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CONSTANT PRINCIPAL STRAIN MAPPINGS ON 2-MANIFOLDS*

MARTIN CHUAQUI† AND JULIAN GEVIRTZ†

Abstract. We study mappings between Riemannian 2-manifolds which have constant principal stretching factors (cps-mappings). Such mappings f can be described in terms of the relationship between the geodesic curvature of the curves of principal strain at p and that of their images at $f(p)$. In the context of local coordinates this relationship takes the form of a nonlinear hyperbolic system, the blow-up properties of which depend on the Gaussian curvatures of the two manifolds. We use the theory of such systems to study global existence when both manifolds are the hyperbolic plane $\mathbb{H}^2$ and obtain a simple description of all cps-mappings of $\mathbb{H}^2$ onto itself. We also obtain a distortion result for disks in $\mathbb{H}^2$ as well as some nonexistence results for cps-mappings of the Euclidean plane onto certain classes of manifolds. In addition, our treatment of cps-mappings in $\mathbb{H}^2$ yields, virtually as a corollary, a generalization of a theorem of Epstein to the effect that a curve in hyperbolic n-space whose geodesic curvature is bounded by 1 must be simple.

Key words. constant principal strains, hyperbolic system, hyperbolic plane

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1. Introduction.

Consider a thin liquid film which upon solidification acquires a cryptocrystalline structure; that is, at each point a suitably oriented infinitesimal square of the original liquid becomes an (again, suitably oriented infinitesimal) rectangular crystal whose side lengths are constant multiples of the side length of the square. Such a process produces a deformation of the surface originally formed by the liquid, and in this paper we examine the class of deformations—those having constant principal strains—that can be realized in this manner. It turns out that the associated mappings are governed by hyperbolic systems of partial differential equations, a circumstance which in retrospect is not surprising since one would expect that singularities, in higher derivatives of the deformation, for example, propagate along the sides of the microscopic crystals, that is, along the associated curves of principal strain. This hyperbolicity in conjunction with the additional element of nonlinearity underlies most of what follows.

To give an idea of some of the relevant issues, we briefly describe the situation in the planar context (see [Ge1] for further details). Let $0 < m_1 < m_2$. A differentiable, orientation preserving mapping f of a domain $U \subset \mathbb{R}^2$ into $\mathbb{R}^2$ has constant principal stretches m_1, m_2 if there are functions $\theta, \overline{\theta}$ on U such that its Jacobian J_f satisfies

$$J_f = T(-\overline{\theta})S(m_1, m_2)T(\theta), \tag{1.1}$$

where

$$T(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad \text{and} \quad S(m_1, m_2) = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}.$$

Throughout, such f will be called (m_1, m_2)-mappings, or less specifically cps-mappings ("cps" for constant principal strain). This direct manner of expressing the

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†Facultad de Matemáticas, P. Universidad Católica de Chile, Casilla 306, Santiago 22, Chile (mchuaqui@mat.puc.cl, jgevirtz@mat.puc.cl).

condition that a mapping has constant principal stretches m_1, m_2 turns out to be rather uninformative, it being far better to work with the compatibility conditions for a matrix function to be a Jacobian; for this reason one adds the additional hypothesis that J_f be locally Lipschitz continuous on U. (See the first paragraph of section 5 for comments about this regularity assumption.) A straightforward calculation shows that a necessary and sufficient condition that locally Lipschitz functions θ and $\overline{\theta}$ give the Jacobian of an (m_1, m_2)-mapping (in a simply connected domain) via the formula (1.1) is that

$$D_1(m_1\theta - m_2\overline{\theta}) = 0 \quad \text{and} \quad D_2(m_2\theta - m_1\overline{\theta}) = 0 \tag{1.2}$$

hold almost everywhere (a.e.), where D_1 and D_2 denote differentiation in the directions $e^{i\theta}$ and $ie^{i\theta}$, respectively. These equations relate the curvatures of the curves (to be referred to henceforth as i-characteristics) along which the stretching factor is m_i and their images. Indeed, if the curvature of the former at $p \in U$ is κ_i and that of the latter at $f(p)$ is $\overline{\kappa}_i$, then (1.2) simply says that $\overline{\kappa}_i = \kappa_i/m_j$, where $\{i, j\} = \{1, 2\}$. These equations constitute a genuinely nonlinear diagonal hyperbolic system for the pair of functions θ, $\overline{\theta}$, so that, in light of a general principle established by Lax [L], one expects cps-mappings to display a marked tendency to form singularities. Specifically, the blow-up law for system (1.2) says, in the case of sufficiently differentiable mappings (and actually for all cps-mappings in the appropriate weak sense), that at each point p the derivative of κ_i in the direction of the j-characteristic through p and toward the concave side of the i-characteristic through this point is κ_i^2, from which it follows at once that the curvatures of both of the characteristics of f at p are bounded above by $1/\text{dist}(p, \partial U)$. Two immediate consequences of this are (i) a cps-analogue of Liouville's theorem—the only cps-mappings of the entire plane onto itself are affine and (ii) the compactness of the class of all (m_1, m_2)-mappings of U into $\mathbb{R}^2$ with respect to the topology of uniform convergence of the first-order derivatives on compact subsets. This blow-up principle also allows one to show that the radius of the largest concentric subdisk of the unit disk Δ whose image under all (m_1, m_2)-mappings $f : \Delta \to \mathbb{R}^2$ is convex is $\left(\frac{m_1}{m_2}\right)^2$. In fact, in conjunction with (1.2) the growth law for the κ_i plays a decisive role in the analysis of other aspects of cps-mappings and of the intimately related "principal strain line inclination function" θ (whose integral curves together with their orthogonal trajectories form what is known in plasticity and optimum structure theory—see [Hil] and [He]—as Hencky–Prandtl nets), such as boundary behavior [Ge3], [Ge4], the nature and distribution of isolated singularities [Ge3], and the determination of all cps-self-homeomorphisms of certain domains [Ge4]. A number of these properties of cps-mappings are strikingly similar to their conformal analogues.

In the present paper we examine some of these issues in the context of 2-dimensional manifolds. We begin in section 2 by establishing the counterparts of (1.2) and the blow-up law, whose formal derivations are somewhat more involved than in the planar case. In section 3 we discuss the analytic details necessary to deal with questions of global existence and behavior, and in addition analyze the relationship between cps-mappings and a generalization of Hencky–Prandtl nets in the constant Gaussian curvature context; more than anything these considerations involve appropriate rewriting of the equations derived in section 2 in coordinate form so as to make manifest the exact nature of the underlying hyperbolicity. In section 4 we apply the results of section 3 first to show that in certain situations there exist no globally defined cps-mappings and then, in the special case of the hyperbolic plane $\mathbb{H}^2$, to do the following: (i) completely describe the (wide) class of cps-mappings of $\mathbb{H}^2$ onto itself, (ii) prove a generalization of a theorem of Epstein [E1], [E2] about the curvature of self-intersecting curves in hyperbolic n-space $\mathbb{H}^n$, and (iii) derive an analogue for $\mathbb{H}^2$ of the planar radius of convexity result mentioned in the preceding paragraph.

In the planar context one could consider in addition to cps-mappings other similarly defined classes such as the one consisting of mappings with Jacobian of the form

$$J_f = T(-\overline{\theta})S(m_1(\theta,\overline{\theta}), m_2(\theta,\overline{\theta}))T(\theta)$$

for any given pair of everywhere distinct positive functions $m_1(\theta,\overline{\theta}), m_2(\theta,\overline{\theta})$ of period π in each variable (that is, mappings for which the principal strains are given functions of the directions of the principal strain lines and their images). Such a generalization is not possible in context of Riemannian 2-manifolds owing to the absence of an absolute reference direction. Indeed, since the principal stretches (and combinations of them) are the only intrinsically definable first-order parameters associated with a mapping between manifolds, in this context there are only two natural classes of mappings defined by point-independent conditions on their Jacobians: conformal mappings and (m_1, m_2)-mappings. (We are considering here only families of mappings for which, loosely speaking, the set of possible Jacobians at each point is governed by two parameters.) For this reason, cps-mappings constitute a natural object of study above and beyond their interpretation as deformations arising in certain physical situations.

2. Formal considerations. Let V and $\overline{V}$ be C^∞ Riemannian 2-manifolds, both metric tensors being denoted by $\langle\cdot,\cdot\rangle$, which we sometimes subscript with V or $\overline{V}$ for additional clarity. Let $U \subset V$ be a domain. The principal stretches (henceforth to be called principal strains in slight abuse of accepted terminology) of a mapping $f : U \to \overline{V}$ at a point $p \in V$ at which the Jacobian transformation $J_f(p)$ is nonsingular are the square roots of the eigenvalues of the transformation $J_f^*(p)J_f(p)$ of the tangent space of V at p onto itself. Let $U \subset V$ be a domain and m_1, m_2 be distinct positive constants. Then $f : U \to \overline{V}$ is an $(m_1,\, m_2)$-mapping if J_f is locally Lipschitz continuous and the principal strains of f are everywhere given by the pair $(m_1,\, m_2)$. As one can imagine from what was said above about the planar case, the direct expression of this condition as a nonlinear 2×2 system of partial differential equations in terms of local coordinate systems for V and $\overline{V}$ is not very revealing, although as we shall explain in section 3 a small amount of information can be gleaned from it. Here also it is much more appropriate to consider a derived higher order system, specifically a second-order one—which has an elegant coordinate-free formulation—in which the geometric structures of V and $\overline{V}$ present themselves in a most transparent way.

In dealing with the differential geometric aspects we shall, apart from minor variations, adhere to the notation of Hicks [Hic] . In general, the counterpart for $\overline{V}$ of any object A associated with V will be denoted by $\overline{A}$. The Lie bracket of two vector fields X_1, X_2 will be denoted as usual by $[X_1, X_2]$. It is clear that if $U \subset V$ is a simply connected domain, then $f : U \to \overline{V}$ is an $(m_1,\, m_2)$-mapping if and only if its Jacobian J_f is locally Lipschitz continuous and there exist locally Lipschitz continuous fields X_1, X_2 on U such that $\langle X_i, X_j\rangle = \delta_{ij}$ and $\langle J_f X_i, J_f X_j\rangle = m_i m_j \delta_{ij}$. The fields X_1, X_2 are *principal direction fields* for f.

The unit vector $J_f X_i/m_i$ will be denoted by $\overline{X}_i$. The covariant derivative in the direction X of the vector field Y will be denoted by $D_X Y$. In addition, D_{X_i} ($D_{\overline{X}_i}$) will be abbreviated by D_i ($\overline{D_i}$), and the same symbols $D_X\alpha$, $D_i\alpha$ will be used to denote the derivative of the scalar function α in the corresponding directions. We shall use the following facts (see [Hic]). If $f : U \to \overline{V}$ is a diffeomorphism and X, Y, and Z are vector fields on V, then

$$J_f[X,Y] = [J_f X, J_f Y], \tag{2.1}$$

$$D_X Y - D_Y X = [X,Y], \tag{2.2}$$

and

$$D_X\langle Y,Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle. \tag{2.3}$$

