PONTIFICIA UNIVERSIDAD CATOLICA DE CHILE

ESCUELA DE INGENIERIA

EMPIRICAL PRICING PERFORMANCE OF COMMODITY DERIVATIVES MODELS: WHEN IS WORTH TO USE A STOCHASTIC VOLATILITY SPECIFICATION?

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Thesis submitted to the Office of Research and Graduate Studies in partial fulfillment of the requirements for the Degree of Master of Science in Engineering

Advisor:

GONZALO CORTAZAR SANZ

Santiago de Chile, April, 2014

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To my beloved family and Constanza who always supported me and stood for me at every moment.

ACKNOWLEDGMENTS

I would like to express my gratitude to my supervisor professor Gonzalo Cortazar for all of his knowledge and guidance through all this process. Without his expertise and patience none of this work could have be done. I also want to thanks to Hector Ortega for sharing with me his knowledge and support from day one in the genesis of this thesis. I am deeply thankful to have had the opportunity to work with them and for the guidance received in this research topic.

I also want to take the opportunity to thank all the present and past crew of Riskamerica-FinlabUC for their support during the developing of my thesis, in particular for helping me with the implementation of my research. To Federico Alonso, Hernan Morales and Ivo Kovacevic, whose advices helped me greatly with the empirical implementation of this work. Finally would like to thanks to Ignacio Badilla and Paula Donoso for giving me the chance to become part of this incredible team and for giving me all the facilities to end up this thesis.

I also thank the financial support of CONICYT through their Fondecyt 1130352 project and of FinanceUC

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ABSTRACT

The valuation of commodities contingent claims depends on the process assumed for the underlying asset. While the drift is important for futures pricing, the volatility is the price driver for options. In this paper we compare the empirical pricing performance of a model with constant volatility (CV) with one with a stochastic volatility specification (SV). These models are applied to oil, copper and gold getting consistent results for all three commodities. First, the CV model is clearly a better alternative to price futures contracts; not only it is simpler, but also it has smaller errors. However, if it is used to price options contracts the error could be considerable higher. Second, the longer the option maturity, the less relevant are the differences in pricing errors. Third, the higher complexity of the SV model is reflected in the, about 10 times, larger execution times. Fourth, the results of the SV model, applied for the first time to gold and copper, strongly suggest the presence of unspanned stochastic volatility components in all three commodities. Choosing the best model to implement in a real situation depends on the objectives pursued and on the tradeoffs between effort and precision. The results presented are then not only new, but also relevant from a practitioner point of view.

Keywords: Commodity derivatives, Stochastic Volatility, Unspanned Stochastic Volatility, USV, Pricing Performance, Options Contracts, Futures Contracts, Crude Oil, Copper, Gold.

RESUMEN

La valoración derivados de *commodity* depende del proceso asumido por el activo subyacente. Mientras que la media de los retornos es importante para los precios futuros, la volatilidad lo es para el precio de las opciones. En este trabajo se compara empíricamente el ajuste de precios de un modelo con volatilidad constante (CV) contra uno con una especificación estocástica de la volatilidad (SV). Estos modelos se aplican al petróleo, el cobre y el oro, consiguiendo resultados consistentes para los tres *commodities*. En primer lugar, el modelo CV es claramente una alternativa mejor para valorizar contratos futuros, no sólo es más simple, sino que también tiene errores más pequeños. Sin embargo, si se utiliza para valorizar contratos de opciones el error es considerablemente mayor. En segundo lugar, cuanto mayor sea la madurez de la opción, menos relevantes son las diferencias en los errores en los precios. En tercer lugar, la mayor complejidad del modelo SV se refleja en tiempos de ejecución, aproximadamente, 10 veces más grandes. En cuarto lugar, los resultados del modelo de SV, por primera vez aplicado al oro y cobre, sugieren con fuerza la presencia de componentes de volatilidad no abarcados por el mercado del *spot* (USV) en todos los tres *commodities*. Elegir el mejor modelo para poner en práctica en una situación real depende de los objetivos que se persiguen y del balance entre esfuerzo y precisión. Los resultados presentados son entonces no sólo nuevos, sino que también relevantes desde el punto de vista práctico-profesional.

Palabras Claves: Derivados de Commodity, Volatilidad Estocástica, USV, Ajuste de Precios, Contrato de opción, Contrato Futuro, Petróleo, Cobre, Oro.

1. ARTICLE INTRODUCTORY CHAPTER

1.1 Introduction

The increasing number of commodity derivatives contracts being traded has come with an incredible growth of the notional value of the contracts itself, and also with an expansion of the research on the market of commodity derivatives. The commodity literature has been fulfilled with models attempting to, adequately, describe the dynamics of commodity spot prices, options, futures and other commodity-linked derivatives. From a practitioner angle, this proliferation of models to price commodity contingent claims presents the challenge of choosing a proper one that satisfies the needs of each particular context. Knowing the tradeoff between different frameworks, quantifying the pricing performance of each, becomes an extremely useful tool to select the specification that suits the proper needs appropriately.

Amongst the frameworks, in the commodity literature, that aim to describe the commodities dynamics and value commodity–linked contingent claims, the main differences arise when it comes to model the volatility of the underlying asset. Earlier studies, primarily focused on the valuation of futures contracts, assume non-stochastic volatility, however, as the interest shift to price more complex derivatives such as options, this assumption became insufficient, and models evolved to account for sophisticated features such as stochastic volatility and price jump diffusions processes.

Empirical evidence regarding the stochastic behavior of the volatility can be found in the literature. For example, Litzenberger and Rabinowitz (1995) study the relationships between volatility, production and the level backwardation in oil prices allowing, though exogenously, time varying volatility, showing a non-decreasing correlation between the latter and backwardation and a non-increasing one with production. Duffie and Gray (1995) make the empirical exercise of computing realized and implied volatility for several commodities and find that the constant volatility hypothesis is rejected at the 95% confidence level.

1.2 Main goals

The main goal of this work is to develop a complete analysis of the empirical differences between existing models in the commodity literature, highlighting the tradeoff between them to help discerning when is worth to use one model despite the others.

Within the diverse range of frameworks in the literature, the focus is put in the differences between models that account for stochastic volatility and the ones that treat it as a constant feature. The objective is to compare this kind of models analyzing the empirical pricing performance on futures and options contracts, but also considering the implementation issues of each. The challenge consists to contrast the models in an extensive empirical framework, which means to build a common panel of data that allows comparing the models under the same conditions for several situations. Following the seminal work of Schwartz (1997), the objective is to calibrate the models over three different commodities, oil, copper and gold, for an extensive panel of data that goes from January 2006 to May 2013. This implicates, most likely, to extend the models involved in the comparison by expanding the dates and commodities used in their original empirical frameworks.

As has been stated, the objective of this work not only aims to compare, empirically, the futures and options pricing performance of both kind of models; it is also focused to analyze the tradeoff between their goodness of fit and the complexity of their implementations. While most of constant volatility models count with futures and options closed-form pricing formulas, stochastic volatility frameworks do not and several numeric approximations have to be done to value some derivatives. The constant volatility assumption allows to derive closed-form Black-Scholes-type formulas (Hilliard and Reis, 1998; Miltersen and Schwartz, 1998). On the other hand, following the seminal work of Heston (1993), stochastic volatility models are able to develop quasi-analytical options pricing formulas based on the Fourier inversion theorem, however they often require solve numerically a system of ordinary differential equations and numerically evaluate complex integrals.

Knowing the strengths and weaknesses of both kind of models and quantifying their pricing errors, allows having a clear picture of when to use each of them. Although it is expected to have a better options pricing performance with the stochastic volatility models, the answer is not clear regarding futures pricing. Even more, it is not clear to what extent they will "perform better" from a practitioner point of view where implementation issues play a role in the decision process.

1.3 Literature review

Most of the work presented in this article is based on existing valuation models to price commodity contingent claims. Within the vast range of models existing in the commodity literature, the attention is focused on those with stochastic volatility and those with multi stochastic factors, but constant volatility.

Following the seminal work of Black and Scholes (1973), on the valuation of equity contingent claims, most of the modern valuation models for commodity-linked securities are based on the assumption that under an equilibrium state there should be an absence of arbitrage opportunities. This implies the existence of a unique measure where all assets have the same expected rate of return, the risk-free rate. This is what has been known as risk-neutral valuation, which states that in a world with only no risk-averse agents, the price of any asset can be taken as the expected discounted future cash flows under a risk-neutral probability measure. Under certain regularity conditions such as the future cash flows being uncorrelated with the risk-free rate, the value of an asset can be calculated as the expected future payoffs, under the risk-neutral measure, discounted at the risk-free rate (Cortazar and Schwartz, 1994).

Equilibrium commodity models often specify a typically affine process, under the risk-neutral world, to model spot price dynamics as they usually have desirable properties such as being Markovian, which means that expected future cash flows can be predicted based only on the present state of the process (Gibson and Schwartz 1990; Schwartz, 1997; Hilliard and Reis, 1998). However, there are some models that specify the commodity price process by modeling the entire forward convenience yield, cost of carry or interest rate curve instead of their spot counterparts. This has the advantage of exactly fit the initial forward curve, but at expenses of commonly loose the Markov property. The no-arbitrage conditions that these models must satisfy are developed in the outstanding work of Heath, et al. (1992) (HJM), reason why models following this kind of specification are usually stated as being developed under the HJM framework (Cortazar and Schwartz, 1994; Miltersen and Schwartz, 1998; Miltersen, 2003).

1.3.1 Constant volatility models of commodity prices

Among the many models found in the literature that attempt to describe the spot price dynamic to value commodity-linked derivatives, there are many that, regardless the number of sources of uncertainty considered, assume the volatility of the commodity price to be constant.

Early models such as Brennan and Schwartz (1985) construct a simple process for a commodity price by assuming only one source of uncertainty, the spot price itself, along with constant drift and volatility. In their model, the spot price follows a geometric Brownian motion, where the growth rate of the commodity price depends only on time and with no mean reversion. Schwartz (1997) also proposes a 1-factor model in similar fashion, but includes mean reversion into the process of the spot price logarithm.

Gibson and Schwartz (1990) develop a 2-factor model where the spot price follows the same process as the one stated by Brennan and Schwartz (1985), but suggesting that is correlated with the process of the convenience yield, which is modeled as mean reverting process. Schwartz and Smith (2000) model the spot price logarithm as the composition of two factors: the short-term deviations in log prices and the equilibrium level for them. This short-term/long-term model assumes a mean-reversion process, towards zero, for the short-term log price deviation, while persistent framework, without mean reversion, to the equilibrium level process.

Models found in the commodity literature that are more sophisticated, often assume 3-factor specifications that extend the aforementioned models, including stochastic interest rates (Schwartz, 1997; Hilliard and Reis, 1998; Casassus and Collin-Dufresne, 2005). Cortazar and Naranjo (2006) go beyond and develop a canonical N-factor model based on the seminal work of Dai and Singleton (2000) on interest rates that assume multiple latent sources of uncertainty to model the process of commodity prices. It has the advantages to partially include mean-reversion and to span many of the existing models in the literature as special cases of it.

Most of these models were figured to adequately price futures contracts, nonetheless, there are some that attempt to price other derivatives such as options by extending Black-Scholes formula to a multi-factor commodity context (Hilliard and Reis, 1998; Miltersen and Schwartz, 2003).

1.3.2 Stochastic volatility models of commodity prices

Since futures contracts are not that sensitive to volatility the constant volatility assumption for commodity prices, despite the evidence (Duffie and Gray, 1995), does not have a greater impact in the pricing performance of futures contracts. However, for valuing more volatility-sensitive derivatives this assumption is at least questionable.

Since the seminal work of Heston (1993) that develops a tractable framework to price options under the stochastic volatility assumption based on the Fourier inversion theorem, commodity literature has been complemented with models that take into account the time varying feature of the volatility for commodity prices. Richter and Sørensen (2002) develop a tractable 3-factor model to value options on agricultural commodities that includes seasonality and stochastic volatility under a strictly affine framework. On the other hand, Nielsen and Schwartz (2004) extends the Gibson and Schwartz (1990) model by letting the volatility to be proportional to the convenience yield and applying it to spot and forward copper data.[1]

Based on the HJM framework, Trolle and Schwartz (2009) develop a model that includes stochastic spot price and forward cost of carry, but also specifying two volatility factors to account for stochastic volatility. Furthermore, it has the advantage to deal with unspanned stochastic volatility, meaning that it has the flexibility to model a market where options are not redundant securities and the spot markets are unable to fully span the volatility structure (Collin-Dufresne and Goldstein, 2002).

1.3.3 Empirical pricing performance.

Empirical performances of alternative pricing models have been treated before in the literature mainly for interest rates derivatives. For example, Bakshi et al. (1997) analyze the term structure options pricing differences between the Black-Scholes model and several other specifications that includes, and combines, stochastic interest rates, stochastic volatility and random jumps diffusion.

In the commodity literature, most papers are focused in demonstrating the statistical significance of their models instead of make empirical comparisons against other benchmarks. Schwartz (1997) is one of the few works where an empirical pricing performance of the different models is conducted for oil, copper and gold markets. However, the scope of his work only includes the pricing performance of futures contracts and considers only models under the constant

[1] Due the lack of readily available options, data Nielsen and Schwartz could not be able to test their model using option prices.

volatility assumption. On other hand, Hughen (2010) compare, in terms of options and futures pricing fit, a stochastic volatility model respect to an affine constant volatility one, but only considers a 3-factor model for the constant volatility model. In addition, he does not use options prices in the calibrating process and only analyze the empirical evidence for crude oil. Since his contribution is focused in the development of a tractable maximal affine stochastic volatility model and not in the empirical demonstration of the pricing performance of the models, only states the overall performance without analyzing the cross-sectional differences that may arise.

1.4 Methodology

In order to carry out the analysis proposed two models representing the stochastic and non-stochastic volatility frameworks are chosen. For the selection of the constant volatility kind of specifications, the N-factor Gaussian model developed by Cortazar and Naranjo (2006) is elected. Between the models that have the ability to replicate the time varying volatility behavior (Nielsen and Schwartz, 2004; Richter and Sørensen 2002; Trolle and Schwartz, 2009), the approach followed by Trolle and Schwartz (2009) is chosen.

The selection of the above-mentioned model for the constant volatility framework paradigm lies on the fact that encompasses most of the, this nature, models present in the literature. Based on the $A_0(N)$ canonical representation of Dai and Singleton (2000) for term structure, the N-factor Gaussian model of Cortazar and Naranjo (2006) generalizes and extends many of the constant volatility models found in the literature. For example two and 3-factor models developed in the works of Gibson and Schwartz (1990), Schwartz (1997), Hilliard and Reis (1998), Schwartz and Smith (2000) and Cortazar and Schwartz (2003), can be represented as special cases of the aforementioned N-factor Gaussian model. On the other hand, the main reason for the stochastic volatility model selection lies in the additional flexibility of Trolle and Schwartz (2009) specification to allow volatility components not to be fully spanned by the spot (futures) market. This phenomenon is what has been called "unspanned stochastic volatility" factors (Collin-Dufresne and Goldstein, 2002). Also, as a complementary objective of this work, the implementation of this framework allows to check whether this characteristic, which would imply that options are not redundant securities, may be present not only for oil but also for copper and gold markets.

The selected models are calibrated using standard statistical approaches in order to subsequently calculate the futures and options contracts pricing errors and compare them on an overall and cross-sectional level. Finally, the empirical results are contrasted with the implementation considerations of each one of the models involved.

Gold, high-grade copper and crude oil are the commodities selected to apply the models. This selection follows from Schwartz (1997) and has the advantage of providing a broad framework to compare the different models. Since both models, Trolle and Schwartz (1997) and Cortazar and Naranjo (2006), have already been applied to the oil market, but in a different date sample, this selection also points to extend the work of both papers and to check whether their conclusions are extendible to other commodities such as gold and copper. For this purposes daily data on settlement prices, open interest and volume, for futures and options, are considered. For oil, futures and options settlement data of the New York Mercantile Exchange (NYMEX) from January 2006 to May 2013 is used, while using the NYMEX division Commodity Exchange (COMEX) data from the same period for gold and copper.

Since settlement, and not transaction, prices are used, liquidity considerations have to be made in order to build faithful data sets. Following a similar approach as the one used in the work of Trolle and Schwartz (2009) futures contracts are chosen according liquidity patterns particular to each commodity. Options over these futures are selected to be as close as possible to the mean of one of several moneyness intervals ranging from 0.74 to 1.22, where moneyness is defined as the ratio between the strike price and the underlying future contract value. Given that the 2008 world financial crisis is part of the data considered, three subsamples of two years each starting from 2006 are built in order to isolate the effect of the crisis and to understand the implication it could have in the estimation of the parameters and the pricing performance of the models. It is important to note that the period from 2012 to 2013 is used to test the out of sample properties of the models.

The calibration procedure is conducted applying the Kalman filter algorithm in conjunction with the maximum likelihood method. In order to compare models with the same number of parameters the 5-factor specification of the Cortazar and Naranjo (2006) model is considered. However, to have a clearer insight of the value of adding more factors to the Cortazar and Naranjo (2006) scheme, the more parsimonious two and 3-factor specifications are also considered. The application of the Kalman filter requires the translation of the models dynamics to their state-space representation. This is accomplished by establishing the relationship between the respective states variables of the system and the observed price vector of futures and options given by the pricing formulas of each model, and by discretizing the dynamics of the state variables. Due the nonlinearity of the Trolle and Schwartz (2009) model an extended version of the Kalman filter is applied linearizing options pricing formulas using a first order Taylor approximation. Since the stochastic volatility assumption imply that innovation errors in the Kalman filter algorithm are no longer normally distributed, the quasi-maximum likelihood method is used to estimate the parameters involved in the Trolle and Schwartz (2009) specification, which means that a Gaussian distribution is used to approximate the true one underlying the innovation errors.

Several numerical considerations are taken into account to implement both kinds of models. The ordinary differential equation systems with no closed-form solutions are solved using a standard fourth order Runge-Kutta algorithm. The complex integral involved in the options pricing formula of the Trolle and

Schwartz (2009) model is numerically evaluated using the Gauss-Legendre quadrature formula with 40 integration points and truncating the integration limits in 400. Increasing the number of points and the truncation limits does not change the likelihood value. In order to maximize the probability of reaching a global optimum the likelihood functions are optimized applying an interior-point algorithm that uses the Broyden-Fletcher-Goldfarb-Shanno algorithm to approximate the Hessian, and using several plausible initial parameters.

The parameters obtained as a result of the calibration process described above are statistically tested to validate, econometrically, the application of each model to each commodity. Time series of daily root mean square errors are calculated for each sample. Futures pricing errors are calculated as the difference between the actual and the fitted price, while options pricing errors are measured as the difference between the actual and fitted lognormal implied volatility, which is the Black-Scholes implied volatility for the actual and fitted options prices. As an overview of the pricing performance, a general analysis of the error is conducted considering the average of all contracts in sample. Nonetheless, to have an idea of the cross section performance of each model, daily time series of root mean square errors are calculated for each contract maturity and then averaged to get the whole sample performance of each model.

As stated, this work is not only focused in the statistical significance of the models and the empirical pricing performance in terms of the measured errors, it is also aimed to analyze the tradeoff between the goodness of fit and the implementations issues of each framework compared. For all three commodities, in addition to the qualitative study of the implementation complexity regarding the pricing formulas and dynamics of each model, a quantitative analysis is made by calculating the execution times taken by the calibration of the parameters of each specification. For comparative purposes, execution times are calculated as the time consumed by each model on each data set per iteration of the calibration process. Considering a gradient base algorithm to find the optimal solution that uses forward finite differences to approximate the derivatives, a N-parameters model includes N+1 valuations of the objective function in each iteration; N valuations to calculate the partial derivatives and one extra for valuing the new point. Although it is not an exact measure of the real time needed by a specific model to get to the optimal solution, since nothing has been said about the velocity of convergence, it is a suitable proxy that gives, though roughly, a notion of the differences in the implementation procedure of each model.

