frac{\lambda^2}{2}B_3 + \frac{\lambda^2}{2}B_4 + \frac{\lambda^2}{2}X_s + \frac{\lambda^2}{2}X_t + \frac{\lambda^2}{2}X_u \tag{4.68}$$

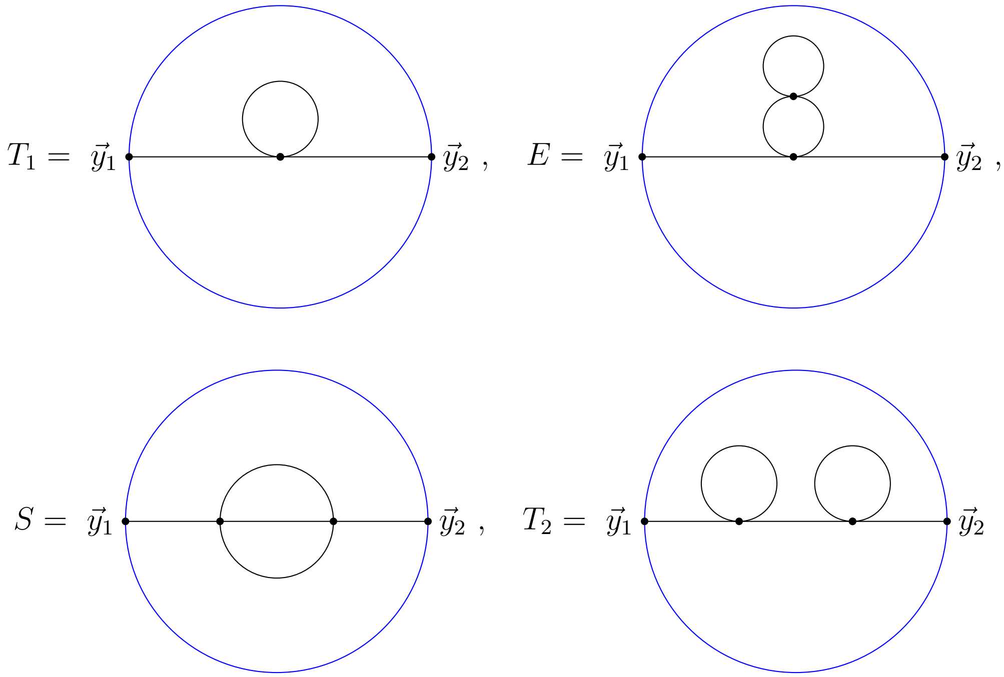


Figure 4.4: Witten diagrams for the loops 2-point function up to order λ^2 in the coupling.

where

$$B_i = \int dx_1 dx_2 K_\Delta(z_1, \vec{x}_1 - \vec{y}_i) G_\Delta(x_1, x_1) G_\Delta(x_1, x_2) \prod_{j=1, j\neq i}^{4} K_\Delta(z_2, \vec{x}_2 - \vec{y}_j), \ i = 1, 2, 3, 4 \tag{4.69}$$

$$X_s = \int dx_1 dx_2 K_\Delta(z_1, \vec{x}_1 - \vec{y}_1) K_\Delta(z_1, \vec{x}_1 - \vec{y}_4) G_\Delta(x_1, x_2)^2 K_\Delta(z_2, \vec{x}_2 - \vec{y}_2) K_\Delta(z_2, \vec{x}_2 - \vec{y}_3), \tag{4.70}$$

$$\vdots$$

The subscript in B denotes the external leg of the loop, and the X diagram denotes the s, t, u channels. Figure 4.5 shows the Witten diagram for X_s and B_1.

Our goal now is to work on each diagram presented and see the effect of the quantum loops in the holographic correlation function.

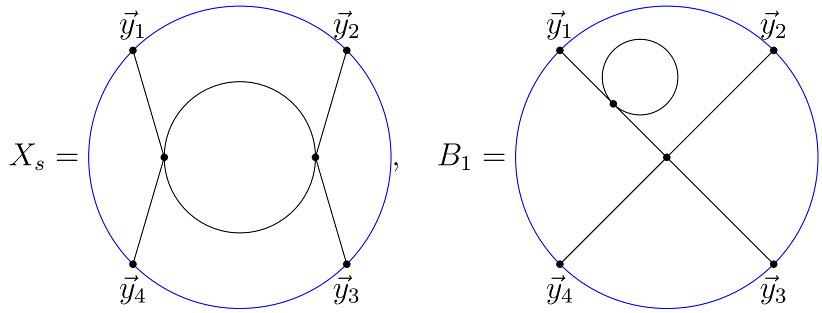


Figure 4.5: 4-point function up to one loop.

From now on, we will use the notation,

$$\int dx = \int_{\epsilon}^{\infty} dz \int d\vec{x}\sqrt{x}, \quad G_{\Delta}(x,y) = G_{\Delta,\kappa}(x,y), \tag{4.71}$$

i.e. denotes the regularised objects.

First, we will work on the 2-point and then the 4-point functions.

4.4.1 2-point function

We have to compute the integrals (4.64), (4.65), (4.66) and (4.67). All these integrals are related to those we studied in section 4.2, and we know they may need regularisation. Indeed, all the loop Witten diagrams for the two-point function will demand IR and UV regularisation.

Writing the regularised 2-point quantum correction in a more suggestive way

$$T_1\left(\kappa,\epsilon,\vec{y}_1,\vec{y}_2\right)=G_{\Delta,\kappa}(1)\int_{z=\epsilon}dx_1K_\Delta(z_1,\vec{x}_1-\vec{y}_1)K_\Delta(z_1,\vec{x}_1-\vec{y}_2), \tag{4.72}$$

$$E\left(\kappa,\epsilon,\vec{y}_1,\vec{y}_2\right)=G_{\Delta,\kappa}(1)\int_{z=\epsilon}dx_1K_\Delta(z_1,\vec{x}_1-\vec{y}_1)K_\Delta(z_1,\vec{x}_1-\vec{y}_2)\int dx_2G_{\Delta,\kappa}(x_1,x_2)^2, \tag{4.73}$$

$$S\left(\kappa,\epsilon,\vec{y}_1,\vec{y}_2\right)=\int_{z=\epsilon}dx_1K_\Delta(z_1,\vec{x}_1-\vec{y}_1)\int dx_2G_{\Delta,\kappa}(x_1,x_2)^3K_\Delta(z_2,\vec{x}_2-\vec{y}_2) \tag{4.74}$$

$$T_2\left(\kappa,\epsilon,\vec{y}_1,\vec{y}_2\right)=G_{\Delta,\kappa}(1)^2\int_{z=\epsilon}dx_1K_\Delta(z_1,\vec{x}_1-\vec{y}_1)\int_{z=\epsilon}dx_2G_{\Delta,\kappa}(x_1,x_2)K_\Delta(z_2,\vec{x}_2-\vec{y}_2). \tag{4.75}$$

Let us write each diagram explicitly.

1-Tadpole diagram

Using (4.11) in (4.72) we have

$$\begin{aligned}T_1\left(\kappa,\epsilon,\vec{y}_1,\vec{y}_2\right)=G_{\Delta,\kappa}(1)\Bigg(&\frac{1}{\epsilon^{2\Delta-d}}\frac{\delta(\vec{y}_1-\vec{y}_2)}{2\Delta-d}+\dots\\&-\frac{2c_\Delta}{|\vec{y}_1-\vec{y}_2|^{2\Delta}}\left(\log\left(\frac{\epsilon}{|\vec{y}_1-\vec{y}_2|}\right)+\frac{\psi\left(\Delta\right)-\psi\left(\Delta-\frac{d}{2}\right)}{2}\right)\Bigg).\end{aligned} \tag{4.76}$$

Notice that this diagram has both IR and UV divergences.

Eight diagram

Using (4.26) we will have the $D_{\Delta,\Delta}$ integral so we have to use (4.11) as well

$$\begin{aligned}
E(\kappa,\epsilon,\vec{y}_1,\vec{y}_2) &= G_{\Delta,\kappa}(1) I_3 \int_{z=\epsilon} dx_1 K_\Delta(z_1,\vec{x}_1-\vec{y}_1) K_\Delta(z_1,\vec{x}_1-\vec{y}_2) \\
&= -G_{\Delta,\kappa}(1) I_3 \left(\frac{1}{\epsilon^{2\Delta-d}} \frac{\delta(\vec{y}_1-\vec{y}_2)}{2\Delta-d} + \dots \right. \\
&\left. -\frac{2c_\Delta}{|\vec{y}_1-\vec{y}_2|^{2\Delta}} \left(\log\left(\frac{\epsilon}{|\vec{y}_1-\vec{y}_2|}\right) + \frac{\psi(\Delta)-\psi\left(\Delta-\frac{d}{2}\right)}{2} \right) \right).
\end{aligned} \tag{4.77}$$

The Sunset diagram

In this case we use (4.24) which will leave the $D_{\Delta,\Delta}$, therefore we use (4.11)

$$\begin{aligned}
S(\kappa,\epsilon,\vec{y}_1,\vec{y}_2) &= c_3 \int_{z=\epsilon} dx_1 K_\Delta(z_1,\vec{x}_1-\vec{y}_1) K_\Delta(z_1,\vec{x}_1-\vec{y}_2) \\
&= c_3 \left(\frac{1}{\epsilon^{2\Delta-d}} \frac{\delta(\vec{y}_1-\vec{y}_2)}{2\Delta-d} + \dots \right. \\
&\left. -\frac{2c_\Delta}{|\vec{y}_1-\vec{y}_2|^{2\Delta}} \left(\log\left(\frac{\epsilon}{|\vec{y}_1-\vec{y}_2|}\right) + \frac{\psi(\Delta)-\psi\left(\Delta-\frac{d}{2}\right)}{2} \right) \right).
\end{aligned}$$

Despite the notation, c_3 and c_Δ are not related. The constant c_3 is defined in (4.24) and c_Δ is defined in (3.35).

The double tadpole

Here we have to notice that there is a double IR divergence. First, working the x_2 integral that corresponds to a GK integral given by (4.18)

$$\begin{aligned}
T_2(\kappa,\epsilon,\vec{y}_1,\vec{y}_2) = -\frac{G^2_{\Delta,\kappa}(1)}{2\Delta-d} \int_{z=\epsilon} dx_1 K_\Delta(z,\vec{x}_1-\vec{y}_1) K_\Delta(z_1,\vec{x}_1-\vec{y}_2) \left[\log\left(\frac{\epsilon z}{z_1^2+(\vec{x}_1-\vec{y}_2)^2}\right) \right. \\
\left. +\psi(\Delta) - \psi\left(\Delta-\frac{d}{2}\right) \right].
\end{aligned} \tag{4.78}$$

This integral is given by (4.34), so the double tadpole diagram is

$$T_2(\kappa,\epsilon,\vec{y}_1,\vec{y}_2) = G_{\Delta,\kappa}(1)^2\frac{\epsilon^{d-2\Delta}\delta(\vec{y}_1-\vec{y}_2)}{(2\Delta-d)^3} + \ldots + \frac{2c_\Delta}{2\Delta-d}\frac{G_{\Delta,\kappa}(1)^2}{|\vec{y}_1-\vec{y}_2|^{2\Delta}}\left[\log\left(\frac{\epsilon}{|\vec{y}_1-\vec{y}_2|}\right)^2 + \right. \tag{4.79}$$

$$\log\left(\frac{\epsilon}{|\vec{y}_1-\vec{y}_2|}\right)\left[\psi(\Delta)-\psi\left(\Delta-\frac{d}{2}\right)\right] + \frac{1}{4}\left[\psi^{(1)}(\Delta-\psi^{(1)}\left(\Delta-\frac{d}{2}\right)\right]$$
$$\left. +\frac{1}{4}\left[\psi(\Delta-\psi\left(\Delta\right)-\frac{d}{2}\right)\right]^2\bigg].$$

Summing and renormalisation

So far, we have computed the loop Witten diagram for the two-point function. Generally, each diagram has the tree-level 2-point function structure times a logarithm. Because of the presence of the logarithm, we see that each loop diagram does not correspond to a CFT 2-point function, nor does the sum of the loop Witten diagram. However, to understand the existence of the logarithmic term in the quantum diagrams, we must remember that the leading term corresponds to the tree-level computation done in section 3.2 and that the loop Witten diagram corresponds to correction to the tree-level computation.

Explicitly this means that the 2-point function up to λ^2 is given by

$$\langle\mathcal{O}_\Delta(\vec{y}_1)\mathcal{O}_\Delta(\vec{y}_2)\rangle = \langle\mathcal{O}_\Delta(\vec{y}_1)\mathcal{O}_\Delta(\vec{y}_2)\rangle_{\text{tree}} - \frac{\lambda}{2}T_1 + \lambda^2\left(\frac{1}{4}E+\frac{1}{6}S+\frac{1}{4}T_2\right), \tag{4.80}$$

which is diagrammatically given by Figure4.6

Just as a matter of introduction, we will first work to order λ. The order λ^2 is analogous.

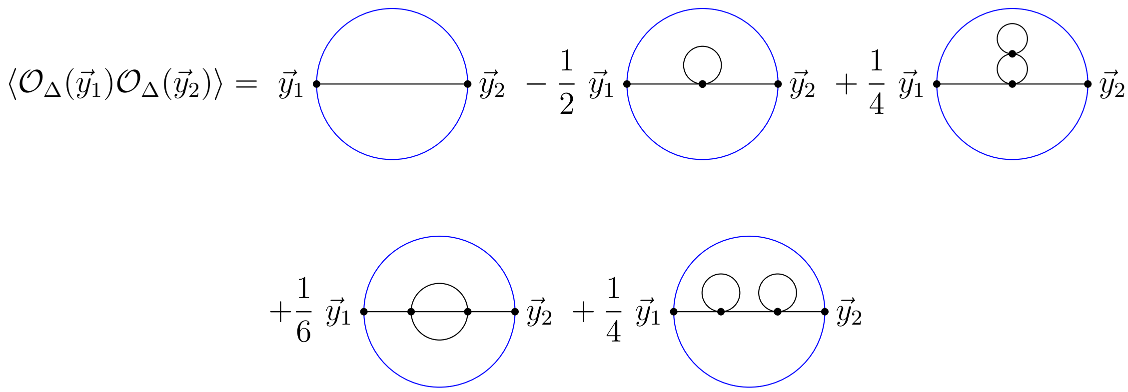


Figure 4.6: Witten diagram for the 2-point function up to two loops

Two-point function up to order λ

We now study the 1-loop correction to the 2-point function. For this purpose, we split the problem by considering the non-local part, which gives the correlation functions, and the local part, which corresponds to the IR divergent terms.

Let us start with the non-local part. The regularised 2-point function (4.80) up to order λ is given by

$$\langle \mathcal{O}_\Delta(\vec{y}_1)\mathcal{O}_\Delta(\vec{y}_2)\rangle_{reg}^{\text{non-local}} = -\frac{(2\Delta-d)c_\Delta}{|\vec{y}_1-\vec{y}_2|^{2\Delta}} - \frac{\lambda}{2}T_1\Big|_{\text{non-local}}. \tag{4.81}$$

Where we made explicit the non-local part obtained from the tree-level computation, and for the quantum correction, we must consider the non-local term. The role of the local IR divergent term in (4.11) will soon become apparent.

Writing explicitly the 1-tadpole diagram (4.76) we will have

$$\langle \mathcal{O}_\Delta(\vec{y}_1)\mathcal{O}_\Delta(\vec{y}_2)\rangle_{reg}^{\text{non-local}} = -\frac{(2\Delta-d)c_\Delta}{|\vec{y}_1-\vec{y}_2|^{2\Delta}}\left(1+2\lambda\gamma_1\left(\log\left(\frac{\epsilon}{|\vec{y}_1-\vec{y}_2|}\right)+\frac{\psi(\Delta)-\psi\left(\Delta-\frac{d}{2}\right)}{2}\right)\right) \tag{4.82}$$

where we defined

$$\gamma_1 \equiv \frac{G_{\Delta,\kappa}(1)}{2(2\Delta - d)} \tag{4.83}$$

which is divergent in the limit of $\kappa \to 0$. The form of this 2-point function is full of divergences. On one side, we have a UV divergent term in γ_1 and IR divergent pieces given by ϵ.

To have a well-defined bulk theory, we must renormalize the UV divergences. To do this, we will understand the mass as a bare parameter, such that

$$m^2 = m_s^2 + \delta_m, \quad \delta_m = a_1\lambda + a_2\lambda^2 + \dots . \tag{4.84}$$

Where δ_m is the mass counterterm. Rather than working with the mass, it is better to use the conformal dimension

$$\Delta = \Delta_s + \lambda\frac{a_1}{2\Delta_s - d} + \mathcal{O}(\lambda^2). \tag{4.85}$$

The mass counterterm can be seen as an interaction, which will give the following term to the action

$$S_{UV,ct} = \int_{z=\epsilon} dx \delta_m \Phi_{cl}^2. \tag{4.86}$$

Taking two derivatives with respect to the source ϕ_0

$$\begin{aligned}\langle \mathcal{O}_\Delta(\vec{y}_1)\mathcal{O}_\Delta(\vec{y}_2)\rangle^{\text{non-local}}_{\text{UV ct}} &= \delta_m \int_{z=\epsilon} dx K_\Delta(z, \vec{x} - \vec{y})K_\Delta(z, \vec{x} - \vec{y}_2) \\ &= -\frac{2\delta_m c_\Delta}{|\vec{y}_1 - \vec{y}_2|^{2\Delta}}\left[\log\left(\frac{\epsilon}{|\vec{y}_1 - \vec{y}_2|}\right) + \frac{\psi(\Delta) - \psi\left(\Delta - \frac{d}{2}\right)}{2}\right].\end{aligned} \tag{4.87}$$

Adding the counterterm, the non-local part of the regularised 2-point function with the UV counterterm is

$$
\begin{aligned}
\langle \mathcal{O}_\Delta(\vec{y}_1)\mathcal{O}_\Delta(\vec{y}_2)\rangle^{\text{non-local}}_{reg,\text{UV ct}} &= -\frac{(2\Delta-d)c_\Delta}{|\vec{y}_1-\vec{y}_2|^{2\Delta}} - \frac{\lambda}{2}T_1\Big|_{non-local} + \langle \mathcal{O}_\Delta(\vec{y}_1)\mathcal{O}_\Delta(\vec{y}_2)\rangle_{\text{UV, ct}} \qquad (4.88)\\
&= -\frac{(2\Delta-d)c_\Delta}{|\vec{y}_1-\vec{y}_2|^{2\Delta}}\left(1+2\lambda\left(\gamma_1+\frac{a_1}{2\Delta-d}\right)\log\left(\frac{\epsilon}{|\vec{y}_1-\vec{y}_2|}+\frac{\psi(\Delta)-\psi\left(\Delta-\frac{d}{2}\right)}{2}\right)\right).
\end{aligned}
$$

Diagrammatically is shown in Figure 4.7

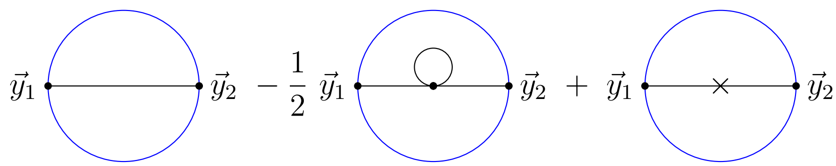
$\langle \mathcal{O}_\Delta(\vec{y}_1)\mathcal{O}_\Delta(\vec{y}_2)\rangle^{\text{non-local}}_{reg,\text{UV ct}} =$ 

Figure 4.7: Witten diagram for the 2-point function up to one loop with UV counterterm

At this point, we are in a position to renormalise the UV divergence. This is done by adequately choosing a_1.

Usually, in standard QFT, the renormalisation scheme is given by defining the value of the mass at some energy scale. In the current computation, we should know the conformal theory that lives in the boundary of a Φ^4 theory in the bulk. So far, we do not have this information and it goes beyond this work's scope.

However, as we can choose the renormalisation scheme, we can choose a_1 to cancel the divergent part of γ_1, or the total value of γ_1. The more straightforward solution corresponds to choosing a_1, which contains the total value of γ_1 and then we will be left with the original conformal dimension and quantum fluctuation has no effect.

We will choose a_1 such that it cancels only the purely divergent piece of γ_1. In standard QFT, this is the minimal subtraction scheme and corresponds to

$$
a_1 = \frac{1}{2}\text{div}\left[G_{\Delta,\kappa}(1)\right]. \qquad (4.89)
$$

Furthermore, we leave only the convergent part of $G_{\Delta,\kappa}(1)$. Thus, we define the renormalised conformal dimension as

$$\Delta_r = \lim_{\kappa \to 0} \left[\Delta_s + \frac{\lambda}{2} \mathrm{conv} \left[G_{\Delta,\kappa}(1) \right] \right] \tag{4.90}$$

Defining the renormalised anomalous dimension as

$$\gamma_{1,r} = \lim_{\kappa \to 0} \left[\frac{\mathrm{conv} \left(G_{\Delta,\kappa}(1) \right)}{2(2\Delta - d)} \right], \tag{4.91}$$

which is finite by construction. So the 2-point function at 1-loop correction is

$$\begin{aligned}
&\langle \mathcal{O}_\Delta(\vec{y}_1) \mathcal{O}_\Delta(\vec{y}_2) \rangle^{\text{non-local}}_{\text{UV ren}} \\
&\qquad = -\frac{(2\Delta - d) c_\Delta}{|\vec{y}_1 - \vec{y}_2|^{2\Delta}} \left(1 + 2\gamma_{1,r} \log \left(\frac{\epsilon}{|\vec{y}_1 - \vec{y}_2|} + \frac{\psi(\Delta) - \psi\left(\Delta - \frac{d}{2}\right)}{2} \right) \right) \\
&\qquad = \frac{(2\Delta_r - d) c_{\Delta_r}}{|\vec{y}_1 - \vec{y}_2|^{2\Delta_r}} \epsilon^{\gamma_{1,r}}.
\end{aligned} \tag{4.92}$$

We have two issues: On the first hand, on the r.h.s, we have to remove the IR regulator. On the other hand, comparing both sides of the equation, we notice that the l.h.s, we have operators of dimension Δ and, keeping aside the ϵ factor, the r.h.s corresponds to the 2-point function for operators with conformal dimension Δ_r. Both problems, the IR regulator and the mismatch among the conformal dimensions are two sides of the same coin.

To obtain the loop diagram, we used the bare source, which has conformal dimension $d - \Delta$. After renormalization, the operator has conformal dimension Δ_r. As we changed the conformal dimension, the problem tells us that we took derivatives with respect to the wrong source ϕ_0. The way to deal with this is by doing the equivalent to wavefunction renormalisation, but the renormalisation has to deal with the IR divergence.

So, defining

$$\phi_0 = \sqrt{Z_\Phi}\phi_0^{(r)} \tag{4.93}$$

where $\phi_0^{(r)}$ is the renormalised source. Then the correlation function are related by

$$\frac{\delta W[\phi_0]}{\delta\phi_0(\vec{y}_1)\delta\phi_0(\vec{y}_2)} = \frac{1}{Z_\phi}\frac{\delta W[\phi_0]}{\delta\phi_0^{(r)}(\vec{y}_1)\delta\phi_0^{(r)}(\vec{y}_2)} \tag{4.94}$$

$$= \frac{(2\Delta_r - d)c_{\Delta_r}}{|\vec{y}_1 - \vec{y}_2|^{2\Delta_r}}\epsilon^{\gamma_{1,r}} \tag{4.95}$$

so choosing $Z_\Phi = \epsilon^{-\frac{\gamma_{1,r}}{2}}$ we will have the correct two-point function

$$\langle \mathcal{O}_{\Delta_r}(\vec{y}_1)\mathcal{O}_{\Delta_r}(\vec{y}_2)\rangle^{\text{non-local}} = \frac{(2\Delta_r - d)c_{\Delta_r}}{|\vec{y}_1 - \vec{y}_2|^{2\Delta_r}} \tag{4.96}$$

which is fully consistent and has exactly the form of the 2-point function.

