PONTIFICIA UNIVERSIDAD CATOLICA DE CHILE
FACULTAD DE FISICA
INSTITUTO DE FISICA

Exploring the landscape of Very Special Relativity

by

Alex Soto Villarroel

Thesis submitted to the Faculty of Physics
of Pontificia Universidad Católica de Chile, as one of the requirements
to qualify for the academic PhD's degree in Physics.

Supervisor	:	Dr. Jorge Alfaro (PUC)
Committee	:	Dr. Benjamin Koch (PUC)
		Dr. Gonzalo Palma (U. de Chile)
		Dr. Alvaro Restuccia (U. de Antofagasta)

September, 2020
Santiago, Chile

¿Quién soy yo para dudar de tu providencia, mostrando así mi ignorancia? Yo estaba hablando de cosas que no entiendo, cosas tan maravillosas que no las puedo comprender.

Job 42:3

Abstract

In this thesis we study the Very Special Relativity (VSR) framework. In particular we put the emphasis in the QED sector. We present the basics of the Lorentz group and the subgroup $SIM(2)$, which is the symmetry of nature in this framework instead of the full Lorentz group. This symmetry allows introducing terms like $n \cdot p/n \cdot q$, where n transforms with a phase under $SIM(2)$ transformations. With this construction, we can explain the neutrino mass without the addition of new particles.

We explore VSR in two dimensions, showing that the Lorentz group allows VSR terms. This fact shows that we can revisit QED_2. We compute the photon self-energy and the axial anomaly, finding differences from the standard result.

In addition, in four dimensions, we review the electron self-energy, and we discuss the importance of a prescription to regulate infrared divergencies in the VSR integrals. We present a prescription to use when we introduce a possible gauge-invariant photon mass in the electron self-energy computation. The Coulomb scattering is presented as an example of a simple process that can be computed, showing a small signal of the vector n.

Ackowledgments

First, thanks to God by the nature that I can study and in my finitude, try to understand his creation and love.

Second, to my wife Nicole for all the words, company, hugs in the despairing time and her patience and love all the time of this thesis. The life with her has dyed of colours my abstract world.

I would like to thank my mother for her support during my life. I have never been who I am without her hard work.

I wish to thank my advisor prof. Jorge Alfaro for all the discussions about the physics, advice for life and for all the guidance along the process. Thanks for taking care of all of the aspects of my academic career, giving me the opportunity to assist in teaching and to do my doctoral visit to London.

In this regard, I would like to thank Jean Alexandre and Nick Mavromatos for all the discussions and help in my visit to London, despite the pandemic issues. It was a real pleasure work with you.

I thank the Institute of Physics at PUC for the financial support for conferences. Furthermore, for all the administrative aspects out of academic life.

Financial support of Conicyt is also acknowledged. My work was supported by the CONICYT-PFCHA/DoctoradoNacional/2017-21171194 and the doctoral visit in London was supported by the Beneficios Complementarios 2020 from Conicyt.

Contents

Chapter 1

Introduction

The Standard Model of particles has been successful in describing the particle content of the universe and its interactions. Despite this success, some facts remain unsolved, which suggests the exploration of new physical ideas. However, we have to stress that these new ideas should contain the Standard Model as a limit case because of the good accuracy of this theory in the nature description as the LHC has shown.

One of the unsolved puzzles is the Neutrino mass. It is well known the fact that the three generations of neutrinos are massless in the Standard Model description. However, this theory alone cannot explain the observational results of the neutrino oscillations[1], which indicates the neutrinos have mass[2].

The most popular idea to explain the neutrino mass is the see-saw mechanism which appeared in four independent papers[3, 4, 5, 6] around the beginning of the '80s. This idea implies the existence of a new particle, called Sterile neutrino. This particle must be massive enough to give a small mass to the neutrinos. However, up to date, there have not been any signal of new particles which can play the role of this sterile neutrino. Considering this absence of experimental results, we can adopt a different approach. Is valid to ask what assumptions in our models could be modified, keeping all the other issues which have been verified by the observations.

One hot topic example is related with hermiticity in quantum mechanics, where the work of Bender and Boettcher[7] points out that we can have a real spectrum in non-hermitian hamiltonians respecting the PT-symmetry. In this context, Schwinger-Dyson analysis on the fermion and axion-like particles mass generation using non-hermitian and PT-symmetric Yukawa interactions[8, 9] was done as part of a doctoral visit (a brief exposition is shown in Appendix B).

Concerning the Standard model, one important cornerstone is the Lorentz symmetry. This symmetry is based on the space isotropy, and the two Einstein postulates that gave birth to Special Relativity[10, 11], the laws of physics are the same for inertial frames of reference and the invariance of the speed of light. Models with possible violations to this symmetry are not new. They have been studied as a signature of the Planck scale physics[12] and in quantum gravity models[13]. The better known is the Standard Model Extension by Colladay and Kostelecky[14]. The main modification of these models lies in the dispersion relation of the particles. Experimental tests put stringent bounds on these modifications[15].

A different class of model built on is the so-called Very Special Relativity (VSR), which appeared in 2006, and this will be the focus of this thesis. The proposal, made by Cohen and Glashow, is the idea that nature could be invariant under a subgroup of the Lorentz group instead of the full group[16]. The subgroup of interest is $SIM(2)$, which has the particularity that does not have invariant tensors and physical phenomena as time dilation and length contraction are unchanged respect to the Lorentz invariant theory. Moreover, the addition of the discrete symmetries P, T, CP or CT extends $SIM(2)$ to the full Lorentz group. This claim ties a violation of Parity or Time-reversal with the breaking of the Lorentz symmetry. CP violations are small[17], so, the Lorentz violations are expected to be small too.

The main feature of this group is the existence of a privileged null vector n^μ, which transforms under $SIM(2)$ with a phase. This feature allows the construction of fractional terms like $n{\cdot}q/n{\cdot}p$ where this phase is cancelled. Thus, new non-local terms are introduced, and they allow the possibility to explain the neutrino mass without new particles[18, 19] and keeping the dispersion relation unchanged. New similar terms for gauge fields were constructed in [20]. Although no new particles are needed, it is possible to make compatible a $SIM(2)$ theory with supersymmetry[21].

This idea received a criticism pointing that VSR could be incompatible with Thomas Precession and therefore it should be ruled out by the observations[22]. However, the work of Alfaro and Rivelles showed that using BMT equations this claim is not true[23]. Hence, the model remains as a viable alternative.

First attempts to construct a VSR QED theory and Feynman rules appeared in the unpublished paper of Dunn and Mehen[24]. In this work was noticed the importance of choosing an appropriate prescription to compute integrals with terms involving $(n \cdot p)^{-1}$. It was commented the possibility to use the Mandelstam-Leibbrandt prescription[25, 26]. However, in this prescription, a new null vector $\bar{n}$ is introduced, breaking the $SIM(2)$ invariance. This issue motivated the authors to give up this method and to essay a new kind of prescription. However, under this prescription was not clear if the gauge invariance was respected. It left open the question to compute these integrals.

Later, a SIM(2) model of electrodynamics appeared[27] and with the application of VSR in non-abelian theories[28] the development of the Electroweak VSR model[29] was possible. With this work, we have the VSR Standard Model description.

The possibility to compute some basic processes and SIM(2) integrals came with the work of Alfaro[30]. In this paper, starting from an easier way to compute the integrals using the Mandelstam-Leibbrandt prescription[31] the $SIM(2)$ invariance was restored in the computation of the photon and electron self energies trading the null vector $\bar{n}$ with a linear combination of n and the only one independent external momentum involved. Thus, Feynman rules were constructed, and the possibility to compute some processes was opened[32].

The simplest processes to be computed are in QED, and they have received particular attention during the last years[33, 34, 35, 36]. In addition, studies of VSR-QED in lower dimensions were carried on. In $1+1$ dimensions there is an important observation that the two dimensional Lorentz group admits the VSR terms[37]. With this, the Schwinger model was studied, adding these new terms. We will analyze this aspect during this thesis. In $2+1$ was checked the validity of the Furry theorem and the apparition of new induced Maxwell-Chern Simons terms[38].

In addition, VSR has been extensively studied in different contexts. Following the observation of $1+1$ dimensions, the meson spectrum in QCD_2 was analyzed [39]. Also, $SIM(2)$ representations[40], Maxwell-Chern-Simons Electrodynamics[41], VSR-Chern-Simons terms [42], Chern Simons generalized term in 4d on the photon polarization[43], Super-Yang-Mills[44], Supergraphs[45], Non-Commutative space-time[46] and connections with Elko and a possible Dark Matter candidate[47, 48] have been studied. A possible extension to a de Sitter space[49] was analyzed starting with local theory with three fermions which is equivalent to the non-local version with one fermionic field. Another point of view of VSR is in the work of Ilderton, who showed that it is possible to get VSR terms considering a laser background[50].

The extension of Very Special Relativity to curved spacetimes and gravity remains incomplete. The first attempts were made by Gibbons et al.[51]. In this work, they studied the deformations of $ISIM(2)$ and related its line element with Finsler Geometry. This work defines a starting point for other papers as [52, 53, 54, 55, 56, 57], which left a new direction in research.

After 14 years of the first VSR paper and the following works that we have seen, there have not been reviews of the subject in the literature. Thus, this work intends to be an introduction to the basics of VSR, mainly the QED sector. We will give some perspectives and open questions to be solved in the future. This exposition tries to cover some aspects of VSR with the natural emphasis on the author own works in the subject and the sources where the author started this research during the last four years.

The outline of this thesis is as follows. In chapter 2 we will review the basics of the Lorentz group, and primordially we will pay attention to an interesting subgroup called $SIM(2)$, which is the basis of the VSR models.

In chapter 3, we will construct the VSR-QED model. We will review first the VSR fermionic lagrangian, where the model of Cohen and Glashow presented the first significant application in the explanation of the neutrino mass. Next, we will move on to the gauge field, where the possibility to add a gauge-invariant mass to the photon is analyzed following part of the discussion that we presented in [33]. Then, we will describe the VSR-QED lagrangian and the Feynman rules.

In chapter 4 we will analyze the Mandelstam-Leibbrandt prescription, and how to use it in the $SIM(2)$ context in order to compute integrals with $(n \cdot p)^{-1}$, where a prescription is required to regulate infrared divergencies. In this part, we will follow the works [30] and [31] closely. Furthermore, we will present the basic integrals to use in the loop computations.

In chapter 5, we will introduce the study of VSR in lower dimensions. In particular, we will describe the two-dimensional case. The first result that we will discuss is the photon self-energy, where we found the first application of the prescription discussed in the previous chapter. The second aspect will be the axial anomaly. Large parts of the discussion in this chapter are based on the results that we presented in [37].

In chapter 6 we will explore the electron self-energy, and we will define a prescription to deal with the non-local term when we consider the photon mass term discussed in the chapter 3. In addition, we will present as an example, the computation of the Coulomb scattering. Part of this discussion is based on the results of [33].

Finally, in chapter 7, we will summarize our results, and we will discuss some open aspects and future directions that could be taken.

Chapter 2

From the Lorentz Group to $SIM(2)$ and Very Special Relativity

Space and time are the basic elements where all the things take place. According to our experience, we can characterize any event with a time of occurrence x^0 and three spatial coordinates x^i, with $i = 1, 2, 3$. From here, some reasonable assumptions should be made. The famous Einstein's work titled "Zur Elektrodynamik bewegter Körper" ("On the Electrodynamics of moving bodies") established as starting point two postulates[10]. The first postulate is the Principle of Relativity, which states that the equations of the fundamental laws of physics are the same in different inertial reference frames. The second is the constancy of the speed of light c.

2.1 Causal Structure and the Lorentz Group

Let be a first event, where we send out a signal with a light beam at time x_1^0 and space coordinates x_1^1, x_1^2 and x_1^3 in a specific frame of reference. In addition, we will consider a second event, the arrival of this signal of light in a time x_2^0 in the position x_2^1, x_2^2 and x_2^3 in the same frame of reference. We will have the following relation

$$c^2(x_2^0 - x_1^0)^2 - (x_2^1 - x_1^1)^2 - (x_2^2 - x_1^2)^2 - (x_2^3 - x_1^3)^2 = 0, \tag{2.1}$$

From the second postulate, using a different frame of reference to measure the time and position we will have

$$c^2(x_2'^0 - x_1'^0)^2 - (x_2'^1 - x_1'^1)^2 - (x_2'^2 - x_1'^2)^2 - (x_2'^3 - x_1'^3)^2 = 0. \tag{2.2}$$

This structure allows us to define a quantity that we will call interval, which is defined for any two events as

$$\Delta s^2 = (x_2^0 - x_1^0)^2 - (x_2^1 - x_1^1)^2 - (x_2^2 - x_1^2)^2 - (x_2^3 - x_1^3)^2. \tag{2.3}$$

where we set $c = 1$ hereafter. This element defines a different geometry respect to the Euclidean case. Thus, we define the Minkowski space-time $\mathcal{M}$ as a four dimensional pseudo Riemmanian manifold with inner product between two elements $x^\mu = (x^0, x^1, x^2, x^3)$ and $y^\mu = (y^0, y^1, y^2, y^3)$ is defined as

$$\eta_{\mu\nu} x^\mu y^\nu = x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3, \tag{2.4}$$

where η is the metric, which allows us to compute distances in this space, and it is defined by $\eta = diag(1, -1, -1, -1)$.

This inner product is not positive definite. Hence, we can classify the vectors in $\mathcal{M}$ according to the sign of the inner product as

- time-like ($x^\mu x_\mu > 0$),
- light-like ($x^\mu x_\mu = 0$),
- space-like ($x^\mu x_\mu < 0$).

The light-like vectors define a double cone in $\mathbb{R}^4$:

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = 0. \tag{2.5}$$

For the sake of drawing and better visualization of the structure, in figure 2.1, we show the double cone considering only two spatial coordinates.

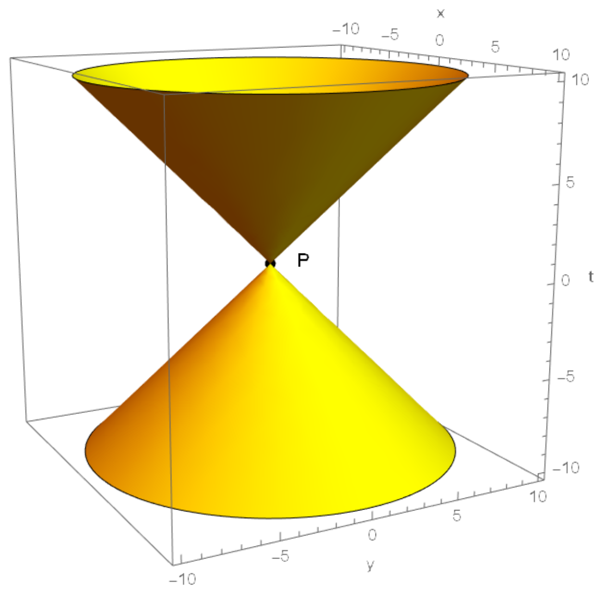

Figure 2.1: Plot of the double cone defined by the equation (2.5). The point P represents an event in the space-time. The upper cone is the future cone of P and the other cone the past. Points inside the cone are causally connected with P, while the points outside are not causally connected.

This cone establishes a causal structure for the point P. Since the speed of light is the maximal attainable velocity, points inside of this cone can be reached from P. Points outside of this cone only could be reached with velocities higher than the speed of light. In this way, we say points inside the cone are causally connected with P while points outside do not.

Mathematically, let be $\bar{x}^\mu = (\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3)$ the point P. Let be x^μ an arbitrary point in $\mathcal{M}$. We can define the boundary of the light cone as the set of points x which satsify

$$\eta_{\mu\nu}(\bar{x} - x)^\mu (\bar{x} - x)^\nu = 0. \tag{2.6}$$

Hence, any point x causally connected with P will satisfy

$$\eta_{\mu\nu}(\bar{x} - x)^\mu (\bar{x} - x)^\nu > 0. \tag{2.7}$$

If we change to different coordinates, the points causally connected must be the same as before. In this way, the accepted coordinate transformations are whose which preserve the cone structure. The first transformation to analyze is a translation of every point x in

$\mathcal{M}$ as $x^\mu \to x^\mu + a^\mu$, with a constant vector.

$$\begin{aligned} \eta_{\mu\nu}(\bar{x}-x)^\mu(\bar{x}-x)^\nu &\to \eta_{\mu\nu}(\bar{x}+a-x-a)^\mu(\bar{x}+a-x-a)^\nu, \\ &= \eta_{\mu\nu}(\bar{x}-x)^\mu(\bar{x}-x)^\nu. \end{aligned} \tag{2.8}$$

The translation have left invariant the object $\eta_{\mu\nu}(\bar{x}-x)^\mu(\bar{x}-x)^\nu$, preserving the cone structure.

Let us take a linear transformation $x^\mu \to \Lambda^\mu{}_\nu x^\nu$. Thus,

$$\eta_{\mu\nu}\Lambda^\mu{}_\alpha(\bar{x}-x)^\alpha\Lambda^\nu{}_\beta(\bar{x}-x)^\beta. \tag{2.9}$$

To preserve the inner product this transformation should satisfy

$$\eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta = \eta_{\alpha\beta}, \tag{2.10}$$

or in matricial notation

$$\Lambda^\top\eta\Lambda = \eta. \tag{2.11}$$

Therefore, we define the Lorentz Group as the set of all the lineal transformations Λ such as the Minkowski metric η is left invariant.

We also define the Poincaré group as the Lorentz group plus translations in the four coordinates of the elements of $\mathcal{M}$.

Let us take the determinant in (2.11).

$$\det\Lambda^T\det\eta\det\Lambda = \det\eta. \tag{2.12}$$

Thus,

$$\det\Lambda = \pm 1. \tag{2.13}$$

We define the proper transformations by the condition $\det\Lambda = 1$. They define a subgroup that we will call the Proper Lorentz group. The transformations which satisfy $\det\Lambda = -1$ are called improper transformations. They do not form a group, since the identity is not part of this set.

We take (2.10) with $\alpha = \beta = 0$. We get

$$(\Lambda^0_0)^2 = 1 + (\Lambda^i_0)^2. \tag{2.14}$$

Hence, $(\Lambda^0_0)^2 \geq 1$. It implies two possibilities, $\Lambda^0_0 \geq 1$ or $\Lambda^0_0 \leq -1$. The set of elements with $\Lambda^0_0 \geq 1$ is a subgroup that we will call orthochronous. On the other hand, the elements with $\Lambda^0_0 \leq -1$ are not a group and we will call this set anti-orthochronous.

We will define the subgroup of elements which satisfies $\Lambda^0_0 \geq 1$ and $\det\Lambda = 1$, the proper and ortochronous set, as the Restricted Lorentz Group. This group is connected. In other words, we can get any element of the group from continuous transformations from the identity.