1.5 Main results

Roughly speaking this paper has three major contributions to the commodity literature: an extensive comparative analysis between existing models in the commodity literature, extension, in several dimensions, of the models used and the practitioner perspective used in the analysis of the existing tradeoffs between the models presented.

The first contribution has to do with the development of an extensive comparative analysis between models that account for stochastic volatility and their constant benchmarks. Beside the aforementioned work conducted by Hughen (2010), commodity literature lacks of comparison of this kind that includes futures and options contracts in their analysis. The second is related with the extension of the work done by the authors of the models selected by considering more than one commodity to conduct the analysis; both models, originally applied only to crude oil, are also applied to gold and copper. In most cases, with the notable exception of Schwartz (1997), existing papers on valuation of commodity contingent claims restrict themselves to statistically testing their models based on the applications of a particular commodity, most commonly, crude oil on a particular set of dates. Also on this line of contribution lies the extension of Cortazar and Naranjo (2006) to include options in the calibration process. The third contribution has to do with the perspective from which the results are presented including the empirical analysis of implementation considerations that helps to highlight the tradeoff made by choosing one model despite the other. As most works tend to focus only on the technical issues of the econometric sense of their models, the practical issues regarding the implementation of its specifications is usually omitted.

The calibration of Trolle and Schwartz (2009) for the data sample considered, supposed a double challenge; the task was not only to analyze the consistency of the model for the oil market based on the previous results for it, but also to apply the framework for two completely different commodities; gold and copper. As expected, based on the results obtained by Trolle and Schwartz (2009) for oil market, the estimates of their model, applied to the sample data used in this work, were significant in all cases, except for the risk premiums, an issue commonly found in the commodity literature (Cortazar, et al., 2013). The results were also consistent with their findings in the sense that the estimates also suggest the presence of unspanned stochastic volatility in oil market for the sample period considered.

Being the first time this model was calibrated for other commodity markets rather than crude oil, all the attention was focused whether the model will be statistically significant and if these markets, copper and gold, will also show the characteristic of having volatility components not fully spanned by the spot (futures) market of the respective commodity. It is important to state, though, that calibration itself is not a proof of the presence of unspanned volatility factors, but strongly suggest it. Having said that, copper estimates obtained, strongly suggest that options, just as for the oil market, are not redundant securities. Most parameters, but risk premiums, were statistically significant, and the estimates showed quite low correlations between the innovations of the volatility components and the innovations of the spot price and cost of carry, indicating that the volatility is not fully spanned by the spot (futures) market.

Empirical evidence found in the commodity literature suggests that gold prices do not exhibit mean reversion under the risk neutral measure (Schwartz, 1997; Casassus and Collin-Dufresne, 2005). Considering this evidence the calibration of

Trolle and Schwartz (2009) was not expected to be statistically significant, nonetheless, at a standard level, most of the parameters turn out to be estimated with relative low standard errors, with the already mentioned exception of the risk premiums. Although this findings contrast with the empirical evidence stated in the literature, it has to be noted that in this work options prices are considered in the calibration process and maybe inducing the mean reversion in gold prices.

As was the case with oil and copper market, gold results also suggest the presence of unspanned stochastic volatility factor, though in a weaker way. Relatively high correlations, opposed to the estimates of oil and copper, between the innovations of the volatility components and the innovations of the spot price and cost of carry were obtained. However, these correlations, still far from being perfect, indicate that the volatility components are only partially spanned by the gold spot market.

Even if the mere estimation of the model parameters is not proof per se, the results found for gold and copper extend the work of Trolle and Schwartz (2009) by suggesting that the presence of unspanned stochastic volatility factors is not a characteristic of the crude oil market only, but also for other commodities quite different as the markets of gold and copper.

The pricing performance results show that for all three commodities the Cortazar and Naranjo (2006) model outperforms Trolle and Schwartz (2009) on attempting to price futures contracts. This is not surprisingly since the Trolle and Schwartz (2009) model only has three factors driving futures prices and the Cortazar and Naranjo (2006) model, with same number of parameters, has five. Futures contracts pricing errors of the 5-factor Cortazar and Naranjo (2006) specifications tend to be the half of the ones committed by Trolle and Schwartz (2009). More interesting is the fact that also less factors specifications of the Cortazar and Naranjo (2006) model also performs better suggesting that simpler models are enough if futures contracts pricing is the main concern.

The constant volatility assumption behind Cortazar and Naranjo (2006) specifications fails to describe options prices adequately. Depending on the data set analyzed, most of the time tends to overestimate (or underestimate) the implied volatilities. Options pricing errors tends to be 5 to 6 times larger than the ones obtained with the Trolle and Schwartz (2009) model. Let it be noted that increasing the number of factors considered do not decrease significantly the options contracts pricing errors of the constant volatility specifications.

The cross section analysis of the options contracts pricing performance show that the differences between both kinds of models get shorter as the maturity of the contract rises. This is more clearly observed in the oil and gold cases where longer contracts were readily available.

To get an idea of the difference in the complexity of implementation of both models, execution times were measured. The results give a clear advantage to the Cortazar and Naranjo (2006) model in this front. The required numeric calculations to evaluate the system of partial differential equations without closed form solutions of the Trolle and Schwartz (2009) model along with the numeric integrations needed to value options contracts results in execution times that are 10 times larger than the ones of the Cortazar and Naranjo (2006) specifications. From a practitioner point of view, this gives a clear picture of the tradeoff involved in the decision of selecting a particular framework.

1.6 Conclusions and further research

Pricing of commodities contingent claims drastically depends on the process assumed for the underlying asset. Particularly for options pricing, whether or not to account for stochastic volatility becomes a major concern. In this article, these two ways of dealing with volatility are contrasted and compared not only from a statistical point of view, but also from a practitioner angle where the trade-off between implementation issues and empirical performance start playing a role in the decision making process. Representative specifications of each kind of models, Trolle and Schwartz (2009) and Cortazar and Naranjo (2006) for stochastic volatility models and not respectively, are chosen and extended to be estimated, and ultimately compared, using an extensive data set on options and futures prices for oil, copper and gold.

For copper and oil markets both models are statistically reliable and stable through all samples, while being a little bit over-specified for gold. As for the first implementation of Trolle and Schwartz (2009), copper and gold results also exhibit unspanned stochastic volatility factors suggesting that this phenomenon is also an important feature on commodity markets less related to crude oil. It would be interesting to investigate if these results are consistent with other model-free regression-based tests that could empirically analyze how much of the changes in the returns of a volatility sensitive portfolio are explained by changes in the futures returns.

The constant volatility assumption does not affect futures pricing performance, what matters most is the number of factors driving futures prices instead; the more these are, the better the fit. Nevertheless, with the same numbers of parameters involved, the 5-factor specification of Cortazar and Naranjo (2006) outperforms Trolle and Schwartz (2009), which only considers three factors for futures prices, on every sample tested. Non-stochastic volatility models fail to describe, adequately, options prices dynamic. Trolle and Schwartz (2009) beat all N-factor Cortazar and Naranjo (2006) at every empirical framework for the three commodities. However, for long-term contracts differences tend to shrink getting to levels where the gap between both models is less than 1 percentage point.

Even though a stochastic volatility framework surpasses the options pricing performance of homoscedastic specifications, is not done at a low cost. The lack of closed form solutions to options pricing formulas require to take several numeric considerations to calibrate this kind of models. Estimated execution times indicates that in order to outcome the limitations of constant volatility models in the pricing of options contracts, ten time more effort has to be invested to implement a stochastic volatility specification. The outstanding options pricing performance of Trolle and Schwartz (2009) is contrasted with the ease-of-use closed form, Black-Scholes alike, options pricing formulas that gives to Cortazar and Naranjo (2006) the ability to be a faster method with remarkable futures pricing performance and acceptable long-term options pricing.

Even when a general and necessary comparison between quite different kinds of models have been made in the commodity context as a first step, it would be interesting to go beyond and to see, for example, how relevant actually is the unspanned stochastic volatility compared with standard stochastic volatility specifications. On the other hand, it would be also interesting to see if other constant volatility models such as the ones that include stochastic interested rates or price jump will shrink even more the differences in options pricing performance against stochastic volatility models.

The evidence presented in this article does not support accounting for stochastic volatility in futures pricing; it is an expensive way to get same, or even worse, results in pricing performance. However, the role that stochastic volatility plays in the pricing of options contracts significantly improves the goodness of fit. While the five to seven times better performance in short-term options contracts seems to worth the implementation effort, for longer contracts the question seems to remain open.

2. EMPIRICAL PRICING PERFORMANCE OF COMMODITY DERIVATIVES MODELS: WHEN IS WORTH TO USE A STOCHASTIC VOLATILITY SPECIFICATION?

Commodity derivative markets have grown at an incredible rate in the past decades. According with the Bank for International Settlements (BIS), the estimate for the notional value of outstanding contracts is about USD 2.46 trillion for over-the-counter (OTC) commodity derivatives in June 2013, more than 3.6 times bigger than the USD 0.67 trillion in June 2001. The nature and distribution of the commodity contingent claims has also changed and options contracts now account for nearly 36% of total notional value.

As a consequence of this market expansion, research on commodity-linked derivatives has increased both in quantity and in sophistication. Starting from simple 1-factor mean reverting, constant-volatility models for pricing futures contracts based on Vasicek (1977), the literature has evolved to include the pricing of more complex derivatives such as options. While for pricing futures modeling the drift of the risk-neutral process is the most important concern, for pricing options, specifying the volatility dynamics becomes crucial.

Despite the empirical evidence of heteroscedasticity in commodity markets, that goes back to the nineties (e.g. Litzenberger and Rabinowitz, 1995; Duffie and Gray, 1995) it is still common to find in the literature models with several risk factors but a constant volatility specification (Schwartz, 1997; Hilliard and Reis, 1998; Schwartz and Smith, 2000; Cortazar and Schwartz, 2003; Cortazar and Naranjo, 2006). In general these models have several desirable properties including closed form solutions for most derivatives and good futures pricing. However, little attention has been put on their performance for options pricing.

While assuming a constant volatility may have little implication for futures pricing in terms of goodness-of-fit of multi-factor models, it is extremely relevant for options pricing. Since the seminal work of Heston (1993), several models that include stochastic volatility have been proposed (Nielsen and Schwartz, 2004; Richter and Sørensen, 2002; Trolle and Schwartz, 2009). The strength of these models is their ability to replicate the time varying volatility behavior of many commodities, thus promising much better pricing of volatility sensitive derivatives, like options.

One of the main drawbacks of stochastic volatility models is their implementation complexity because there are no closed-form solutions for derivative prices and intensive numerical methods must be used. Given this difficulty, to obtain results in a reasonable amount of time, most stochastic volatility models are implemented restricting the number of risk factors, which could take a toll in the model performance for pricing some derivatives.

We are interested in comparing the performance of these two kinds of models on several grounds. Up to now most papers on valuation of commodity contingent claims have focused mainly in testing the statistical significance of its models based on the application of a particular commodity, most commonly, crude oil. Although stochastic volatility models have shown to be statistically significative and consistent with the empirical evidence (Duffie and Gray, 1995), it is not clear to what extent and how they "perform better" than their constant volatility counterparts from the view of a practitioner, where implementation issues have to be considered. This is why it is critical to understand the magnitude and distribution of pricing errors and the effort required in implementing both kinds of models for different commodities and contracts in order to be able to have a clear picture about the tradeoffs between them.

In this paper we compare the futures and options pricing performance of constant and stochastic volatility models for several commodities. In order to do this we use the N-factor Gaussian model developed by Cortazar and Naranjo (2006) to represent a constant volatility framework and the Trolle and Schwartz (2009) for the stochastic volatility approach. We choose Cortazar and Naranjo (2006) because its canonical representation for N-factor Gaussian models nests several of the existing Gaussian models in literature as special cases (Brennan and Schwartz, 1985; Gibson and Schwartz, 1990; Schwartz, 1997; Schwartz and Smith, 2000; Cortazar and Schwartz, 2003). On the other hand the

Trolle and Schwartz (2009) specification is selected because, having already been applied to oil, has the added flexibility to allow for volatility components not to be fully spanned by the spot (futures) market in what has been called “unspanned stochastic volatility” factors (Collin-Dufresne and Goldstein, 2002). Using this specification will allow us to check also if this characteristic, which would imply that options are not redundant securities, may be present not only for oil, but also eventually for other commodities.

Following Schwartz (1997) we make the analysis for three different commodities: oil, copper and gold. We not only focus on the statistical significance, and the futures versus options pricing errors, but also consider implementation issues to analyze the strengths of each model specification. Execution times are calculated as a proxy measure of the implementation complexity of each model. This quantitative analysis may help, from the perspective of a practitioner, to balance the tradeoffs between a better fit with a more time consuming and harder implementation.

Empirical performance of alternative commodity models that do not restrict themselves to statistical tests are limited. In a seminal work Schwartz (1997) analyzes futures pricing performance of three models accounting for mean reversion: crude oil, high-grade copper and gold. Also, Hughen (2010) compares, in terms of options and futures pricing fit, a stochastic volatility model with an affine constant volatility. However the paper only analyzes the 3-factor model specification, does not use options prices in the calibration process, restricts the analysis only to oil and does not analyze cross-sectional performance differences. In an analysis for interest rate derivatives Bakshi et al. (1997) analyze the term structure options pricing differences between the Black-Scholes model and several other specifications that includes, and combines, stochastic interest rates, stochastic volatility and random jumps diffusion.

In this paper we follow Schwartz (1997) and chose to analyze crude oil, copper and gold. For oil we use futures and options settlement data of the New York Mercantile Exchange (NYMEX) from January 2006 to May 2013, while using the NYMEX division Commodity Exchange (COMEX) data from the same period for gold and copper. In order to maximize the data available on each day of the sample, the Kalman filter, in its traditional and extended version, is used together with maximum likelihood as the estimation procedure.

To our knowledge this paper is the first to compare stochastic volatility models against their constant benchmarks for oil, copper and gold using options and futures to calibrate the respective parameters. It also extends Trolle and Schwartz (2009), by expanding dates and commodities, and Cortazar and Naranjo (2006), by including options, and not only futures, in the analysis.

This paper is structured as follows. Section 2.1 describes the benchmark models. Section 2.2 describes the crude oil, copper and gold data. Section 2.3 analyzes and discusses the empirical results. Section 2.4 concludes.

2.1 Models

In this section the two benchmark models used to compare the empirical performance of the constant versus the stochastic volatility models, are described.

2.1.1 Stochastic Volatility Model: Trolle and Schwartz (2009) (TS).

We now briefly describe the Trolle and Schwartz (2009), TS model which, besides its ability to account for stochastic volatility, offers a tractable framework to price commodity derivatives in the presence of unspanned stochastic volatility. It has the flexibility to model a market where options are not redundant securities, meaning that the spot markets are unable to fully span the volatility structure (Collin-Dufresne and Goldstein, 2002).

The TS model is chosen, among other reasons, because it is based on the Heath et al. (1992) (HJM) model, which has the advantage, over the typical affine models, of making it easier to include the parameter restrictions on volatility to be unspanned[2]. Being the first stochastic volatility HJM-type model used for pricing

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[2]See Cortazar and Schwartz (1994), Miltersen and Schwartz (1998) and Miltersen (2003) for others HJM-type models for pricing commodity derivatives

commodity derivatives has, in its most general form, five factors driving the prices: three factors for futures prices and two for the volatility process.

Following Cortazar and Schwartz (1994), a process for the spot price and the forward cost- of-carry is specified. Let $S(t)$ denote the spot price of the commodity at time $t$and $\delta(t)$ the spot cost-of-carry. Let $y(t,T)$ denote the t-time instantaneous cost-of-carry curve at time T ($\delta(t) = y(t,t)$). To account for stochastic volatility let $\nu_1(t)$ and $\nu_2(t)$be two volatility factors affecting $S(t)$ and $y(t,T)$.

The processes for $S(t)$, $y(t,T)$, $\nu_1(t)$ and $\nu_2(t)$, under the risk-neutral measure, are:

$$\frac{dS(t)}{S(t)} = \delta(t)dt + \sigma_{S1}\sqrt{\nu_1(t)}dW_1^Q(t) + \sigma_{S2}\sqrt{\nu_2(t)}dW_2^Q(t) \tag{2.1}$$

$$dy(t,T) = \mu_y(t,T)dt + \sigma_{y1}(t,T)\sqrt{\nu_1(t)}dW_3^Q(t)$$

$$+\sigma_{y2}(t,T)\sqrt{\nu_2(t)}dW_4^Q(t) \tag{2.2}$$

$$d\nu_1(t) = \big(\eta_1 - \kappa_1\nu_1(t) - \kappa_{12}\nu_2(t)\big)dt + \sigma_{\nu 1}\sqrt{\nu_1(t)}dW_5^Q(t) \tag{2.3}$$

$$d\nu_2(t) = \big(\eta_2 - \kappa_{21}\nu_1(t) - \kappa_2\nu_2(t)\big)dt + \sigma_{\nu 2}\sqrt{\nu_2(t)}dW_6^Q(t) \tag{2.4}$$

$$\big(dW(t)\big)\big(dW(t)\big)^T = \begin{pmatrix} 1 & 0 & \rho_{13} & 0 & \rho_{15} & 0 \\ 0 & 1 & 0 & \rho_{24} & 0 & \rho_{26} \\ \rho_{13} & 0 & 1 & 0 & \rho_{35} & 0 \\ 0 & \rho_{24} & 0 & 1 & 0 & \rho_{46} \\ \rho_{15} & 0 & \rho_{35} & 0 & 1 & 0 \\ 0 & \rho_{26} & 0 & \rho_{46} & 0 & 1 \end{pmatrix}$$

where $dW_i^Q(t)$ corresponds to Wiener processes under the risk-neutral measure.

The futures price of a contract expiring at time T, $F(t,T)$, is:

$$F(t,T) = S(t)\exp\left\{\int_t^T y(t,u)du\right\} \tag{2.5}$$

Under no-arbitrage conditions the drift of the instantaneous futures return should be zero, thus:

$$\frac{dF(t,T)}{F(t,T)} = \sqrt{\nu_1(t)}\left(\sigma_{S1} dW_1^Q(t) + \int_t^T \sigma_{y1}(t,u)du\, dW_3^Q(t)\right) + \sqrt{\nu_2(t)}\left(\sigma_{S2} dW_2^Q(t) + \int_t^T \sigma_{y2}(t,u)du\, dW_4^Q(t)\right) \quad (2.6)$$

The ability of the TS model to account for unspanned stochastic volatility can be seen in Equation (2.6) where volatility of futures prices is shown to depend on $\nu_1(t)$ and $\nu_2(t)$ but not on $dW_5^Q(t)$ and $dW_6^Q(t)$. Therefore if $dW_5^Q(t)$ and $dW_6^Q(t)$ have a low correlation with the risk processes of the spot and forward cost-of-carry curve ($dW_i^Q(t)$, $i = 1, \dots, 4$) then it can be seen that options (which are highly sensitive to volatilities) cannot be hedged using only futures, and we are in the presence on what is called “unspanned volatility”.

