A Nitsche-based domain decomposition method for hypersingular integral equations

Article in Numerische Mathematikk · February 2011

DOI: 10.1007/s00211-012-0451-2 · Source: arXiv

Franz Chouly
University of Burgundy

Norbert Heuer
Pontificia Universidad Católica de Chile

A Nitsche-based domain decomposition method for hypersingular integral equations *

Franz Chouly † Norbert Heuer ‡

Abstract

We introduce and analyze a Nitsche-based domain decomposition method for the solution of hypersingular integral equations. This method allows for discretizations with non-matching grids without the necessity of a Lagrangian multiplier, as opposed to the traditional mortar method. We prove its almost quasi-optimal convergence and underline the theory by a numerical experiment.

Key words: boundary element method, domain decomposition, Nitsche method.
AMS Subject Classification: 65N38, 65N55.

1 Introduction

We propose and analyze the Nitsche method as a simple domain decomposition method for the solution of hypersingular boundary integral equations. In this context, simple means that (i) its implementation is not more difficult than a conforming approach and (ii) its numerical analysis avoids mathematical difficulties inherent to usual domain decomposition approaches. Still, a thorough analysis of our method, given in this paper, faces the problem of non-existence of a well-posed continuous counterpart for the discrete formulation. This is due to the low regularity of the underlying energy space. Main attraction of the Nitsche method, apart from its relative simplicity, is that it can maintain ellipticity and symmetry of the original problem.

We study the hypersingular integral equation governing the Laplacian in $\mathbb{R}^3$ exterior to an open surface, subject to a Neumann boundary condition. In principle, our domain decomposition approach is applicable to more realistic problems like linear elasticity and acoustics. Nevertheless, whereas a generalization to the Helmholtz equation is not difficult (it is a compact perturbation of the Laplace case) there are major difficulties in case of the operator governing the Lamé equation. This remains an open problem.

*Supported by FONDECYT project 1080044

†Laboratoire de Mathématiques de Besançon, CNRS UMR 6623, Université de Franche-Comté, 16 route de Gray, 25030 Besançon Cedex, France. email: `franz.chouly@univ-fcomte.fr`

‡Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Avenida Vicuña Mackenna 4860, Macul, Santiago, Chile. email: `nheuer@mat.puc.cl`

For the solution of partial differential equations, domain decomposition is a classical strategy. It is used mainly for parallelization and the solution of linear systems. A variety of techniques exist, such as alternating Schwarz methods (see e.g. [26]). Of particular interest are methods that allow for non-matching meshes at the interface between sub-domains. They facilitate to a great extent mesh generation for complicated geometries. The so-called mortar method has been designed for this purpose [8, 9]. It consists in introducing an unknown Lagrangian multiplier on the interface and adding interface conditions in a weak sense. For an analysis of the Laplacian in two and three space dimensions see [7]. This method transforms the original problem into a saddle-point structure, so that any numerical scheme requires a discrete *inf-sup* condition, i.e. compatibility between approximation spaces on sub-domains and the interface.

An alternative to the mortar method is Nitsche's method, originally published in [25, 1], and adapted in [6] to a domain decomposition framework. The interface condition is again treated weakly; not as an additional equation but like a penalization term in the (discrete) variational formulation. Other terms are added to the formulation to achieve consistency and ellipticity. Moreover, symmetry can be maintained for symmetric problems. As a result, Nitsche's method differs from classical penalization methods where consistency is lost [6].

In conclusion, main advantages of Nitsche's method are that

1. no additional unknown is needed on the interface,
2. no *inf-sup* condition must be satisfied among discrete spaces (except for the global ones, of course), and that
3. discrete problems are elliptic and can be symmetric for symmetric problems, so that
4. standard linear solvers can be used.

Nitsche's method is closely related to the stabilized method of Barbosa & Hughes [3, 4], which also circumvents the *inf-sup* compatibility condition that arises when a Dirichlet boundary condition is imposed weakly through Lagrangian multipliers. The connection between the two methods is established in [28].

In the context of partial differential equations, the Nitsche method has been applied successfully to a variety of problems such as linear elasticity [14, 5], two-phase flows [27], and fluid-structure interaction [17, 11, 2].

In the context of boundary integral equations and the use of non-matching grids or weakly imposed boundary (or interface) conditions, we only know of the results [15, 18]. Both analyze a setting based on Lagrangian multipliers. The former reference provides the basic results like an integration-by-parts formula for the hypersingular integral operator, and analyzes the implementation of Dirichlet boundary conditions in a fractional order Sobolev space of order 1/2. The latter reference proposes and analyzes the mortar domain decomposition approach for the hypersingular integral equation. An extreme case, the use of discontinuous basis functions for hypersingular operators, is studied in [20].

Let us also mention that there are several papers on domain decomposition involving boundary elements, e.g. [21] where standard boundary elements are used for problems on sub-domains of the PDE problem, and [30, 19] which analyze domain decomposition for boundary elements in the construction of preconditioners. These papers do not deal with the problem of approximating functions (of fractional order Sobolev spaces) in a non-conforming way.

In this paper we propose and analyze a Nitsche domain decomposition variant for the hypersingular integral equation governing the Laplacian. Although this approach is simpler than mortar strategies in important aspects, as explained before, there are some non-trivial obstacles in its numerical analysis. Energy spaces of hypersingular operators are fractional order Sobolev spaces of order $1/2$. These spaces form the natural basis for variational formulations. Now, domain decomposition introduces interfaces where discontinuities arise. In the variational setting, these discontinuities are not well posed, simply because no well-defined trace operator exists. Therefore, we analyze the discrete Nitsche method without using a corresponding variational formulation. This is very much in the spirit of Strang's second lemma for non-conforming methods. The difficulty of non-existence of a well-posed trace operator reappears in the analysis of the discrete problem. We deal with this problem by making use of a whole scale of Sobolev spaces (of higher regularity than $1/2$) and by using inverse properties of discrete functions. The result is an almost quasi-optimal error estimate for the Nitsche method. Here, "almost" refers to perturbations which are only logarithmic in the mesh size.

The rest of this paper is organized as follows. In §2 we define some Sobolev spaces and our model problem. We also briefly recall the standard boundary element approximation. In §3 we introduce a domain decomposition (for simplicity only into two sub-domains; but this generalizes to more sub-domains in a straightforward way), the Nitsche-based discretization, and present our main result (Theorem 3.1). Technical details and the proof of Theorem 3.1 are given in §4. In §5 we present some numerical experiments that confirm the theoretical result.

Throughout the article, we will use the symbols "$\lesssim$" and "$\gtrsim$" in the usual sense. In short $a_h(v) \lesssim b_h(v)$ when there exists a constant $C > 0$ independent of v, the mesh size h and a fractional Sobolev index ε (if present), such that: $a_h(v) \leq C\, b_h(v)$.

2 Sobolev spaces and model problem

First let us briefly define the needed Sobolev spaces. We consider standard Sobolev spaces where the following norms are used: For $\Omega \subset \mathbb{R}^n$ and $0 < s < 1$ we define

$$\|u\|^2_{H^s(\Omega)} := \|u\|^2_{L^2(\Omega)} + |u|^2_{H^s(\Omega)}$$

with semi-norm

$$|u|_{H^s(\Omega)} := \Big(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{2s+n}}\, dx\, dy\Big)^{1/2}.$$

For a Lipschitz domain Ω and $0 < s < 1$, the space $\tilde{H}^s(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{\tilde{H}^s(\Omega)} := \Big(|u|^2_{H^s(\Omega)} + \int_\Omega \frac{u(x)^2}{(\mathrm{dist}(x, \partial\Omega))^{2s}}\, dx\Big)^{1/2}.$$

For $s \in (0,1/2)$, $\|\cdot\|_{\tilde{H}^s(\Omega)}$ and $\|\cdot\|_{H^s(\Omega)}$ are equivalent norms whereas for $s \in (1/2,1)$ there holds $\tilde{H}^s(\Omega) = H_0^s(\Omega)$, the latter space being the completion of $C_0^\infty(\Omega)$ with norm in $H^s(\Omega)$. Also we note that functions from $\tilde{H}^s(\Omega)$ are continuously extendible by zero onto a larger domain. For all these results we refer to [22, 16]. For $s > 0$ the spaces $H^{-s}(\Omega)$ and $\tilde{H}^{-s}(\Omega)$ are the dual spaces of $\tilde{H}^s(\Omega)$ and $H^s(\Omega)$, respectively.

Let Γ be a plane open surface with polygonal boundary. In the following we will identify Γ with a domain in $\mathbb{R}^2$, thus referring to sub-domains of Γ rather than sub-surfaces. The boundary of Γ is denoted by $\partial\Gamma$.