From this Restricted Lorentz Group we can get all the elements of the full Lorentz group adding the discrete transformations Parity (P), which reverts the spatial orientation of a vector, and Time-reversal (T), which reverts the time direction. We notice both transformations have determinant equal to -1. Applying Parity to the Restricted Lorentz group, we get the objects of the improper and orthochronous set, and the application of Time reversal let us get the elements of the improper anti-orthochonous set. Taking both, we get the proper and anti-orthochronous set. In figure 2.2, we observe a diagram with this transformations. For the sake of simplicity, we call the Lorentz group to the Restricted Lorentz group because we can get the full group under the application of Parity and Time reversal.

Now, let us consider a Lorentz transformation as a small deviation of the identity. Thus,

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu, \tag{2.15}$$

where the $\omega^\mu{}_\nu$ are reals and continuous. The transformation Λ satisfies (2.11). There-

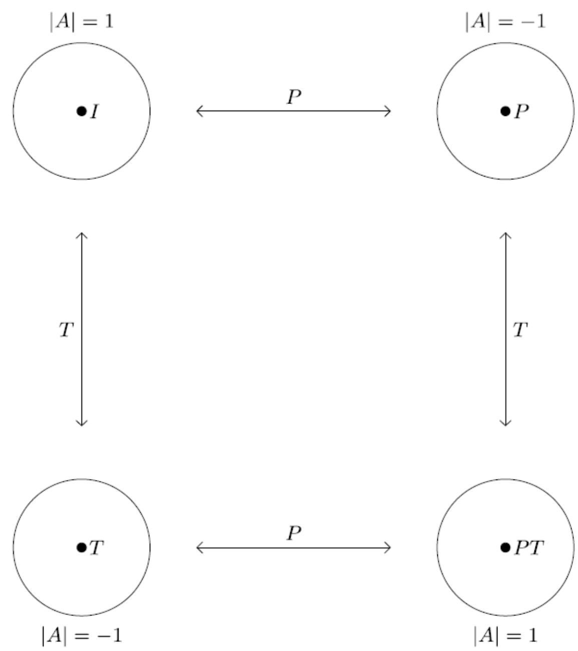

Figure 2.2: The image, obtained from [58], contains the four components of the Lorentz Group with its determinants. The proper and orthochronous part is in the upper left corner. With Parity, we get the Improper and orthochronous set in the upper right. Using Time reversion, we get the improper and anti-orthochronous set. The application of P and T yields to the proper and anti-orthochronous set.

fore,

$$\eta_{\mu\nu} = \eta_{\alpha\beta}(\delta^{\alpha}{}_{\mu} + \omega^{\alpha}{}_{\mu})(\delta^{\beta}{}_{\nu} + \omega^{\beta}{}_{\nu}). \tag{2.16}$$

Keeping up to first order in ω:

$$\eta_{\mu\nu} = \eta_{\mu\nu} + \eta_{\mu\beta}\omega^{\beta}{}_{\nu} + \eta_{\alpha\nu}\omega^{\alpha}{}_{\mu} + \mathcal{O}(\omega^2). \tag{2.17}$$

Hence,

$$\omega_{\mu\nu} = -\omega_{\nu\mu}. \tag{2.18}$$

We notice $\omega_{\mu\nu}$ is antisymmetric. This object is represented by a 4×4 matrix because the Lorentz transformations act in four-dimensional space-time. Therefore, the matrix ω has 16 elements. However, four of them, when $\mu = \nu$, are zero due to the antisymmetry. We are left with 12 parameters. Nevertheless, by the same antisymmetry, there are only six free parameters as we see in (2.19).

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & \omega_{01} & \omega_{02} & \omega_{03} \\ -\omega_{01} & 0 & \omega_{12} & \omega_{13} \\ -\omega_{02} & -\omega_{12} & 0 & \omega_{23} \\ -\omega_{03} & -\omega_{13} & -\omega_{23} & 0 \end{pmatrix}. \tag{2.19}$$

Therefore, the transformations of the Lorentz group are characterized by six parameters. Considering this, we can write any matrix $\omega_{\rho\nu}$ using as basis six 4×4 antisymmetric matrices $(M^{\alpha\beta})_{\rho\nu}$:

$$\omega_{\rho\nu} = \omega_{01}(M^{01})_{\rho\nu} + \omega_{02}(M^{02})_{\rho\nu} + \omega_{03}(M^{03})_{\rho\nu} + \omega_{23}(M^{23})_{\rho\nu} - \omega_{13}(M^{13})_{\rho\nu} + \omega_{12}(M^{12})_{\rho\nu}, \tag{2.20}$$

with the most obvious choice of

$$(M^{01})_{\rho\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (M^{02})_{\rho\nu} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (M^{03})_{\rho\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$
$$(M^{23})_{\rho\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad (M^{13})_{\rho\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad (M^{12})_{\rho\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.21}$$

We use the metric to get $\omega^{\mu}{}_{\nu} = \eta^{\mu\rho}\omega_{\rho\nu}$, which is the element in (2.15). Thus,

$$\omega^{\mu}{}_{\nu} = \omega_{01}(K_1)^{\mu}{}_{\nu} + \omega_{02}(K_2)^{\mu}{}_{\nu} + \omega_{03}(K_3)^{\mu}{}_{\nu} + \omega_{23}(J_1)^{\mu}{}_{\nu} + \omega_{13}(J_2)^{\mu}{}_{\nu} + \omega_{12}(J_3)^{\mu}{}_{\nu}. \tag{2.22}$$

with

$$
\begin{aligned}
(K_1)^\mu{}_\nu &= \eta^{\mu\rho}(M^{01})_{\rho\nu}, && (2.23)\\
(K_2)^\mu{}_\nu &= \eta^{\mu\rho}(M^{02})_{\rho\nu}, && (2.24)\\
(K_3)^\mu{}_\nu &= \eta^{\mu\rho}(M^{03})_{\rho\nu}, && (2.25)\\
(J_1)^\mu{}_\nu &= \eta^{\mu\rho}(M^{23})_{\rho\nu}, && (2.26)\\
(J_2)^\mu{}_\nu &= \eta^{\mu\rho}(M^{13})_{\rho\nu}, && (2.27)\\
(J_3)^\mu{}_\nu &= \eta^{\mu\rho}(M^{12})_{\rho\nu}. && (2.28)
\end{aligned}
$$

These six matrices are called the generators of the group because using these matrices and different parameters multiplying them; we can get any transformation of the Lorentz Group. With our choices, these matrices read

$$
J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad
J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad
J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
$$
$$
K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad
K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad
K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad (2.29)
$$

These generators satisfy the following commutation relations:

$$
\begin{aligned}
[J_i, J_j] &= \epsilon_{ijk} J_k, && (2.30)\\
[K_i, K_j] &= -\epsilon_{ijk} J_k, && (2.31)\\
[J_i, K_j] &= \epsilon_{ijk} K_k. && (2.32)
\end{aligned}
$$

These commutation relations define the Lie algebra of the Lorentz group. We notice the commutation relation (2.30) is the $SO(3)$ algebra. Thus, the J_i's are the generators of rotations, which preserve the time component of a four-vector. The K_i's correspond to the boosts generators.

2.2 $SIM(2)$ properties and Very Special Relativity

The choice of basis to write (2.19) in (2.20) was the most obvious and this is the standard way, but it is not the only possibility. It is not forbidden to make another choice. We will write (2.19) using a different basis,

$$\begin{aligned}\omega_{\rho\nu} &= \left(\frac{\omega_{01}+\omega_{13}}{2}\right)(N_1)_{\rho\nu} + \left(\frac{\omega_{02}-\omega_{23}}{2}\right)(N_2)_{\rho\nu} + \left(\frac{\omega_{13}-\omega_{01}}{2}\right)(N_3)_{\rho\nu} \\ &\quad - \left(\frac{\omega_{02}+\omega_{23}}{2}\right)(N_4)_{\rho\nu} + \omega_{12}(N_5)_{\rho\nu} + \omega_{03}(N_6)_{\rho\nu},\end{aligned} \tag{2.33}$$

where

$$(N_1)_{\rho\nu} = (M^{01})_{\rho\nu} + (M^{13})_{\rho\nu}, \tag{2.34}$$

$$(N_2)_{\rho\nu} = (M^{02})_{\rho\nu} - (M^{23})_{\rho\nu}, \tag{2.35}$$

$$(N_3)_{\rho\nu} = -(M^{01})_{\rho\nu} + (M^{13})_{\rho\nu}, \tag{2.36}$$

$$(N_4)_{\rho\nu} = -(M^{02})_{\rho\nu} - (M^{23})_{\rho\nu}, \tag{2.37}$$

$$(N_5)_{\rho\nu} = (M^{12})_{\rho\nu}, \tag{2.38}$$

$$(N_6)_{\rho\nu} = (M^{03})_{\rho\nu} \tag{2.39}$$

We use the metric to raise the index ρ, and we get

$$\begin{aligned}\omega^{\mu}{}_{\nu} &= -\left(\frac{\omega_{01}+\omega_{13}}{2}\right)(Y_1)^{\mu}{}_{\nu} + \left(\frac{\omega_{02}-\omega_{23}}{2}\right)(T_2)^{\mu}{}_{\nu} - \left(\frac{\omega_{13}-\omega_{01}}{2}\right)(T_1)^{\mu}{}_{\nu} \\ &\quad - \left(\frac{\omega_{02}+\omega_{23}}{2}\right)(Y_2)^{\mu}{}_{\nu} + \omega_{12}(J_3)^{\mu}{}_{\nu} + \omega_{03}(K_3)^{\mu}{}_{\nu}.\end{aligned} \tag{2.40}$$

We notice this new choice of generators keeps J_3 and K_3 unchanged and the new elements T_1, T_2, Y_1 and Y_2 can be related with the standard basis by

$$T_1 = K_1 + J_2, \tag{2.41}$$

$$T_2 = K_2 - J_1, \tag{2.42}$$

$$Y_1 = -K_1 + J_2, \tag{2.43}$$

$$Y_2 = -K_2 - J_1. \tag{2.44}$$

This new construction has an interesting advantage. Let us see the action of the discrete operators parity P and time reversal T on J_i and K_i. We observe that

$$PJ_iP^{-1} = J_i, \tag{2.45}$$

$$PK_iP^{-1} = -K_i, \tag{2.46}$$

$$TJ_iT^{-1} = J_i, \tag{2.47}$$

$$TK_iT^{-1} = -K_i. \tag{2.48}$$

Hence, applying P or T on T_1 we get Y_1. The same operation on T_2 gives Y_2. So, we can start with the subgroup whose generators are T_1, T_2, J_3 y K_3. The addition of the discrete symmetries enlarges this subgroup to the full Lorentz group. Thus, Parity or Time-reversal breaking is tied with a Lorentz violation. In consideration with this fact, we can think that nature is invariant under this four-parameter group called $SIM(2)$. This statement was the main idea of Cohen and Glashow that defines Very Special Relativity[16].

From here, we will focus on the group $SIM(2)$ and its main features. Using the representation (2.29) for the J_i's and K_i's is easy to check the commutation relations between the generators of $SIM(2)$. They are

$$\begin{aligned} &[T_1, T_2] = 0, \qquad [T_1, J_3] = -T_2, \qquad [T_2, J_3] = T_1, \\ &[T_1, K_3] = -T_1, \qquad [T_2, K_3] = T_2, \qquad [J_3, K_3] = 0. \end{aligned} \tag{2.49}$$

It defines a closed algebra for this group.

It is known that the only invariant tensor in the Lorentz Group is the metric. There have been attempts to construct theories where invariant tensors under a specific symmetry have been introduced. For instance, in [59] was introduced a time-like four-vector, invariant under $SO(3)$. The introduction of this invariant tensors, called spurions, affects the particle propagation and the kinematics of particle decays. In order to have a model that does not depart from these well constrained limits[15], we should not have invariant tensors in $SIM(2)$. Let us see the vectors first. A vector n^μ is invariant under the action of a group if

$$n^\mu = \Omega^\mu{}_\nu n^\nu, \tag{2.50}$$

for any transformation Ω in the group. In our case, we write an infinitesimal transformation $\Omega^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ in $SIM(2)$ to see the action of the generators. It yields

$$\omega^\mu{}_\nu n^\nu = 0. \tag{2.51}$$

The vectors which satisfy this relation are the vectors of the kernel of $\omega^\mu{}_\nu$. The most general $SIM(2)$ transformation can be written as

$$\omega^\mu{}_\nu = \alpha_1 T_1 + \alpha_2 T_2 + \alpha_3 J_3 + \alpha_4 K_3. \tag{2.52}$$

In our specific representation of matrices,

$$\omega^\mu{}_\nu = \begin{pmatrix} 0 & \alpha_1 & \alpha_2 & \alpha_4 \\ \alpha_1 & 0 & -\alpha_3 & \alpha_1 \\ \alpha_2 & \alpha_3 & 0 & \alpha_2 \\ \alpha_4 & -\alpha_1 & -\alpha_2 & 0 \end{pmatrix}. \tag{2.53}$$

There is no vectors in the kernel of this matrix. Therefore, there is no invariant vectors in $SIM(2)$. However, a special case is the null vector $n = (1, 0, 0, 1)$. Is easy to see that

$$\begin{aligned}(T_1)^{\mu}{}_{\nu}n^{\nu} = (T_2)^{\mu}{}_{\nu}n^{\nu} = (J_3)^{\mu}{}_{\nu}n^{\nu} = 0,\\ (K_3)^{\mu}{}_{\nu}n^{\nu} = -n^{\nu}.\end{aligned} \tag{2.54}$$

The last line in (2.54) shows that the vector n keeps invariant the direction under K_3. In a general $SIM(2)$ transformation n will transform as $n \to e^{\phi}n$, with ϕ a phase. Although there are no invariant vectors, this null vector can be used to construct terms like $n \cdot p/n \cdot q$, where the phase is cancelled.

Analogously, a tensor $T^{\mu\nu}$ is invariant under $SIM(2)$ if

$$T^{\mu\nu} = \Omega^{\mu}{}_{\alpha}\Omega^{\nu}{}_{\beta}T^{\alpha\beta}. \tag{2.55}$$

Writing Ω as an infinitesimal transformation as before, we get

$$\omega^{\mu}{}_{\alpha}T^{\alpha\nu} + \omega^{\nu}{}_{\beta}T^{\mu\beta} = 0. \tag{2.56}$$

Written in matrix form

$$\omega T + T\omega^{\top} = 0. \tag{2.57}$$

The only tensor which satisfies this relation is the metric. Therefore, there are no invariant tensors.

We should highlight an important comment here. Notice that the specific null vector $n = (1, 0, 0, 1)$ has the properties that we reviewed in the specific representation used for the generators of the group. We can choose generators in different directions. As a result, the null vector will be different, although in this new choice it will transform as we saw before. To keep the discussion independent of the choice of the generators, we will define the null vector n^{μ} as the vector which transforms under $SIM(2)$ transformations as $n^{\mu} \to e^{\phi}n^{\mu}$ without specifying its coordinates.

Chapter 3

Very Special Relativity QED

3.1 Fermionic Field

The observation in the previous chapter, that ratios containing the null vector n are invariant under $SIM(2)$, allows the construction of the fermionic lagrangian

$$\mathcal{L}_f = \bar{\psi}\left(i\not{\partial} - M + i\frac{m^2}{2}\frac{\not{n}}{n\cdot\partial}\right)\psi, \tag{3.1}$$

where the slash notation denotes $\not{a} = \gamma^\mu a_\mu$ for any vector a and γ^μ are the gamma matrices. The element $\bar{\psi}$ is defined by $\bar{\psi} = \psi^\dagger\gamma^0$. The first two terms in (3.1) are standard. However, the third is not Lorentz invariant but $SIM(2)$ invariant due to the null vector n in the numerator and denominator. This element contains a derivative in the denominator. Therefore, to have the right units the new term is multiplied by a factor, m^2, with mass square units. The one half is inserted by later convenience. Since this new term violates the Lorentz invariance this factor should be small in agreement with the observations. The way to handle this new operator, which is non-local, is using the following definition

$$\frac{1}{n\cdot\partial} = \int_0^\infty d\alpha e^{-\alpha n\cdot\partial}. \tag{3.2}$$

The equation of motion of ψ is given by

$$\left(i\not{\partial} - M + i\frac{m^2}{2}\frac{\not{n}}{n\cdot\partial}\right)\psi = 0. \tag{3.3}$$

Going to Fourier space

$$\left(\not{p} - M - \frac{m^2}{2}\frac{\not{n}}{n\cdot p}\right)\psi(p) = 0. \tag{3.4}$$

Multiplying by the operator $\left(\not{p} + M - \frac{m^2}{2}\frac{\not{n}}{n\cdot p}\right)$ in the left in (3.4) under the consideration that $\not{n}\not{n} = 0$ we get

$$p^2 = M_e^2, \tag{3.5}$$

where we have defined $M_e^2 = M^2 + m^2$. If we set $M = 0$ we obtain from (3.5) the standard dispersion relation for a particle with mass m. Thus, a neutrino, without a mass M coming from the spontaneous symmetry breaking, can have mass with the introduction of a $SIM(2)$ invariant term without invoking new particles. This fact was the observation of Cohen and Glashow in [18]. Neutrino masses are expected to be small, and this is consistent with our initial comment of the smallness of the parameter m^2, which parameterizes the deviations of the Lorentz invariance.

The equation (3.1) can be written using (3.2) and partial integrations as

$$\mathcal{L} = \bar{\psi}\left(-i\overleftarrow{\not{\partial}} - M - i\frac{m^2}{2}\frac{\not{n}}{n\cdot\overleftarrow{\partial}}\right)\psi, \tag{3.6}$$

where the arrow pointing to the left indicates that the derivatives act on the elements in the left. Thus, the equation of motion for $\bar{\psi}$ is

$$\bar{\psi}\left(\overleftarrow{\not{\partial}} + M + \frac{m^2}{2}\frac{\not{n}}{n\cdot\overleftarrow{\partial}}\right) = 0. \tag{3.7}$$

Multiplying by $\bar{\psi}$ in the left in (3.3) and by ψ in the right in (3.7) and adding both we get

$$\partial_\mu \left(\bar{\psi}\gamma^\mu\psi + \frac{1}{2}m^2 \left(\frac{1}{n\cdot\partial}\bar{\psi}\right) \not{n} n^\mu \left(\frac{1}{n\cdot\partial}\psi\right)\right) = 0. \tag{3.8}$$

This defines a conserved current j^μ by

$$j^\mu = \bar{\psi}\gamma^\mu\psi + \frac{1}{2}m^2 \left(\frac{1}{n\cdot\partial}\bar{\psi}\right) \not{n} n^\mu \left(\frac{1}{n\cdot\partial}\psi\right). \tag{3.9}$$

Setting $m = 0$ we recover the standard fermionic vector current. This association allows us to say that (3.9) corresponds to the VSR current related with the invariance to transformations $\psi \to e^{i\alpha}\psi$: the symmetry $U(1)$.