Furthermore, if Y is a vector field and α, β are scalar functions, then

$$D_X(\alpha Y) = (D_X\alpha)Y + \alpha D_X Y \tag{2.4}$$

and

$$D_{\alpha X+\beta Z}(Y) = \alpha D_X Y + \beta D_Z Y. \tag{2.5}$$

Let $\{X_1, X_2\}$ be an orthonormal pair of locally Lipschitz vector fields on some domain U in V. The covariant derivative $D_l X_k$ exists a.e., and the equations appearing in this paragraph hold a.e. in U. As a consequence of (2.3) we have that

$$0 = D_l\langle X_j, X_k\rangle = \langle D_l X_j, X_k\rangle + \langle X_j, D_l X_k\rangle$$

so that

$$\langle D_l X_j, X_j\rangle = 0 \quad \text{and} \quad \langle D_l X_j, X_k\rangle = -\langle D_l X_k, X_j\rangle \tag{2.6}$$

and with the convention that $\{i,j\} = \{1,2\}$, which will be in force throughout, this means that there are locally bounded measurable scalar functions κ_i such that

$$D_i X_i = \kappa_i X_j \ \text{ and } \ D_i X_j = -\kappa_i X_i. \tag{2.7}$$

At a point p at which it exists (and it does so a.e. on U), $\kappa_i(p)$ is the geodesic curvature of the integral curve through p of the field X_i. Now consider the pairs of orthonormal fields $\{X_1, X_2\}$ and $\{\overline{X}_1, \overline{X}_2\}$ associated with an (m_1, m_2)-mapping $f : U \to \overline{V}$. It follows from (2.2) and (2.7) that

$$[X_i, X_j] = D_i X_j - D_j X_i = \kappa_j X_j - \kappa_i X_i \tag{2.8}$$

so that

$$\kappa_j = \langle [X_i, X_j], X_j\rangle.$$

By (2.8) and (2.1), which may be applied since f is a local diffeomorphism,

$$\begin{aligned}
\overline{\kappa}_j &= \langle [\overline{X}_i, \overline{X}_j], \overline{X}_j\rangle = \langle [J_f X_i/m_i, J_f X_j/m_j], \overline{X_j}\,\rangle = \langle [J_f X_i, J_f X_j], \overline{X_j}\,\rangle/m_i m_j \\
&= \langle J_f[X_i, X_j], \overline{X_j}\,\rangle/m_i m_j = \langle J_f(\kappa_j X_j - \kappa_i X_i), \overline{X_j}\,\rangle/m_i m_j \\
&= \langle \kappa_j J_f X_j - \kappa_i J_f X_i, \overline{X_j}\,\rangle/m_i m_j \\
&= \langle \kappa_j m_j \overline{X}_j - \kappa_i m_i \overline{X}_i, \overline{X_j}\,\rangle/m_i m_j = \kappa_j/m_i.
\end{aligned}$$

We thus have the fundamental *curvature equations*

$$\overline{\kappa}_j = \kappa_j/m_i \ \text{ a.e. in } U, \qquad j = 1, 2. \tag{2.9}$$

We next consider how the curvatures change as we move along characteristics, and for the time being we shall assume that the mapping in question is of class C^3. (We shall explain in section 3—see Theorem 3.2—in what way this additional regularity requirement is in fact superfluous.) We use the fact that the Gaussian curvature of a 2-dimensional manifold V at a point p is given by $\langle R(X,Y)Y, X\rangle$ for all orthonormal pairs X, Y of vectors in the tangent space of V at p, where

$$R(X,Y)Y = D_X D_Y Y - D_Y D_X Y - D_{[X,Y]}Y.$$

In particular we have from (2.7) and (2.8)

$$\begin{aligned} R(X_1,X_2)X_2 &= D_1 D_2 X_2 - D_2 D_1 X_2 - D_{[X_1,X_2]}X_2 \\ &= D_1(\kappa_2 X_1) + D_2(\kappa_1 X_1) - D_{\kappa_2 X_2 - \kappa_1 X_1} X_2, \end{aligned}$$

so that upon taking into account (2.4), (2.5), and (2.7) again, we have

$$R(X_1,X_2)X_2 = \kappa_1\kappa_2 X_2 + (D_1\kappa_2)X_1 - \kappa_1\kappa_2 X_2 + (D_2\kappa_1)X_1 - \kappa_2^2 X_1 - \kappa_1^2 X_1.$$

Thus, if K and $\overline{K}$ denote Gaussian curvature on V and $\overline{V}$, we have

$$K = \langle R(X_1,X_2)X_2, X_1\rangle = D_1\kappa_2 + D_2\kappa_1 - \kappa_2^2 - \kappa_1^2 \tag{2.10}$$

and

$$\overline{K} = \langle R(\overline{X}_1,\overline{X}_2)\overline{X}_2,\overline{X}_1\rangle = \overline{D}_1\overline{\kappa}_2 + \overline{D}_2\overline{\kappa}_1 - \overline{\kappa}_2^2 - \overline{\kappa}_1^2. \tag{2.11}$$

In light of the fundamental relations (2.9) and the fact that $\overline{X}_i = J_f X_i/m_i$, it then follows that $\overline{D}_i\overline{\kappa}_j(f(p)) = (D_i\kappa_j(p))/m_i^2$, so that (2.11) may be written as

$$\overline{K} = (D_1\kappa_2)/m_1^2 + (D_2\kappa_1)/m_2^2 - \kappa_2^2/m_1^2 - \kappa_1^2/m_2^2. \tag{2.12}$$

Upon solving the linear system for $D_1\kappa_2$ and $D_2\kappa_1$ given by (2.10) and (2.12), we obtain

$$D_j\kappa_i = \kappa_i^2 + c_i, \qquad i = 1, 2, \tag{2.13}$$

where

$$c_i = m_j^2 \frac{m_i^2\overline{K} - K}{m_i^2 - m_j^2}. \tag{2.14}$$

We emphasize that when these *blow-up equations* (2.13) are written out fully in coordinate form the functions giving the mapping itself appear as arguments of $\overline{K}$, so that they do not in general characterize the net of principal strain lines in an intrinsic fashion. Although they purport to tell us something about how far along a characteristic from a given point a singularity—a point where the mapping fails to be locally Lipschitz—must lie, their content in this regard is meaningless unless one has information about K and $\overline{K}$. For this reason, the most interesting cases by far are those in which at least one of these curvatures is constant.

Given an orthonormal pair of fields X_1, X_2 on $U \subset V$ we refer to arcs of the integral curves of the field X_k as k-arcs. A domain $Q \subset U$ will be said to be a *characteristic quadrilateral* of X_1, X_2 (or of an associated cps-mapping) if ∂Q is a Jordan curve lying in D containing four points a, b, c, d occurring in that order when ∂D is traversed (in one direction or the other) and such that ab and cd are i-arcs, and bc and da are j-arcs. For such a Q we denote by Q_i^+ the i-side (i.e., ab or cd) along which X_j points toward the inside of Q. This i-side of Q will be referred to as the positive i-side. The other, negative, i-side will be denoted by Q_i^-. For an i-arc C we write

$$\Delta(C) = \int_C \kappa_i ds,$$

the unoriented arc length integral of κ_i along C. Let $U \subset V$ be simply connected, and let $f : U \to \overline{V}$ be an (m_1, m_2)-homeomorphism. For each characteristic quadrilateral $Q \subset U$ the positive sides of Q are mapped onto the positive sides of the image quadrilateral $\overline{Q}$. Because the exterior angles of a characteristic quadrilateral are all $\pi/2$, the Gauss–Bonnet formula says

$$\Delta(Q_1^+) - \Delta(Q_1^-) + \Delta(Q_2^+) - \Delta(Q_2^-) = -\int_Q K dA. \tag{2.15}$$

However, the cps-conditions and (2.9) together imply that

$$\Delta(\overline{Q}_i^\sigma) = \frac{m_i}{m_j}\Delta(Q_i^\sigma), \qquad i = 1, 2,\ \sigma = +, -,$$

so that application of the Gauss–Bonnet formula to $\overline{Q}$ gives

$$\frac{m_1}{m_2}\big(\Delta(Q_1^+) - \Delta(Q_1^-)\big) + \frac{m_2}{m_1}\big(\Delta(Q_2^+) - \Delta(Q_2^-)\big) = -m_1 m_2 \int_Q \overline{K}(f) dA. \tag{2.16}$$

Upon solving the system (2.15), (2.16) for the $\Delta(Q_i^+) - \Delta(Q_i^-)$, we obtain

$$\Delta(Q_i^+) - \Delta(Q_i^-) = -\int_Q c_i dA \tag{2.17}$$

for every closed characteristic quadrilateral $Q \subset U$, where c_i is given in (2.14). Although we have only shown that (2.17) holds for quadrilaterals on whose closure f is one-to-one, these equations can easily be seen to hold for any characteristic quadrilateral by the standard process of breaking them up into smaller quadrilaterals. We note that in light of (2.15) and the fact that $c_1 + c_2 = K$ the validity of (2.17) with one value of i implies it with the other one.

In the planar context a Hencky–Prandtl (HP) net on a simply connected domain D consists of two mutually orthogonal one-parameter families of curves covering D with the property that for any two fixed curves C_1, C_2 belonging to one of the families, the change in the inclination of the tangent is the same along all subarcs of curves of the other family which join a point of C_1 to a point of C_2. For simply connected domains, an orthogonal pair of curve families is an HP net if and only if it is the net of principal strain lines of a cps-mapping. This gives an intrinsic characterization of principal strain lines that, unlike one based (2.13), does not make reference to third-order derivatives. In order to obtain such an intrinsic characterization in the nonplanar context, one needs to assume that the curvature $\overline{K}$ of the image manifold $\overline{V}$ is constant, and in order to avoid a clumsy formulation as well as to preserve the symmetry of the discussion, we shall assume that the curvature K of V is constant as well. Thus for such a V we will say that two mutually orthogonal locally Lipschitz unit vector fields X_1, X_2 on a simply connected domain $U \subset V$ are an $(m_1, m_2, \overline{K})$-HP pair if either of the equations

$$\Delta(Q_i^+) - \Delta(Q_i^-) = -c_i A(Q),$$

where $A(Q)$ is the area of Q, is satisfied for all relevant quadrilaterals; here, of course, the curvatures κ_i are defined by the first equation in (2.7). Thus we have derived the following.

THEOREM 2.1. *If V and $\overline{V}$ have constant Gaussian curvature K and $\overline{K}$, respectively, and $f : V \to \overline{V}$ is an (m_1, m_2)-mapping, then the corresponding principal fields X_1, X_2 are an $(m_1, m_2, \overline{K})$-HP pair.*

In the next section (see Theorem 3.4) we show that, conversely, given an $(m_1, m_2, \overline{K})$-HP pair on a simply connected domain U in V, there is an (m_1, m_2)-mapping f of U into a manifold $\overline{V}$ with constant Gaussian curvature $\overline{K}$, and that this mapping is unique up to rigid motions in $\overline{V}$.

3. Analytic considerations. In investigating cps-mappings two fundamental directions are to be pursued. On the one hand, one would like to say something about the global behavior of all possible cps-mappings of a given domain, that is, to develop some elements of a distortion theory for such mappings. This aspect of the theory is to be based on the three fundamental relations derived in the preceding section: the curvature equations, the blow-up equations, and the HP property, and an example will be discussed in section 4. On the other hand, one should also be able to manufacture such mappings, that is, to construct solutions to the corresponding differential equations, and this is the point we address in this section.

The most straightforward approach is that of DeTurck and Yang [DY] in which one considers the differential equations which state that the eigenvalues of the transformation $J_f^*(p)J_f(p)$ (of the tangent space at p onto itself) are the m_i^2. Specifically, we consider coordinates (u_1, u_2) and $(\overline{u}_1, \overline{u}_2)$ for neighborhoods $U, \overline{U}$ in $V, \overline{V}$, respectively. For convenience we further assume that $U = \{(u_1, u_2) \,\big|\, |u_1|, |u_2| < \epsilon\}$. In terms of these coordinate systems let $(f_1, f_2) = f : U \to \overline{U}$ be an (m_1, m_2)-mapping for which the length change produced by f on the arc corresponding to $u_2 = 0$ is everywhere strictly between m_1 and m_2. DeTurck and Yang showed that there are four pairs of real-analytic functions F_k^σ, $1 \le \sigma \le 4$, $k = 1, 2$, of twelve variables such that for one of the four values of σ,

$$\frac{\partial f_k}{\partial u_2}(u) = F_k^\sigma \left(\frac{\partial f}{\partial u_1}, m_1, m_2, G(u), \overline{G}(f(u)) \right), \qquad k = 1, 2,$$

where each of G and $\overline{G}$ stands for the four elements of the metric tensors of V and $\overline{V}$ evaluated as indicated. Each of these systems makes the required statement about the eigenvalues of $J_f^*(p)J_f(p)$, and that there are four of them is simply a reflection of the fact that for any given m strictly between m_1 and m_2, and any nonzero $e \in \mathbb{R}^2$, there are four distinct linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ with principal stretches m_1, m_2 for which $Te = me$ (two orientation preserving and two orientation reversing). Conversely, in the analytic category the Cauchy–Kowalewski theorem implies that for each of these four systems the initial value problem $f(u_1, 0) = f_0(u_1)$ has a unique local solution provided that along the curve $u_2 = 0$ the given initial mapping f_0 changes arc length by factors lying strictly between m_1 and m_2. DeTurck and Yang made the additional very important observation that the linearizations of these four systems are diagonal hyperbolic, and this allowed them to deduce local existence in the C^∞ category. (Their work is actually considerably more general in that it deals with mappings with distinct principal strains on manifolds of arbitrary dimension.) In [Ge2] we dubbed the four Cauchy problems collectively the *DeTurck–Yang initial value problem*, a term we shall employ in what follows to refer to any one of them.