Our model problem is: *For a given function $f \in L^2(\Gamma)$ find $u \in \tilde{H}^{1/2}(\Gamma)$ such that*

$$Wu(x) := -\frac{1}{4\pi}\frac{\partial}{\partial \mathbf{n}_x}\int_\Gamma u(y)\frac{\partial}{\partial \mathbf{n}_y}\frac{1}{|x-y|}\,dS_y = f(x), \quad x \in \Gamma. \tag{2.1}$$

Here, $\mathbf{n}$ is a normal unit vector on Γ, e.g. $\mathbf{n} = (0,0,1)^T$. Note that W maps $\tilde{H}^{1/2}(\Gamma)$ continuously onto $H^{-1/2}(\Gamma)$ (see [12]). The variational formulation of (2.1) is: *Find $u \in \tilde{H}^{1/2}(\Gamma)$ such that*

$$\langle Wu, v\rangle_\Gamma = \langle f, v\rangle_\Gamma \quad \forall v \in \tilde{H}^{1/2}(\Gamma). \tag{2.2}$$

Here, $\langle\cdot,\cdot\rangle_\Gamma$ denotes the duality pairing between $H^{-1/2}(\Gamma)$ and $\tilde{H}^{1/2}(\Gamma)$. Throughout, this generic notation will be used for the L^2-inner product as well as for other dualities, the domain being mentioned by the index.

A standard boundary element method for the approximate solution of (2.2) is to select a piecewise polynomial subspace $\tilde{H}_h \subset \tilde{H}^{1/2}(\Gamma)$ and to define an approximant $\tilde{u}_h \in \tilde{H}_h$ by

$$\langle W\tilde{u}_h, v\rangle_\Gamma = \langle f, v\rangle_\Gamma \quad \forall v \in \tilde{H}_h. \tag{2.3}$$

Such a scheme is known to converge quasi-optimally in the energy norm. In §5 we will compare such a conforming approximation with our proposed Nitsche approach and a Lagrangian multiplier variant.

3 Discrete variational formulation with Nitsche coupling

In this section, we introduce the Nitsche-based boundary element method for the approximate solution of problem (2.2), and present the main result, Theorem 3.1.

3.1 Some preliminaries

We consider a decomposition of Γ into two non-intersecting polygonal sub-domains Γ_1 and Γ_2. The extension to an arbitrary number N of sub-domains is straightforward. We will denote this partition of Γ as

$$\mathcal{T} := \{\Gamma_1, \Gamma_2\}.$$

The interface between the sub-domains is denoted by $\gamma := \bar{\Gamma}_1 \cap \bar{\Gamma}_2$. Throughout the paper, we will use the notation v_i for the restriction of a function v to a sub-domain Γ_i. Also, as in [6], we will use the following notation for the jump on γ:

$$[v] := \quad (v_1 - v_2)|_\gamma.$$

Corresponding to the decomposition of Γ, we will need product Sobolev spaces, e.g.

$$H^s(\mathcal{T}) := H^s(\Gamma_1) \times H^s(\Gamma_2),$$

with usual product norm. This notation (putting the decomposition $\mathcal{T}$ instead of Γ) is used generically, i.e. also for the spaces $\tilde{H}^s(\mathcal{T})$. We introduce the following inner product

$$\langle v, w \rangle_\mathcal{T} := \langle v_1, w_1 \rangle_{\Gamma_1} + \langle v_2, w_2 \rangle_{\Gamma_2}$$

for $v, w \in L^2(\mathcal{T})(= L^2(\Gamma))$ and its extension by duality to $\tilde{H}^s(\mathcal{T}) \times H^{-s}(\mathcal{T})$.

For $1/2 \le s \le 1$, we introduce the following (semi-)norms, that are needed for the error analysis:

$$\begin{cases} |v|^2_{H^s(\mathcal{T})} & := \displaystyle\sum_{i=1}^{2} |v|^2_{H^s(\Gamma_i)}, \\ \|v\|^2_{H^s_*(\mathcal{T})} & := \displaystyle\sum_{i=1}^{2} |v|^2_{H^s(\Gamma_i)} + \|[v]\|^2_{L^2(\gamma)}, \end{cases} \tag{3.1}$$

where $|v|_{H^s(\Gamma_i)}$ is the Sobolev-Slobodeckij semi-norm as previously defined. The case $s = 1/2$ will be used only for discrete functions where the jump across γ is well defined.

To introduce the discrete scheme, let us define regular, quasi-uniform meshes $\mathcal{T}_i$, $i = 1, 2$, of shape regular elements (quadrilaterals or triangles): $\bar{\Gamma}_i = \cup_{K \in \mathcal{T}_i} \bar{K}$. The maximum, respectively minimum, diameter of the elements of $\mathcal{T}_i$ is denoted by h_i, respectively $\underline{h}_i$, and we define:

$$h := \max\,\{h_1, h_2\}, \qquad \underline{h} := \min\{\underline{h}_1, \underline{h}_2\}.$$

Throughout this paper we assume that $h < 1$. This is no restriction of generality and is just needed to simplify the writing of logarithmic terms. We introduce discrete spaces on sub-domains consisting of piecewise (bi)linear functions:

$$X_{h,i} := \{v \in C^0(\Gamma_i);\ v|_K \text{ is a polynomial of degree one for all } \ K \in \mathcal{T}_i;\ v|_{\partial\Gamma \cap \partial\Gamma_i} = 0\},$$

for $i = 1, 2$. We define a global discrete space on Γ:

$$X_h := X_{h,1} \times X_{h,2}.$$

Note that functions $v \in X_h$ do satisfy the homogeneous boundary condition along $\partial\Gamma$ but are in general discontinuous across the interface γ. Therefore $X_h \not\subset \tilde{H}^{1/2}(\Gamma)$, and this discrete space cannot be used directly for the discretization (2.3). Instead, we reformulate (2.3) as a Nitsche variant so that X_h can be used to approximate the continuous problem (2.2).

3.2 Setting of the Nitsche-based domain decomposition

For the setup of the Nitsche method let us introduce the following surface differential operators:

$$\mathbf{curl}\,\varphi := \big(\partial_{x_2}\varphi, -\partial_{x_1}\varphi, 0\big), \quad \mathrm{curl}\,\boldsymbol{\varphi} := \partial_{x_1}\varphi_2 - \partial_{x_2}\varphi_1 \quad \text{for} \quad \boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3).$$

The definitions of the surface curl operators are appropriate just for flat surfaces (as in our case) but can be extended to open and closed Lipschitz surfaces (see e.g. [10, 15]). We define corresponding piecewise differential operators $\mathbf{curl}_{\mathcal{T}}$ and $\mathrm{curl}_{\mathcal{T}}$ as follows:

$$\mathbf{curl}_{\mathcal{T}}\,\varphi := \sum_{i=1}^{2} (\mathbf{curl}_{\Gamma_i}\,\varphi_i)^0, \quad \mathrm{curl}_{\mathcal{T}}\,\boldsymbol{\varphi} := \sum_{i=1}^{2} (\mathrm{curl}_{\Gamma_i}\,\boldsymbol{\varphi}_i)^0,$$

where $\mathbf{curl}_{\Gamma_i}$ and curl_{Γ_i} refer to the restrictions of $\mathbf{curl}$ and curl, respectively, to Γ_i, and $(\cdot)^0$ indicates extension by zero to Γ. We made use of the notation introduced before $\varphi_i = \varphi|_{\Gamma_i}$, $\boldsymbol{\varphi}_i = \boldsymbol{\varphi}|_{\Gamma_i}$. Furthermore, we need the single layer potential operator V defined by:

$$V\boldsymbol{\varphi}(x) := \frac{1}{4\pi}\int_{\Gamma} \frac{\boldsymbol{\varphi}(y)}{|x-y|}\,dS_y, \quad \boldsymbol{\varphi} \in (\tilde{H}^{-1/2}(\Gamma))^3, \; x \in \Gamma.$$

We define the following bilinear form on $X_h \times X_h$:

$$\begin{aligned} A_{\mathcal{T}}(u_h, v_h) := {} & \langle V\,\mathbf{curl}_{\mathcal{T}}\,u_h, \mathbf{curl}_{\mathcal{T}}\,v_h\rangle_{\mathcal{T}} \\ & + \frac{1}{2}\langle T_1u_h - T_2u_h, [v_h]\rangle_\gamma + \frac{\sigma}{2}\langle [u_h], T_1v_h - T_2v_h\rangle_\gamma \\ & + \nu\langle [u_h], [v_h]\rangle_\gamma, \end{aligned} \tag{3.2}$$

where $\nu > 0$ and $\sigma \in \{-1, 1\}$ are numerical parameters. The operators T_i are defined as follows:

$$T_iv = [(V\,\mathbf{curl}_{\mathcal{T}}\,v)|_{\Gamma_i}\cdot \mathbf{t}_i]|_\gamma$$

for $i = 1, 2$. Here, $\mathbf{t}_i$ is the unit tangential vector on $\partial\Gamma_i$ (in mathematically positive orientation when identifying Γ_i with a subset of $\mathbb{R}^2$ which is compatible with the identification of Γ as a subset of $\mathbb{R}^2$). Note that T_iv is not well defined for $v \in \tilde{H}^{1/2}(\Gamma)$ in general since there is no well-defined trace from $H^{1/2}(\Gamma_i)$ to $\partial\Gamma_i$.