It is known that the mass M breaks invariance under the axial symmetry $\psi \to e^{i\beta\gamma^5}\psi$. Setting $M = 0$, multiplying by $-\bar{\psi}\gamma^5$ in the left in (3.3) and by $\gamma^5\psi$ in the right in (3.7) and adding both we get

$$\partial_\mu \left(\bar{\psi}\gamma^\mu\gamma^5\psi + \frac{1}{2}m^2 \left(\frac{1}{n\cdot\partial}\bar{\psi}\right) \not{n}\gamma^5 n^\mu \left(\frac{1}{n\cdot\partial}\psi\right)\right) = 0. \tag{3.10}$$

Hence, the conserved axial current $j^{\mu 5}$ is defined by

$$j^{\mu 5} = \bar{\psi}\gamma^\mu\gamma^5\psi + \frac{1}{2}m^2 \left(\frac{1}{n\cdot\partial}\bar{\psi}\right) \not{n}\gamma^5 n^\mu \left(\frac{1}{n\cdot\partial}\psi\right). \tag{3.11}$$

3.2 Gauge Field

As in the fermionic case we can construct a $SIM(2)$ invariant term for the gauge fields. We can define the following tensor

$$\tilde{F}_{\mu\nu} = F_{\mu\nu} - \frac{m_\gamma^2}{2}\left(n_\mu \frac{1}{(n\cdot\partial)^2} n^\alpha F_{\nu\alpha} - n_\nu \frac{1}{(n\cdot\partial)^2} n^\alpha F_{\mu\alpha}\right), \tag{3.12}$$

where $F_{\mu\nu}$ is defined as usual

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu. \tag{3.13}$$

Our new construction (3.12) has a new parameter m_γ, which has units of mass to preserve the dimensions. If we set this parameter to zero we recover the usual electromagnetic tensor. Thus, m_γ is other quantity that parameterizes the deviation of the Lorentz symmetry. This constrains this parameter to be small, as in the neutrino mass parameter. In addition, we see that (3.12) is gauge invariant under transformations $A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda$. With this element we can construct the VSR gauge field lagrangian

$$\mathcal{L}_g = -\frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}. \tag{3.14}$$

Using the expression (3.12) we get

$$\mathcal{L}_g = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}(n^\alpha F_{\mu\alpha})\frac{1}{(n\cdot\partial)^2}(n_\beta F^{\mu\beta}). \tag{3.15}$$

The equation of motion is

$$\partial_\mu F^{\mu\nu} + m_\gamma^2 n^\nu \frac{1}{(n\cdot\partial)^2}\partial_\alpha(n_\beta F^{\alpha\beta}) + m_\gamma^2 \frac{1}{n\cdot\partial}(n_\beta F^{\beta\nu}) = 0. \tag{3.16}$$

We write equation (3.16) in terms of the gauge field as

$$\partial^2 A^\nu - \partial^\nu\partial_\mu A^\mu + m_\gamma^2 n^\nu \frac{1}{(n\cdot\partial)^2}[\partial^2(n\cdot A) - (n\cdot\partial)(\partial_\alpha A^\alpha)] + m_\gamma^2\frac{1}{n\cdot\partial}(n\cdot\partial A^\nu - \partial^\nu n\cdot A) = 0. \tag{3.17}$$

If we contract equation (3.17) with n_ν we get

$$\partial^2 n\cdot A - n\cdot\partial(\partial\cdot A) = 0. \tag{3.18}$$

We will fix the Lorenz gauge $\partial_\mu A^\mu = 0$. Then equation (3.18) implies the condition

$$\partial^2(n\cdot A) = 0. \tag{3.19}$$

However, a gauge degree of freedom remains, since $\partial^\mu A'_\mu = \partial^\mu A_\mu + \partial^2\Lambda_1$ implies $\partial^2\Lambda_1 = 0$, if both A'_μ and A_μ are in the Lorenz gauge. We will use the remaining gauge freedom to impose the additional condition $n\cdot A = 0$. To see the consistency of our condition we have that

$$n\cdot A' = n\cdot A + n\cdot\partial\Lambda_1 = 0 \tag{3.20}$$

has the solution for Λ given by

$$\Lambda_1 = -\frac{1}{n\cdot\partial} n \cdot A. \tag{3.21}$$

Notice that if we apply the operator ∂^2 in (3.21) we have

$$\partial^2\Lambda_1 = -\frac{1}{n\cdot\partial}\partial^2 n \cdot A = 0, \tag{3.22}$$

where we used the equation (3.19). Hence, the condition $n \cdot A = 0$ is consistent. Therefore, we apply the Lorenz gauge plus the subsidiary condition $n \cdot A = 0$ in the equation (3.17) and we obtain

$$(\partial^2 + m_\gamma^2)A^\nu = 0. \tag{3.23}$$

We see from equation (3.23) that A^ν is a field with mass m_γ. Hence, in VSR we have the possibility to add a photon mass coming from a term that is gauge invariant, unlike the standard case where the mass term of the type $m_\gamma^2 A^\mu A_\mu$ is forbidden because it breaks the gauge invariance.

Now, we use a plane wave solution for A^ν, as $A^\nu = \varepsilon^\nu e^{-ikx}$, so we have

$$k^2 - m_\gamma^2 = 0 \tag{3.24}$$

with the conditions $k \cdot \varepsilon = 0$ and $n \cdot \varepsilon = 0$. Thus, these conditions left only two degrees of freedom. Therefore, despite the mass, the gauge field has two polarizations, as in the standard case.

3.3 Coupling fermions and photons and Feynman rules

Now we can couple fermions and gauge fields considering the $SIM(2)$ invariant terms. Using the covariant derivative $D_\mu = \partial_\mu + ieA_\mu$ we have

$$\mathcal{L} = \bar{\psi}\left(i\not{D} - M + i\frac{m^2}{2}\frac{\not{n}}{n\cdot D}\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m_\gamma^2}{2}(n^\alpha F_{\mu\alpha})\frac{1}{(n\cdot\partial)^2}(n_\beta F^{\mu\beta}) - \frac{1}{2\xi}(\partial_\mu A^\mu)^2. \tag{3.25}$$

We observe that setting $m = m_\gamma = 0$ we recover the standard QED. The last term in (3.25) was introduced to fix the gauge. Perhaps the most natural choice of gauge fixing term is the light cone one $n^\mu A_\mu = 0$. Nevertheless, in order to make comparisons between our results and the standard result easily to find in textbooks, we will use the Lorenz gauge.

The operator $(n \cdot D)^{-1}$ contains the field A_μ in the denominator. In order to manage this object, we will expand perturbatively in series for e small. Thus,

$$\begin{aligned}\frac{1}{n \cdot D} &= \frac{1}{n \cdot \partial}\left(1 - ien \cdot A\frac{1}{n \cdot \partial} - e^2 n \cdot A\frac{1}{n \cdot \partial}n \cdot A\frac{1}{n \cdot \partial}\right.\\ &\left.\quad + ie^3 n \cdot A\frac{1}{n \cdot \partial}n \cdot A\frac{1}{n \cdot \partial}n \cdot A\frac{1}{n \cdot \partial}\right). \end{aligned} \tag{3.26}$$

We notice this operator is in the middle of $\bar{\psi}$ and ψ. Therefore, the series expansion will generate interaction terms between the fermion and different fields A, which increases when we take more terms. Therefore, the operator $(n \cdot D)^{-1}$ will generate an infinite number of vertices with an increasing number of external photon legs in the Feynman diagrams to compute. In figure 3.1, we show the first three vertices.

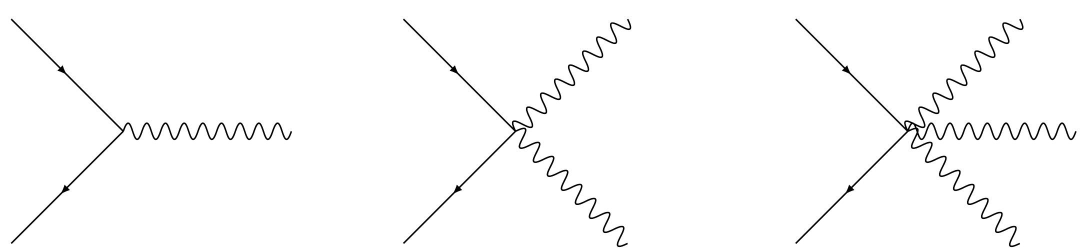

Figure 3.1: Vertices with one, two and three external photonic legs.

Using this expansion in (3.25) we can deduce the Feynman rules for the propagators and any vertex. These rules are listed in the table 3.1 only considering the first two vertices. We can obtain the other vertices taking more terms in the expansion (3.26). For a detailed computation of these rules, we refer the reader to Appendix A.

Notice that the operator $(n \cdot \partial)^{-1}$ in Fourier space gives a $(n \cdot p)^{-1}$. In the limit $p \to 0$ generates divergencies that should be treated using a prescription. We inserted a subindex

Electron propagator	$$S_F(p,\bar{n}) = i\frac{\not{p} + M - \frac{m^2}{2}\frac{\not{n}}{(n\cdot p)_{\bar{n}}}}{p^2 - M_e^2 + i\varepsilon} \tag{3.27}$$
Photon propagator	$$\begin{aligned}\Delta_{\mu\nu}(p,\bar{n}) &= -\frac{i}{p^2 - m_\gamma^2}\left[g_{\mu\nu} + \frac{m_\gamma^2}{(n\cdot p)_{\bar{n}}^2}n_\mu n_\nu \right.\\ &\left. - \frac{m_\gamma^2}{p^2(n\cdot p)_{\bar{n}}}(p_\mu n_\nu + p_\nu n_\mu)\right]\end{aligned} \tag{3.28}$$
One photon leg Vertex	$$V_{1\mu}[(p,\bar{n}_1),(p+q,\bar{n}_2)] = -ie\left(\gamma_\mu + \frac{1}{2}m^2\not{n}\frac{n_\mu}{(n\cdot p)_{\bar{n}_1}[n\cdot(p+q)]_{\bar{n}_2}}\right) \tag{3.29}$$
Two photon leg Vertex	$$\begin{aligned}&V_{2\mu\nu}[(p,\bar{n}_1),(p',\bar{n}_2),(p+q_1,\bar{n}_3),(p+q_2,\bar{n}_4)] =\\ &-ie^2\frac{1}{2}m^2\not{n}\frac{n_\mu n_\nu}{(n\cdot p)_{\bar{n}_1}(n\cdot p')_{\bar{n}_2}}\left(\frac{1}{[n\cdot(p+q_1)]_{\bar{n}_3}} + \frac{1}{[n\cdot(p+q_2)]_{\bar{n}_4}}\right)\end{aligned} \tag{3.30}$$

Table 3.1: Table with the Feynman rules for the Lagrangian in the equation (3.25).

$\bar{n}$ in each term $(n \cdot p)^{-1}$ in the propagators and vertices to indicates this aspect. Since a priori the prescriptions used in each non-local term could be different, we use different $\bar{n}$ when more than one $(n \cdot p)^{-1}$ appear. The rationale to use the name $\bar{n}$ will be clarified in the next chapter. By now, it merely indicates the use of a prescription to deal with these infrared divergencies.

We can prove that any vertex with a specific number of external photon legs contracted with one momentum of the external leg can be related with a difference of two vertices with one less external photon leg.

First, we analyze the vertex with one external photon leg, which is the diagram in the left in figure 3.1. Defining as q the photon momentum, p and $p' = p + q$ the incoming and outcoming fermionic momenta respectively, we contract the expression for this vertex, equation (3.29), with q^μ. Inserting a convenient zero, we get

$$q^\mu V_{1\mu} u = \left(\not{p} + \not{q} - M - \frac{1}{2} m^2 \frac{\not{n}}{[n \cdot (p+q)]_{\bar{n}_2}} \right) - \left(\not{p} - M - \frac{1}{2} m^2 \frac{\not{n}}{(n \cdot p)_{\bar{n}_1}} \right). \tag{3.31}$$

We recognize the inverse of the electron propagator and

$$q^\mu V_{1\mu}[(p, \bar{n}_1), (p+q, \bar{n}_2)] = S_F^{-1}(p+q, \bar{n}_2) - S_F^{-1}(p, \bar{n}_1). \tag{3.32}$$

This expression is familiar in the standard QED and here it holds too.

For the vertex with two external photon legs, we define the incoming and outcoming fermion momentum as p and $p' = p + q_1 + q_2$, with q_1 and q_2 the momenta of the photons. We contract this vertex with an external photon leg, for instance q_1 and we get

$$q_1^\mu V_{2\mu\nu}[(p, \bar{n}_1), (p', \bar{n}_2), (p+q_1, \bar{n}_3), (p+q_2, \bar{n}_4)] = V_{1\nu}[(p, \bar{n}_1), (p+q_2, \bar{n}_4)] - V_{1\nu}[(p', \bar{n}_2), (p+q_1, \bar{n}_3)]. \tag{3.33}$$

For the contraction with q_2 the result is

$$q_2^\nu V_{2\mu\nu}[(p, \bar{n}_1), (p', \bar{n}_2), (p+q_1, \bar{n}_3), (p+q_2, \bar{n}_4)] = V_{1\mu}[(p, \bar{n}_1), (p+q_1, \bar{n}_3)] - V_{1\mu}[(p', \bar{n}_2), (p+q_2, \bar{n}_4)]. \tag{3.34}$$

Doing the same with vertices with more photon legs, we can write the contraction between a vertex with n external photon legs with one external momentum of a photon as the difference between two terms. The first one, a vertex with $n-1$ external legs, without the leg whose momentum is the contracted, and the second, another vertex with $n-1$ external legs, whose inner fermionic leg has momentum as the sum of the original inner fermionic momentum and the momentum contracted, as is shown in figure 3.2.

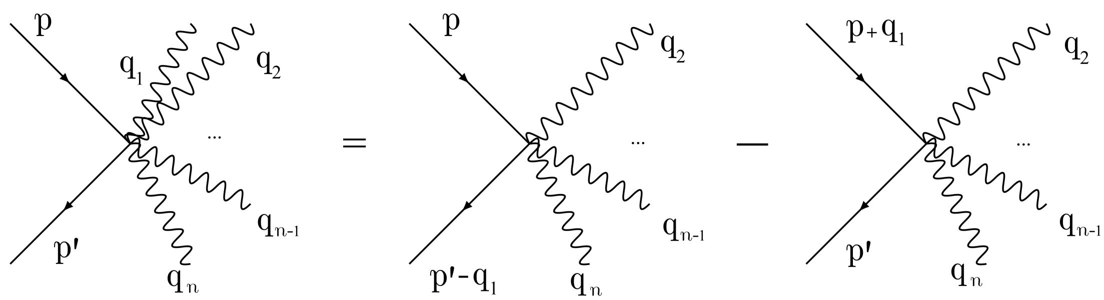

Figure 3.2: Vertex with n external photonic legs written as difference between two diagrams with one less leg.

Chapter 4

Computing the integrals

In the previous chapter, we found as in the propagators as in the vertices, the non-local element $(n \cdot p)^{-1}$. We made the comment that in the infrared limit $p \to 0$ divergencies appear. Hence, how to deal with this divergencies will play an important role in the computation of Feynman diagrams. In order to tackle this problem, we recognize that this kind of integration is not a new issue. It is known in the literature the possibility to use non-covariant gauges in Yang-Mills and Chern-Simons theories (for more details on it see [60, 61, 62, 63] and references therein). One of these non-covariant gauges is the light-cone gauge, where $n \cdot A = 0$. The recognized advantage of this gauge is that ghost diagrams do not contribute to the cross section. Hence, they do not need to be evaluated. This gauge shares the existence of a null vector with our case in study. Therefore, similar integrals appear in VSR and light-cone gauge computations. In order to deal with the divergencies in the light-cone gauge, we need to use a prescription. Mandelstam[25] and Leibbrandt[26] developed independently the following prescription

$$\frac{1}{(n \cdot p)_{\bar{n}}} = \lim_{\epsilon \to 0} \frac{\bar{n} \cdot p}{n \cdot p \bar{n} \cdot p + i\epsilon}, \tag{4.1}$$

where we have introduced a new null vector $\bar{n}$ such as

$$n \cdot \bar{n} = 1. \tag{4.2}$$

Thus, the subindex in each $(n \cdot p)^{-1}$ that we inserted in the chapter 3 corresponds to the null vector $\bar{n}$ that we will use. The appearance of this new null vector in (4.1) complicates a little the computations. However, the work of Alfaro[31] presents a simple way to do the integrals. We will review this method.

Consider the following integral in d dimensions

$$I_\mu = \int d^d p \frac{r(p^2) p_\mu}{n \cdot p}, \tag{4.3}$$

where r is an arbitrary function. To compute this integral we point out the following symmetry of the null vectors n and $\bar{n}$:

$$n_\mu \to \lambda n_\mu, \qquad \bar{n}_\mu \to \lambda^{-1} \bar{n}_\mu, \tag{4.4}$$

for $\lambda \in \mathbb{R} - \{0\}$. We notice this change of scale preserves $n^2 = \bar{n}^2 = 0$ and the relation (4.2).

We notice from (4.4)

$$\frac{1}{n \cdot p} \to \frac{1}{n \cdot p} \lambda^{-1}. \tag{4.5}$$

The integral (4.3) is a Lorentz vector which scales under (4.4) as λ^{-1}. There are only two vectors available, n and $\bar{n}$. The symmetry only allows $\bar{n}$. Hence,

$$I_\mu = \alpha \bar{n}_\mu. \tag{4.6}$$

To find α we contract with n^μ. Therefore,

$$I_\mu = \int d^d p \frac{r(p^2) p_\mu}{n \cdot p} = \bar{n}_\mu \int d^d p\, r(p^2). \tag{4.7}$$

Using the same symmetry arguments, we can compute a general integral

$$I = \int d^d p \frac{F(p^2, p \cdot q)}{n \cdot p}, \tag{4.8}$$

where F is an arbitrary function that depends only on the scalars p^2 and/or $p \cdot q$. The momentum p is a loop momentum to be integrated and q is an external momentum. Using the symmetry (4.4) the integral is given by

$$\int d^d p \frac{F(p^2, p \cdot q)}{n \cdot p} = (\bar{n} \cdot q) f(q^2, n \cdot q \bar{n} \cdot q), \tag{4.9}$$

where the function f is a scalar function that should be scale invariant. To determine the explicit form of f we derive both sides in (4.9) by q^μ and we contract with n_μ. Thus,

$$\int d^d p \frac{\partial F(u,v)}{\partial v} = f(x,y) + 2y \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}, \tag{4.10}$$

where we have defined $v = p \cdot q$, $x = q^2$ and $y = n \cdot q \bar{n} \cdot q$. Assuming that the solution and its partial derivatives are finite at $y = 0$ we have that $f(x, 0) = g(x)$, where

$$g(x) = \int d^d p \frac{\partial F(u,v)}{\partial v}. \tag{4.11}$$

We use this result for a function that will be continuously used in our computations later. Thus, choosing $F = (p^2 + 2p \cdot q - m^2)^{-a}$, with a integer, is clear that

$$g(x) = -2a \int d^d p \frac{1}{(p^2 + 2p \cdot q - m^2)^{a+1}}. \tag{4.12}$$

The solution of f in (4.10) with this function g is given by

$$f(x,y) = -\frac{1}{y}\left(\int d^d p \frac{1}{(p^2 - x - m^2)^a} - \int d^d p \frac{1}{(p^2 - x - 2y - m^2)^a}\right). \tag{4.13}$$

In the same way we can get the whole family of loop integrals using dimensional regularization:

$$\begin{aligned}\int d^d p \frac{1}{(p^2 + 2p \cdot q - m^2)^a} \frac{1}{(n \cdot p)^b} &= (-1)^{a+b} i \pi^\omega (-2)^b \frac{\Gamma(a+b-\omega)}{\Gamma(a)\Gamma(b)} (\bar{n} \cdot q)^b \\ &\times \int_0^1 dt t^{b-1} \frac{1}{[m^2 + q^2 - 2(n \cdot q)(\bar{n} \cdot q) t]^{a+b-\omega}},\end{aligned} \tag{4.14}$$

where $\omega = d/2$.