Even though we have obtained the renormalised correlation function, we have to understand the role of the IR divergent parts of the (4.11) integral. By looking at the regularised action (3.25) and taking two derivatives with respect to the source ϕ_0, we will find

$$\frac{\delta S_{reg}}{\delta\phi_0(\vec{y}_1)\delta\phi_0(\vec{y}_2)} = (\Delta - d)\delta(\vec{y}_1 - \vec{y}_2)\epsilon^{d-2\Delta} + \dots \tag{4.97}$$

which has the same structure as the divergent piece of (4.11).

The local part of the correlation function up to 1-loop with the mass counterterm corresponds to

$$\langle \mathcal{O}_\Delta(\vec{y}_1)\mathcal{O}_\Delta(\vec{y}_2)\rangle^{\text{local}} = \epsilon^{d-2\Delta}\left((d-\Delta) - \lambda\frac{G_{\Delta,\kappa}(1)}{2(2\Delta-d)} - \lambda\frac{2\delta_m}{2\Delta - d}\right)\delta(\vec{y}_1 - \vec{y}_2) + \dots \tag{4.98}$$

in the bracket, the first term comes from the classical computation, the second term from the loop, and the third is the mass counterterm. It exactly agrees with the non-local computation. We choose $\delta_m = a_1\lambda$, such that cancel the UV divergent part of $G_{\Delta,\kappa}(1)$. Doing so and using the renormalised conformal dimensions (4.90) we have

$$\langle \mathcal{O}_\Delta(\vec{y}_1)\mathcal{O}_\Delta(\vec{y}_2)\rangle^{\text{local}} = \epsilon^{d-2\Delta}(d-\Delta_r)\delta(\vec{y}_1-\vec{y}_2)+\dots. \tag{4.99}$$

Again we find a mismatch between both sides of the equality. Source renormalisation solves this, and we have

$$\langle \mathcal{O}_{\Delta_r}(\vec{y}_1)\mathcal{O}_{\Delta_r}(\vec{y}_2)\rangle^{\text{local}} = \epsilon^{d-2\Delta_r}(d-\Delta_r)\delta(\vec{y}_1-\vec{y}_2)+\dots \tag{4.100}$$

From holographic renormalisation, all local terms are cancelled by the local boundary counterterm

Two-point function up order λ^2

Going to two-loops, the procedure is similar to the 1-loop computation, so we will skip some details.

The 2-point function is given by (4.80). Similarly, as we did for the 1-loop computation, we first focus on the non-local part,

$$\langle \mathcal{O}_\Delta(\vec{y}_1)\mathcal{O}_\Delta(\vec{y}_2)\rangle^{\text{non-local}}_{reg} = -\frac{(2\Delta-d)c_\Delta}{|\vec{y}_1-\vec{y}_2|^{2\Delta}} - \frac{\lambda}{2}T_1 + \lambda^2\left(\frac{1}{4}E+\frac{1}{6}S+\frac{1}{4}T_2\right). \tag{4.101}$$

The diagram is shown in Figure 4.6. The way to deal with this problem is exactly as was done for the 1-loop problem, but now we have

$$\langle \mathcal{O}_\Delta(\vec{y}_1)\mathcal{O}_\Delta(\vec{y}_2)\rangle_{reg}^{\text{non-local}} = -\frac{(2\Delta-d)c_\Delta}{|\vec{y}_1-\vec{y}_2|^{2\Delta}}\left(1+2\lambda\left(\gamma_1-\lambda\gamma_2\right)\left(\log\left(\frac{\epsilon}{|\vec{y}_1-\vec{y}_2|}\right)\right.\right.$$
$$\left.+\frac{\psi(\Delta)-\psi\left(\Delta-\frac{d}{2}\right)}{2}\right)-2\lambda^2\bar{\gamma}_2\left[\log\left(\frac{\epsilon}{|\vec{y}_1-\vec{y}_2|}\right)^2\right.$$
$$\left.\left.+\frac{\psi^{(1)}(\Delta)-\psi^{(1)}\left(\Delta-\frac{d}{2}\right)}{4}+\frac{\left(\psi(\Delta)-\psi\left(\Delta-\frac{d}{2}\right)\right)^2}{4}\right]\right), \tag{4.102}$$

where we called

$$\gamma_1 = \frac{G_{\Delta,\kappa}(1)}{2(2\Delta-d)}, \tag{4.103}$$
$$\gamma_2 = -\frac{c_3}{6(2\Delta-d)}-\frac{G_{\Delta,\kappa}(1)}{4(2\Delta-d)}+\frac{G_{\Delta,\kappa}(1)I_3}{4(2\Delta-d)^2}, \tag{4.104}$$
$$\bar{\gamma}_2 = \frac{G_{\Delta,\kappa}(1)^2}{4(2\Delta-d)^2}. \tag{4.105}$$

We may notice that $\gamma_1^2=\bar{\gamma}_2$. Defining

$$\gamma=\gamma_1+\lambda\gamma_2 \tag{4.106}$$

and using $\gamma^2=\bar{\gamma}_2+\mathcal{O}(\lambda)$ we may write,

$$\langle \mathcal{O}_\Delta(\vec{y}_1)\mathcal{O}_\Delta(\vec{y}_2)\rangle_{reg}^{\text{non-local}} = -\frac{(2\Delta-d)c_\Delta}{|\vec{y}_1-\vec{y}_2|^{2\Delta}}\left(1+2\lambda\gamma\left(\log\left(\frac{\epsilon}{|\vec{y}_1-\vec{y}_2|}\right)+\frac{\psi(\Delta)-\psi\left(\Delta-\frac{d}{2}\right)}{2}\right)\right.$$
$$\left.-2\lambda^2\gamma^2\left[\log\left(\frac{\epsilon}{|\vec{y}_1-\vec{y}_2|}\right)^2+\frac{\psi^{(1)}(\Delta)-\psi^{(1)}\left(\Delta-\frac{d}{2}\right)}{4}+\frac{\left(\psi(\Delta)-\psi\left(\Delta-\frac{d}{2}\right)\right)^2}{4}\right]\right).$$

This has the same structure of (4.82), but now it considers corrections up to the second order in the coupling. Following precisely the same steps, we must understand the mass as a bare parameter such that $m^2=m_s^2+\delta_m$, with $\delta_m=a_1\lambda+a_2\lambda^2$ is the mass counterterm,

At the level of the action, the mass redefinition leads to new action on the path integral

$$S_{UV,ct} = \int_{z=\epsilon} dx \delta_m \Phi^2, \tag{4.107}$$

which is the UV counterterm action.

Now we proceed exactly as at the beginning of this section. We consider the field to be composed of a classical field that solves the equation of the motion with mass m_s^2 and a quantum fluctuation. Consider the UV counterterm as an interaction rather than part of the kernel. Following the holographic dictionary, we will find the UV counterterm two-point function

$$\begin{aligned}\langle \mathcal{O}_\Delta(\vec{y}_1)\mathcal{O}_\Delta(\vec{y}_2)\rangle^{\text{non-local}}_{UVct} &= -\delta_m \int_{z=\epsilon} dx K_\Delta(z,\vec{x}-\vec{y}_1)K_\Delta(z,\vec{x}-\vec{y}_2) \\ &+ \lambda\delta_m G_{\Delta,\kappa}(1)\int_{z=\epsilon} dx_1 dx_2 K_\Delta(z_1,\vec{x}_1-\vec{y}_1)G_\Delta(x_1,x_2)K_\Delta(z_1,\vec{x}_1-\vec{y}_2) \\ &+ \delta_m^2 \int dx_1 dx_2 K_\Delta(z_1,\vec{x}_1-\vec{y}_1)G_\Delta(x_1,x_2)K_\Delta(z_1,\vec{x}_1-\vec{y}_2).\end{aligned}$$

Through the integrals (4.11) and (4.34), we notice that δ_m will sum to the correlation function (4.102).

So, we define the regularised plus UV counterterm correlation function as

$$\begin{aligned}&\langle \mathcal{O}_\Delta(\vec{y}_1)\mathcal{O}_\Delta(\vec{y}_2)\rangle^{\text{non-local}}_{reg,UVct} = \langle \mathcal{O}_\Delta(\vec{y}_1)\mathcal{O}_\Delta(\vec{y}_2)\rangle^{\text{non-local}}_{reg} + \langle \mathcal{O}_\Delta(\vec{y}_1)\mathcal{O}_\Delta(\vec{y}_2)\rangle^{\text{non-local}}_{UVct} \\ &= -\frac{(2\Delta-d)c_\Delta}{|\vec{y}_1-\vec{y}_2|^{2\Delta}}\left(1+\left(2\lambda\gamma+\frac{2a_1}{2\Delta-d}\lambda+\frac{2a_2}{2\Delta-d}\lambda^2\right)\left(\log\left(\frac{\epsilon}{|\vec{y}_1-\vec{y}_2|}\right)\right.\right. \\ &\left.\frac{\psi(\Delta)-\psi\left(\Delta-\frac{d}{2}\right)}{2}\right)-\left(2\lambda^2\gamma^2-a_1\lambda^2\frac{G_{\Delta,\kappa}(1)}{2\Delta-d}-\frac{a_1^2\lambda^2}{2\Delta-d}\right)\left[\log\left(\frac{\epsilon}{|\vec{y}_1-\vec{y}_2|}\right)^2\right. \\ &\left.\left.+\frac{\psi^{(1)}(\Delta)-\psi^{(1)}\left(\Delta-\frac{d}{2}\right)}{4}+\frac{\left(\psi(\Delta)-\psi\left(\Delta-\frac{d}{2}\right)\right)^2}{4}\right]\right).\end{aligned} \tag{4.108}$$

To renormalize the UV theory means to find a_1 and a_2. Having renormalized, the 1-loop correction immediately fixes the a_1 counterterm. On the other hand, a_2 cancels the divergent term of γ_2. Then we have a UV finite theory. So defining

$$\gamma^{(r)} = \gamma_1^{(r)} + \lambda\gamma_2^{(r)}, \tag{4.109}$$

we will have the equivalent of (4.92), but now up to two-loops

$$\langle \mathcal{O}_\Delta(\vec{y}_1)\mathcal{O}_\Delta(\vec{y}_2)\rangle^{\text{non-local}}_{\text{UV ren}} = \frac{(2\Delta_r - d)c_{\Delta_r}}{|\vec{y}_1 - \vec{y}_2|^{2\Delta_r}}\epsilon^{\gamma_{1,r}},$$

where,

$$\Delta_r = \lim_{\kappa\to 0}\left[\Delta_s + \lambda\gamma^{(r)}\right]. \tag{4.110}$$

Having found $\gamma^{(r)}$, we may claim that the UV problem is solved. We are still left with the IR regulator and the mismatch between conformal dimensions. These problems are solved by source renormalisation.

The renormalisation of the boundary terms proceeds precisely as the 1-loop problem. This is through local boundary counterterms with renormalized conformal dimensions.

4.4.2 4-point function

Following the same procedure dictated by (4.62) we have for the 4-point function up to order λ^2 the exchange diagram and the tadpole diagram. We will start with the former and then work on the exchange diagram.

Tadpole

For simplicity, we will work on the tadpole diagram with the loop in one leg. The other tadpole diagrams in the 4-point function are equivalent.

The tadpole diagram is given by

$$B_1 = \int dx_1 dx_2 K_\Delta(z_1, \vec{x}_1 - \vec{y}_1) G_\Delta(1) G_\Delta(x_1, x_2) \prod_{j=2}^{4} K_\Delta(z_2, \vec{x}_2 - \vec{y}_j). \tag{4.111}$$

It is easy to check that the presence of the bulk-to-bulk propagator at coincident points demands UV regularisation. Also, we may notice that the x_1 integral corresponds to a KG integral, so it needs IR regularisation. The regularised KG integral is given by (4.18). Then the tadpole in the 4-point function is,

$$B_1(\epsilon, \vec{y}_i) = -\frac{G_{\Delta,\kappa}(1)}{2\Delta - d} \int dx \prod_{j=1}^{4} K_\Delta(z, \vec{x} - \vec{y}_j) \left[\log\left(\frac{\epsilon z}{z^2 + (\vec{x} - \vec{y}_1)^2} \right) + \psi(\Delta) - \psi\left(\Delta - \frac{d}{2} \right) \right]. \tag{4.112}$$

It becomes evident that the tadpole in the 4-point function correlation function is IR divergent and UV divergent. This also happens on each 2-point function finding that renormalises the mass. Then we may expect that this diagram gives corrections to the mass.

Using (3.53) we have,

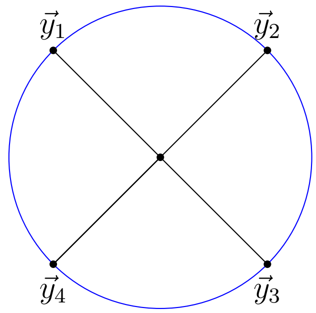
$$\begin{aligned} B_1(\epsilon, \vec{y}_i) = &-\frac{G_{\Delta,\kappa}(1)}{2\Delta - d} \int dx \left[\prod_{j=1}^{4} K_\Delta(z, \vec{x} - \vec{y}_j) \log\left(\frac{\epsilon z}{z^2 + (\vec{x} - \vec{y}_1)^2} \right) \right] \\ &- \left(\psi(\Delta) - \psi\left(\Delta - \frac{d}{2} \right) \right) \frac{G_{\Delta,\kappa}(1)}{2\Delta - d} D_{\Delta\Delta\Delta\Delta}(\vec{y}_1, \vec{y}_2, \vec{y}_3, \vec{y}_4). \end{aligned} \tag{4.113}$$

As we already know, this diagram may correct the mass. Let us consider the tree-level result plus the 4-point tadpole diagram,

$$\langle \prod_{i=1}^{4} \mathcal{O}_\Delta(\vec{y}_i) \rangle_{1,reg} = \langle \mathcal{O}_\Delta(\vec{y}_1) \ldots \mathcal{O}_\Delta(\vec{y}_1) \rangle_{\text{tree}} + \frac{\lambda^2}{2} B_1. \tag{4.114}$$

The subscript 1 denotes that the tadpole is in the leg that connects the point boundary point $\vec{y}_1$ to the bulk. The Witten diagram is given in Figure 4.8.

$$\langle \mathcal{O}_\Delta(\vec{y}_1)\dots\mathcal{O}_\Delta(\vec{y}_4)\rangle_{1,reg} =$$



$$+\,\frac{1}{2}$$

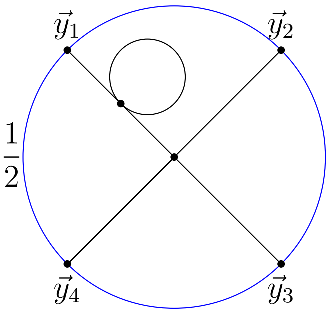


Figure 4.8: The tadpole correction to the 4-point function

Using the tree level result for the 4-point function (3.52) and reorganising the terms,

$$\langle\prod_{i=1}^{4}\mathcal{O}_\Delta(\vec{y}_i)\rangle_{1,reg} = \lambda\int\prod_{j=1}^{4}K_\Delta(z,\vec{x}-\vec{y}_j)\left(1-\lambda\frac{G_{\Delta,\kappa}(1)}{\cdot 2(2\Delta-d)}\left[\log\left(\frac{\epsilon z}{z^2+(\vec{x}-\vec{y}_1)^2}\right)\right.\right. \tag{4.115}$$

$$\left.\left.+\psi(\Delta)-\psi\left(\Delta-\frac{d}{2}\right)\right]\right).$$

The term in brackets has the same form as the expansion of the two-point function up to one loop, but here we have more bulk-to-boundary propagators. Let us write the former equation in a known way,

$$\langle\prod_{i=1}^{4}\mathcal{O}_\Delta(\vec{y}_i)\rangle_{1,reg} = \lambda\int\prod_{j=1}^{4}K_\Delta(z,\vec{x}-\vec{y}_j)\left(1-\lambda\gamma_1\left[\log\left(\frac{\epsilon z}{z^2+(\vec{x}-\vec{y}_1)^2}\right)\right.\right. \tag{4.116}$$
$$\left.\left.+\psi(\Delta)-\psi\left(\Delta-\frac{d}{2}\right)\right]\right),$$

with γ_1 defined as was defined in the 2-point function analysis,

$$\gamma_1 = \frac{G_{\Delta,\kappa}(1)}{2(2\Delta-d)}. \tag{4.117}$$

As γ_1 is UV divergent, we need to add the counterterm to cancel the divergence. The mass counterterm does this job. Therefore, we shall follow exactly the steps done for the

2-point function. Understanding the mass as a bare parameter, such that $m^2 = m_s^2 + \delta_m$, where the counterterm δ_m leads to the new interaction to the action given by the UV counterterm action,

$$S_{UVct} = \delta_m \int dx\Phi,\ \delta_m = a_1\lambda + \dots . \tag{4.118}$$

In an identical procedure, we will find that the counterterm action will add a UV counterterm 4-point function,

$$\langle \prod_{i=1}^{4} \mathcal{O}_\Delta(\vec{y}_i)\rangle_{1,UVct} = \lambda\delta_m \int dx_1 dx_2 G(x_1, x_2) \prod_{i=1}^{4} K_\Delta(z_2, \vec{x}_2 - \vec{y}_i), \tag{4.119}$$

up to a constant, this integral is given by (4.116). Then, considering the regularised 4-point function plus the UV counterterm 4-point function we have,

$$\begin{aligned}\langle \prod_{i=1}^{4} \mathcal{O}_\Delta(\vec{y}_i)\rangle_{1,reg,\text{UV ct}} &= \langle \prod_{i=1}^{4} \mathcal{O}_\Delta(\vec{y}_i)\rangle_{1,reg} + \langle \prod_{i=1}^{4} \mathcal{O}_\Delta(\vec{y}_i)\rangle_{1,UVct} \\ &= \lambda \int dx \left[\prod_{j=1}^{4} K_\Delta(z, \vec{x} - \vec{y}_j)\left(1 - \lambda\left(\gamma_1 + \frac{a_1}{2\Delta - d}\right)\right)\right. \\ &\left.\left[\log\left(\frac{\epsilon z}{z^2 + (\vec{x} - \vec{y}_1)^2}\right) + \psi(\Delta) - \psi\left(\Delta - \frac{d}{2}\right)\right]\right].\end{aligned} \tag{4.120}$$

In the minimal subtraction scheme, we will define a_1 to cancel the purely divergent term of γ_1. This means,

$$a_1 = -\text{div}(\gamma_1) = -\frac{1}{2}\text{div}\left[G_\Delta(1)\right], \tag{4.121}$$

which is exactly the counterterm found in the 2-point function renormalisation.

Defining the renormalised conformal dimension as

$$\Delta_r = \lim_{\kappa \to 0}\left[\Delta_s + \frac{\lambda}{2}\text{conv}\left[G_{\Delta,\kappa}(1)\right]\right], \quad \gamma_{1,r} = \lim_{\kappa \to 0}\text{conv}\left[\frac{G_{\Delta,\kappa}(1)}{2(2\Delta - d)}\right], \tag{4.122}$$

we will have,

$$\langle\prod_{i=1}^{4}\mathcal{O}_{\Delta}(\vec{y}_i)\rangle_{1,ren}=\lambda\int K_{\Delta_r}(z,\vec{x}-\vec{y}_1)\prod_{j=2}^{4}K_{\Delta}(z,\vec{x}-\vec{y}_j)\epsilon^{-\lambda\gamma_{1,r}}. \tag{4.123}$$

The ϵ regulator is removed by source renormalisation. The source that needs renormalisation is the one that gives the operator at the point $\vec{y}_1$ because $\gamma_{1,r}$ depends on the conformal dimension of the operator at this point. With the renormalised source, both sides of the equality hold and the 4-point function is consistent with the expected result.

Generalising the procedure to loops on the other legs leads to the same conclusion.

So the tadpole diagram in the 4-point function renormalises the mass. Even more, just renormalising the 2-point function up to 1-loop, we will immediately renormalise the tadpole diagram in the 4-point function.

The exchange diagram

There are three exchange diagrams, one for each channel. We will study the diagram shown in Figure 4.5 for simplicity. We shall consider the three diagrams to study the renormalisation of this diagram. However, for simplicity, we will work on one channel only, and the other will follow by doing permutations of the boundary points $\vec{y}$.

Let us consider the exchange diagram given by

$$X_s=\int dx_1K_{\Delta}(z_1,\vec{x}_1-\vec{y}_1)K_{\Delta}(z_1,\vec{x}_1-\vec{y}_2)\int dx_2K_{\Delta}(z_2,\vec{x}_2-\vec{y}_3)K_{\Delta}(z_2,\vec{x}_2-\vec{y}_4)G(x_2,x_1)^2 \tag{4.124}$$

the x_2 integral is exactly the one that worked in the section 4.2. Keeping in mind that

α_1 can be UV divergent, we consider the regularised version of α_1, which is given by,

$$\alpha_i(\kappa) = (-1)^i(d-1)\frac{(\Delta)_i(\Delta)_i(d)_i}{(2)_i i!}\frac{\Gamma\left(2\Delta-\frac{d}{2}+1+i\right)}{\Gamma\left(\Delta+\frac{3}{2}+i\right)\Gamma(\Delta+1+i)}\left(\frac{1}{1+\kappa}\right)^{2\Delta+2i+2} \tag{4.125}$$
$$ {}_4F_3\left(2\Delta-\frac{d}{2}+1+i,1,1,d+i;\Delta+\frac{3}{2}+i,\Delta+i+1,2+i,\left(\frac{1}{1+\kappa}\right)^2\right).$$

Just quoting the result, we will have,

$$X_s = \pi^{\frac{d+1}{2}}\left(\frac{2^{-2\Delta}c_\Delta}{2\Delta-d}\right)^2\int dx_1 K_\Delta(z_1,\vec{x}_1-\vec{y}_1)K_\Delta(z_1,\vec{x}_1-\vec{y}_2) \tag{4.126}$$
$$K_\Delta(z_1,\vec{x}_1-\vec{y}_3)K_\Delta(z_1,\vec{x}_1-\vec{y}_4)\sum_{i=0}^{\infty}\left(\frac{z_1}{z_1^2+(\vec{x}_1-\vec{y}_3)^2}\frac{z_1}{z_1^2+(\vec{x}_1-\vec{y}_4)^2}|\vec{y}_{34}|^2\right)^i[\alpha_i(\kappa)$$
$$+\beta_i+\sigma_i\log\left[\frac{z}{z^2+(\vec{x}-\vec{y}_3)^2}\frac{z_1}{z_1^2+(\vec{x}_1-\vec{y}_4)^2}|\vec{y}_{34}|^2\right]]$$

The first term of the series is proportional to a product of two bulk-to-boundary propagator. Arranging,

$$X_s = \pi^{\frac{d+1}{2}}\left(\frac{2^{-2\Delta}c_\Delta^2}{2\Delta-d}\right)^2\sum_{i=0}^{\infty}\frac{|\vec{y}_{34}|^{2i}}{c_{\Delta+i}^2}\int dx_1 K_\Delta(z_1,\vec{x}_1-\vec{y}_1)K_\Delta(z_1,\vec{x}_1-\vec{y}_2)$$
$$K_{\Delta+i}(z_1,\vec{x}_1-\vec{y}_3)K_{\Delta+i}(z_1,\vec{x}_1-\vec{y}_4)\left[\alpha_i(\kappa)+\beta_i\right.$$
$$\left.+\sigma_i\log\left[\frac{z}{z^2+(\vec{x}-\vec{y}_3)^2}\frac{z_1}{z_1^2+(\vec{x}_1-\vec{y}_4)^2}|\vec{y}_{34}|^2\right]\right]$$
$$= \pi^{\frac{d+1}{2}}\left(\frac{2^{-2\Delta}c_\Delta^2}{2\Delta-d}\right)^2\sum_{i=0}^{\infty}\frac{|\vec{y}_{34}|^{2i}}{c_{\Delta+i}^2}\left[D_{\Delta\Delta,\Delta+i,\Delta+i}(\vec{y}_1,\vec{y}_2,\vec{y}_3,\vec{y}_4)\left[\alpha_i(\kappa)+\beta_i\right]\right. \tag{4.127}$$
$$+\int dx_1 K_\Delta(z_1,\vec{x}_1-\vec{y}_1)K_\Delta(z_1,\vec{x}_1-\vec{y}_2)K_{\Delta+i}(z_1,\vec{x}_1-\vec{y}_3)$$
$$\left.\times K_{\Delta+i}(z_1,\vec{x}_1-\vec{y}_4)\sigma_i\log\left[\frac{z}{z^2+(\vec{x}-\vec{y}_3)^2}\frac{z_1}{z_1^2+(\vec{x}_1-\vec{y}_4)^2}|\vec{y}_{34}|^2\right]\right].nonumber \tag{4.128}$$

To work on the renormalisation problem, we must also consider tree-level computation. We will keep just the UV divergent piece of X given by the α_i coefficient to do that. The

	$a_0(\kappa)$	$a_1(\kappa)$	$a_2(\kappa)$	$a_3(\kappa)$
$d=2$	Convergent	Convergent	Convergent	Convergent
$d=3,4$	Divergent	Convergent	Convergent	Convergent
$d=5,6$	Divergent	Divergent	Convergent	Convergent
$d\geq 7$	Divergent	Divergent	Divergent	Divergent

Table 4.1: Convergence behaviour of the a_i coefficient for $\kappa \to 0$ expansion in different dimensions.

