Orthogonality relations and supercharacter formulas of $U(m|n)$ representations

Jorge Alfaro *
Facultad de Física
Universidad Católica de Chile
Casilla 306, Santiago 22, Chile

Ricardo Medina †
Instituto de Física Teórica
Universidade Estadual Paulista
Rua Pamplona 145
01405-900 São Paulo, Brasil

and

Luis F. Urrutia
Instituto de Ciencias Nucleares
Universidad Nacional Autónoma de México
Circuito Exterior, C.U.
Apartado Postal 70-543, 04510 México, D.F.

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*e-mail address: jalfaro@lascar.puc.cl .
†e-mail address: rmedina@power.ift.unesp.br

Abstract

In this paper we obtain the orthogonality relations for the supergroup $U(m|n)$, which are remarkably different from the ones for the $U(N)$ case. We extend our results for ordinary representations, obtained some time ago, to the case of complex conjugated and mixed representations. Our results are expressed in terms of the Young tableaux notation for irreducible representations. We use the supersymmetric Harish-Chandra-Itzykson-Zuber integral and the character expansion technique as mathematical tools for deriving these relations. As a byproduct we also obtain closed expressions for the supercharacters and dimensions of some particular irreducible $U(m|n)$ representations. A new way of labeling the $U(m|n)$ irreducible representations in terms of $m + n$ numbers is proposed. Finally, as a corollary of our results, new identities among the dimensions of the irreducible representations of the unitary group $U(N)$ are presented.

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1 Introduction

In recent times there has been an enormous amount of work devoted to the understanding of random surfaces and statistical systems on random surfaces. The range of application of these ideas include non-critical string theory as well as Quantum Chromodynamics (QCD) in the large N limit. Progress in this area has been possible because the mathematical knowledge on random matrices has increased dramatically in the last fifteen years [1].

An important mathematical concept that appears naturally in the discussion of random matrices is the integration over the unitary group, which basis are well understood in the literature. A distinguished particular case of such integrals, the Harish-Chandra-Itzykson-Zuber (HCIZ) integral [2, 3], has been applied to the solution of the Two matrix model [3, 4] and, more recently, to the Migdal-Kazakov model of "induced QCD" [5]. In a different context, it has also been applied to the study of phase transitions in nematic liquids [6] . The HCIZ integral can also be considered a powerful alternative tool for deriving results regarding the representation theory of the group $U(N)$.

On the other hand, since its discovery, there has been considerable expectation that supersymmetry might play an important role in the physical world . This hope has motivated, on one hand, the extension of many important physical ideas to the supersymmetric world [7]. A related example of direct interest to us is the case of random supermatrices and supermatrix models [8]. On the other hand, this expectation has also contributed to the study and development of the associated mathematical tools: supermanifolds, differential and integral calculus over a Grassmann algebra, differential geometry over a supermanifold, superalgebras and Lie supergroups among others [9, 10].

This paper deals with the integration properties of the unitary supergroup $U(m|n)$. For our purposes we will work in a representation of this supergroup given by the set of all $(m+n)\times(m+n)$ supermatrices $U=[U_{AB}]$, such that $UU^\dagger=1$, endowed with the operation of supermatrix multiplication.

The issue of defining an invariant integral for supergroups has been discussed previously in [9], [11] and [12], among other references. In Ref.[11] the problem is solved by defining the invariant integral over a Lie supergroup as equal to that over its related Lie group and subsequently using the theory of invariant (Haar) integrals for topological groups [13, 14]. References [9] and [12] are on the line of a physicist approach, by keeping the grassmannian character of the integration volume element. We adhere to the last point of view and we introduce an integration measure $[dU]$ based on the Berezin integration properties of the independent elements of the supermatrix. As first noted by Berezin [9] and also in the paper by Yost in Ref. [8], the unusual property $\int[dU]=0$ will hold, thus making the calculation of the orthogonality relations a much more involved issue. Orthogonality relations for $U(m|n)$ have been previously obtained by Berezin [9] and formulated in terms of the classification of the representations of the supergroup via the Cartan approach . Instead, we use the Young tableaux method for classifying the irreducible representations for $U(m|n)$ [15]. We have not studied the relation between Berezin's result and our way of presenting the orthogonality relations, which are derived using a completely different approach. Our method of calculation is based on the result obtained for the supersymmetric extension of the HCIZ integral [16, 17], together with the use of character expansion techniques.