Tensorial integrals can be easily obtained deriving (4.14) respect to q^μ. Therefore, two useful integrals with one and two indices, are given by

$$
\begin{aligned}
\int d^d p \frac{p_\mu}{(p^2+2p\cdot q-m^2)^{a+1}} \frac{1}{(n\cdot p)^b} &= (-1)^{a+b} i\pi^\omega (-2)^{b-1} \frac{\Gamma(a+b-\omega)}{\Gamma(a+1)\Gamma(b)} (\bar{n}\cdot q)^{b-1} b\bar{n}_\mu \\
&\times \int_0^1 dt t^{b-1} \frac{1}{[m^2+q^2-2(n\cdot q)(\bar{n}\cdot q)t]^{a+b-\omega}} + \\
&(-1)^{a+b} i\pi^\omega (-2)^{b} \frac{\Gamma(a+b+1-\omega)}{\Gamma(a+1)\Gamma(b)} (\bar{n}\cdot q)^{b} \\
&\times \int_0^1 dt t^{b-1} \frac{q_\mu - t(n\cdot q\bar{n}_\mu + \bar{n}\cdot q n_\mu)}{[m^2+q^2-2(n\cdot q)(\bar{n}\cdot q)t]^{a+b+1-\omega}},
\end{aligned}
\tag{4.15}
$$

$$
\begin{aligned}
\int d^d p \frac{p_\mu p_\nu}{(p^2+2p\cdot q-m^2)^{a+2}} \frac{1}{(n\cdot p)^b} &= (-1)^{a+b} i\pi^\omega (-2)^{b-2} \Bigg\{ \frac{\Gamma(a+b-\omega)}{\Gamma(a+2)\Gamma(b-1)} (\bar{n}\cdot q)^{b-2} b\bar{n}_\mu \bar{n}_\nu \\
&\times \int_0^1 dt t^{b-1} \frac{1}{(m^2+q^2-2(n\cdot q)(\bar{n}\cdot q)t)^{a+b-\omega}} \\
&-2\frac{\Gamma(a+b+1-\omega)}{\Gamma(a+2)\Gamma(b)} (\bar{n}\cdot q)^{b-1} b\bar{n}_\mu \\
&\times \int_0^1 dt t^{b-1} \frac{q_\nu - t(n\cdot q\bar{n}_\nu + \bar{n}\cdot q n_\nu)}{(m^2+q^2-2(n\cdot q)(\bar{n}\cdot q)t)^{a+b+1-\omega}} \\
&-2\frac{\Gamma(a+b+1-\omega)}{\Gamma(a+2)\Gamma(b)} (\bar{n}\cdot q)^{b-1} b\bar{n}_\nu \\
&\times \int_0^1 dt t^{b-1} \frac{q_\mu - t(n\cdot q\bar{n}_\mu + \bar{n}\cdot q n_\mu)}{(m^2+q^2-2(n\cdot q)(\bar{n}\cdot q)t)^{a+b+1-\omega}} \\
&+4\frac{\Gamma(a+b+2-\omega)}{\Gamma(a+2)\Gamma(b)} (\bar{n}\cdot q)^{b} \\
&\times \int_0^1 dt t^{b-1} \frac{[q_\nu - t(n\cdot q\bar{n}_\nu + \bar{n}\cdot q n_\nu)][q_\mu - t(n\cdot q\bar{n}_\mu + \bar{n}\cdot q n_\mu)]}{(m^2+q^2-2(n\cdot q)(\bar{n}\cdot q)t)^{a+b+2-\omega}} \\
&-2\frac{\Gamma(a+b+1-\omega)}{\Gamma(a+2)\Gamma(b)} (\bar{n}\cdot q)^{b} \\
&\times \int_0^1 dt t^{b-1} \frac{g_{\mu\nu} - t(n_\nu \bar{n}_\mu + \bar{n}_\nu n_\mu)}{(m^2+q^2-2(n\cdot q)(\bar{n}\cdot q)t)^{a+b+1-\omega}} \Bigg\}.
\end{aligned}
\tag{4.16}
$$

We notice the result of these integrals depends on $\bar{n}$. It breaks the $SIM(2)$ invariance of Very Special Relativity. To fix it a clever idea was found in [30] where the null vector $\bar{n}$ is traded by a linear combination of n and the external momentum q as

$$\bar{n}_\mu = \alpha n_\mu + \beta q_\mu, \tag{4.17}$$

with α and β constants. To determine the value of these constants we use the conditions $\bar{n} \cdot \bar{n} = 0$ and (4.2). Thus,

$$\bar{n}_\mu = -\frac{q^2}{2(n \cdot q)^2} n_\mu + \frac{q_\mu}{n \cdot q}. \tag{4.18}$$

Then, we replace in (4.14), (4.15) and (4.16) any $\bar{n}$ with (4.18) to obtain a $SIM(2)$ invariant integral. This method was proven to be useful in the computation of the photon self-energy as we will review in the next chapter. However, we need to rethink a new definition of a $SIM(2)$ prescription when photon mass propagators are involved, as we will see later in the electron self-energy, and for cases where more than one independent external momentum appear.

Chapter 5

Lower dimensional VSR: The case $1+1$

We have reviewed in the chapters 2 and 3 how to construct a model from a subgroup of the Lorentz group in four dimensions and the possibilities that it has. In order to gain a better comprehension of the differences with the standard case and how these new elements work, we can use as laboratories lower-dimensional cases. In the standard case, two-dimensional models have been widely studied (a detailed exposition of models can be found in [64]). In QED_2, the two-dimensional realisation of QED, an exact solution was discovered by Schwinger[65]. In this chapter, we will revisit QED_2 and the Schwinger solution under the light of VSR. The content in this chapter is based on the results that we presented in [37].

5.1 Lorentz group in $1+1$ dimensions

First, we will analyse the Lorentz group in $1+1$ dimensions. We use the metric as $\eta = diag(1, -1)$. Thus, the metric and any Lorentz transformation, which satisfy (2.10), can be represented by 2×2 matrices. From (2.15) we got that $\omega_{\mu\nu}$ is antisymmetric. Due to this antisymmetry, there is only one free parameter. In consequence, we have one generator that we will call K, defined as

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5.1}$$

This observation shows us there are no subgroups of the Lorentz group in $1+1$, because there is no groups with less parameters than one. However, let us see the general transformation from the generator. Any Lorentz transformation can be constructed from K through

$$\Lambda(\theta) = e^{K\theta}, \tag{5.2}$$

for an arbitrary parameter θ. From here, we get

$$\Lambda = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}. \tag{5.3}$$

It is straightforward to check that (5.3) satisfies (2.10). Consider the null vector

$$n = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{5.4}$$

We observe that

$$\Lambda n = e^{\theta} n. \tag{5.5}$$

Since this vector transforms with a phase, exactly as in the VSR four dimensional case, we can add in the lagrangian the same terms reviewed in the chapter 3. In the four dimensional case these new terms are invariant under $SIM(2)$ but no under Lorentz. Nevertheless, here the new terms are Lorentz invariant. This observation introduces the possibility to reexplore some known two dimensional models since a priori these terms are equally possible than the standards.

5.2 Classical aspects in QED_2

First, we begin with the free fermion without standard mass. Thus, the equation (3.1) now reads

$$\mathcal{L}_0 = \bar{\psi}\left(i\partial\!\!\!/ + i\frac{m^2}{2}\frac{n\!\!\!/}{n\cdot\partial}\right)\psi. \tag{5.6}$$

We will use the following representation of the gamma matrices in $1+1$,

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{5.7}$$

Also, we will define $\gamma^5 = \gamma^0\gamma^1$. In this representation, γ^5 is given by

$$\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.8}$$

In chapter 3 we saw when there is no standard mass M there are two conserved currents, vector and axial, (3.9) and (3.11) respectively. We can relate both currents in this case, since for two-dimensional gamma matrices the following relation is satisfied

$$\gamma^\mu\gamma^5 = -\epsilon^{\mu\nu}\gamma_\nu, \tag{5.9}$$

where $\epsilon^{\mu\nu}$ is the Levi-Civita symbol in two dimensions. Therefore, using (5.9) in (3.11) we can show that

$$j^{\mu 5} = -\epsilon^{\mu\nu}j_\nu + \frac{m^2}{2}\left(\frac{1}{n\cdot\partial}\bar{\psi}\right)\left(\epsilon^{\mu\nu}n\!\!\!/ n_\nu + n\!\!\!/\gamma^5 n^\mu\right)\left(\frac{1}{n\cdot\partial}\psi\right). \tag{5.10}$$

Since $n^0 = n^1 = 1$ and $n_0 = -n_1 = 1$ we found $\epsilon^{\mu\nu}n\!\!\!/ n_\nu + n\!\!\!/\gamma^5 n^\mu = 0$. Thus

$$j^{\mu 5} = -\epsilon^{\mu\nu}j_\nu, \tag{5.11}$$

as in the standard case.

Now, we add an external electromagnetic field. To couple our fermion to this field we use the covariant derivative. Now, the lagrangian is

$$\mathcal{L} = \bar{\psi}\left(i\not{D} + \frac{i}{2}m^2\frac{\not{n}}{n\cdot D}\right)\psi. \tag{5.12}$$

Doing integration by parts the lagrangian (5.12) can be written as

$$\mathcal{L} = -i(D^\dagger_\mu\bar{\psi})\gamma^\mu\psi - \frac{i}{2}m^2\left(\frac{1}{n\cdot D^\dagger}\bar{\psi}\right)\not{n}\psi, \tag{5.13}$$

where $D^\dagger_\mu = \partial_\mu - ieA_\mu$.

With (5.12) and (5.13) we get the equations of motion

$$\left(\not{D} + \frac{1}{2}m^2\frac{\not{n}}{n\cdot D}\right)\psi = 0, \tag{5.14}$$

$$(D^\dagger_\mu\bar{\psi})\gamma^\mu + \frac{1}{2}m^2\left(\frac{1}{n\cdot D^\dagger}\bar{\psi}\right)\not{n} = 0. \tag{5.15}$$

Multiplying by $\bar{\psi}$ in the left in (5.14) and by ψ in the right in (5.15) and adding both we get

$$\partial_\mu(\bar{\psi}\gamma^\mu\psi) + \frac{1}{2}m^2\left[\bar{\psi}\not{n}\left(\frac{1}{n\cdot D}\psi\right) + \left(\frac{1}{n\cdot D^\dagger}\bar{\psi}\right)\not{n}\psi\right] = 0. \tag{5.16}$$

Expanding the operator $(n\cdot D)^{-1}$ to the first order in e we get

$$\partial_\mu\left[\bar{\psi}\gamma^\mu\psi + \frac{1}{2}m^2\left(\frac{1}{n\cdot\partial}\bar{\psi}\right)\not{n}n^\mu\left(\frac{1}{n\cdot\partial}\psi\right) + \frac{1}{2}m^2 ie\left(\frac{1}{n\cdot\partial}n\cdot A\frac{1}{n\cdot\partial}\bar{\psi}\right)\not{n}n^\mu\left(\frac{1}{n\cdot\partial}\psi\right)\right.$$
$$\left.-\frac{1}{2}m^2 ie\left(\frac{1}{n\cdot\partial}\bar{\psi}\right)\not{n}n^\mu\left(\frac{1}{n\cdot\partial}n\cdot A\frac{1}{n\cdot\partial}\psi\right)\right] = 0. \tag{5.17}$$

In this way we define the current j^μ as

$$\begin{aligned} j^\mu \;=\;& \bar{\psi}\gamma^\mu\psi + \frac{1}{2}m^2\left(\frac{1}{n\cdot\partial}\bar{\psi}\right)\not{n}n^\mu\left(\frac{1}{n\cdot\partial}\psi\right) + \frac{1}{2}iem^2\left(\frac{1}{n\cdot\partial}n\cdot A\frac{1}{n\cdot\partial}\bar{\psi}\right)\not{n}n^\mu\left(\frac{1}{n\cdot\partial}\psi\right) \\ & -\frac{1}{2}iem^2\left(\frac{1}{n\cdot\partial}\bar{\psi}\right)\not{n}n^\mu\left(\frac{1}{n\cdot\partial}n\cdot A\frac{1}{n\cdot\partial}\psi\right). \end{aligned} \tag{5.18}$$

The addition of the electromagnetic field modifies the current due to the existence of the non-local operator $(n\cdot D)^{-1}$. We notice new terms appear. The first two terms in (5.18) are the free current (3.9), while the next terms come from the addition of the external field. Despite this new addition, this current is conserved as we see in (5.17).

A similar treatment can be done to get the axial current

$$\begin{aligned} j^{\mu 5} &= \bar{\psi}\gamma^{\mu}\gamma^{5}\psi + \frac{1}{2}m^2\left(\frac{1}{n\cdot\partial}\bar{\psi}\right)\not{n}n^{\mu}\gamma^{5}\left(\frac{1}{n\cdot\partial}\psi\right) + \frac{1}{2}iem^2\left(\frac{1}{n\cdot\partial}n\cdot A\frac{1}{n\cdot\partial}\bar{\psi}\right)\not{n}n^{\mu}\gamma^{5}\left(\frac{1}{n\cdot\partial}\psi\right) \\ &\quad -\frac{1}{2}iem^2\left(\frac{1}{n\cdot\partial}\bar{\psi}\right)\not{n}n^{\mu}\gamma^{5}\left(\frac{1}{n\cdot\partial}n\cdot A\frac{1}{n\cdot\partial}\psi\right). \end{aligned} \tag{5.19}$$

The first two terms in (5.19) are the axial current in the free case (3.11). As in the vector current case, despite the new terms that appear, the axial current is conserved. Moreover, the new terms do not modify the relation (5.11), that we will use in the quantum computation.

5.3 Photon Self-Energy

We will move on to the quantum level. To proceed, we use the path integral formalism. In this way, for the fermion under the influence of an external electromagnetic field we have

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left\{i\int d^2x\bar{\psi}\left(i\not{D} + i\frac{m^2}{2}\frac{\not{n}}{n\cdot D}\right)\psi\right\}. \tag{5.20}$$

We proceed to compute the photon self-energy. In chapter 3, we observed that $(n\cdot D)^{-1}$ generates an infinite number of vertices after expanding in series this operator. The vertex with two external photonic legs plays an important role here. Now, a new diagram can be constructed in addition to the standard, as we show in figure 5.1.

Figure 5.1: The diagrams corresponding to the photon self-energy in VSR.

Using the rules in table 3.1 we get

$$\begin{aligned} i\Pi_{\mu\nu} &= \int \frac{d^2p}{(2\pi)^2} tr\{V_{1\mu}[(p-q,\bar{n}_2),(p,\bar{n}_1)]S_F(p,\bar{n}_1)V_{1\nu}[(p,\bar{n}_1),(p-q,\bar{n}_2)]S_F(p-q,\bar{n}_2)\} \\ &+ \int \frac{d^2p}{(2\pi)^2} tr\{V_{2\mu\nu}[(p,\bar{n}_3),(p,\bar{n}_3),(p+q,\bar{n}_4),(p-q,\bar{n}_5)]S_F(p,\bar{n}_3)\}. \end{aligned} \tag{5.21}$$

The first line corresponds to the left diagram in figure 5.1. The second line to the diagram in the right. The momentum q is the photon momentum. As we said in the previous chapter, a priori, we could use different $\bar{n}$ for each non-local term. Hence, we put different $\bar{n}$ in both diagrams when the momenta are different.

Since the photon field possesses gauge invariance the Ward identity should be respected. In consequence the prescriptions used should preserve it. We will use the Ward identity to define all the $\bar{n}$ used. We contract (5.21) with q^μ and we get

$$\begin{aligned} iq^\mu\Pi_{\mu\nu} &= \int \frac{d^2p}{(2\pi)^2} tr\{q^\mu V_{1\mu}[(p-q,\bar{n}_2),(p,\bar{n}_1)]S_F(p,\bar{n}_1)V_{1\nu}[(p,\bar{n}_1),(p-q,\bar{n}_2)]S_F(p-q,\bar{n}_2)\} \\ &+ \int \frac{d^2p}{(2\pi)^2} tr\{q^\mu V_{2\mu\nu}[(p,\bar{n}_3),(p,\bar{n}_3),(p+q,\bar{n}_4),(p-q,\bar{n}_5)]S_F(p,\bar{n}_3)\}. \end{aligned} \tag{5.22}$$

We use the relations (3.32) and (3.33) and after a change of variable we get

$$\begin{aligned} iq^\mu\Pi_{\mu\nu} &= \int \frac{d^2p}{(2\pi)^2} tr\{V_{1\nu}[(p,\bar{n}_1),(p-q,\bar{n}_2)]S_F(p-q,\bar{n}_2)\} \\ &- \int \frac{d^2p}{(2\pi)^2} tr\{S_F(p,\bar{n}_1)V_{1\nu}[(p,\bar{n}_1),(p-q,\bar{n}_2)]\} \\ &+ \int \frac{d^2p}{(2\pi)^2} tr\{S_F(p,\bar{n}_3)V_{1\nu}[(p,\bar{n}_3),(p-q,\bar{n}_5)]\} \\ &- \int \frac{d^2p}{(2\pi)^2} tr\{V_{1\nu}[(p,\bar{n}_4),(p-q,\bar{n}_3)]S_F(p-q,\bar{n}_3)\}. \end{aligned} \tag{5.23}$$

To get $q^\mu\Pi_{\mu\nu}=0$, the relation between all the $\bar{n}$ is

$$\bar{n}_1 = \bar{n}_3 = \bar{n}_4 = \bar{n}_2 = \bar{n}_5. \tag{5.24}$$

It means we use the same $\bar{n}$ in all the integrations in the photon self-energy. The same can be done for more legs in a loop. Therefore, the rule for a loop computation is that we use the same $\bar{n}$ in all the integrals.