In order to estimate the model it is necessary to specify the drift and instantaneous volatility of the forward cost of carry curve. A drift condition, analogous to the one developed by Heath et al. (1992) in forward rate term structure models, can be obtained and the volatility is chosen in a way that the forward cost of carry curve can be expressed as a linear function of a finite number of state variables.[3] Then it follows

$$\mu_y(t,T) = -\left(\nu_1(t)\sigma_{y1}(t,T)\left(\rho_{13}\sigma_{S1} + \int_t^T \sigma_{y1}(t,u)du\right) + \nu_2(t)\sigma_{y2}(t,T)\left(\rho_{24}\sigma_{S2} + \int_t^T \sigma_{y2}(t,u)du\right)\right) \quad (2.7)$$

where

$$\sigma_{yi}(t,T) = \alpha_i e^{-y_i(T-t)} \quad (2.8)$$

Under such conditions $F(t,T)$ is given by

[3] Bhar and Chiarella (1997) investigate the conditions under HJM-type models are Markovian respect to a finite number of states variables.

$$F(t,T) = S(t)\frac{F(0,T)}{F(0,t)}\exp\left\{\sum_{i=1}^{2}\left(x_i(t)\frac{\alpha_i}{\gamma_i}\left(1-e^{-\gamma_i(T-t)}\right)+\phi_i(t)\frac{\alpha_i}{2\gamma_i}\left(1-e^{-2\gamma_i(T-t)}\right)\right)\right\} \tag{2.9}$$

where $x_i(t)$ and $\phi_i(t), i = 1,2$, are the resulting state variables from the transformation of the HJM-type model to an affine Markovian specification. They follow a particular process detailed in the Appendix A.1.

Defining the new state variable $s(t) = \log S(t)$, the log futures prices follows an affine function of the state variables:

$$\log\left(F(t,T)\right) = \log F(0,T) - \log F(0,T) + s(t) + \sum_{i=1}^{2}\left(x_i(t)\frac{\alpha_i}{\gamma_i}\left(1-e^{-\gamma_i(T-t)}\right)+\phi_i(t)\frac{\alpha_i}{2\gamma_i}\left(1-e^{-2\gamma_i(T-t)}\right)\right) \tag{2.10}$$

where the dynamics of $s(t)$ is given by

$$ds(t) = \left(y(0,t)+\sum_{i=1}^{2}\alpha_i\left(x_i(t)+\phi_i(t)\right)-\frac{1}{2}\left(\sigma_{S1}^2\nu_1(t)+\sigma_{S2}^2\nu_2(t)\right)\right)dt + \sigma_{S1}\sqrt{\nu_1(t)}dW_1^Q(t)+\sigma_{S2}\sqrt{\nu_2(t)}dW_2^Q(t) \tag{2.11}$$

Based on Heston (1993) European options on futures are priced applying the Fourier inversion theorem. A similar approach have been used by Collin-Dufresne and Goldstein (2003) and Richter and Sørensen (2002), among others. Letting K be the strike price and T_0 the option expiration time on a future contract expiring at T_1 the price of put is given by

$$\mathcal{P}(t,T_0,T_1,K) = P(t,T_0)\left(KG_{0,1}(\log(K)) - G_{1,1}(\log(K))\right) \tag{2.12}$$

where $P(t,T_0)$ is the price of a zero-coupon bond with $T_0 - t$ maturity and $G_{a,b}(y)$ is defined as

$$G_{a,b}(y) = \frac{\Psi(a,t,T_0,T_1)}{2} - \frac{1}{\pi}\int_0^{\infty} \frac{Im\left(\Psi(a+\mathrm{i}ub,t,T_0,T_1)e^{-\mathrm{i}uy}\right)}{u} du \tag{2.13}$$

Here $\Psi(u,t,T_0,T_1) = E_t^Q\left[e^{u\log(F(T_0,T_1))}\right]$represents the transform of $F(T_0,T_1)$ and has an affine representation given by the solution of a non-trivial system of ordinary differential equations with no closed form solution (See Appendix A.2).

For calibration purposes the dynamics of the commodity has to be stated under the actual probability measure. In order to do this a market price of risk is specified. Based on an affine formulation widely used in the literature, the market price of risk is defined as:

$$\Lambda_i(t) = \lambda_i\sqrt{\nu_1(t)},\ i = 1{,}3{,}5$$
$$\Lambda_i(t) = \lambda_i\sqrt{\nu_2(t)},\ i = 2{,}4{,}6 \tag{2.14}$$

Then, the processes under the actual probability measure can be obtained by substituting $dW^Q(t)$ by

$$dW_i^Q(t) = dW_i^P(t) - \Lambda_i(t)dt \tag{2.15}$$

Roughly speaking in order to implement the TS model for options pricing it is necessary first, to numerically solve for each contract an ordinary differential equations system to get the above mentioned transform, then, also for each contract, to apply numerical integration algorithms twice to finally get the option price. This theoretically complex framework and also numerically sophisticated formulation, has practical consequences that leads to the question of when is it worth to make the effort, as we will see in further sections.[4]

[4]The model presented here is time-inhomogeneous and fits the initial futures curve by construction. In order to estimate the model, the initial forward cost of carry curve is assumed to be flat and equal to a constant ϕ. Also, the model is over identified, therefore $\eta_i, i = 1{,}2$, are normalized to one, to achieve identification.

2.1.2 Constant Volatility Model: Cortazar and Naranjo (2006) (CN).

We now describe the main features of the Cortazar and Naranjo (2006) N-factor Gaussian model for a commodity spot price that will later be used to analyze the empirical performance of a constant volatility approach. This model generalizes 2- and 3-factor models found in the literature (Gibson and Schwartz, 1990; Schwartz, 1997; Schwartz and Smith, 2000; Cortazar and Schwartz, 2003). It provides a framework based on the $A_0(N)$ canonical representation of Dai and Singleton (2000) for term structure, extending existing commodity pricing models to an arbitrary number of factors. One of the main advantages of this model, compared to stochastic volatility ones, is its relatively simple implementation and the existence of closed-form analytic formulas for futures and options prices.

Let $S(t)$be the spot price of the commodity at time t, and μ the long-term growth rate. Then the process for the spot price of the commodity is:

$$\log S(t) = 1^T X(t) + \mu t \tag{2.16}$$

where $X(t)$ is a $N \times 1$ vector of unobservable state variables with a process, under the actual probability measure, given by:

$$dX(t) = -AX(t)dt + \Sigma dW(t) \tag{2.17}$$

where

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_N \end{bmatrix}, \Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_N \end{bmatrix}$$

are $N \times N$ matrices with positive entries and $dW(t)$ is a $N \times 1$ vector of correlated Wiener processes such that

$$\left(dw(t)\right)\left(dW(t)\right)^T = \Omega dt = \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1N} \\ \rho_{12} & 1 & \cdots & \rho_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1N} & \rho_{2N} & \cdots & 1 \end{bmatrix} dt$$

where $\rho_{ij} \in [-1, 1]$are the instantaneous correlation between state variables i and j.[5]

Without specifying a stochastic process for Σ the model implies that the state variables follow a multivariate Normal distribution where each variable, except for the first one, reverts to zero at a speed rate a_i.

Futures pricing formulas are easily obtained under the risk-neutral measure by assuming constant risk premiums λ_i:

$$dX(t) = -\big(\lambda + AX(t)\big)dt + \Sigma dW^Q(t) \tag{2.18}$$

Using no-arbitrage arguments, the futures price becomes:

$$F(X(t), t, T) = \exp\Bigg(X_1(t) + \sum_{i=2}^{N} e^{-a_i(T-t)}X_i(t) + \mu t + \Big(\mu - \lambda_i + \frac{1}{2}\sigma_1^2\Big)(T-t) - \sum_{i=2}^{N} \frac{1-e^{-a_{i}(T-t)}}{a_i}\lambda_i + \frac{1}{2}\sum_{i,j\neq 1} \sigma_i\sigma_j\rho_{ij}\,\frac{1-e^{-(a_i+a_j)(T-t)}}{a_i+a_j}\Bigg) \tag{2.19}$$

Following Hilliard and Reis (1998) and Miltersen and Schwartz (1998) a Black-Scholes-type formula can be derived for European future options under the CN specification. Let σ_F be the instantaneous volatility of the returns on futures, then the price of a European put option at time t expiring at T_0 and with strike price K over a future contract maturing at T_1 is given by

$$\mathcal{P}(t, T_0, T_1, K) = P(t, T_0)\big(KN(-d_2) - F(t, T_1)N(-d_1)\big) \tag{2.20}$$

where

$$d_1 = \frac{\log\left(\frac{F(t,T_1)}{K}\right) + \frac{1}{2}\nu^2}{\nu},\ d_2 = d_1 - \nu \tag{2.21}$$

$N(\cdot)$ is the cumulative standard normal distribution function and $P(t, T_0)$is the price of a zero-coupon bond at time t, expiring at T_0 and ν is the volatility term.

Then

[5]It is important to note that a_1 has been fixed at zero in order to have a non-stationary process for the underlying spot price as it is commonly assumed in the literature.

$$\nu^2 = \int_t^{T_0} \sigma_F(u, T_1)^2 du = \int_t^{T_0} \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_i \sigma_j \rho_{ij} e^{-(a_i + a_j)(T_1 - u)} du \qquad (2.22)$$

where ν^2 is the average, over the life of the options, of the instantaneous variance of the futures returns innovations.

In contrast with the TS model, the CN framework provides closed form solutions for futures and options pricing formulas making it easier to apply standard estimation procedures for empirical implementations.

2.2 Data

This section describes the data that will be used later to analyze the empirical performance of the two previously described commodity models for oil, copper and gold. The data consists of daily observations between January 2006 and May 2013 of settlement prices, open interest and volume for futures and options. For oil we consider the West Texas Intermediate crude oil data (WTI) from the New York Mercantile Exchange (NYMEX), For copper and gold we use high grade copper (HG) and gold (GC) traded at Commodity Exchange (COMEX).

Tables II-1, II-2 and II-3 describe the data used for oil, copper and gold respectively. Following Trolle and Schwartz (2009) liquidity considerations for each commodity are taken into account in building the data sets. Daily observations on contracts prices are selected according to the level of open interest and specific liquidity patterns for each commodity. This procedure leaves twelve generic futures contracts for oil: the first 6-month contracts (F1-F6), the following two contracts with expiration either in March, June, September and December (MD1-MD2), and the next four contracts with expiration in December (D1-D4). For copper and gold the selection process leaves eight and eleven generic futures contracts, respectively: the first 6-month contracts (F1-F6) and the following two first contracts with expiration either in March, May, September and December (MD1-MD2) for copper, and the first 6-month contracts (F1-F6) and the first five contracts with expiration either in June and December (JD1-JD5) after a year, for gold.

Given that the commodity markets trade only American options and that for simplicity they are priced using European options formulas, only at-the-money and out-of-the-money options are considered to reduce the size of the early exercise premium. The options are classified in eleven moneyness intervals, ranging from 0.78 to 1.22 years, and the closest contract to the mean of each interval is selected.

Figure II-1 shows the spot price evolution of the nearest contract for the three commodities for the whole period. It is important to note the impact of the financial crisis of 2008 on the spot prices of the three commodities. Figure II-2 shows that this impact is not only on the price levels, but also on the Black-Scholes implied volatility of the closest-to-maturity contract.

In the following section results of the two models for the three commodities are presented. For better analyzing the empirical performance of the model the period from January 2012 to May 2013 is defined as the out-of-data set. Also in order to isolate the effect of the financial crisis data, sub-samples are constructed: Panel A, will represent the full in-sample period, from January 2006 to December 2011, Panel B from January 2006 to December 2007, Panel C from January 2008 to December 2009 and Panel D from January 2010 to December 2011.Panel E will represent the out-of-data set.

Table II-1: Oil Data

From January 2006 to May 2013. Daily Observations

	FUTURES			OPTIONS			
Future Contract	Avg. Price	Avg. Maturity	Avg. Open Interest	Nº Puts	Nº Calls	Avg. Put price	Avg. Call price
F1	82,310	0,083	287016	8350	8748	1,106	1,142
F2	82,973	0,166	132036	9907	10006	1,844	1,928
F3	83,481	0,249	84095	9944	9925	2,577	2,738
F4	83,884	0,333	66127	9778	9520	3,204	3,463
F5	84,205	0,416	53623	9502	8979	3,792	4,061
F6	84,461	0,500	46728	8869	8402	4,288	4,617
MD1	84,827	0,665	73521	9415	8936	5,194	5,470
MD2	85,157	0,915	59311	7824	7177	6,232	6,536
D1	85,248	1,546	105709	10002	9609	7,668	8,060
D2	84,929	2,547	55105	9809	8695	8,983	9,934
D3	84,659	3,548	33066	8830	7850	10,012	11,200
D4	84,710	4,549	20473	6136	5270	10,905	12,868

The Future Contract column: Fi denotes the first i-month contracts; MDi denotes the i-following contract with expiration either in March, June, September or December; Di denotes the i-following contract with expiration in December. Prices are expressed in US$ and maturities in years

Table II-2: Copper Data

From January 2006 to May 2013. Daily Observations

	FUTURES			OPTIONS			
Future Contract	Avg. Price	Avg. Maturity	Avg. Open Interest	Nº Puts	Nº Calls	Avg. Put price	Avg. Call price
F1	3,276	0,082	8546	1246	1363	0,027	0,028
F2	3,276	0,165	29621	5772	6035	0,046	0,050
F3	3,277	0,249	29956	5729	5499	0,067	0,081
F4	3,278	0,332	17979	4205	4069	0,090	0,113
F5	3,277	0,416	9389	3092	2898	0,107	0,139
F6	3,275	0,499	5718	2096	2017	0,129	0,163
MD1	3,267	0,646	6520	2426	2100	0,170	0,219
MD2	3,253	0,848	2623	769	444	0,202	0,232

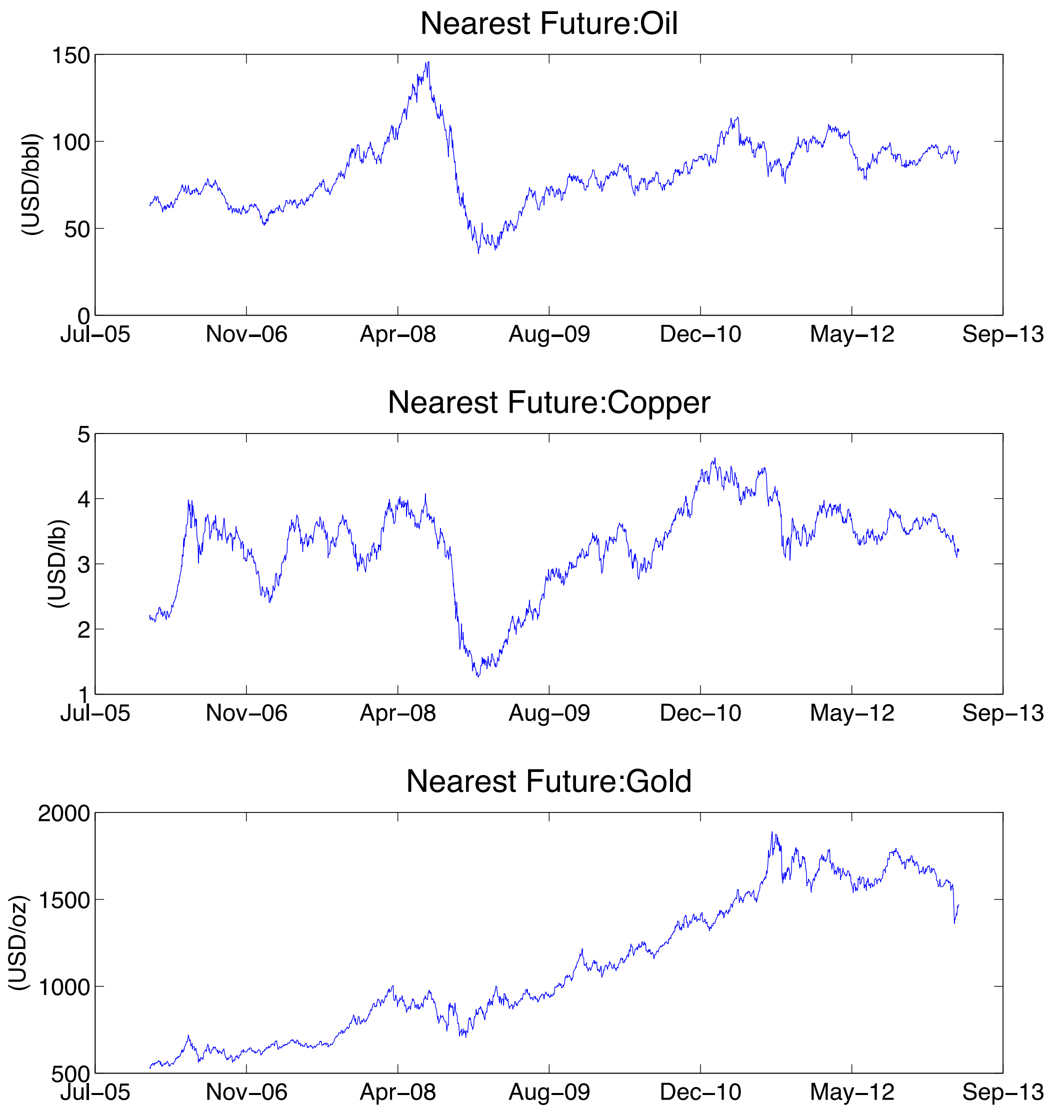
The Future Contract column: Fi denotes the first i-month contracts; MDi denotes the i-following contract with expiration either in March, May, September or December. Prices are expressed in US$ and maturities in years.