The paper is organized as follows : Sections 2 and 3 are basically a brief review of supermatrices and supergroup representations, respectively, designed to make the presentation self-contained and also to introduce our notation and conventions.

Sections 4 and 5 deal with the orthogonality relations for the irreducible representations of $U(m|n)$. Besides the expected product of Kronecker deltas, these relations include a representation dependent coefficient $\alpha_{\{t\}}$ which calculation, for all three types of representations of the supergroup (ordinary, complex conjugated and mixed), is the main subject of this section. We show that in the case of the mixed representations these coefficients can be written in terms of those corresponding to the ordinary representations. Some examples are presented in the Appendix 8.5 The unusual property of the integration measure mentioned above has also the consequence that this coefficient is non-zero only for a class of representations which are completely identified in our approach. Some preliminary results regarding this issue were previously presented in Ref.[18]. Here we have completed the determination of the coefficients $\alpha_{\{t\}}$ for the cases that were missing in [18] and we also give a more detailed version of our calculation. Closed formulas for the dimensions and supercharacters of ordinary and complex conjugate representations with $\alpha_{\{t\}} \neq 0$ are also presented.

The restriction $U(m|n) \rightarrow U(m)$ correctly reproduces the result $\alpha_{\{t\}} \rightarrow \frac{1}{d_{\{t\}}}$ in the orthogonality relations, where $d_{\{t\}}$ is the dimension of the corresponding representation. In this way, our expressions for the $\alpha_{\{t\}}$ coefficients of the mixed representation in terms of those of the ordinary representations, turn into identities for the corresponding dimensions of the $U(m)$ representations. Up to our knowledge, these identities were not known before and they are presented in Section 6. Some specific examples can be read off in the Appendix 8.5, after the replacement $\alpha \rightarrow \frac{1}{d}$ is made.

Section 7 contains a proposal to label the irreducible representations of $U(m|n)$, in terms of a finite array of $m+n$ numbers, not all necessarily independent, instead of giving an arbitrary large array of numbers (that can contain infinite numbers in principle) corresponding to the number of boxes in the rows of the associated Young tableau. The possible advantages of this relabelling are not further explored.

Appendix 8.1 contains a brief review of the supersymmetric HCIZ integral which result is the basic tool used in our calculations. The remaining Appendices are the detailed calculations of some expressions in the main text, together with the statement of useful relations which are also used along the paper.

Finally, Tables I (II) in the Appendix 8.4 contain a list of characters and dimensions of representations of the group $GL(N)$ (supercharacters and dimensions of representations of the supergroup $GL(m|n)$) which are an extended version of those found in Ref. [3].

2 Introduction to supermatrices

Supergroups can be conveniently represented by matrices acting on a superspace (supermatrices). To this end we briefly review some of the basic properties of the linear algebra defined over a Grassmann algebra. This sets the stage for the rest of the paper and also fixes our notation. For a more detailed and complete discussion on these matters the reader is referred to Refs.[9, 10].

Let us consider a superspace with coordinates $z^P = (q^i, \theta^\alpha), i = 1, \dots m$, $\alpha = 1, \dots, n$ such that the q^i's (θ^α's) are even (odd) elements of a Grassmann algebra. This means that $z^P z^Q = (-1)^{\epsilon(P)\epsilon(Q)} z^Q z^P$, where $\epsilon(P)$ is the Grassmann parity of the index P defined by $\epsilon(i) = 0,\ mod(2); \epsilon(\alpha) = 1,\ mod(2)$.The above multiplication rule implies in particular that any odd element of the Grassmann algebra has zero square, i.e. it is nilpotent. Also we have that $\epsilon(z^{P_1} z^{P_2} \dots z^{P_k}) = \sum \epsilon(P_i)$.