Now we proceed to compute explicitly the self-energy. Thus, with the expressions of the vertices and propagators (5.21) reads

$$\begin{aligned} i\Pi_{1\mu\nu} &= -e^2\int\frac{d^2p}{(2\pi)^2}\frac{1}{(p^2-m^2+i\varepsilon)((p-q)^2-m^2+i\varepsilon)}tr\left\{\left(\gamma_\mu+\frac{1}{2}m^2\frac{\not{n}n_\mu}{n\cdot pn\cdot(p-q)}\right)\right. \\ &\left.\times\left(\not{p}-\frac{m^2}{2}\frac{\not{n}}{n\cdot p}\right)\left(\gamma_\nu+\frac{1}{2}m^2\frac{\not{n}n_\nu}{n\cdot pn\cdot(p-q)}\right)\left(\not{p}-\not{q}-\frac{m^2}{2}\frac{\not{n}}{n\cdot(p-q)}\right)\right\}, \end{aligned} \tag{5.25}$$

$$\begin{aligned} i\Pi_{2\mu\nu} &= -\frac{1}{2}e^2m^2n_\mu n_\nu\int\frac{d^2p}{(2\pi)^2}\frac{1}{(n\cdot p)^2}\left(\frac{1}{n\cdot(p+q)}+\frac{1}{n\cdot(p-q)}\right)\frac{1}{p^2-m^2+i\varepsilon} \\ & tr\left\{\not{n}\left(\not{p}-\frac{m^2}{2}\frac{\not{n}}{n\cdot p}\right)\right\}, \end{aligned} \tag{5.26}$$

where $\Pi_{1\mu\nu}$ is the expression corresponding to the diagram in the left in figure 5.1 and $\Pi_{2\mu\nu}$ the diagram in the right.

In order to compute the integrals, we will use dimensional regularization. Therefore, we will do the integrals in d dimensions. To use the integrations that we reviewed in chapter 4 we need to split products like $[(n\cdot p)(n\cdot(p-q))]^{-1}$. To do it, we will use the decomposition formula

$$\frac{1}{(n\cdot(p+k_i))(n\cdot(p+k_j))}=\frac{1}{n\cdot(k_i-k_j)}\left(\frac{1}{(n\cdot(p+k_j)}-\frac{1}{n\cdot(p+k_i)}\right). \tag{5.27}$$

We use the integrals (4.14), (4.15) and (4.16) and as we have a unique $\bar{n}$, we write this vector as in (4.18) where the external momentum used here is the only momentum available, q. Thus, we get

$$i\Pi_{\mu\nu}=\alpha(q^2)\left[q^2\eta_{\mu\nu}-q_\mu q_\nu\right]+\beta(q^2)\left[-\eta_{\mu\nu}+\frac{n_\nu q_\mu+n_\mu q_\nu}{n\cdot q}-q^2\frac{n_\mu n_\nu}{(n\cdot q)^2}\right], \tag{5.28}$$

where

$$\alpha(q^2) = -\frac{ie^2}{\pi}\int_0^1 dx\left(\frac{x(1-x)}{m^2-x(1-x)q^2-i\varepsilon}\right), \tag{5.29}$$

$$\beta(q^2) = \frac{ie^2m^2}{2\pi}\int_0^1 dx\frac{xq^2}{(m^2-xq^2-i\varepsilon)(m^2-x(1-x)q^2-i\varepsilon)}. \tag{5.30}$$

We observe that (5.28) satisfies the Ward identity $q_\mu\Pi^{\mu\nu} = q_\nu\Pi^{\mu\nu} = 0$. This implies the prescription used preserve the gauge invariance and our result is gauge invariant. In addition, $\alpha(0)$ is a finite number and $\beta(0) = 0$. This result shows that there is no pole in $q = 0$ and in consequence the photon does not receive mass. This is an interesting aspect of the model with respect to the standard. Setting $m = 0$ we recover the standard computation. In this case $\Pi_{\mu\nu}$ reads

$$i\Pi_{\mu\nu} = \frac{ie^2}{\pi}\left(\eta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right). \tag{5.31}$$

This means the photon receives a mass e^2/π. This is the only case in the standard computation where this situation occurs. In any other dimension the photon remains massless. The addition of the new invariant terms keep in the two dimensional case the photon massless unless we add the new VSR mass term for the photon. However, we recall this term from (3.25),

$$\frac{m_\gamma^2}{2}(n^\alpha F_{\mu\alpha})\frac{1}{(n\cdot\partial)^2}(n_\beta F^{\mu\beta}) = \frac{m_\gamma^2}{2}(n^\alpha F_{0\alpha})\frac{1}{(n\cdot\partial)^2}(n_\beta F^{0\beta}) + \frac{m_\gamma^2}{2}(n^\alpha F_{1\alpha})\frac{1}{(n\cdot\partial)^2}(n_\beta F^{1\beta}). \tag{5.32}$$

Using the antisymmetry of F and $n^0 = n_0 = n^1 = -n_1 = 1$ we get

$$\frac{m^2}{2}(n^\alpha F_{\mu\alpha})\frac{1}{(n\cdot\partial)^2}(n_\beta F^{\mu\beta}) = 0. \tag{5.33}$$

Hence, in the two-dimensional case, there is no way to give mass to the photon.

The existence of $m \neq 0$ adds another new feature. In the limit $\epsilon \to 0$ we observe in (5.29) there is a branch cut where $m^2 - x(1-x)q^2 < 0$. The product $x(1-x)$ is at most $1/4$. Hence, the branch cut begins at $q^2 = 4m^2$, which corresponds to the threshold for the creation of an electron-positron pair. Hence, in this model, we have the possibility of pair production.

5.4 Axial Anomaly

Now, we come back to the currents. We reviewed previously that the vector current as well as the axial current, are classically conserved. In the standard case at the quantum level, the axial current is no longer conserved. We call anomaly when a quantity is classically conserved, but it is not conserved in the quantum case. Here, we want to see how the new terms affect the anomaly.

First, we will compute the expectation value for the vector current, defined as

$$\langle j^\mu \rangle = \frac{1}{Z}\int \mathcal{D}\bar{\psi}\mathcal{D}\psi j^\mu \exp\left[i\int d^2x\mathcal{L}\right]. \tag{5.34}$$

We notice if we work in the light-cone gauge $n\cdot A=0$ the non-local operator $(n\cdot D)^{-1}$ turns into $(n\cdot\partial)^{-1}$. In consequence, there is only one vertex, with one external photonic leg. The other vertices vanishes in this gauge. This observation simplifies the computation. Hence, operating in the light cone gauge,

$$\langle j^\mu \rangle = \frac{1}{Z}\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\left(\bar{\psi}\left(\gamma^\mu+\frac{m^2}{2}\frac{\not{n}n^\mu}{(n\cdot\overleftarrow{\partial})(n\cdot\partial)}\right)\psi\right)\exp\left[i\int d^2x\mathcal{L}_0\right]\exp\left[-ie\int d^2x\bar{\psi}\not{A}\psi\right], \tag{5.35}$$

where $\mathcal{L}_0$ is the free fermion lagrangian (5.6).

In the standard case, the axial anomaly is exact at one loop by topological reasons[66]. The argument holds here. Since topological quantities cannot change continuously, perturbative corrections at higher than one loop should not appear. Then, we will compute only to one loop. Therefore, the equation (5.35) reads

$$\begin{aligned}\langle j^\mu(x)\rangle &= \lim_{x'\to x} tr\left[\left(\gamma^\mu+\frac{1}{2}m^2\frac{1}{n\cdot\partial_{x'}}\not{n}n^\mu\frac{1}{n\cdot\partial_x}\right)S_F(x-x')\right]\\ &- \lim_{x'\to x} ie\int d^2y \quad tr\left[\left(\gamma^\mu+\frac{1}{2}m^2\frac{1}{n\cdot\partial_{x'}}\not{n}n^\mu\frac{1}{n\cdot\partial_x}\right)S_F(x-y)\not{A}(y)S_F(y-x')\right].\end{aligned} \tag{5.36}$$

Here, we have replaced the dependence on x in $\bar{\psi}$ with the limit $x' \to x$ only to distinguish where the non-local operator $n\cdot\partial$ acts. Moreover, within this spirit, we indicate with a subscript in the partial derivatives the variable to be derived. Now, we pass to the Fourier space and we get

$$\begin{aligned} \langle j^\mu(q)\rangle &= (-ie)\int \frac{d^2p}{(2\pi)^2} tr\left[\left(\gamma^\mu + \frac{1}{2}m^2\frac{\not{n}n^\mu}{(n\cdot(p-q))(n\cdot p)}\right)\frac{i}{\not{p} - \frac{m^2}{2}\frac{\not{n}}{n\cdot p}}(\gamma^\nu)\right. \\ &\quad \left.\times\frac{i}{\not{p} - \not{q} - \frac{m^2}{2}\frac{\not{n}}{n\cdot(p-q)}}\right] A_\nu(q). \end{aligned} \tag{5.37}$$

The right hand side of (5.37) corresponds to the photon self-energy after using the condition $n\cdot A = 0$ and multiplies by a factor i/e. Hence, we can write

$$\langle j^\mu(q)\rangle = \frac{i}{e}\Pi^{\mu\nu}A_\nu(q). \tag{5.38}$$

Using (5.28) the equation (5.38) reads

$$\langle j^\mu(q)\rangle = \frac{i}{e}\alpha(q^2)\left[q^2A^\mu - q^\mu q\cdot A\right] + \frac{i}{e}\beta(q^2)\left[-A^\mu + \frac{n^\mu q\cdot A}{n\cdot q}\right]. \tag{5.39}$$

Since the photon self-energy satisfies the Ward identity, the expectation value of the vector current is quantum conserved, $q_\mu\langle j^\mu(q)\rangle = 0$.

For the axial current $j^{\mu 5}$ we use the relation (5.11) to get its expectation value. Using the equation (5.39) we get

$$j^{\mu 5} = -\frac{i}{e}\alpha(q^2)\epsilon^{\mu\nu}\left[q^2A_\nu - q_\nu q\cdot A\right] - \frac{i}{e}\beta(q^2)\epsilon^{\mu\nu}\left[-A_\nu + \frac{n_\nu q\cdot A}{n\cdot q}\right]. \tag{5.40}$$

We contract (5.40) with q_μ and we write it in terms of $F_{\mu\nu}$. Therefore,

$$q_\mu\langle j^{\mu 5}\rangle = -\frac{i}{e}\left[(\alpha(q^2)q^2 - \beta(q^2))\frac{1}{2}\varepsilon^{\mu\nu}F_{\mu\nu}(q) - \beta(q^2)\frac{n^\alpha q^\beta F_{\alpha\beta}}{(n\cdot q)^2}\epsilon^{\mu\nu}q_\mu n_\nu\right]. \tag{5.41}$$

Since we are working in $1+1$ dimensions, the following identity holds

$$n^\alpha q^\beta F_{\alpha\beta} = \frac{1}{2}\varepsilon^{\mu\nu} n_\mu q_\nu \varepsilon^{\alpha\beta} F_{\alpha\beta}. \tag{5.42}$$

Using (5.42) in (5.41) we get

$$q_\mu \langle j^{\mu 5}\rangle = -\frac{i}{e}\left[(\alpha(q^2)q^2 - \beta(q^2))\frac{1}{2}\varepsilon^{\mu\nu}F_{\mu\nu}(q) + \beta(q^2)\frac{(\varepsilon^{\alpha\beta}n_\alpha q_\beta)^2}{2(n\cdot q)^2}\varepsilon^{\mu\nu}F_{\mu\nu}\right]. \tag{5.43}$$

In addition, using $n_0 = -n_1 = 1$ we get as result

$$\frac{\varepsilon^{\alpha\beta}n_\alpha q_\beta}{n\cdot q} = 1, \tag{5.44}$$

Therefore, the terms with $\beta(q^2)$ cancel out and we get

$$q_\mu \langle j^{\mu 5}\rangle = -\frac{i}{2e}\alpha(q^2)q^2\varepsilon^{\mu\nu}F_{\mu\nu}(q), \tag{5.45}$$

Notice that the anomaly term $\varepsilon^{\mu\nu}F_{\mu\nu}$ remains unchanged respect to the standard case. It should not be a surprise, because it is the only element that we can construct contracting F and the Levi-Civita symbol. The only modification is in the coefficient of $\varepsilon^{\mu\nu}F_{\mu\nu}$. An interesting possibility is to explore the anomaly in higher dimensions since there are more indices to contract and new terms involving the vector n can be constructed. Observing the equation (5.45), in the limit $m \to 0$ we recover the standard result. In the limit $q \to 0$ we notice the right hand side of (5.45) vanishes. The result above presents an interpolation between two different cases. The first one, in the large momentum regime (short distances), where $q \to \infty$, which is equivalent to take $m \to 0$ in (5.45), the axial anomaly is present. The second case, corresponding to a low momentum regime (large distances), there is no anomaly.

Chapter 6

Computations in 4D

After our excursion in lower dimensions, we come back to the four dimensional case. In the previous chapter, we observe that there is no photon mass in two dimensions. In four dimensions, the mass term for the photon will be relevant, and it has some subtleties. The first computation to see this aspect will be the electron self-energy, where the photon propagator appears. Later, we will review the cancellation of infrared divergences.

6.1 Electron self-energy

As in the photon self-energy, the existence of new vertices implies a new diagram to compute in the electron self-energy, as we see in figure 6.1

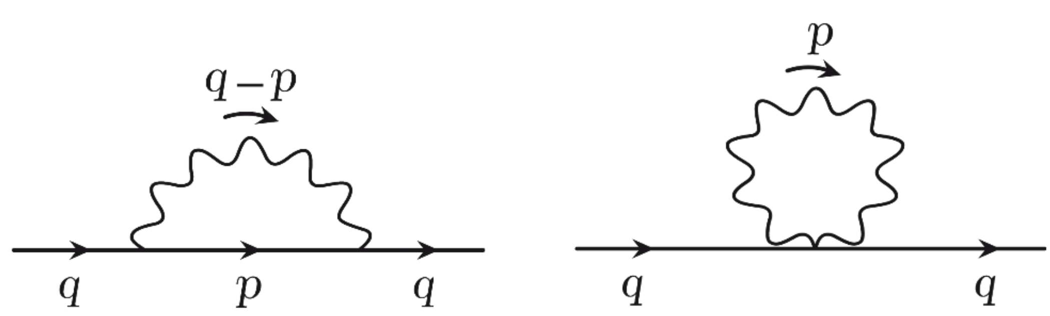

Figure 6.1: Electron Self Energy diagrams in VSR.

Using the Feynman rules in table 3.1 and working in dimensional regularization we have

$$
\begin{aligned}
-i\Sigma(q) \;=\; & (-ie)^2 \int \frac{d^d p}{(2\pi)^{2\omega}} \frac{1}{(p^2 - M_e^2 + i\varepsilon)((q-p)^2 - m_\gamma^2 + i\varepsilon)} \left(\gamma_\mu + \frac{1}{2} m^2 \frac{\not{n} n_\mu}{n \cdot q n \cdot p}\right) \times \\
& \times \left(\not{p} + M - \frac{1}{2} m^2 \frac{\not{n}}{n \cdot p}\right) \left(\gamma_\nu + \frac{1}{2} m^2 \frac{\not{n} n_\nu}{n \cdot q n \cdot p}\right) g_{\mu\nu} \\
& (-ie)^2 m_\gamma^2 \int \frac{d^d p}{(2\pi)^{2\omega}} \frac{1}{(p^2 - M_e^2 + i\varepsilon)((q-p)^2 - m_\gamma^2 + i\varepsilon)} \left(\gamma_\mu + \frac{1}{2} m^2 \frac{\not{n} n_\mu}{n \cdot q n \cdot p}\right) \times \\
& \times \left(\not{p} + M - \frac{1}{2} m^2 \frac{\not{n}}{n \cdot p}\right) \left(\gamma_\nu + \frac{1}{2} m^2 \frac{\not{n} n_\nu}{n \cdot q n \cdot p}\right) \\
& \times \left[\frac{n_\mu n_\nu}{[n \cdot (q-p)]^2} - \frac{(q_\mu - p_\mu) n_\nu + (q_\nu - p_\nu) n_\mu}{(q-p)^2 [n \cdot (q-p)]}\right]. \qquad (6.1)
\end{aligned}
$$

In the expression (6.1) we have omitted the subscripts $\bar{n}$ to make easier the reading. We notice that setting $m = 0$ and $m_\gamma = 0$ we recover the standard computation. Here the parameter m_γ plays the role of the small photon mass introduced by hand in the standard case to regularize the infrared divergences. Here it is a parameter that comes from the theory, not a mathematical trick ad hoc.

In order to find a relation between each $\bar{n}$ in (6.1), we will use a similar procedure as we did in the photon self-energy in chapter 5. We will use the Ward identity in the vertex correction. In the standard case, the contraction between the momentum of the external photon and the vertex is a difference between two electron self energies. Here, since we have new possible vertices, now we have additional diagrams that we show in figure 6.2. However, we can write this difference too.

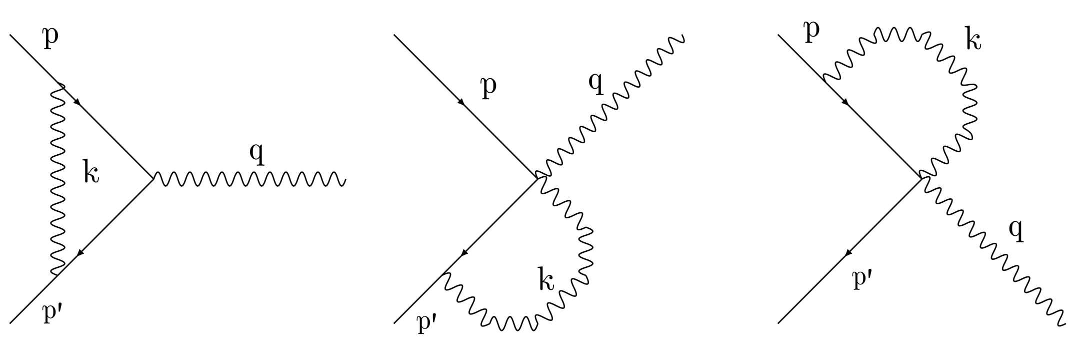

Figure 6.2: Loop correction to the vertex.