Table II-3: Gold Data

From January 2006 to May 2013. Daily Observations

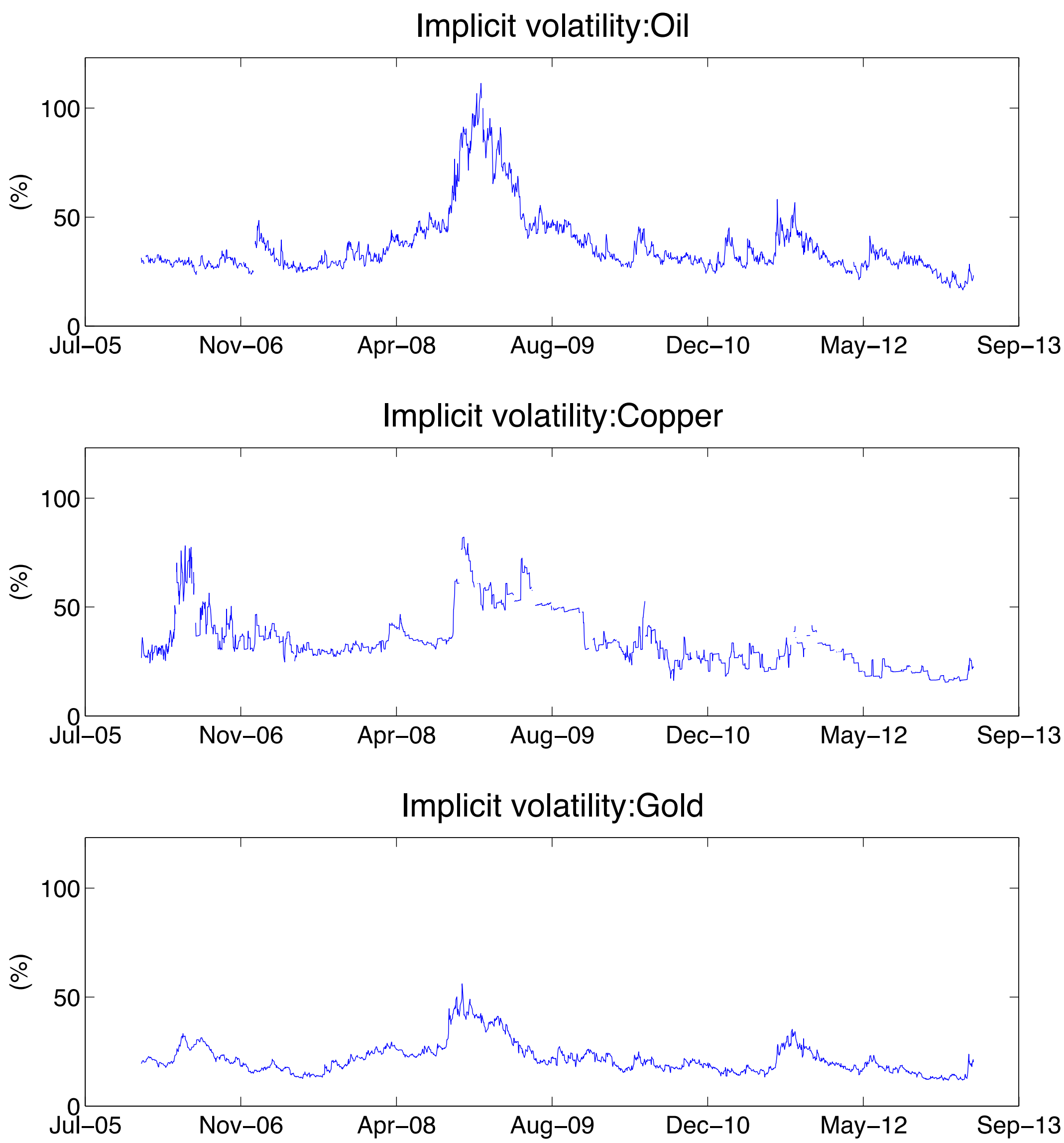
	FUTURES			OPTIONS			
Future Contract	Avg. Price	Avg. Maturity	Avg. Open Interest	Nº Puts	Nº Calls	Avg. Put price	Avg. Call price
F1	1111,278	0,080	148564	4608	5152	5,117	5,671
F2	1114,707	0,241	138392	9631	10129	11,519	14,681
F3	1117,883	0,408	46464	9572	9997	20,900	27,159
F4	1120,963	0,575	21960	8874	9360	30,096	39,582
F5	1124,037	0,742	16560	7721	8519	40,216	51,692
F6	1127,172	0,908	12790	6090	6780	50,507	63,131
JD1	1141,244	1,578	10112	3655	4708	83,081	104,012
JD2	1153,732	2,078	8160	1235	1924	129,536	149,678
JD3	1168,120	2,579	6441	444	760	134,755	176,939
JD4	1183,991	3,079	4164	295	66	128,333	187,092
JD5	1201,505	3,579	4427	254	137	154,651	253,023

The Future Contract column: Fi denotes the first i-month contracts; JDi denotes the first i contracts with expiration either in June or December after a year from date. Prices are expressed in US$ and maturity in years.



Spot price proxied by the closest-to-maturity future contract from January 2006 to May 2013.

Figure I-1: Spot price of Oil, Copper and Gold



Black-Scholes implied volatilities for the closest-to-maturity option contract between January2006 and May 2013

Figure II-2: Lognormal Implied Volatility.

2.3 Results

In this section we present the results of applying the two models to each of the panels of data for every commodity. We start by analyzing the parameter values for each model and data set. Next we compare pricing performances of each model to finally present a measure of implementation complexity to provide all information needed to choose among the models.

2.3.1 Parameter values.

Tables II-4 to II-9 present the results of applying Kalman filter and maximum likelihood to each model and data set. To make both models comparable, a 5-factor specification for the CN model is used so it has also to estimate 27 parameters, as in the TS specification. Nevertheless, 4 and 3-factor CN models are also considered in order to study the trade-offs with simpler models. Appendix B presents a detailed explanation of the estimation procedure.

Table II-4 presents the estimates of the TS parameters for oil. The model is statistically significant for most parameters with the exception of the risk premiums, λ_i, that are estimated with relatively large standard errors. This is a common issue for most of models in the literature (Cortazar, et al., 2013) but does not affect the pricing of futures and options. In addition, the parameter results are consistent with those in Trolle and Schwartz (2009). Volatility factors, $\nu_1(t)$ and $\nu_2(t)$, exhibit low correlation with the spot price (ρ_{15}, ρ_{26}) and forward cost-of-carry curve (ρ_{35}, ρ_{46}), which strongly suggest the presence of unspanned stochastic volatility. The estimates of the mean reverting coefficients, κ_1, κ_{12}, κ_{21} and κ_2, of the volatility components indicate, as well, that most of the transitory shocks to the volatility are absorbed by $\nu_1(t)$, which strongly reverts to $\nu_2(t)$. As expected, observation standard errors, σ_{fut} and σ_{op} are highly significant and relatively low, being larger for the panels that include the 2008 financial crisis.

Table II-5 summarizes the parameters of the Cortazar and Naranjo (2006) 5-factor model for oil. Results for the 3 and 4-factor models are presented in Appendix C. It can be seen that most parameters are stable through the different panels. All mean reversion parameters, a_i, as well as the volatility ones, σ_i, are significant and show strong mean reversion in oil prices, which gets stronger for the panels that includes the 2008 crisis (Panels A and C). Most correlation estimates, ρ_{ij}, are also statistically different from zero. As the case for the TS model, risk premiums, λ_i, are not significant for most panels and the same happens for the long-term growth rate parameter, μ, which is consistent with Schwartz (1997). The standard deviation observation errors, σ_{fut} and σ_{op}, are small, but highly significant. Results for fewer factor specifications shows that a 4-factor specification seems to be statistically better specified than the 5-factor (Appendix C).

Tables II-6 and II-7 show the TS and CN 5-factor specification for copper. Table II-6 shows that all parameters are stable through time and most of them are also statistically significant.[6] As in the case for oil, correlations between the volatility components and other variables are quite low for all panels, which is consistent with the existence of unspanned stochastic volatility for copper. It is important to note that this is the first time, as far as we know, that results for the TS model for copper have been reported in the literature. Also both volatility components present relatively high mean reversion coefficients, κ_1 and κ_2, compared to oil estimates, being $\nu_1(t)$ the volatility parameter that most strongly reverts and therefore the variable that account the majority of the transitory shocks to volatility. Standard deviation of the measurement errors, σ_{fut} and σ_{op}, are very low, but highly significant and tend to peak in the periods of higher volatility, such as Panels B and C.

Table II-7 shows the results for the CN 5-factor specification for copper, which are consistent with those for oil. Most parameters are significant and stable through

[6] Risk premiums are the only parameters that are not significant in most cases. As has been said this is a common issue in the existing models in the commodity literature.

time. The main difference is the stronger mean reversion in copper. This is consistent with the results reported by Schwartz (1997) and by Cortazar and Naranjo (2006). Similarly to the results for the TS specification, higher volatility periods have larger measurement errors standard deviations, σ_{fut} and σ_{op}.

Tables II-8 and II-9 summarize the results for the TS and 5-factor CN specifications for gold. Even though Schwartz (1997) found that mean-reverting prices did not seem to hold for gold, our estimates of the TS model, being lower than oil and copper estimates, are still significant at standard levels. It must be noted, however, that we do not use only futures, like Schwartz (1997), but also options in the calibration process. Also, correlations between the spot price and the forward cost-of-carry curve, although small, are significant, contrasting the evidence found by Casassus and Collin-Dufresne (2005). It must be noted that both Schwartz (1997) and Casassus and Collin-Dufresne (2005) use only futures. Maybe adding options into the calibration captures mean reversion in gold prices.

Being this the first time that results are report from the application of the TS model to gold prices, it is interesting to note that low correlations between the volatility components and the spot price and forward cost-of-carry curve are consistent with the presence of unspanned stochastic volatility in this market but in a weaker way than for oil and copper. Just as with oil, the first volatility component $\nu_1(t)$ accounts for most of the transitory shocks to volatility and is the most volatile. Observation error standard deviations, σ_{fut} and σ_{op}, are quite small but highly significant in all samples.

Table II-9 presents the estimates for the 5-factor CN model applied to gold market. With the aforementioned exception of risk premiums, λ_i, and long-term growth rate, μ, most of parameters are significant. However, parameter estimates seem not to be stable across panels, suggesting that 5-factor specification may be over-specified. Indeed, the high correlation between the most reverting state variables, ρ_{45}, and the negligible ones between them and some of the rest of variables, ρ_{24}, ρ_{15}, reinforces the idea of reducing the number of factors considered. As shown in the Appendix C, a 3 or less-factor specification seems more suitable to the gold case. Whichever specification is considered the mean reverting parameters, a_i, are quite small compared to those for oil and copper, indicating a weaker mean reversion in gold. Measurement errors standard deviations for future and options, σ_{fut} and σ_{op}, are all significant.

Table II-4: TS model parameters: Oil

	Panel A *(2006-2011)*	**Panel B** *(2006-2007)*	**Panel C** *(2008-2009)*	**Panel D** *(2010-2011)*
ϕ	0.0191*	0.0086*	0.0192*	0.0074*
S_{S1}	0.2789*	0.3196*	0.2784*	0.5683*
S_{S2}	0.1225*	0.1116*	0.1220*	0.4266*
S_{v1}	8.4207*	6.0299*	7.5059*	2.4962*
S_{v2}	1.1229*	1.5478*	1.1617*	2.4935*
ρ_{13}	-0.8586*	-0.6931*	-0.8549*	-0.9484*
ρ_{15}	-0.2109*	0.0319*	-0.1828*	0.2395*
ρ_{35}	0.1095*	-0.1617*	0.1109*	-0.2818*
ρ_{24}	-0.3015*	0.3204*	-0.3206*	-0.6897*
ρ_{26}	-0.3087*	-0.5451*	-0.3130*	-0.7809*
ρ_{46}	-0.0282	-0.2350*	-0.0624	0.3434*
λ_1	0.7697	-13914	0.2598	0.3320
λ_2	1.5534*	0.0179	1.6376*	0.4064
λ_3	-1.1086*	0.0807*	-0.7319	-0.5272
λ_4	-0.0402	0.7860*	0.2533	-0.0310*
λ_5	-0.1559	0.1543	-0.1333	-16369
λ_6	0.5306*	-0.1622	0.4807*	0.4601*
α_1	0.2639*	0.2635*	0.2663*	0.5520*
α_2	0.0368*	0.0190*	0.0369*	0.1415*
γ_1	1.1167*	1.1910*	1.1147*	0.8872*
γ_2	0.2929*	0.2312*	0.2917*	0.3258*
κ_1	6.8588*	8.3363*	7.0539*	3.4196*
κ_{12}	-5.1711*	-2.0699*	-49693	-1.1486*
κ_{21}	-0.1710*	-0.2145	-0.1777	-0.0000
κ_2	0.8214*	0.9365*	0.7914*	1.9052*
σ_{fut}	0.0042*	0.0028*	0.0049*	0.0034*
σ_{op}	0.0182*	0.0097*	0.0181*	0.0134*

Maximum-likelihood estimates for all data sets. The (*) symbol indicates the significance at a 95% level of confidence. Panel A represents the full-sample period, Panels B, C and D the corresponding subsamples.

Table II-5: CN 5-factor model parameters: Oil

	Panel A *(2006-2011)*	**Panel B** *(2006-2007)*	**Panel C** *(2008-2009)*	**Panel D** *(2010-2011)*
a_2	0.0356*	0.2170*	0.0360*	0.0583*
a_3	0.9115*	1.3468*	0.8505*	0.8661*
a_4	1.1543*	1.9469*	1.1883*	1.5068*
a_5	8.6570*	5.1132*	9.6454*	6.1454*
σ_1	0.3986*	0.2365*	0.3624*	0.2907*
σ_2	0.4288*	0.2941*	0.3514*	0.2840*
σ_3	0.6531*	0.4346*	0.6348*	0.5375*
σ_4	0.5755*	0.3327*	0.5993*	0.3345*
σ_5	0.1840*	0.1891*	0.3130*	0.1532*
ρ_{12}	-0.8227*	-0.6579*	-0.6981*	-0.6315*
ρ_{13}	0.0474*	0.2109*	-0.0722*	-0.2594*
ρ_{14}	-0.0263*	-0.1057	0.0187	0.1360*
ρ_{15}	-0.0797*	-0.2832*	-0.1685*	0.0764
ρ_{23}	0.0080	-0.1784*	-0.0866*	0.2545*
ρ_{24}	0.0013	0.1285	0.2117*	-0.1799*
ρ_{25}	0.0664*	0.2627*	0.0128	-0.1814*
ρ_{34}	-0.9409*	-0.8800*	-0.8516*	-0.8419*
ρ_{35}	0.0832*	0.0608	0.2565*	-0.3352*
ρ_{45}	-0.0707*	-0.3746*	-0.1403*	0.1196
λ_1	-0.0250	0.0203*	0.0067	-0.0233
λ_2	-0.6813*	-0.1680	0.2351	-0.5575*
λ_3	-0.2949	0.3016	-0.6122	0.0107*
λ_4	-0.2677	-0.2322	-0.3490	-0.5176*
λ_5	-0.3195*	0.0558	-0.0368*	0.0554
μ	-0.0152	0.0024	0.0118	-0.0131
σ_{fut}	0.0012*	0.0010*	0.0014*	0.0011*
σ_{op}	0.1052*	0.0285*	0.1325*	0.0517*

Maximum-likelihood estimates for all data sets. The (*) symbol indicates the significance at a 95% level of confidence. Panel A represents the full-sample period, Panels B, C and D the corresponding subsamples.

Table II-6: TS model parameters: Copper

	Panel A *(2006-2011)*	**Panel B** *(2006-2007)*	**Panel C** *(2008-2009)*	**Panel D** *(2010-2011)*
ϕ	0.0362*	0.0535	0.0375*	0.0043*
S_{S1}	0.4957*	0.5513*	0.5224*	0.3476*
S_{S2}	0.1115*	0.2206*	0.1617*	0.0833*
S_{v1}	2.2427*	2.8379*	2.8898*	2.1645*
S_{v2}	2.0139*	2.5886*	2.6449*	2.1109*
ρ_{13}	-0.1767*	-0.2410*	-0.2192*	-0.1966*
ρ_{15}	-0.0734*	-0.1446*	-0.0410*	-0.0340*
ρ_{35}	0.0088	0.0704	0.1323	0.0404
ρ_{24}	-0.6121*	-0.6425*	-0.6583*	-0.8595*
ρ_{26}	-0.3523*	-0.2739*	-0.1702*	-0.4096*
ρ_{46}	0.0148	-0.0362	0.0655	-0.0305
λ_1	1.8692*	-0.0486	12752	3.6456*
λ_2	0.3325	16639	0.3117	13040
λ_3	3.1032*	14770	17472	0.3361
λ_4	-3.3923*	-3.8514*	-3.1161*	-12501
λ_5	-0.3190	0.3410	-0.5111	2.5467*
λ_6	-0.8076*	-0.6370	-0.1540	-1.5705*
α_1	0.2187*	0.2946*	0.2824*	0.1071*
α_2	0.1120*	0.1857*	0.1754*	0.0673*
γ_1	1.7400*	1.7241*	1.8192*	0.8613*
γ_2	0.1428*	0.2042*	0.3261*	0.1722*
κ_1	10.8751*	11.2664*	11.4430*	12.1396*
κ_{12}	-7.3825*	-7.1476*	-68282	-9.7343*
κ_{21}	-0.6891*	-1.0067*	-0.7793	-0.7795*
κ_2	1.6236*	2.3051*	2.6418*	1.0675*
σ_{fut}	0.0014*	0.0035*	0.0013*	0.0006*
σ_{op}	0.0229*	0.0292*	0.0215*	0.0155*

Maximum-likelihood estimates for all data sets. The (*) symbol indicates the significance at a 95% level of confidence. Panel A represents the full-sample period, Panels B, C and D the corresponding subsamples.

Table II-7: CN 5-factor model parameters: Copper

	Panel A *(2006-2011)*	**Panel B** *(2006-2007)*	**Panel C** *(2008-2009)*	**Panel D** *(2010-2011)*
a_2	0.4501*	0.3829*	0.3228*	0.3069*
a_3	1.2737*	1.3026*	0.9676*	0.9825*
a_4	2.7140*	2.8489*	6.2585*	2.2704*
a_5	13.0672*	14.9944*	16.1435*	3.8863*
σ_1	0.4662*	0.4955*	0.4569*	0.3395*
σ_2	0.7159*	0.7319*	0.4978*	0.4485*
σ_3	0.6523*	0.5887*	0.2621*	0.3855*
σ_4	0.3034*	0.3071*	0.1278*	0.2613*
σ_5	0.1326*	0.1821*	0.1938*	0.1276*
ρ_{12}	-0.6893*	-0.6732*	-0.5161*	-0.5848*
ρ_{13}	0.6035*	0.4524*	0.4872*	0.5760*
ρ_{14}	-0.3904*	-0.1825*	-0.0686	-0.2666*
ρ_{15}	-0.1490*	-0.2347*	-0.1107*	-0.0411
ρ_{23}	-0.8805*	-0.7550*	-0.8327*	-0.8986*
ρ_{24}	0.5918*	0.3038*	0.2640*	0.5116*
ρ_{25}	-0.0051	0.0311	-0.0241	-0.1587
ρ_{34}	-0.8861*	-0.8139*	-0.6190*	-0.7933*
ρ_{35}	0.2256*	0.1606*	0.2210*	0.4347*
ρ_{45}	-0.4743*	-0.3179*	-0.6852*	-0.8735*
λ_1	0.1269*	0.0027	0.0380*	0.0889*
λ_2	0.0368	0.1459	0.0870	0.0447
λ_3	-0.0998	-0.2781	0.0733	0.0861
λ_4	-0.0824	-0.1865	-0.0068*	-0.2374*
λ_5	0.1416	0.2599	0.2419	0.1509
μ	0.0074	-0.0106	-0.0039	0.0108
σ_{fut}	0.0008*	0.0012*	0.0006*	0.0003*
σ_{op}	0.1032*	0.0905*	0.1118*	0.0605*

Maximum-likelihood estimates for all data sets. The (*) symbol indicates the significance at a 95% level of confidence. Panel A represents the full-sample period, Panels B, C and D the corresponding subsamples.

Table II-8: TS model parameters: Gold

	Panel A *(2006-2011)*	**Panel B** *(2006-2007)*	**Panel C** *(2008-2009)*	**Panel D** *(2010-2011)*
ϕ	0.0897*	0.0823*	0.0679*	0.1007*
S_{S1}	0.1895*	0.1845*	0.1977*	0.2951*
S_{S2}	0.0592*	0.0793*	0.0620*	0.1084*
S_{v1}	4.8334*	6.2097*	5.9416*	6.0650*
S_{v2}	1.2108*	1.2177*	1.2247*	0.8362*
ρ_{13}	-0.2647*	-0.3408*	-0.2786*	-0.1847*
ρ_{15}	0.1584*	0.1725*	0.2161*	-0.0497*
ρ_{35}	0.5253*	0.5104*	0.2183*	0.5892*
ρ_{24}	-0.1284*	-0.1856*	-0.0809	0.3346*
ρ_{26}	-0.3221*	-0.0766*	0.1878*	0.1749*
ρ_{46}	-0.2698*	-0.2640*	0.1941*	0.0659
λ_1	2.8480*	3.3353*	4.0267*	40342
λ_2	1.3974*	-2.0845*	2.7694*	2.6203*
λ_3	-3.9397*	0.0320	-18984	-17666
λ_4	-2.1685*	-0.3278	0.5443	-4.1242*
λ_5	-1.1147*	-2.4592*	-4.6311*	-0.9058
λ_6	-4.0419*	-0.7736*	0.0703	-2.0516*
α_1	0.0898*	0.1529*	0.1582*	0.2336*
α_2	0.0180*	0.0548*	0.0256*	0.0278*
γ_1	0.3555*	0.6989*	0.5717*	1.0168*
γ_2	0.0955*	0.1775*	0.0760*	0.1065*
κ_1	2.8250*	5.3231*	4.8868*	5.6819*
κ_{12}	-1.8907*	-1.5291*	-38587	-1.5064*
κ_{21}	-0.0400	-1.6668*	-0.2334	-0.0401
κ_2	0.5933*	1.4470*	1.1351*	0.6574*
σ_{fut}	0.0006*	0.0005*	0.0006*	0.0011*
σ_{op}	0.0099*	0.0078*	0.0094*	0.0076*

Maximum-likelihood estimates for all data sets. The (*) symbol indicates the significance at a 95% level of confidence. Panel A represents the full-sample period, Panels B, C and D the corresponding subsamples.