This approach to the construction of cps-mappings as solutions to first-order systems, however, throws no light on global existence because it reveals nothing about how, where, or why singularities form. Information of this nature is, on the other hand, implicit in the blow-up equations and can be put to use by basing the construction of cps-mappings either directly on them or, better still, on the analytically simpler system of curvature equations. We pursue this latter option, but because there are only two distinct characteristics and we are interested in working with the absolutely minimal condition of locally Lipschitz continuity of J_f, we do so via the method of characteristic coordinates. We begin by deriving the necessary equations.

Let U be a (small) neighborhood in V and let (u_1, u_2) be local coordinates for U; in what follows we freely identify points $p \in U$ with the corresponding $(u_1, u_2) \in \mathbb{R}^2$. We denote by $e_k = e_k(p)$ the Euclidean unit vectors at $p \in U$. A right-hand orthonormal pair (with respect to the metric of V) of vectors X_1, X_2 at $u \in U$ is completely specified by the inclination θ of X_1 to the positive u_1-axis. In other words, there are functions $\alpha_k^{(i)}(u, \theta)$ such that in terms of θ

$$X_i = \sum_{k=1}^{2} \alpha_k^{(i)}(u, \theta) e_k = F_i(u, \theta). \tag{3.1}$$

If we are dealing with a real-analytic manifold, then the $\alpha_k^{(i)}$ are, of course, real-analytic. In the discussion to follow, β will denote specific but not explicitly calculated (vector- or scalar-valued) functions of arguments to be indicated; these functions will easily be seen to be real-analytic when we are in that category and to be independent of the particular fields X_1, X_2. It is to be borne in mind that the functions denoted by this symbol may change from line to line and that the symbol D_i ($\overline{D}_i$) is used to denote both differentiation of scalar functions and covariant differentiation of vector fields in the direction X_i ($\overline{X}_i$). In the calculations to follow we use covariant differentiation rules (2.4) and (2.5). We have

$$\begin{aligned} D_i X_i = D_i \left(\sum_{k=1}^{2} \alpha_k^{(i)}(u, \theta) e_k \right) &= \sum_{k=1}^{2} (D_i \alpha_k^{(i)}(u, \theta)) e_k + \beta(u, \theta) \\ &= (D_i \theta\,) \sum_{k=1}^{2} \frac{\partial \alpha_k^{(i)}(u, \theta)}{\partial \theta} e_k + \beta(u, \theta). \end{aligned}$$

Since $\kappa_i = \langle D_i X_i, X_j \rangle$, it follows that

$$\kappa_i = \left\langle \sum_{k=1}^{2} \frac{\partial \alpha_k^{(i)}}{\partial \theta} e_k, X_j \right\rangle D_i \theta + \beta(u, \theta) = P_i(u, \theta) D_i \theta + \beta(u, \theta), \tag{3.2}$$

where $P_i(u, \theta) = \langle \frac{\partial X_i}{\partial \theta}, X_j \rangle$. Since $\frac{\partial \langle X_i, X_j \rangle}{\partial \theta} = 0$, it follows that

$$P_j(u, \theta) = -P_i(u, \theta). \tag{3.3}$$

Because X_1 is of the form $\beta(u,\theta)(\cos\theta e_1 + \sin\theta e_2)$,

$$P_1 = \left\langle \frac{\partial X_1}{\partial\theta}, X_2 \right\rangle = \left\langle \frac{\partial\beta}{\partial\theta}(u,\theta)(\cos\theta e_1 + \sin\theta e_2) + \beta(u,\theta)(-\sin\theta e_1 + \cos\theta e_2), X_2 \right\rangle$$
$$= \beta(u,\theta)\langle -\sin\theta e_1 + \cos\theta e_2, X_2\rangle \neq 0,$$

since $-\sin\theta e_1 + \cos\theta e_2$ is not a multiple of X_1. Thus, in light of (3.3) we have

$$D_i\theta = R_i(u,\theta)\kappa_i + S_i(u,\theta), \tag{3.4}$$

where $R_i(p,\theta)$ and $S_i(p,\theta)$ are functions which for given V depend only on the arguments $p \in V$ and θ.

Let X_1, X_2 be an orthonormal pair of Lipschitz continuous fields on $U \subset V$ and let $S_\epsilon = \{(t_1,t_2) : -\epsilon < t_1, t_2 < \epsilon\}$. A bi-Lipschitz homeomorphism $u : S_\epsilon \to U$ is a characteristic coordinate mapping if each segment $t_i =$ constant is carried onto a j-characteristic. The Lipschitz continuity of the X_i imply that such mappings exist locally. With reference to such a mapping, in what follows Y_i will denote the tangent field $J_u e_i$, where the e_i are the Euclidean unit vector fields on S_ϵ; more concretely, $(D_{Y_i}w)(u(t_1,t_2)) = \partial w(u(t_1,t_2))/\partial t_i$ for scalar functions w. Obviously, $[Y_i, Y_j] = 0$. Furthermore, we define $y_i(t_1,t_2)$ by

$$Y_i(u(t_1,t_2)) = y_i(t_1,t_2)X_i(u(t_1,t_2)) = y_i(t_1,t_2)F_i(u(t_1,t_2),\theta(u(t_1,t_2))), \tag{3.5}$$

where F_i is the vector-valued function appearing in (3.1). Note that Y_i and y_i only exist a.e. on $u(S_\epsilon)$ and S_ϵ, respectively.

Assuming for the moment that u has enough regularity for the calculations to make sense, we have from the rules (2.4) and (2.5) of covariant differentiation together with (2.7) that

$$D_{Y_j}Y_i = (D_{Y_j}y_i)X_i - \kappa_j y_i y_j X_j \tag{3.6a}$$

and by symmetry that

$$D_{Y_i}Y_j = (D_{Y_i}y_j)X_j - \kappa_i y_j y_i X_i. \tag{3.6b}$$

(In these formulas $y_k = y_k(u^{-1}(p))$.)

Rule (2.2) and the fact $[Y_i, Y_j] = 0$ imply equality of the right-hand sides of (3.6a) and (3.6b) from which it follows that

$$\frac{\partial y_i}{\partial t_j} = -\kappa_i\, y_i y_j. \tag{3.7}$$

For pairs of functions $\eta = (\eta_1,\eta_2)$, $y = (y_1,y_2)$ we define

$$I_1(\eta,y) = \int_0^{t_2} \eta_1(t_1,t)y_1(t_1,t)y_2(t_1,t)dt, \tag{3.8a}$$

$$I_2(\eta,y) = \int_0^{t_1} \eta_2(t,t_2)y_1(t,t_2)y_2(t,t_2)dt. \tag{3.8b}$$

We need the following lemma which says in what sense (3.7) holds in general.

LEMMA 3.1. *For almost all $t_1 \in (-\epsilon, \epsilon)$, y_1 as a function of t_2 satisfies*

$$y_1(t_1, t_2) = y_1(t_1, 0) - I_1(\kappa, y) \tag{3.9}$$

for almost all $t_2 \in (-\epsilon, \epsilon)$ and analogously for y_2. (Here,$\kappa_i = \kappa_i(u(t_1, t_2))$.)

Proof. It is enough to show that this is the case for sufficiently small ϵ, since one can then patch together small squares to conclude that it is so in the original square. If u is a characteristic coordinate mapping, then so is $v(t_1, t_2) = u(f_1(t_1), f_2(t_2))$ for any pair of bi-Lipschitz functions f_1, f_2. If w_k is the counterpart of y_k for v, then

$$w_k(t_1, t_2) = y_k(f_1(t_1), f_2(t_2))f_k'(t_k),$$

from which one sees that it is sufficient to prove the statement in the case that y_1 and y_2 are identically 1 on the lines $t_2 = 0$ and $t_1 = 0$, respectively. By working with a sequence of smooth approximations to the θ which gives X_1, X_2, we can approximate the pair X_1, X_2 by sequences $X_1^{(n)}, X_2^{(n)}$of orthonormal C^∞ fields which converge uniformly to the X_i in a neighborhood U of the closure of $u(S_\epsilon)$, for which the corresponding curvatures $\kappa_i^{(n)}$ are uniformly bounded and converge to the κ_i in $L^1(S_\epsilon)$, and such that $X_i^{(n)}(u(0,0)) = X_i(u(0,0))$. We consider the corresponding characteristic coordinate mappings $u^{(n)}$ with corresponding $Y_i^{(n)}$ and $y_i^{(n)}$, where $y_i^{(n)}$ is identically 1 on the line $t_j = 0$. Since the $y_i^{(n)}$ are smooth they satisfy (3.7) and consequently

$$y_i^{(n)} = 1 - I_i(\kappa_i^{(n)}, y^{(n)}), \qquad i = 1, 2. \tag{3.10}$$

Clearly, $u^{(n)} \to u$ uniformly on S_ϵ. Since the fields $X_1^{(n)}, X_2^{(n)}$ are uniformly Lipschitz continuous it follows from elementary facts about the continuous dependence of solutions of ordinary differential equations on the initial conditions (see [Hille, Theorem 3.1.1, p. 76]) that the $u^{(n)}$ are also uniformly Lipschitz continuous, so that the $y_i^{(n)}$ are uniformly bounded on S_ϵ. Since $y_i^{(n)}$ is identically 1 on the line $t_j = 0$, (3.10) implies that for sufficiently small ϵ

$$0.9 < |Y_i^{(n)}(u(t_1, t_2))|_V < 1.1$$

on S_ϵ for all n, so that by reducing ϵ, if necessary, we may assume that the $u^{(n)}$ are uniformly bi-Lipschitz on S_ϵ. From this it follows that $\kappa_i{}^{(n)}(u^{(n)}(t_1, t_2))$ tends to $\kappa_i(u(t_1, t_2))$ in $L^1(S_\epsilon)$. For sufficiently small $\epsilon > 0$, the system made up of (3.9) and its counterpart for y_2 can easily be seen to have a unique solution in $L^\infty(S_\epsilon)$. Indeed, this solution is the L^∞ limit of the sequence generated by the iteration

$$y_0 = (1, 1),\ y_{n+1} = (1, 1) - (I_1(\kappa, y_n), I_2(\kappa, y_n)). \tag{3.11}$$

Using this we can easily estimate $\|y - z\|_{L^1} = \|y_1 - z_1\|_{L^1} + \|y_2 - z_2\|_{L^1}$, where y and z are the solutions corresponding to kernels κ and η, respectively. Let M be an upper bound for the L^∞ norms of the components of κ and η. It follows immediately from (3.11) that for appropriately small $\epsilon > 0$ the L^∞ norms of the components of the y_n and z_n are all at most 2. We have

$$\|y_{1,n+1} - z_{1,n+1}\|_{L^1} = \int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}\int_0^{t_2} |\kappa_1 y_{1,n} y_{2,n} - \eta_1 z_{1,n} z_{2,n}| d\tau dt_1 dt_2,$$

where all the functions in the integrands are evaluated at (t_1,τ). Thus,

$$\|y_{1,n+1}-z_{1,n+1}\|_{L^1}\le\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}|\kappa_1y_{1,n}y_{2,n}-\eta_1z_{1,n}z_{2,n}|d\tau dt_1dt_2$$

$$\le 2\epsilon\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}|\kappa_1y_{1,n}y_{2,n}-\eta_1z_{1,n}z_{2,n}|d\tau dt_1$$

$$\le 8\epsilon\|\kappa-\eta\|_{L^1}+2\epsilon M\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}|y_{1,n}y_{2,n}-z_{1,n}z_{2,n}|d\tau dt_1$$

$$\le 8\epsilon\|\kappa-\eta\|_{L^1}+4\epsilon M\|y_n-z_n\|_{L^1}.$$

Obviously, the same bound holds for $\|y_2-z_2\|_{L^1}$, so that

$$\|y_{n+1}-z_{n+1}\|_{L^1}\le 16\epsilon\|\kappa-\eta\|_{L^1}+8\epsilon M\|y_n-z_n\|_{L^1}.$$