We write the sum of the three diagrams in figure 6.2 as Λ^ρ and it is

$$
\begin{aligned}
\bar{u}(p')\Lambda^\rho u(p) \;=\;& \bar{u}(p',\bar{n}_1)V_1^\mu[(p'+k,\bar{n}_2),(p',\bar{n}_1)]S_F(p'+k,\bar{n}_2)V_1^\rho[(p'+k,\bar{n}_2),(p+k,\bar{n}_3)] \\
& \times S_F(p+k,\bar{n}_3)V_1^\nu[(p,\bar{n}_4),(p+k,\bar{n}_3)]\Delta_{\mu\nu}(k,\tilde{\bar{n}}_1)u(p,\bar{n}_4)+ \\
& +\bar{u}(p',\bar{\bar{n}}_1)V_1^\mu[(p'+k,\bar{\bar{n}}_2),(p',\bar{\bar{n}}_1)]S_F(p'+k,\bar{\bar{n}}_2) \\
& \times V_2^{\nu\rho}[(p,\bar{\bar{n}}_3),(p'+k,\bar{\bar{n}}_2),(p',\bar{\bar{n}}_1),(p+k,\bar{\bar{n}}_4)]\Delta_{\mu\nu}(k,\tilde{\bar{n}}_2)u(p,\bar{\bar{n}}_3)+ \\
& +\bar{u}(p',\tilde{n}_1)V_2^{\mu\rho}[(p-k,\tilde{n}_2),(p',\tilde{n}_1),(p,\tilde{n}_3),(p'-k,\tilde{n}_4)]S_F(p-k,\tilde{n}_2) \\
& \times V_1^\nu[(p,\tilde{n}_3),(p-k,\tilde{n}_2)]\Delta_{\mu\nu}(k,\tilde{\tilde{n}}_3)u(p,\tilde{n}_3),
\end{aligned}
\tag{6.2}
$$

where the $u(p)$ and $\bar{u}(p')$ are the external incoming/outcoming fermionic legs. This expression contains an integral in k that we have omitted only by simplicity in the writing. Now, we contract (6.2) with the external photonic leg q. Using the relations for the vertices (3.32) and (3.34) we get

$$
\begin{aligned}
& \bar{u}(p')\Lambda^\rho u(p) = \bar{u}(p',\bar{n}_1)V_1^\mu[(p'+k,\bar{n}_2),(p',\bar{n}_1)]S_F(p+k,\bar{n}_3)V_1^\nu[(p,\bar{n}_4),(p+k,\bar{n}_3)] \\
& \times\Delta_{\mu\nu}(k,\tilde{\bar{n}}_1)u(p,\bar{n}_4) \\
& -\bar{u}(p',\bar{n}_1)V_1^\mu[(p'+k,\bar{n}_2),(p',\bar{n}_1)]S_F(p'+k,\bar{n}_2)V_1^\nu[(p,\bar{n}_4),(p+k,\bar{n}_3)]\Delta_{\mu\nu}(k,\tilde{\bar{n}}_1)u(p,\bar{n}_4) \\
& \bar{u}(p',\bar{\bar{n}}_1)V_1^\mu[(p'+k,\bar{\bar{n}}_2),(p',\bar{\bar{n}}_1)]S_F(p'+k,\bar{\bar{n}}_2)V_1^\nu[(p,\bar{\bar{n}}_3),(p+k,\bar{\bar{n}}_4)]\Delta_{\mu\nu}(k,\tilde{\bar{n}}_2)u(p,\bar{\bar{n}}_3) \\
& -\bar{u}(p',\bar{\bar{n}}_1)V_1^\mu[(p'+k,\bar{\bar{n}}_2),(p',\bar{\bar{n}}_1)]S_F(p'+k,\bar{\bar{n}}_2)V_1^\nu[(p',\bar{\bar{n}}_1),(p'+k,\bar{\bar{n}}_2)]\Delta_{\mu\nu}(k,\tilde{\bar{n}}_2)u(p,\bar{\bar{n}}_3) \\
& \bar{u}(p',\tilde{n}_1)V_1^\mu[(p+k,\tilde{n}_2),(p,\tilde{n}_3)]S_F(p+k,\tilde{n}_2)V_1^\nu[(p,\tilde{n}_3),(p+k,\tilde{n}_2)]\Delta_{\mu\nu}(k,\tilde{\tilde{n}}_3)u(p,\bar{n}_3) \\
& -\bar{u}(p',\tilde{n}_1)V_1^\mu[(p'+k,\tilde{n}_4),(p',\tilde{n}_1)]S_F(p+k,\tilde{n}_2)V_1^\nu[(p,\tilde{n}_3),(p+k,\tilde{n}_2)]\Delta_{\mu\nu}(k,\tilde{\tilde{n}}_3)u(p,\tilde{n}_3),
\end{aligned}
\tag{6.3}
$$

where in the last two lines we did a change of variables $k \to -k$.

To satisfy the Ward Identity, the relation between all the $\bar{n}$ is given by

$$\begin{aligned}
\bar{n}_1 &= \bar{\bar{n}}_1 = \tilde{n}_1, \\
\bar{n}_2 &= \bar{\bar{n}}_2 = \tilde{n}_4, \\
\bar{n}_3 &= \bar{\bar{n}}_4 = \tilde{n}_2, \\
\bar{n}_4 &= \bar{\bar{n}}_3 = \tilde{n}_3,
\end{aligned} \tag{6.4}$$

and

$$\tilde{\tilde{n}}_1 = \tilde{\tilde{n}}_2 = \tilde{\tilde{n}}_3. \tag{6.5}$$

With this relations we get

$$\begin{aligned}
q_\rho \Lambda^\rho = {} & V_1^\mu[(p+k,\bar{n}_3),(p,\bar{n}_4)]S_F(p+k,\bar{n}_3)V_1^\nu[(p,\bar{n}_4),(p+k,\bar{n}_3)]\Delta_{\mu\nu}(k,\tilde{\tilde{n}}_1) \\
& -V_1^\mu[(p'+k,\bar{n}_2),(p',\bar{n}_1)]S_F(p'+k,\bar{n}_2)V_1^\nu[(p',\bar{n}_1),(p'+k,\bar{n}_2)]\Delta_{\mu\nu}(k,\tilde{\tilde{n}}_1),
\end{aligned} \tag{6.6}$$

where we have omitted the external legs u and $\bar{u}$. Diagramatically, equation (6.6) is represented in figure 6.3.

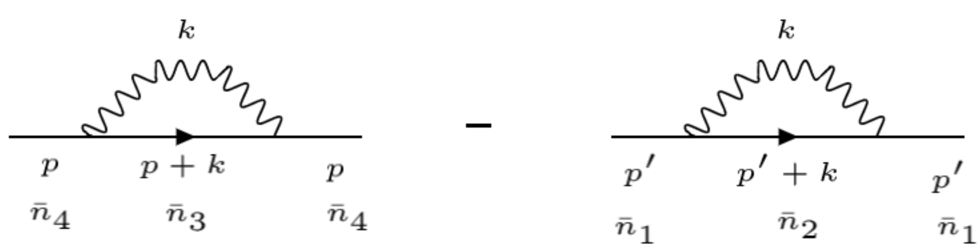

Figure 6.3: Ward identity, vertex written as difference between two propagators with different electron momentum.

In the case of the photon self-energy we observed all the $\bar{n}$ are the same and this allows to trade this unique $\bar{n}$ as a linear combination of n and the external momentum. In this situation, from the figure 6.3, we observe that as $\bar{n}_3$ as $\bar{n}_4$ have the same external momentum p and $\bar{n}_1$ with $\bar{n}_2$ share the external momentum $p' = p + k$. We adopt the same idea that we used in the photon self-energy, so $\bar{n}_1 = \bar{n}_2$ and $\bar{n}_3 = \bar{n}_4$. With this rule, and considering $m_\gamma = 0$, there is no problem to compute the electron self-energy using the integrals (4.14), (4.15) and (4.16) and replacing $\bar{n}$ with the definition (4.18).

The problem appears when we allow the photon mass, since $\bar{n}$ is required in the photon propagator. Furthermore, as we saw before, the relation (6.5) is satisfied, corresponding to the photon propagator in each diagram in figure 6.2. With this, the two diagrams in figure 6.3 have the same $\bar{n}$ for their photon propagators, but the external momenta are different. The only common momentum for both is the vector zero, but this vector does not respect the condition $n \cdot \bar{n} = 1$.

To solve this problem we will proceed in the limit sense. We will use an arbitrary and common momentum P for both and at the end we will use the limit $P \to 0$ to eliminate the arbitrariness. Therefore, the Mandelstam-Leibbrandt prescription in the equation (4.1) after the replacement of $\bar{n}$ using the P vector reads

$$\frac{1}{n \cdot k} = \lim_{\eta \to 0} \lim_{P \to 0} \frac{-P^2 n \cdot k + 2P \cdot k n \cdot P}{(n \cdot k)(-P^2 n \cdot k + 2P \cdot k n \cdot P) + i\eta}, \tag{6.7}$$

where we have defined $\eta = 2\epsilon (n \cdot P)^2$ which satisfies $\eta > 0$. Notice that for $(n \cdot P) \neq 0$, $lim_{\epsilon \to 0}$ coincides with $lim_{\eta \to 0}$, so we use as a definition the expression (6.7). In the limit $P \to 0$ and keeping η not zero, the fraction $1/n \cdot k$ vanishes without problem.

With these considerations, we compute all the integrals in 6.1 and we get

$$-i\Sigma(q) = C \frac{\not{n}}{n \cdot q} + D \not{q} + E, \tag{6.8}$$

where the coefficients C, D and E are

$$\begin{aligned}
C &= (-ie)^2 m^2 \left[-\frac{i}{16\pi^2} \int_0^1 dx \frac{1}{x} \log\left(1 + \frac{x^2 q^2}{(1-x)M_e^2 - xq^2 + xm_\gamma^2 - i\varepsilon}\right) \right. \\
&\quad + \frac{2i}{(4\pi)^\omega} \int_0^1 dx \frac{\Gamma(2-\omega)}{[(1-x)M_e^2 - x(1-x)q^2 + xm_\gamma^2 - i\varepsilon]^{2-\omega}} \\
&\quad \left. + \frac{i}{8\pi^2} \int_0^1 dx \log\left(1 + \frac{m_\gamma^2 (1-x)}{xM_e^2 - x(1-x)q^2}\right) \right], \qquad (6.9)\\
D &= -2(-ie)^2(\omega - 1) \frac{i}{(4\pi)^\omega} \int_0^1 dx \frac{x\Gamma(2-\omega)}{[(1-x)M_e^2 - x(1-x)q^2 + xm_\gamma^2 - i\varepsilon]^{2-\omega}} \\
&\quad + \frac{i}{8\pi^2} \int_0^1 dx \log\left(1 + \frac{m_\gamma^2 (1-x)}{xM_e^2 - x(1-x)q^2}\right), \qquad (6.10)\\
E &= (-ie)^2 2\omega M \frac{i}{(4\pi)^\omega} \int_0^1 dx \frac{\Gamma(2-\omega)}{[(1-x)M_e^2 - x(1-x)q^2 + xm_\gamma^2 - i\varepsilon]^{2-\omega}} \\
&\quad + M \frac{i}{8\pi^2} \int_0^1 dx \log\left(1 + \frac{m_\gamma^2 (1-x)}{xM_e^2 - x(1-x)q^2}\right). \qquad (6.11)
\end{aligned}$$

The process to get the result (6.8) showed that a revision of the prescription for $(n \cdot p)^{-1}$ should be done. It is not straightforward, and it required the introduction of a new definition. This change was due to the existence of the photon mass term. It could be possible to avoid this aspect setting $m_\gamma = 0$ from the beginning. However, the introduction of this parameter allows us to get a natural way to regulate infrared divergencies in the coefficients integrals. In the standard case, a fictitious photon mass is introduced as a trick to regulate; however, here, this mass plays this role naturally.

6.2 A simple process: Coulomb Scattering

After the discussion about the prescription, we will compute a simple example of a process. We will compute the Coulomb scattering, where an electron is deflected by an external electromagnetic field. First, we compute the diagram at tree level that we show in figure 6.4.

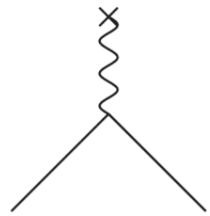

Figure 6.4: Tree level diagram for the Coulomb scattering.

The mathematical expression reads

$$i\mathcal{M} = \bar{u}(p') \left[(-ie) \left(\gamma^0 + \frac{1}{2} m^2 \frac{\not{n} n^0}{n \cdot p n \cdot p'} \right) A_0(q) \right] u(p), \tag{6.12}$$

where for this case the only non zero component in A_μ will be $A_0 = \frac{Ze^2}{|\vec{q}|^2}$. Squaring the matrix $\mathcal{M}$

$$\begin{aligned} |\mathcal{M}|^2 &= \frac{2Ze^4}{|\vec{q}|^4} \left[p^0 p'^0 + \vec{p} \cdot \vec{p'} + M_e^2 - m^2 + \frac{1}{2} m^2 \left(\frac{n \cdot p}{n \cdot p'} + \frac{n \cdot p'}{n \cdot p} \right) \right. \\ & \left. + m^2 n^0 (p^0 - p'^0) \left(\frac{1}{n \cdot p} - \frac{1}{n \cdot p'} \right) + m^2 (M_e^2 - p \cdot p') \frac{(n^0)^2}{n \cdot p n \cdot p'} \right]. \end{aligned} \tag{6.13}$$

The external field only changes the direction of the momentum, but it does not change its magnitude. Hence, $|\vec{p}| = |\vec{p'}|$. As $\vec{p'} = \vec{p} + \vec{q}$, we will have

$$|\vec{q}|^2 = 4 |\vec{p}|^2 \sin^2 \frac{\theta}{2}, \tag{6.14}$$

where θ is the deflection angle. Moreover, since the energy is conserved, $p^0 = p'^0 = E$ and $p^2 = M_e^2$, then, considering the frame of reference where $n = (1, 0, 0, 1)$, the equation (6.13) reads

$$\begin{aligned} |\mathcal{M}|^2 = \frac{Ze^4}{8|\vec{p}|^4 \sin^4\left(\frac{\theta}{2}\right)} & \left[2E^2 - 2|\vec{p}|^2 \sin^2\left(\frac{\theta}{2}\right) - m^2 + \frac{1}{2} m^2 \left(\frac{E - |\vec{p}| \sin\eta \sin\phi}{E - |\vec{p}| \sin\eta \sin(\phi - \theta)} \right. \right. \\ + \left. \frac{E - |\vec{p}| \sin\eta \sin(\phi - \theta)}{E - |\vec{p}| \sin\eta \sin\phi} \right) & \left. - m^2 |\vec{p}|^2 \sin^2\left(\frac{\theta}{2}\right) \frac{1}{(E - |\vec{p}| \sin\eta \sin\phi)(E - |\vec{p}| \sin\eta \sin(\phi - \theta))} \right], \end{aligned} \tag{6.15}$$

where η is the angle between $\vec{n}$ and the normal vector to the plane where the scattering occurs. In addition, ϕ is the angle between the projection of $\vec{n}$ in the plane where the scattering takes place and $-\vec{p}$. With this, the cross section is given by

$$\frac{d\sigma}{d\Omega}=\frac{Z\alpha^2}{4|\vec{p}|^2\beta^2\sin^4\left(\frac{\theta}{2}\right)}\left[1-\beta^2\sin^2\left(\frac{\theta}{2}\right)-\frac{m^2}{2M_e^2}+\frac{m^2}{4M_e^2}\left(\frac{1-\beta\sin\eta\sin\phi}{1-\beta\sin\eta\sin(\phi-\theta)}\right.\right.$$
$$\left.\left.+\frac{1-\beta\sin\eta\sin(\phi-\theta)}{1-\beta\sin\eta\sin\phi}\right)-\frac{m^2}{2M_e^2}\beta^2\sin^2\left(\frac{\theta}{2}\right)\frac{1}{(1-\beta\sin\eta\sin\phi)(1-\beta\sin\eta\sin(\phi-\theta))}\right]. \tag{6.16}$$

Notice that in the limit $m\to 0$ we recover the standard case, the Mott formula. Here we have found a VSR generalization, where a signal of the direction n is found in terms of the angles η and ϕ. However, the magnitude of this signal is proportional to m^2/M_e^2.

Chapter 7

Conclusions and open questions

We reviewed the basic aspects of Very Special Relativity with an emphasis in the QED sector. The invariance under $SIM(2)$ instead of the Lorentz group provides mass to the neutrinos without introducing new particles. This aspect is an interesting option considering the absence of particle discoveries in the colliders. Moreover, the dispersion relation for the neutrino and other fermions remains as the standard. This fact is in agreement with the experiments[67]. The main feature of the model is the non-local operator $(n \cdot \partial)^{-1}$, which contains the null vector n that defines a privileged direction.

We showed that the Lorentz group in the two dimensional case allows the same terms considered in $SIM(2)$ invariant theories. This situation motivated the study of the Schwinger model with these new possible terms. The photon self-energy has substantial differences compared to the standard case. Here, there is no pole in $q^2 = 0$ due to the parameter m^2, which makes finite the integral (5.29). Also, this parameter introduces the possibility of pair production, which is absent in the standard case.

The axial anomaly was also studied. We showed the anomaly term $\varepsilon^{\mu\nu} F_{\mu\nu}$ is the same as in the standard case. However, the coefficient changes. This change produces two different regimes. In the higher momentum case, we recover the standard form, while in the low momentum regime, we observed that in the limit $q \to 0$, the anomaly vanishes.

Further research can be done in this line in higher dimensions. New anomaly terms containing the null vector could be constructed. Since these terms are related to the topology of the system, it may have interesting consequences that could be tested. Besides, in three dimensions we have shown recently that new induced Maxwell-Chern-Simons terms appear from the photon self-energy computation[38].

We have presented a simple example, the Coulomb scattering, where we found a small signal of n. However, more work should be done in this direction. In order to test this model, we require to compute different processes. However, to reach this, one crucial challenge must be solved, the full construction of a prescription to cope with the divergencies in $(n \cdot p)^{-1}$.

Mandelstam-Leibbrandt prescription, described in equation (4.1), allows to deal with integrals with $(n \cdot p)^{-1}$. However, the method involves the introduction of a new null vector $\bar{n}$, which breaks the $SIM(2)$ invariance. We reviewed the Ward identity allowed to define that all the $\bar{n}$ are the same in the photon self-energy computation and the $SIM(2)$ invariance is easily restored writing the common $\bar{n}$ as a linear combination of n and the external momentum as we showed in (4.18). The same can be done for the electron self-energy if the photon mass is fixed to zero. Nevertheless, the introduction of the photon mass term requires a revision of this rule, and we introduced the definition (6.7) to compute the integrals associated. The problem becomes bigger when we look at diagrams with more than one independent external leg. For instance, in photon-photon scattering, under the same consideration applied in the photon self-energy, where all the $\bar{n}$ are the same, the differential cross section presents divergencies. This problem shows that further analysis in the Ward identity should be done in order to elucidate the right prescription. In this work, we have opted by the possibility to extract information and features of the model bypassing this aspect considering only one independent external leg[1]. Nevertheless, the solution to this issue is crucial either to explore new VSR extensions as QCD or to compute additional processes.

[1]Despite a right prescription for more external legs is needed, we have shown recently that the computation of three photon legs diagram in $2+1$ dimensions is zero as in the standard case without writing $\bar{n}$ as a linear combination of n and external photonic momenta[38].