UV-regulated 4-point function is,

$$\begin{aligned}\langle\prod_{i=1}^{4}\mathcal{O}(\vec{y}_i)\rangle_{\text{UV reg}} &= \langle\prod_{i=1}^{4}\mathcal{O}(\vec{y}_i)\rangle_{tree} + \frac{\lambda^2}{2}X|_{\text{UV reg}} \\ &= \lambda D_{\Delta\Delta\Delta\Delta}(\vec{y}_1,\vec{y}_2,\vec{y}_3,\vec{y}_4) + \\ &\lambda^2\frac{\pi^{\frac{d+1}{2}}}{2}\left(\frac{2^{-2\Delta}c_\Delta^2}{2\Delta-d}\right)^2\sum_{i=0}^{\infty}\frac{|\vec{y}_{34}|^{2i}}{c_{\Delta+i}^2}D_{\Delta\Delta,\Delta+i,\Delta+i}(\vec{y}_1,\vec{y}_2,\vec{y}_3,\vec{y}_4)\left[\alpha_i(\kappa)+\beta_i\right].\end{aligned} \tag{4.129}$$

As we check in the section 4.2, the divergence term is hidden in α_i. A summary of the behaviour for the different coefficients is shown in table 4.1

The $d=2$ case is finite, and no renormalisation is needed. In higher dimensions, divergences demand renormalisation. From (4.129), we notice that the only way to renormalise the exchange diagram is by considering the coupling constant to be,

$$\lambda = \lambda_s + \delta_\lambda, \quad \delta_\lambda = b_2\lambda^2 + \dots \tag{4.130}$$

where δ_λ is the coupling counterterm. In the same way, as we did for the mass, the modification of the coupling can be seen as adding a new counterterm action given by

$$S_{UV,ct} = \delta_\lambda\int dx\Phi^4. \tag{4.131}$$

To deal with the renormalisation, we must consider the three diagrams X_s, X_t and X_u. In practice, this means that we ought to consider exactly (4.129) plus the combinations $\vec{y}_2 \leftrightarrow \vec{y}_3$ and $\vec{y}_2 \leftrightarrow \vec{y}_4$

$$
\begin{aligned}
&\langle\prod_{i=1}^{4}\mathcal{O}(\vec{y}_i)\rangle_{\text{UV reg, UV ct}} = \langle\prod_{i=1}^{4}\mathcal{O}(\vec{y}_i)\rangle_{tree} + \frac{\lambda^2}{2}X|_{\text{UV reg}} + \delta_\lambda\langle\prod_{i=1}^{4}\mathcal{O}(\vec{y}_i)\rangle_{\text{UV ct}} \\
&= \lambda D_{\Delta\Delta\Delta\Delta}(\vec{y}_1,\vec{y}_2,\vec{y}_3,\vec{y}_4) + b_2\lambda^2 D_{\Delta\Delta\Delta\Delta}(\vec{y}_1,\vec{y}_2,\vec{y}_3,\vec{y}_4) \\
&+ \lambda^2\frac{\pi^{\frac{d+1}{2}}}{2}\left(\frac{2^{-2\Delta}c_\Delta^2}{2\Delta-d}\right)^2\sum_{i=0}^{\infty}\frac{|\vec{y}_{34}|^{2i}}{c_{\Delta+i}^2}D_{\Delta\Delta,\Delta+i,\Delta+i}(\vec{y}_1,\vec{y}_2,\vec{y}_3,\vec{y}_4)\left[\alpha_i(\kappa)+\beta_i\right] \\
&+ (\vec{y}_2\leftrightarrow\vec{y}_3) + (\vec{y}_2\leftrightarrow\vec{y}_4). \qquad (4.132)
\end{aligned}
$$

We shall work on the renormalisation in different dimensions.

- For $d = 3, 4$, the term α_0 of the sum is divergent. This is proportional to $D_{\Delta\Delta\Delta\Delta}$, so choosing b_2 such that it cancels the divergence, we have renormalised the theory.

- For $d = 5, 6$, the divergent pieces come from α_0 and α_1. To cancel the α_0 part is direct because it is proportional to $D_{\Delta\Delta\Delta\Delta}$. To deal with the $i = 1$ term of the sum, we must work the $D_{\Delta,\Delta,\Delta+1,\Delta+1}$ such that it can be written in terms of the $D_{\Delta\Delta\Delta\Delta}$. This relation between D-functions is fairly well-known [67] and is given by

$$\frac{\partial}{\partial\left(|\vec{y}_{34}|^2\right)}D_{\Delta\Delta\Delta\Delta} = -\frac{2\Delta^2}{4\Delta-d}D_{\Delta,\Delta,\Delta+1,\Delta+1}. \qquad (4.133)$$

  We may change the $D_{\Delta,\Delta,\Delta+1,\Delta+1}$ by derivatives of the D function. This relation is not useful. However, in [67] is also proven,

$$\left(\vec{y}_{12}^2\frac{\partial}{\partial\left(\vec{y}_{12}^2\right)} + \vec{y}_{13}^2\frac{\partial}{\partial\left(\vec{y}_{13}^2\right)} + \vec{y}_{14}^2\frac{\partial}{\partial\left(\vec{y}_{14}^2\right)}\right)\bar{D}_{\Delta,\Delta,\Delta,\Delta}(u,v) = 0 \qquad (4.134)$$

  where $\bar{D}$ is given in (3.54) and u and v the crossing ratios (2.30). So, at this point, is where becomes clear why we need to consider the three possible exchange diagrams.

Then, we can write,

$$D_{\Delta,\Delta,\Delta+1,\Delta+1} + D_{\Delta,\Delta+1,\Delta,\Delta+1} + D_{\Delta,\Delta+1,\Delta+1,\Delta} = \frac{2\Delta - \frac{d}{2}}{\Delta} D_{\Delta\Delta\Delta\Delta}. \tag{4.135}$$

So, the divergent term in the α_1 coefficient can be cancelled by the δ_λ countertem.

- For dimension $d \geq 7$ it is not possible to re-write the $D_{\Delta,\Delta,\Delta+2,\Delta+2}$ as $D_{\Delta,\Delta,\Delta,\Delta}$, then the counterterm δ_λ cannot cancel the divergent term α_2.

In summary, it is possible to renormalise the coupling constant only up to $d = 6$ or equivalently up to AdS_7. In [37], by studying the CFT, they find that bulk Φ^4 theory is renormalisable only up to AdS_7.

4.5 Quantum Effective action

In the previous section, we used the connected diagram generator to obtain the quantum correction to the holographic correlation function. However, regarding the functional generator, we can go another step forward and work on the same problem using the Quantum Effective action.

The quantum effective action was introduced as perturbative series in [72] and the non-perturbative definition was given by the equation (4.140) by B. de Witt in [73]. Following this approach, we can work the loop correction and renormalisation by looking at the quantum effective action and the equation of motion. This will significantly simplify the whole renormalisation problem.

As before, the starting point is the action for a Φ^4 interacting theory,

$$S = \int dx \left[\frac{1}{2} g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + \frac{1}{2}m^2\Phi^2 + \frac{\lambda}{4!}\Phi^4\right] + \int d\vec{x} B. \tag{4.136}$$

The partition function with the bulk source is

$$Z[J] = \int D\Phi e^{-S+\int dx J\Phi}. \tag{4.137}$$

In contrast with section 4.4, we will not split the field into a classical field plus a quantum fluctuation. Instead, by using standard QFT tricks, we add a bulk source J such that we can take out of the path integral the interaction,

$$\begin{aligned} Z[J] &= e^{-\frac{\lambda}{4!}\int dx \frac{\delta^4}{\delta J(x)^4}} \int Dh e^{-\int dx\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi+\frac{1}{2}m^2\Phi^2+J\Phi\right]} \\ &= e^{-\frac{\lambda}{4!}\int dx \frac{\delta^4}{\delta J(x)^4}} e^{-\frac{1}{2}\int dx_1 dx_2 J(x_1)G_\Delta(x_1,x_2)J(x_2)}, \end{aligned} \tag{4.138}$$

where $G_\Delta(x_1, x_2)$ is the bulk-to-bulk propagator (3.48). Then, we can compute the connected diagram generator. Up to order λ is,

$$\begin{aligned} W[J] = -\int d^d\vec{x}B + \frac{1}{2}\int dydzJ(y)G(y,z)J(z) + \frac{\lambda}{4!}\left[\int dx\left(\int dx_1 G(x,x_1)J(x_1)\right)^4\right. \\ \left. +6\int dxG(x,x)\left(\int dx_1\sqrt{g_1}J(x_1)G(x,x_1)\right)^2 + 3\int dxG(x,x)^2\right]. \end{aligned} \tag{4.139}$$

The connected diagram function $W[J]$ is a functional of the source.

To obtain the quantum effective action, we must find a functional that depends on the field. To do this, we perform a Legendre transformation

$$\Gamma[\Phi] = \int dx J_\Phi(x)\Phi(x) - W[J_\Phi] \tag{4.140}$$

where $J_\Phi = J[\Phi]$ is defined as a solution of

$$\left.\frac{\delta W[J]}{\delta J(x)}\right|_{J=J_\Phi} = \Phi(x). \tag{4.141}$$

So we will have

$$\frac{\delta W[J]}{\delta J(x)}\Big|_{J=J_\Phi} = \int dy G(x,y)J(y) + \frac{\lambda}{4!}\left[4\int dx'\left(\int dx_1 G(x',x_1)J(x_1)\right)^3 G(x,x')\right.$$
$$\left. +12\int dx' G(1)\left(\int dx_1\sqrt{g_1}J(x_1)G(x',x_1)\right)G(x,x')\right]$$
$$= \Phi(x).$$

We must invert this equation such that J depends on Φ. The former equation suggests that $\Phi(x) = \Phi_0 + \lambda\Phi_1 + \ldots$, meaning that the source J has its own expansion on the coupling $J = J_0 + \lambda J_1 + \ldots$.

Applying the wave operator we have

$$\left(-\Box + m^2\right)\Phi = \left(-\Box + m^2\right)\Phi_0 + \lambda\left(-\Box + m^2\right)\Phi_1 + \ldots \tag{4.142}$$
$$= J_0(x) + \lambda J_1(x) + \frac{\lambda}{3!}\left(\int dx_1 G(x,x_1)J_0(x_1)\right)^3$$
$$+\frac{\lambda}{2!}G(x,x)\int dx_1 G(x,x_1)J_0(x_1).$$

Solving order by order

$$J_0(x) = (-\Box + m^2)\Phi_0. \tag{4.143}$$

The order λ of (4.142) is

$$(-\Box + m^2)\Phi_1 = J_1(x) + \frac{1}{3!}\Phi_0(x)^3 + \frac{1}{2!}G(x,x)\Phi_0(x) \tag{4.144}$$
$$\Rightarrow J_1(x) = (-\Box + m^2)\Phi_1 - \frac{1}{3!}\Phi_0^3 - \frac{\lambda}{2!}G(x,x)\Phi_0(x). \tag{4.145}$$

Then we can write the source up to order λ as

$$\begin{aligned} J[\Phi]\Big|_{\mathcal{O}(\lambda)} &= J_0 + \lambda J_1 \\ &= (-\Box + m^2)\Phi_0 + \lambda(-\Box + m^2)\Phi_1 - \frac{\lambda}{3!}\Phi_0^3 - \frac{\lambda}{2!}G(1)\Phi_0 \\ &= (-\Box + m^2)\Phi - \frac{\lambda}{3!}\Phi^3 - \frac{\lambda}{2!}G(1)\Phi. \end{aligned}$$

So, building the effective action,

$$\Gamma[\Phi] = \int d^d\vec{x}B + \int dx \left[\frac{1}{2}\Phi(-\Box + m^2)\Phi + \frac{\lambda}{4}G_\Delta(1)\Phi^2 + \frac{\lambda}{4!}\Phi^4 + \frac{\lambda}{4}G(1)\right]. \tag{4.146}$$

This gives the effective action to $\mathcal{O}(\lambda)$. The renormalisation of the vacuum term $\lambda G(1)$ is trivial by adding a constant term into the Lagrangian, so we will not consider it in the following discussion.

Following the same steps, the effective quantum action up to order λ^2 is

$$\begin{aligned} \Gamma[\Phi] = &\int dx \left[\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + \frac{1}{2}m^2\Phi^2 + \frac{\lambda}{4}G_\Delta(1)\Phi^2 + \frac{\lambda}{4!}\Phi^4\right] \\ &- \frac{\lambda^2}{12}\int dx dx_1 \Phi(x)G_\Delta(x,x_1)^3\Phi(x_1) - \frac{\lambda^2}{8}G_\Delta(1)\int dx dx_1 \Phi^2(x)G_\Delta(x,x_1)^2 \\ &- \frac{\lambda^2}{16}\int dx dx_1 \Phi^2(x)G_\Delta(x,x_2)^2\Phi^2(x_1) + \int d\vec{x}B, \end{aligned} \tag{4.147}$$

where $\Delta(\Delta - d) = m^2$ and B is a boundary term.

In the first line of the quantum effective action, it is tempting to absorb the UV divergent piece $G_\Delta(1)$ into the mass parameter and renormalise the theory without further information about the field. Despite this being correct, it is just an accident of the Φ^4 up to order λ and cannot be generalised to higher order in the coupling or other interaction. Indeed, by looking at the λ^2, there are non-local terms that cannot be renormalised just by looking at the off-shell action. To deal with the renormalization problem, we will demand a well-defined variational problem for the quantum effective action.

An arbitrary variation of the quantum effective action is,

$$\delta\Gamma = \int d^{d+1}x \mathcal{E}\delta\Phi + \int_{\partial AdS} d^d x \left[A\delta\phi\right], \tag{4.148}$$

where the equation of motion $\mathcal{E}$ has to be finite.

The equation of motion is,

$$\begin{aligned}(-\Box + m^2)\Phi + \frac{\lambda}{3!}\Phi^3 + \frac{\lambda}{2}G_\Delta(1)\Phi + \frac{\lambda^2}{6}\int dx_2 G(x,x_2)^3\Phi(x_2) \\ + \frac{\lambda^2}{4}G(1)\Phi \int dx_2 G(x,x_2)^2 + \frac{\lambda^2}{4}\Phi \int dx_2 G(x,x_2)^2\Phi(x_2)^2 = 0.\end{aligned} \tag{4.149}$$

If the theory is renormalisable, then every UV divergence has to be absorbed in the theory's parameters, in this case, m^2 and λ, and eventually, the field.

So, we understand the parameters as bare quantities such that,

$$m^2 = m_s^2 + \delta_m, \quad \delta_m = a_1\lambda_s + a_2\lambda_s^2 + \ldots, \tag{4.150}$$

$$\lambda = \lambda_s + \delta_\lambda, \quad \delta_\lambda = b_2\lambda_s^2 + b_3\lambda_s^3 + \ldots, \tag{4.151}$$

where δ_m and δ_λ are the counterterms and the coefficient $a_1,\ a_2,\ b_2$ are the counterterms coefficients. If we find the coefficient, then we have solved the UV divergence problem.

The equation of motion for the quantum effective action with the UV counterterm is

$$\begin{aligned}&(-\Box + m_s^2)\Phi + \delta_m\Phi + \frac{\lambda_s}{3!}\Phi^3 + \frac{\delta_\lambda}{3!}\Phi^3 + \frac{\lambda_s}{2}G_{\Delta_s}(x,x)\Phi + \frac{\lambda_s\delta_m}{2(2\Delta_s - d)}G'_{\Delta_s}(1)\Phi \\ &+ \frac{\delta_\lambda}{2}G_{\Delta_s}(1)\Phi - \frac{\lambda_s^2}{6}\int dx_2 G_{\Delta_s}(x,x_2)^3\Phi(x_2) - \frac{\lambda_s^2}{4}G_{\Delta_s}(1)\Phi\int dx_2 G_{\Delta_s}(x,x_2)^2 \\ &- \frac{\lambda_s^2}{4}\Phi\int dx_2 G_{\Delta_s}(x,x_2)^2\Phi(x_2)^2 = 0\end{aligned} \tag{4.152}$$

where ′ denotes derivative with respect to Δ_s[2] As the equation is non-linear, so we solve it perturbatively in the field

$$\Phi = \Phi_0 + \lambda_s \Phi_1 + \lambda_s^2 \Phi_2 + \dots, \tag{4.154}$$

this leads to the following equation of motions

$$(-\Box + m_s^2)\Phi_0 = 0, \tag{4.155}$$

$$(-\Box + m_s^2)\Phi_1 = -a_1 \Phi_0 - \frac{1}{2} G(1) \Phi_0 - \frac{1}{3!} \Phi_0^3, \tag{4.156}$$

$$\begin{aligned}(-\Box + m_s^2)\Phi_2 = & -\frac{1}{2} G(1)\Phi_1 - \frac{a_1}{2(2\Delta_s - d)} G'_{\Delta_s}(1)\Phi_0 - \frac{b_2}{2} G(1)\Phi_0 - \frac{1}{2}\Phi_1\Phi_0^2 \\ & - \frac{1}{3!}\int dx_2 G_{\Delta_s}(x, x_2)^3 \Phi_0(x_2) - \frac{1}{2!} G_{\Delta_s}(1)\Phi_0 \int dx_2 G_{\Delta_s}(x, x_2)^2 \\ & - \frac{1}{4}\Phi_0 \int dx_2 G(x, x_2)^2 \Phi_0(x_2)^2 - \frac{1}{6} b_2 \Phi_0^3 - a_2 \Phi_0 - a_1 \Phi_1\end{aligned} \tag{4.157}$$

As we want to compute holographic correlation functions, the boundary condition is $\Phi_0(z \to 0, \vec{x}) \sim z^{d-\Delta}\phi_0$.

4.5.1 Order 0 in the coupling

The equation of motion of order λ^0 is solved by the bulk-to-boundary propagator related to the mass m_s^2. This is

$$\Phi_0(z, \vec{x}) = \int d\vec{y} K_{\Delta_s}(z, \vec{x} - \vec{y})\phi_0(\vec{y}). \tag{4.158}$$

[2] In appendix C.3 is proven that

$$\frac{1}{2\Delta - d}\frac{d}{d\Delta} G_\Delta(x, x) = -\int dx_2 G_\Delta(x, x_2)^2, \tag{4.153}$$

.

4.5.2 Order 1 in the coupling

Up to order λ, now we have two ways. Either to solve (4.155) by itself and sum it with the solution (4.158) or to solve (4.155)+ (4.156) as one equation. For simplicity, we will follow the second option

$$(-\Box + m_s^2)(\Phi_0 + \lambda_s \Phi_1) = -\lambda_s \left(a_1 + \frac{1}{2} G_{\Delta_s}(1) \right) \Phi_0 - \frac{\lambda_s}{3!} \Phi_0^3, \tag{4.159}$$

which is nothing but

$$(-\Box + m_s^2)\Phi + \lambda_s \left(a_1 + \frac{1}{2} G_{\Delta_s}(1) \right) \Phi = -\frac{\lambda_s}{3} \Phi^3. \tag{4.160}$$

From here, we can find a_1 such that the equation of motion is finite. As matter of consistency with the previous sections, we will use minimal subtraction as renormalization scheme

$$a_1 = -\frac{1}{2} \mathrm{div} \left[G_{\Delta_s}(1) \right]. \tag{4.161}$$

Defining the renormalised mass as

$$m_r^2 = \lim_{\kappa \to 0} \left[m_s^2 + \frac{\lambda_s}{2} \mathrm{conv} \left[G_{\Delta_s}(1) \right] \right], \tag{4.162}$$

we will have

$$(-\Box + m_r^2)\Phi = -\frac{\lambda_s}{3!} \Phi^3. \tag{4.163}$$

So the field is

$$\Phi_r = \Phi_{0,r} + \lambda_s \Phi_{1,r}, \tag{4.164}$$

$$\Phi_{0,r} = \int d\vec{y} K_{\Delta_r}(z, \vec{x} - \vec{y}) \phi_0^{(r)}(\vec{y}), \tag{4.165}$$

$$\Phi_{1,r} = -\frac{1}{3!} \int dx' G_{\Delta_r}(x, x') \left(\Phi_{0,r}(x')\right)^3, \tag{4.166}$$

we labelled the field with the subscript (r) to denote that it is built from the renormalised mass m_r^2 and the renormalised source $\phi_0^{(r)}$.

4.5.3 Order 2 in the coupling

Having a finite Φ_0 and Φ_1, we move to the equation of motion of order λ^2, where a_1 is already fixed, so we have to work with a_2 and b_2.

Consider (4.152) and place each linear term on Φ to the l.h.s and the non-linear term to the right

$$\begin{aligned}&\left(-\Box + m_s^2 + \lambda_s \left(a_1 + \frac{1}{2} G_{\Delta_s}(1)\right) + \lambda_s^2 \left(a_2 - \frac{1}{2}\left(a_1 + \frac{1}{2} G_{\Delta_s}(1)\right) I_3 + \frac{b_2}{2} G_{\Delta_s}(1)\right)\right)\Phi \\ &- \frac{\lambda_s^2}{6} \int dx_2 G_{\Delta_s}(x, x_2)^3 \Phi(x_2) = -\frac{\lambda_s + b_2\lambda^2}{3!}\Phi^3 - \frac{\lambda_s^2}{4}\Phi \int dx_2 G_{\Delta_s}(x, x_2)^2 \Phi(x_2)^2\end{aligned} \tag{4.167}$$

where we used (4.153) and I_3 is (4.26). Working the l.h.s first.

Keeping in mind that $\Phi = \Phi_0 + \lambda_s \Phi_1 + \lambda_s^2 \Phi_2$, with Φ_0 and Φ_1 known we can compute,

$$\begin{aligned}\frac{\lambda_s^2}{6} \int dx_2 G_{\Delta_s}(x, x_2)^3 \Phi(x_2) &= \frac{\lambda_s^2}{6} \int dx_2 G_{\Delta_s}(x, x_2)^3 \Phi_{0,s}(x_2) \\ &= \frac{\lambda_s^2}{6} g_3 \Phi(x)\end{aligned} \tag{4.168}$$

with g_3 given by (4.24).

So, the equation of motion up to the second order in the coupling is,

$$\left(-\Box + m_s^2 + \lambda_s \left(a_1 + \frac{1}{2}G_{\Delta_s}(1)\right) + \lambda_s^2 \left(a_2 - \frac{1}{2}\left(a_1 + \frac{1}{2}G_{\Delta_s}(1)\right) I_3 + \frac{b_2}{2}G_{\Delta_s}(1) - \frac{1}{6}c_3\right)\right)\Phi$$
$$= -\frac{\lambda_s}{3!}\Phi^3 - \lambda_s^2\Phi\left(\frac{b_2}{3!}\Phi^2 + \frac{1}{4}\int dx_2 G_{\Delta_s}(x,x_2)^2\Phi(x_2)^2\right). \tag{4.169}$$

We see that a_1 already renormalises some divergence. This aspect is known from standard QFT, whereby we renormalise the tadpole and automatically renormalise the loop on top of the eight diagram shown in Figure 4.4. We can determine a_2 as well

$$a_1 = -\frac{1}{2}\mathrm{div}\left[G_{\Delta_s}(1)\right], \tag{4.170}$$
$$a_2 = \frac{1}{4}\mathrm{div}\left[I_3 \mathrm{conv}\left[G_{\Delta_s}(1)\right]\right] - \frac{1}{2}\mathrm{div}\left[b_2 G_{\Delta_s}(1)\right] + \frac{1}{6}\mathrm{div}\left[c_3\right]. \tag{4.171}$$

On the r.h.s of equation (4.169) in the bracket, we can consider $\Phi = \Phi_0 + \mathcal{O}(\lambda_s)$. To renormalise the theory, i.e. find b_2, the divergent part of $\int G^2K^2$ has to be proportional to $\int K^2$. If the divergent part is not proportional to $\int K^2$, then the theory is non-renormalisable.