Since $y_0=z_0=(1,1)$, it follows from this that

$$\|y_{n+1}-z_{n+1}\|_{L^1}\le 16\epsilon\|\kappa-\eta\|_{L^1}/(1-8\epsilon M),$$

so that

$$\|y-z\|_{L^1}\le 16\epsilon\|\kappa-\eta\|_{L^1}/(1-8\epsilon M). \tag{3.12}$$

Because, as we have explained, $\kappa^{(n)}=\kappa^{(n)}(u^{(n)}(t_1,t_2))$ tends to $\kappa=\kappa(u(t_1,t_2))$ in $L^1(S_\epsilon)$, it follows from (3.12) that $y^{(n)}$ tends in the $L^1(S_\epsilon)$ norm to the (unique) solution $\overline{y}$ in $L^\infty(S_\epsilon)$ of the system (3.9) with the original κ_i's. But then, by replacing the $\kappa^{(n)}$ by an appropriate subsequence, we can assume that for almost all fixed $T\in(-\epsilon,\epsilon)$, $y^{(n)}(T,t_2)$ and $\kappa^{(n)}(T,t_2)$ converge to $\overline{y}(T,t_2)$ and $\kappa(T,t_2)$, respectively, in $L^1(-\epsilon,\epsilon)$. Thus, for such T it follows from (3.8a) and (3.10) that

$$\overline{y}_1(T,t_2)=1-\int_0^{t_2}\kappa_1(T,t)\overline{y}_1(T,t)\overline{y}_2(T,t)dt,$$

for almost all $t_2\in(-\epsilon,\epsilon)$ and analogously for $\overline{y}_2$. Finally, we must show that these $\overline{y}_i$ are our original y_i, defined by $Y_i=y_iX_i$. In other words, we have to show that the $y_i^{(n)}$ converge to the y_i. As we have seen, $u^{(n)}\to u$ and $X_k^{(n)}(u^{(n)}(t_1,t_2))\to X_k(u(t_1,t_2))$ uniformly on S_ϵ, so that if we denote by $\theta^{(n)}$ the θ corresponding to $X_1^{(n)}$, $\theta^{(n)}(u^{(n)}(t_1,t_2))$ converges uniformly to $\theta(u(t_1,t_2))$ on S_ϵ. We have by (3.5)

$$u^{(n)}(t_1,b)-u^{(n)}(t_1,a)=\int_a^b y_2^{(n)}(t_1,\tau)F_2(u^{(n)}(t_1,\tau),\theta^{(n)}(u^{(n)}(t_1,\tau)))d\tau.$$

But, as we saw, on almost all of the lines $t_1=T$, $y^{(n)}(T,t_2)$ tends to $\overline{y}(T,t_2)$ in $L^1(-\epsilon,\epsilon)$, so that for such T we have by letting $n\to\infty$ that

$$\int_a^b y_2(T,\tau)F_2(u(T,\tau),\theta(u(T,\tau)))d\tau=u(T,b)-u(T,a)$$

$$=\int_a^b\overline{y}_2(T,\tau)F_2(u(T,\tau),\theta(u(T,\tau)))d\tau,$$

from which we conclude that $\overline{y}_2(T, t_2) = y_2(T, t_2)$ for almost all $t_2 \in (-\epsilon, \epsilon)$ and analogously for y_1. This yields the desired conclusion. □

Let U and $\overline{U}$ be (small) neighborhoods in V and $\overline{V}$, and let $f : U \to \overline{U}$ be an (m_1, m_2)-mapping. Let (u_1, u_2) and $(\overline{u}_1, \overline{u}_2)$ be corresponding local coordinates, so that f is given by $\overline{u} = f(u) = (f_1(u), f_2(u))$. We consider a characteristic coordinate mapping u of S_ϵ into U for the pair X_1, X_2 of principal direction fields. Obviously, $f \circ u$ is a characteristic coordinate mapping for the pair $\overline{X}_1, \overline{X}_2$. Without loss of generality we can assume that the y_k as well as the corresponding $\overline{y}_k$ for $f \circ u$ are all positive. Clearly, $\overline{y}_k = m_k y_k$. Let θ and $\overline{\theta}$ be the inclination functions for these pairs of fields. We derive equations satisfied by the ten functions

$$u_k,\ y_k, \lambda_k = \kappa_k y_k,\ \overline{u}_k, \theta, \overline{\theta} \tag{3.13}$$

of (t_1, t_2), $k = 1, 2$. Note that by the curvature (2.9) the counterpart $\overline{\lambda}_k = \overline{\kappa}_k \overline{y}_k$ of λ_k is equal to $m_i \lambda_i / m_j$. In what follows, when we say that $\partial w / \partial t_i = w'$ for some functions w, w' defined a.e. on S_ϵ we mean that there is a function v equal to w a.e. on S_ϵ such that for almost all $T \in (-\epsilon, \epsilon)$, v is absolutely continuous on the line $t_j = T$ and $\partial v / \partial t_i = w'$ holds in the strict sense a.e. on it. In particular, the preceding lemma says that (3.7) holds in this sense.

Consider a rectangle $\alpha_k \le t_k \le \beta_k, k = 1, 2$, in S_ϵ. Then since the arc length element $ds = y_1 dt_1$ (a.e. along 1-characteristics) and $dA = y_1 y_2 dt_1 dt_2$, (2.17) says that

$$\int_{\alpha_1}^{\beta_1} \kappa_1(t_1, \alpha_2) y_1(t_1, \alpha_2) dt_1 - \int_{\alpha_1}^{\beta_1} \kappa_1(t_1, \beta_2) y_1(t_1, \beta_2) dt_1 = -\int_{\alpha_2}^{\beta_2} \int_{\alpha_1}^{\beta_1} c_1 y_1 y_2 dt_1 dt_2$$

and

$$\int_{\alpha_2}^{\beta_2} \kappa_2(\alpha_1, t_2) y_2(\alpha_1, t_2) dt_2 - \int_{\alpha_2}^{\beta_2} \kappa_2(\beta_1, t_2) y_2(\beta_1, t_2) dt_2 = -\int_{\alpha_2}^{\beta_2} \int_{\alpha_1}^{\beta_1} c_2 y_1 y_2 dt_1 dt_2,$$

where

$$c_i(t_1, t_2) = c_i(u, \overline{u}) = m_j^2 \frac{m_i^2 \overline{K}(\overline{u}) - K(u)}{m_i^2 - m_j^2}.$$

Thus, the following equations hold a.e. on S_ϵ:

$$\lambda_1(t_1, t_2) = \lambda_1(t_1, 0) + \int_0^{t_2} c_1(t_1, \tau) y_1(t_1, \tau) y_2(t_1, \tau) d\tau$$

and

$$\lambda_2(t_1, t_2) = \lambda_2(0, t_2) + \int_0^{t_1} c_2(\tau, t_2) y_1(\tau, t_2) y_2(\tau, t_2) d\tau$$

or in derivative form

$$\frac{\partial \lambda_i}{\partial t_j} = c_i y_1 y_2. \tag{3.14}$$

We also have that

$$\frac{\partial u}{\partial t_1} = y_1 F_1(u, \theta) \tag{3.15}$$

and since $\overline{y}_1 = m_1 y_1$

$$\frac{\partial \overline{u}}{\partial t_1} = m_1 y_1 \overline{F}_1(\overline{u}, \overline{\theta}), \tag{3.16}$$

where $F_i(u, \theta)$ is defined in (3.1).

As an immediate consequence of Lemma 3.1 we also have

$$\frac{\partial y_i}{\partial t_j} = -\lambda_i y_j \tag{3.17}$$

(in the sense explained above, of course). Finally, in light of (3.4), we have $D_1\theta = R_1(u,\theta)\kappa_1 + S_1(u,\theta)$, so that since $D_1\theta = \frac{\partial\theta}{\partial t_1}/y_1$ we conclude

$$\frac{\partial \theta}{\partial t_1} = \lambda_1 R_1(u,\theta) + y_1 S_1(u,\theta), \tag{3.18}$$

and analogously, using the fact that $\overline{\lambda}_k = m_i\lambda_i/m_j$,

$$\frac{\partial \overline{\theta}}{\partial t_1} = \frac{m_i\lambda_i}{m_j}\overline{R}_1(\overline{u},\overline{\theta}) + \overline{y}_1\overline{S}_1(\overline{u},\overline{\theta}). \tag{3.19}$$

We are now able to analyze the sense in which the blow-up equations are satisfied for cps-mappings which are not necessarily C^3. (The argument to follow contains an alternate derivation of these equations based on the Gauss–Bonnet formula.) Let w be a finite valued measurable function on an open set $D \subset \mathbb{R}^2$. Then for almost all $p \in D$ it is true that for all $\eta > 0$

$$\frac{1}{\pi\delta^2}\lim_{\delta\to 0} A(\{\xi : |w(\xi) - w(p)| > \eta\} \cap \Delta(p,\delta)) = 0, \tag{3.20}$$

where A denotes 2-dimensional measure, and $\Delta(p,\delta)$ is the disk of radius δ about p. A point p for which (3.20) holds will be called a *point of approximate continuity* of w. For an orthonormal pair X_1, X_2 of Lipschitz continuous fields on U we denote by $E_i = E_i(X_1, X_2)$ the image under u of the set of points of approximate continuity of $\kappa_i \circ u$, and it is immediate that this definition is independent of the coordinate system used. It is easy to see that if $\kappa = \kappa_i$ a.e. in U and p is a point of approximate continuity of κ, then $\kappa_i(p)$ exists and is equal to $\kappa(p)$.

Theorem 3.2. *Let $f : U \to \overline{U}$ be an (m_1, m_2)-mapping. Then for almost all $p \in U$, κ_i (as defined by (2.7)) exists on the entire j-characteristic C through p, and the restriction of κ_i to C is differentiable and satisfies the blow-up equation $D_j\kappa_i = \kappa_i^2 + c_i$ along it, where D_j is to be interpreted as arc length differentiation along C.*

Proof. It is clearly enough to establish the conclusion in $u(S_\epsilon)$ for any characteristic coordinate mapping u. For convenience let $i = 1$. There is a set $B \subset (-\epsilon, \epsilon)$ of measure 2ϵ and functions κ and y which coincide with $\kappa_1 \circ u$ and y_1 a.e. on each line $t_1 = T \in B$ and are such that y and $\lambda = \kappa y$ are absolutely continuous on each of these lines and satisfy $\frac{\partial\lambda}{\partial t_2} = c_1 y y_2$ and $\frac{\partial y}{\partial t_2} = -\lambda y_2$ in the strict sense a.e. on them. We can assume in addition that for all $T \in B$ almost all points of the 2-arc C_T corresponding to $t_1 = T$ are in E_1. Then at all points (T, t_2) at which the equations are satisfied, we have

$$\frac{\partial\kappa}{\partial t_2} = \frac{\partial(\lambda/y)}{\partial t_2} = \frac{y^2 c_1 y_2 + \lambda^2 y_2}{y^2} = (c_1 + \kappa^2)y_2,$$

or, in other words,

$$D_2(\kappa \circ u^{-1}) = c_1 + (\kappa \circ u^{-1})^2, \tag{3.21}$$

where D_2 is interpreted as arc length differentiation. Since κ is absolutely continuous (3.21) holds everywhere on C_T. It follows easily from this and the fact that almost all points of C_T are of points of approximate continuity of $\kappa \circ u^{-1}$ that in fact *all* points of C_T are points of approximate continuity of $\kappa \circ u^{-1}$(since the same equation holds on almost all nearby 2-characteristics). But then from the comment contained in the last sentence immediately preceding the statement of the theorem we conclude that (3.21) holds everywhere on C_T with $\kappa \circ u^{-1}$ replaced with κ_i, as desired. ☐

Theorem 3.2 has the following important corollary.