The Nitsche-based boundary element method associated to problem (2.2) then reads:
Find $u_h \in X_h$ such that

$$A_{\mathcal{T}}(u_h, v_h) = \langle f, v_h\rangle_\Gamma \tag{3.3}$$

for all $v_h \in X_h$.

Remark 3.1. *For any function $u \in H^s(\Gamma)$ $(s > 1/2)$, in particular for the solution of* (2.1)*, there holds $\langle [u], [v]\rangle_\gamma = \langle [u], T_1v - T_2v\rangle_\gamma = 0$ for sufficiently smooth v. Therefore the terms $\langle [u_h], [v_h]\rangle_\gamma$ and $\langle [u_h], T_1v_h - T_2v_h\rangle_\gamma$ are not required for consistency of* (3.3)*. However, the* *additional term $\frac{\sigma}{2}\langle [u_h], T_1 v_h - T_2 v_h\rangle_\gamma$ in the Nitsche-based formulation is of interest for two reasons [6, Remark 2.11]. First, for $\sigma = 1$, the bilinear form $A_{\mathcal{T}}(\cdot,\cdot)$ becomes symmetric, as in the standard case* (2.2)*. This allows in particular to make use of fast linear solvers for symmetric matrices. Also, for $\sigma = -1$, symmetry is lost, but we recover ellipticity of $A_{\mathcal{T}}(\cdot,\cdot)$ for any value of the parameter $\nu > 0$ (see Lemma 4.4 (i)). In fact, any value of σ (including $\sigma = 0$) can be chosen though only values -1 and 1 lead to interesting particular cases.*

The main result of this paper is:

Theorem 3.1. *Let $u \in H^r(\Gamma)$ with $r \in (1/2, 1)$ be the solution of* (2.1)*. In the case $\sigma = -1$ let $\nu > 0$ be arbitrary and in the case $\sigma = 1$ let $\nu \geq C_1 |\log \underline{h}|^3$ for a sufficiently large constant $C_1 > 0$. Then, the discrete problem* (3.3) *is uniquely solvable and there exists a constant $C_2 > 0$, depending on ν and r, but not on u and the actual mesh, such that there holds the error estimate*

$$\|u - u_h\|_{H^{1/2}_*(\mathcal{T})} \leq C_2 |\log \underline{h}|^{3/2} h^{r-1/2} \|u\|_{H^r(\Gamma)}.$$

A proof of this result will be given at the end of Section 4.

Remark 3.2. *It is known that $u \in H^r(\Gamma)$ for any $r < 1$, see, e.g.,* [29]*. Using this regularity, Theorem* 3.1 *proves a convergence which is close to $O(h^{1/2})$, the optimal one. The reduction to $h^{r-1/2}$ for any $r < 1$ is due to the assumed regularity in standard Sobolev spaces, and not a sub-optimality of the method. On the other hand, the logarithmic perturbation $|\log \underline{h}|^{3/2}$ is due to the Nitsche coupling, and is also present in non-conforming approaches (the same exponent* 3/2 *appears in the Lagrangian multiplier approach* [15]*, and in the mortar coupling* [18] *the exponent is* 2*). It is unknown whether these logarithmic terms in the upper bounds are optimal.*

4 Technical results and the proof of the main theorem

In §4.1, we present some preliminary results and lemmas. In §4.2 we then prove the consistency of the method, using an integration-by-parts formula coming from [15, 18]. Discrete continuity and discrete ellipticity are studied in §4.3. We conclude with the proof of the main theorem in §4.4.

The steps followed in the error analysis are quite similar to those of the analysis of a Nitsche-based method for finite elements (see e.g. [6]). The main difficulty in the case of boundary elements consist in the non-existence of a well-posed variational Nitsche formulation. Error estimates are wanted in spaces related to $H^{1/2}(\Gamma)$ where no well-defined trace operator exists. Therefore, the numerical analysis of (3.3) makes use of a whole family of Sobolev spaces H^r with r close to 1/2. Additional difficulty in our case is that operators are non-local in contrast to the finite element setting. In opposition to the mortar boundary element method [18], no *inf-sup* condition needs to be checked since no Lagrangian multipliers are introduced.

4.1 Preliminary results

We first introduce the following spaces for the definition of the single layer potential operator V (see [15]):

$$\begin{aligned}\tilde{\boldsymbol{H}}_t^{s-1}(\Gamma) &:= \{\boldsymbol{\psi} \in \big(\tilde{H}^{s-1}(\Gamma)\big)^3;\ \boldsymbol{\psi}\cdot\mathbf{n} = 0\},\\ \boldsymbol{H}_t^{s}(\Gamma) &:= \{\boldsymbol{\psi} \in \big(H^{s}(\Gamma)\big)^3;\ \boldsymbol{\psi}\cdot\mathbf{n} = 0\},\end{aligned} \tag{4.1}$$

where $0 \le s \le 1$ and the normal vector $\mathbf{n}$ has been defined previously. We will make use of the continuity (see [12]):

$$V : \tilde{\boldsymbol{H}}_t^{s-1}(\Gamma) \to \boldsymbol{H}_t^{s}(\Gamma), \quad 0 \le s \le 1. \tag{4.2}$$

Lemma 4.1. *For $i = 1, 2$ there holds*

$$\|T_i v\|_{L^2(\gamma)} \lesssim (s-1/2)^{-1/2}\ \|\,\mathbf{curl}_{\mathcal{T}}\, v\|_{\tilde{\boldsymbol{H}}_t^{s-1}(\Gamma)} \qquad \forall v \in H^s(\mathcal{T}), \quad 1/2 < s \le 1, \tag{4.3}$$

$$\|T_i v\|_{L^2(\gamma)} \lesssim (s-1/2)^{-3/2}\ |v|_{H^s(\mathcal{T})} \qquad \forall v \in H^s(\mathcal{T}), \quad 1/2 < s \le 1, \tag{4.4}$$

$$\|T_i v_h\|_{L^2(\gamma)} \lesssim |\log \underline{h}\,|^{3/2}\ |v_h|_{H^{1/2}(\mathcal{T})} \qquad \forall v_h \in X_h. \tag{4.5}$$

Proof. Let $v \in H^s(\mathcal{T})$, with $1/2 < s \le 1$. We use the trace theorem [15, Lemma 4.3] and the continuity of V (4.2) to bound

$$\|T_i v\|^2_{L^2(\gamma)} \lesssim \frac{1}{s-1/2}\|V\,\mathbf{curl}_{\mathcal{T}}\, v\|^2_{\boldsymbol{H}_t^{s}(\Gamma)} \lesssim \frac{1}{s-1/2}\|\,\mathbf{curl}_{\mathcal{T}}\, v\|^2_{\tilde{\boldsymbol{H}}_t^{s-1}(\Gamma)}.$$

This proves (4.3). Further, by using the equivalence of the $\boldsymbol{H}_t^{s-1}(\Gamma_i)$ and $\tilde{\boldsymbol{H}}_t^{s-1}(\Gamma_i)$ norms for $|s-1| < 1/2$ [19, Lemma 5] and the boundedness of $\mathbf{curl}_{\Gamma_i} : H^s(\Gamma_i) \to \boldsymbol{H}_t^{s-1}(\Gamma_i)$ [18, Lemma 3.4]), we obtain

$$\begin{aligned}\|\,\mathbf{curl}_{\mathcal{T}}\, v\|^2_{\tilde{\boldsymbol{H}}_t^{s-1}(\Gamma)} &\lesssim \sum_{i=1}^{2}\|\,\mathbf{curl}_{\Gamma_i}\, v\|^2_{\tilde{\boldsymbol{H}}_t^{s-1}(\Gamma_i)}\\ &\lesssim \frac{1}{(s-1/2)^2}\sum_{i=1}^{2}\|\,\mathbf{curl}_{\Gamma_i}\, v\|^2_{\boldsymbol{H}_t^{s-1}(\Gamma_i)} \lesssim \frac{1}{(s-1/2)^2}\sum_{i=1}^{2}\|v\|^2_{H^s(\Gamma_i)}.\end{aligned}$$

Combining these two estimates and using a quotient space argument proves (4.4).