Supermatrices are arrays that act linearly on the supercoordinates leaving invariant the partition among even and odd coordinates. To be more specific, the supercoordinates can be thought as forming an $(m+n) \times 1$ column vector with the first m entries (last n entries) being even (odd) elements of the Grassmann algebra. In this way, an $(m+n) \times (m+n)$ supermatrix is an array written in the partitioned block form

$$M = \begin{pmatrix} A_{m\times m} & B_{m\times n} \\ C_{n\times m} & D_{n\times n} \end{pmatrix}, \tag{1}$$

where the constituent matrices have components $A_{ij},\ B_{i\alpha},\ C_{\alpha i}$ and $D_{\alpha\beta}$. Besides, $A_{ij},\ D_{\alpha\beta}$ ($B_{i\alpha},\ C_{\alpha i}$) are even (odd) elements of the Grassmann algebra in such a way that the parity array of the supercoordinate vector columnn is preserved under supermatrix multiplication of that vector. The parity of any supermatrix element is $\epsilon(M_{PQ}) = \epsilon(P) + \epsilon(Q)$. The addition and multiplication of supermatrices according to the rules

$$(M_1 + M_2)_{PQ} = (M_1)_{PQ} + (M_2)_{PQ}, \quad (M_1 M_2)_{PQ} = \sum_R (M_1)_{PR} (M_2)_{RQ},$$

is such that it produces again a supermatrix. The inverse of a supermatrix can be constructed in block form, in complete analogy with the classical case and it is well defined provided A^{-1} and D^{-1} exist. The inverse of these even matrices is calculated in the standard way.

The basic invariant of a supermatrix under similarity transformations is the supertrace

$$Str(M) = Tr(A) - Tr(D) = \sum_{P=1}^{m+n} (-1)^{\epsilon(P)} M_{PP},$$

which is defined so that the cyclic property $Str(M_1M_2) = Str(M_2M_1)$ is fullfilled for arbitrary supermatrices M_1, M_2. The above definition of the supertrace leads to the construction of the superdeterminant in the form $Sdet(M) = exp[Str(lnM)]$, which is explicitly given by the following two equivalent forms [19]

$$Sdet(M) = \frac{det(A - BD^{-1}C)}{det(D)} = \frac{det(A)}{det(D - CA^{-1}B)}. \tag{2}$$

The above expression is written only in terms of determinants of even matrices in such a way that the determinant has its usual meaning. The superdeterminat has the multiplicative property $Sdet(M_1M_2) = Sdet(M_1)Sdet(M_2)$.

The definition of the adjoint supermatrix follows the usual steps by requiring the identity $(y^{P*}M_{PQ}z^Q)^* = z^{P*}M^\dagger{}_{PQ}y^Q$, for an arbitrary bilinear form in the complex supercoordinates y^P, where $*$ denotes complex conjugation. Since the usual definition of complex conjugation in a Grassmann algebra, $(y^Py^Q)^* = y^{Q*}y^{P*}$, reverses the order of the factors without introducing any sign factor , we have the result $M^\dagger{}_{PQ} = M_{QP}{}^*$ as in the standard case.

A hermitian $(m+n)\times(m+n)$ supermatrix M is such that $M^\dagger = M$ and it has $(m+n)^2$ real independent components. The following properties are also fullfilled: (i) $(M^\dagger)^\dagger = M$, (ii) $(M_1M_2)^\dagger = M_2{}^\dagger M_1{}^\dagger$ and (iii) $Sdet(M^\dagger) = Sdet(M)^*$.

A unitary $(m+n)\times(m+n)$ supermatrix U is such that $UU^\dagger = U^\dagger U = I$ (where I is the identity supermatrix) and also has $(m+n)^2$ real independent components, which have the additional property that $(SdetU)(SdetU)^* = 1$. The set of all $(m+n)\times(m+n)$ unitary supermatrices, under the operation of supermatrix multiplication, constitutes the supergroup $U(m|n)$. Under very general conditions [21], hermitian supermatrices can be diagonalized by superunitary transformations, thus introducing the corresponding eigenvalues. Our notation is such that the first m eigenvalues of an $(m+n)\times(m+n)$ hermitian supermatrix M are denoted by λ_i, while the remaining n eigenvalues are denoted by $\bar{\lambda}_\alpha$. Such partition is characterized by the following parity assignment of the corresponding eigenvector components $V_P, \bar{V}_P : \epsilon(V_P) = \epsilon(P), \epsilon(\bar{V}_P) = \epsilon(P) + 1$, which are called eigenvectors of the first and second class respectively. Thus, a diagonalizable hermitian supermatrix can be decomposed as $M = U\Lambda U^\dagger$, where U is a unitary supermatrix (which is built from the eigenvectors of M) and

$$\Lambda = \begin{pmatrix} \lambda_{m\times m} & 0 \\ 0 & \bar{\lambda}_{n\times n} \end{pmatrix}, \tag{3}$$

is a diagonal supermatrix, $\lambda_{m\times m}$ ($\bar{\lambda}_{n\times n}$) being an $m\times m$ ($n\times n$) diagonal matrix with components λ_i ($\bar{\lambda}_\alpha$).