COROLLARY (COMPACTNESS PRINCIPLE). *Let U be a domain in V and let $P \subset \overline{V}$ be compact. Then the class of all (m_1, m_2)-mappings of U into $\overline{V}$ for which $f(U) \subset P$ is compact in the topology of uniform convergence of first derivatives on compact sets.*

Proof. It is enough to see that any $p \in V$ and $\overline{p} \in \overline{V}$ have (small) coordinate neighborhoods U_1 and $\overline{U}_1$ such that the (m_1, m_2)-mappings $f : U \to P$ for which $f(U_1) \subset \overline{U}_1$ have, when expressed in coordinate form, uniformly Lipschitz first derivatives on U_1. For sufficiently small U_1 Theorem 3.2 implies that κ_1 and κ_2 must be uniformly bounded and the curvature equations then say that the same must be true for $\overline{\kappa}_1$ and $\overline{\kappa}_2$. But then (3.4) and its counterpart for the $\overline{\kappa}_k$ and $\overline{\theta}$ imply that the first derivatives of θ and $\overline{\theta}$ are uniformly bounded on U_1 and $\overline{U}_1$ and in light of (3.1) and the fact that the Jacobian of f is completely determined by the X_k and $\overline{X}_k$ it follows that the first derivatives of the $f \in \mathcal{C}$ are indeed uniformly Lipschitz. ☐

We now examine the DeTurck–Yang initial value problem from the point of view of (3.14)–(3.19). Let C be a curve in U with Lipschitz continuous unit tangent and let $(g_1, g_2) = g : C \to \overline{U}$ have locally Lipschitz continuous derivative. We assume that the factor by which g changes arc length (when calculated with respect to the metrics in U and $\overline{U}$) is everywhere strictly between m_1 and m_2. We want to find the (m_1, m_2)-mappings of a neighborhood of C onto a neighborhood of $g(C)$ which coincide with g on C. We limit consideration to mappings which are orientation preserving with respect to the coordinate systems u and $\overline{u}$; trivial modifications cover the orientation-reversing mappings. Let T be a unit tangent field to C and let $\overline{T}$ be the corresponding unit tangent field $J_g T/|J_g T|$ to $\overline{C} = g(C)$. Let X_1, X_2 and $\overline{X}_1, \overline{X}_2$ be the fields associated with an (m_1, m_2)-extension f of g. Let ϕ denote the angle, calculated with respect to the metric of V, between X_1 and T; without loss of generality we can assume that $0 < \phi < \pi$. Let $\overline{\phi} \in (0, \pi)$ be the angle between $\overline{X}_1$ and $\overline{T}$. Then

$$m_1^2 \cos^2 \phi + m_2^2 \sin^2 \phi = |J_g T|_{\overline{V}} \quad \text{and} \quad \tan \overline{\phi} = \frac{m_2}{m_1} \tan \phi, \tag{3.22}$$

so that there are two possible choices for continuous X_1 along C, that is, two possibilities for θ corresponding to an (m_1, m_2)-mapping of a neighborhood of C onto a neighborhood of $g(C)$ and coinciding with g on C. The second equation in (3.22) means that $\overline{X}_1$ (i.e., $\overline{\theta}$) is determined once one of these θ is selected. It follows from the first of these equations that θ is a Lipschitz continuous function of arc length along C, and then from the second equation that $\overline{\theta}$ is also.

In order to proceed with the present discussion as well as to carry out some of the derivations in section 4 it is necessary to examine the relationship between the curvature of the curve C, that of its image under the (m_1, m_2)-mapping f, and the values along C of the κ_i associated with f, which by Theorem 3.2 exist a.e. on C. For the moment we assume that J_f is differentiable (as a function of two variables) at almost all points of C. The following calculation will be valid a.e. on C. By reversing the direction of some of the vectors $X_1, X_2, \overline{X}_1, \overline{X}_2$, if necessary, we can assume that

$$T = \cos\phi X_1 + \sin\phi X_2 \quad \text{and} \quad \overline{T} = \cos\overline{\phi}\,\overline{X}_1 + \sin\overline{\phi}\,\overline{X}_2. \tag{3.23}$$

Let $N = -\sin\phi X_1 + \cos\phi X_2$ be the unit normal to C and let $\kappa = \kappa(p)$ denote the geodesic curvature of C defined by $\kappa N = D_T T$. Applying (2.4), (2.5), and (2.7) we see that a.e. on C there holds

$$\begin{aligned}
\kappa N = D_T T &= D_T\phi(-\sin\phi X_1 + \cos\phi X_2) + \cos\phi D_T X_1 + \sin\phi D_T X_2 \\
&= D_T\phi N + \cos\phi(\cos\phi D_1 X_1 + \sin\phi D_2 X_1) + \sin\phi(\cos\phi D_1 X_2 + \sin\phi D_2 X_2) \\
&= D_T\phi N + \kappa_1\cos^2\phi X_2 - \kappa_2\cos\phi\sin\phi X_2 - \kappa_1\sin\phi\cos\phi X_1 + \kappa_2\sin^2\phi X_1 \\
&= D_T\phi N + (\kappa_1\cos\phi - \kappa_2\sin\phi)N,
\end{aligned}$$

so that

$$\kappa_1\cos\phi - \kappa_2\sin\phi = \kappa - D_T\phi. \tag{3.24}$$

If $\overline{\kappa}$ and $\overline{N}$ are the analogous entities on $\overline{V}$, then we also have

$$\overline{\kappa}_1\cos\overline{\phi} - \overline{\kappa}_2\sin\overline{\phi} = \overline{\kappa} - D_{\overline{T}}\overline{\phi},$$

so that in light of the curvature (2.9)

$$\frac{\kappa_1}{m_2}\cos\overline{\phi} - \frac{\kappa_2}{m_1}\sin\overline{\phi} = \overline{\kappa} - D_{\overline{T}}\overline{\phi}.$$

In addition, it follows from the second equation in (3.22) that

$$\cos\overline{\phi} = \frac{m_1\cos\phi}{\sqrt{m_1^2\cos^2\phi + m_2^2\sin^2\phi}}$$

and

$$\sin\overline{\phi} = \frac{m_2\sin\phi}{\sqrt{m_1^2\cos^2\phi + m_2^2\sin^2\phi}}.$$

Since we also have

$$D_{\overline{T}}\overline{\phi} = \frac{1}{\sqrt{m_1^2\cos^2\phi + m_2^2\sin^2\phi}} D_T\overline{\phi}(f(p)),$$

it therefore follows that

(3.25)
$$\frac{m_1}{m_2}\kappa_1\cos\phi - \frac{m_2}{m_1}\kappa_2\sin\phi = \sqrt{m_1^2\cos^2\phi + m_2^2\sin^2\phi}\,\overline{\kappa} - D_T\tan^{-1}\left(\frac{m_2}{m_1}\tan\phi\right).$$

Finally, we point out that this holds for all curves C with Lipschitz continuous tangent, as can be seen by a simple approximation argument using Theorem 3.2.

It is now easy to cast the DeTurck–Yang initial value problem in a characteristic coordinate setting. Let C be a curve in U with Lipschitz continuous unit tangent and let $(g_1, g_2) = g : C \to \overline{U}$ have locally Lipschitz continuous derivative. We associate (a small piece) of C with the diagonal $L_\epsilon = \{(t, -t) \,:\, -\epsilon < t < \epsilon\}$ of S_ϵ via a one-to-one bi-Lipschitz function $u(t, -t)$ of L_ϵ into C having Lipschitz continuous derivative. The functions $\phi, \overline{\phi}$ and consequently $\theta, \overline{\theta}$ also are determined on C via (3.22) and then κ_1 and κ_2 are determined uniquely a.e. on C as the solution of the system (3.24), (3.25), so that in effect these six functions, as well as $u = (u_1, u_2)$ and $\overline{u} = (\overline{u}_1, \overline{u}_2)$, are determined on L_ϵ. By interchanging the roles of m_1 and m_2 and/or reversing the orientations of the corresponding X_i's as necessary, we can assume that $0 < \phi < \pi/2$, so that on the initial line $y_1, y_2 > 0$. Simple geometry implies that on the initial line

$$y_1(t,-t) = \left|\frac{du(t,-t)}{dt}\right|_V \cos\phi, \qquad y_1(t,-t) = \left|\frac{du(t,-t)}{dt}\right|_V \sin\phi.$$

Thus, all of the functions (3.13) are given on the initial line; these initial values for u_k, y_k, $\overline{u}_k$, $k = 1, 2$, and $\theta, \overline{\theta}$ are continuous; but for $\lambda_k = \kappa_k y_k$, they are merely bounded measurable functions. It is well known that for a system of equations of the form

$$\frac{\partial v}{\partial t_1} = A(v, w), \qquad \frac{\partial w}{\partial t_2} = B(v, w), \tag{3.26}$$

where $v = (v_1, \ldots, v_r)$ and $w = (w_1, \ldots, w_s)$ are functions of (t_1, t_2), and where A and B are Lipschitz continuous, the initial value problem with bounded measurable initial data $v(t, -t) = v_0(t), w(t, -t) = w_0(t)$, $|t| < \epsilon$, is locally well posed. Here the solutions are bounded measurable functions. The neighborhood of L_ϵ in which the solution is guaranteed to exist depends, for a given system (3.26), on the range of the initial functions $\{(v_0(t), w_0(t)) \,:\, -\epsilon < t < \epsilon\}$. Furthermore, if we are in the C^∞ or analytic category (i.e., A, B, and the initial data belong to one of these categories) then the solutions belong to the same category in any domain in which they exist.

The only thing one must do to complete this treatment of the DeTurck–Yang initial value problem is to show that the function $f = \overline{u} \circ u^{-1}$ which maps a neighborhood of the piece $u(L_\epsilon)$ of C onto a neighborhood of $g(u(L_\epsilon))$ is an (m_1, m_2)-mapping. One would expect such to be the case, but this has in fact been substantially obscured by the calculations used to arrive at the system. It is, however, not necessary to show directly that for a solution of this system, with initial data arising from a mapping g of C into $\overline{V}$ in the way described above, f is necessarily an (m_1, m_2)-mapping. Indeed, for C^∞ data (i.e., C and g) one can conclude this solely from the basic principles governing hyperbolic systems, as is explained fully in [Ge2, section 3]. (It is because this argument is based on polynomial approximation and the principle of permanence of functional equations for analytic functions that we have pointed out in several places that certain functions arising in the calculations were analytic.) One can conclude in general that f is an (m_1, m_2)-mapping simply by approximating the initial data by data in the C^∞ category and using the compactness principle together with the uniqueness of the solution of the initial value problem.

Theorem 3.2 tells us that a solution to a DeTurck–Yang initial value problem will exist in the entire (two-sided) domain of dependence unless the solution of one of the ordinary differential equations $D_j\kappa_i = \kappa_i^2 + c_i$ blows up along one of the characteristics along which this equation is valid; with obvious modifications, an analogous statement holds for the characteristic initial value problem (see discussion immediately following

Lemma 3.3 below). In particular, we will have such global existence if the initial values of the κ_i and the values of m_1, m_2, K, and $\overline{K}$ are such that the solutions of these equations never blow up.

We will need the following.