Appendix A

Derivation of the Feynman Rules

Expanding the equation (3.25) using (3.26) up to second order in e we get

$$\begin{aligned} \mathcal{L} &= \bar{\psi}\left(i\partial\!\!\!/ - M + i\frac{m^2}{2}n\!\!\!/\frac{1}{n\cdot\partial}\right)\psi \\ &\quad -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}(n^\alpha F_{\mu\alpha})\frac{1}{(n\cdot\partial)^2}(n_\beta F^{\mu\beta}) - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \\ &\quad -e\bar{\psi}\left(A\!\!\!/ - \frac{m^2}{2}n\!\!\!/\frac{1}{n\cdot\partial}\left(n\cdot A\frac{1}{n\cdot\partial}\right)\right)\psi \\ &\quad -ie^2\bar{\psi}\left(\frac{m^2}{2}n\!\!\!/\frac{1}{n\cdot\partial}n\cdot A\frac{1}{n\cdot\partial}n\cdot A\frac{1}{n\cdot\partial}\right)\psi. \end{aligned} \tag{A.1}$$

The first line corresponds to the free fermion that we have denoted by $\mathcal{L}_f$. The second to the free gauge part, that we called $\mathcal{L}_g$. The third line is the interaction part that generates the vertex with one external photonic leg (the diagram in the left in figure 3.1). The fourth line corresponds to the diagram in the centre in figure 3.1.

A.1 Fermionic propagator

We start from the generating functional for the free fermion

$$Z_F = \int \mathcal{D}[\psi\bar{\psi}]\exp\left\{i\int d^4x\left[\bar{\psi}\left(i\gamma^\mu\partial_\mu - M + i\frac{m^2}{2}\frac{n\!\!\!/}{n\cdot\partial}\right)\psi + \bar{\psi}\eta + \bar{\eta}\psi\right]\right\}. \tag{A.2}$$

Completing square we have

$$
\begin{aligned}
Z_F &= \int \mathcal{D}[\psi\bar{\psi}] \exp\left\{ i \int d^4x \left[\left(\bar{\psi} - \bar{\eta}\left(i\gamma^\mu \partial_\mu - M + i\frac{m^2}{2}\frac{\not{n}}{n\cdot\partial} \right)^{-1} \right) \left(i\gamma^\mu \partial_\mu - M + i\frac{m^2}{2}\frac{\not{n}}{n\cdot\partial} \right) \right.\right. \\
&\quad \left.\left. \times \left(\psi - \left(i\gamma^\mu \partial_\mu - M + i\frac{m^2}{2}\frac{\not{n}}{n\cdot\partial} \right)^{-1} \eta \right) - \bar{\eta}\left(i\gamma^\mu \partial_\mu - M + i\frac{m^2}{2}\frac{\not{n}}{n\cdot\partial} \right)^{-1} \eta \right] \right\}. \qquad \text{(A.3)}
\end{aligned}
$$

We make a change of variables $\psi - \left(i\gamma^\mu \partial_\mu - M + i\frac{m^2}{2}\frac{\not{n}}{n\cdot\partial} \right)^{-1} \eta \to \psi$ and we get

$$
Z_F = \mathcal{N}_F \exp\left\{ -i \int d^4x d^4y \left[\bar{\eta}(x) \left(i\gamma^\mu \partial_\mu - M + i\frac{m^2}{2}\frac{\not{n}}{n\cdot\partial} \right)^{-1} \eta(y) \right] \right\}, \qquad \text{(A.4)}
$$

where we have defined

$$
\mathcal{N}_F = \int \mathcal{D}[\psi\bar{\psi}] \exp\left\{ i \int d^4x \left[\bar{\psi} \left(i\gamma^\mu \partial_\mu - M + i\frac{m^2}{2}\frac{\not{n}}{n\cdot\partial} \right) \psi \right] \right\}. \qquad \text{(A.5)}
$$

From (A.2) we obtain

$$
\frac{1}{Z}\frac{\delta^2 Z}{\delta\bar{\eta}(x)\delta\eta(y)}|_{\eta=\bar{\eta}=0} = \langle \psi(x)\bar{\psi}(y) \rangle. \qquad \text{(A.6)}
$$

From (A.4) we have

$$
\frac{1}{Z}\frac{\delta^2 Z}{\delta\bar{\eta}(x)\delta\eta(y)}|_{\eta=\bar{\eta}=0} = \frac{i}{i\gamma^\mu \partial_\mu - M + i\frac{m^2}{2}\frac{\not{n}}{n\cdot\partial}}. \qquad \text{(A.7)}
$$

Equating (A.6) with (A.7) and passing to Fourier space we get

$$
S_F(p) = i\frac{\not{p} + M - \frac{m^2}{2}\frac{\not{n}}{n\cdot p}}{p^2 - M_e^2 + i\varepsilon} \qquad \text{(A.8)}
$$

A.2 Gauge sector

For the gauge field we start from the generating functional of the free field

$$Z_A = \int \mathcal{D}[A_\mu] \exp\left\{ i \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{m_\gamma^2}{2} (n^\alpha F_{\mu\alpha}) \frac{1}{(n\cdot\partial)^2} (n_\beta F^{\mu\beta}) - \frac{1}{2\xi} (\partial^\mu A_\mu)^2 + J^\mu A_\mu \right] \right\} \tag{A.9}$$

Writing (A.9) in terms of A we have

$$Z_A = \int \mathcal{D}[A_\mu] \exp\left\{ i \int d^4x \left[\frac{1}{2} A_\mu (\Delta^{-1})^{\mu\nu} A_\nu + J^\mu A_\mu \right] \right\}, \tag{A.10}$$

where the operator $(\Delta^{-1})^{\mu\nu}$ is defined as

$$(\Delta^{-1})^{\mu\nu} = \left[(\partial^2 + m_\gamma^2) g^{\mu\nu} - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu + m_\gamma^2 \frac{\partial^2 n^\nu n^\mu}{(n\cdot\partial)^2} - m_\gamma^2 \frac{\partial^\mu n^\nu + \partial^\nu n^\mu}{n\cdot\partial} \right]. \tag{A.11}$$

We recognize this object as the inverse of the propagator. We write (A.11) in Fourier space

$$\Delta^{-1} = \left[(-p^2 + m_\gamma^2) g^{\mu\nu} + \left(1 - \frac{1}{\xi}\right) p^\mu p^\nu + m_\gamma^2 \frac{p^2 n^\nu n^\mu}{(n\cdot p)^2} - m_\gamma^2 \frac{p^\mu n^\nu + p^\nu n^\mu}{n\cdot p} \right]. \tag{A.12}$$

To get the propagator Δ we use $\Delta^{-1}\Delta = i\delta^\mu_\sigma$. Therefore,

$$\begin{aligned} i\delta^\mu_\sigma \quad = \quad & \left[(-p^2 + m_\gamma^2) g^{\mu\nu} + \left(1 - \frac{1}{\xi}\right) p^\mu p^\nu + m_\gamma^2 \frac{p^2 n^\nu n^\mu}{(n\cdot p)^2} - m_\gamma^2 \frac{p^\mu n^\nu + p^\nu n^\mu}{n\cdot p} \right] \\ & \times [A g_{\nu\sigma} + B p_\nu p_\sigma + C n_\nu n_\sigma + D(p_\nu n_\sigma + p_\sigma n_\nu)]. \end{aligned} \tag{A.13}$$

We do the products and comparing the tensorial structures we get the coefficients

$$A = \frac{-i}{p^2 - m_\gamma^2}, \tag{A.14}$$

$$B = \frac{i}{p^4}(1-\xi), \tag{A.15}$$

$$C = \frac{-im_\gamma^2}{(p^2 - m_\gamma^2)(n\cdot p)^2}, \tag{A.16}$$

$$D = \frac{im_\gamma^2}{(p^2 - m_\gamma^2)p^2}\frac{1}{n\cdot p}. \tag{A.17}$$

Thus, the propagator is

$$\Delta_{\mu\nu} = -\frac{i}{p^2 - m_\gamma^2}\left[g_{\mu\nu} - \frac{1-\xi}{p^4}(p^2 - m_\gamma^2)p_\mu p_\nu + \frac{m_\gamma^2}{(n\cdot p)^2}n_\mu n_\nu - \frac{m_\gamma^2}{p^2 n\cdot p}(p_\mu n_\nu + p_\nu n_\mu)\right]. \tag{A.18}$$

We choose the Feynman gauge $\xi = 1$. Hence,

$$\Delta_{\mu\nu} = -\frac{i}{p^2 - m_\gamma^2}\left[g_{\mu\nu} + \frac{m_\gamma^2}{(n\cdot p)^2}n_\mu n_\nu - \frac{m_\gamma^2}{p^2 n\cdot p}(p_\mu n_\nu + p_\nu n_\mu)\right]. \tag{A.19}$$

A.3 Vertices

The vertices come from the interaction part. Writing the third and the fourth line of (A.1) in Fourier space and deriving respect the fields we get

$$V_{1\mu}(p, p+q) = -ie\left(\gamma_\mu + \frac{1}{2}m^2 \not{n}\frac{n_\mu}{(n\cdot p)[n\cdot(p+q)]}\right), \tag{A.20}$$

$$V_{2\mu\nu}(p, p', p+q_1, p+q_2) = -ie^2\frac{1}{2}m^2\not{n}\frac{n_\mu n_\nu}{(n\cdot p)(n\cdot p')}\left(\frac{1}{n\cdot(p+q_1)} + \frac{1}{n\cdot(p+q_2)}\right). \tag{A.21}$$

Appendix B

Dynamical neutrino masses and axions

In this appendix we show a brief presentation of part of the work done during the doctoral visit in London, which appears in [8, 9]. This work is not part of the Very Special Relativity framework. However, it is another possibility in the quest of ideas about the neutrino mass. In this case the neutrino mass requires the existence of pseudo-scalar fields (axion-like particles) that appear in string models[68, 69] and heavy right handed fermions whose masses are generated radiatively by shift symmetry breaking Yukawa interactions with the axions[70]. Here, we analyzed using Schwinger-Dyson equations the possibility to have dynamically induced masses for axion and the sterile neutrino via hermitian and anti-hermitian Yukawa interaction. The lagrangian which describes the model is

$$\mathcal{L} = \frac{1}{2}\partial_\mu a \partial^\mu a + \bar{\psi} i \partial\!\!\!/ \psi - \frac{\gamma}{f_b}(\partial_\mu a)\bar{\psi}\gamma^\mu\gamma^5\psi + i\lambda a \bar{\psi}\gamma^5\psi - \frac{g^2}{2f_b^2}(\bar{\psi}\gamma^5\psi)^2. \tag{B.1}$$

We will analyze the lagrangian (B.1) in parts. We will proceed in this way to see the effect of each term separately.

B.1 Only Yukawa interaction

We start from the simplest case. Considering $\gamma = 0$ and $g = 0$, that is no anomaly term and four fermion interaction. In this case, the lagrangian (B.1) reads

$$\mathcal{L} = \frac{1}{2}\partial_\mu a \partial^\mu a + \bar{\psi} i \partial\!\!\!/ \psi + i\lambda a \bar{\psi}\gamma^5\psi. \tag{B.2}$$

If there is dynamical mass generation, we should take into account all the possible mass terms, which are

$$\frac{1}{2}M^2 a^2 \ , \quad m\overline{\psi}\psi \ , \tag{B.3}$$

where M^2 and m are real.

From (B.2), the Schwinger-Dyson (SD) equations obtained are

$$G_f^{-1}(k) - S_f^{-1}(k) = \lambda\gamma^5 \int_p G_f(p)\Gamma^{(3)}(p,k)G_s(p-k), \tag{B.4}$$

$$G_s^{-1}(k) - S_s^{-1}(k) = \mathrm{Tr}\left\{\lambda\gamma^5 \int_p G_f(p)\Gamma^{(3)}(p,k)G_f(p-k)\right\}, \tag{B.5}$$

where the index s refers to the scalar and the index f refers to the fermion. $G_{s,f}$ denote the dressed propagators, $S_{s,f}$ denote the bare propagators, $\Gamma^{(3)}(p,k)$ is the dressed vertex, and we abbreviated the four-momentum integrals by $\int_p \equiv \int \frac{d^4p}{(2\pi)^4}$. We use the standard propagators and working in the lowest order approximation, we neglect corrections to the vertex (rainbow approximation). The momentum integrals are regulated using an UV cut off Λ, which will play the role of the mass scale of the system. With these considerations, we arrived to the solutions

$$1 = \frac{\lambda^2}{16\pi^2}\frac{1}{M^2 - m^2}\left[M^2 \ln\left(1 + \frac{\Lambda^2}{M^2}\right) - m^2 \ln\left(1 + \frac{\Lambda^2}{m^2}\right)\right], \tag{B.6}$$

$$M^2 = -\frac{\lambda^2}{4\pi^2}\left[\Lambda^2 - m^2 \ln\left(1 + \frac{\Lambda^2}{m^2}\right)\right]. \tag{B.7}$$

From these equations we observe that there is no scalar mass generation since $\Lambda^2 > m^2$

because Λ is the highest mass scale in the problem. Setting $M = 0$ we get

$$\left(\frac{\Lambda}{m}\right)^2 \simeq \ln\left(1 + \left(\frac{\Lambda}{m}\right)^2\right) \simeq \frac{16\pi^2}{\lambda^2}. \tag{B.8}$$

These equalities are incompatible. It means there is no dynamical mass generation for the fermion when $M = 0$. The only possibility to get dynamical mass for both is assuming a bare mass for the scalar field $M_0 \neq 0$. Thus, the SD equations read

$$\begin{aligned} 1 &= \frac{\lambda^2}{16\pi^2}\frac{1}{M^2 - m^2}\left[M^2 \ln\left(1 + \frac{\Lambda^2}{M^2}\right) - m^2 \ln\left(1 + \frac{\Lambda^2}{m^2}\right)\right], &\text{(B.9)} \\ M^2 &= M_0^2 - \frac{\lambda^2}{4\pi^2}\left[\Lambda^2 - m^2\ \ln\left(1 + \frac{\Lambda^2}{m^2}\right)\right]. &\text{(B.10)} \end{aligned}$$

It yields

$$\begin{aligned} m^2 &\simeq \exp\Big(-\frac{16\,\pi^2}{\lambda^2}\Big)\,\Lambda^2, \quad |\lambda| \ll 1, \\ M^2 &\simeq m^2 = M_0^2 - \frac{\lambda^2}{4\pi^2}\,\Lambda^2, \quad M_0^2 = \frac{\lambda^2}{4\pi^2}\,\Lambda^2 + \exp\Big(-\frac{16\,\pi^2}{\lambda^2}\Big)\,\Lambda^2. \end{aligned} \tag{B.11}$$

which indicates a non-perturbative (in the Yukawa coupling λ) small dynamical fermion and scalar masses.

If we consider a non-hermitian Yukawa coupling the lagrangian (B.2) is

$$\mathcal{L} = \frac{1}{2}\partial_\mu a \partial^\mu a + \bar{\psi} i \partial\!\!\!/ \psi + \lambda a \bar{\psi}\gamma^5\psi. \tag{B.12}$$

Here, the solutions of the SD equations are

$$\begin{aligned} 1 &= -\frac{\lambda^2}{16\pi^2}\frac{1}{M^2 - m^2}\left[M^2 \ln\left(1 + \frac{\Lambda^2}{M^2}\right) - m^2 \ln\left(1 + \frac{\Lambda^2}{m^2}\right)\right] &\text{(B.13)} \\ M^2 &= \frac{\lambda^2}{4\pi^2}\left[\Lambda^2 - m^2\ \ln\left(1 + \frac{\Lambda^2}{m^2}\right)\right]\ . \end{aligned}$$

Considering solutions $m \simeq M \ll \Lambda$ we get

$$-\frac{16\pi^2}{\lambda^2} = \ln\left(1 + \frac{\Lambda^2}{M^2}\right), \tag{B.14}$$

which is inconsistent. Moreover, is easy to see that $M = 0$ does not lead to fermion dynamical mass. Hence, it is impossible to get a mass dynamically for the fermion with anti-hermitian Yukawa coupling.

B.2 Adding attractive four fermion interaction

Now we will look the behavior under the inclusion of the four fermion interaction. Hence, $g \neq 0$ in (B.1). Therefore for hermitian Yukawa coupling,

$$\mathcal{L} = \frac{1}{2}\partial_\mu a \partial^\mu a + \bar{\psi} i \partial\!\!\!/ \psi + i\lambda a \bar{\psi}\gamma^5\psi - \frac{g^2}{2f_b^2}(\bar{\psi}\gamma^5\psi)^2. \tag{B.15}$$

After linearising the four fermion interaction with the help of an auxiliary pseudoscalar field σ the solution to the SD equations are

$$1 = \frac{\lambda^2}{16\pi^2}\frac{1}{M^2 - m^2}\left[M^2 \ln\left(1 + \frac{\Lambda^2}{M^2}\right) - m^2 \ln\left(1 + \frac{\Lambda^2}{m^2}\right)\right] + \frac{g^2}{16\pi^2 f_b^2}\left(\Lambda^2 - m^2 \ln\left(1 + \frac{\Lambda^2}{m^2}\right)\right), \tag{B.16}$$

$$M^2 = M_0^2 - \frac{\lambda^2}{4\pi^2}\left[\Lambda^2 - m^2 \ln\left(1 + \frac{\Lambda^2}{m^2}\right)\right], \tag{B.17}$$

where we have introduced the bare mass M_0 as in the previous case. Now, we search for solutions $m \simeq M$ and we get that dynamical masses for fermions and scalars of order

$$\begin{aligned} m^2 &\simeq M^2 \simeq M_0^2 - \frac{\lambda^2\,\Lambda^2}{4\pi^2} \simeq \lambda^2\,\frac{f_b^2}{g2} + \mathcal{O}(\lambda^4 \ln \lambda^2) = \lambda^2\,\frac{\Lambda^2}{16\pi^2} + \mathcal{O}(\lambda^4\,\ln \lambda^2) \ll \Lambda^2, \\ M_0^2 &\simeq \lambda^2\,\frac{5\,\Lambda^2}{16\pi^2} + \mathcal{O}(\lambda^4\,\ln \lambda^2)\,, \end{aligned} \tag{B.18}$$

can be generated, which are much larger than the masses (B.11) in the pure Yukawa case where $g \to 0$. For consistency,

$$\frac{f_b}{g} \sim \frac{\Lambda}{4\pi}\,. \tag{B.19}$$

In the antihermitian Yukawa interaction case, we get dynamical fermion and scalar mass generation, without bare mass. The results obtained are of the same order as in the corresponding hermitian-Yukawa case (B.18):

$$m^2 \simeq M^2 \simeq 4\,\lambda^2\,\frac{f_4^2}{g^2} + \left|\mathcal{O}(\lambda^4\,\ln\lambda^2)\right| \simeq \frac{\lambda^2}{4\pi^2}\Lambda^2 + \left|\mathcal{O}(\lambda^4\,\ln\lambda^2)\right| \tag{B.20}$$

with the four fermion coupling given by (B.19), as in the hermitian Yukawa interaction case.