In section 4.4.2, we checked that $\int G^2KK$ is proportional to $\int KK$ only up to $d = 6$, meaning that the Φ^4 theory in the bulk is renormalisable only up to AdS_7. So, if we are in $d \leq 6$, we may find the renormalisation coefficient for the coupling, and it is given by

$$\begin{aligned} b_2 &= \frac{3}{2}\mathrm{div}\left[\int dx_2 G_{\Delta_s}(x,x_2)\Phi(x_2)^2\right] \\ &= \frac{3}{2}\mathrm{div}\left[\int dx_2 G_{\Delta_s}(x,x_2)\Phi_0(x_2)^2\right], \end{aligned}$$

with Φ_0 given by (4.158).

Having found the counterterm a_1, a_2 and b_2 we may define the renormalised parameters as

$$m_r^2 = m_s^2 + \frac{\lambda_s}{2}\mathrm{conv}\left[G_{\Delta_s}(1)\right] + \lambda_s^2\left[-\frac{1}{4}\mathrm{conv}\left[I_3\mathrm{conv}\left[G_{\Delta_s}(1)\right]\right] + \frac{1}{2}\mathrm{conv}\left[b_2 G_{\Delta_s}(1)\right] - \frac{1}{6}\mathrm{conv}\left[c_3\right]\right] \tag{4.172}$$

$$\lambda_r = \lambda_s + \frac{\lambda_s^2}{4}\mathrm{conv}\left[\int dx_2 G_{\Delta_s}(x,x')^2\Phi(x')^2\right] \tag{4.173}$$

By demanding a finite equation of motion with a given Dirichlet boundary condition, we found only UV divergent terms and the corresponding counterterm such that bulk equations of motion are finite. As we already know, the problem of obtaining the CFT correlation function is not yet solved. To solve this, we have to do Holographic Renormalisation

4.5.4 UV Renormalised Quantum Effective Action

We have found the UV counterterms by demanding finiteness of the equation of motion, so the quantum effective action (4.147) is UV finite on-shell. The on-shell field is given

$$\Gamma^{\mathrm{ren}}_{o-s}[\Phi] = \int_{AdS_{d+1}} dx\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + \frac{1}{2}m_r^2\Phi^2 + \frac{\lambda_r}{4!}\Phi^4\right] + \int d\vec{x}B, \quad d \leq 6. \tag{4.174}$$

We are sure that the bulk action is UV finite. So now we have to work on the IR divergence that will naturally appear. To do this, we follow exactly the holographic renormalisation presented in section 3.1, but now we impose as a boundary condition

$$\Phi(z \to 0, \vec{x}) \to z^{d-\Delta_r}\phi_0^{(r)}, \tag{4.175}$$

where the renormalised conformal dimension, Δ_r is the positive solution of $\Delta_r(\Delta_r - d) = m_r^2$. The next steps are exactly as presented in section 3.1 but replacing $\Delta \to \Delta_r$. While the computation of the correlation function is the same as the one done in section 3.2 but replacing $\Delta \to \Delta_r$ and $\lambda \to \lambda_r$. However, in contrast with the tree-level computation, now we can only compute the 4-point function if we consider a bulk theory of dimension seven or lower.

Chapter 5

Holographic Correlator: Momentum space

In section 2.1, we found the general structure of 2- and 3-point functions using conformal symmetry. However, these relations only hold at separated points, and correlation functions should be well-defined distributions, i.e. they must have a well-defined Fourier transformation. If we have operators with conformal dimension $\Delta = \frac{d}{2}+k,\ k = 0, 1, 2, \ldots$, the 2-point function (2.27) in position space does not have a well-defined Fourier transform.

Rather than taking the Fourier transform of the well-known position space correlation function, we may use Ward identities to obtain correlation functions. In momentum space, to solve the 2-point function, regularisation and renormalisation are necessary and lead to new conformal anomalies [74, 75, 76]. For the 3-point function, the problem becomes more involved. We refer to [55] for the 3-point function for scalars, and [54, 77, 78] for the 3-point function of tensor fields. Recently has been some progress on working the 4-point function for CFT in momentum space [79, 80, 56]

In general, the problem of the correlation function for CFT in momentum space is not trivial, mainly because Ward identities in momentum spaces become a second-order

PDE that may be difficult to solve. However, we can use the AdS/CFT correspondence to work the computation of the holographic correlation function.

In momentum space, at tree level, the general scheme is essentially the same as presented in section 3.1, so in the asymptotic expansion of the on-shell field, we must find the term $z^{\Delta}\phi_{2\Delta-d}$ and $\langle\mathcal{O}_{\Delta}\rangle \propto \phi_{2\Delta-d}$. From the former equation, we may compute holographic correlation functions. Going to loop correction leads to the Witten diagrams found in section 4.4, with similar integrals to be solved. However, now the propagators are given in momentum space along the transverse coordinate.

In contrast with the study of the correlation function in position space, in this chapter, we will condense the complete analysis of holographic correlation functions in momentum space. This includes the tree level and quantum computation of the dual correlation functions. This is because most of the discussion is equivalent to the discussion on position space, so we will quote the necessary results and translate them into momentum space.

The structure of the chapter is as follows: in section 5.1, we work on the solution of the equation of motion and introduce the bulk-to-boundary and bulk-to-bulk propagator in momentum space. In particular, we compute the 2-point function and give an example of calculating the 3-point function for a Φ^3 interacting theory. In section 5.2, we move to work on the quantum correction in the momentum space following the background field method and solve the 1-loop correction to the 2-point function.

We will follow the notation

$$\langle\mathcal{O}_{\Delta}(\vec{p}_1)\ldots\mathcal{O}_{\Delta}(\vec{p}_n)\rangle = (2\pi)^d\delta(\vec{p}_1+\ldots\vec{p}_n)\langle\langle\mathcal{O}_{\Delta}(\vec{p}_1)\ldots\mathcal{O}_{\Delta}(\vec{p}_n)\rangle\rangle, \tag{5.1}$$

and mainly focus on $\langle\langle\ldots\rangle\rangle$ part.

As we will see, both propagators are composed of modified Bessel functions; therefore, in this chapter, we will refer to the integral according to what Bessel function composes. For example, an IK-integral is the integral of the I-Bessel times K-Bessel function.

5.1 Holographic Correlator in Momentum space: Tree level

To work the background field, we must understand the semi-classical approach in momentum space. To do this, we solve the classical equation of motion with the bulk-to-boundary and bulk-to-bulk propagator in momentum space along the transverse coordinates. Furthermore, following the holographic dictionary, we find the corresponding tree-level holographic correlation function.

5.1.1 Perturbative solutions

We will consider the same scalar field with action (4.48). As we did before, we solve the equation perturbatively in the coupling $\Phi = \Phi_{\{0\}} + \lambda\Phi_{\{1\}} + \mathcal{O}(\lambda^2)$, where each term satisfies the equation (3.43) and (3.44) respectively. As we are interested in the correlation function in momentum space we will do a Fourier transformation to the field along the transverse coordinate $\vec{x}$ while the radial direction z remains unchanged; this is

$$\Phi(z, \vec{p}) = \mathcal{F}(\Phi(z, \vec{x})). \tag{5.2}$$

The equations of motions in momentum space are

$$L_{d,\Delta}(z, \vec{p})\Phi_{\{0\}}(z, \vec{p}) = 0, \quad \Phi(z, \vec{p}) \sim z^{d-\Delta}\phi_0(\vec{p}), \tag{5.3}$$

$$L_{d,\Delta}(z, \vec{p})\Phi_{\{1\}}(z, \vec{p}) = -\frac{1}{3!}\left(\Phi_{\{0\}}\right)^3, \tag{5.4}$$

$$\vdots$$

where

$$L_{d,\Delta}(z,\vec{p}) = -z^2\partial_z^2 + (d-1)z\partial_z + m^2 + z^2p^2. \tag{5.5}$$

In the field, we will demand regularity in $z \to \infty$. Similarly, as we did for the position space problem, the perturbative solution of the equation of motion is given by the bulk-to-boundary propagator and the bulk-to-bulk propagator

$$\mathcal{K}_\Delta(z,\vec{p}) = \frac{2^{\frac{d}{2}-\Delta+1}}{\Gamma\left(\Delta-\frac{d}{2}\right)} p^{\Delta-\frac{d}{2}} z^{\frac{d}{2}} K_{\Delta-\frac{d}{2}}(pz), \tag{5.6}$$

$$\mathcal{G}_\Delta(z_1,\vec{p},z_2) = \begin{cases} (z_1z_2)^{\frac{d}{2}} I_{\Delta-\frac{d}{2}}(pz_1)K_{\Delta-\frac{d}{2}}(pz_2) & \text{for } z_1 \leq z_2, \\ (z_1z_2)^{\frac{d}{2}} I_{\Delta-\frac{d}{2}}(pz_2)K_{\Delta-\frac{d}{2}}(pz_1) & \text{for } z_1 > z_2, \end{cases} \tag{5.7}$$

where $\mathcal{K}_\Delta(z,\vec{p})$ and $\mathcal{G}_\Delta(z_1,\vec{p},z_2)$ represents the bulk-to-boundary and the bulk-to-bulk propagator respectively and are given in terms of the K and I-Bessel functions [1] which depends on $p = |\vec{p}|$. The derivation of both propagators is in appendix B.

The solution to the equation of motions are

$$\Phi_{\{0\}}(z,\vec{p}) = \mathcal{K}_\Delta(z,p)\phi_0(\vec{p}), \tag{5.8}$$

$$\begin{aligned}\Phi_{\{1\}}(z,\vec{p}) = \frac{1}{3!}\int_0^\infty dz_1\sqrt{g}\,\mathcal{G}_\Delta(z,p,z_1)\int \frac{d\vec{k}_1 d\vec{k}_1}{(2\pi)^d}\mathcal{K}_\Delta(z_2,\vec{k}_1)\mathcal{K}_\Delta(z_2,\vec{k}_2) \\ \times\, \mathcal{K}_\Delta(z_1,\vec{p}-\vec{k}_1-\vec{k}_2)\phi_0(\vec{p}-\vec{k}_1-\vec{k}_2)\phi_0(\vec{k}_1)\phi_0(\vec{k}_2).\end{aligned} \tag{5.9}$$

Compared with the solution in position space, both have the same structure, but the homogeneous solution does not have integration in momentum space.

To obtain the tree-level holographic correlation function, we will consider (3.30)

$$\langle\mathcal{O}\rangle_s = -(2\Delta-d)\phi_{2\Delta-d}(\phi_0) + X[\phi_0], \tag{5.10}$$

where $\phi_{2\Delta-d}$ is the coefficient of z^Δ in the asymptotic expansion of Φ and $X[\phi_0]$ are local

[1] Do not confuse with the K_Δ presented in the position space discussion

functions of ϕ_0.

5.1.2 Tree Level: 2 point function on CFT

To compute the 2-point function, we only need the homogeneous solution of the equation of motion. If we consider $\Delta - \frac{d}{2} = k,\ k = 0, 1, 2, \ldots$ the coefficient of z^{Δ} is

$$\phi_{2\Delta-d} = \alpha_{\Delta} p^{2k} \log(p)\phi_0 + \beta p^{2k}\phi_0. \tag{5.11}$$

So, it is direct to compute the 2-point function by computing the derivative with respect to ϕ_0. Logarithmic terms are found at the tree-level computation for $\Delta - \frac{d}{2} = k$ with k integer.

For $\Delta - \frac{d}{2}$ non-integer, there is no logarithmic term. Explicitly this is

$$\langle\langle \mathcal{O}_{\Delta}\mathcal{O}_{\Delta}\rangle\rangle = -(2\Delta - d)\frac{2^{d-2\Delta}\Gamma\left(\frac{d}{2} - \Delta\right)}{\Gamma\left(\Delta - \frac{d}{2}\right)} p^{2\Delta-d}. \tag{5.12}$$

5.1.3 Tree level: 3-point function

Most of the discussion presented in this subsection is based on [55]. For example, let us consider a Φ^3 bulk interaction.

In a Φ^3 interaction theory, the solution of the equation of motions with the non-normalisable boundary condition is

$$\begin{aligned}\Phi(z,\vec{x}) &= \mathcal{K}_{\Delta}(z,p)\phi_0(\vec{x}) \\ &+ \frac{\lambda}{2}\int_0^{\infty} dz_1\sqrt{g}\mathcal{G}_{\Delta}(z,p,z_1)\int d\vec{k}\mathcal{K}_{\Delta}(z_1,k)\mathcal{K}_{\Delta}(z_1,p-k)\phi_0(\vec{k})\phi_0(\vec{p}-\vec{k}) + \mathcal{O}(\lambda^2).\end{aligned} \tag{5.13}$$

The first line is the solution to the homogeneous equation, and the second line is the order λ solution.

From (3.30) follows that the 3-point function will come from the term of Φ that depends quadratically on the source ϕ_0. This is given in the second line of (5.13). Expanding this equation such that we find the z^Δ dependence and computing a double functional derivative to $\langle \mathcal{O} \rangle_s = -(2\Delta - d)\Phi_{(\Delta)}$ we will find

$$\langle\langle \mathcal{O}_{\Delta_1}(\vec{p}_1)\mathcal{O}_{\Delta_2}(\vec{p}_2)\mathcal{O}_{\Delta_3}(\vec{p}_3)\rangle\rangle = -\lambda(2\Delta - d)\left(I(z,\vec{p}_1,\vec{p}_2) + I(z,\vec{p}_1,\vec{p}_3)\right)_{(\Delta)} \tag{5.14}$$

$$+ \frac{\delta X(\vec{p}_1)}{\delta\phi_0(\vec{p}_2)\phi_0(\vec{p}_3)}, \tag{5.15}$$

where the subscript (Δ) denotes that we look for the z^Δ term, and

$$I(z,\vec{p},\vec{k}) = \int_0^\infty dz_1\sqrt{g}\mathcal{G}(z,p,z_1)\mathcal{K}_\Delta(z_1,k)\mathcal{K}(z_1,|\vec{p}-\vec{k}|). \tag{5.16}$$

This integral is IR divergent, so we regulate by adding a cut-off, ϵ, in the lower limit of the integral. In the early days of AdS/CFT, computation of the 3-point function [65] did not consider the IR structure of the holographic 3-point function. We realise from the current computation that IR is divergent, and we must regularise. We will denote $I_\epsilon(z,\vec{p},\vec{k})$ to the regularised KKK-integral.

Then, the 3-point function to study is

$$\langle\langle \mathcal{O}_{\Delta_1}(\vec{p}_1)\mathcal{O}_{\Delta_2}(\vec{p}_2)\mathcal{O}_{\Delta_3}(\vec{p}_3)\rangle\rangle = -\lambda(2\Delta - d)\lim_{\epsilon\to 0}\left[I_\epsilon(z,\vec{p}_1,\vec{p}_2) + I_\epsilon(z,\vec{p}_1,\vec{p}_3) + I_\epsilon^{ct}\right]_{(\Delta)} \tag{5.17}$$

$$+ \frac{\delta^2 X(\vec{p}_1)}{\delta\phi_0(\vec{p}_2)\delta\phi_0(\vec{p}_3)}, \tag{5.18}$$

with I_ϵ^{ct} is a suitable counterterm.

The I_ϵ integral depends on the bulk-to-bulk propagator (5.7), so the integral will split into two regions: a near boundary region $z_1 < z$, that we will denote as $I_\epsilon^<$; and an inner region $z_1 > z$ denoted as $I_\epsilon^>$.

The near boundary integral, $\epsilon \leq z_1 \leq z$ is

$$I^<_\epsilon(z,\vec{p},\vec{k}) = z^{\frac{d}{2}}_2 K_{\Delta-\frac{d}{2}}(pz)\int_\epsilon^z dz_1 z_1^{-\frac{d}{2}-1} I_{\Delta-\frac{d}{2}}(pz_1)\mathcal{K}_\Delta(z_1,\vec{k})\mathcal{K}_\Delta(z_1,|\vec{p}-\vec{k}|). \tag{5.19}$$

Any divergences that arise are regulated with ϵ and can be removed via holographic counterterm. To obtain the 3-point function, we must find the z^Δ term. By power expanding the integrand (look at Appendix A for Bessel functions expansions), we find that the z^Δ term exists if

$$\frac{d}{2} \pm \beta \pm \beta \pm \beta = -2k, \quad \beta = \Delta - \frac{d}{2}, \quad k = 0, 1, 2\ldots. \tag{5.20}$$

Consider now the contribution to the 3-point function from the inner region, $z < z_1$, whose integral is

$$I^>(z,\vec{p},\vec{k}) = z^{\frac{d}{2}}_2 I_{\Delta-\frac{d}{2}}(pz)\int_z^\infty dz_1 z_1^{-\frac{d}{2}-1} K_{\Delta-\frac{d}{2}}(pz_1)\mathcal{K}_\Delta(z_1,\vec{k})\mathcal{K}_\Delta(z_1,|\vec{p}-\vec{k}|). \tag{5.21}$$

When $z \to 0$, the expansion of the bulk-to-bulk propagator is

$$\mathcal{G}_\Delta(z,\vec{p},z_1) = \frac{z^\Delta}{2\Delta-d}\mathcal{K}_\Delta(z_1,\vec{p}) + \mathcal{O}(z^{\Delta+2}), \quad \text{for } z_1 > z \tag{5.22}$$

the inner region integral gives

$$I^>(z,\vec{p},\vec{k}) = \frac{z^\Delta}{2\Delta-d}\int_z^\infty dz_1 z^{-\frac{d}{2}-1}\mathcal{K}_\Delta(z_1,p)\mathcal{K}_\Delta(z_1,k)\mathcal{K}_\Delta(z_1,|p-\vec{k}|) + \mathcal{O}(z^{\Delta+2}) \tag{5.23}$$

When we use the explicit form of the bulk-to-boundary propagator, this integral is proportional to a triple-K-integral with a lower cut-off.

We have to study how this integral behaves when $z \to 0$. The convergence at

$z \to 0$ demands

$$\left|\Delta - \frac{d}{2}\right| + \left|\Delta - \frac{d}{2}\right| + \left|\Delta - \frac{d}{2}\right| < \frac{d}{2}. \tag{5.24}$$

However, we can extend the treatment of these integrals considering the analytical continuation. From the expansion (A.4) we may consider the analytic continuation if

$$\frac{d}{2} \pm \left(\Delta - \frac{d}{2}\right) \pm \left(\Delta - \frac{d}{2}\right) \pm \left(\Delta - \frac{d}{2}\right) \neq 2j \tag{5.25}$$

holds for any choice of independent signs and non-negative integer j. When equality holds, the triple-K-integral has poles, and we must keep the cut-off.

If (5.25) is satisfied, the boundary integral, $I^<$ vanishes and does not add local terms, while the inner region integral, $I^>$, reduces to a triple-K- integral and the 3-point function is

$$\langle\langle \mathcal{O}_\Delta(\vec{p}_1)\mathcal{O}_\Delta(\vec{p}_2)\mathcal{O}_\Delta(\vec{p}_3)\rangle\rangle = -2\lambda \left(\frac{2^{\frac{d}{2}-\Delta+1}}{\Gamma\left(\Delta - \frac{d}{2}\right)}\right)^3 \int_0^\infty dz z^{\frac{d}{2}-1} \prod_{j=1}^{3} p_j^{\Delta-\frac{d}{2}} K_{\Delta-\frac{d}{2}}(zp_j). \tag{5.26}$$

This triple-K-integral is finite, but it may be necessary to use the analytic continuation to define its precise value.

Meanwhile, if (5.25) does not hold, the boundary region is expected to contribute with a non-local part of the correlation function.

For further details and some examples, we refer to [55].

5.2 Quantum Correction

The holographic dictionary does not change whether we work in position or momentum space. Thus, the discussion presented in section 4.1 holds, and we will not repeat it now.

In agreement with the space position, we have to check whether the relevant integrals converge. The subsection 5.2.1 studies the convergence of integrals composed of Bessel functions.

Having the relevant integral, we move on to compute the corresponding Witten diagram. For this purpose, the strategy is to take the integrals obtained in position space, for instance, the diagrams dictated by (4.64)-(4.67) and perform the Fourier transform along the transverse coordinate. The associated Witten diagram and the computation of the one loop 2-point function are done in subsection 5.2.2.

5.2.1 Relevant integrals

The KK integral

We will work the general KK-integral, where K stands for the $K-$Bessel function

$$I^{KK}_{\alpha\{\beta_1,\beta_2\}}(p_1,p_2)=\int_0^\infty dx\; x^\alpha K_{\beta_1}(p_1x)K_{\beta_2}(p_2x)p_1^{\beta_1}p_2^{\beta_2}. \tag{5.27}$$

The variables p_1 and p_2 will be relevant for the coming discussion of the renormalisability of the Witten diagram, and the superscript KK denotes that we are considering the double-K-integral. Along the subsection, we will not use the superscript to saturate the notation, but we will recover it when we study the corresponding Witten diagram.

The integral converges if

$$\alpha>|\beta_1|+|\beta_2|-1. \tag{5.28}$$

We want to check whether we can go beyond the convergence region. To do so, we will split the x integral into a lower and upper region and see how it behaves. Each integral is defined as

$$I_{\alpha\{\beta_1,\beta_2\}} = I^{\text{lower}}_{\alpha\{\beta_1,\beta_2\}} + I^{\text{upper}}_{\alpha\{\beta_1,\beta_2\}}, \tag{5.29}$$

with

$$I^{\text{lower}}_{\alpha\{\beta_1,\beta_2\}} = \int_0^{\mu^{-1}} dx\; x^\alpha K_{\beta_1}(p_1 x) K_{\beta_2}(p_2 x) p_1^{\beta_1} p_2^{\beta_2}, \tag{5.30}$$

$$I^{\text{upper}}_{\alpha\{\beta_1,\beta_2\}} = \int_{\mu^{-1}}^{\infty} dx\; x^\alpha K_{\beta_1}(p_1 x) K_{\beta_2}(p_2 x) p_1^{\beta_1} p_2^{\beta_2}, \tag{5.31}$$

where $p_1,\ p_2 > 0$ and μ some positive constant.

From eq (A.7) and eq (A.16), we can prove that the upper integral has no poles. Therefore, we say that the KK-integral has no UV divergence.

The lower integral demands more attention. Assuming the index of the Bessel function to be an integer, i.e. $\beta = n,\ n = 0, 1, 2, \dots$ and using the eq.(A.6) we will find the following kind of integrals

$$\int_0^{\mu^{-1}} x^a \log^n(x) = \frac{1}{a+1} \Big[\log^n\left(\mu^{-1}\right) \left(-(a+1)\log\left(\mu^{-1}\right)\right)^{-n} \\ \left(\Gamma\left(n+1, -(a+1)\log\left(\mu^{-1}\right)\right) - \Gamma(n+1)\right) + (-1)^n (a+1)^{-n} \Gamma(n+1) \Big]. \tag{5.32}$$

where $\Gamma(n, x)$ is the incomplete gamma function. The l.h.s integral is convergent for $a > -1$.

Using the representation (A.17) give by, $\Gamma(n, x) = (n-1)! e^{-x} \sum_{k=0}^{n-1} x^k/k!$, we have

$$\int_0^{\mu^{-1}} x^a \log^n(x) dx = \frac{(-1)^n n!}{(a+1)^{n+1}} \mu^{-(a+1)} \sum_{j=0}^{n} \frac{(a+1)^j \log^j \mu}{j!},\ n = 0, 1, 2, \dots. \tag{5.33}$$

We may notice that $a = -1$ is a pole of order $n + 1$. Apart from this value, the r.h.s is analytic. Expanding around $a = -1$, the integral behaves as

$$\int_0^{\mu^{-1}} x^a \log^n x dx = \frac{(-1)^n n!}{(a+1)^{n+1}} + \frac{(-1)^n \log^{n+1}(\mu^{-1})}{n+1} + \mathcal{O}(a+1) \tag{5.34}$$

so any possible pole at $a = -1$ does not depend on μ.