LEMMA 3.3. *Let $C \subset V$ be an arc with Lipschitz continuous tangent and let $p \in C$. Let κ_1 and κ_2 be any two bounded measurable functions on C. Let $\overline{p} \in \overline{V}$ and let $\overline{S}$ be any tangent vector to $\overline{V}$ at $\overline{p}$ with $|\overline{S}| \in (m_1, m_2)$. Then there is an open subarc C' containing p on which there are exactly two $f : C' \to \overline{V}$ with Lipschitz continuous derivative for which $f(p) = \overline{p}$ and $J_f(p)T = \overline{S}$ and such that along C the solutions to the corresponding DeTurck–Yang initial value problems have these κ_i as the curvatures of the corresponding curves of principal strain.*

Proof. Let $z = z(s)$, $-\epsilon < s < \epsilon$, be an arc length parametrization of a subarc of C with $z(0) = p$. If $\phi(s) = \phi(z(s))$, then (3.24) is simply the differential equation

$$\phi' = \kappa - \kappa_1 \cos\phi(s) + \kappa_2 \sin\phi(s).$$

If we add the initial condition $\phi(0) = \phi_0$, where $\phi_0 \in (0, \pi)$ is either of the solutions of

$$m_1^2 \cos^2\phi_0 + m_2^2 \sin^2\phi_0 = |\overline{S}|,$$

then there is a unique Lipschitz continuous solution of the corresponding initial value problem on some interval $(-\delta, \delta)$. Let $C' = z((-\delta, \delta))$. Then it is easy to see that there is an $f : C' \to \overline{V}$ with $f(p) = \overline{p}$ and $J_f(p)T = \overline{S}$ such that the geodesic curvature $\overline{\kappa}(s)$ at $f(z(s))$ as stipulated above is determined by (3.25), that is,

$$\overline{\kappa}(s) = \left(\frac{m_1}{m_2}\kappa_1 \cos\phi - \frac{m_2}{m_1}\kappa_2 \sin\phi + D_T \tan^{-1}\left(\frac{m_2}{m_1}\tan\phi\right)\right) / \sqrt{m_1^2\cos^2\phi + m_2^2\sin^2\phi}.$$

But since (3.24) and (3.25) uniquely define κ_1 and κ_2 once ϕ, κ, and $\overline{\kappa}$ are given, the solution of the DeTurck–Yang problem corresponding to initial mapping f (with the $X_k, \overline{X}_k$ chosen in accordance with the normalizing stipulations implicit in (3.23)) will have principal strain line curvatures coinciding along C' with the given κ_1 and κ_2. $\square$

We now discuss the characteristic initial value problem for (m_1, m_2)-mappings, which is often easier to apply and more appropriate for the description of certain classes of such mappings as well as of individual ones. Let $C_k, k = 1, 2$, be curves on V with arc length parametrizations $w_k : [\alpha_k, \beta_k] \to V$, $\alpha_k < 0 < \beta_k$, such that the unit tangent vector fields $T_k(s)$ are Lipschitz continuous, $C_1 \cap C_2 = \{p\}$, where $p = w_1(0) = w_2(0)$, and $\langle T_1(0), T_2(0)\rangle = 0$. Given $\overline{p} \in \overline{V}$ and orthonormal tangent vectors $\overline{T}_1, \overline{T}_2$ to $\overline{V}$ at $\overline{p}$, the characteristic initial value problem for (m_1, m_2)-mappings consists of finding such a mapping f for which the C_k are m_k-characteristics and such that $f(p) = \overline{p}$ and $J_f T_k(0) = \overline{T}_k$. Of course, the possibility of high curvatures of the initial curves C_k in general precludes the existence of a solution even in a neighborhood of $C_1 \cup C_2$, but it is a relatively straightforward matter to see, by formulating this problem in terms of characteristic coordinates via the system (3.14)–(3.19), that it is well posed in a neighborhood of p. As with the Cauchy problem (i.e., the DeTurck–Yang problem) the key requirement is that the initial data for the ten functions (3.13) be Lipschitz continuous, which will clearly be the case for the data we have described. Here again one must make sure that the solution corresponds to an (m_1, m_2)-mapping. However, as we have seen, one can avoid the possibly cumbersome calculations implicit in a direct verification by appealing to the theory of hyperbolic systems. Specifically, in this case the desired conclusion is a consequence of the fact that by using the blow-up (2.13) together with Lemma 3.3 we can arrange initial data for a DeTurck–Yang initial value problem along a C^∞ curve through p whose tangent at p is orthogonal to neither of the $T_k(0)$ in such a way that its solution will have the desired characteristics.

The remainder of this section deals with the generalization of HP nets discussed at the end of section 2. Specifically, we shall prove the following.

THEOREM 3.4. *Let U be a simply connected domain on a 2-manifold with constant Gaussian curvature K, and let X_1, X_2 be an orthonormal pair of Lipschitz continuous fields on U with curvatures κ_1, κ_2 defined by* (2.7). *Let $m_1, m_2 > 0$ and $\overline{K}$ be constants. Then the following are equivalent.*

(i) *For almost all $p \in U$, κ_i is a differentiable function of arc length on the entire j-characteristic through p along which it satisfies the ordinary differential equation $D_j\kappa_i = \kappa_i^2 + c_i$, where c_i is as given in* (2.14). *(Note that we are assuming only that one of the two equations in* (2.13) *is satisfied; that the other also holds will follow as a consequence.)*

(ii) *X_1, X_2 is an $(m_1, m_2, \overline{K})$-HP pair.*

(iii) *There is an (m_1, m_2)-mapping of U into a 2-manifold $\overline{V}$ with Gaussian curvature $\overline{K}$ whose principal strain fields are X_1 and X_2.*

Proof. (i)⇒(ii). Assume that the fields X_1, X_2 satisfy (i). For notational convenience we deal with the case $i = 1$. Let $u : S_\epsilon \to U$ be a characteristic coordinate mapping for these fields corresponding a small characteristic quadrilateral for which Lemma 3.1 holds; again without loss of generality we may assume that the y_i are positive. Then for $i = 1, 2$ there exist functions z_i which are equal to y_i a.e. on S_ϵ, which are absolutely continuous on almost all lines $t_j =$ constant, and satisfy (3.7) in the strict sense a.e. on them. Let T be such that the differential equation for κ_1 holds on the 2-arc corresponding to $t_1 = T$ and z_1 satisfies (3.7) a.e. on this segment. Let $\kappa(t) = \kappa_1(u(T, t))$ and $z(t) = z_1(T, t)$. Then the equations say

$$\kappa' = y_2(T, t)(c_1 + \kappa^2)$$

and

$$z' = -\kappa z y_2(T, t)$$

a.e. on $(-\epsilon, \epsilon)$. Thus,

$$\frac{d(\kappa z)}{dt} = \kappa' z + \kappa z' = z y_2(c_1 + \kappa^2) - \kappa^2 z y_2 = c_1 z y_2,$$

a.e. on $(-\epsilon, \epsilon)$, so that since κz is Lipschitz continuous on $(-\epsilon, \epsilon)$, it follows that for almost all T, $\alpha_2, \beta_2 \in (-\epsilon, \epsilon)$ with $\alpha_2 < \beta_2$ there holds

$$\kappa_1(u(T, \beta_2))y_1(T, \beta_2) - \kappa_1(u(T, \alpha_2))y_1(T, \alpha_2) = c_1 \int_{\alpha_2}^{\beta_2} y_1(T, t)y_2(T, t)dt.$$

Since $dA = y_1 y_2 dt_1 dt_2$ and $|du/dt_1| = y_1 dt_1$, integration with respect to T tells us that for almost all $\alpha_2 < \beta_2$ and any $\alpha_1 < \beta_1$ in $(-\epsilon, \epsilon)$, (2.17) holds with $i = 1$ for the characteristic quadrilateral $u([\alpha_1, \beta_1] \times [\alpha_2, \beta_2])$. Since by hypothesis κ_1 is continuous on almost all 2-characteristics, this is then true for all $\alpha_2 < \beta_2$. This shows that (ii) is true locally; that it is true globally follows by breaking large quadrilaterals into smaller ones.

(ii)⇒(iii) Let X_1, X_2 be an $(m_1, m_2, \overline{K})$-HP pair, and again let $u : S_\epsilon \to U$ be a characteristic coordinate mapping for these fields. Equation (3.14) holds since it was shown to follow from the HP-condition (2.17), and (3.15), (3.17), and (3.18) hold since they are consequences of the definitions of the u_k, y_k, λ_k, and θ. None of these equations involves any of the barred functions $\overline{u}_k, \overline{\theta}$; indeed, the only place any of these functions could enter these equations is in the c_i appearing in (3.14), and this does not happen because of our assumption that the Gaussian curvatures are constant. Uniqueness for characteristic initial value problems tells us that the only solution of the system (3.14), (3.15), (3.17), (3.18) for the $u_k, y_k, \lambda_k, \theta$ is the one associated with the given pair X_1, X_2. If we add (3.16) and (3.19) to the system and solve the corresponding characteristic initial value problem with the same initial data, we get an (m_1, m_2)-mapping of a neighborhood of $u(0,0)$. But the X_1, X_2 so arising are still the original fields. This shows that the desired mapping exists in a neighborhood of each point of U; that it exists in all of this simply connected domain will then follow from the monodromy principle.

(iii)⇒(i). This is a special case of Theorem 3.2. □

4. Some applications.

4.1. Nonexistence of cps-mappings. We shall use the blow-up equations to show that there is no cps-mapping f of the Euclidean plane onto certain (complete, noncompact) manifolds $\overline{V}$. First of all, one notes that the solutions of the ordinary differential equation $y' = y^2$ regular at 0 are

$$y(x) = \frac{y(0)}{1 - y(0)x},$$

so that if $y(0) \neq 0$, the solution blows up to the right or left of 0 accordingly as $y(0)$ is positive or negative. From this it easily follows that if $c(x)$ is a nonnegative continuous function on $\mathbb{R}$ which is not identically 0, then the equation $y' = y^2 + c(x)$ has no solutions on all of $\mathbb{R}$.

We begin by noting that, as indicted in the introduction, there are no cps-mappings f of all of $\mathbb{R}^2$ onto itself other than the linear ones. In this case K as well as $\overline{K}$ are identically zero, so that both blow-up equations reduce to $\kappa_i' = \kappa_i^2$. From the above comments together with Theorem 3.2, for any such f it follows that $\kappa_i = 0$ a.e. on each i-characteristic, which means that all characteristics are straight lines. The linearity easily follows from this.

More interesting, perhaps, are situations in which there exist no cps-mappings of $\mathbb{R}^2$ onto $\overline{V}$ at all. In light of the interpretation of such mappings as deformations produced by the cryptocrystalline solidification of a planar lamina, this rules out the attainment of certain configurations as the result of such a process. Since $V = \mathbb{R}^2$, K is identically 0. To facilitate the discussion we assume that $m_1 < m_2$. From (2.14) we have $c_i = \frac{m_i^2 m_j^2}{m_i^2 - m_j^2} \overline{K}$, so that

$$\operatorname{sgn}(c_1) = -\operatorname{sgn}(\overline{K}) \quad \text{and} \quad \operatorname{sgn}(c_2) = \operatorname{sgn}(\overline{K}). \tag{4.1}$$

We have the following.

(1) *If $\overline{K}$ does not change sign on $\overline{V}$ and is not identically* 0, *then there are no cps-mappings* $f : \mathbb{R}^2 \to \overline{V}$. This follows immediately from the foregoing since if such an f were to exist in the case of nonnegative $\overline{K}$, for example, then by Theorem 3.2 and (4.1) there would exist a 1-characteristic with arc length parametrization $z =$

$z(s)$, $-\infty < s < \infty$, along which $c_2(z(s))$ is nonnegative but not identically 0, and along which $d\kappa_2(z(s))/ds = (\kappa_2(z(s)))^2 + c_2(z(s))$, which is impossible, as indicated in the opening paragraph of this section.

For the next case we consider $\overline{V}$ such that there is a C^∞ homeomorphism $u : \mathbb{R}^2 \to \overline{V}$ for which there are a finite number of disjoint closed disks $\overline{\Delta}_k = \overline{\Delta}(p_k, r_k), k = 1, \ldots, n$ (where $\overline{\Delta}(a, r)$ is the disk $|p - a| \leq r$), such that u is an isometry on $\mathbb{R}^2 \setminus \overline{\Delta}_1 \cup \cdots \cup \overline{\Delta}_n$ and there is some $\epsilon > 0$ for which $K(u(p)) < 0$ for $r_k - \epsilon < |p - p_k| < r_k$, $1 \leq k \leq n$. We regard the interiors of the $u(\overline{\Delta}_k)$ as being bumps on an otherwise planar surface. One can produce such a bump by replacing a disk of radius r by the surface obtained by rotating the graph of $y = q(x)$, $0 \leq x \leq r$, about the y-axis, where $q \in C^\infty(\mathbb{R})$ is even and both $q''(x) > 0$ and $q'(x) < 0$ on some (r', r). These bumps have the desired negative curvature in a vicinity of the boundary circle and any number of them can be grafted into the plane, provided the corresponding closed disks are disjoint.

(2) *There exist no cps-mappings f of $\mathbb{R}^2$ onto such a "bumpy" plane $\overline{V}$.* Again assume that such an f existed. Let $z = z_k(s), k = 1, 2$, be arc length parametrizations of the characteristics through some point p_0 lying inside the preimage of one of the bumps with $z_k(0) = p_0$ and increasing s corresponding to the direction of X_k. The curvatures of the lines of principal strain are bounded since the preimage W of the union of the bumps is compact, and from the above discussion of blow-up in the planar case $|D_i\kappa_j(p)| \leq 1/\mathrm{dist}(p, W)$ a.e. in $\mathbb{R}^2 \setminus W$. From this it follows by a simple compactness argument on the family of 2-characteristics that there is a 2-characteristic C which just touches ∂W but is disjoint from W (for example, by minimizing the area of the part of W to one side of C). It then follows from Theorem 3.2 that there are 2-characteristics C' arbitrarily close to C along which κ_1 exists and satisfies the corresponding blow-up equation. However, in light of the hypothesis, along such a C' sufficiently close to C, $c_1 \geq 0$ but is not identically 0, which is impossible by the comment at the end of the first paragraph of this section.