Table II-9: CN 5-factor model parameters: Gold

	Panel A *(2006-2011)*	**Panel B** *(2006-2007)*	**Panel C** *(2008-2009)*	**Panel D** *(2010-2011)*
a_2	0.1044*	0.0609*	0.3065*	0.0548*
a_3	0.4262*	0.4359*	0.7272*	0.5040*
a_4	1.0328*	2.9586*	1.7031*	1.0662*
a_5	1.3809*	5.4130*	2.5441*	1.5808*
σ_1	0.2853*	0.2458*	0.3345*	0.3342*
σ_2	0.2550*	0.2320*	0.2852*	0.2818*
σ_3	0.1756*	0.0674*	0.2380*	0.3167*
σ_4	0.2025*	0.0255*	0.1807*	0.3271*
σ_5	0.1414*	0.0282*	0.0995*	0.2186
ρ_{12}	-0.4059*	-0.5837*	-0.4427*	-0.3928*
ρ_{13}	-0.1381*	0.1661	0.2002*	-0.4763*
ρ_{14}	0.1233*	-0.1675	0.2141*	-0.0822
ρ_{15}	-0.0440	0.2782*	-0.3642*	0.3300*
ρ_{23}	-0.5265*	-0.5853*	-0.8341*	-0.0922
ρ_{24}	-0.0625	0.4106*	0.3012*	-0.1657
ρ_{25}	0.1486*	-0.3374*	-0.0929	0.0096
ρ_{34}	-0.4863*	-0.5079*	-0.7412*	-0.4179
ρ_{35}	0.2006*	0.0123	0.5267*	0.0081
ρ_{45}	-0.9418*	-0.8182*	-0.9452*	-0.8757*
λ_1	-0.0145	-0.4450*	-0.4817*	-0.0037
λ_2	0.0953	0.0869	-0.0451	0.0279*
λ_3	0.0975	0.0508	0.1947*	0.1833
λ_4	-0.0321	-0.0301	-0.1528	0.0706
λ_5	-0.0018	0.0208	0.0788	0.1603
μ	-0.0242	-0.4424*	-0.5000*	0.0583
σ_{fut}	0.0003*	0.0003*	0.0002*	0.0003*
σ_{op}	0.0627*	0.0438*	0.0676*	0.0363*

Maximum-likelihood estimates for all data sets. The (*) symbol indicates the significance at a 95% level of confidence. Panel A represents the full-sample period, Panels B, C and D the corresponding subsamples.

2.3.2 Pricing Performance Comparison.

We now compare the pricing performance of the two models. For each commodity and model specification we construct time series of daily futures and options root mean square pricing errors (RMSE). Errors are defined as the percentage difference between actual and fitted prices for futures, and as the difference between the fitted and the actual lognormal implied volatilities, for options. First, results are presented computing the average RMSE for futures and options, for each data Panel. Then a figure with the time series for the errors is shown. Finally, a cross section error analysis is presented showing which contracts are worse priced.

These analyses are shown for each of the three commodities: oil, copper and gold.

a) Oil

Table II-10 summarizes the RMSE for Oil futures and options on all specifications for each data set. For futures contracts 5 and 4-factor CN models perform better than the TS model, while for the 3-factor specification there are no significant differences. This is not surprising given that futures prices in the TS model are driven only by 3 factors, instead of the 4 and 5 factors in the CN specifications. The futures pricing errors in Panel A for the TS model are 3.5 and 1.8 times higher than for 5 and 4 factors CN models. These differences are quite stable through the different panels, being a little smaller for the out-of-sample Panel E. Thus for futures pricing, how volatility is modeled appears not to be relevant. For options contracts, on the other hand, the volatility specification seems to have a great impact. Table II-10 shows that, on average, the TS model substantially outperforms every CN model in all data sets, in and out-of-sample. Also it shows that adding an extra factor to the CN specification does not significantly improve the options pricing performance. Regardless of the number of factors used by CN models, the RMSE for Panel A in CN models is more than 5 times larger than in the TS model. This difference is even larger for the financial crisis period (Panel C), as expected.

Table II-10: Overall RMSE: Oil

	Panel A *(2006-2011)*	**Panel B** *(2006-2007)*	**Panel C** *(2008-2009)*	**Panel D** *(2010-2011)*	**Panel E** *(2012-5'2013)*
			Futures		
TS	0.35	0.24	0.43	0.29	0.27
5F	0.10	0.08	0.12	0.09	0.12
4F	0.19	0.12	0.25	0.13	0.24
3F	0.34	0.24	0.41	0.31	0.33
			Options		
TS	1.62	0.91	1.64	1.28	1.35
5F	8.50	2.45	10.70	4.68	7.88
4F	8.49	2.46	10.72	4.68	7.85
3F	8.56	2.48	10.67	4.71	7.99

RMSE for all data sets. Panel E is the out of sample period between January 2012 and May 2013. Here 5F, 4F and 3F represent the CN models of 5, 4 and 3 factors respectively. Errors are expressed in percentages.

Figure II-3 presents the time series of the RMSEs for options contracts for the full sample period, both in and out-of-sample. It can be seen that during the whole period the TS model outperforms the CN models. This performance advantage is greater when volatility increases in the peak of the financial crisis. This highlights the advantages of accounting for stochastic volatility, showing that TS model absorbs the shocks through the volatility state variables themselves and not only through the parameters, as is the case of the CN models.

Finally, Table II-11 presents a cross section error analysis for options with different maturities. For all data panels, the RMSE error differences between the CN and TS models are much greater for shorter than for longer maturity contracts, ranging from more than 6 times higher for the CN model for the shortest-maturity contract to less than twice for the longest-maturity contract (around 4 years). Moreover, for periods without a financial crisis, like Panel B and D, the RMSE for long maturity contracts in both models is similar.

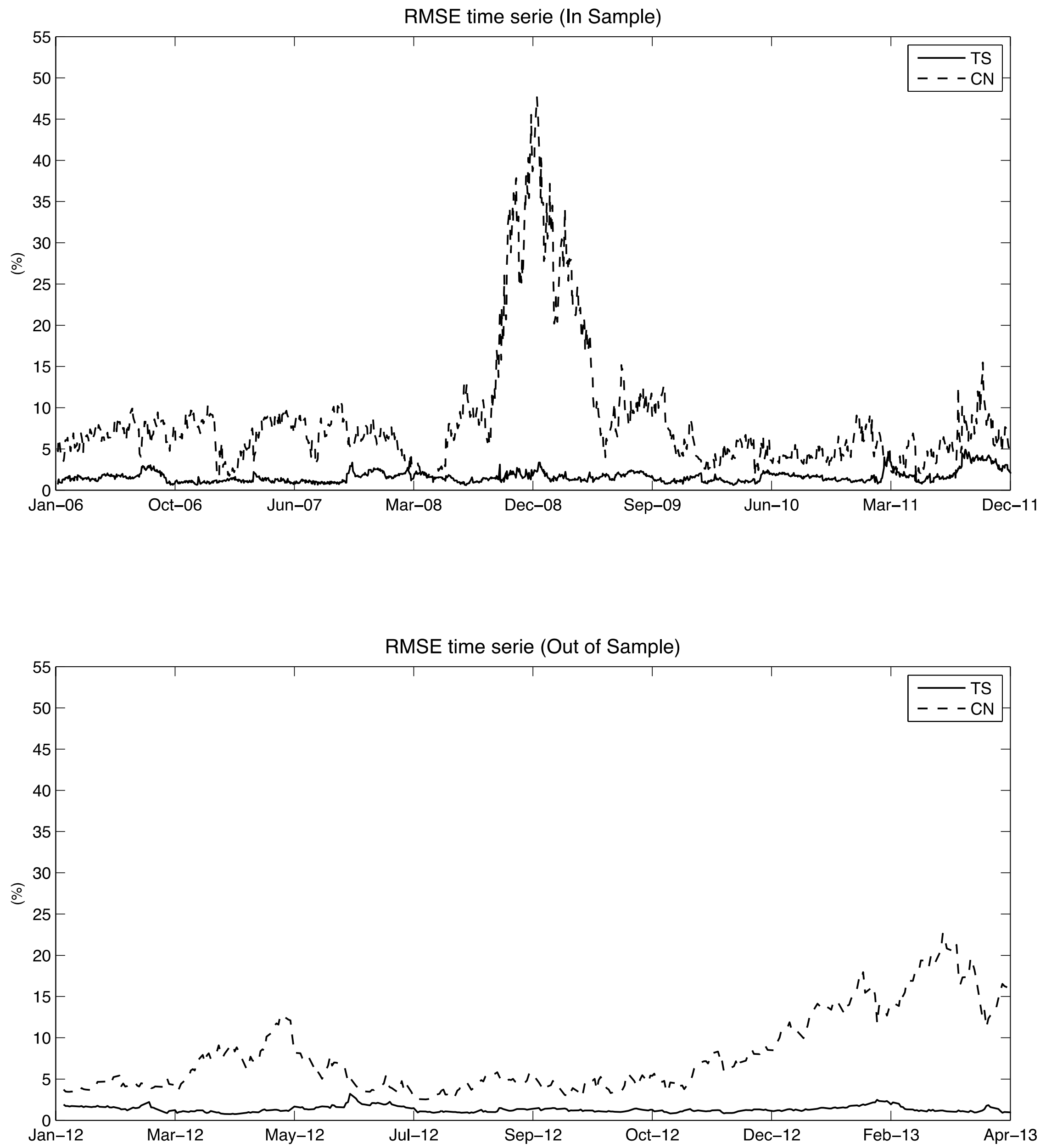
Table II-11: Cross-section RMSE: Oil Options

		Panel A (2006-2011)		Panel B (2006-2007)		Panel C (2008-2009)		Panel D (2010-2011)		Panel E (2012-5'2013)	
	τ	TS	CN	TS	CN	TS	CN	TS	CN	TS	CN
F1	0.07	2.38	14.49	1.22	3.95	2.10	17.96	2.38	6.39	1.73	11.64
F2	0.15	1.98	12.21	1.02	3.01	1.59	14.89	1.25	6.03	1.35	10.84
F3	0.24	1.74	10.63	0.89	2.62	1.45	12.87	1.05	5.63	1.30	10.10
F4	0.32	1.47	9.64	0.73	2.50	1.39	11.55	0.94	5.14	1.28	9.28
F5	0.40	1.30	9.14	0.58	2.27	1.38	10.88	0.93	4.94	1.34	9.09
F6	0.49	1.24	8.14	0.63	2.18	1.37	9.85	0.89	4.70	1.35	7.82
MD1	0.64	1.19	7.77	0.70	2.25	1.36	8.94	1.01	4.43	1.35	7.73
MD2	0.89	1.34	6.82	0.89	2.56	1.38	7.71	0.92	3.33	1.46	5.71
D1	1.45	1.67	5.50	1.24	2.17	1.91	4.82	1.21	3.34	1.21	5.42
D2	2.47	1.86	4.23	1.67	2.29	2.11	4.49	1.50	2.35	1.30	5.40
D3	3.46	2.24	3.89	2.52	2.59	2.32	3.76	1.78	2.27	1.53	4.80
D4	4.39	1.84	3.18	1.79	2.29	1.88	4.77	2.21	2.36	1.66	4.67

RMSE of the TS and 5-factor CN models cross maturity. Panel E is the out of sample period between January 2012 and May 2013. Fi denotes the first i-month contracts, MDi denotes the i-following contract with expiration either in March, June, September or December, Di denotes the i-following contract with expiration in December. Maturity of each contract, τ, is expressed in years, errors in percentages.

b) Copper

Table II-12 summarizes the RMSE for Copper futures and options on all specifications for each data set. It can be seen that just like in the oil case, CN models tends to outperform TS model as the number of factors increases in futures pricing, however all model specifications have small errors ranging from 0.05% to 0.09% depending on the model and factor specification.



In sample goes from January 2006 to December 2011 and out of sample from January 2012 to May 2013

Figure II-3: RMSE time series: Oil Options (Panel A)

Table II-12: Overall RMSE: Copper

	Panel A *(2006-2011)*	**Panel B** *(2006-2007)*	**Panel C** *(2008-2009)*	**Panel D** *(2010-2011)*	**Panel E** *(2012-5'2013)*
			Futures		
TS	0.09	0.15	0.08	0.05	0.04
5F	0.05	0.08	0.04	0.02	0.02
4F	0.06	0.09	0.05	0.03	0.02
3F	0.09	0.13	0.08	0.05	0.03
			Options		
TS	1.70	2.51	1.25	1.32	1.73
5F	8.76	7.00	10.67	5.16	19.39
4F	8.76	6.94	10.67	5.27	19.40
3F	8.76	6.96	10.68	5.16	19.37

RMSE for all data sets. Panel E is the out of sample period between January 2012 and May 2013. Here 5F, 4F and 3F represent the CN models of 5, 4 and 3 factors respectively. Errors are expressed in percentages.

For options contracts, just as was the case for oil, the volatility specification is important, and regardless of the number of factors in the CN models, again the RMSE for Panel A in CN models is more than 5 times larger than in the TS model. This is even larger for the financial crisis period (Panel C), like the oil case.

Figure II-4 is similar to Figure II-3, but now presents the time series of the RMSEs for copper options contracts for the whole period. It shows that always the TS model outperforms the CN models for options, and also that the worst performance of the CN model is during the financial crisis.

Finally, Table II-13 presents a cross section error analysis for copper options with different maturities. Again, the RMSE differences between the CN and TS models are much greater for shorter than for longer maturity contracts and for periods of higher financial distress, like those in Panel C. Finally it is important to note that the longest copper option contract is less than a year, so copper options errors, on average, are higher than those corresponding to oil contracts, due to the different average maturity.

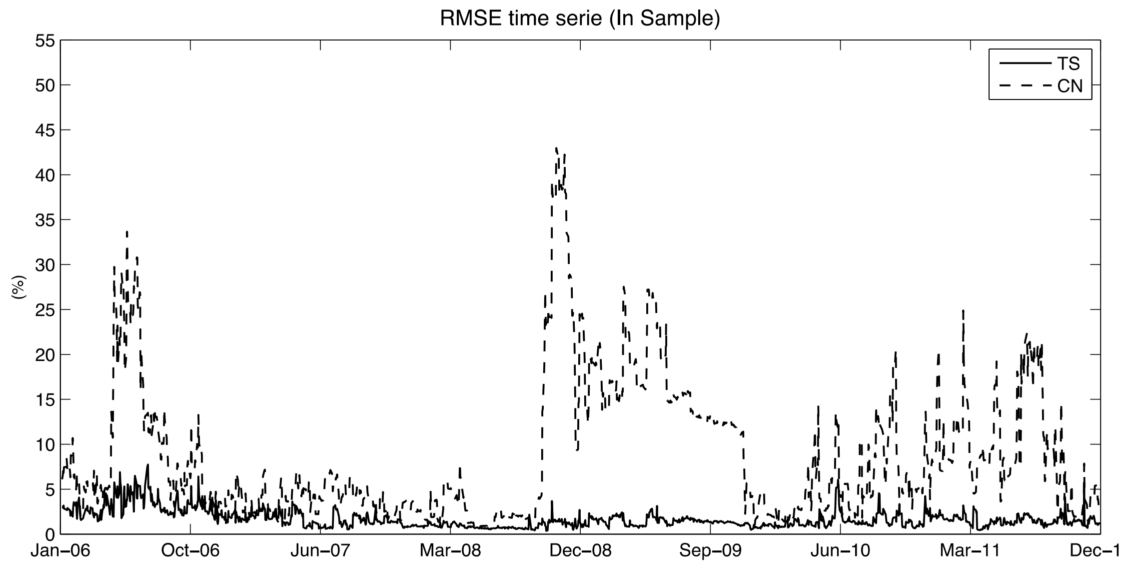
Table II-13: Cross-section RMSE: Copper Options

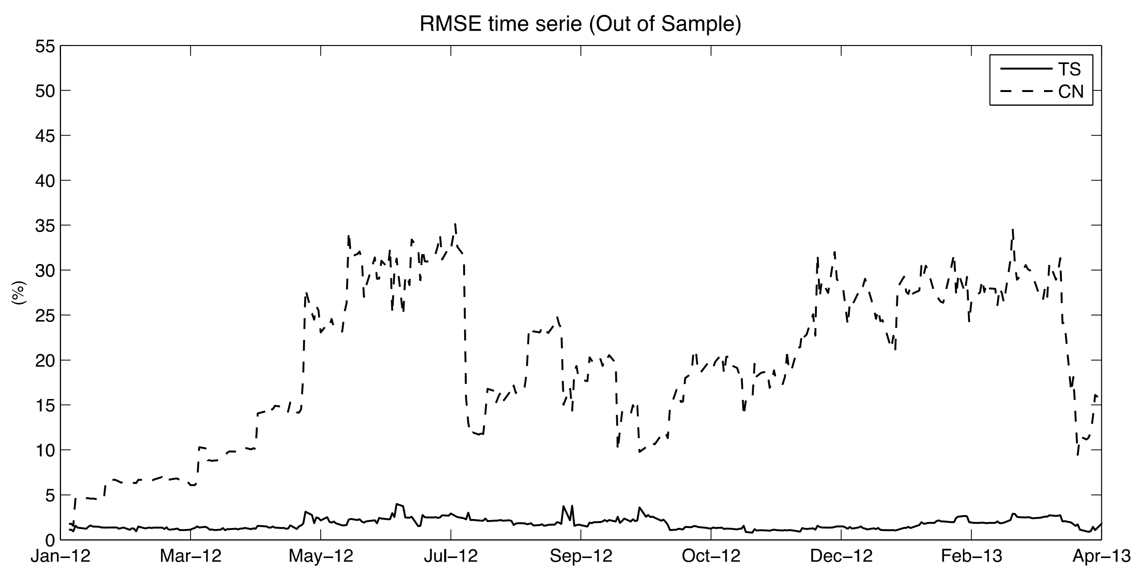
		Panel A (2006-2011)		**Panel B** (2006-2007)		**Panel C** (2008-2009)		**Panel D** (2010-2011)		**Panel E** (2012-5'2013)	
	Tau	TS	CN	TS	CN	TS	CN	TS	CN	TS	CN
F1	0,08	3,30	13,84	3,26	12,96	2,41	14,77	2,82	6,82	1,46	16,72
F2	0,17	2.06	11,78	2.94	10,40	1.23	12,41	1.67	6,20	1.48	17,18
F3	0,25	2.27	10,39	2.99	9,66	1.29	10,64	1.56	5,56	1.54	18,18
F4	0,33	2.14	10,06	2.61	9,38	1.77	11,51	1.49	6,06	1.52	17,97
F5	0,42	2.78	10,06	2.31	8,14	1,28	13,05	1.17	5,43	1.46	17,61
F6	0,50	1.76	7,97	2.21	7,74	1.02	7,99	1.55	5,99	1.88	17,55
MD1	0,65	2.09	8,28	2.48	8,23	1.49	7,95	1.57	6,14	3.10	15,79
MD2	0,85	1.96	5,15	2.49	6,14	1.08	8,15	1.84	3,77	4.31	15,31

RMSE of the TS and 5-factor CN models cross maturity. Panel E is the out of sample period between January 2012 and May 2013. Fi denotes the first i-month contracts; MDi denotes the i-following contract with expiration either in March, May, September or December. Maturity of each contract, τ, is expressed in years, errors in percentages

c) Gold

Tables II-14 and II-15, and Figure II-5, repeats for gold the exercise previously done for oil and copper. Results for gold are very similar to those from the other commodities: the CN model performs better than the TS model for futures, and much worse options; during the financial crisis the CN model behaves particularly bad for options; and finally, that the shorter the option contract maturity, the worse the performance of the CN model relative to the TS model.