For the double-K- integral (5.27)

$$a = \alpha \pm \beta_1 \pm \beta_2 + 2k, \quad k = 0, 1, 2, \ldots \tag{5.35}$$

with k representing the power in the series of the Bessel function. This is the same condition found for the triple K-Bessel integral (5.20), but for the double K-Bessel integral.

To avoid poles, we must demand

$$\alpha \pm \beta_1 \pm \beta_2 + 1 \neq -2k. \tag{5.36}$$

This equation has to be read, keeping in mind that it must hold for every combination of signs. If at least one combination of signs satisfies the equality, there will be poles in the integral.

So the existence of pole is given by the $\sigma_i \in \{\pm 1\}$ such that

$$\alpha + 1 + \sigma_1 \beta_1 + \sigma_2 \beta_2 = -2k, \quad k = 0, 1, 2, \ldots \tag{5.37}$$

The combination of sign that satisfies (5.37) will be called $(+,+)$, $(-,-)$ and $(+,-)$ depending on which holds. More than one combination may be satisfied giving more divergences.

The KI integral

Another very important integral is the KI integral. We will proceed similarly as we did with the KK- integral.

Consider the general integral

$$I^{KI}_{\{\alpha,\beta_1,\beta_2\}}(p_1,p_2) = \int_0^\infty dz z^\alpha K_{\beta_1}(p_1 z) I_{\beta_2}(p_2 z), \tag{5.38}$$

with $\alpha, \beta_1, \beta_2 > 0$. To simplify the notation, we will omit these labels.

In the same way, as we did for the KK integral, we will split the range of the integrals

$$I = I^{\text{lower}} + I^{\text{upper}} \tag{5.39}$$

with,

$$I^{\text{lower}} = \int_0^{\mu^{-1}} dz z^\alpha K_{\beta_1}(p_1 z) I_{\beta_2}(p_2 z), \tag{5.40}$$

$$I^{\text{upper}} = \int_{\mu^{-1}}^\infty dz k^\alpha K_{\beta_1}(p_1 z) I_{\beta_2}(p_2 z), \tag{5.41}$$

and study each integral separately.

In the **lower** integral, we will consider (A.4) or (A.6) depending on whether β_1 is an integer or not; and (A.3). So we find the following kind of integral

$$\int_0^{\mu^{-1}} dz z^a \log^N(z). \tag{5.42}$$

The result of this integral is given by (5.33) for $a > -1$. Thus following the same steps as we did for the KK-integral, we may consider the analytical continuation and check that there is a pole at $a = -1$ of order $N + 1$.

The pole existence condition for (5.38) is

$$\alpha + \sigma\beta_1 + \beta_2 + 1 = -2n, \quad n = 0, 1, 2, \ldots \tag{5.43}$$

for some $\sigma \in \{\pm 1\}$. If no-combination of $\sigma \in \{\pm 1\}$ gives the equality, then there is no pole, and we can consider the result of the analytical continuation.

In the **upper** integral using (A.7) and (A.8) we will find the following kind of integral

$$\int_{\mu^{-1}}^{\infty} dz e^{-(p_1-p_2)z} z^{\alpha-n_1-n_2-1} = \frac{1}{(p_1-p_2)^{\alpha-n_1-n_2}} \Gamma\left(\alpha - n_1 - n_2, \frac{p_1-p_2}{\mu}\right).$$

Where n_1, n_2 are the powers in the power expansion of each Bessel function, then n_1, n_2 are non-negative integers. The z integral is convergent for $\alpha < n_1 + n_2$, or equivalently $\alpha < 0$ and $p_1 - p_2 \geq 0$; or $p_1 > p_2$. [81].

However, we shall check whether we can go beyond the convergence regime.

The possible poles arise for $p_1 = p_2$.

- For $\alpha - n_1 - n_2 > 0$, the pole is due to the denominator. In this case, expanding around $p_1 \to p_2$ we find

$$\frac{\Gamma(\alpha - n_1 - n_2)}{(p_1-p_2)^{\alpha-n_1-n_2}} - \frac{1}{\mu^{\alpha-n_1-n_2}(\alpha - n_1 - n_2)} + \mathcal{O}(p_1 - p_2). \tag{5.44}$$

  Then, the pole of order $\alpha - n_1 - n_2$ at $p_1 = p_2$ does not depend on μ.

- On the other hand, if $\alpha - n_1 - n_2 < 0$, then the pole comes from the incomplete Gamma function.

While if $p_1 \neq p_2$, the incomplete gamma function is well behaved as long as $p_1 > p_2$ [82].

The $\mathcal{G}^2$ integral

We will study how to work the integral

$$I_3(z_1) = \int \frac{dz_2}{z_2^{d+1}} \int d\vec{k}_1 \mathcal{G}_\Delta(z_1, -\vec{k}_1, z_2)\mathcal{G}_\Delta(z_1, -\vec{k}_1, z_2). \tag{5.45}$$

Explicitly this integral is

$$\begin{aligned} I_3(z_1) = z_1^d S_d \Bigg[&\int_0^\infty dk_1 k_1^{d-1} K_{\Delta-\frac{d}{2}}(k_1 z_1)^2 \int_0^{z_1} \frac{dz_2}{z_2} I_{\Delta-\frac{d}{2}}(k_1 z_2)^2 \\ &+ \int_0^\infty dk_1 k_1^{d-1} I_{\Delta-\frac{d}{2}}(k_1 z_1)^2 \int_{z_1}^\infty \frac{dz_2}{z_2} K_{\Delta-\frac{d}{2}}(k_1 z_2)^2 \Bigg] \end{aligned} \tag{5.46}$$

Where S_d is the area of the d dimensional sphere. On each z_2 integral, we can do a change of variables $z_2 \to u = k_1 z_2$ for fixed k_1

$$\begin{aligned} I_3(z_1) = z_1^d S_d \Bigg[&\int_0^\infty dk_1 k_1^{d-1} K_{\Delta-\frac{d}{2}}(k_1 z_1)^2 \int_0^{k_1 z_1} \frac{du}{u} I_{\Delta-\frac{d}{2}}(u)^2 \\ &+ \int_0^\infty dk_1 k_1^{d-1} I_{\Delta-\frac{d}{2}}(k_1 z_1)^2 \int_{k_1 z_1}^\infty \frac{du}{u} K_{\Delta-\frac{d}{2}}(u)^2 \Bigg] \end{aligned} \tag{5.47}$$

Notice that in the second line, we have a KK integral where the lower limit works as a cut-off, ensuring the finiteness of this integral. Working for the k_1 integral we may change $k_1 \to w = k_1 z_1$, such that

$$\begin{aligned} I_3 = S_d \Bigg[&\int_0^\infty dw w^{d-1} K_{\Delta-\frac{d}{2}}(w)^2 \int_0^{w} \frac{du}{u} I_{\Delta-\frac{d}{2}}(u)^2 \\ &+ \int_0^\infty dw w^{d-1} I_{\Delta-\frac{d}{2}}(w)^2 \int_{w}^\infty \frac{du}{u} K_{\Delta-\frac{d}{2}}(u)^2 \Bigg]. \end{aligned} \tag{5.48}$$

So, in exact agreement with (4.26), the I_3 integral is just a constant.

5.2.2 2-point function

Having studied integrals of Bessel functions, we focus on computing the 1-loop correction to the 2-point function for a Φ^4 theory in the bulk. The corresponding Witten diagram is exactly like Feynman's diagram in standard QFT, where momentum in the loop is conserved.

The tadpole

By considering the Witten diagram 4.4

$$\langle\langle \mathcal{O}_\Delta(\vec{p_1})\mathcal{O}_\Delta(\vec{p_2})\rangle\rangle_q = \int_0^\infty \frac{dz}{z^{d+1}} \int d\vec{k}\mathcal{K}_\Delta(\vec{p_1},z)\mathcal{G}_\Delta(z,\vec{k},z)\mathcal{K}_\Delta(\vec{p_1},z), \tag{5.49}$$

whose Witten diagram is in figure 5.1,

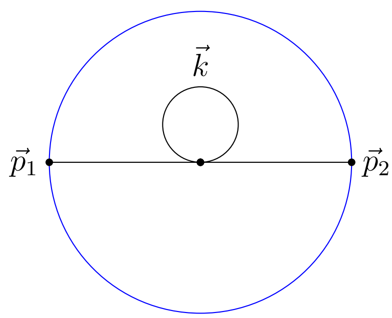
$T_1 =$ 

Figure 5.1: Tadpole Witten diagram in momentum space.

Using the propagators (5.6) and (5.7), we will have

$$\begin{aligned}\langle \mathcal{O}_\Delta(p_1)\mathcal{O}_\Delta(p_2)\rangle_q = \delta(\vec{p_1}+\vec{p_2})\frac{2^{d-2\Delta+2}}{\Gamma\left(\Delta-\frac{d}{2}\right)^2}p_1^{2\Delta-d} \\ \int_0^\infty dz K_{\Delta-\frac{d}{2}}\left(zp_1\right)K_{\Delta-\frac{d}{2}}\left(zp_1\right)\int d\vec{k}z^{d-1}I_{\Delta-\frac{d}{2}}(kz)K_{\Delta-\frac{d}{2}}(kz).\end{aligned} \tag{5.50}$$

First, we will work the $\vec{k}$ integral. Using spherical coordinates $\vec{k} = k\hat{\rho}$ we will have

$$\int d\vec{k} I_{\Delta-\frac{d}{2}}(kz) K_{\Delta-\frac{d}{2}}(kz) = S_d \int_0^\infty dk k^{d-1} I_{\Delta-d/2}(kz) K_{\Delta-d/2}(kz) \tag{5.51}$$

$$= S_d I^{IK}_{\{d-1,\Delta-\frac{d}{2},\Delta-\frac{d}{2}\}}(z,z), \tag{5.52}$$

with $S_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}$ the area of the sphere.

Comparing with equation (5.51), we identify $\alpha = d-1$, so the integral is convergent for $0 < d < 1$. The integral is finite in this range so we can manipulate it. Doing a change of variable $u = kz$, we will have

$$I^{IK}_{\{d-1,\Delta-\frac{d}{2},\Delta-\frac{d}{2}\}}(z,z) = \frac{S_d}{z^d} \int_0^\infty du u^{d-1} K_{\Delta-\frac{d}{2}}(u) I_{\Delta-\frac{d}{2}}(u) \tag{5.53}$$

$$= \frac{S_d}{z^d} I^{IK}_{\{d-1,\Delta-\frac{d}{2},\Delta-\frac{d}{2}\}}(1,1).$$

So, we have isolated the z-dependence and we are left with a constant integral for $d < 1$.

However, we may still be able to consider the analytic continuation to $d > 1$. Indeed,

$$I^{IK}_{\{-1,\Delta-\frac{d}{2},\Delta-\frac{d}{2}\}}(1,1) = \int_0^\infty du u^{d-1} I_{\Delta-\frac{d}{2}}(u) K_{\Delta-\frac{d}{2}}(u)$$

$$= \frac{\Gamma\left(\frac{1}{2}-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)\Gamma(\Delta)}{4\sqrt{\pi}\Gamma(\Delta-d+1)}, \quad \text{for } d < 1. \tag{5.54}$$

In the last line, we shall notice that it is possible to consider the analytic continuation to $d > 1$. It turns out that it is finite for d : *even* but may diverge for d : *odd*. In the former case, we can use dimensional regularisation such that $d \to d - \kappa$, where κ is the UV regulator. The former formula is the result in (4.45).

Having regularised the UV problem, we now focus on checking whether the IK integral has IR divergences.

The pole condition (5.43) determines the behaviour of the lower integral. For $\beta = \Delta - d/2 > 0$ is

$$\frac{d}{2}(1-\sigma) + \Delta(1+\sigma) = -2n, \quad \sigma = \{\pm 1\}. \tag{5.55}$$

The different conditions come from the sign of σ,

- If $\sigma = -1$ then $d = -2n$, this never happens.
- If $\sigma = +1$ we have $\Delta = -n$ and this never happens because we are considering $\Delta > d/2 > 0$.

So, for $\Delta > \frac{d}{2}$, the KI-integral is always IR convergent. This is the conformal dimensions that we are working

---

Aside Commentary It is well known that unitarity allows having $\Delta > \frac{d-2}{2}$. So considering $\frac{d-2}{2} < \Delta < \frac{d}{2}$ this is having negative β. To work this, we will take $\beta \to -\beta$, keeping β positive. The function $K_\beta(x) = K_{-\beta}(x)$, while $I_\beta(x) = I_{-\beta}(x)$ for $\beta \in \mathbb{N}$ and we notice that the pole existence (5.43) in the lower limit becomes

$$\alpha + \sigma\beta_1 - \beta_2 + 1 = -2n. \tag{5.56}$$

- If $\sigma = 1$ we have $d = -2n$ which is not possible.
- If $\sigma = -1$,

$$\Delta - d = n, \quad n = 0, 1, 2, \ldots, \tag{5.57}$$

  which may hold depending on certain values of d and Δ.

In summary, when $\Delta > d/2$, the loop integral (5.51) is IR finite but UV divergent. While if $(d-2)/2 < \Delta \leq d/2$ there can be IR divergence.

In [52, 53], they notice that for $d = 3$ and $\Delta = 2$ there are no IR divergences, and for $d = 3$ and $\Delta = 1$ there are IR divergences. This is in complete agreement with our conclusions.

We will only consider $\Delta > \frac{d}{2}$

We are still left to work on the KK integral. As we saw in section 5.2.1, the integral is UV convergent,

$$I^{KK}_{\{-1,\Delta-\frac{d}{2},\Delta-\frac{d}{2}\}} = p_1^{2\Delta-d}\int_0^\infty dz z^{-1} K_{\Delta-\frac{d}{2}}\left(zp_1\right) K_{\Delta-\frac{d}{2}}\left(zp_1\right). \tag{5.58}$$

Through a change of variables, $w = zp_1$, we can prove that the integral, in principle, does not depend on p_1. However, this is incorrect because the pole condition (5.37) tells us that there can be IR divergences. In this case, the pole condition is

$$\left(\Delta - \frac{d}{2}\right)(\sigma_1 + \sigma_2) = -2k,\ k = 0, 1, 2\dots \tag{5.59}$$

despite $\beta_1 = \beta_2 = \Delta - \frac{d}{2}$, the σ's are different because they come from the series expansion of each Bessel function. Here are three different situations that we have to check case by case,

$(+,+)$ This correspond to

$$\Delta - \frac{d}{2} = -k, \qquad k = 0, 1, 2, \dots. \tag{5.60}$$

For $\Delta > \frac{d}{2}$ is never satisfied. But if we are in $\frac{d-2}{2} < \Delta \leq \frac{d}{2}$ it can be satisfied for some values of k.

$(+,-)$ It is direct to see from the l.h.s of (5.59) is satisfied identically for $k = 0$.

$(-,-)$ This is

$$\Delta - \frac{d}{2} = k, \quad k = 0, 1, 2, \dots. \tag{5.61}$$

Is satisfied for $\Delta = \frac{d}{2} + k,\ k = 0, 1, 2, \ldots$.

From the condition $(+,-)$, we see that there is at least one pole. From condition $(+,+)$ there will be more poles for $\Delta - d/2 \in \mathbb{Z}$.

As at least one pole existence condition is always satisfied, IR divergence will always exist. To regulate the divergence, we use the same regulator used in tree-level holography; this is a cut-off ϵ in the lower limit. Defining the regulated KK- integral as

$$I^{KK}_{\{-1,\Delta-\frac{d}{2},\Delta-\frac{d}{2}\}}(\epsilon, p_1) = \lim_{\epsilon\to 0} p_1^{2\Delta-d} \int_{\epsilon}^{\infty} dz z^{-1} K_{\Delta-\frac{d}{2}}(zp_1) K_{\Delta-\frac{d}{2}}(zp_1). \tag{5.62}$$

Doing a change of variable $w = zp_1$, we will have

$$I^{KK}_{\{-1,\Delta-\frac{d}{2},\Delta-\frac{d}{2}\}} = \lim_{\epsilon\to 0} p_1^{2\Delta-d} \int_{\epsilon p_1}^{\infty} dw w^{-1} K_{\Delta-\frac{d}{2}}(w) K_{\Delta-\frac{d}{2}}(w). \tag{5.63}$$

Here is explicit the importance of performing the change of variable on the regulated integral rather than the original integral (5.58). The regulated integral depends on the momentum p_1 through the lower limit.

To compute the KK integral, we can follow the idea of the $D_{\Delta,\Delta}$ in position space by taking a derivative with respect to the lower limit of the integral and integrating it back. However, it is possible to compute the full integral using some software, for instance, Mathematica [83]. The result is given in Appendix C.4. Assuming $\Delta - \frac{d}{2} \neq n,\ n = 0, 1, 2, \ldots$, for simplicity, and expanding around $\epsilon \to 0$ the I^{KK} is

$$\begin{aligned} I^{KK}_{\{-1,\Delta-\frac{d}{2},\Delta-\frac{d}{2}\}} = \lim_{\epsilon\to 0} \Bigg[& \epsilon^{d-2\Delta} \left(-\frac{2^{2\Delta-d}\Gamma\left(-\frac{d}{2}+\Delta+1\right)^2}{(d-2\Delta)^3} + O\left(\epsilon^2\right) \right) \\ & + \frac{1}{4} p_1^{2\Delta-d} \Gamma\left(\frac{d}{2}-\Delta\right) \Gamma\left(\Delta-\frac{d}{2}\right) \left(\psi^{(0)}\left(\frac{d}{2}-\Delta\right) + \psi^{(0)}\left(\Delta-\frac{d}{2}\right) + \log\left(\frac{4}{p^2\epsilon^2}\right) \right) + \ldots \Bigg] \end{aligned} \tag{5.64}$$

The first terms are local divergences that can be removed with local counterterms. On the other hand, the second line is a non-local term that comes from the regulated integral and plays a crucial role. For the case $\Delta - \frac{d}{2} = n,\ n = 0, 1, 2, \ldots$, there is an additional term that goes as $\log^2$ in the series. This can be seen from the pole existence condition for the KK integral where two conditions are satisfied. Either we have just a log or a $\log^2$ term; both play the same role in interpreting the result.

Then, the finite 1-loop correction to the 2-point function is

$$\langle\langle \mathcal{O}_\Delta(p_1)\mathcal{O}_\Delta(p_2)\rangle\rangle^{\text{1-Tadpole}}_{\text{finite}} = \frac{2^{d-2\Delta}\Gamma\left(\frac{d}{2}-\Delta\right)p^{2\Delta-d}}{\Gamma\left(\Delta-\frac{d}{2}\right)} I^{IK}\left(-2\log(p\epsilon)+\log(4)\right. \\ \left.\psi^{(0)}\left(\frac{d}{2}-\Delta\right)+\psi^{(0)}\left(\Delta-\frac{d}{2}\right)\right), \tag{5.65}$$

where I^{IK} is a constant given by equation (5.54) and is a constant, in position is the equivalent of $G_{\Delta,\kappa}(1)$.

Renormalisation

The renormalisation problem is very similar to what is done in position space. We will sketch the important feature; for further details, we refer to section 4.4.1.

The 2-point function is given by the tree level computation (5.12) plus the 1-tadpole correction (5.65)

$$\begin{aligned}\langle\langle \mathcal{O}_\Delta(\vec{p}_1)\mathcal{O}_\Delta(\vec{p}_2)\rangle\rangle &= \langle\langle \mathcal{O}_\Delta(\vec{p}_1)\mathcal{O}_\Delta(\vec{p}_2)\rangle\rangle^{\text{tree}}_{\text{IR finite}} - \frac{\lambda}{2}\langle\langle \mathcal{O}_\Delta(\vec{p}_1)\mathcal{O}_\Delta(\vec{p}_2)\rangle\rangle^{\text{1-Tadpole}}_{\text{finite}} \\ &= -(2\Delta-d)\frac{2^{d-2\Delta}\Gamma\left(\frac{d}{2}-\Delta\right)p^{2\Delta-d}}{\Gamma\left(\Delta-\frac{d}{2}\right)}\left(1+2\lambda\gamma\left(\log(p\epsilon)-\log(2)\right.\right. \\ &\left.\left.-\frac{\psi^{(0)}\left(\frac{d}{2}-\Delta\right)+\psi^{(0)}\left(\Delta-\frac{d}{2}\right)}{2}\right)\right)\end{aligned} \tag{5.66}$$

where we called

$$\gamma = \frac{I^{IK}}{2(2\Delta-d)}, \tag{5.67}$$

which is UV divergent.

Equation (5.66) has the same structure as (4.82). So, as we did in subsection 4.4.1, we renormalise the UV divergence by introducing a mass counterterm.

Having renormalised the mass, we still have the IR regulator in the $\log(\epsilon)$ term. This divergence cannot be renormalised by adding a boundary counterterm. Nevertheless, the $\log(\epsilon)$ term is renormalised by source renormalisation.

The renormalised holographic two-point function up to 1-loop for operators with conformal dimension $\Delta - \frac{d}{2}$ non-integer is

$$\langle\langle \mathcal{O}_{\Delta_r}(\vec{p}_1)\mathcal{O}_{\Delta_r}(\vec{p}_2)\rangle\rangle = -\left(2\Delta_r - d\right)\frac{2^{d-2\Delta_r}\Gamma\left(\frac{d}{2}-\Delta_r\right)p^{2\Delta_r-d}}{\Gamma\left(\Delta_r-\frac{d}{2}\right)}. \tag{5.68}$$

5.2.3 Further Diagrams

As we did in position space, it is natural to go beyond order λ^2 for the 2-point and 4-point functions. Similarly, as we did in position space, we can perform the eight-diagram. In other diagrams, such as the double tadpole or the sunset, the bulk-to-boundary propagators are connected through bulk-to-bulk propagators leading to difficulties in working these diagrams.

However, let us start with the eight-diagram.

The eight-diagram

The eight-diagram in momentum space is dictated by

$$E = \int_0^\infty \frac{dz_1}{z_1^{d+1}}\mathcal{K}_\Delta(z_1,\vec{p}_1)\mathcal{K}_\Delta(z_1,-\vec{p}_1)I(z_1), \tag{5.69}$$

where

$$I(z_1) = \int_0^\infty \frac{dz_2}{z_2^{d+1}} \int d\vec{k}_1 \mathcal{G}_\Delta(z_1, -\vec{k}_1, z_2)\mathcal{G}_\Delta(z_1, \vec{k}_1, z_2) \int d\vec{k}_2 \mathcal{G}_\Delta(z_2, -\vec{k}_2, z_2)$$
$$= \int_0^\infty \frac{dz_2}{z_2} \int d\vec{k}_1 \mathcal{G}_\Delta(z_1, -\vec{k}_1, \bar{z}_2)\mathcal{G}_\Delta(z_1, \vec{k}_1, \bar{z}_2) \int d\vec{k}_2 K_{\Delta-\frac{d}{2}}(z_2k_2) I_{\Delta-\frac{d}{2}}(z_2k_2) \quad (5.70)$$

are the loops. The Witten diagram is shown in Figure 5.2. The $\vec{k}_2$ integral is given by

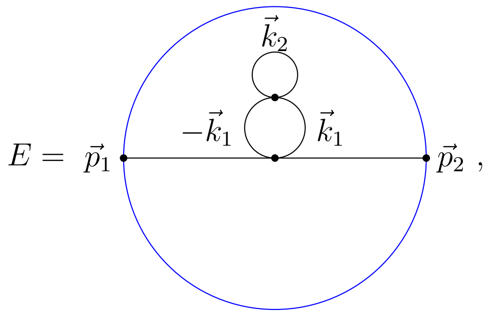


Figure 5.2: eight-diagram in momentum space. The bottom loop is composed of two bulk-to-bulk propagators, one with momentum $\vec{k}_1$ and the other with momentum $-\vec{k}_1$; the top loop has momentum $\vec{k}_2$.