B.3 Inclusion of anomaly term

Now, we introduce the anomaly term. With $\gamma \neq 0$ we have for hermitian Yukawa coupling and hermitian anomaly term

$$\mathcal{L} = \frac{1}{2}\partial_\mu a\partial^\mu a + \bar{\psi}i\partial\!\!\!/\psi - \frac{\gamma}{f_b}(\partial_\mu a)\bar{\psi}\gamma^\mu\gamma^5\psi + i\lambda a\bar{\psi}\gamma^5\psi - \frac{g^2}{2f_b^2}(\bar{\psi}\gamma^5\psi)^2. \tag{B.21}$$

Workin in the following regime

$$|g| \gtrsim 1 \gg |\gamma| > |\lambda| > 0, \tag{B.22}$$

and after linearisation of the four fermion interaction we get the solutions

$$\begin{aligned} 1 \;=\; & \frac{\lambda^2}{16\pi^2}\frac{1}{M^2-m^2}\left[M^2\ln\left(1+\frac{\Lambda^2}{M^2}\right) - m^2\ln\left(1+\frac{\Lambda^2}{m^2}\right)\right] \\ & -\frac{\gamma^2}{16\pi^2 f_b^2}\frac{1}{(M^2-m^2)}\left[-\Lambda^2(M^2-m^2) + M^4\ln\left(1+\frac{\Lambda^2}{M^2}\right) - m^4\ln\left(1+\frac{\Lambda^2}{m^2}\right)\right] \\ & +\frac{g^2}{16\pi^2 f_b^2}\left(\Lambda^2 - m^2\ln\left(1+\frac{\Lambda^2}{m^2}\right)\right), \end{aligned} \tag{B.23}$$

$$M^2 = M_0^2 - \frac{\lambda^2}{4\pi^2}\left[\Lambda^2 - m^2\ln\left(1+\frac{\Lambda^2}{m^2}\right)\right], \tag{B.24}$$

where we have introduced the bare mass for the scalar field as before. Looking solutions $m \simeq M$ we get

$$\begin{aligned}
\frac{f_b}{\sqrt{\gamma^2+g^2}} &\simeq \frac{\Lambda}{4\pi} \\
M^2 \simeq m^2 = \lambda^2 \frac{f_b^2}{(2\gamma^2+g^2)} &\simeq \frac{\lambda^2}{16\pi^2}\, \frac{\gamma^2+g^2}{2\gamma^2+g^2}\, \Lambda^2 \ll \Lambda^2, \\
M_0^2 &= \frac{\lambda^2}{16\pi^2}\left(\frac{9\gamma^2+5g^2}{2\gamma^2+g^2}\right)\Lambda^2 + \mathcal{O}\Big(\lambda^4\, \ln(\lambda^2)\Big).
\end{aligned} \tag{B.25}$$

The result here shows that the bare mass is required to have a dynamical mass for the scalar field. The term γ^2 appears with the same sign than g^2. It means the anomaly term plays a similar role to the four fermion interaction.

If we consider non-hermitian Yukawa coupling and non-hermitian anomaly term the solutions are

$$\begin{aligned}
1 \;=\; & -\frac{\lambda^2}{16\pi^2}\frac{1}{M^2-m^2}\left[M^2\ln\left(1+\frac{\Lambda^2}{M^2}\right) - m^2\ln\left(1+\frac{\Lambda^2}{m^2}\right)\right] \\
& +\frac{\gamma^2}{16\pi^2 f_b^2}\frac{1}{M^2-m^2}\left[-\Lambda^2(M^2-m^2) + M^4\ln\left(1+\frac{\Lambda^2}{M^2}\right) - m^4\ln\left(1+\frac{\Lambda^2}{m^2}\right)\right] \\
& +\frac{g^2}{16\pi^2 f_b^2}\left(\Lambda^2 - m^2\ln\left(1+\frac{\Lambda^2}{m^2}\right)\right),
\end{aligned} \tag{B.26}$$

$$M^2 = \frac{\lambda^2}{4\pi^2}\left[\Lambda^2 - m^2\ln\left(1+\frac{\Lambda^2}{m^2}\right)\right]. \tag{B.27}$$

Here we observe it is not necessary the introduction of the bare mass to have dynamical mass for the scalar field. Thus, the consistent solutions with $m \simeq M$ are

$$\frac{f_b}{\sqrt{g^2-\gamma^2}} \simeq \frac{\Lambda}{4\pi} - |\mathcal{O}(\lambda^2)|, \tag{B.28}$$

$$M^2 \simeq m^2 \simeq \frac{4 f_b^2 \lambda^2}{g^2-\gamma^2} + |\mathcal{O}\Big((\lambda^4, \lambda^2\gamma^2)\, \ln(\lambda^2)\Big)|. \tag{B.29}$$

In order to have dynamical mass generation, one must have $g^2 > \gamma^2$, which is satisfied under (B.22). We notice in this case, contrary to the hermitian case, the anomaly term resists the dynamical mass generation.

Appendix C

Presentations during the PhD work

During the time of completion of the program, the following talks were given:

- “One loop electron Self energy in VSR QED with a VSR photon mass”, La Parte y el todo, Afunalhue, Chile, 3-5 January 2018
- “Aspects on QED in Very Special Relativity”, SOCHIFI, Antofagasta, Chile, 14-16 November 2018
- “QED Process in Very Special Relativity”, XII Latin American Symposium on High Energy Physics, Lima, Peru, 26-30 November 2018
- “QED Process in Very Special Relativity”, La Parte y el todo, Afunalhue, Chile, 8-11 January 2019
- “Two dimensional chiral anomaly in Very Special Relativity”, Workshop Cosmology and Particles, Chillán, Chile, 12-14 June 2019
- “Looking QED through the glasses of Very Special Relativity, University of Sussex, UK, 6th April 2020
- “Looking QED through the glasses of Very Special Relativity, University of Edinburgh, UK, 27th May 2020

- “Looking QED through the glasses of Very Special Relativity, University of Cambridge, UK, 25th June 2020

Bibliography

[1] Y. Fukuda et al. Evidence for oscillation of atmospheric neutrinos. *Phys. Rev. Lett.*, 81:1562–1567, 1998. arXiv:hep-ex/9807003.

[2] B. Pontecorvo. Neutrino Experiments and the Problem of Conservation of Leptonic Charge. *Sov. Phys. JETP*, 26:984–988, 1968.

[3] Murray Gell-Mann, Pierre Ramond, Richard Slansky, P Van Nieuwenhuizen, and DZ Freedman. Supergravity. 1979.

[4] Osamu Sawada and Akio Sugamoto, editors. *Proceedings: Workshop on the Unified Theories and the Baryon Number in the Universe: Tsukuba, Japan, February 13-14, 1979*, Tsukuba, Japan, 1979. Natl.Lab.High Energy Phys.

[5] S.L. Glashow. The Future of Elementary Particle Physics. *NATO Sci. Ser. B*, 61:687, 1980.

[6] Rabindra N. Mohapatra and Goran Senjanovic. Neutrino Mass and Spontaneous Parity Nonconservation. *Phys. Rev. Lett.*, 44:912, 1980.

[7] Carl M. Bender and Stefan Boettcher. Real spectra in nonHermitian Hamiltonians having PT symmetry. *Phys. Rev. Lett.*, 80:5243–5246, 1998. arXiv:physics/9712001.

[8] Jean Alexandre, Nick E. Mavromatos, and Alex Soto. Dynamical Majorana Neutrino Masses and Axions. arXiv:2004.04611 [hep-ph].

[9] Nick E. Mavromatos and Alex Soto. Dynamical Majorana Neutrino Masses and Axions II: Inclusion of Axial Background and Anomaly Terms. arXiv:2006.13616 [hep-ph].

[10] Albert Einstein. On the electrodynamics of moving bodies. *Annalen Phys.*, 17:891–921, 1905.

[11] S. Dhasmana and Z.K. Silagadze. Finsler spacetime in light of Segal's principle. *Mod. Phys. Lett. A*, 35(06):2050019, 2019. arXiv:1911.04233 [gr-qc].

[12] Ralf Lehnert. CPT and lorentz violation as signatures for planck-scale physics. *Journal of Physics: Conference Series*, 171:012036, jun 2009.

[13] Petr Horava and Charles M. Melby-Thompson. General Covariance in Quantum Gravity at a Lifshitz Point. *Phys. Rev. D*, 82:064027, 2010. arXiv:1007.2410 [hep-th].

[14] Don Colladay and V.Alan Kostelecky. Lorentz violating extension of the standard model. *Phys. Rev. D*, 58:116002, 1998. arXiv:hep-ph/9809521.

[15] Ted Jacobson, Stefano Liberati, and David Mattingly. TeV astrophysics constraints on Planck scale Lorentz violation. *Phys. Rev. D*, 66:081302, 2002. arXiv:hep-ph/0112207.

[16] Andrew G. Cohen and Sheldon L. Glashow. Very special relativity. *Phys. Rev. Lett.*, 97:021601, 2006. arXiv:hep-ph/0601236.

[17] J.H. Christenson, J.W. Cronin, V.L. Fitch, and R. Turlay. Evidence for the 2π Decay of the K_2^0 Meson. *Phys. Rev. Lett.*, 13:138–140, 1964.

[18] Andrew G. Cohen and Sheldon L. Glashow. A Lorentz-Violating Origin of Neutrino Mass? arXiv:hep-ph/0605036.

[19] A.E. Bernardini. Dirac neutrino mass from the beta-decay end point modified by the dynamics of a Lorentz-violating equation of motion. *Phys. Rev. D*, 75:097901, 2007. arXiv:0706.3932 [hep-ph].

[20] Ulf Lindstrom and Martin Rocek. SIM(2) and Superspace. arXiv:hep-th/0606093.

[21] Andrew G. Cohen and Daniel Z. Freedman. SIM(2) and SUSY. *JHEP*, 07:039, 2007. arXiv:hep-th/0605172.

[22] Suratna Das and Subhendra Mohanty. Very Special Relativity is incompatible with Thomas precession. *Mod. Phys. Lett. A*, 26:139–150, 2011. arXiv:0902.4549 [hep-ph].

[23] Jorge Alfaro and Victor O. Rivelles. Very Special Relativity and Lorentz Violating Theories. *Phys. Lett. B*, 734:239–244, 2014. arXiv:1306.1941 [hep-ph].

[24] Alan Dunn and Thomas Mehen. Implications of SU(2)(L) x U(1) symmetry for SIM(2) invariant neutrino masses. arXiv:hep-ph/0610202.

[25] Stanley Mandelstam. Light Cone Superspace and the Ultraviolet Finiteness of the N=4 Model. *Nucl. Phys. B*, 213:149–168, 1983.

[26] George Leibbrandt. The Light Cone Gauge in Yang-Mills Theory. *Phys. Rev. D*, 29:1699, 1984.

[27] Sangmo Cheon, Choonkyu Lee, and Seung Jae Lee. SIM(2)-invariant Modifications of Electrodynamic Theory. *Phys. Lett. B*, 679:73–76, 2009. arXiv:0904.2065 [hep-th].

[28] Jorge Alfaro and Victor O. Rivelles. Non Abelian Fields in Very Special Relativity. *Phys. Rev. D*, 88:085023, 2013. arXiv:1305.1577 [hep-th].

[29] Jorge Alfaro, Pablo Gonzalez, and Ricardo Avila. Electroweak standard model with very special relativity. *Phys. Rev. D*, 91:105007, 2015. arXiv:1504.04222 [hep-ph].

[30] Jorge Alfaro. A $Sim(2)$ invariant dimensional regularization. *Phys. Lett. B*, 772:100–104, 2017. arXiv:1704.02299 [hep-th].

[31] J. Alfaro. Mandelstam-Leibbrandt prescription. *Phys. Rev. D*, 93(6):065033, 2016. arXiv:1603.06453 [hep-th].

[32] Jorge Alfaro. Feynman Rules, Ward Identities and Loop Corrections in Very Special Relativity Standard Model. *Universe*, 5(1):16, 2019.

[33] Jorge Alfaro and Alex Soto. Photon mass in very special relativity. *Phys. Rev. D*, 100(5):055029, 2019. arXiv:1901.08011 [hep-th].

[34] R. Bufalo and M. Ghasemkhani. Thermal effects of Very Special Relativity Quantum Electrodynamics. *Phys. Rev. D*, 100(6):065024, 2019. arXiv:1909.06466 [hep-th].

[35] R. Bufalo and T. Cardoso e Bufalo. Tree-level processes in very special relativity. *Phys. Rev. D*, 100(12):125017, 2019. arXiv:1911.08386 [hep-th].

[36] A.F. Santos and Faqir C. Khanna. Bhabha scattering in Very Special Relativity at finite temperature. ArXiv:2007.13133 [hep-th].

[37] Jorge Alfaro and Alex Soto. Schwinger model a la Very Special Relativity. *Phys. Lett. B*, 797:134923, 2019. arXiv:1907.06273 [hep-th].

[38] R. Bufalo, M. Ghasemkhani, Z. Haghgouyan, and A. Soto. Induced Maxwell-Chern-Simons Effective Action in Very Special Relativity. arXiv:2004.02176 [hep-th].

[39] Jorge Alfaro and Alex Soto. Meson spectrum in QCD_2 revisited. arXiv:1908.07660 [hep-th].

[40] Gustavo Salinas de Souza. The representations of HOM(2) and SIM(2) in the context of Very Special Relativity, 2015.

[41] R. Bufalo. SIM(1)–VSR Maxwell–Chern–Simons electrodynamics. *Phys. Lett. B*, 757:216–222, 2016. arXiv:1604.00213 [hep-th].

[42] Jiri Vohanka and Mir Faizal. Chern–Simons theory in SIM(1) superspace. *Eur. Phys. J. C*, 75(12):592, 2015. arXiv:1503.04761 [hep-th].

[43] Alekha C. Nayak, Ravindra K. Verma, and Pankaj Jain. Effect of VSR invariant Chern-Simon Lagrangian on photon polarization. *JCAP*, 07:031, 2015. arXiv:1504.04921 [hep-ph].

[44] Jiri Vohanka and Mir Faizal. Super-Yang-Mills Theory in SIM(1) Superspace. *Phys. Rev. D*, 91(4):045015, 2015. arXiv:1409.6334 [hep-th].

[45] Stanislav Petras, Rikard von Unge, and Jiri Vohanka. SIM(2) and supergraphs. *JHEP*, 07:015, 2011. arXiv:1102.3856 [hep-th].

[46] Sudipta Das, Subir Ghosh, and Salvatore Mignemi. Noncommutative Spacetime in Very Special Relativity. *Phys. Lett. A*, 375:3237–3242, 2011. arXiv:1004.5356 [hep-th].

[47] D.V. Ahluwalia and S.P. Horvath. Very special relativity as relativity of dark matter: The Elko connection. *JHEP*, 11:078, 2010. arXiv:1008.0436 [hep-ph].

[48] R.T. Cavalcanti, R. da Rocha, and J.M. Hoff da Silva. Could Elko Spinor Fields Induce VSR Symmetry in the DKP (Meson) Algebra? *Adv. Appl. Clifford Algebras*, 27(1):267–277, 2017.

[49] Enrique Alvarez and Roberto Vidal. Very Special (de Sitter) Relativity. *Phys. Rev. D*, 77:127702, 2008. arXiv:0803.1949 [hep-th].

[50] Anton Ilderton. Very Special Relativity as a background field theory. *Phys. Rev. D*, 94(4):045019, 2016. arXiv:1605.04967 [hep-th].

[51] G.W. Gibbons, Joaquim Gomis, and C.N. Pope. General very special relativity is Finsler geometry. *Phys. Rev. D*, 76:081701, 2007. arXiv:0707.2174 [hep-th].

[52] Wolfgang Muck. Very Special Relativity in Curved Space-Times. *Phys. Lett. B*, 670:95–98, 2008. arXiv:0806.0737 [hep-th].

[53] A.P. Kouretsis, M. Stathakopoulos, and P.C. Stavrinos. The General Very Special Relativity in Finsler Cosmology. *Phys. Rev. D*, 79:104011, 2009. arXiv:0810.3267 [gr-qc].

[54] A.P. Kouretsis, M. Stathakopoulos, and P.C. Stavrinos. Imperfect fluids, Lorentz violations and Finsler Cosmology. *Phys. Rev. D*, 82:064035, 2010. arXiv:1003.5640 [gr-qc].

[55] Xin Li, Zhe Chang, and Xiaohuan Mo. Isometric group of (α, β)-type Finsler space and the symmetry of Very Special Relativity. arXiv:1001.2667 [gr-qc].

[56] Lei Zhang and Xun Xue. The Finsler Type of Space-time Realization of Deformed Very Special Relativity. arXiv:1205.1134 [math-ph].

[57] Lei Zhang and Xun Xue. The deformation of Poincare subgroups concerning very special relativity. *Sci. China Phys. Mech. Astron.*, 57:859–874, 2014. arXiv:1204.6425 [math-ph].

[58] Nadir Jeevanjee. *An introduction to tensors and group theory for physicists.* Birkhauser, New York, NY, 2011.

[59] Sidney R. Coleman and Sheldon L. Glashow. High-energy tests of Lorentz invariance. *Phys. Rev. D*, 59:116008, 1999. arXiv:hep-ph/9812418.

[60] P. Gaigg, W. Kummer, and M. Schweda, editors. *Physical and nonstandard gauges. Proceedings, Workshop, Vienna, Austria, September 19-23, 1989*, volume 361, 1990.

[61] A. Bassetto, G. Nardelli, and R. Soldati. *Yang-Mills theories in algebraic noncovariant gauges: Canonical quantization and renormalization.* 1991.

[62] G. Leibbrandt. *Noncovariant gauges: Quantization of Yang-Mills and Chern-Simons theory in axial type gauges.* 1994.

[63] Andre Burnel. *Noncovariant gauges in canonical formalism*, volume 761. 2008.

[64] E. Abdalla, M.C.B. Abdalla, and K.D. Rothe. *Nonperturbative methods in two-dimensional quantum field theory.* 1991.

[65] Julian S. Schwinger. Gauge Invariance and Mass. 2. *Phys. Rev.*, 128:2425–2429, 1962.

[66] Jeffrey A. Harvey. TASI 2003 lectures on anomalies. arXiv:hep-th/0509097.

[67] V. Vasileiou, A. Jacholkowska, F. Piron, J. Bolmont, C. Couturier, J. Granot, F.W. Stecker, J. Cohen-Tanugi, and F. Longo. Constraints on Lorentz Invariance Violation from Fermi-Large Area Telescope Observations of Gamma-Ray Bursts. *Phys. Rev. D*, 87(12):122001, 2013. arXiv:1305.3463 [astro-ph.HE].

[68] Asimina Arvanitaki, Savas Dimopoulos, Sergei Dubovsky, Nemanja Kaloper, and John March-Russell. String Axiverse. *Phys. Rev. D*, 81:123530, 2010. arXiv:0905.4720 [hep-th].

[69] Michele Cicoli, Mark Goodsell, and Andreas Ringwald. The type IIB string axiverse and its low-energy phenomenology. *JHEP*, 10:146, 2012. arXiv:1206.0819 [hep-th].

[70] Nick E. Mavromatos and Apostolos Pilaftsis. Anomalous Majorana Neutrino Masses from Torsionful Quantum Gravity. *Phys. Rev. D*, 86:124038, 2012. arXiv:1209.6387 [hep-ph].