In sample goes from January 2006 to December 2011 and out of sample from January 2012 to May 2013

Figure II-4: RMSE time series: Copper Options (Panel A)

Table II-14: Overall RMSE: Gold

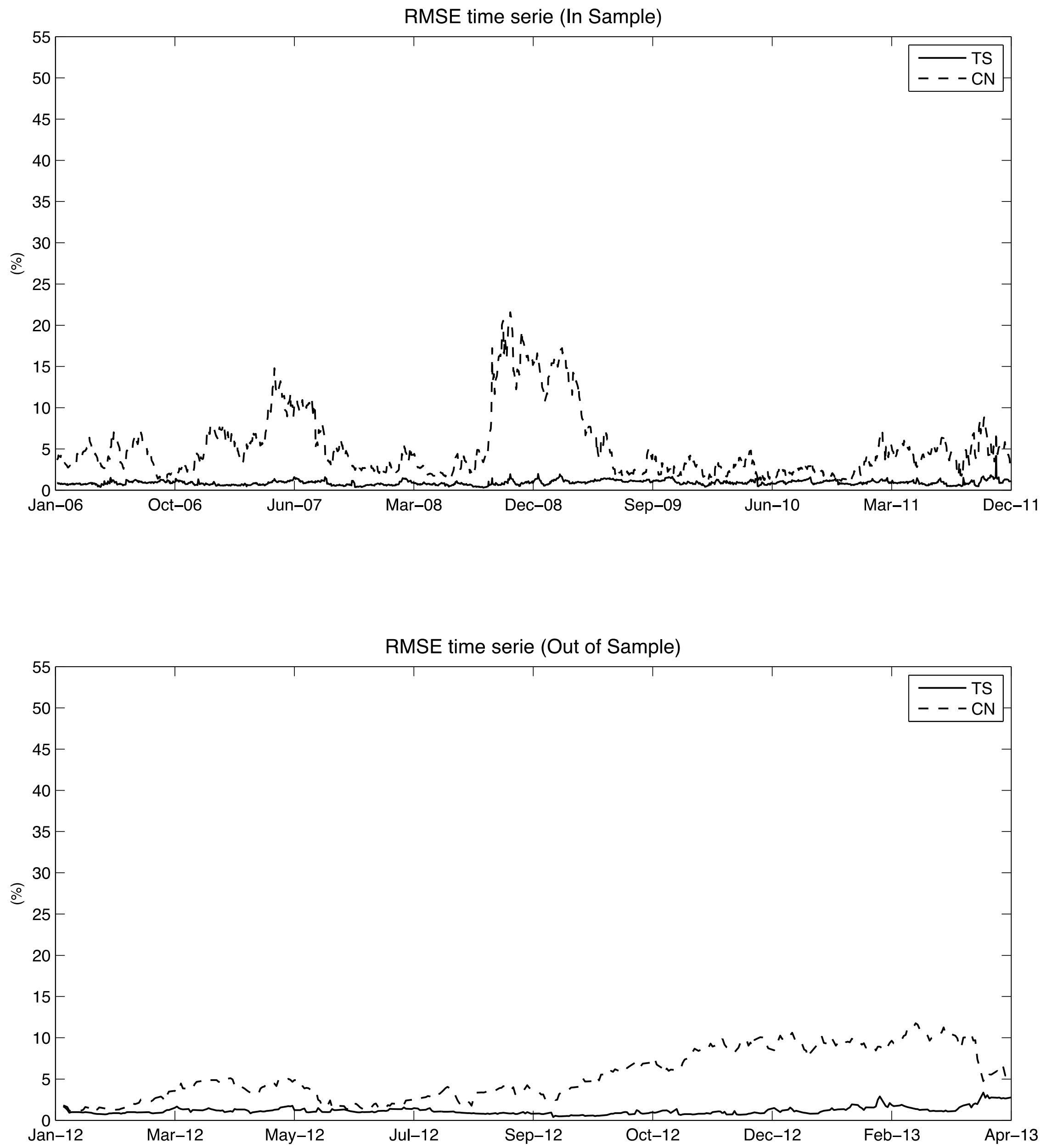
	Panel A *(2006-2011)*	**Panel B** *(2006-2007)*	**Panel C** *(2008-2009)*	**Panel D** *(2010-2011)*	**Panel D** *(2012-5'2013)*
			Futures		
TS	0.05	0.04	0.04	0.07	0.03
5F	0.02	0.02	0.02	0.02	0.02
4F	0.03	0.03	0.02	0.03	0.02
3F	0.05	0.04	0.04	0.05	0.04
			Options		
TS	0.92	0.74	0.86	0.75	1.21
5F	5.17	3.68	5.90	3.23	5.28
4F	5.16	3.70	5.90	3.52	5.26
3F	5.08	3.72	5.86	3.27	5.15

RMSE for all data sets. Panel E is the out of sample period between January 2012 and May 2013. Here 5F, 4F and 3F represent the CN models of 5, 4 and 3 factors respectively. Errors are expressed in percentages.

Table II-15: Cross-section RMSE: Gold Options

		Panel A (2006-2011)		**Panel B** (2006-2007)		**Panel C** (2008-2009)		**Panel D** (2010-2011)		**Panel E** (2012-5'2013)	
	τ	TS	CN	TS	CN	TS	CN	TS	CN	TS	CN
F1	0.08	1.74	6.67	1.18	4.48	1.18	8.36	1.77	4.69	2.52	7.36
F2	0.20	1.22	6.66	0.87	4.51	0.92	7.63	1.11	4.23	1.74	7.41
F3	0.36	0.88	6.55	0.72	4.21	0.81	7.05	0.65	3.81	1.25	6.68
F4	0.53	0.77	6.35	0.65	4.41	0.71	6.44	0.51	3.59	0.95	5.89
F5	0.69	0.67	6.12	0.61	4.32	0.65	6.06	0.46	3.45	0.79	5.39
F6	0.85	0.64	6.22	0.68	4.54	0.68	5.95	0.46	3.41	0.72	5.11
JD1	1.49	0.89	5.17	0.86	4.47	1.29	5.33	0.51	2.83	0.72	3.45
JD2	2.00	0.89	4.60	1.22	4.45	1.93	6.88	0.69	2.87	0.77	2.85
JD3	2.48	0.79	2.96	-	-	1.84	4.43	0.91	2.44	0.77	1.72
JD4	3.03	0.75	3.24	-	-	0.77	1.59	1.12	1.98	-	-
JD5	3.53	1.47	7.54	-	-	1.77	5.37	1.41	1.55	-	-

RMSE of the TS and 5-factor CN models cross maturity. Panel E is the out of sample period between January 2012 and May 2013. Fi denotes the first i-month contracts; JDi denotes the first i contracts with expiration in June or December. Maturity of each contract, τ, is expressed in years, errors in percentages.



In sample goes from January 2006 to December 2011 and out of sample from January 2012 to May 2013

Figure II-5: RMSE time series: Gold Options (Panel A)

2.3.3 Implementation complexity

In addition to the pricing performance just reported, a quantitative analysis of the tradeoff between goodness-of-fit and complexity of implementation is proxied by calculating the execution times of each model for all three commodities. For comparative purposes execution times are computed as the time required by a model per each iteration of the calibration process.

Considering a gradient base algorithm to find the optimal solution that uses forward finite differences to approximate the derivatives, an N-parameter model includes N+1 valuations of the objective function in each iteration: N valuations for calculating the partial derivatives plus one for valuing the new point. Results reported in the previous section show very clearly that the TS model outperforms the CN models in options pricing in many settings, but in this section we are interested on measuring the cost of a better performance in terms of the model implementation complexity.

Table II-16 shows the execution times measured for the TS and CN specification over all data sets.[7] It can be seen that the TS model are between 8 and 15 times slower than the CN specifications. This is due to the lack of closed form solutions, which required the implementation of numerical methods required for evaluating options pricing formulas. For illustration purposes on the effect that this could have, if we consider that on average each starting point takes about 15 iterations to get to an optimum and we use a grid of 100 different starting points to maximize the probability of reaching a global optimum, the process would take in a standard system around 8.3, 4.3 or 6.2 days to run TS model on the data sample for oil, copper or gold, respectively. Note than on the same computer the CN model would take only 0.71, 0.59 or 0.69 days, respectively, depending on the commodity. Thus, in order to improve the 5 time larger errors of the CN model on short-term options using the TS model, 10 times more effort is required.

[7] All measurements are done based on an Intel Core i5, 2.4 GHz processor, 8 GB RAM, OS X system.

Table II-16: Execution times per iteration

	Panel A *(2006-2011)*	**Panel B** *(2006-2007)*	**Panel C** *(2008-2009)*	**Panel D** *(2010-2011)*
Oil				
TS	477.16	228.71	220.11	200.20
5F	59.44	18.66	21.07	20.62
4F	41.03	13.50	14.46	14.44
3F	29.02	9.50	9.51	9.76
Copper				
TS	247.39	179.64	160.86	107.88
5F	45.54	15.38	18.38	17.02
4F	34.57	11.55	11.74	13.22
3F	27.86	7.25	6.49	8.88
Gold				
TS	358.53	164.34	203.52	271.32
5F	55.84	19.17	18.48	21.35
4F	40.70	11.04	12.58	14.56
3F	22.66	6.87	8.40	8.53

Execution times are calculated base on N+1 valuations of the objective function. Times are expressed in seconds per iteration. Here 5F, 4F and 3F represent the CN models of 5, 4 and 3 factors respectively

2.4 Conclusions

Prices of commodities contingent claims depend on the process assumed for the underlying asset. For futures, a good specification of the drift is very important, but for options, the volatility specification is crucial. Roughly speaking, commodity models in the literature can be classified in two types: those with a constant volatility and those with a stochastic volatility specification. In this paper, these two ways of dealing with volatility are contrasted in a way relevant to practitioners eliciting the trade-offs between the empirical performance and the implementation effort required for each model and commodity contract.

To make this comparative analysis, the Cortazar and Naranjo (2006) CN model to represent a constant volatility specification and the Trolle and Schwartz (2009) TS model as the stochastic volatility one, are chosen. Both models, specified with the same number of parameters (to make them comparable), are then applied to futures and options data for oil, copper and gold during different time periods. Pricing errors are calculated and execution times measured.

Results for all commodities are, in general, consistent. First, for futures pricing it is clearly better to use the CN model, because not only it is simpler but also errors are smaller. Also the higher the number of risk factors in the model, the better. Second, options pricing errors are considerably higher using the CN model increasing at the most by a factor of 6. Third, the longer the option maturity, the less relevant is the difference in pricing errors. For long maturity contracts the error difference is small. Fourth, the TS is much more complex to implement and its execution times are about 10 times higher. Fifth, our results of implementing the TS model for copper and gold are consistent with unspanned stochastic volatility for both commodities.

Results presented in this paper are new and relevant for practitioners. Up to now it is, to our knowledge, the first work to empirically test the pricing performance, using futures and options contracts, of stochastic volatility models against constant volatility benchmarks for oil, copper and gold. Also it is the first to apply the TS model to copper and gold markets. Choosing the best model to implement in a real situation depends on the objectives pursued and in the tradeoffs between effort and precision.

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APPENDIXES

APPENDIX A : TROLLE AND SCHWARTZ MODEL DETAILS

A.1 Affine transformation of the HJM process

Given the specification of $\sigma_{yi}(t,T), i = 1,2$, and $\mu_y(t,T)$ described in section 2.1.1 the t-time instantaneous forward cost of carry at time T, $y(t,T)$, is given by

$$y(t,T) = y(0,T) + \sum_{i=1}^{2}\left(\alpha_i e^{-\gamma_i(T-t)}x_i(t) + \alpha_i e^{-2\gamma_i(T-t)}\phi_i(t)\right) \quad \text{(A.1)}$$

where $x_i(t)$ and $\phi_i(t)$, $i = 1,2$, evolve according to following system of differential equations:

$$dx_1(t) = \left(-\gamma_1 x_1(t) - \left(\frac{\alpha_1}{\gamma_1} + \rho_{13}\sigma_{S1}\right)\nu_1(t)\right)dt + \sqrt{\nu_1(t)}dW_3^Q(t) \quad \text{(A.2)}$$

$$dx_2(t) = \left(-\gamma_2 x_2(t) - \left(\frac{\alpha_2}{\gamma_2} + \rho_{24}\sigma_{S2}\right)\nu_2(t)\right)dt + \sqrt{\nu_2(t)}dW_4^Q(t) \quad \text{(A.3)}$$

$$d\phi_i(t) = \left(-2\gamma_i\phi_i(t) + \frac{\alpha_i}{\gamma_i}\nu_i(t)\right)dt,\ i = 1,2 \quad \text{(A.4)}$$

Subject to $x_i(0) = \phi_i(0) = 0$, $i = 1,2$.

A.2 Transform equations.

To price options on futures, a transform of $F(t,T)$ is introduced in Section 2.1.1

$$\Psi(u,t,T_0,T_1) = E_t^Q\left[e^{u\log\left(F(T_0,T_1)\right)}\right] \quad \text{(A.5)}$$

This transform has an affine solution given by

$$\Psi(u,t,T_0,T_1) = e^{M(T_0-t)+N_1(T_0-t)\nu_1(t)+N_2(T_0-t)\nu_2(t)+u\log\left(F(t,T_1)\right)} \quad \text{(A.6)}$$

where,$M(\tau)$, $N_1(\tau)$ and $N_2(\tau)$ solve the following system of ordinary differential equations

$$\frac{dM(\tau)}{d\tau} = N_1(\tau)\eta_1 + N_2(\tau)\eta_2 \quad \text{(A.7)}$$

$$\frac{dN_1(\tau)}{d\tau} = -N_2(\tau)\kappa_{21} + N_1(\tau)\left(-\kappa_1 + u\sigma_{v1}\left(\rho_{15}\sigma_{S1} + \rho_{35}\frac{\alpha_1}{\gamma_1}\left(1 - e^{-\gamma_1(T_1-t)}\right)\right)\right) + \frac{1}{2}N_1(\tau)^2\sigma_{v1}^2 + \frac{1}{2}(u^2-u)\left(\sigma_{S1}^2 + \left(\frac{\alpha_1}{\gamma_1}\left(1-e^{-\gamma_1(T_1-t)}\right)\right)^2 + 2\rho_{13}\sigma_{S1}\frac{\alpha_1}{\gamma_1}\left(1-e^{-\gamma_1(T_1-t)}\right)\right) \quad \text{(A.8)}$$

$$\frac{dN_2(\tau)}{d\tau} = -N_1(\tau)\kappa_{12} + N_2(\tau)\left(-\kappa_2 + u\sigma_{v2}\left(\rho_{26}\sigma_{S2} + \rho_{46}\frac{\alpha_2}{\gamma_2}\left(1 - e^{-\gamma_2(T_1-t)}\right)\right)\right) + \frac{1}{2}N_2(\tau)^2\sigma_{v2}^2 + \frac{1}{2}(u^2-u)\left(\sigma_{S2}^2 + \left(\frac{\alpha_2}{\gamma_2}\left(1-e^{-\gamma_2(T_1-t)}\right)\right)^2 + 2\rho_{24}\sigma_{S2}\frac{\alpha_2}{\gamma_2}\left(1-e^{-\gamma_2(T_1-t)}\right)\right) \quad \text{(A.9)}$$

subject to the boundary conditions $M(0) = N_1(0) = N_2(0) = 0$.

APPENDIX B : ESTIMATION PROCEDURE

For estimation and calibration purposes of the parameters involved in the aforementioned models, the Kalman filter (KF) is applied in conjunction with the method of maximum likelihood (ML). This requires translating the CN and TS dynamics to their state-space representation. This is accomplished establishing the relationship between the respective state variables of the system and the observed price vector of futures and options (Measurement Equation), and discretizing the dynamics of the same (Transition Equation).

In particular, for TS specification, given the nonlinearity of the USV model, the approach used is the extended version of the Kalman filter (EKF), which linearizes the Measurement Equation, and applies the method of quasi-maximum likelihood (QML) for parameters calibration, which uses a Gaussian distribution to approximate the true distribution of the innovation errors.

The Kalman filter is a widely used estimation methodology in the commodity literature (Schwartz, 1997; Pindyck, 2004; Schwartz and Smith, 2000; Cortazar and Naranjo, 2006; Richter and Sørensen, 2002; Trolle and Schwartz, 2009) that calculates, recursively, optimal estimates of unobservable variables using all past information. Then, parameter estimates can be obtained by maximizing the likelihood function of its innovation errors.

In order to apply the Kalman filter, models have to be expressed in their state-space representation. The first step is to relate the vector of observables variables, options and futures prices, z_t, to the vector of state variables, X_t. Let X_t^{TS} and X_t^{CN} be the vector of states variables of the TS and CN specification respectively, and let h^M be the functional form that summarizes the pricing formulas of model Mthen

$$z_t = h^M(X_t^M) + u_t, u_t \sim \text{iid } N(0, \Omega) \tag{B.1}$$

where z_t is a vector of $m_t \times 1$ observations that may vary through time,X_t^M denotes the vector of state variables of model M, and

$$X_t^{TS} = \big(s(t), x_1(t), x_2(t), \phi_1(t), \phi_2(t), \nu_1(t), \nu_2(t)\big) \tag{B.2}$$

$$X_t^{CN} = \big(x_1(t), \cdots, x_N(t)\big) \tag{B.3}$$

The measurement equation (B.1) requires the existence of a linear relation between observed variables and the state variables and since nonlinear options prices are considered in the observations vector, z_t, the h-function must be linearized.[8] Let $\hat{X}_{t|s}^M = E_s[X_t^M]$ the expectations of X_t including the information until z_s. Then we have

$$z_t = c_t^M + H^M X_t^M + u_t,\, u_t \sim \text{iid } N(0, \Omega_t^M) \tag{B.4}$$

where $c_t^M = \big(h^M\big(\hat{X}_{t|t-1}^M\big) - H^M \hat{X}_{t|t-1}^M\big)$ and

$$H^M = \frac{\partial h^M(X_t^M)}{\partial X_t^M}\bigg|_{X_t^M = \hat{X}_{t|t-1}^M} \tag{B.5}$$

The transition equation describes the stochastic process followed by the states variables and it can be obtained from the risk-neutral dynamic along with the market price of risk specifications described in section 2.1.1 and 2.1.2 for each model:

$$X_{t+1}^M = \Phi_0^M + \Phi_X^M X_t^M + \omega_{t+1}^M,\, \omega_{t+1}^M \sim \text{iid} \tag{B.6}$$

$$E[\omega_{t+1}^M] = 0 \tag{B.7}$$

$$Var[\omega_{t+1}^{TS}] = Q_0^{TS} + Q_{\nu_1}\nu_1(t) + Q_{\nu_2}\nu_2(t) \tag{B.8}$$

$$Var[\omega_{t+1}^{CN}] = Q_0^{CN} \tag{B.9}$$

where Φ_0^M, Φ_X^M, Q_0^M, Q_{ν_1}, Q_{ν_2} can be computed in closed form following Fisher and Gilles (1996).