(5.51). Using this, we have

$$\langle\langle \mathcal{O}_\Delta(\vec{p}_1)\mathcal{O}_\Delta(\vec{p}_2)\rangle\rangle_E = S_d I^{IK}_{\{d-1,\Delta-\frac{d}{2},\Delta-\frac{d}{2}\}} \int_0^\infty \frac{dz_1}{z_1^{d+1}} \mathcal{K}_\Delta(z_1, \vec{p}_1)\mathcal{K}_\Delta(z_1, -\vec{p}_1) \quad (5.71)$$
$$\times \int_0^\infty \frac{dz_2}{z_2^{d+1}} \int d\vec{k}_1 \mathcal{G}_\Delta(z_1, -\vec{k}_1, z_2)\mathcal{G}_\Delta(z_1, \vec{k}_1, z_2). \quad (5.72)$$

The former line corresponds to a constant given by (5.2.1).

Then, the structure of the eight-diagram in momentum space is

$$\langle\langle \mathcal{O}_\Delta(\vec{p}_1)\mathcal{O}_\Delta(\vec{p}_2)\rangle\rangle_E = S_d I^{IK}_{\{d-1,\Delta-\frac{d}{2},\Delta-\frac{d}{2}\}} I_2 \int_0^\infty \frac{dz_1}{z_1^{d+1}} \mathcal{K}_\Delta(z_1, \vec{p}_1)\mathcal{K}_\Delta(z_1, -\vec{p}_1) \quad (5.73)$$

which has the same structure obtained in the position space (4.77).

Other diagrams

Consider, for example, the sunset diagram

$$S = \int_0^\infty \frac{dz_1}{z_1^{d+1}} \frac{dz_2}{z_2^{d+1}} \mathcal{K}_\Delta(z_1, \vec{p}_1) \mathcal{K}_\Delta(z_2, -\vec{p}_1) \int d\vec{k}_1 \vec{k}_2 \Big[\mathcal{G}_\Delta(z_1, \vec{k}_1, z_2) \\ \mathcal{G}_\Delta(z_1, \vec{k}_2, z_2) \mathcal{G}_\Delta(z_1, \vec{p} - \vec{k}_1 - \vec{k}_2, z_2) \Big] \tag{5.74}$$

We know from the position space treatment that this diagram is proportional to the $\mathcal{K}^2$ integral. However, we have not been able to isolate the structure of this integral. This is because the bulk-to-boundary propagator does not coincide in a single point, so they are connected through a bulk-to-bulk propagator that exchanges momentum. The same problem arises in bulk Φ^3 interacting theory, where the holographic 4-point function at tree level has not been explicitly computed [2] yet.

Other diagrams, like the double tadpole or the loop 4-point function, have the same problems.

[2] Private communication with Kostas Skenderis.

Chapter 6

Introduction to Graviton Propagator

We have worked on loop corrections for scalars on an AdS background. In this sense, it corresponds to a toy model that shows the structure of how to work quantum correction to holographic correlation functions and how to deal with the renormalisation problem.

However, as we know from GR, spacetime interacts with matter. Therefore, to work with more realistic models, we must include the interaction between scalar fields and gravitons. Additionally, it is well-known [84, 85] that type IIB SUGRA contains scalar fields interacting with gravity and has a CFT dual.

As we have seen in previous chapters, propagators are crucial for studying quantum correction for holographic correlation functions. To include gravitons, we need the bulk-to-boundary propagator [85] and the bulk-to-bulk propagator. The former has been reported on-shell only[84, 86]. This means that it is obtained by inverting linearised Einstein equations that do not have an arbitrary source because it must be covariantly conserved. To include loops of gravitons, we need the off-shell graviton propagator.

As gravity is a gauge theory, we will use standard QFT techniques to quantise gauge theories. Following the *Becchi-Rouet-Stora-Tyutin* (BRST) quantisation [59, 87, 88, 89] for gravitons.

This chapter is still a work in progress. However, the general idea of how to obtain the off-shell graviton propagator in AdS is proposed.

6.1 Short introduction to BRST

The path integral can be seen as the sum over all possible configurations; these include those related through a gauge transformation. So in the path integral, we will be over-counting different configurations that are physically the same.

Consider a gauge transformation $\epsilon^\alpha \delta_\alpha$, where α is not necessarily a discrete label. A gauge transformation satisfies an algebra $[\delta_\alpha, \delta_\beta] = f^{\gamma}_{\alpha\beta}\delta_\gamma$.

The way to deal with the gauge invariance in the path integral is by fixing the gauge condition $F^A(\Phi) = 0$ and consider

$$Z = \int \frac{D\Phi_i}{V_{gauge}} e^{-S}, \tag{6.1}$$

where V_{gauge} is the volume of gauge transformation, Φ_i denotes all the gauge invariants fields, and S is the gauge invariant action.

Then, following the Faddeev-Popov procedure, the path integral becomes

$$Z = \int D\Phi_i DB_A Db_A Dc^\alpha e^{-S-S_{gf}-S_{FP}}, \tag{6.2}$$

where B_A is an auxiliary field, called Nakanishi-Lautrup; b_A is the anti-ghost, and c^α is a ghost, S is the original gauge invariant action, S_{gf} is the gauge fixing action and S_{FP} is the Faddeev-Popov action. Each new action is given by

$$S_{GF} = -i \int d^{d+1}x \sqrt{g} B_A F^A, \tag{6.3}$$

$$S_{FP} = \int d^{d+1}x \sqrt{g} b_A c^\alpha \delta_\alpha F^A(\Phi), \tag{6.4}$$

where $\delta_\alpha F^A$ is a gauge transformation over the gauge fixing function. Notice that the integration over B_A gives a Dirac delta that ensures the gauge fixing. By fixing the gauge, we are breaking the gauge invariance

Becchi-Rouet-Stora-Tyutin (BRST) notice that the path integral has a new symmetry, larger than the gauge symmetry, called the BRST symmetry. The BRST symmetry has associated BRST transformations

$$\delta_B \Phi_i = -i\epsilon c^\alpha \delta_\alpha \Phi_i, \tag{6.5}$$

$$\delta_B B_A = 0, \tag{6.6}$$

$$\delta_B b_A = \epsilon B_A, \tag{6.7}$$

$$\delta_B c^\alpha = \frac{i}{2}\epsilon f^\alpha_{\ \beta\gamma} c^\beta c^\gamma, \tag{6.8}$$

that leaves the whole path integral unchanged. The field change is just a gauge symmetry but exchanging the parameter by a ghost. As BRST symmetry is a symmetry of the path integral, the physical state must be BRST invariant. The proof is in [59] and referenced therein.

In the next section, we will compute the quadratic expansion of the Einstein-Hilbert action and include the gauge fixing action.

6.2 Einstein Hilbert action up to quadratic order

To obtain the graviton propagator, we start from the Einstein-Hilbert action with cosmological constant

$$S = \int d^{d+1}x\sqrt{g}\,(R - 2\Lambda)\,, \tag{6.9}$$

and consider the metric to be split into a classical part and a quantum fluctuation, $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where $\bar{g}_{\mu\nu}$ solves free Einstein equations $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$ and $h_{\mu\nu}$ does not follow the equation of motion. The expansion reads,

$$S[g] = S[\bar{g}] + \frac{\delta S}{\delta g^{\mu\nu}}\Big|_{g=\bar{g}} h^{\mu\nu} + \frac{1}{2}\frac{\delta^2 S}{\delta g^{\mu\nu}\delta g^{\alpha\beta}}\Big|_{g=\bar{g}} h^{\mu\nu}h^{\alpha\beta}. \tag{6.10}$$

It is well known that

$$\delta g_{\alpha\beta} = -g_{\mu\alpha}g_{\nu\beta}\delta g^{\mu\nu},\ \delta R = \delta g^{\mu\nu}R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}, \tag{6.11}$$

$$\delta R_{\mu\nu} = \nabla_\rho\left(\delta\Gamma^\rho{}_{\mu\nu}\right) - \nabla_\nu\left(\delta\Gamma^\rho{}_{\rho\mu}\right),\ \delta\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(\nabla_\mu\delta g_{\nu\lambda} + \nabla_\nu\delta g_{\lambda\mu} - \nabla_\lambda\delta g_{\mu\nu}\right). \tag{6.12}$$

Considering that the linear term in h vanishes because of the equation of motion, we will have

$$S = S[\bar{g}] + \int d^{d+1}x\sqrt{\bar{g}}h^{\mu\nu}D_{\mu\nu\alpha\beta}h^{\alpha\beta} \tag{6.13}$$

with

$$S[\bar{g}] = \frac{4}{d-1}\int d^{d+1}x\sqrt{\bar{g}}\Lambda, \tag{6.14}$$

$$\begin{aligned} D_{\mu\nu\alpha\beta} = &\frac{1}{4}\bar{g}_{\mu\nu}\bar{g}_{\alpha\beta}\nabla^2 - \frac{1}{4}\bar{g}_{\mu\alpha}\bar{g}_{\nu\beta}\nabla^2 + \frac{1}{2}\bar{g}_{\nu\beta}\nabla_\alpha\nabla_\mu - \frac{1}{2}\bar{g}_{\alpha\beta}\nabla_\nu\nabla_\mu \\ &- \frac{1}{4}\bar{g}_{\mu\alpha}\bar{g}_{\nu\beta}(\bar{R} - 2\Lambda) + \frac{1}{4}\bar{g}_{\mu\nu}\bar{R}_{\alpha\beta} \end{aligned} \tag{6.15}$$

where the covariant derivative act with the background metric.

Using the Ricci tensor in Poincare coordinates, Ricci scalar and cosmological constant for AdS_{d+1} we have

$$D_{\mu\nu\alpha\beta} = \frac{1}{4}\bar{g}_{\mu\nu}\bar{g}_{\alpha\beta}\nabla^2 - \frac{1}{4}\bar{g}_{\mu\alpha}\bar{g}_{\nu\beta}\nabla^2 + \frac{1}{2}\bar{g}_{\nu\beta}\nabla_\alpha\nabla_\mu - \frac{1}{2}\bar{g}_{\alpha\beta}\nabla_\nu\nabla_\mu + \frac{d}{2}\bar{g}_{\mu\alpha}\bar{g}_{\nu\beta} - \frac{d}{4}\bar{g}_{\mu\nu}\bar{g}_{\alpha\beta} \tag{6.16}$$

The quadratic action is invariant under local change of coordinates $\delta x^\mu = \xi^\mu(x)$, which leads to the finding that the change of the perturbation is given by

$$\delta h_{\mu\nu} = -\nabla_\mu \xi_\nu - \nabla_\mu \xi_\nu \tag{6.17}$$

where the covariant derivatives act with the AdS_{d+1} metric.

6.2.1 Fixing the Gauge

We will fix the gauge. Demanding the axial gauge $h_{\mu 0} = 0$ [90] the metric is

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= \frac{1}{z^2} dz^2 + \left(\frac{1}{z^2} \bar{g}_{ij} + h_{ij} \right) dx^i dx^j, \end{aligned} \tag{6.18}$$

where we used the Poincare coordinates of AdS and fixed the axial gauge over the fluctuation.

Although we have already fixed a gauge, there remains a gauge invariance. To see this consider $\delta h_{\mu\nu} = -\nabla_\nu \epsilon_\mu - \nabla_\mu \epsilon_\nu$, such that

$$\epsilon_0 = 0, \quad \epsilon_i = \frac{1}{z^2} \omega_i(x) \Rightarrow \delta h_{ij} = -\partial_i \omega_j - \partial_j \omega_i. \tag{6.19}$$

To fix the remaining gauge freedom, we use the de-Donder gauge over the transverse coordinates,

$$\nabla_j h^j_{\ i} - \frac{1}{2} \nabla_i h = \partial_j h^j_{\ i} - \frac{1}{2} \partial_i h = 0, \tag{6.20}$$

where we explicitly compute the covariant derivatives in the Poincaré coordinates.

In summary, the gauge fixing we are going to use

$$\begin{aligned} h_{\mu 0} &= 0, && \text{Axial gauge} \\ \partial_j h^j{}_i - \frac{1}{2}\partial_i h &= 0 && \text{De-Donder gauge.} \end{aligned} \tag{6.21}$$

To work the path integral, we will impose the axial gauge to the action (6.13) and fix the De-Donder gauge via the gauge fixing action.

6.2.2 The path integral

Once we have obtained the Einstein-Hilbert to second order and fixed the gauge, we will build the path integral. As we are looking for the graviton propagator, we will not compute the ghost action.

$$Z = \int Dh_{\mu\nu} DBDbDce^{-S-S_{GF}-S_{FP}}, \tag{6.22}$$

where B is the Nakanishi-Lautrup fields, b anti-ghost and c ghost. The ghost action is S_{FP}, S is given by (6.13) together with (6.14) and (6.15). While the gauge fixing action is

$$S_{GF} = -i \int d^{d+1}x \sqrt{g} \left[B^i \left(\partial_j h^j{}_i - \frac{1}{2}\partial_i h \right) \right]. \tag{6.23}$$

To integrate the Nakanishi-Lautrup field we shall add the BRST invariant term

$$\delta S = \frac{1}{2}\xi \int d^{d+1}x \sqrt{g} \bar{g}^{ij} B_i B_j. \tag{6.24}$$

The path integral over B^i is gaussian so we can integrate it.

Then, imposing $h_{\mu 0} = 0$, the partition function is

$$Z = \int Dh_{ij} DbDce^{-S-G_{GF}-S_{FP}} \tag{6.25}$$

with

$$S_{GF} = \frac{1}{2\xi}\int d^{d+1}x\sqrt{\bar{g}}\bar{g}^{ij}\left[\partial_m h^m_i - \frac{1}{2}\partial_i h\right]\left[\partial_n h^n_j - \frac{1}{2}\partial_j h\right] \tag{6.26}$$

$$S = \int d^{d+1}x\sqrt{\bar{g}}\left[h^{ij}D_{ijkl}h^{kl} + \frac{4\Lambda}{d-1}\right] \tag{6.27}$$

The S_{GF} is quadratic on h so we may write the path integral as

$$Z = \int Dh_{ij} DbDc\, e^{-S_{h^2}-S_{FP}} \tag{6.28}$$

where

$$S_{h^2} = \int d^{d+1}x\sqrt{g}\left[h^{ij}\bar{D}_{ijkl}h^{kl} + \frac{4\Lambda}{d-1}\right] \tag{6.29}$$

with

$$\bar{D}_{ijkl} = D_{ijkl} - \frac{1}{2\xi}\left(\bar{g}_{il}\nabla_j\nabla_k - \bar{g}_{kl}\nabla_i\nabla_j + \frac{1}{4}\bar{g}_{ij}\bar{g}_{kl}\nabla^2\right). \tag{6.30}$$

The action S_{h^2} is the Einstein-Hilbert action plus the gauge fixing action.

To compute the propagator, we will consider the functional

$$Z[T] = \frac{1}{Z[0]}\int Dh_{ij}\, e^{-S_{h^2}+\int dxT^{ij}h_{ij}} \tag{6.31}$$

$$= e^{\frac{1}{4}\int dxdyT^{ij}G_{ijkl}T^{kl}}, \tag{6.32}$$

where the operator $\bar{D}$ satisfies

$$\bar{D}^{mnkl}G_{klij}(x_1, x_2) = \delta^m_{(i}\delta^n_{j)}\delta(x_1, x_2) \tag{6.33}$$

with G_{ijkl} the graviton propagator.

To solve the differential equation (6.33), we will consider the Fourier transformation

$$G_{ijkl}(x_1, x_2) = \frac{1}{(2\pi)^d}\int d\vec{k}e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}G_{ijkl}(z_1, \vec{k}, z_2). \tag{6.34}$$

Similarly, as in [91, 92], we write the graviton propagator in the most general bi-tensor basis in momentum along the transverse coordinate. Then we proceed to solve the ordinary differential equation for each basis.

Chapter 7

Conclusions

In this thesis, we worked on the role of bulk quantum fields in holographic correlation functions. According to the holographic dictionary, each bulk field corresponds to a gauge-invariant conformal operator. The boundary condition for the bulk field corresponds to the source of the dual conformal operator theory. An operator with conformal dimension Δ has a source with conformal dimension $d - \Delta$.

Using the strong/weak duality, we study the bulk's semi-classical approximation and compute the dual theory's correlation functions. However, the infinite volume of AdS leads to IR divergence that demands regularisation. We regularise the bulk theory by adding a cut-off when approaching the boundary. All IR divergences are local terms of the source and depend on the conformal dimension; then, all IR divergences are renormalized by adding local counterterms to the boundary.

Having solved the classical problem, we presented the general procedure to deal with quantum corrections in the bulk. As we deal with quantum fields in the bulk, both UV and IR divergences are present. From the holographic dictionary, IR divergences on one side corresponding to UV divergences on the other. In the CFT, we do not expect low-energy anomalies. From the point of view of the bulk, this means that we must regularise the UV divergences with an AdS invariant regulator. To renormalize the UV divergences, we re-defined the mass and the coupling of the bulk theory such that the corresponding countertem cancels the UV divergence. Having finite bulks parameters, we shall deal with the IR divergences. The renormalization is done by modifying the semi-classical boundary counterterms and, eventually, the source.

To better understand how to proceed with loops in the bulk, we consider interacting quantum scalar fields over an AdS background. We solved the problem of renormalization in three different ways,

In position space, we solved the problem in two ways, i) background field method and ii) quantum effective action. In momentum, we solved the renormalization problem using the background field method.

In the background field method, we consider the field to have a classical part and a quantum fluctuation. The classical part of the partition function demands standard holographic renormalization, therefore adding boundary counterterms. Meanwhile, for the quantum correction, we used the GKPW rules finding that the corrections to the tree-level correlation function correspond to loops Witten diagrams. In general, the two-point function at loop order is IR and UV divergent. UV divergences demand mass renormalization or equivalent conformal dimension renormalization. The boundary counterterms in the semi-classical approach depend on the conformal dimension; they must also be renormalized. Thus, any 2-point function at loop order has to be IR divergent to correct the boundary term. Even though we have renormalized the mass and corrected the boundary counterterm, an IR divergent piece remains. To renormalize this divergence, we have to renormalize the boundary condition for the bulk field. The source renormalization gives the source with the correct renormalized conformal dimension, such that it corresponds to the source of an operator with renormalized conformal dimension. In the one-loop 4-point function, the tadpole 4-point function demands mass renormalization. The counterterm is precisely the one obtained in the 2-point function single tadpole. The exchange diagram demands coupling renormalization and is UV renormalizable only up to $d + 1 = 7$.

From the point of view of quantum effective action we study the same problem of quantum correction to holographic correlation functions. Using the standard techniques of QFT, we obtain the quantum effective action up to the second order in the coupling constant. The quantum effective action is UV divergent, so we need to renormalize the bulk. To do that, we need to have a well-defined variational problem. In particular, the equation of motion must be UV finite. To solve the scalar equation of motion, we demand a non-normalizable boundary condition for the bulk field. Solving the equation of motion order by order, we found that each UV divergence can be cancelled by adding the corresponding bulk counterterm. In particular, we shall renormalize the mass and the coupling constant. The coupling constant is only renormalizable up to $d + 1 = 7$. After UV renormalization of the mass and the coupling, the bulk action is finite. We can do standard holographic renormalization with the renormalized conformal dimension and coupling constant.

Both methods, background field, and quantum effective action lead to the same conclusions. However, both share different aspects of the problem. In the background field methods, we realised that the renormalization of the source is a must, while in the quantum effective approach, this is implicit. However, the effective quantum approach does not lead (explicitly) to the Witten diagram, while the background field conduces to a nice diagrammatic formulation of the problem.

To work the holographic CFT correlation function, we used a third method. We computed quantum corrections to the correlation function in momentum space. Using the background field method adapted to the field in momentum space, we computed the 1-loop correction to the 2-point function. Similarly, as in position space, the 1-loop 2-point function Witten diagram is both IR and UV divergent. We used dimensional regularisation to deal with the UV divergences. The regularisation of the IR divergence is done with the standard cut-off. The loop UV divergence is cancelled by mass renormalization.

Furthermore, with this, we renormalize the semi-classical boundary counterterm.

To renormalize the leftover IR divergence, we use source renormalization. However, the problem of working quantum corrections in momentum space up to the second order in the coupling is still not solved. We do not have a systematic way to deal with exchanges of scalar fields.

We presented a systematic renormalization procedure for loop diagrams in AdS, and we illustrated the method using the scalar Φ^4 theory, which is only renormalisable up to $d+1 = 7$. Bulk renormalisation is consistent with expectations based on the AdS/CFT, supporting the duality.

We obtain the result considering a scalar field over an AdS background. However, it is known that the scalar field interacts with the background. So in a more realistic model, the computation of the holographic correlation function for scalar fields in the bulk must include the interaction of bulk graviton. The first step to this objective is to obtain the off-shell graviton propagator.

To obtain the off-shell AdS graviton propagator, we follow the BRST quantisation method. Introducing the gauge fixing action to the path integral, we fix the gauge and must include ghost action. We fix the axial gauge directly to the graviton action, while the De-Donder gauge is fixed through the gauge fixing action. Merging the gauge fixing action with graviton action, the quadratic graviton action is no longer gauge invariant. However, the whole path integral is invariant under BRST transformations. Solving the gaussian path integral naturally demands finding the inverse of the kernel that corresponds to the propagator. To obtain the explicit form of the graviton, we will consider the Fourier transformation and solve the corresponding differential equation. This is a work in progress.

7.0.1 Future Work and open questions

Some future work, in no particular order, corresponds to,

- In the current work, we studied $\Delta > \frac{d}{2}$. However, CFT unitarity allows to consider $\frac{d}{2} - 1 < \Delta < \frac{d}{2}$. This case was not studied, and it would be interesting to know the behaviour of each integral in this regime and how renormalisation works.

- To go beyond on loops, consider other bulk interactions and higher point functions to study the renormalisation problem.

- To finish the computation of the off-shell graviton propagator. The off-shell propagator will allow us to work on the interaction between matter and gravity to study the bulk gravitons loop corrections.

- The fact that we can use the semi-classical approximation on the bulk side relies on the large N expansion on the CFT side. Then, computing loops in the bulk corresponds to the correction of order $\frac{1}{N^2}$ and beyond. It is worth comparing the loop in bulk results with known models.

- The most studied example of AdS/CFT corresponds to the duality between type IIB SUGRA on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ Super Yang-Mills in $d = 4$. In the CFT theory, are known operators with protected conformal dimensions. Then, from the point of view of the bulk, the quantum correction should vanish.

- Recently [93, 94] proposed that the Witten diagram can be related to amplitudes in celestial holography. It would be interesting to relate the loops Witten diagram and their renormalisation with celestial amplitudes.

Appendix A

Special Functions

A.1 Modified Bessel fucntions

By considering the differential equation [95, 96]

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2+\nu^2)y = 0, \tag{A.1}$$

the solution is,

$$y(x) = AK_\nu(x) + BI_\nu(x), \tag{A.2}$$

with A and B constant and the functions K and I are known as modified Bessel functions. The series representation of each is [96],

$$I_\nu(x) = \sum_{j=0}^{\infty} \frac{1}{j!\Gamma(\nu+j+1)}\left(\frac{x}{2}\right)^{\nu+2j}, \quad \nu \neq -1,-2,-3,\ldots \tag{A.3}$$

$$K_\nu(x) = \sum_{j=0}^{\infty}\left[a_j^-(\nu)x^{-\nu+2j} + a_j^+(\nu)x^{\nu+2j}\right], \quad \nu \notin \mathbb{Z} \tag{A.4}$$

with

$$a_j^\sigma(\nu) = \frac{(-1)^j \Gamma(-\sigma\nu - j)}{2^{\sigma\nu+2j+1} j!}, \sigma \in \{\pm 1\}. \tag{A.5}$$

For non-negative integer n, the expansion is,

$$\begin{aligned} K_n(x) = & \frac{1}{2}\left(\frac{x}{2}\right)^{-n} \sum_{j=0}^{n-1} \frac{(n-j-1)!}{j!}(-1)^j \left(\frac{x}{2}\right)^{2j} \\ & + (-1)^{n+1} \log\left(\frac{x}{2}\right) I_n(x) \\ & + (-1)^n \frac{1}{2}\left(\frac{x}{2}\right)^n \sum_{j=0}^{\infty} \frac{\psi(j+1) + \psi(n+j+1)}{j!(n+j)!}\left(\frac{x}{2}\right)^{2j}, \end{aligned} \tag{A.6}$$

with $\psi(x)$ the digamma function.