Now picking $v_h \in X_h$, we have:

$$\begin{aligned}\|T_i v_h\|_{L^2(\gamma)} &\lesssim (s-1/2)^{-3/2}|v_h|_{H^s(\mathcal{T})}\\ &\lesssim \underline{h}^{1/2-s}(s-1/2)^{-3/2}|v_h|_{H^{1/2}(\mathcal{T})},\end{aligned}$$

using (4.4) and then the inverse property $|v|_{H^s(\Gamma_i)} \lesssim \underline{h}^{1/2-s}|v|_{H^{1/2}(\Gamma_i)}$ (see, e.g., [19, Lemma 4] together with a quotient space argument). With the choice $s = 1/2 + |\log \underline{h}|^{-1}$, this proves (4.5). ∎

4.2 Consistency of the Nitsche formulation

In this part, we show that the Nitsche formulation (3.3) for the hypersingular operator is consistent, a classical result for the Nitsche method in the standard case (see e.g. [6]). One difficulty here is that the boundary operator T_i is not well defined for $v \in H^{1/2}(\Gamma)$. Nevertheless, we can take advantage of previous results proven in [15].

First, we need to start from an appropriate integration-by-parts formula for the hypersingular operator. For the convenience of the reader we recall the setting from [15, 18]. For a smooth scalar function v and a smooth tangential vector field $\boldsymbol{\varphi}$, integration by parts on Γ_i gives

$$\langle \boldsymbol{\varphi} \cdot \mathbf{t}_i, v_i \rangle_{\partial\Gamma_i} = \langle \mathrm{curl}_{\Gamma_i} \boldsymbol{\varphi}, v_i \rangle_{\Gamma_i} - \langle \mathbf{curl}_{\Gamma_i} v_i, \boldsymbol{\varphi} \rangle_{\Gamma_i},$$

for $i = 1, 2$. We apply this formula to $\boldsymbol{\varphi} = (V \,\mathbf{curl}\, u)|_{\Gamma_i}$, so that:

$$\langle (V \,\mathbf{curl}\, u)|_{\Gamma_i} \cdot \mathbf{t}_i, v_i \rangle_{\partial\Gamma_i} = \langle \mathrm{curl}_{\Gamma_i} (V \,\mathbf{curl}\, u), v_i \rangle_{\Gamma_i} - \langle \mathbf{curl}_{\Gamma_i} v_i, V \,\mathbf{curl}\, u \rangle_{\Gamma_i}.$$

Recalling the definition of T_i, and using a function v that vanishes on $\partial\Gamma$, we obtain

$$\langle T_i u, v_i \rangle_{\gamma} = \langle \mathrm{curl}_{\Gamma_i} (V \,\mathbf{curl}\, u), v_i \rangle_{\Gamma_i} - \langle \mathbf{curl}_{\Gamma_i} v_i, V \,\mathbf{curl}\, u \rangle_{\Gamma_i}. \tag{4.6}$$

Let us recall the following lemma from [18] (Lemma 3.5):

Lemma 4.2. *For $u \in \tilde{H}^{1/2}(\Gamma)$ with $Wu = f \in L^2(\Gamma)$, the equation* (4.6) *defines $T_i u = (V \,\mathbf{curl}\, u)|_{\Gamma_i} \cdot \mathbf{t}_i \in H^{-s}(\gamma)$, with $0 < s \leq 1/2$.*

As a result, we can state:

Lemma 4.3. *Let $\nu > 0$ and $|\sigma| = 1$. Then, the Nitsche formulation is consistent, i.e. the solution u of* (2.1) *($Wu = f \in L^2(\Gamma)$) solves the discrete setting* (3.3)*,*

$$A_{\mathcal{T}}(u, v_h) = \langle f, v_h \rangle_{\Gamma} \qquad \forall v_h \in X_h.$$

Proof. Let $u \in \tilde{H}^{1/2}(\Gamma)$ be the solution of (2.1). It is well known that $u \in \tilde{H}^{s}(\Gamma)$ for any $s < 1$, see, e.g., [29], so that the trace of u on γ is well defined and $[u] = 0$.

By Lemma 4.2, $T_1 u$ and $T_2 u \in H^{-s}(\gamma)$ $(0 < s \leq 1/2)$. Moreover, since $[(V \,\mathbf{curl}\, u)|_{\Gamma_1} - (V \,\mathbf{curl}\, u)|_{\Gamma_2}]|_{\gamma} = 0$ and $\mathbf{t}_1 = -\mathbf{t}_2$ on γ, there holds

$$T_1 u + T_2 u = 0 \quad \text{on } \gamma. \tag{4.7}$$

We obtain for $v_h \in X_h$

$$\begin{aligned} A_{\mathcal{T}}(u, v_h) =& \langle V \,\mathbf{curl}_{\mathcal{T}}\, u, \mathbf{curl}_{\mathcal{T}}\, v_h \rangle_{\mathcal{T}} \quad + \frac{1}{2} \langle T_1 u - T_2 u, [v_h] \rangle_{\gamma} + \frac{\sigma}{2} \langle [u], T_1 v_h - T_2 v_h \rangle_{\gamma} \\ &+ \nu \langle [u], [v_h] \rangle_{\gamma} \\ =& \langle V \,\mathbf{curl}_{\Gamma}\, u, \mathbf{curl}_{\mathcal{T}}\, v_h \rangle_{\mathcal{T}} \quad + \frac{1}{2} \langle T_1 u - T_2 u, [v_h] \rangle_{\gamma}. \end{aligned}$$

Using (4.7) to write $T_i u = \frac{1}{2}T_i u - \frac{1}{2}T_j u$ $(i \neq j)$, and rearranging terms, we obtain

$$\frac{1}{2}\langle T_1 u - T_2 u, [v_h]\rangle_\gamma = \sum_{i=1}^{2}\langle T_i u, v_{h,i}\rangle_\gamma$$

so that, together with the integration-by-parts formula (4.6),

$$\begin{aligned} A_{\mathcal{T}}(u, v_h) =& \sum_{i=1}^{2}\left[\langle V\,\mathbf{curl}_\Gamma\, u, \mathbf{curl}_{\Gamma_i}\, v_{h,i}\rangle_{\Gamma_i} + \langle T_i u, v_{h,i}\rangle_\gamma\right] = \sum_{i=1}^{2}\langle \mathrm{curl}_{\Gamma_i}(V\,\mathbf{curl}_\Gamma\, u), v_{h,i}\rangle_{\Gamma_i} \\ =& \langle \mathrm{curl}_\Gamma(V\,\mathbf{curl}_\Gamma\, u), v_h\rangle_\Gamma = \langle Wu, v_h\rangle_\Gamma = \langle f, v_h\rangle_\Gamma, \end{aligned}$$

Here, we have also used the relation

$$Wu = \mathrm{curl}_\Gamma\, V\,\mathbf{curl}_\Gamma\, u,$$

see [23, 24]. This proves the lemma. □

4.3 Discrete ellipticity and continuity

Main advantage of the Nitsche method is that it yields elliptic bilinear forms in the case of elliptic problems. In the boundary element setting, we do not have an appropriate variational formulation. Nevertheless, discrete ellipticity is still achievable. This is contents of the first lemma. Afterwards, we briefly state discrete continuity without giving a bound for the continuity constant. This bound is studied in more detail in the proof of the main theorem in §4.4.

Lemma 4.4. (i) *Let $\sigma = -1$. For all $\nu > 0$ there exists a constant $C(\nu) > 0$ such that*

$$A_{\mathcal{T}}(v_h, v_h) \geq C(\nu)\,\|v_h\|^2_{H_*^{1/2}(\mathcal{T})} \quad \forall v_h \in X_h$$

and

$$A_{\mathcal{T}}(v_h, v_h) \geq C(\nu)\left(\|\,\mathbf{curl}_{\mathcal{T}}\, v_h\|_{\tilde{H}^{-1/2}(\Gamma)} + \|[v_h]\|_{L^2(\gamma)}\right)\|v_h\|_{H_*^{1/2}(\mathcal{T})} \quad \forall v_h \in X_h.$$

(ii) *Let $\sigma = 1$. There exists a constant $C_1 > 0$ such that, if $\nu \geq C_1\,|\log \underline{h}|^3$, then there exists a constant $C_2 > 0$ independent of ν such that*

$$A_{\mathcal{T}}(v_h, v_h) \geq C_2\,\|v_h\|^2_{H_*^{1/2}(\mathcal{T})} \quad \forall v_h \in X_h$$

and

$$A_{\mathcal{T}}(v_h, v_h) \geq C_2\left(\|\,\mathbf{curl}_{\mathcal{T}}\, v_h\|_{\tilde{H}^{-1/2}(\Gamma)} + \|[v_h]\|_{L^2(\gamma)}\right)\|v_h\|_{H_*^{1/2}(\mathcal{T})} \quad \forall v_h \in X_h.$$

Proof. (i) Case $\sigma = -1$.