The Kalman Filter recursively calculate the optimal estimates of $\hat{X}_t^M$ and the variance-covariance matrix $P_t^M = E\left[\big(X_t^M - \hat{X}_t^M\big)\big(X_t^M - \hat{X}_t^M\big)'\right]$ by minimizing the prediction error, $\epsilon_t^M = \big(z_t - \hat{z}_{t|t-1}^M\big)$, in each step. Given $\hat{X}_{t-1}^M$ and P_{t-1}^M, the first step is to compute

[8]Note that only TS model is linearized since options are priced based on the actual, rather that the fitted, futures prices. As consequence CN option prices do not depend directly on the state variables and TS specification only on $\nu_1(t)$ and $\nu_2(t)$.

the predictions at time t for the states variables, $\hat{X}^M_{t|t-1}$, and for the variance-covariance matrix, $P_{t|t-1}$, given all the information up to time $t-1$:

$$\hat{X}^M_{t|t-1} = \Phi^M_0 + \Phi^M_X \hat{X}^M_{t-1} \tag{B.10}$$

$$P^M_{t|t-1} = \Phi_{X_M} P^M_{t-1} \Phi^{M\prime}_X + Var[\omega^M_t] \tag{B.11}$$

Then predictions on the observed variables are done and prediction or innovation errors ϵ^M_t, along with their associated variance-covariance matrix F^M_t, are calculated:

$$\hat{z}_{t|t-1} = h^M\left(\hat{X}^M_{t|t-1}\right) \tag{B.12}$$

$$\epsilon^M_t = \left(z_t - h^M\left(\hat{X}^M_{t|t-1}\right)\right) \tag{B.13}$$

$$F^M_t = H^M_t P_{t|t-1} H^{M\prime}_t + \Omega^M_t \tag{B.14}$$

This is what is known as the prediction step in the Kalman filter. Once the prediction step is conducted, it follows the update step where optimal solutions for the state variables vector and the variance-covariance matrix are calculated:

$$\hat{X}^M_t = \hat{X}^M_{t|t-1} + P^M_{t|t-1} H^{M\prime}_t {F^M_t}^{-1} \epsilon^M_t \tag{B.15}$$

$$P^M_t = P^M_{t|t-1} + P^M_{t|t-1} {H^M_t}' {F^M_t}^{-1} H^M_t P^M_{t|t-1} \tag{B.16}$$

The estimation of the model parameters, Θ^M, is obtained by maximizing the log-likelihood function of innovations:[9]

$$\log L(\Theta^M) = \frac{1}{2}\sum_t^T \log|F^M_t| - \frac{1}{2}\sum_t^T {\epsilon^M_t}' \, {F^M_t}^{-1} \epsilon^M_t \tag{B.17}$$

where T is number of observation dates and Θ^M is the vector of unknown parameters, of model M, to be estimated.

[9]For the CN model the innovations distribution is Gasusian provided by the fact that futures are log-normally distributed. The distribution for the TS innovations is not Gaussian, since its variance-covariance matrix depend on the volatility factors, $\nu_i(t)$, but is approximated by it. This is what is known as the QML method.

APPENDIX C : DETAILED PARAMETERS DESCRIPTION

a) Oil

Table C-1 show the results for the TS model applied to the entire sub samples for crude oil. Mean reverting parameters for the volatility process are highly significant for all data sets except κ_{21} that it is not statistically distinct from zero in most cases. From Equations (2.3) and (2.4) it can be seen that, if κ_{12} is close to $-\kappa_1$, the latter is somehow the mean reversion coefficient of $\nu_1(t)$ towards $\nu_2(t)$. Taking this into account, the results for oil show that $\nu_1(t)$ is highly mean reverting towards $\nu_2(t)$ ($\kappa_1 = 6.8588$ and $\kappa_{12} = -5.1711$), while the latter, considering that $\kappa_{21} \approx 0$, is relative persistent towards a level of $\frac{1}{\kappa_2}$($\kappa_2 = 0.8214$). In this way $\nu_1(t)$ it can be taken as the component of the volatility that accounts for the more transitory shocks and $\nu_2(t)$ as the one that capture the more persistent shocks to volatility, which means that meanwhile $\nu_2(t)$ affects the price of all options, $\nu_1(t)$ affects mainly the short-term ones. As expected the transitory component is more volatile than $\nu_2(t)$; $\sigma_{\nu 1}$ is about 5 and 7 times larger than $\sigma_{\nu 2}$.

One of the major features in the TS model is its ability to allow volatility factors to be partially spanned by the spot (futures) market. The empirical evidence presented by Trolle and Schwartz (2009) strongly suggests the presence of unspanned stochastic volatility on the crude oil market, and the results for the oil displayed on Table C-1 are consistent with this[10]. Oil volatility is primarily unspanned; $\rho_{15} = -0.2109$ and $\rho_{35} = 0.1095$ are the correlations between the spot price and cost of carry curve with the transitory component of volatility while $\rho_{26} = -0.3087$ and $\rho_{46} = -0.0282$ are the ones with the more persistent component.

[10]Trolle and Schwartz (2009) found very low correlations with $\rho_{15} = -0.039$, $\rho_{35} = -0.103$, $\rho_{26} = -0.131$and $\rho_{46} = -0.001$ for the 1990-2006 period.

Table C-1: TS model parameters: Oil

	Panel A *(2006-2011)*	**Panel B** *(2006-2007)*	**Panel C** *(2008-2009)*	**Panel D** *(2010-2011)*
ϕ	0.0191*	0.0086*	0.0192*	0.0074*
S_{S1}	0.2789*	0.3196*	0.2784*	0.5683*
S_{S2}	0.1225*	0.1116*	0.1220*	0.4266*
$S_{\nu1}$	8.4207*	6.0299*	7.5059*	2.4962*
$S_{\nu2}$	1.1229*	1.5478*	1.1617*	2.4935*
ρ_{13}	-0.8586*	-0.6931*	-0.8549*	-0.9484*
ρ_{15}	-0.2109*	0.0319*	-0.1828*	0.2395*
ρ_{35}	0.1095*	-0.1617*	0.1109*	-0.2818*
ρ_{24}	-0.3015*	0.3204*	-0.3206*	-0.6897*
ρ_{26}	-0.3087*	-0.5451*	-0.3130*	-0.7809*
ρ_{46}	-0.0282	-0.2350*	-0.0624	0.3434*
λ_1	0.7697	-13914	0.2598	0.3320
λ_2	1.5534*	0.0179	1.6376*	0.4064
λ_3	-1.1086*	0.0807*	-0.7319	-0.5272
λ_4	-0.0402	0.7860*	0.2533	-0.0310*
λ_5	-0.1559	0.1543	-0.1333	-16369
λ_6	0.5306*	-0.1622	0.4807*	0.4601*
α_1	0.2639*	0.2635*	0.2663*	0.5520*
α_2	0.0368*	0.0190*	0.0369*	0.1415*
γ_1	1.1167*	1.1910*	1.1147*	0.8872*
γ_2	0.2929*	0.2312*	0.2917*	0.3258*
κ_1	6.8588*	8.3363*	7.0539*	3.4196*
κ_{12}	-5.1711*	-2.0699*	-49693	-1.1486*
κ_{21}	-0.1710*	-0.2145	-0.1777	-0.0000
κ_2	0.8214*	0.9365*	0.7914*	0.9052*
σ_{fut}	0.0042*	0.0028*	0.0049*	0.0034*
σ_{op}	0.0182*	0.0097*	0.0181*	0.0134*

Maximum-likelihood estimates for all data sets. The (*) symbol indicates the significance at a 95% level of confidence. Panel A represents the full-sample period, Panels B, C and D the corresponding subsamples.

Differences between the estimates of Table C-1 and Trolle and Schwartz (2009) arise; the spot price seems to span a little more the components of volatility, nevertheless volatility is still mainly unspanned. The most likely reason for this is that in this article, different panel of data is being used and longer contracts are considered in the calibration of parameters.

Since $\nu_1(t)$ is much more volatile than $\nu_2(t)$, $\sigma_{S1} > \sigma_{S2}$, $\gamma_1 > \gamma_2$ and $\alpha_1 > \alpha_2$, most of the instantaneous volatility for the spot price and the front end of the forward cost of carry curve is driven by $\nu_1(t)$. This explains, in part, the high correlation between them ($\rho_{13} = -0.8586$). As is common in this type of analysis, standard deviations of the measurement errors, σ_{fut} and σ_{op}, are highly significant and consistent through samples.

Tables C-2, C-3 and C-4 display the results for the CN models for the oil case. For the 3 and 4-factor specifications, most parameters are significant and stables to standard considerations, all but market prices of risk, λ_i, and the long-term growth rate, μ, which is consistent with the literature (Schwartz, 1997). Although, mean reverting parameters are all significant for the 5-factor model, the correlations between the second and the rest of the factors are estimated with high standard errors. In addition, the high correlations between the third and fourth variables ($\rho_{34} = -0.9409$) along with the negligible correlation of some of the rest of the variables with them, suggest that the 5-factor model could be over specified and less factors should be considered. The standard deviation measurement error parameters, σ_{fut} and σ_{op}, are small, but highly significant, being the futures one 400 times lower than the one for options as opposed to the 4 to 1 relation in the TS specification.

b) Copper

Table C-5 displays the results for TS model on all samples for the copper case. As usual most of market prices of risk are not statistically significant in contrast with the highly significant, but low standard deviation of measurement errors σ_{fut} and σ_{op}.

Table C-2: CN 3-factor model parameters: Oil

	Panel A *(2006-2011)*	**Panel B** *(2006-2007)*	**Panel C** *(2008-2009)*	**Panel D** *(2010-2011)*
a_2	0.2079*	0.2844*	0.1720*	0.3272*
a_3	1.1755*	1.2262*	1.1208*	1.4271*
σ_1	0.2645*	0.1795*	0.2856*	0.2050*
σ_2	0.2196*	0.1642*	0.3118*	0.2147*
σ_3	0.4263*	0.2952*	0.5051*	0.1348*
ρ_{12}	-0.0552*	0.1263*	-0.4769*	0.2728*
ρ_{13}	-0.4459*	-0.2303*	0.0075	-0.0642*
ρ_{23}	-0.1896*	-0.2653*	-0.2603*	-0.0182
λ_1	0.0035	-0.0056	0.0708	-0.0235*
λ_2	0.0254	-0.1003	-0.1096	0.3473
λ_3	-0.0080	0.1724*	-0.1263*	-0.1347
μ	-0.0136	-0.0146	0.0500	-0.0260*
σ_{fut}	0.0044*	0.0028*	0.0056*	0.0037*
σ_{op}	0.1054*	0.0282*	0.1286*	0.0525*

Maximum-likelihood estimates for all data sets. The (*) symbol indicates the significance at a 95% level of confidence. Panel A represents the full-sample period, Panels B, C and D the corresponding subsamples.

Table C-3: CN 4-factor model parameters: Oil

	Panel A *(2006-2011)*	**Panel B** *(2006-2007)*	**Panel C** *(2008-2009)*	**Panel D** *(2010-2011)*
a_2	0.0950*	0.2689*	0.0354*	0.0668*
a_3	0.7155*	0.9293*	0.7043*	0.5866*
a_4	2.7221*	3.6527*	5.4243*	3.9534*
σ_1	0.2751*	0.1894*	0.4169*	0.2377*
σ_2	0.2690*	0.1771*	0.4334*	0.2946*
σ_3	0.2814*	0.2438*	0.3363*	0.5264*
σ_4	0.1838*	0.1259*	0.2497*	0.1572*
ρ_{12}	-0.5701*	-0.3730*	-0.8245*	-0.4439*
ρ_{13}	0.0897*	0.2905*	-0.2412*	-0.1376*
ρ_{14}	-0.0690*	-0.1083*	-0.1005	0.1676*
ρ_{23}	-0.1059*	-0.3050*	0.2790*	-0.4323*
ρ_{24}	0.1221*	0.1675*	0.0811	0.0662
ρ_{34}	-0.3036*	-0.3139*	0.3058*	-0.7010*
λ_1	0.0224*	-0.0107	0.0473*	-0.0100
λ_2	0.0387	-0.1207	0.3765*	0.3804
λ_3	-0.0210	0.1507*	-0.8974*	0.3475
λ_4	-0.0808	-0.1375	-0.1583	-0.3499*
μ	0.0191*	-0.0221	0.0500*	0.0044
σ_{fut}	0.0025*	0.0015*	0.0031*	0.0015*
σ_{op}	0.1051*	0.0281*	0.1283*	0.0510*

Maximum-likelihood estimates for all data sets. The (*) symbol indicates the significance at a 95% level of confidence. Panel A represents the full-sample period, Panels B, C and D the corresponding subsamples.

Table C-4: CN 5-factor model parameters: Oil

	Panel A *(2006-2011)*	**Panel B** *(2006-2007)*	**Panel C** *(2008-2009)*	**Panel D** *(2010-2011)*
a_2	0.0356*	0.2170*	0.0360*	0.0583*
a_3	0.9115*	1.3468*	0.8505*	0.8661*
a_4	1.1543*	1.9469*	1.1883*	1.5068*
a_5	8.6570*	5.1132*	9.6454*	6.1454*
σ_1	0.3986*	0.2365*	0.3624*	0.2907*
σ_2	0.4288*	0.2941*	0.3514*	0.2840*
σ_3	0.6531*	0.4346*	0.6348*	0.5375*
σ_4	0.5755*	0.3327*	0.5993*	0.3345*
σ_5	0.1840*	0.1891*	0.3130*	0.1532*
ρ_{12}	-0.8227*	-0.6579*	-0.6981*	-0.6315*
ρ_{13}	0.0474*	0.2109*	-0.0722*	-0.2594*
ρ_{14}	-0.0263*	-0.1057	0.0187	0.1360*
ρ_{15}	-0.0797*	-0.2832*	-0.1685*	0.0764
ρ_{23}	0.0080	-0.1784*	-0.0866*	0.2545*
ρ_{24}	0.0013	0.1285	0.2117*	-0.1799*
ρ_{25}	0.0664*	0.2627*	0.0128	-0.1814*
ρ_{34}	-0.9409*	-0.8800*	-0.8516*	-0.8419*
ρ_{35}	0.0832*	0.0608	0.2565*	-0.3352*
ρ_{45}	-0.0707*	-0.3746*	-0.1403*	0.1196
λ_1	-0.0250	0.0203*	0.0067	-0.0233
λ_2	-0.6813*	-0.1680	0.2351	-0.5575*
λ_3	-0.2949	0.3016	-0.6122	0.0107*
λ_4	-0.2677	-0.2322	-0.3490	-0.5176*
λ_5	-0.3195*	0.0558	-0.0368*	0.0554
μ	-0.0152	0.0024	0.0118	-0.0131
σ_{fut}	0.0012*	0.0010*	0.0014*	0.0011*
σ_{op}	0.1052*	0.0285*	0.1325*	0.0517*

Maximum-likelihood estimates for all data sets. The (*) symbol indicates the significance at a 95% level of confidence. Panel A represents the full-sample period, Panels B, C and D the corresponding subsamples.

Table C-5: TS model parameters: Copper

	Panel A *(2006-2011)*	**Panel B** *(2006-2007)*	**Panel C** *(2008-2009)*	**Panel D** *(2010-2011)*
ϕ	0.0362*	0.0535	0.0375*	0.0043*
S_{S1}	0.4957*	0.5513*	0.5224*	0.3476*
S_{S2}	0.1115*	0.2206*	0.1617*	0.0833*
$S_{\nu 1}$	2.2427*	2.8379*	2.8898*	2.1645*
$S_{\nu 2}$	2.0139*	2.5886*	2.6449*	2.1109*
ρ_{13}	-0.1767*	-0.2410*	-0.2192*	-0.1966*
ρ_{15}	-0.0734*	-0.1446*	-0.0410*	-0.0340*
ρ_{35}	0.0088	0.0704	0.1323	0.0404
ρ_{24}	-0.6121*	-0.6425*	-0.6583*	-0.8595*
ρ_{26}	-0.3523*	-0.2739*	-0.1702*	-0.4096*
ρ_{46}	0.0148	-0.0362	0.0655	-0.0305
λ_1	1.8692*	-0.0486	12752	3.6456*
λ_2	0.3325	16639	0.3117	13040
λ_3	3.1032*	14770	17472	0.3361
λ_4	-3.3923*	-3.8514*	-3.1161*	-12501
λ_5	-0.3190	0.3410	-0.5111	2.5467*
λ_6	-0.8076*	-0.6370	-0.1540	-1.5705*
α_1	0.2187*	0.2946*	0.2824*	0.1071*
α_2	0.1120*	0.1857*	0.1754*	0.0673*
γ_1	1.7400*	1.7241*	1.8192*	0.8613*
γ_2	0.1428*	0.2042*	0.3261*	0.1722*
κ_1	10.8751*	11.2664*	11.4430*	12.1396*
κ_{12}	-7.3825*	-7.1476*	-68282	-9.7343*
κ_{21}	-0.6891*	-1.0067*	-0.7793	-0.7795*
κ_2	1.6236*	2.3051*	2.6418*	1.0675*
σ_{fut}	0.0014*	0.0035*	0.0013*	0.0006*
σ_{op}	0.0229*	0.0292*	0.0215*	0.0155*

Maximum-likelihood estimates for all data sets. The (*) symbol indicates the significance at a 95% level of confidence. Panel A represents the full-sample period, Panels B, C and D the corresponding subsamples.

Even though $\kappa_1 = 10.8751$ is significantly higher than $\kappa_2 = 1.6236$, volatility mean reverting coefficients differs from the estimates for oil, mainly because the speed of adjustment for $v_1(t)$ is higher and there is not a moderately persistent component as clear as is for oil. Both volatility factors are quite mean reverting: $v_1(t)$ towards $v_2(t)$ and more slowly $v_2(t)$ towards a level of 0.62.[11] The lack of long-term contracts involved in the estimation procedure is probably the reason begin this since transitory shocks to volatility affects primarily short-term option prices while more persistent ones affects all. Having only short-term options may be overestimating the reverting coefficient of the supposed more persistent component, along with causing higher estimates for the reversion of transitory shocks. Since both components capture transitory shocks to volatility, it is not strange to have similar estimates for the variability, σ_{vi}, of them.

Supported by the fact of a larger estimate for σ_{S1}, relative to σ_{S2}, it is clear that $v_1(t)$ is the component that drive most of the instantaneous volatility of the spot price. In addition, it is the main driver of instantaneous variance of the front end of the forward cost of carry curve provided by $\alpha_1 > \alpha_2$ and the fact that the maturity-decreasing coefficient associated with the more transitory component of the volatility, γ_1, is larger than γ_2.[12] As maturity increases, the proportion of the instantaneous volatility of the forward cost of carry curve accounted by $v_1(t)$ decreases.