For $x >> 1$, we have the asymptotic series,

$$K_{\beta_1}(x) \sim \left(\frac{\pi}{2x}\right)^{1/2} e^{-x} \sum_{m=0}^{\infty} A_m(\beta_1) \left(\frac{1}{2x}\right)^m \tag{A.7}$$

$$I_{\beta_2}(x) \sim \frac{1}{(2\pi x)^{1/2}} e^{x} \sum_{m=0}^{\infty} B_m(\beta_2) \left(\frac{1}{2x}\right)^m \tag{A.8}$$

with $B_m(\beta) = (-1)^m A_m(\beta)$ and,

$$A_m(\beta) = \frac{\Gamma(1/2 + \beta + m)}{m! \Gamma(1/2 + \beta - m)}, \tag{A.9}$$

and for $x << 1$ each modified Bessel function is,

$$I_\nu(x) = x^\nu \left(\frac{2^{-\nu}}{\Gamma(\nu+1)} + O\left(x^2\right)\right), \tag{A.10}$$

$$K_\nu(x) = x^\nu \left(2^{-\nu-1}\Gamma(-\nu) + O\left(x^2\right)\right) + x^{-\nu}\left(2^{\nu-1}\Gamma(\nu) + O\left(x^2\right)\right). \tag{A.11}$$

A.2 The (Incomplete-)Gamma Function

The Gamma function may be defined as,

$$\Gamma(z) = \int_0^\infty e^{-u} u^{z-1} du, \tag{A.12}$$

and is analytic for any z except for $z = 0, -1, -2, \ldots$.

The logarithmic derivative of the Gamma function is known as the digamma function,

$$\psi^{(0)}(z) = \frac{d}{dz} \log(\Gamma(z)). \tag{A.13}$$

While polygamma functions are defined as,

$$\psi^{(n)}(z) = \frac{d^n}{dz^n} \psi^{(0)}(z). \tag{A.14}$$

We may define the upper incomplete Gamma function as,

$$\Gamma(n, x) = \int_x^\infty e^{-t} t^{n-1} dt. \tag{A.15}$$

For $n = 0, 1, 2, \ldots$, we can write,

$$a^{-n-1}\Gamma(n+1, ax) = e^{ax} \sum_{k=0}^{n} \frac{n!}{k!} \frac{a^k}{x^{n-k+1}} = \int_x^\infty t^n e^{-ax} dt, \quad x > 0. \tag{A.16}$$

We also have,

$$\Gamma(-n, x) = \frac{(-1)^n}{n!} \left[E_1(x) - e^{-x} \sum_{m=0}^{n-1} (-1)^m \frac{m!}{x^{m+1}} \right], \tag{A.17}$$

with

$$
\begin{aligned}
E_1(x) = -\mathrm{Ei}(-x) &= \int_x^\infty e^{-t}t^{-1}dt = \Gamma(0,x) \\
&= \gamma + \log x + \sum_{n=1}^{\infty} \frac{(-x)^n}{n!n}.
\end{aligned} \tag{A.18}
$$

with γ the Euler-Mascheroni constant.

By using $\Gamma(z+1) = z\Gamma(z)$ we can write,

$$
\Gamma(\epsilon - n) = \frac{\Gamma(\epsilon+1)}{\epsilon(\epsilon-1)\ldots(\epsilon-n)}
$$

Expanding around $\epsilon \to 0$ we have,

$$
\Gamma(\epsilon - n) = \frac{(-1)^n}{\epsilon n!} + \frac{(-1)^n}{n!}\psi^{(0)}(n+1) + +O\left(\epsilon\right),
$$

where $\psi^{(n)}$ is the n-polygamma function.

A.3 Hypergeometric function

The hypergeometric function is defined as,

$$
{}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \quad |z| < 1 \tag{A.19}
$$

, where $(a)_n$ is the rising Pochhammer symbol defined as,

$$
(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{A.20}
$$

The Euler transformation allows us to write,

$$
{}_2F_1(a, b; c; z) = (1 - z)^{c-a-b}\, {}_2F_1(c - a, c - b, c, z). \tag{A.21}
$$

For $z = 1$, the hypergeometric function is

$$
{}_2F_1(a, b; c; 1) = \frac{\Gamma(c - a - b)\Gamma(c)}{\Gamma(c - a)\Gamma(c - b)}, \quad c > a + b. \tag{A.22}
$$

However, we may use the analytic continuation to $c - a - b$ non-negative integer.

Appendix B

Propagators in momentum space

We have two types of propagators. The bulk to boundary propagator is the solution to

$$(-\Box + m^2)\Phi(z, \vec{x}) = 0, \tag{B.1}$$

with boundary condition, in Poincare coordinates, $\Phi(z \to 0, \vec{x}) = z^{d-\Delta}\phi_0(\vec{x})$, where $\phi_0(\vec{x})$ is an arbitrary function, we also demand regularity in the interior.

On the other hand, the bulk-to-bulk propagator comes from solving the non-homogeneous equation $(-\Box + m^2)\Phi(z, \vec{x}) = f(\Phi)$ by solving the Green function equation

$$(-\Box + m^2)G(x_1, x_2) = \frac{1}{\sqrt{g}}\delta(x_1 - \vec{x}_2). \tag{B.2}$$

demanding that the Green function $G(x_1, x_2)$ vanishes when $z_1 \to 0$ and $z_1 \to \infty$.

Considering the field $\Phi(z, \vec{x})$ as the Fourier transform of the field $\Phi(z, \vec{p})$. The equation becomes

$$L_{d,\Delta}\Phi(z, \vec{p}) = 0, \tag{B.3}$$

$$L_{d,\Delta}\Phi(z, \vec{p}) = f(\Phi(z, \vec{p})) \tag{B.4}$$

where

$$L_{d,\Delta} = -z^2\partial_z^2 + (d-1)z\partial_z + m^2 + z^2p^2. \tag{B.5}$$

B.1 Bulk to Boundary Propagator

Doing $\Phi(z,\vec{p}) = z^{d/2}f(z,\vec{p})$ the equation (B.3) becomes a Bessel-type equation over $f(z,\vec{p})$.

The explicit solution to (B.3) is

$$\Phi(z,\vec{p}) = C_\Delta(\vec{p})z^{d/2}K_{\Delta-d/2}(pz) + B_\Delta(\vec{p})z^{d/2}I_{\Delta-d/2}(pz), \tag{B.6}$$

where K and I are the modified Bessel function, and B_Δ and C_Δ to be determined. The Δ parameter solves $\Delta(\Delta-d) = m^2$. As a quadratic equation, it has two solutions, namely Δ_+ and Δ_- where $\Delta_+ = d - \Delta_+$. Along the next line, we will call $\Delta_+ = \Delta > \frac{d}{2}$.

For the free scalar field, we demand Dirichlet boundary condition and regularity in the interior

$$\Phi(z\to 0,\vec{p}) = z^{d-\Delta}\phi_0(\vec{p}) + \mathcal{O}(z^2), \quad \Phi(z\to\infty,\vec{p}) = 0. \tag{B.7}$$

When $z\to\infty$, the $I(z,\vec{p}) \sim e^{pz}$ so to ensure regularity in the interior we have $B_\Delta(\vec{p}) = 0$. So, we are left with

$$\Phi(z,\vec{p}) = C_\Delta(\vec{p})z^{d/2}K_{\Delta-d/2}(pz), \tag{B.8}$$

looking at the limit of $z \to 0$, we have

$$\begin{aligned}\Phi(z \to 0, \vec{p}) &= C_\Delta p^{\Delta - \frac{d}{2}} z^\Delta \left(2^{\frac{d}{2}-\Delta-1}\Gamma\left(\frac{d}{2}-\Delta\right) + \mathcal{O}\left(z^2\right)\right) \\ &+ C_\Delta z^{\frac{d}{2}} p^{\frac{d}{2}-\Delta}\left(2^{-\frac{d}{2}+\Delta-1}\Gamma\left(\Delta-\frac{d}{2}\right)+\mathcal{O}\left(z^2\right)\right) \\ &= z^{d-\Delta}\phi_0(\vec{p}).\end{aligned} \tag{B.9}$$

The first line is sub-leading in z, so the second line gives the relevant term in the limit. From here, we can read the constant to be

$$C_\Delta = p^{\Delta-\frac{d}{2}} \frac{2^{\frac{d}{2}-\Delta+1}}{\Gamma\left(\Delta-\frac{d}{2}\right)}. \tag{B.10}$$

Finally, the solution to the homogeneous equation with the appropriate boundary condition is

$$\Phi(z, \vec{p}) = \mathcal{K}_\Delta(z, p)\phi_0(\vec{p}), \tag{B.11}$$

where

$$\mathcal{K}_\Delta(z, p) = \frac{2^{d/2-\Delta+1}}{\Gamma\left(\Delta-\frac{d}{2}\right)} p^{\Delta-\frac{d}{2}} z^{\frac{d}{2}} K_{\Delta-\frac{d}{2}}(pz) \tag{B.12}$$

is the bulk to boundary propagator and satisfies,

$$L_{d\Delta} K_\Delta = 0, \tag{B.13}$$

$$\lim_{z\to 0} \mathcal{K}_\Delta(z, p) = z^{d-\Delta}, \tag{B.14}$$

$$\mathcal{K}_\Delta(z \to \infty, p) = 0. \tag{B.15}$$

B.2 The bulk to bulk propagator

To solve (B.4), we will consider the equation

$$L^z_{d,\Delta}\mathcal{G}_\Delta(z,\vec{p},\zeta) = \frac{\delta(z-\zeta)}{\sqrt{g_\zeta}}, \quad g_\zeta = g(\zeta), \tag{B.16}$$

with the conditions

$$\lim_{z\to 0}\left[z^{\Delta-d}\mathcal{G}_\Delta(z,p;\zeta)\right] = 0, \quad \mathcal{G}_\Delta(z\to\infty,p;\zeta) = 0 \tag{B.17}$$

where $L^z_{d,\Delta}$ denotes the operator $L_{d,\Delta}$ acting with the z coordinate.

First consider $z \neq \zeta$, then the equation becomes

$$L^z_{d,\Delta}\mathcal{G}_\Delta(z,\vec{p},\zeta) = 0 \tag{B.18}$$

which is essentially (B.3). The solution is

$$\mathcal{G}_\Delta(z,\vec{p};\zeta) = A_\Delta(\zeta,\vec{p})z^{d/2}I_{\Delta-d/2}(pz) + B_\Delta(\zeta,\vec{p})z^{d/2}K_{\Delta-d/2}(pz). \tag{B.19}$$

From the condition $z\to 0$ and $z\neq\zeta$ we have $z<\zeta$. In this regime, the non-zero solution is

$$\mathcal{G}^{z<\zeta}_\Delta(z,\vec{p};\zeta) = A_\Delta(\zeta,\vec{p})z^{d/2}I_{\Delta-d/2}(pz), \tag{B.20}$$

while if $z\to\infty$, lead us to

$$\mathcal{G}^{z>\zeta}_\Delta(z,\vec{p};\zeta) = B_\Delta(\zeta,\vec{p})z^{d/2}K_{\Delta-d/2}(pz), \tag{B.21}$$

demanding continuity in $z=\zeta$ we have

$$\mathcal{G}_\Delta(z,\vec{p},z) = A_\Delta(z,\vec{p})z^{d/2}I_{\Delta-d/2}(pz) = B_\Delta(z,\vec{p})z^{d/2}K_{\Delta-d/2}(pz). \tag{B.22}$$

Here is easy to read

$$B_\Delta(z,\vec{p}) = A_\Delta(z,\vec{p})\frac{I_{\Delta-d/2}(pz)}{K_{\Delta-d/2}(pz)}. \tag{B.23}$$

To find A consider (B.16). It is convenient to define $\mathcal{G}_\Delta(z,\vec{p},\zeta) = z^{d/2}g_\Delta(z,\vec{p},\zeta)$, where g_Δ can be read from, either (B.20) or (B.21). So (B.16) becomes

$$\partial_z(z\partial_z g_\Delta) - z\left(\frac{m^2+d^2/4}{z}+p^2\right)g_\Delta = -z^{-d/2-1}\zeta^{d+1}\delta(\zeta-z). \tag{B.24}$$

Integrating over z between $\zeta-\epsilon$ and $\zeta+\epsilon$, we find the discontinuity slope

$$z\partial_z g_\Delta(z,p,\zeta)\Big|_{z=\zeta+\epsilon} - z\partial_z g_\Delta(z,p,\zeta)\Big|_{z=\zeta-\epsilon} = -\zeta^{d/2}. \tag{B.25}$$

Taking the limit of $\epsilon\to 0$, we find

$$A_\Delta(\zeta,\vec{p}) = A_\Delta(\zeta,p) = \zeta^{d/2}K_{\Delta-\frac{d}{2}}(p\zeta). \tag{B.26}$$

So the function $\mathcal{G}_\Delta(z,p,\zeta)$ has been fully determined and is,

$$\mathcal{G}_\Delta(z,k,\zeta) = \begin{cases} (z\zeta)^{d/2}I_{\Delta-d/2}(kz)K_{\Delta-d/2}(k\zeta) & \text{for } z\le\zeta, \\ (z\zeta)^{d/2}I_{\Delta-d/2}(k\zeta)K_{\Delta-d/2}(kz) & \text{for } z>\zeta. \end{cases} \tag{B.27}$$

Appendix C

Integrals

Along the main text, we used several integrals. In this appendix, we will prove or quote some of our principal integrals. For some, we give an alternative proof of the result presented in the main text.

C.1 Position space: Master integral

We want to compute the following integral

$$I = I_{\alpha\beta\gamma}(\vec{a}, \vec{b}, \vec{c}) = \int_0^\infty dz \int_{\mathbb{R}^d} d\vec{x} \frac{z^\alpha}{[z^2 + |\vec{x} - \vec{a}|^2]^\beta \, [z^2 + b^2 + |\vec{x} - \vec{c}|^2]^\gamma}. \tag{C.1}$$

doing a Schwinger parametrization which is given by

$$\frac{1}{A^n} = \frac{1}{\Gamma(n)} \int_0^\infty du u^{n-1} e^{-uA}, \tag{C.2}$$

we have

$$I = \frac{1}{\Gamma(\beta)\Gamma(\gamma)} \int_0^\infty dudwdz \int_{\mathbb{R}^d} d\vec{x} z^\alpha u^{\beta-1} w^{\gamma-1} e^{-z^2(u+w)} e^{-wb^2} e^{-u(\vec{x}-\vec{a})^2 - w(\vec{x}-\vec{c})^2}. \tag{C.3}$$

The z integral is, after a change of variable, a Gamma function, and the $\vec{x}$ integral can be done by completing squares. The result of each integral is

$$\int dz z^{\alpha} e^{-z^2(u+w)} = \frac{1}{2}\Gamma\left(\frac{\alpha+1}{2}\right)\frac{1}{(u+w)^{\frac{\alpha}{2}+\frac{1}{2}}}, \tag{C.4}$$

$$\int_{\mathbb{R}^d} d\vec{x} e^{-u(\vec{x}-\vec{a})^2-w(\vec{x}-\vec{c})^2} = \frac{\pi^{\frac{d}{2}}}{(u+w)^{\frac{d}{2}}} e^{-\frac{uw(a-c)^2}{u+w}}. \tag{C.5}$$

Then we are left with

$$I = \frac{\pi^{\frac{d}{2}}\Gamma\left(\frac{\alpha+1}{2}\right)}{2\Gamma(\beta)\Gamma(\gamma)}\int_0^{\infty} dudw \frac{u^{\beta-1}w^{\gamma-1}}{(u+w)^{\frac{\alpha+d+1}{2}}} e^{-wb^2-\frac{uw(a-c)^2}{u+w}}. \tag{C.6}$$

For simplicity lets call $A = \frac{\alpha+d+1}{2}$. Now we work the double integral

$$f = \int_0^{\infty} dudw \frac{u^{\beta-1}w^{\gamma-1}}{(u+w)^A} e^{-wb^2-\frac{uw(a-c)^2}{u+w}}, \tag{C.7}$$

doing $u \to z = \frac{u}{w+u}$ we have

$$f = \int_0^1 dz z^{\beta-1}(1-z)^{A-\beta-1} \underbrace{\int_0^{\infty} dw w^{\beta+\gamma-A-1} e^{-w(b^2+(c-a)^2 z)}}_{\Gamma(-A+\beta+\gamma)(z(a-c)^2+b^2)^{A-\beta-\gamma}} \tag{C.8}$$

$$= \Gamma\left(\beta+\gamma-A\right)\int_0^1 dz z^{\beta-1}\frac{(1-z)^{A-\beta-1}}{(b^2+(a-c)^2 z)^{\beta+\gamma-A}}. \tag{C.9}$$

Using the following representation of the hypergeometric function

$${}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 dz z^{b-1}\frac{(1-z)^{-b+c-1}}{(1-xz)^a}, \tag{C.10}$$

we finally have

$$
\begin{aligned}
I_{\alpha\beta\gamma}(\vec{a},\vec{b},\vec{c}) &= \int_0^\infty dz \int_{\mathbb{R}^d} d\vec{x} \frac{z^\alpha}{\left[z^2+|\vec{x}-\vec{a}|^2\right]^\beta \left[z^2+b^2+|\vec{x}-\vec{c}|^2\right]^\gamma} \\
&= \frac{\pi^{\frac{d}{2}}\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\beta+\gamma-\frac{\alpha+d+1}{2}\right)\Gamma\left(\frac{\alpha+d+1}{2}-\beta\right)}{2\Gamma(\gamma)\Gamma\left(\frac{\alpha+d+1}{2}\right)b^{2\beta+2\gamma-\alpha-d-1}} \\
&\quad {}_2F_1\left(\beta+\gamma-\frac{\alpha+d+1}{2},\beta,\frac{\alpha+d+1}{2},\frac{(a-c)^2}{b^2}\right).
\end{aligned}
\tag{C.11}
$$

C.2 Alternative Proof of GK integral

This proof relies upon using the quantum effective action worked in 4.5.2.

If we directly solve the equation (4.156), the solution is

$$
\Phi_{1,s}(z,\vec{x}) = \Phi_{1,s}^{\{1\}} + \Phi_{1,s}^{\{3\}}, \tag{C.12}
$$

$$
\Phi_{1,s}^{\{1\}} = -\int dx' G_{\Delta_s}(x,x')\left(a_1+\frac{1}{2}G_{\Delta_s}(1)\right)\Phi_{0,s}, \tag{C.13}
$$

$$
\Phi_{1,s}^{\{3\}} = -\frac{1}{3!}\int dx' G_{\Delta_s}(x,x')\left(\Phi_{0,s}(x')\right)^3. \tag{C.14}
$$

The superscript $\{\cdot\}$ denotes the power on Φ_0. Here, we can choose the counterterm a_1 such that Φ_1 is finite. This depends on the reduction scheme. For instance, if we choose to have

$$
a_1 = -\frac{1}{2}\text{div}\left[G_{\Delta_s}(1)\right], \tag{C.15}
$$

i.e a_1 only cancel the purely divergent piece of $G_{\Delta_s}(1)$. We will still have the finite part of $G_{\Delta_s}(1)$. Then, we will have

$$\Phi_{1,s} = \Phi_{1,s}^{\{1\}} + \Phi_{1,s}^{\{3\}}, \tag{C.16}$$

$$\Phi_{1,s}^{\{1\}} = -\text{conv}\left[\frac{1}{2}G_{\Delta_s}(1)\right]\int dx' G_{\Delta_s}(x,x')\Phi_{0,s}(x'), \tag{C.17}$$

$$\Phi_{1,s}^{\{3\}} = -\frac{1}{3!}\int dx' G_{\Delta_s}(x,x')\left(\Phi_{0,s}(x')\right)^3. \tag{C.18}$$

The full field is (at this point $\lambda = \lambda_s = \lambda_r$)

$$\begin{aligned}\Phi_s &= \Phi_{0,s} + \lambda_s \Phi_{1,s} \\ &= \int d\vec{y} K_{\Delta_s}(z, \vec{x}-\vec{y})\phi_0^{(s)}(\vec{y}) - \frac{\lambda_s}{2}\text{conv}\left[G_{\Delta_s}(1)\right]\int dx' G_{\Delta_s}(x,x')\Phi_{0,s}(x') \\ &- \frac{\lambda_s}{3!}\int dx' G_{\Delta_s}(x,x')\left(\Phi_{0,s}(x')\right)^3,\end{aligned} \tag{C.19}$$

where $\Phi^{(s)}$ is the field built with the propagator given by the subtracted mass Δ_s. This field is already UV renormalized and, therefore, UV finite.

It turns out to be that (C.19), and (4.164) must be equals, but their solutions are built with different sources, i.e. $\phi_0^{(s)}$ and $\phi_0^{(s)}$.

So, we have to find how both sources are related. This is done by comparing the asymptotic behaviour of both fields

$$\Phi_s(\epsilon \to 0, \vec{x}) = \epsilon^{d-\Delta_s}\phi_0^{(s)} + \ldots, \quad \Phi_r(\epsilon \to 0, \vec{x}) = \epsilon^{d-\Delta_r}\phi_0^{(r)} + \ldots. \tag{C.20}$$

As $\Phi_s = \Phi_r$,

$$\phi_0^{(s)} = \epsilon^{\Delta_s - \Delta_r}\phi_0^{(r)}, \tag{C.21}$$

using (4.162)

$$m_s^2 = m_r^2 - \frac{\lambda}{2}\text{conv}\left[G_{\Delta_r}(1)\right] \Rightarrow \Delta_s = \Delta_r - \lambda\frac{\text{conv}\left[G_{\Delta_r}(1)\right]}{2(2\Delta_r - d)}, \tag{C.22}$$

then,

$$\phi_0^{(s)} = \phi_0^{(r)} - \lambda_s\frac{\text{conv}\left[G_{\Delta_r}(1)\right]}{2(2\Delta_r - d)}\log(\epsilon)\phi_0^{(r)} + \mathcal{O}(\lambda_s^2). \tag{C.23}$$

So, expanding the field $\Phi^{(s)}$ given by (C.19) in terms of the renormalized parameters and source we find

$$\begin{aligned}\Phi_s = \Phi_0^{(r)} &- \frac{\lambda_s}{3!}\int dx' G_{\Delta_r}(x,x')\left(\Phi_{0,r}(x')\right)^3 \\ &- \lambda_s\frac{\text{conv}\left[G_{\Delta_r}(1)\right]}{2(2\Delta_r - d)}\log(\epsilon)\int d\vec{y}K_{\Delta_r}(z,\vec{x}-\vec{y})\phi_0^{(r)}(\vec{y}) \\ &- \lambda_s\frac{\text{conv}\left[G_{\Delta_r}(1)\right]}{2(2\Delta_r - d)}\int d\vec{y}\frac{d}{d\Delta_s}K_{\Delta_s}(z,\vec{x}-\vec{y})\Big|_{\Delta_s=\Delta_r}\phi_0^{(r)}(\vec{y}) \\ &- \frac{\lambda_s}{2}\text{conv}\left[G_{\Delta_r}(1)\right]\int d\vec{y}\int dx' G_{\Delta_r}(x,x')K_{\Delta_r}(z',\vec{x}'-\vec{y})\phi_0^{(r)}(\vec{y}).\end{aligned}$$

Comparing with (4.164) and computing the derivative of the bulk-to-boundary propagator, we will have

$$\begin{aligned}&\int d\vec{y}\phi_0^{(r)}(\vec{y})\left[\int dx' G_{\Delta_r}(z,\vec{x};z',\vec{x}')K_{\Delta_r}(z',\vec{x}'-\vec{y})\right. \\ &\left.+\frac{1}{2\Delta_r - d}K_{\Delta_r}(z,\vec{x}-\vec{y})\left[\log\left(\frac{z\epsilon}{(\vec{x}-\vec{y})^2+z^2}\right) - \psi^{(0)}\left(\Delta_r - \frac{d}{2}\right) + \psi^{(0)}(\Delta_r)\right]\right] = 0.\end{aligned}$$

As $\phi_0(\vec{y})$ is arbitrary, then we find

$$\begin{aligned}&\int dx' G_{\Delta_r}(z,\vec{x};z',\vec{x}')K_{\Delta_r}(z',\vec{x}'-\vec{y}) \\ &= -\frac{1}{2\Delta_r - d}K_{\Delta_r}(z,\vec{x}-\vec{y})\left[\log\left(\frac{z\epsilon}{(\vec{x}-\vec{y})^2+z^2}\right) - \psi^{(0)}\left(\Delta_r - \frac{d}{2}\right) + \psi^{(0)}(\Delta_r)\right]\end{aligned} \tag{C.24}$$

which is exactly the result obtained in (4.18).