4.2. The hyperbolic plane $\mathbb{H}^2$. We begin by examining blow-up of the solutions of the ordinary differential equations to which the equations (2.13) reduce when both of the Gaussian curvatures K and $\overline{K}$ are constant. Upon writing

$$\gamma_i^2 = |c_i| = m_j^2 \left| \frac{m_i^2 \overline{K} - K}{m_i^2 - m_j^2} \right|,$$

(2.13) becomes $D_j\kappa_i = \kappa_i^2 \pm \gamma_i^2$, so we have only to look at the solutions of the elementary equations $\kappa' = \kappa^2 + \gamma^2$ and $\kappa' = \kappa^2 - \gamma^2$, $\gamma > 0$, $\kappa = \kappa(s)$. The general solution of the first is $\kappa(s) = \gamma \tan(\gamma s + C)$, so that the longest open interval in which a regular solution can exist has length π/γ. On the other hand, the solutions of $\kappa' = \kappa^2 - \gamma^2$ are of the form

$$\kappa(s) = \gamma \frac{1 + C\, e^{2\gamma s}}{1 - C\, e^{2\gamma s}}, \tag{4.2}$$

which is regular on the entire s-axis with range $(-\gamma, \gamma)$ when $C < 0$, reduces to the constant γ when $C = 0$, and has singularity at $s_0 = -(1/2\gamma)\log C$ when $C > 0$, in which case the range consists of the intervals $(-\infty, -\gamma)$ for $s > s_0$ and (γ, ∞) for $s < s_0$. In particular, the solution exists on all of $\mathbb{R}$ if and only if $|\kappa(0)| \leq \gamma$.

Henceforth $V = \overline{V} = \mathbb{H}^2$, so that $K = \overline{K} = -1$. For convenience we also assume that $m_1 < m_2$, which of course constitutes no loss of generality. We have

$$c_i = \frac{m_j^2(1-m_i^2)}{m_i^2 - m_j^2},$$

so that both of the equations (2.13) will be of the form $\kappa' = \kappa^2 - \gamma^2$ ($\gamma \geq 0$) if and only if

$$m_1 \leq 1 \leq m_2. \tag{4.3}$$

Specifically, for such m_1, m_2 they are

$$D_j\kappa_i = \kappa_i^2 - \gamma_i^2, \text{ where } \gamma_i = \sqrt{\frac{m_j^2(m_i^2-1)}{m_i^2 - m_j^2}}. \tag{4.4}$$

Consider the characteristic initial value problem with initial m_k-characteristic $C_k, k = 1, 2$. Let C_k have the arc length parametrization $w = w_k(s)$, $\alpha_k < s < \beta_k$, where $0 \in (\alpha_k, \beta_k)$ and where $p = w_1(0) = w_2(0)$. As was pointed out in the discussion of this problem in section 3, we are in general guaranteed a solution only in a neighborhood of p. However, if we assume (4.3) and that the curvatures κ_k of the initial curves satisfy

$$|\kappa_k(s)| \leq \gamma_k \text{ a.e. on } (\alpha_k, \beta_k), k = 1, 2,$$

then the comment in the paragraph immediately preceding the statement of Lemma 3.3 implies that the solution exists in the entire characteristic quadrilateral determined by the C_k. Among other things this means that C_1 and C_2 are simple curves and $C_1 \cap C_2 = \{p\}$. Thus, in light of the facts that $\gamma_1^2 + \gamma_2^2 = 1$ and that γ_1 can take any value in $[0, 1]$ with appropriate m_1 and m_2 satisfying (4.3), we have established the following.

Theorem 4.1. *Let C_1 and C_2 be curves in $\mathbb{H}^2$ whose arc length parametrizations have locally Lipschitz derivatives and which meet orthogonally at p. Let $\lambda_1, \lambda_2 > 0$ satisfy $\lambda_1^2 + \lambda_2^2 \leq 1$. If the (unsigned) geodesic curvature of C_k is bounded above by λ_k, $k = 1, 2$, then these curves are both simple and p is their only common point.*

With exactly the same hypotheses on the curves this theorem holds in the n-dimensional hyperbolic space $\mathbb{H}^n$ as well. To prove this, it suffices to show that p is the only common point when C_1 and C_2 are both simple curves. Indeed, if we have established this and C_1 and C_2 satisfy the hypotheses but C_1 is not simple, then we can replace C_2 by a geodesic E_2 which joins two points of a simple subarc E_1 of C_1 and thereby obtain a contradiction since the curvature of E_2 is everywhere 0 and that of E_1 is bounded by λ_1. Thus we shall assume that C_1 and C_2 are both simple. Assume that $n \geq 3$ and that they have a second point of intersection q. Let $w = w_k(s)$ be corresponding arc length parametrizations with $w_k(0) = p$ and $w_k(a_k) = q$, $k = 1, 2$. A simple compactness argument allows us to assume that the pair C_1, C_2 minimizes $a_1 + a_2$, i.e., the sum of the lengths of the two arcs pq. Henceforth $\text{dist}(z_1, z_2)$ will denote the geodesic distance between points $z_1, z_2 \in \mathbb{H}^n$. Then $\frac{d}{ds}\text{dist}(p, w_k(s)) > 0$ on $(0, a_k)$, since if it were equal to 0 for some $b \in (0, a_k)$, then C_k would be orthogonal to the geodesic joining p to $w_k(b)$, and this would give us a new pair of simple curves for which the sum of lengths of the two arcs joining the two intersection points is smaller than $a_1 + a_2$. Let $\epsilon > 0$. Then there exists a new pair of simple curves C_1'

and C_2' with C^∞ arc length parametrizations v_k on $(-1, l_k+1)$ for which

(i) $l_k \le a_1 + a_2 + \epsilon$;
(ii) the corresponding curvatures $\kappa_k(s)$ satisfy $\kappa_k(s) \le \lambda_k + \epsilon$ on $(-1, l_k+1)$, $k = 1, 2$;
(iii) $v_k(0) = p$, $k = 1, 2$;
(iv) $v_k(l_k) = q$, $k = 1, 2$;
(v) $\operatorname{dist}(p, v_k(s))$ increases on $(0, l_k)$;
(vi) C_1' and C_2' are orthogonal at their common initial point p;
(vii) for no $s \in (0, l_k]$ is the geodesic which joins p to $v_k(s)$ tangent to C_k' at $v_k(s)$.

Let $V_k(s)$, $s \in (0, l_k]$ be the unit tangent vector at O to the geodesic ray emanating from p and passing through $v_k(s)$. It follows from (vii) that $|V_k'(s)| > 0$ on $(0, l_k]$. It is also easy to see that $\lim_{s\to 0^+} |V_k'(s)|$ exists. We claim that there exist $s_k \in (0, l_k]$ such that

$$A(s_1, s_2) = \int_0^{s_1} |V_1'(s)|\, ds + \int_0^{s_2} |V_2'(s)|\, ds = \pi/2 \tag{4.5}$$

and

$$\operatorname{dist}(p, v_1(s_1)) = \operatorname{dist}(p, v_2(s_2)). \tag{4.6}$$

To see this, consider $A(t_1, t_2)$ for $(t_1, t_2) \in Q = [0, l_1] \times [0, l_2]$. Then $A(0, 0) = 0$ and, because C_1' and C_2' are orthogonal at p, $A(l_1, l_2) \ge \pi/2$. Furthermore, since $|V_k'(s)| > 0$ on $(0, l_k]$, $A(t_1, t_2)$ is strictly increasing in each of its arguments. Thus the set $S = \{(t_1, t_2) \big| A(t_1, t_2) = \pi/2\}$ is a curve which joins the union of the left-hand side and bottom of Q to the union of its right-hand side and top. (This curve could degenerate to the point (l_1, l_2).) But (v) implies that there are increasing continuous functions $\tau_k, k = 1, 2$, which map $[0, 1]$ onto $[0, l_k]$ such that $\operatorname{dist}(p, v_1(\tau_1(t))) = \operatorname{dist}(p, v_2(\tau_2(t)))$, $t \in [0, 1]$. This means that there must be a $t \in (0, 1]$ such that $(\tau_1(t), \tau_2(t)) \in S$, so that (4.5) and (4.6) hold with $(s_1, s_2) = (\tau_1(t), \tau_2(t))$.

We consider the following mappings from a domain in $\mathbb{H}^2$ into $\mathbb{H}^n$. Let $O \in \mathbb{H}^2$ and T^* be a fixed unit vector in the tangent space of $\mathbb{H}^2$ at O. We define the continuous function T_k from the interval $[0, t_k]$ to the set of unit tangent vectors to $\mathbb{H}^2$ at O by $T(0) = T^*$ and $|T_k'(s)| = |V_k'(s)|$, where $T_k(s)$ moves in the positive sense as s increases when $k = 1$, and in the negative sense when $k = 2$. Let $G_k(s)$ be the geodesic ray emanating from O in the direction $T_k(s)$ and let $G_k(s, \sigma)$ be the point on $G_k(s)$ at distance σ from O, $0 \le s \le s_k$, $0 < \sigma$. Let F_k map the sector of $\mathbb{H}^2$ made up of the $G_k(s)$, $0 \le s \le s_k$, into $\mathbb{H}^n$ in such a way that $F_k(G_k(s, \sigma))$ is the point on the geodesic ray emanating from p through $v_k(s)$ whose distance from p is σ. One easily sees that F_k is an isometry (as a mapping between surfaces) and that it is locally one-to-one, so that F_k^{-1} is well defined. Let $\overline{C}_k$ be the preimage of C_k' under F_k. Since F_k is an isometry, the curvature of $\overline{C}_k$ at $F_k(v_k(s))$ is the curvature of C_k at $v_k(s)$ when calculated from the point of view of C_k' as a curve in the submanifold made up of the geodesics joining its points to p; this curvature is at most $\kappa_k(s)$. Thus, the curvature of $\overline{C}_k$ is bounded above by $\lambda_k + \epsilon$. Let $\overline{C}_2'$ be the curve onto which $\overline{C}_2$ is carried when $\mathbb{H}^2$ is rotated about O through a positive angle of $\pi/2$. Then from our construction $\overline{C}_1$ and $\overline{C}_2'$ are simple arcs in $\mathbb{H}^2$ which meet orthogonally at O, intersect again at their other endpoint, have lengths bounded by $a_1 + a_2 + \epsilon$, and have curvatures bounded, respectively, by $\lambda_1 + \epsilon$ and $\lambda_2 + \epsilon$. If we allow ϵ to tend to 0, then a simple compactness argument will provide curves in $\mathbb{H}^2$ which satisfy the hypotheses of Theorem 4.1 in $\mathbb{H}^2$ but not the conclusion. This contradiction proves that the theorem is indeed true in $\mathbb{H}^n$. As an immediate consequence we obtain the following result due to Epstein [E1], [E2].

COROLLARY. *A curve in* $\mathbb{H}^n$ *whose curvature is everywhere bounded by* 1 *cannot intersect itself.*

We now give a very simple and quite explicit description of all of the cps-mappings of the entire space $\mathbb{H}^2$ into itself. Actually, it is easy to see that if $f : \mathbb{H}^2 \to \mathbb{H}^2$ is an (m_1, m_2)-mapping, then f is one-to-one and onto, so that we shall speak of the cps-self-homeomorphisms of $\mathbb{H}^2$. Fix a point $O \in \mathbb{H}^2$, and consider any (m_1, m_2)-mapping $f : \mathbb{H}^2 \to \mathbb{H}^2$, again with the nonrestrictive assumption that $m_1 < m_2$. Let C_k be the k-characteristic passing through O parametrized with respect to arc length by w_k, where $w_1(0) = w_2(0) = O$. Since all characteristics of f have infinite length in both directions, it follows from the above discussion that (4.3) holds. It furthermore follows from the initial comments that we must have $|\kappa_k| \le \gamma_k$ a.e. on C_k, and conversely, the discussion of existence and blow-up of the preceding section shows that if these bounds are satisfied then there exists a corresponding (m_1, m_2)-mapping, which, moreover, is uniquely determined by the two functions κ_1, κ_2 once we assign the image of O and directions corresponding in the image to the tangent directions of the C_k at O. (Note that by Theorem 4.1 the conditions $|\kappa_k| \le \gamma_k$ automatically imply that C_1 and C_2 are simple and only cross at O.) Thus we have the following.