Let $v_h \in X_h$. By the ellipticity of V and [15, Lemma 4.1] there holds

$$\begin{aligned}\langle V\,\mathbf{curl}_{\mathcal{T}}\, v_h, \mathbf{curl}_{\mathcal{T}}\, v_h\rangle_{\mathcal{T}} &\gtrsim \|\,\mathbf{curl}_{\mathcal{T}}\, v_h\|^2_{\tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)} \gtrsim \sum_{i=1}^{2} \|\,\mathbf{curl}_{\Gamma_i}\, v_h\|^2_{\boldsymbol{H}_t^{-1/2}(\Gamma_i)}\\ &\gtrsim \sum_{i=1}^{2} |v_h|^2_{H^{1/2}(\Gamma_i)} = |v_h|^2_{H^{1/2}(\mathcal{T})}.\end{aligned}$$

This proves that

$$\begin{aligned}A_{\mathcal{T}}(v_h, v_h) &= \langle V\,\mathbf{curl}_{\mathcal{T}}\, v_h, \mathbf{curl}_{\mathcal{T}}\, v_h\rangle_{\mathcal{T}} + \nu\langle [v_h], [v_h]\rangle_\gamma\\ &\gtrsim |v_h|^2_{H^{1/2}(\mathcal{T})} + \|[v_h]\|^2_{L^2(\gamma)} = \|v_h\|^2_{H_*^{1/2}(\mathcal{T})}\end{aligned}\tag{4.8}$$

(which is the first assertion) and also

$$A_{\mathcal{T}}(v_h, v_h) \gtrsim \|\,\mathbf{curl}_{\mathcal{T}}\, v_h\|^2_{\tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)} + \|[v_h]\|^2_{L^2(\gamma)}.$$

Both estimates together prove the second assertion.

(ii) In the case $\sigma = 1$ we obtain for $v_h \in X_h$

$$A_{\mathcal{T}}(v_h, v_h) = \langle V\,\mathbf{curl}_{\mathcal{T}}\, v_h, \mathbf{curl}_{\mathcal{T}}\, v_h\rangle_{\mathcal{T}} + \nu\langle [v_h], [v_h]\rangle_\gamma + \langle T_1 v_h - T_2 v_h, [v_h]\rangle_\gamma.$$

Using the Cauchy-Schwarz and Young's inequalities, and (4.4), we bound

$$\begin{aligned}\langle T_1 v_h - T_2 v_h, [v_h]\rangle_\gamma &\lesssim \delta\|T_1 v_h - T_2 v_h\|^2_{L^2(\gamma)} + \frac{1}{\delta}\|[v_h]\|^2_{L^2(\gamma)}\\ &\lesssim \frac{\delta}{(s-1/2)^3}|v_h|^2_{H^s(\mathcal{T})} + \frac{1}{\delta}\|[v_h]\|^2_{L^2(\gamma)} \quad \forall \delta > 0.\end{aligned}$$

Combination of these two relations with (4.8), and making use of the inverse property $|v|_{H^s(\Gamma)} \lesssim \underline{h}^{1/2-s}|v|_{H^{1/2}(\Gamma)}$ (see, e.g., [19, Lemma 4] together with a quotient space argument), yields

$$A_{\mathcal{T}}(v_h, v_h) \gtrsim \Big(1 - \delta C_1 \frac{\underline{h}^{1-2s}}{(s-1/2)^3}\Big)|v_h|^2_{H^{1/2}(\mathcal{T})} + (\nu - \frac{C_2}{\delta})\|[v_h]\|^2_{L^2(\gamma)} \quad \forall \delta > 0$$

for two unknown constants $C_1, C_2 > 0$. Selecting

$$s = \frac{1}{2}(1 + |\log \underline{h}|^{-1}) \qquad \text{and} \qquad \delta = \frac{ce}{8C_1}|\log \underline{h}|^{-3} \quad \text{for} \quad c \in (0,1)$$

this yields

$$A_{\mathcal{T}}(v_h, v_h) \gtrsim |v_h|^2_{H^{1/2}(\mathcal{T})} + \|[v_h]\|^2_{L^2(\gamma)} = \|v_h\|^2_{H_*^{1/2}(\mathcal{T})}$$

for $\nu \geq \frac{C_2}{\delta} + c = \frac{8C_1C_2}{ce}|\log \underline{h}|^3 + c$. This proves the first estimate in (ii). As in the case $\sigma = -1$, and using (4.3) in addition to (4.4), one proves the second assertion under the same condition on ν. $\square$

Lemma 4.5. *Let $\nu > 0$ and $|\sigma| = 1$. The bilinear form $A_{\mathcal{T}}$ is continuous:*

$$A_{\mathcal{T}}(v_h, w_h) \lesssim C(\nu, \underline{h}) \|v_h\|_{H_*^{1/2}(\mathcal{T})} \|w_h\|_{H_*^{1/2}(\mathcal{T})} \quad \forall v_h, w_h \in X_h,$$

with $C(\nu, \underline{h}) > 0$ a number that depends on ν and on the mesh parameter $\underline{h}$.

Proof. This estimate follows by using the mapping properties of the involved operators V, $\mathbf{curl}_{\mathcal{T}}$, T_i and inverse properties of discrete functions.

□

4.4 Proof of the main theorem

By Lemma 4.4, and under the stated assumptions, the bilinear form $A_{\mathcal{T}}$ is elliptic. Moreover, by Lemma 4.5, this bilinear form is also continuous on X_h (with bound depending on the mesh) so that problem (3.3) has a unique solution. It remains to bound the error. To this end we follow the lines of a Strang estimate for non-conforming methods. By Lemma 4.4 there holds for any $v_h \in X_h$

$$\begin{aligned} \|u - u_h\|_{H_*^{1/2}(\mathcal{T})} &\le \|u - v_h\|_{H_*^{1/2}(\mathcal{T})} + \|u_h - v_h\|_{H_*^{1/2}(\mathcal{T})} \\ &\lesssim \|u - v_h\|_{H_*^{1/2}(\mathcal{T})} + \sup_{w_h \in X_h \setminus \{0\}} \frac{A_{\mathcal{T}}(u_h - v_h, w_h)}{\|\mathbf{curl}_{\mathcal{T}}\, w_h\|_{\tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)} + \|[w_h]\|_{L^2(\gamma)}}. \end{aligned} \tag{4.9}$$

Now, by the consistency (see Lemma 4.3) we obtain $A_{\mathcal{T}}(u_h - v_h, w_h) = A_{\mathcal{T}}(u - v_h, w_h)$ so that we continue bounding (using duality estimates and the continuity of V (4.2))

$$\begin{aligned} A_{\mathcal{T}}(u_h - v_h, w_h) &= \langle V\, \mathbf{curl}_{\mathcal{T}}(u - v_h), \mathbf{curl}_{\mathcal{T}}\, w_h \rangle_{\mathcal{T}} + \frac{1}{2} \langle T_1(u - v_h) - T_2(u - v_h), [w_h] \rangle_\gamma \\ &\quad + \frac{\sigma}{2} \langle [u - v_h], T_1 w_h - T_2 w_h \rangle_\gamma + \nu \langle [u - v_h], [w_h] \rangle_\gamma \\ &\lesssim \|\mathbf{curl}_{\mathcal{T}}(u - v_h)\|_{\tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)} \|\mathbf{curl}_{\mathcal{T}}\, w_h\|_{\tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)} \\ &\quad + \sum_{i=1}^{2} \Big(\|T_i(u - v_h)\|_{L^2(\gamma)} \|[w_h]\|_{L^2(\gamma)} + \|T_i w_h\|_{L^2(\gamma)} \|[u - v_h]\|_{L^2(\gamma)} \Big) \\ &\quad + \|[u - v_h]\|_{L^2(\gamma)} \|[w_h]\|_{L^2(\gamma)}. \end{aligned} \tag{4.10}$$

We bound the terms on the right-hand side.

In the following let s be a small positive number. Using [19, Lemma 5] and the continuity of $\mathbf{curl}_{\Gamma_i} : H^{s+1/2}(\Gamma_i) \to \boldsymbol{H}_t^{s-1/2}(\Gamma_i)$, together with a quotient space argument, yields

$$\begin{aligned} \|\mathbf{curl}_{\mathcal{T}}(u - v_h)\|^2_{\tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)} &\lesssim \sum_{i=1}^{2} \|\mathbf{curl}_{\mathcal{T}}(u - v_h)\|^2_{\tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma_i)} \\ &\lesssim \frac{1}{s^2} \sum_{i=1}^{2} \|\mathbf{curl}_{\mathcal{T}}(u - v_h)\|^2_{\boldsymbol{H}_t^{s-1/2}(\Gamma_i)} \lesssim \frac{1}{s^2} \sum_{i=1}^{2} |u - v_h|^2_{H^{s+1/2}(\Gamma_i)}, \end{aligned} \tag{4.11}$$

and estimate (4.4) proves

$$\|T_i(u-v_h)\|_{L^2(\gamma)} \lesssim \frac{1}{s^{3/2}}|u-v_h|_{H^{s+1/2}(\mathcal{T})}. \tag{4.12}$$