3 Basic Properties of supergroup representations

Supergroups will be represented by linear operators $\tilde{D}(g)$ acting on some vector space with basis $\{\Phi_I\}$. Linearity is defined by $\tilde{D}(g)\left(\Phi_I\alpha + \Phi_J\beta\right) = \left(\tilde{D}(g)\Phi_I\right)\alpha + \left(\tilde{D}(g)\Phi_J\right)\beta$, where α and β are arbitrary Grassmann numbers. An alternative choice is produced by having the factors to the left.

The action

$$\tilde{D}(g)\left(\Phi_I\right) = \sum_J \Phi_J \mathcal{D}^{(t)}_{JI}(g), \tag{4}$$

defines a representation (t) of the supergroup.

In spite of the use of Grassmann variables in our definition of linearity, the representation property $\mathcal{D}^{(t)}_{JI}(g_1 * g_2) = \sum_K \mathcal{D}^{(t)}_{JK}(g_1)\mathcal{D}^{(t)}_{KI}(g_2)$ is verified, thus showing that the representation of the supergroup elements in terms of supermatrices respects the multiplication rule of supermatrices.

The above linearity convention applies also to the action of group operators acting upon vectors of the space. Let us consider the vector $\Psi = \sum_K \Phi_K \alpha_K$ with components α_K. Then we have

$$\tilde{D}(g)\,(\Psi) = \sum_K \tilde{D}(g)\,(\Phi_K)\,\alpha_K = \sum_{K,L} \Phi_L \mathcal{D}^{(t)}_{LK}(g)\alpha_K, \tag{5}$$

which is consistent with the representation of a vector as a column with entries α_K, together with the representation of the action of a group element upon such vector as the multiplication of the corresponding supermatrix by the respective columm.

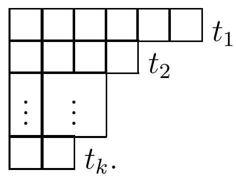
Now, let us recall that there are two fundamental representations of $U(m|n)$: The ordinary one (or undotted), $\mathcal{D}^{\square}_{ij}(U) = U_{ij}$, and the complex conjugate one (or dotted), $\mathcal{D}^{\dot{\square}}_{ij}(U) = \bar{U}_{ij} = (-1)^{\epsilon_i(\epsilon_i+\epsilon_j)}U^*_{ij}$ [15]. It is a direct calculation to show that $\bar{U}$ is a unitary supermatrix and also that $(\bar{UV}) = \bar{U}\bar{V}$ for arbitrary $U(m|n)$ supermatrices, thus showing that the bar operation constitutes indeed a representation of the supergroup.

Using the fundamental representations, three types of irreducible representations $\{t\}$ are built : ordinary (undotted) $\{u\}$, complex conjugated (dotted) $\{\dot{v}\}$ and mixed $\{\dot{v}\}|\{u\}$, which we do, in analogy to the $U(N)$ case, according to the conventions in Ref. [22]. In particular we have $\{u\}$=$\{\dot{0}\}|\{u\}$ and $\{\dot{v}\} = \{\dot{v}\}|\{0\}$.

Contrary to what happens in the $SU(N)$ case, the dotted and undotted representations cannot be related through an epsilon symbol [22], so they are not equivalent.

We will label the irreducible representations by means of the Young tableaux notation. Thus, an undotted irreducible representation $\{t\}$ will be characterized by the non negative integers $(t_1, t_2, \ldots, t_k)$, where $t_1 \geq t_2 \geq \ldots \geq t_k$ are the number of boxes in the corresponding rows of the tableau.