THEOREM 4.2. *Let* $O \in \mathbb{H}^2$ *be fixed. There is a one-to-one correspondence between cps-self-homeomorphisms of* $\mathbb{H}^2$ *and* 6*-tuples* $(m_1, m_2, C_1, C_2, \overline{O}, T)$*, such that*

(i) $m_1 \le 1 \le m_2$;

(ii) C_1 *and* C_2 *are curves, of infinite length in both directions, with Lipschitz continuous unit tangent vectors and whose (unsigned) geodesic curvatures* κ_k *are bounded by the numbers* γ_k *defined in* (4.4);

(iii) $\overline{O} \in \mathbb{H}^2$;

(iv) T *is an orthogonal transformation of the tangent space at* O *onto the tangent space at* $\overline{O}$.

For each such 6*-tuple the mapping is the solution of the corresponding characteristic initial value problem.*

Before continuing we point out that the blow-up conditions allow one to completely answer the following question: Given simple curves C and $\overline{C}$ on $\mathbb{H}^2$, of infinite length in both directions, and whose arc length parametrizations have locally Lipschitz continuous derivatives, give necessary and sufficient conditions on a mapping $f : C \to \overline{C}$ for which $|df/ds|$ is locally Lipschitz continuous and satisfies $m_1 < |df/ds| < m_2$ a.e. on C, such that the corresponding DeTurck–Yang initial value problems have global solutions. To do this we proceed as follows. Let $m_1 \le 1 \le m_2$, since otherwise there are no global (m_1, m_2)-mappings of $\mathbb{H}^2$ onto itself by Theorem 4.2. Let $z(s)$, $-\infty < s < \infty$, be an arc length parametrization of a simple curve C in V and let $\overline{z}(s) = f(z(s))$. Let $T = T(s)$ be the corresponding unit tangent vector to C at $z(s)$, $\overline{S} = \overline{S}(s) = J_f T(s)$, and $\overline{T} = \overline{S}/|\overline{S}|$. We assume that $|\overline{S}(s)|$ lies everywhere between m_1 and m_2 and shall apply the notation, normalizations, and calculations of the paragraph immediately following the proof of the corollary to Theorem 3.2 in section 3. Rewriting (3.24) and (3.25) slightly we have

$$\kappa_1 \cos\phi - \kappa_2 \sin\phi = \kappa - \phi' \tag{4.7}$$

and

$$\begin{aligned} \frac{m_1}{m_2}\kappa_1 \cos\phi - \frac{m_2}{m_1}\kappa_2 \sin\phi &= |\overline{S}|\overline{\kappa} - D_T \tan^{-1}\left(\frac{m_2}{m_1}\tan\phi\right) \\ &= |\overline{S}|\overline{\kappa} - m_1 m_2 \phi'/|\overline{S}|^2, \end{aligned} \tag{4.8}$$

so that solving for κ_1 and κ_2 we find

$$\begin{bmatrix}\kappa_1\\ \kappa_2\end{bmatrix} = \frac{1}{D}\begin{bmatrix}-\frac{m_2\sin\phi}{m_1} & \sin\phi\\ -\frac{m_1\cos\phi}{m_2} & \cos\phi\end{bmatrix}\begin{bmatrix}\kappa-\phi'\\ |\overline{S}|\overline{\kappa}-m_1m_2\phi'/|\overline{S}|^2\end{bmatrix},$$

where $D = (m_1^2 - m_2^2)\,\frac{\sin\phi\cos\phi}{m_1m_2}$. Thus we find from our analysis of the blow-up of the κ_i that a necessary and sufficient condition for the solution of the DeTurck–Yang initial value problem to exist in all of $\mathbb{H}^2$ is that the following hold a.e. for $-\infty < s < \infty$:

$$|(\kappa-\phi')\frac{m_2\sin\phi}{m_1} - (|\overline{S}|\overline{\kappa} - m_1m_2\phi'/|\overline{S}|^2)\sin\phi| \le m_2|D|\left(\frac{(1-m_1^2)}{m_2^2-m_1^2}\right)^{\frac{1}{2}},$$

$$|(\kappa-\phi')\frac{m_1\cos\phi}{m_2} - (|\overline{S}|\overline{\kappa} - m_1m_2\phi'/|\overline{S}|^2)\cos\phi| \le m_1|D|\left(\frac{(m_2^2-1)}{m_2^2-m_1^2}\right)^{\frac{1}{2}}.$$

These bounds are, admittedly, not particularly revealing but they become considerably more so when we limit ourselves to the case in which ϕ is constant, that is, when the initial mapping of the curve C onto $\overline{C}$ has length change $|\overline{S}| = \sigma$, a constant. Since in this case we have $\phi' = 0$, the conditions simplify to

$$|m_2\kappa - m_1\overline{\kappa}\sigma| \le \sqrt{(m_2^2-m_1^2)(1-m_1^2)}\,|\cos\phi|,$$

$$|m_1\kappa - m_2\overline{\kappa}\sigma| \le \sqrt{(m_2^2-m_1^2)(m_2^2-1)}\,\sin\phi.$$

Finally, we derive some sharp values for the radius of convexity for cps-mappings in $\mathbb{H}^2$. Returning to (4.7) and (4.8) above, we see that

$$\overline{\kappa} = \frac{1}{|\overline{S}|}\Big(\frac{m_1}{m_2}\kappa_1\cos\phi - \frac{m_2}{m_1}\kappa_2\sin\phi + m_1m_2\phi'/|\overline{S}|^2\Big),$$

so that, since $|\overline{S}|^2 = m_1^2\cos^2\phi + m_2^2\sin^2\phi$, we have by (4.7) that

$$|\overline{S}|^3\overline{\kappa} = m_1m_2(\kappa_2\sin^3\phi - \kappa_1\cos^3\phi + \kappa) + \frac{m_1^3}{m_2}\kappa_1\cos^3\phi - \frac{m_2^3}{m_1}\kappa_2\sin^3\phi.$$

Thus, writing $\mu = (\frac{m_2}{m_1})^2$ we have

$$\frac{|\overline{S}|^3\overline{\kappa}}{m_1m_2} = \kappa - (\mu-1)\left(\frac{\kappa_1}{\mu}\cos^3\phi + \kappa_2\sin^3\phi\right) \text{ a.e. on } C.$$

Let $\Delta = \Delta(R,a)$ denote the disk of radius R and centered at a in $\mathbb{H}^2$ and let $f : \Delta \to \mathbb{H}^2$ be an (m_1, m_2)-mapping, which, without loss of generality we assume to be orientation preserving. We apply the above calculations to the curve $\partial\Delta$ with positive orientation so that N and $\overline{N}$ are inward pointing normals (see (3.23) and the sentence which follows it). The curve $\partial f(\Delta)$ is convex if and only if $\overline{\kappa} = \langle D_{\overline{T}}\overline{T}, \overline{N}\rangle \ge 0$ a.e. on $\partial\Delta$, that is, if and only if

$$\kappa \ge (\mu-1)\left(\frac{\kappa_1}{\mu}\cos^3\phi + \kappa_2\sin^3\phi\right). \tag{4.9}$$

If p is a point of Δ at distance d from $\partial\Delta$, then it follows from (4.2) that the greatest value that $\kappa_i(p)$ can have is

$$\kappa_i^{\max} = \gamma_i \frac{e^{2\gamma_i d}+1}{e^{2\gamma_i d}-1} = \gamma_i \coth(\gamma_i d), \tag{4.10}$$

since otherwise f would have to have a singularity inside Δ. It is well known and easily calculated that the hyperbolic geodesic curvature $k(r)$ of a circle of hyperbolic radius r is given by

$$k(r) = \frac{1+\tanh^2(r/2)}{2\tanh(r/2)}.$$

It then follows from (4.9) that $f(\Delta(r,a))$ is convex provided that

$$k(r) \geq (\mu-1)\max\left\{\frac{\gamma_1}{\mu}\coth(\gamma_1(R-r)), \gamma_2\coth(\gamma_2(R-r))\right\}. \tag{4.11}$$

For fixed $m_1 \leq 1 \leq m_2$, $R > 0$ the right-hand side is increasing, so that since the left-hand side is decreasing, there is a unique $\rho = \rho(m_1, m_2, R)$ for which they coincide.

THEOREM 4.3. *Let $m_1 \leq 1 \leq m_2$, $R > 0$. Then $\rho(m_1, m_2, R)$ is the largest r such that all (m_1, m_2)-mappings of $\Delta(R,a)$ into $\mathbb{H}^2$ map $\Delta(r,a)$ onto simply covered convex domains.*

Proof. That the images of the concentric disk of radius $\rho(m_1, m_2, r)$ are all convex follows from the preceding discussion. Thus we have only to show that this ρ cannot be replaced by any larger number. Let i be the index corresponding to the maximum in (4.11). Let C_j be a geodesic through a and let $q \in C_j$ be at distance ρ from a. Let $d > R-\rho$, and let C_i be a curve orthogonal to C_j at q whose geodesic curvature is 0 everywhere except on a small neighborhood N of q along which it is given by the expression in (4.10), with the "concave side" of N towards the shorter of the two arcs into which q divides C_j. It is clear then that for sufficiently small N the solution f to the characteristic initial value problem for (m_1, m_2)-mappings with these characteristics exists in all of $\Delta(R,a)$. But given any $r > \rho$, for a $d > R-\rho$ sufficiently close to $R-\rho$, (4.9) will be violated for the circle centered at a and of radius r, that is, the image of the interior of this circle will not be a convex domain. ◻

5. Comments. In closing we touch on a few of the many questions about cps-mappings that naturally suggest themselves. First of all, there are reasons to believe that the Jacobian of a C^1-mapping between 2-manifolds having constant principal strains is necessarily locally Lipschitz continuous. A partial result in this direction was given in [Ge1], where it was shown that in the planar case this conclusion is valid under the stronger assumption that the derivatives of the mapping satisfy a Hölder condition with exponent $\alpha > (\sqrt{5}-1)/2$, and the arguments given there can be strengthened to extend this result to the general manifold context with the lower bound decreased to 1/2.

In section 4 we considered only the radius of convexity problem in $\mathbb{H}^2$ under the assumption that $m_1 \leq 1 \leq m_2$ because for other values of the principal stretches there are no (m_1, m_2)-mappings of $\Delta(R,a)$ into $\mathbb{H}^2$ when R is sufficiently large. This leads us to the problem of determining the radius of the largest disk on a complete manifold of constant Gaussian curvature K on which there exist (m_1, m_2)-mappings into a manifold of constant Gaussian curvature $\overline{K}$. In light of the opening sentences of section 1 the answer to this question, and more generally the determination of maximal domains of existence for cps-mappings on manifolds, would have an obvious bearing on the appearance of flaws in cryptocrystalline films.

Theorem 4.2 gives a complete description of all cps-mappings of $\mathbb{H}^2$ onto itself, and we have done the same [Ge4] for two planar domains (the half-plane and the exterior of a disk), but it would appear that the nonlinear hyperbolic nature of the underlying equations precludes such a description in any appreciable generality. Moreover, it is most likely that even for many "nice" domains in $\mathbb{R}^2$ there are no such mappings at all. (Although we believe this to be the case for disks, we have as yet been unable to come up with a proof.) These circumstances suggest two problems: (1) Find other manifolds for which it is possible to describe all the cps-self-homeomorphisms. (2) Find some simple conditions on a manifold which imply that this class is vacuous.

We end with a few words about cps-mappings in higher dimensions, that is, about mappings with distinct constant principal stretches between n-dimensional manifolds. The treatment of section 2 can be carried over to this more general context, but the equations that result are vastly more complicated. In the first place, the higher dimensional counterpart of the system (2.9) of curvature equations, although hyperbolic, is not diagonal, and in the second place the analogues of the blow-up equations (2.13) involve not only the principal strain line curvatures but functions that give the rate of rotation of the frames of principal strain directions as well (see [Ge2]). An example of Yin [Y] shows that there are nonaffine cps-self-homeomorphisms of $\mathbb{R}^3$, and it would be of interest to determine all such mappings. Indeed, most of the questions we have touched on in this paper can be examined in the higher dimensional context as well.

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