Eventually, by (4.3) and the inverse property (see [19, Lemma 4]),

$$\|T_i w_h\|^2_{L^2(\gamma)} \lesssim \frac{1}{s}\|\,\mathbf{curl}_{\mathcal{T}}\, w_h\|^2_{\tilde{\boldsymbol{H}}_t^{s-1/2}(\Gamma)} \lesssim \frac{1}{s\underline{h}^{2s}}\|\,\mathbf{curl}_{\mathcal{T}}\, w_h\|^2_{\tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)}. \tag{4.13}$$

Combination of (4.10)–(4.13) proves that for any $w_h \in X_h \setminus \{0\}$ and any small $s > 0$ there holds

$$\frac{A_{\mathcal{T}}(u_h - v_h, w_h)}{\|\,\mathbf{curl}_{\mathcal{T}}\, w_h\|_{\tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)} + \|[w_h]\|_{L^2(\gamma)}} \lesssim \underline{h}^{-s} s^{-3/2}\Big(|u-v_h|_{H^{s+1/2}(\mathcal{T})} + \|[u-v_h]\|_{L^2(\gamma)}\Big).$$

By a standard approximation result we have that, for $r \in (1/2;1)$,

$$\inf_{v_h\in X_h}\Big(|u-v_h|_{H^{s+1/2}(\mathcal{T})} + \|[u-v_h]\|_{L^2(\gamma)}\Big) \lesssim h^{r-s-1/2}\|u\|_{H^r(\Gamma)},$$

so that referring to (4.9) this proves that

$$\|u-u_h\|_{H_*^{1/2}(\mathcal{T})} \lesssim \underline{h}^{-2s} s^{-3/2} h^{r-1/2}\|u\|_{H^r(\Gamma)}.$$

Selecting $s = |\log \underline{h}|^{-1}$ this proves Theorem 3.1.

5 Numerical results

We consider the model problem (2.1) with $\Gamma = (0,1)\times(0,1)$ and $f = 1$. For the sake of simplicity, we only deal with the case of one sub-domain and in where the homogeneous Dirichlet condition on the boundary $\gamma = \partial\Gamma$ (implicitly present in the energy space $\tilde{H}^{1/2}(\Gamma)$) is imposed weakly (in the discrete case) through a Nitsche formulation.

This situation is identical to the one described in [15, Section V] where a Lagrangian multiplier is used to impose the homogeneous boundary condition. Below we compare numerical results from both methods (Figure 5.5).

To obtain the Nitsche formulation of this problem, we formally extend u by 0 onto $\mathbb{R}^2$ and decompose $\mathbb{R}^2$ into $\Gamma_1 = \Gamma$ and $\Gamma_2 = \mathbb{R}^2 \setminus \Gamma$. The extension of u by 0 is continuous in $H^{1/2}(\mathbb{R}^2)$ since $u \in \tilde{H}^{1/2}(\Gamma)$. As a result, the Nitsche formulation is a particular case of the one studied in this paper, and the corresponding bilinear form is obtained from (3.2) by using that jumps across $\gamma = \partial\Gamma$ are identical to traces on γ (taking the exact approximation 0 of the solution exterior to Γ).

We use uniform meshes $\mathcal{T}_h$ on Γ which consist of squares of side-length h. The discrete spaces X_h are made of continuous piecewise bilinear polynomials on $\mathcal{T}_h$. Then the Nitsche-based formulation reads: *Find $u_h \in X_h$ such that*

$$\langle V\,\mathbf{curl}_\Gamma\, u_h, \mathbf{curl}_\Gamma\, v_h\rangle_\Gamma + \langle Tu_h, v_h\rangle_\gamma + \sigma\langle u_h, Tv_h\rangle_\gamma + \nu\langle u_h, v_h\rangle_\gamma = \langle f, v_h\rangle_\Gamma \tag{5.1}$$

for all $v_h \in X_h$. Here, the operator T is defined by

$$Tv := \mathbf{t} \cdot V\,\mathbf{curl}_\Gamma\, v|_\gamma$$

with $\mathbf{t}$ being the tangential unit vector along γ. Since the exact solution u of (2.1) is unknown, the error

$$\|u - u_h\|^2_{H^{1/2}_*(\Gamma)} = |u - u_h|^2_{H^{1/2}(\Gamma)} + \|u_h\|^2_{L^2(\gamma)}$$

cannot be computed directly (note that $u = 0$ on γ). Instead, we approximate an upper bound to the semi-norm $|u - u_h|_{H^{1/2}(\Gamma)}$ as follows.

First, note that there holds

$$|u - u_h|^2_{H^{1/2}(\Gamma)} \lesssim \langle V\,\mathbf{curl}_\Gamma(u - u_h), \mathbf{curl}_\Gamma(u - u_h)\rangle_\Gamma,$$

due to ellipticity of V. Taking into account that u is solution of (2.1) and u_h is solution of (5.1), we find:

$$\begin{aligned}
|u - u_h|^2_{H^{1/2}(\Gamma)} &\lesssim \langle V\,\mathbf{curl}_\Gamma\, u, \mathbf{curl}_\Gamma\, u\rangle_\Gamma + \langle V\,\mathbf{curl}_\Gamma\, u_h, \mathbf{curl}_\Gamma\, u_h\rangle_\Gamma - 2\langle V\,\mathbf{curl}_\Gamma\, u, \mathbf{curl}_\Gamma\, u_h\rangle_\Gamma \\
&= \langle Wu, u\rangle_\Gamma + \langle f, u_h\rangle_\Gamma - \langle Tu_h, u_h\rangle_\gamma - \sigma\langle u_h, Tu_h\rangle_\gamma \\
&\quad - \nu\langle u_h, u_h\rangle_\gamma - 2\langle Wu, u_h\rangle_\Gamma + 2\langle Tu, u_h\rangle_\gamma \\
&\lesssim \langle Wu, u\rangle_\Gamma - \langle f, u_h\rangle_\Gamma - (1+\sigma)\langle Tu_h, u_h\rangle_\gamma - \nu\langle u_h, u_h\rangle_\gamma + 2\langle Tu, u_h\rangle_\gamma.
\end{aligned}$$

Then, from $\langle u_h, u_h\rangle_\gamma \geq 0$, the Cauchy-Schwarz inequality and Lemma 4.1 (inequality (4.5)), we obtain

$$\begin{aligned}
|u - u_h|^2_{H^{1/2}(\Gamma)} &\lesssim \langle Wu, u\rangle_\Gamma - \langle f, u_h\rangle_\Gamma + \|Tu_h\|_{L^2(\gamma)}\|u_h\|_{L^2(\gamma)} + \|Tu\|_{L^2(\gamma)}\|u_h\|_{L^2(\gamma)} \\
&\lesssim \langle Wu, u\rangle_\Gamma - \langle f, u_h\rangle_\Gamma + \Big(|\log h|^{3/2}|u_h|_{H^{1/2}(\Gamma)} + \|Tu\|_{L^2(\gamma)}\Big)\,\|u_h\|_{L^2(\gamma)}.
\end{aligned}$$

Note that for this specific problem, $Tu \in L^2(\gamma)$. Furthermore, since the method is stable, $|u_h|_{H^{1/2}(\Gamma)}$ is bounded independently of h. This proves that

$$|u - u_h|_{H^{1/2}(\Gamma)} \lesssim |\langle Wu, u\rangle_\Gamma - \langle f, u_h\rangle_\Gamma|^{1/2} + |\log h|^{3/4}\|u_h\|^{1/2}_{L^2(\gamma)}.$$

Moreover, since $\|u_h\|_{L^2(\gamma)} \lesssim \|u_h\|^{1/2}_{L^2(\gamma)}$, there also holds

$$\|u - u_h\|_{H^{1/2}_*(\Gamma)} \lesssim |\langle Wu, u\rangle_\Gamma - \langle f, u_h\rangle_\Gamma|^{1/2} + |\log h|^{3/4}\|u_h\|^{1/2}_{L^2(\gamma)}.$$

The terms $\langle f, u_h\rangle_\Gamma$ and $\|u_h\|_{L^2(\gamma)}$ are easy to compute. The energy norm $\langle Wu, u\rangle^{1/2}_\Gamma$ of u can be approximated through extrapolation, denoted by $\|u\|_{ex}$ in the following, see [13]. Therefore,

$$\Big(\big|\|u\|^2_{\mathrm{ex}} - \langle f, u_h\rangle_\Gamma\big|^{1/2} + |\log h|^{3/4}\|u_h\|^{1/2}_{L^2(\gamma)}\Big)/\|u\|_{\mathrm{ex}}$$

is a computable and reasonable measure for an upper bound of the error $\|u - u_h\|_{H_*^{1/2}(\Gamma)}$ normalized by $\|u\|_{\tilde{H}^{1/2}(\Gamma)}$. Below we present numerical results for the two contributions

$$\big|\|u\|_{\rm ex}^2 - \langle f, u_h\rangle_\Gamma\big|^{1/2} / \|u\|_{\rm ex} \tag{5.2}$$

(referred to as "$H^{1/2}$" error in the figures) and

$$\|u_h\|_{L^2(\gamma)}^{1/2} / \|u\|_{\rm ex} \tag{5.3}$$

(referred to as "L^2" error).