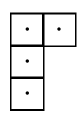
For the moment we assume that there is no restriction upon the number of t_i's characterizing the tableau. Pictorically the tableau will look like



(6)

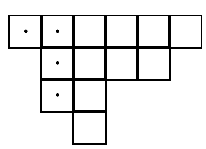
The supermatrix representation $\mathcal{D}^{\{t\}}(g)$ will then be an $(m_{\{t\}}+n_{\{t\}})\times(m_{\{t\}}+n_{\{t\}})$ supermatrix written in the standard form (1), consisting of elements $\mathcal{D}^{\{t\}}_{JI}(g)$.

So besides the undotted representations pictorically shown in (6) the dotted and mixed ones will look like

$\{\dot{v}\} =$ 

(7)

and

$\{\dot{v}\}|\{u\} =$  .

(8)

Now, we observe that in the case of ordinary groups the determinant $det(U)$ provides a one dimensional representation which can be constructed as a completely antisymmetrized product of fundamental representations. In the case of supergroups, the superdeterminant $Sdet(U)$ provides also a one dimensional representation which, nevertheless, cannot be constructed in terms of the fundamental representations. This is because the superdeterminant is a non polynomial function of the eigenvalues.

When considering tensor products of the fundamental representations we define

$$\underbrace{(\mathcal{D}^{\square} \otimes \ldots \otimes \mathcal{D}^{\square})}_{p\ times}(U) = \oplus_{\{t\},|t|=p}\ \sigma_{\{t\}} \mathcal{D}^{\{t\}}(U), \tag{9}$$

where $|t|$ represents the number of boxes of the representation $\{t\}$ and $\sigma_{\{t\}}$ is a Clebsch-Gordan coefficient which represents the number of times that the irreducible representation $\{t\}$ is contained in the above tensor product. It may be calculated using the Young tableaux rules for the tensor product of representations in (9) or alternatively using the formula (see Chapter 7, formula (5.21) of Ref. [25])

$$\sigma_{\{t\}} = |t|! \frac{\Delta(t_1 + k - 1, t_2 + k - 2, \ldots, t_k)}{\prod_{p=1}^{k}(t_p + k - p)!}, \tag{10}$$

in terms of the tableau labels given in (6), where

$$\Delta(l_1, \ldots, l_k) = \prod_{i>j=1}^{k} (l_i - l_j) \tag{11}$$

is the Vandermonde determinant.

Some values of $\sigma_{\{t\}}$ are given in the tables in Appendix 8.4. In particular, Eq.(9) implies that

$$(strU)^p = \sum_{\{t\},|t|=p} \sigma_{\{t\}} s\chi_{\{t\}}(U), \tag{12}$$

where the supercharacter of representation $\{t\}$ is

$$s\chi_{\{t\}}(U) = str(\mathcal{D}^{\{t\}}(U)), \tag{13}$$

and where we have also used the property that the supercharacter of tensor product of representations equals the product of the corresponding supercharacters. An explicit formula for $s\chi_{\{t\}}(U)$ in terms of supermatrix U is given in Appendix 8.4.

Let us emphasize that the Grassmannian character of the supermatrices involved introduces further sign factors with respect to the classical case in the case of tensor products. Let us illustrate this point with the direct product of two fundamental undotted representations. The corresponding basis vectors are $\Psi_i \Phi_j$ which are rotated to $\Psi'_k \Phi'_l$ by the independent actions of the supergroup $\Psi'_k = \Psi_i U_{ik}$ and $\Phi'_l = \Phi_j U_{jl}$. Looking for the transformation of the product we have

$$\Psi'_k \Phi'_l = (\Psi_i U_{ik})(\Phi_j U_{jl}) = \Psi_i \Phi_j \left((-1)^{\epsilon_j(\epsilon_k + \epsilon_i)} U_{ik} U_{jl} \right), \tag{14}$$

which identifies

$$\left(\mathcal{D}^{\square} \times \mathcal{D}^{\square} \right)_{ij,kl}(U) = (-1)^{\epsilon_j(\epsilon_k + \epsilon_i)} U_{ik} U_{jl}. \tag{15}$$

It is a direct calculation to verify that this assigment constitutes indeed a representation of the supergroup.