C.3 Proof of equation (4.153)

The proof relies on the free path integral. We can take the mass counterterm as part of the propagator or as an interaction.

The starting point is the partition function

$$Z[J] = \int D\Phi e^{-\int dx \frac{1}{2}\Phi(-\Box+m^2)\Phi+J\Phi}. \tag{C.25}$$

We will work it in two different ways:

1. Compute the path integral for the mass m^2 that gives

$$Z_1[J] = \frac{1}{\sqrt{\det(-\Box+m^2)}} e^{\frac{1}{2}\int dx_1 dx_2 J(x_1)G(x_1,x_2)J(x_2)}, \tag{C.26}$$

   where $G(x_1, x_2)$ is defined through, $(-\Box_1 + m^2)G(x_1, x_2) = \delta(x_1, x_2)$.

2. Consider $m^2 = m_0^2 + \delta_m$, and use the $\delta_m \Phi^2$ term in the action as interaction

$$Z_2[J] = \frac{1}{\sqrt{\det(-\Box+m_0^2)}} e^{-\frac{\delta_m}{2}\int dx \frac{\delta^2}{\delta J^2}} e^{\frac{1}{2}\int dx_1 dx_2 J(x_1)G_0(x_1,x_2)J(x_2)} \tag{C.27}$$

   where $G_0(x_1, x_2)$ is defined through, $(-\Box_1 + m_0^2)G_0(x_1, x_2) = \delta(x_1, x_2)$.

Notice that the normalisation factor is different in both cases. It is easier to work with the connected diagram generator, $W[J] = \log Z[J]$.

Expanding the propagator

First, we will solve the path integral using the mass parameter m^2. The partition function (C.26)

$$W_1[J] = -\frac{1}{2}\log\left(\det\left(-\Box + m^2\right)\right) + \frac{1}{2}\int dxdyJ(x)G_\Delta(x,y)J(y), \tag{C.28}$$

where $G_\Delta(x,y)$ is the bulk-to-bulk propagator with parameter $\Delta(\Delta - d) = m^2$. The mass is $m^2 = m_0 + \delta_m$. So we have to expand the determinant and the bulk-to-bulk propagator.

- The determinant becomes

$$\begin{aligned}\det\left(-\Box + m_0^2 + \delta_m\right) &= \det\left(\left(-\Box_x + m_0^2\right)\left(1 + \delta_m \int dyG_{\Delta_0}(x,y)\right)\right) \\ &= \det\left(-\Box_x + m_0^2\right)\det\left(1 + \delta_m\int dyG_{\Delta_0}(x,y)\right)\end{aligned} \tag{C.29}$$

  where we made explicit that the differential operator acts on the x coordinates. So, using $\log(\det(A)\det(B)) = \log(\det(A)) + \log(\det(B))$ and using $\log(\det(B)) = \mathrm{tr}(\log(B))$ on the second term, we find

$$\log\left(\det(-\Box + m^2)\right) = \log\left(\det(-\Box + m_0^2)\right) + \mathrm{tr}\left(\log\left(1 + \delta_m\int dyG_{\Delta_0}(x,y)\right)\right). \tag{C.30}$$

  Expanding in small δ_m and using the linearity on the trace, we finally get

$$\log\left(\det(-\Box + m^2)\right) = \log\left(\det(-\Box + m_0^2)\right) + \delta_m\mathrm{tr}\left(\int dyG_{\Delta_0}(x,y)\right).$$

- Expanding the solution for Δ

$$\Delta = \Delta_0 + \frac{\delta_m}{2\Delta_0 - d}, \tag{C.31}$$
$$G_\Delta(x,y) = G_{\Delta_0}(x,y) + \frac{\delta_m}{2\Delta_0 - d}\frac{d}{d\Delta}G_\Delta(x,y)\Big|_{\Delta=\Delta_0}.$$

So, the partition function is

$$W_1[J] = -\frac{1}{2}\log\left(\det(-\Box + m_0^2)\right) + \frac{1}{2}\int dxdyJ(x)G_{\Delta_0}(x,y)J(y) \tag{C.32}$$
$$-\frac{\delta_m}{2}\left(\int dxG_{\Delta_0}(x,x) - \frac{1}{(2\Delta_0 - d)}\frac{d}{d\Delta}\left[\int dxdyJ(x)G_\Delta(x,y)J(y)\right]\Big|_{\Delta=\Delta_0}\right).$$

As interaction

If we consider the mass parameter to be $m^2 = m_0^2 + \delta_m$, we have,

$$Z[J] = \frac{1}{\sqrt{\det\left(-\Box + m_0^2\right)}}e^{-\frac{\delta_m}{2}\int dx\frac{\delta^2}{\delta J^2}}\int D\Phi e^{-\frac{1}{2}\int dx\Phi(-\Box+m_0^2)\Phi + J\Phi} \tag{C.33}$$
$$= \frac{1}{\sqrt{\det\left(-\Box + m_0^2\right)}}e^{-\frac{\delta_m}{2}\int dx\frac{\delta^2}{\delta J^2}}e^{\frac{1}{2}\int dxdyJ(x)G_{\Delta_0}(x,y)J(y)},$$

where $\Delta_0(\Delta_0 - d) = m_0^2$. Working the exponential with the counterterm, we have,

$$W_2[J] = -\frac{1}{2}\log\left(\det\left(-\Box + m_0^2\right)\right) + \int dxdyJ(x)G_{\Delta_0}(x,y)J(y) \tag{C.34}$$
$$-\frac{\delta_m}{2}\left[\int dxG_{\Delta_0}(x,x) + \int dxdx_1dx_2G_{\Delta_0}(x,x_1)G_{\Delta_0}(x,x_2)J(x_1)J(x_2)\right].$$

$W_1 = W_2$

By construction, $W_1[J] = W_2[J]$, then to first order in δ_m and computing two derivatives with respect to the source J we find

$$\frac{1}{(2\Delta_0 - d)} \frac{d}{d\Delta} G_\Delta(x_1, x_2)\Big|_{\Delta=\Delta_0} = -\int dx G_{\Delta_0}(x, x_1) G_{\Delta_0}(x, x_2). \tag{C.35}$$

An important case is to consider $x_1 = x_2$. Of course, this demands to consider the regulated bulk-to-boundary propagator

$$-\frac{1}{(2\Delta - d)} \frac{d}{d\Delta} G(1) = \int dx G_\Delta(x_1, x) G_\Delta(x, x_1). \tag{C.36}$$

The r.h.s integral is the known integral that appears in the eight-diagram.

This computation gives an ultra-simple way to compute the integral (C.35) that will be useful for the triple tadpole diagram. However, this computation goes beyond the scope of this work.

C.4 Integrals over Bessel functions

A useful integral consist is the integral (5.2.1) for $\alpha = -1$. This integral diverges in the lower limit, so we consider the regulated integral. The result is obtained using Mathematica [83].

$$\int_{\epsilon}^{\infty} x^{-1} K_\nu(x) K_\nu(x)\, dx = \frac{\csc(\pi\nu)}{8\nu} \left(\pi^2 \nu \epsilon^2 \csc(\pi\nu)\, {}_3\tilde{F}_4 \left(1, 1, \frac{3}{2}; 2, 2, 2-\nu, \nu+2; \epsilon^2\right) \right. \tag{C.37}$$

$$+4^{-\nu} \sin(\pi\nu) \epsilon^{-2\nu} \left(16^\nu \Gamma(\nu)^2\, {}_2F_3 \left(\frac{1}{2} - \nu, -\nu; 1-2\nu, 1-\nu, 1-\nu; \epsilon^2\right)\right.$$

$$\left. -\Gamma(-\nu)^2 \epsilon^{4\nu}\, {}_2F_3 \left(\nu, \nu + \frac{1}{2}; \nu+1, \nu+1, 2\nu+1; \epsilon^2\right)\right)$$

$$\left. -2\pi(\psi^{(0)}(-\nu) + \psi^{(0)}(\nu) - 2\log(\epsilon) + \log(4))\right).$$

We shall notice that the result only holds for $\nu \notin \mathbb{Z}$.

This integral is useful for studying the regulated integral of two bulk-to-boundary propagators colliding into a bulk point. To obtain the position space result (4.11) we have to take the limit of $\epsilon \to 0$ and use[97]

$$\int d\vec{k} k^{2\nu} e^{i\vec{k}\cdot\vec{x}} = \frac{\Gamma\left(\nu + \frac{d}{2}\right)}{\Gamma(-\nu)} \frac{\pi^{\frac{d}{2}} 2^{2\nu+d}}{|\vec{x}|^{2\nu+2}}, \tag{C.38}$$

$$\int d\vec{k} k^{2\nu} \log(k) e^{i\vec{k}\cdot\vec{x}} = \frac{\Gamma\left(\nu + \frac{d}{2}\right)}{\Gamma(-\nu)} \frac{\pi^{\frac{d}{2}} 2^{2\nu+d}}{|\vec{x}|^{2\nu+2}} \left(\log(2) - \log(|\vec{x}|) + \frac{\psi\left(\nu + \frac{d}{2}\right) + \psi(-\nu)}{2}\right).$$

From here we can obtain the result of (4.11).

Bibliography

[1] I. Newton. *Philosophiae naturalis principia mathematica.* J. Societatis Regiae ac Typis J. Streater, 1687.

[2] Albert Einstein. Über Gravitationswellen. *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin*, pages 154–167, January 1918.

[3] Karl Schwarzschild. Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie. *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin*, pages 189–196, January 1916.

[4] Roger Penrose. Gravitational collapse and space-time singularities. *Phys. Rev. Lett.*, 14:57–59, Jan 1965.

[5] A. M. Ghez, B. L. Klein, M. Morris, and E. E. Becklin. High proper motion stars in the vicinity of Sgr A*: Evidence for a supermassive black hole at the center of our galaxy. *Astrophys. J.*, 509:678–686, 1998.

[6] A. Eckart and R. Genzel. Observations of stellar proper motions near the Galactic Centre. *Nature*, 383:415–417, 1996.

[7] Kazunori Akiyama et al. First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole. *Astrophys. J. Lett.*, 875:L1, 2019.

[8] Albert Einstein. Näherungsweise Integration der Feldgleichungen der Gravitation. *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin*, pages 688–696, January 1916.

[9] Abbott and et al. Observation of gravitational waves from a binary black hole merger. *Phys. Rev. Lett.*, 116:061102, Feb 2016.

[10] C. W. F. Everitt and et al. Gravity probe b: Final results of a space experiment to test general relativity. *Physical Review Letters*, 106(22), may 2011.

[11] Peter W. Higgs. Broken symmetries and the masses of gauge bosons. *Phys. Rev. Lett.*, 13:508–509, Oct 1964.

[12] F. Englert and R. Brout. Broken symmetry and the mass of gauge vector mesons. *Phys. Rev. Lett.*, 13:321–323, Aug 1964.

[13] G. S. Guralnik, C. R. Hagen, and T. W. B. Kibble. Global conservation laws and massless particles. *Phys. Rev. Lett.*, 13:585–587, Nov 1964.

[14] G. Aad. and et al. Observation of a new particle in the search for the standard model higgs boson with the ATLAS detector at the LHC. *Physics Letters B*, 716(1):1–29, sep 2012.

[15] S. Chatrchyan. and et al. Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC. *Physics Letters B*, 716(1):30–61, sep 2012.

[16] Stephen Hawking and W. Israel. *General Relativity: an Einstein Centenary Survey.* 2010.

[17] J. D. Bekenstein. Black holes and the second law. 4, 737-740, 1972.

[18] Jacob D. Bekenstein. Black holes and entropy. *Phys. Rev. D*, 7:2333–2346, Apr 1973.

[19] S. W. Hawking. Particle creation by black holes. 43, 199-220, 1975.

[20] Andrew Strominger and Cumrun Vafa. Microscopic origin of the bekenstein-hawking entropy. *Physics Letters B*, 379(1-4):99–104, jun 1996.

[21] G. 't Hooft. Dimensional reduction in quantum gravity. 930308, 284-286, 1993.

[22] L. Susskind. The world as a hologram. 36, 6377, 1995.

[23] J. M. Maldacena. The large-n limit of superconformal field theories and supergravity. 38, 1113-1133, 1999.

[24] E. Witten. Anti-de sitter space and holography. 2, 253-291, 1998.

[25] S. S. Gubser, Igor R. Klebanov, and Alexander M. Polyakov. Gauge theory correlators from noncritical string theory. *Phys. Lett. B*, 428:105–114, 1998.

[26] Igor R. Klebanov and Edward Witten. AdS / CFT correspondence and symmetry breaking. *Nucl. Phys. B*, 556:89–114, 1999.

[27] M. Henningson and K. Skenderis. The Holographic Weyl anomaly. *JHEP*, 07:023, 1998.

[28] Mans Henningson and Kostas Skenderis. Holography and the Weyl anomaly. *Fortsch. Phys.*, 48:125–128, 2000.

[29] Ioannis Papadimitriou and Kostas Skenderis. Thermodynamics of asymptotically locally AdS spacetimes. *JHEP*, 08:004, 2005.

[30] Marc Henneaux. Boundary terms in the AdS / CFT correspondence for spinor fields. In *International Meeting on Mathematical Methods in Modern Theoretical Physics (ISPM 98)*, pages 161–170, 9 1998.

[31] Tomas Andrade, Maximo Banados, and Francisco Rojas. Variational Methods in AdS/CFT. *Phys. Rev. D*, 75:065013, 2007.

[32] K. Skenderis. Lecture notes on holographic renormalization. 19, 5849-5876, 2002.

[33] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz. Large n field theories, string theory and gravity. 323, 183-386, 2000.

[34] Horatiu Nastase. *Introduction to the AdS/CFT Correspondence*. Cambridge University Press, 2015.

[35] Joao Penedones. Writing CFT correlation functions as AdS scattering amplitudes. *JHEP*, 03:025, 2011.

[36] A. Liam Fitzpatrick and Jared Kaplan. Unitarity and the Holographic S-Matrix. *JHEP*, 10:032, 2012.

[37] Ofer Aharony, Luis F. Alday, Agnese Bissi, and Eric Perlmutter. Loops in AdS from Conformal Field Theory. *JHEP*, 07:036, 2017.

[38] Luis F. Alday and Agnese Bissi. Loop Corrections to Supergravity on $AdS_5 \times S^5$. *Phys. Rev. Lett.*, 119(17):171601, 2017.

[39] F. Aprile, J. M. Drummond, P. Heslop, and H. Paul. Quantum Gravity from Conformal Field Theory. *JHEP*, 01:035, 2018.

[40] F. Aprile, J. M. Drummond, P. Heslop, and H. Paul. Unmixing Supergravity. *JHEP*, 02:133, 2018.

[41] Kausik Ghosh. Polyakov-Mellin Bootstrap for AdS loops. *JHEP*, 02:006, 2020.

[42] Alexandria Costantino and Sylvain Fichet. Opacity from Loops in AdS. *JHEP*, 02:089, 2021.

[43] Simone Giombi, Charlotte Sleight, and Massimo Taronna. Spinning AdS Loop Diagrams: Two Point Functions. *JHEP*, 06:030, 2018.

[44] Ellis Ye Yuan. Loops in the Bulk. 10 2017.

[45] Ellis Ye Yuan. Simplicity in AdS Perturbative Dynamics. 1 2018.

[46] Dmitry Ponomarev. From bulk loops to boundary large-N expansion. *JHEP*, 01:154, 2020.

[47] Dean Carmi. Loops in AdS: From the Spectral Representation to Position Space. *JHEP*, 06:049, 2020.

[48] David Meltzer, Eric Perlmutter, and Allic Sivaramakrishnan. Unitarity Methods in AdS/CFT. *JHEP*, 03:061, 2020.

[49] Soner Albayrak and Savan Kharel. Spinning loop amplitudes in anti–de Sitter space. *Phys. Rev. D*, 103(2):026004, 2021.

[50] Dean Carmi. Loops in AdS: From the Spectral Representation to Position Space II. 4 2021.

[51] Sylvain Fichet. Dressing in AdS and a Conformal Bethe-Salpeter Equation. 6 2021.

[52] Igor Bertan and Ivo Sachs. Loops in Anti–de Sitter Space. *Phys. Rev. Lett.*, 121(10):101601, 2018.

[53] Igor Bertan, Ivo Sachs, and Evgeny D. Skvortsov. Quantum ϕ^4 Theory in AdS_4 and its CFT Dual. *JHEP*, 02:099, 2019.

[54] Adam Bzowski, Paul McFadden, and Kostas Skenderis. Implications of conformal invariance in momentum space. *Journal of High Energy Physics*, 2014(3), mar 2014.

[55] Adam Bzowski, Paul McFadden, and Kostas Skenderis. Scalar 3-point functions in CFT: renormalisation, beta functions and anomalies. *JHEP*, 03:066, 2016.

[56] Adam Bzowski, Paul McFadden, and Kostas Skenderis. A handbook of holographic 4-point functions. 7 2022.

[57] Máximo Bañados, Ernesto Bianchi, Iván Muñoz, and Kostas Skenderis. Bulk renormalization and the AdS/CFT correspondence. 8 2022.

[58] Philippe Di Francesco, Pierre Mathieu, and David Sénéchal. *Conformal Field Theory*. Springer, 1997.

[59] Joseph Polchinski. *String Theory*, volume 1 of *Cambridge Monographs on Mathematical Physics*. Cambridge University Press, 1998.

[60] Paul H. Ginsparg. APPLIED CONFORMAL FIELD THEORY. In *Les Houches Summer School in Theoretical Physics: Fields, Strings, Critical Phenomena*, 9 1988.

[61] A. A. Belavin, Alexander M. Polyakov, and A. B. Zamolodchikov. Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory. *Nucl. Phys. B*, 241:333–380, 1984.

[62] J. David Brown and M. Henneaux. Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity. *Commun. Math. Phys.*, 104:207–226, 1986.

[63] Sebastian de Haro, Sergey N. Solodukhin, and Kostas Skenderis. Holographic reconstruction of space-time and renormalization in the AdS / CFT correspondence. *Commun. Math. Phys.*, 217:595–622, 2001.

[64] E. D'Hoker and D. Z. Freedman. Supersymmetric gauge theories and the ads/cft correspondence. 2001.

[65] D. Z. Freedman, S. D. Mathur, A. Matusis, and L. Rastelli. Correlation functions in the cft(d)/ads(d+1) correspondence. 546, 96-118, 1999.

[66] Daniel Z. Freedman, Samir D. Mathur, Alec Matusis, and Leonardo Rastelli. Comments on 4-point functions in the CFT/AdS correspondence. *Physics Letters B*, 452(1-2):61–68, apr 1999.

[67] E. D'Hoker, D. Z. Freedman, S. D. Mathur, A. Matusis, and L. Rastelli. Graviton exchange and complete four-point functions in the ads/cft correspondence. 562 (1-2), 353-394, 1999.

[68] W. Muck and K. S. Viswanathan. Conformal field theory correlators from classical scalar field theory on anti-de sitter space. 58, 041901, 1998.

[69] F. A. Dolan and Hugh Osborn. Conformal four point functions and the operator product expansion. 599 (1-2), 459-496, 2001.

[70] E. D'Hoker, D. Z. Freedman, and L. Rastelli. Ads/cft four-point functions: How to succeed at z-integrals without really trying. 562 (1-2), 395-411, 1999.

[71] T. S. Bunch and P. C. W. Davies. Quantum Field Theory in de Sitter Space: Renormalization by Point Splitting. *Proc. Roy. Soc. Lond. A*, 360:117–134, 1978.

[72] Jeffrey Goldstone, Abdus Salam, and Steven Weinberg. Broken symmetries. *Phys. Rev.*, 127:965–970, Aug 1962.

[73] C. DeWitt-Morette and B.S. DeWitt. *Relativite, Groupes Et Topologie: Lectures Delivered at Les Houches During the 1963 Session of the Summer School of Theoretical Physics, University of Grenoble.* École d'Été de Physique Théorique LesHouches: Session. Gordon and Breach, 1964.

[74] H. Osborn. Weyl consistency conditions and a local renormalisation group equation for general renormalisable field theories. *Nuclear Physics B*, 363(2):486–526, 1991.

[75] H. Osborn and A. Petkou. Implications of conformal invariance in field theories for general dimensions. *Annals of Physics*, 231(2):311–362, may 1994.

[76] Anastasios Petkou and Kostas Skenderis. A Nonrenormalization theorem for conformal anomalies. *Nucl. Phys. B*, 561:100–116, 1999.

[77] Adam Bzowski, Paul McFadden, and Kostas Skenderis. Renormalised 3-point functions of stress tensors and conserved currents in CFT. *Journal of High Energy Physics*, 2018(11), nov 2018.

[78] Adam Bzowski, Paul McFadden, and Kostas Skenderis. Renormalised CFT 3-point functions of scalars, currents and stress tensors. *Journal of High Energy Physics*, 2018(11), nov 2018.

[79] Adam Bzowski, Paul McFadden, and Kostas Skenderis. Conformal n-point functions in momentum space. *Phys. Rev. Lett.*, 124(13):131602, 2020.

[80] Adam Bzowski, Paul McFadden, and Kostas Skenderis. Conformal correlators as simplex integrals in momentum space. *JHEP*, 01:192, 2021.

[81] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products.* Elsevier/Academic Press, Amsterdam, seventh edition, 2007. Translated from the Russian, Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger, With one CD-ROM (Windows, Macintosh and UNIX).

[82]

[83] Wolfram Research, Inc. Mathematica, Version 13.1. Champaign, IL, 2022.

[84] Hong Liu and Arkady A. Tseytlin. On four point functions in the CFT / AdS correspondence. *Phys. Rev. D*, 59:086002, 1999.

[85] Hong Liu and Arkady A. Tseytlin. D = 4 superYang-Mills, D = 5 gauged supergravity, and D = 4 conformal supergravity. *Nucl. Phys. B*, 533:88–108, 1998.

[86] Eric D'Hoker, Daniel Z. Freedman, Samir D. Mathur, Alec Matusis, and Leonardo Rastelli. Graviton and gauge boson propagators in AdS(d+1). *Nucl. Phys. B*, 562:330–352, 1999.

[87] Michael Edward Peskin and Daniel V. Schroeder. *An Introduction to Quantum Field Theory*. Westview Press, 1995. Reading, USA: Addison-Wesley (1995) 842 p.

[88] Steven Weinberg. *The quantum theory of fields. Vol. 2: Modern applications*. 1996. Cambridge, UK: Univ. Pr. (1996) 489 p.

[89] Matthew D. Schwartz. *Quantum Field Theory and the Standard Model*. Cambridge University Press, 3 2014.

[90] Oliver DeWolfe and Daniel Z. Freedman. Notes on fluctuations and correlation functions in holographic renormalization group flows. 2 2000.

[91] Bruce Allen and Michael Turyn. An Evaluation of the Graviton Propagator in De Sitter Space. *Nucl. Phys. B*, 292:813, 1987.

[92] Michael Turyn. The Graviton Propagator in Maximally Symmetric Spaces. *J. Math. Phys.*, 31:669, 1990.

[93] Eduardo Casali, Walker Melton, and Andrew Strominger. Celestial Amplitudes as AdS-Witten Diagrams. 4 2022.

[94] Lorenzo Iacobacci, Charlotte Sleight, and Massimo Taronna. From Celestial Correlators to AdS, and back. 8 2022.

[95] Bateman Manuscript Project, H. Bateman, A. Erdélyi, and United States. Office of Naval Research. *Higher Transcendental Functions*. Number v. 2 in Higher Transcendental Functions. McGraw-Hill, 1953.

[96] G.N. Watson. *A Treatise on the Theory of Bessel Functions*. Cambridge Mathematical Library. Cambridge University Press, 1995.

[97] Monica Guica, Kostas Skenderis, Marika Taylor, and Balt C. van Rees. Holography for Schrodinger backgrounds. *JHEP*, 02:056, 2011.