We first consider some tests in the skew-symmetric case ($\sigma = -1$), for different values of ν. The corresponding results are given in Figures 5.1 and 5.2. A double logarithmic scale is chosen and the errors (5.2) are plotted versus the dimension of the discrete space X_h. Figure 5.1 presents results for the term (5.2) of the error and indicates that the Nitsche-based method converges for all the tested values of ν, with a logarithmic perturbation of the convergence, as expected by the theory (Theorem 3.1). As a consequence, the convergence is asymptotically a bit slower (by a factor of $|\log h|$) than in the case of the conforming BEM. The latter method converges like $O(h^{1/2})$, and for comparison we have given the curve $|\log h| h^{1/2}$ as well (with a constant factor for adjustment). For all studied values of ν the curves exhibit the same asymptotic convergence order, though their initial behavior differ. In particular, for $\nu \geq 2$, a minimum is reached quickly, after which the asymptotic behavior is recovered. Apparently, for any particular mesh, there simply is an optimal value of ν for term (5.2).

Figure 5.2 shows that the other part of the error (given by (5.3)) also behaves as predicted. All the curves are parallel and the parameter ν does not seem to have a great influence, except for shifting the curves which corresponds to multiplication of the error by a constant.

Next we study the symmetric case ($\sigma = 1$). The corresponding results are given in Figures 5.3 and 5.4. As expected, if the value of ν is not sufficiently large, the method does not converge (see the curve for $\nu = 1$ in Figures 5.3 and 5.4). Indeed, if ν is too small, discrete ellipticity of $A_{\mathcal{T}}(\cdot, \cdot)$ cannot be guaranteed (see Lemma 4.4). Taking higher values of ν ensures convergence of the method. In particular, if ν is not taken as a constant but a power of $|\log h|$ the asymptotic behavior improves. For $\nu = |\log h|^2$, the behavior of the conforming BEM method is recovered, with quasi-optimal convergence. Note that theoretically, a sufficient condition to guarantee discrete ellipticity and convergence (in the symmetric case) is $\nu \gtrsim |\log h|^3$ (cf. Lemma 4.4 and Theorem 3.1). The same conclusions hold when one looks at the $L^2(\gamma)$-error (5.3) in Figure 5.4.

In Figure 5.5 we compare the Nitsche method (symmetric and skew-symmetric versions) with the Lagrangian multiplier-based method [18]. One observes that the symmetric Nitsche method and the Lagrangian multiplier method have the same asymptotic convergence, which is quasi-optimal (without logarithmic perturbation) in this example. The skew-symmetric method, on the other hand, remains almost quasi-optimal, i.e. with logarithmic perturbation.

Concluding, the numerical experiments are in good agreement with the theory, and illustrate the applicability of the Nitsche-based domain decomposition method for hypersingular integral equations, e.g. as a possible alternative to a Lagrangian multiplier approach which requires

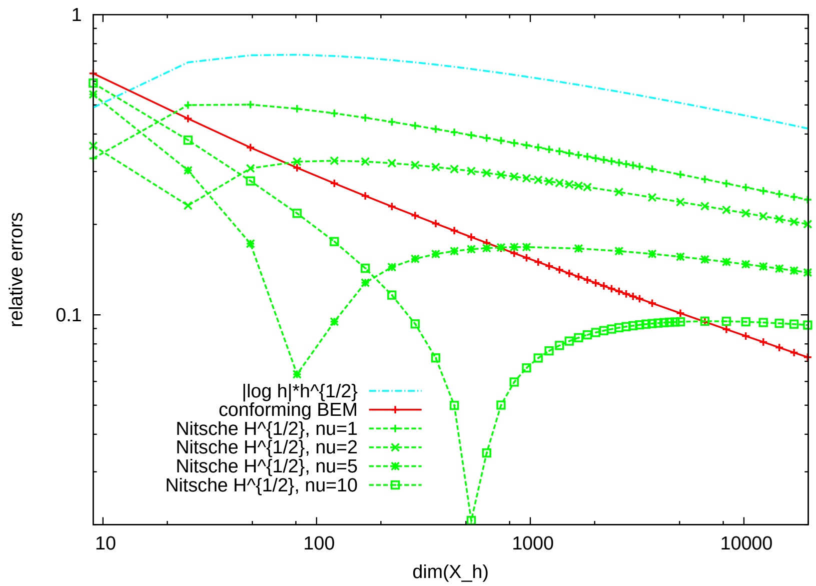

Figure 5.1: Skew-symmetric Nitsche method ($\sigma = -1$): relative error curves (upper bound (5.2)). Comparison with conforming BEM.

an additional unknown and destroys ellipticity. In particular, the symmetric case seems to be more appealing due to its competitive convergence for large values of ν, and since it maintains symmetry.

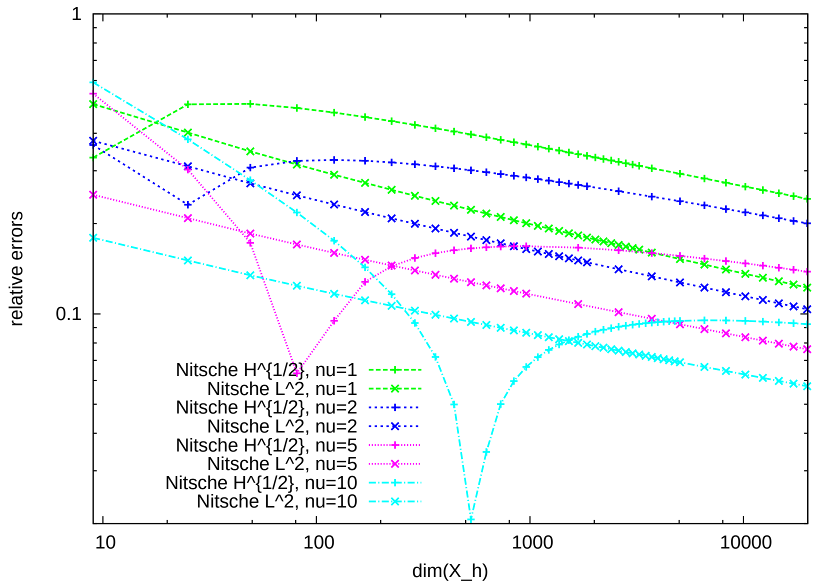

Figure 5.2: Skew-symmetric Nitsche method ($\sigma = -1$): relative error curves (upper bounds (5.2) and (5.3)).

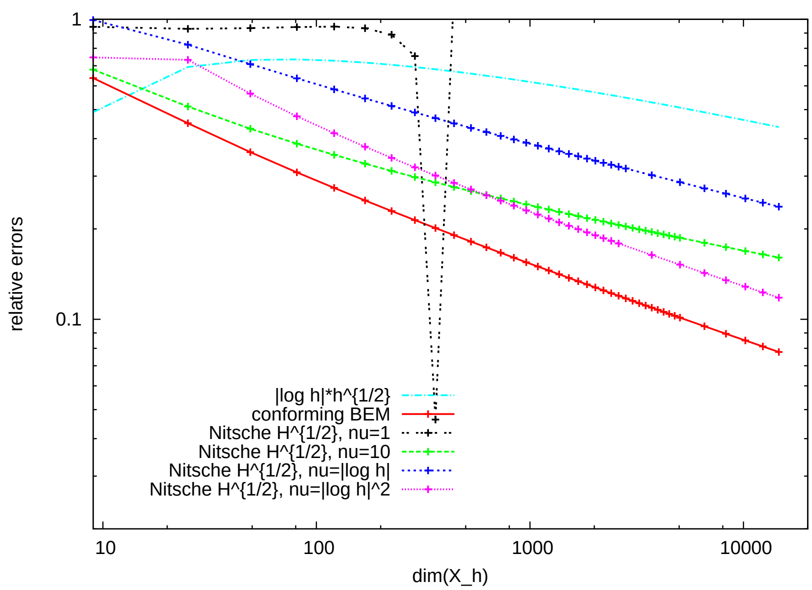

Figure 5.3: Symmetric Nitsche method ($\sigma = 1$): relative error curves (upper bound (5.2)). Comparison with conforming BEM.