The expression (15) can be generalized for an arbitrary tensor product

$$\underbrace{(\mathcal{D}^{\square} \times \ldots \times \mathcal{D}^{\square})_{IJ}}_{p\ times}(U) = (-1)^{\epsilon_{j_1}(\epsilon_{i_2} + \epsilon_{j_2} + \ldots + \epsilon_{i_p} + \epsilon_{j_p}) + \ldots}$$
$$\times (-1)^{\epsilon_{j_{p-1}}(\epsilon_{i_p} + \epsilon_{j_p})} U_{i_1 j_1} U_{i_2 j_2} \ldots U_{i_p j_p}, \tag{16}$$

with $I = \{i_1, i_2, \ldots, i_p\}$, $J = \{j_1, j_2, \ldots, j_p\}$.

As we mentioned before, the construction of the irreducible tensor representations symmetrized according to a specific Young tableau, to which we referred in (6), (7) and (8), proceeds in complete analogy to the $U(N)$ case, as stated in Ref. [24]. In particular, the corresponding supercharacters are exactly those of $U(N)$ with the trace replaced by a supertrace (see appendix 8.4 for some examples).

4 Orthogonality relations for $U(m|n)$

4.1 Unitary supergroup measure

For finding the orthogonality relations we will make use of the Schur's lemma, extended to the case of continuous supergroups. We will have to deal with supergroup integration and for this reason we briefly refer to the unitary supergroup measure.

In general, the supergroup measure must be left and right-invariant under the supergroup action. In the case of $U(m|n)$ it is defined by

$$[dU] = \mu \prod_{P,Q=1}^{m+n} dU_{PQ} dU^*_{PQ} \; \delta(UU^\dagger - I), \tag{17}$$

where the δ-function really means the product of $(m+n)^2$ unidimensional δ-functions corresponding to the independent constraints set by the condition $UU^\dagger = I$. The integration over each Grassmann valued element dU_{PQ} is defined according to the standard Berezin's rules. The arbitrary non null constant μ will be fixed from the convention adopted for our normalization of the supersymmetric HCIZ integral. It is important to observe that although the above measure contains odd differentials and odd variables, it has 0 Grasmann parity and therefore behaves as an even Grassmann variable (commutes with everything).

4.2 General Form of the Orthogonality relations

In order to derive the general form of the orthogonality relations we apply Schur's lemma to the quantity $\mathcal{X}_{IL}^{\{s\},\{t\}} = \int [dU] \mathcal{D}_{IJ}^{\{s\}}(U) X_{JK} \mathcal{D}_{KL}^{\{t\}}(U^{-1})$, where X_{JK} is an arbitrary supermatrix. We are assuming sum over repeated indices. Multiplying this expression to the left by the arbitrary element $\mathcal{D}_{RI}^{\{s\}}(S)$ and using the composition property of the representation we obtain

$$\mathcal{D}_{RI}^{\{s\}}(S) \mathcal{X}_{IL}^{\{s\},\{t\}} = \int [dU] \mathcal{D}_{RJ}^{\{s\}}(SU) X_{JK} \mathcal{D}_{KL}^{\{t\}}(U^{-1}). \tag{18}$$

Here we used the fact that $[dU]$ behaves like an even Grassmann variable. Next we rewrite $\mathcal{D}_{KL}^{\{t\}}(U^{-1}) = \mathcal{D}_{KM}^{\{t\}}((SU)^{-1}) \mathcal{D}_{ML}^{\{t\}}(S)$ and substitute this expression in the previous equation, obtaining

$$\mathcal{D}_{RI}^{\{s\}}(S) \mathcal{X}_{IL}^{\{s\},\{t\}} = \left(\int [dU] \mathcal{D}_{RJ}^{\{s\})}(SU) X_{JK} \mathcal{D}_{KM}^{\{t\}}((SU)^{-1}) \right) \mathcal{D}_{ML}^{\{t\}}(S). \tag{19}$$

From the invariance of the measure under left multiplications we realize that the quantity in brackets is precisely $\mathcal{X}_{RM}^{\{s\},\{t\}}$ and therefore we obtain that $\mathcal{D}^{\{s\}}(S) \mathcal{X}^{\{s\},\{t\}} = \mathcal{X}^{\{s\},\{t\}} \mathcal{D}^{\{t\}}(S)$. Then, in analogy with the ordinary case, we have that : (i) if $\{s\} \neq \{t\}$ then $\mathcal{X}^{\{s\},\{t\}} = 0$, and (ii) if $\{s\} = \{t\}$ then $\mathcal{X}^{\{s\},\{s\}}$ is a multiple of the $sd_{\{s\}}$-dimensional identity supermatrix (where $sd_{\{s\}}$ is the dimension of the $\{s\}$ representation). Thus,