Although parameter calibration is not a prove per se and a non-dependent model test, like the one realized in the NBER version of the Trolle and Schwartz (2009) paper, should be done, estimates for the correlations between the innovations of the volatility and innovations of the spot price and forward cost of carry curve suggest that volatility

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[11] Actually $v_1(t)$ reverts to $\frac{1}{\kappa_1} + \left(\frac{-\kappa_{12}}{\kappa_1}\right) v_2(t)$, but considering the high estimates for κ_1 and that $\kappa_1 \approx -\kappa_{12}$, it is said that reverts towards $v_2(t)$. Since $\kappa_{21} = -0.6891$ is quite small, $v_2(t)$ reverts towards $\frac{1}{\kappa_2} + \left(-\frac{\kappa_{21}}{\kappa_2}\right) v_1(t) \approx \frac{1}{\kappa_2}$.

[12] Recalling from Equations (2.2) and (2.8), if $\gamma_1 > \gamma_2$ then $\sigma_{y1}(t,T)$ goes to zero faster than $\sigma_{y2}(t,T)$. Particularly for the copper case $\sigma_{y1}(t,T)$ vanishes ten times faster as maturity of contracts increase.

may have unspanned components for copper markets as well as for the oil case. This correlations are quite low; $\rho_{15} = -0.0734$, $\rho_{35} = 0.0088$, $\rho_{26} = -0.3523$ and $\rho_{46} = 0.0148$. Since correlations of the variables with the first volatility component are all nearly zero, the model could be simplified by assuming one of the two volatility factors to be completely unspanned without affecting the prices too much.

The results of the CN model for copper, presented on Tables C-6, C-7 and C-8 show, for all specifications, highly significant parameters for all samples except for the market prices of risk, λ_i, and the long-term growth rate μ. The standard deviations of futures prices measurement errors, σ_{fut}, are very low. However, compared to the TS estimates, the observation standard error for option contracts, σ_{op}, is relatively high. In contrast to the oil case, all CN specification seems to be well specified and stables trough data sets. For the 3-factor model, this is consistent with the results reported by Schwartz (1997) and Cortazar and Naranjo (2006). The main differences between the subsamples estimates have to do with stronger mean reverting parameters for the periods with larger volatility shocks. Copper data has two main shocks in volatility, the first one at the beginning of the 2006, and the latter for the 2008 crisis (see Figure II-2), which explains why Panel B and C show stronger mean reverting parameters than Panel D for the copper case.

Table C-6: CN 3-factor model parameters: Copper

	Panel A *(2006-2011)*	**Panel B** *(2006-2007)*	**Panel C** *(2008-2009)*	**Panel D** *(2010-2011)*
a_2	0.6283*	0.5603*	0.2706*	0.5877*
a_3	1.1480*	1.1047*	3.5537*	0.8312*
σ_1	0.3536*	0.4594*	0.3892*	0.2666*
σ_2	0.6290*	0.9989*	0.3643*	0.5404*
σ_3	0.4241*	0.6396*	0.0807*	0.4238*
ρ_{12}	-0.3093*	-0.5716*	-0.3131*	-0.0363
ρ_{13}	0.2791*	0.4782*	0.3308*	0.0553
ρ_{23}	-0.9761*	-0.9742*	-0.7197*	-0.9962*
λ_1	0.0340	0.0196	0.1042*	0.0099
λ_2	0.6000*	0.7755	-0.0216*	1.0000*
λ_3	-0.5547*	-0.8427	-0.0511	-0.8595*
μ	-0.0499	-0.0500	0.0077	-0.0500*
σ_{fut}	0.0013*	0.0019*	0.0011*	0.0006*
σ_{op}	0.1033*	0.0908*	0.1119*	0.0609*

Maximum-likelihood estimates for all data sets. The (*) symbol indicates the significance at a 95% level of confidence. Panel A represents the full-sample period, Panels B, C and D the corresponding subsamples.

Table C-7: CN 4-factor model parameters: Copper

	Panel A *(2006-2011)*	**Panel B** *(2006-2007)*	**Panel C** *(2008-2009)*	**Panel D** *(2010-2011)*
a_2	0.7500*	0.5844*	0.3850*	0.6295*
a_3	1.2673*	1.2119*	3.5038*	1.0292*
a_4	10.2706*	12.3345*	11.8035*	2.4306*
σ_1	0.3668*	0.4410*	0.3630*	0.3579*
σ_2	0.6132*	0.8784*	0.2427*	0.5938*
σ_3	0.4760*	0.5953*	0.0895*	0.5448*
σ_4	0.1051*	0.1672*	0.1225*	0.1957*
ρ_{12}	-0.3729*	-0.5870*	0.0064	-0.4393*
ρ_{13}	0.3861*	0.5561*	0.1516*	0.2270*
ρ_{14}	-0.3893*	-0.5229*	-0.0981*	0.2274*
ρ_{23}	-0.9726*	-0.9669*	-0.5351*	-0.8661*
ρ_{24}	0.4104*	0.2874*	0.0767	0.2143
ρ_{34}	-0.5172*	-0.3540*	-0.5917*	-0.6664*
λ_1	0.0410	0.0907	0.0208	0.0869*
λ_2	0.0561	0.8732	-0.0075*	0.1917
λ_3	-0.1981	-0.9169*	-0.0793	0.0746
λ_4	0.0439	0.2840	0.1476	0.1518
μ	-0.0492	0.0500	-0.0500	-0.0023
σ_{fut}	0.0010*	0.0013*	0.0007*	0.0004*
σ_{op}	0.1033*	0.0908*	0.1120*	0.0595*

Maximum-likelihood estimates for all data sets. The (*) symbol indicates the significance at a 95% level of confidence. Panel A represents the full-sample period, Panels B, C and D the corresponding subsamples.

Table C-8: CN 5-factor model parameters: Copper

	Panel A *(2006-2011)*	**Panel B** *(2006-2007)*	**Panel C** *(2008-2009)*	**Panel D** *(2010-2011)*
a_2	0.4501*	0.3829*	0.3228*	0.3069*
a_3	1.2737*	1.3026*	0.9676*	0.9825*
a_4	2.7140*	2.8489*	6.2585*	2.2704*
a_5	13.0672*	14.9944*	16.1435*	3.8863*
σ_1	0.4662*	0.4955*	0.4569*	0.3395*
σ_2	0.7159*	0.7319*	0.4978*	0.4485*
σ_3	0.6523*	0.5887*	0.2621*	0.3855*
σ_4	0.3034*	0.3071*	0.1278*	0.2613*
σ_5	0.1326*	0.1821*	0.1938*	0.1276*
ρ_{12}	-0.6893*	-0.6732*	-0.5161*	-0.5848*
ρ_{13}	0.6035*	0.4524*	0.4872*	0.5760*
ρ_{14}	-0.3904*	-0.1825*	-0.0686	-0.2666*
ρ_{15}	-0.1490*	-0.2347*	-0.1107*	-0.0411
ρ_{23}	-0.8805*	-0.7550*	-0.8327*	-0.8986*
ρ_{24}	0.5918*	0.3038*	0.2640*	0.5116*
ρ_{25}	-0.0051	0.0311	-0.0241	-0.1587
ρ_{34}	-0.8861*	-0.8139*	-0.6190*	-0.7933*
ρ_{35}	0.2256*	0.1606*	0.2210*	0.4347*
ρ_{45}	-0.4743*	-0.3179*	-0.6852*	-0.8735*
λ_1	0.1269*	0.0027	0.0380*	0.0889*
λ_2	0.0368	0.1459	0.0870	0.0447
λ_3	-0.0998	-0.2781	0.0733	0.0861
λ_4	-0.0824	-0.1865	-0.0068*	-0.2374*
λ_5	0.1416	0.2599	0.2419	0.1509
μ	0.0074	-0.0106	-0.0039	0.0108
σ_{fut}	0.0008*	0.0012*	0.0006*	0.0003*
σ_{op}	0.1032*	0.0905*	0.1118*	0.0605*

Maximum-likelihood estimates for all data sets. The (*) symbol indicates the significance at a 95% level of confidence. Panel A represents the full-sample period, Panels B, C and D the corresponding subsamples.

c) Gold

Table C-9 show, for the entire and all sub samples, the parameter estimates for the TS model applied to gold. Even when for all data sets parameters are statistically significant, except for the risk premiums, the results show to be quite unstable through sample periods indicating that the model could be over-specified. For example, mean reverting estimates for the volatility process show that for all samples $\nu_1(t)$ accounts for the transitory shocks to volatility, with relative larger estimations of κ_1 against κ_2. However, the role of $\nu_2(t)$ component as the one that capture the more persistent shocks to volatility does not seems to hold for all panels, being moderately persistent for Panel A and D, but reverting for Panel B and C.

As well as for the oil and copper case, the results suggest the presence of unspanned stochastic volatility components for the gold market; however, the magnitudes of the correlations implicated are larger than the ones obtained for oil and copper. It is important to note that the lack of stability, in the estimation of the correlations, suggests that the model is over-specified and should not be taken as a clear proof of the presence of unspanned stochastic volatility factors for gold. Nonetheless, within each panel the results obtained for gold seems to follow a similar logic of the ones obtained for oil in the sense that $\nu_1(t)$ is the more volatile component and accounts for most of the instantaneous volatility of the spot price and front end of the forward cost of carry curve.

Tables C-10, C-11 and C-12 display the estimates for the CN model applied to gold market. As was the case for the oil, higher factors specification seems to over-specify the gold dynamics.

Table C-9: TS model parameters: Gold

	Panel A *(2006-2011)*	**Panel B** *(2006-2007)*	**Panel C** *(2008-2009)*	**Panel D** *(2010-2011)*
ϕ	0.0897*	0.0823*	0.0679*	0.1007*
S_{S1}	0.1895*	0.1845*	0.1977*	0.2951*
S_{S2}	0.0592*	0.0793*	0.0620*	0.1084*
S_{v1}	4.8334*	6.2097*	5.9416*	6.0650*
S_{v2}	1.2108*	1.2177*	1.2247*	0.8362*
ρ_{13}	-0.2647*	-0.3408*	-0.2786*	-0.1847*
ρ_{15}	0.1584*	0.1725*	0.2161*	-0.0497*
ρ_{35}	0.5253*	0.5104*	0.2183*	0.5892*
ρ_{24}	-0.1284*	-0.1856*	-0.0809	0.3346*
ρ_{26}	-0.3221*	-0.0766*	0.1878*	0.1749*
ρ_{46}	-0.2698*	-0.2640*	0.1941*	0.0659
λ_1	2.8480*	3.3353*	4.0267*	40342
λ_2	1.3974*	-2.0845*	2.7694*	2.6203*
λ_3	-3.9397*	0.0320	-18984	-17666
λ_4	-2.1685*	-0.3278	0.5443	-4.1242*
λ_5	-1.1147*	-2.4592*	-4.6311*	-0.9058
λ_6	-4.0419*	-0.7736*	0.0703	-2.0516*
α_1	0.0898*	0.1529*	0.1582*	0.2336*
α_2	0.0180*	0.0548*	0.0256*	0.0278*
γ_1	0.3555*	0.6989*	0.5717*	1.0168*
γ_2	0.0955*	0.1775*	0.0760*	0.1065*
κ_1	2.8250*	5.3231*	4.8868*	5.6819*
κ_{12}	-1.8907*	-1.5291*	-38587	-1.5064*
κ_{21}	-0.0400	-1.6668*	-0.2334	-0.0401
κ_2	0.5933*	1.4470*	1.1351*	0.6574*
σ_{fut}	0.0006*	0.0005*	0.0006*	0.0011*
σ_{op}	0.0099*	0.0078*	0.0094*	0.0076*

Maximum-likelihood estimates for all data sets. The (*) symbol indicates the significance at a 95% level of confidence. Panel A represents the full-sample period, Panels B, C and D the corresponding subsamples.

Along with relatively low mean reverting coefficients, compared to oil and copper, the 4 and 5-factor specifications show several correlations not statistically significant in all samples. For example, the 5-factor strong reverting state variables are strongly correlated ρ_{45}, but the correlations of one of them with other variables are not. This indicates that one or more factors are over-specified for gold. The same analysis could be done for the 4 and even 3-factor specifications, where high correlations between the strongest reverting variables contrast with not significant correlations of the rest with them. Only the 3-factor CN model is significant for the Panel A and C provided that this sets includes the crisis period, which may induce, though weakly, the mean reversion behavior of state variables for gold.[13] In order to study if a specification with less factor is more consistent, a 2-factor model is considered; results are shown in Table C-13. The estimates are all significant and stable through panels, even in the periods that not include the 2008 crisis, indicating that a 2-factor specification is statistically more suitable to gold prices.

Even though Schwartz (1997) found that mean-reverting prices did not seem to hold for gold, our estimates of the TS and CN models are still significant at standard levels. It must be noted, however, that we do not use only futures, like Schwartz (1997), but also options in the calibration process. Maybe adding options into the calibration captures mean reversion in gold prices.

[13]This complements the hypothesis that the inclusion of option prices in the calibration process also induce mean reversion in gold prices as opposed to the results found in the literature (Schwartz, 1997; Casassus and Collin-Dufresne, 2005)

Table C-10: CN 3-factor model parameters: Gold

	Panel A *(2006-2011)*	**Panel B** *(2006-2007)*	**Panel C** *(2008-2009)*	**Panel D** *(2010-2011)*
a_2	0.1348*	0.0845*	0.1603*	0.1317*
a_3	0.2461*	0.2033*	0.6293*	0.2752*
σ_1	0.3934*	0.2585*	0.3049*	0.3844*
σ_2	0.6004*	0.4675*	0.5947*	0.6063*
σ_3	0.4817*	0.5326*	0.4181*	0.4936
ρ_{12}	-0.3451*	-0.2496	-0.3565*	-0.3602
ρ_{13}	-0.0703*	-0.1534	-0.1436	-0.0542
ρ_{23}	-0.8753*	-0.8445*	-0.7614*	-0.8757*
λ_1	-0.4145*	-0.2440	-0.2382	-0.4280*
λ_2	0.9997*	0.0981	0.6049*	0.9997
λ_3	-0.8762*	-0.1568	-0.1909	-0.8329
μ	-0.3895*	-0.2901	-0.1851	-0.3690*
σ_{fut}	0.0007*	0.0007*	0.0006*	0.0009*
σ_{op}	0.0620*	0.0454*	0.0744*	0.0606*

Maximum-likelihood estimates for all data sets. The (*) symbol indicates the significance at a 95% level of confidence. Panel A represents the full-sample period, Panels B, C and D the corresponding subsamples.

Table C-11: CN 4-factor model parameters: Gold

	Panel A *(2006-2011)*	**Panel B** *(2006-2007)*	**Panel C** *(2008-2009)*	**Panel D** *(2010-2011)*
a_2	0.2505*	0.1599*	0.3061*	0.2405*
a_3	0.5020*	0.5109*	0.6149*	0.4862*
a_4	0.8349*	1.0368*	1.1691*	0.9937*
σ_1	0.2760*	0.1849*	0.3223*	0.2954*
σ_2	0.3435*	0.2586*	0.4108*	0.3310*
σ_3	0.3907*	0.2229*	0.4821*	0.4138*
σ_4	0.1594*	0.1004	0.2483*	0.1542*
ρ_{12}	-0.2369*	-0.1182	-0.2281*	-0.2628*
ρ_{13}	-0.0586	-0.0745	-0.1213*	-0.0399
ρ_{14}	0.2055*	0.1878	0.2547*	0.1598*
ρ_{23}	-0.8623*	-0.8520*	-0.6397*	-0.8593*
ρ_{24}	0.6330*	0.5592*	0.0861	0.5679*
ρ_{34}	-0.9325*	-0.8967*	-0.8074*	-0.9008*
λ_1	0.3498*	0.0720	0.2874*	0.3255*
λ_2	0.1542*	0.1228	0.1310	0.1082
λ_3	0.0114	0.0587	-0.0053*	-0.0289
λ_4	-0.0417	-0.0230	-0.0909	-0.0799
μ	0.3574*	0.1006	0.2645*	0.3337*
σ_{fut}	0.0004*	0.0004*	0.0003*	0.0004*
σ_{op}	0.0627*	0.0444*	0.0679*	0.0602*

Maximum-likelihood estimates for all data sets. The (*) symbol indicates the significance at a 95% level of confidence. Panel A represents the full-sample period, Panels B, C and D the corresponding subsamples.

Table C-12: CN 5-factor model parameters: Gold

	Panel A *(2006-2011)*	**Panel B** *(2006-2007)*	**Panel C** *(2008-2009)*	**Panel D** *(2010-2011)*
a_2	0.1044*	0.0609*	0.3065*	0.0548*
a_3	0.4262*	0.4359*	0.7272*	0.5040*
a_4	1.0328*	2.9586*	1.7031*	1.0662*
a_5	1.3809*	5.4130*	2.5441*	1.5808*
σ_1	0.2853*	0.2458*	0.3345*	0.3342*
σ_2	0.2550*	0.2320*	0.2852*	0.2818*
σ_3	0.1756*	0.0674*	0.2380*	0.3167*
σ_4	0.2025*	0.0255*	0.1807*	0.3271*
σ_5	0.1414*	0.0282*	0.0995*	0.2186
ρ_{12}	-0.4059*	-0.5837*	-0.4427*	-0.3928*
ρ_{13}	-0.1381*	0.1661	0.2002*	-0.4763*
ρ_{14}	0.1233*	-0.1675	0.2141*	-0.0822
ρ_{15}	-0.0440	0.2782*	-0.3642*	0.3300*
ρ_{23}	-0.5265*	-0.5853*	-0.8341*	-0.0922
ρ_{24}	-0.0625	0.4106*	0.3012*	-0.1657
ρ_{25}	0.1486*	-0.3374*	-0.0929	0.0096
ρ_{34}	-0.4863*	-0.5079*	-0.7412*	-0.4179
ρ_{35}	0.2006*	0.0123	0.5267*	0.0081
ρ_{45}	-0.9418*	-0.8182*	-0.9452*	-0.8757*
λ_1	-0.0145	-0.4450*	-0.4817*	-0.0037
λ_2	0.0953	0.0869	-0.0451	0.0279*
λ_3	0.0975	0.0508	0.1947*	0.1833
λ_4	-0.0321	-0.0301	-0.1528	0.0706
λ_5	-0.0018	0.0208	0.0788	0.1603
μ	-0.0242	-0.4424*	-0.5000*	0.0583
σ_{fut}	0.0003*	0.0003*	0.0002*	0.0003*
σ_{op}	0.0627*	0.0438*	0.0676*	0.0363*

Maximum-likelihood estimates for all data sets. The (*) symbol indicates the significance at a 95% level of confidence. Panel A represents the full-sample period, Panels B, C and D the corresponding subsamples.

Table C-13: CN 2-factor model parameters: Gold

	Panel A *(2006-2011)*	**Panel B** *(2006-2007)*	**Panel C** *(2008-2009)*	**Panel D** *(2010-2011)*
a_2	0.2542*	0.1546*	0.1617*	0.0821*
σ_1	0.4048*	0.2987*	0.3982*	0.7104*
σ_2	0.2514*	0.2586*	0.1896*	0.5313*
ρ_{12}	-0.8696*	-0.7476*	-0.7515*	-0.9823*
λ_1	0.2997*	0.2862*	-0.0270	-0.2749*
λ_2	0.1476*	0.1650	0.1345	0.1140
μ	0.2777*	0.2963*	-0.0338	-0.3147*
σ_{fut}	0.0024*	0.0008*	0.0019*	0.0027*
σ_{op}	0.0619*	0.0444*	0.0676*	0.0363*

Maximum-likelihood estimates for all data sets. The (*) symbol indicates the significance at a 95% level of confidence. Panel A represents the full-sample period, Panels B, C and D the corresponding subsamples.