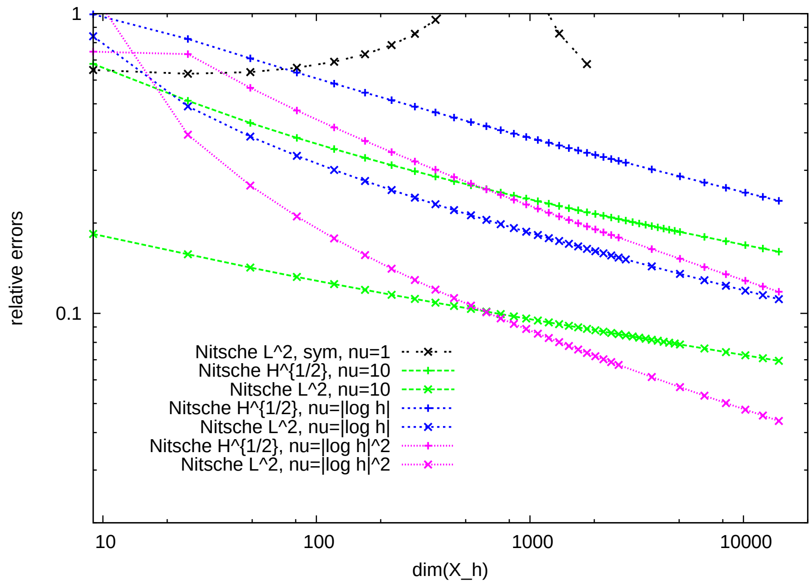

Figure 5.4: Symmetric Nitsche method ($\sigma = -1$): relative error curves (upper bounds (5.2), except for $\nu = 1$, and (5.3)).

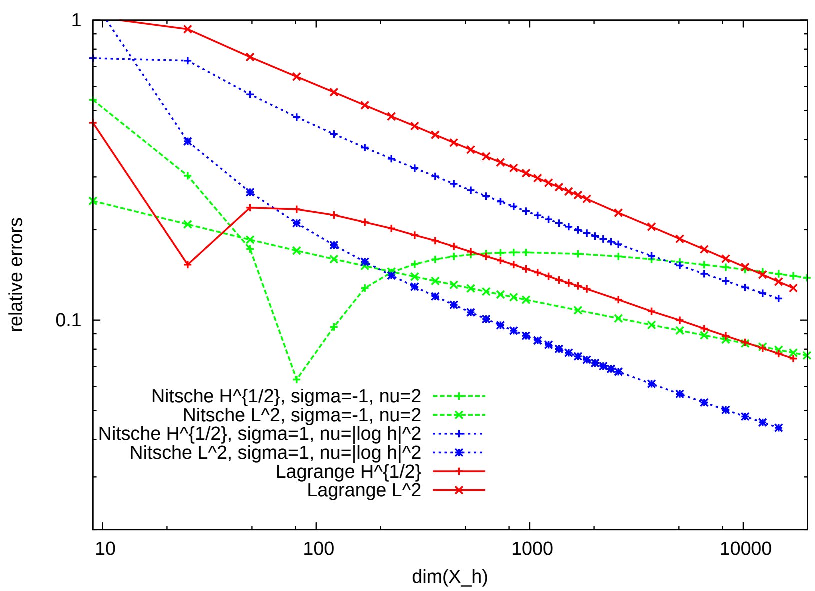

Figure 5.5: Comparing Nitsche and Lagrangian multiplier methods: relative error curves (upper bounds (5.2) and (5.3) for all cases).

Acknowledgments. Part of this work has been carried out during research stays of Franz Chouly at the Universidad Técnica Federico Santa María, Valparaíso, and the Pontificia Universidad Católica de Chile, Santiago.

References

[1] D. Arnold, *An interior penalty finite element method with discontinuous elements*, SIAM Journal on Numerical Analysis, 19 (1982), pp. 742–760.

[2] M. Astorino, F. Chouly, and M. A. Fernández, *Robin based semi-implicit coupling in fluid-structure interaction: stability analysis and numerics*, SIAM Journal on Scientific Computing, 31 (2009/10), pp. 4041–4065.

[3] H. J. C. Barbosa and T. J. R. Hughes, *The finite element method with Lagrange multipliers on the boundary: circumventing the Babuška-Brezzi condition*, Computer Methods in Applied Mechanics and Engineering, 85 (1991), pp. 109–128.

[4] ——, *Boundary Lagrange multipliers in finite element methods: error analysis in natural norms*, Numerische Mathematik, 62 (1992), pp. 1–15.

[5] R. Becker, E. Burman, and P. Hansbo, *A Nitsche extended finite element method for incompressible elasticity with discontinuous modulus of elasticity*, Computer Methods in Applied Mechanics and Engineering, 198 (2009), pp. 3352–3360.

[6] R. Becker, P. Hansbo, and R. Stenberg, *A finite element method for domain decomposition with non-matching grids*, M2AN Mathematical Modelling and Numerical Analysis, 37 (2003), pp. 209–225.

[7] F. Ben Belgacem, *The mortar finite element method with Lagrange multipliers*, Numerische Mathematik, 84 (1999), pp. 173–197.

[8] C. Bernardi, Y. Maday, and A. T. Patera, *Domain decomposition by the mortar element method*, in Asymptotic and numerical methods for partial differential equations with critical parameters (Beaune, 1992), vol. 384 of NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., Kluwer Acad. Publ., Dordrecht, 1993, pp. 269–286.

[9] ——, *A new nonconforming approach to domain decomposition: the mortar element method*, in Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. XI (Paris, 1989–1991), vol. 299 of Pitman Res. Notes Math. Ser., Longman Sci. Tech., Harlow, 1994, pp. 13–51.

[10] A. Buffa, M. Costabel, and D. Sheen, *On traces for H(curl,Ω) in Lipschitz domains*, Journal of Mathematical Analysis and Applications, 276 (2002), pp. 845–867.

[11] E. Burman and M. A. Fernández, *Stabilization of explicit coupling in fluid-structure interaction involving fluid incompressibility*, Computer Methods in Applied Mechanics and Engineering, 198 (2009), pp. 766–784.

[12] M. Costabel, *Boundary Integral Operators on Lipschitz Domains: Elementary Results*, SIAM Journal on Mathematical Analysis, 19 (1988), pp. 613–626.

[13] V. J. Ervin, N. Heuer, and E. P. Stephan, *On the h-p version of the boundary element method for symm's integral equation on polygons*, Computer Methods in Applied Mechanics and Engineering, 110 (1999), pp. 25–38.

[14] A. Fritz, S. Hüeber, and B. I. Wohlmuth, *A comparison of mortar and Nitsche techniques for linear elasticity*, Calcolo, 41 (2004), pp. 115–137.

[15] G. N. Gatica, M. Healey, and N. Heuer, *The boundary element method with lagrangian multipliers*, Numerical Methods for Partial Differential Equations, 25 (2009), pp. 1303–1319.

[16] P. Grisvard, *Elliptic problems in nonsmooth domains*, Pitman Advanced Pub. Program, Boston, 1985.

[17] P. Hansbo, J. Hermansson, and T. Svedberg, *Nitsche's method combined with space-time finite elements for ALE fluid-structure interaction problems*, Computer Methods in Applied Mechanics and Engineering, 193 (2004), pp. 4195–4206.

[18] M. Healey and N. Heuer, *Mortar boundary elements*, SIAM Journal on Numerical Analysis, 48 (2010), pp. 1395–1418.

[19] N. Heuer, *Additive Schwarz method for the p-version of the boundary element method for the single layer potential operator on a plane screen*, Numerische Mathematik, 88 (2001), pp. 485–511.

[20] N. Heuer and F.-J. Sayas, *Crouzeix–Raviart boundary elements*, Numerische Mathematik, 112 (2009), pp. 381–401.

[21] G. C. Hsiao and W. L. Wendland, *Domain decomposition in boundary element methods*, in Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, Y. A. Kuznetsov, G. A. Meurant, J. Périaux, and O. B. Widlund, eds., Philadelphia, 1991, SIAM, pp. 41–49.

[22] J. L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications*, Springer-Verlag, Berlin, New York, 1972.

[23] A.-W. Maue, *Zur Formulierung eines allgemeinen Beugungsproblems durch eine Integralgleichung*, Zeitschrift für Physik, 126 (1949), pp. 601–618.

[24] J.-C. Nédélec, *Integral equations with nonintegrable kernels*, Integral Equations Operator Theory, 5 (1982), pp. 562,572.

[25] J. Nitsche, *Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg, 36 (1971), pp. 9–15.

[26] A. Quarteroni and A. Valli, *Domain decomposition methods for partial differential equations*, Numerical Mathematics and Scientific Computation, The Clarendon Press - Oxford University Press, 1999. Oxford Science Publications.

[27] A. Reusken and T. H. Nguyen, *Nitsche's method for a transport problem in two-phase incompressible flows*, The Journal of Fourier Analysis and Applications, 15 (2009), pp. 663–683.

[28] R. Stenberg, *On some techniques for approximating boundary conditions in the finite element method*, Journal of Computational and Applied Mathematics, 63 (1995), pp. 139–148. International Symposium on Mathematical Modelling and Computational Methods Modelling 94 (Prague, 1994).

[29] E. P. Stephan, *A boundary integral equation method for three-dimensional crack problems in elasticity*, Mathematical Methods in the Applied Sciences, 8 (1986), pp. 236–257.

[30] T. Tran and E. P. Stephan, *Additive Schwarz method for the h-version boundary element method*, Applicable Analysis, 60 (1996), pp. 63–84.