$$\mathcal{X}_{IL}^{\{s\},\{t\}}(X) = \int [dU] \mathcal{D}_{IJ}^{\{s\}}(U) X_{JK} \mathcal{D}_{KL}^{\{t\}}(U^{-1}) = \alpha^{\{s\}}(X) \delta^{\{s\},\{t\}} \delta_{IL}^{\{s\}}, \tag{20}$$

where the coefficient α depends upon the arbitrary supermatrix X. We can prove that the above equation is invariant under the rotation $X' = \mathcal{D}^{\{s\}} X {\mathcal{D}^{\{s\}}}^{-1}$, for a given representation $\{s\}$, in virtue of the composition properties of a representation together with the invariance of the measure with respect to right multiplication. This means that $\alpha^{\{s\}}(X)$ must be an invariant under similarity transformations, which is linear in X. The only possibility is that $\alpha^{\{s\}}(X) = \alpha^{\{s\}} str X$, where $\alpha^{\{s\}}$ is now a numerical coefficient. So, in (20) we have obtained an equality between two linear expressions of the X_{IK}'s. Taking care of the Grasmannian character of the indices involved, the comparison of the coefficients of the fully independent variables X_{IK} leads to the general form of the orthogonality relations

$$\int [dU] \mathcal{D}^{\{s\}}_{IJ}(U) \mathcal{D}^{\{t\}*}_{KL}(U) = (-1)^{\epsilon^{\{s\}}_J} \alpha_{\{t\}} \delta^{\{s\},\{t\}} \delta^{\{s\}}_{IK} \delta^{\{t\}}_{JL}, \tag{21}$$

where $(U^\dagger)_{ij} = (U^{-1})_{ij} = (U^*)_{ji}$. Our notation is such that the fundamental representation is labeled with lower case indices $i_1, i_2, \ldots, i_q$ and capital letter indices denote a family of lower case indices, i.e $I = \{i_1, i_2, \ldots, i_p\}$, for example.

In equation (21) we have restricted ourselves to the supergroup $U(m|n)$. Except for the $(-1)^{\epsilon^{\{s\}}_J}$ factor that appears as a consequence of dealing with Grassmann numbers, the general form of the orthogonality relations (21) does not apparently differ from that of the $U(N)$ case. However, as we will see in the sequel, the determination of the $\alpha_{\{t\}}$ coefficients will be crucial in stating their difference.

4.3 Null integral over the $U(m|n)$ measure

As opposed to what happens in the $U(N)$ case, the determination of the coefficients $\alpha_{\{t\}}$'s will be much more involved in our case. The reason for this is the unexpected normalization condition which is used for the determination of these coefficients. In the $U(N)$ case this normalization is $\int [dU] = 1$, while in our case it turns out to be

$$\int [dU] = 0, \quad U \in U(m|n). \tag{22}$$

Although this result was known before [9] it emerges naturally when we deal with the SUSY HCIZ integral (see Appendix 8.1). On one hand, by setting $\beta = 0$ on its definition, we directly obtain the integral over the measure of the supergroup. On the other hand, using its explicit result we have to calculate $\lim_{\beta \to 0}[\beta^{mn} I(\lambda_1, \lambda_2, \beta) I(\bar{\lambda}_1, \bar{\lambda}_2, -\beta)]$, where $I(\lambda_1, \lambda_2, \beta)$ is the standard HCIZ for $U(m)$. Since $I(\lambda_1, \lambda_2, \beta = 0) = 1$, we obtain the desired result.

An important application in this work will be the characterization of the undotted and dotted representations $\{s\}$ of $U(m|n)$ for which $\alpha_{\{s\}} \neq 0$. But before going ahead with the determination of the $\alpha_{\{t\}}$'s we briefly show two immediate consequences of (22).

(i) Choosing a fixed representation $\{s\}$ and summing with respect to $J = L$ in equation (21) we are left with the constraint

$$\int [dU] = 0 = \alpha_{\{s\}} str I_{(m_{\{s\}}+n_{\{s\}})\times(m_{\{s\}}+n_{\{s\}})}. \tag{23}$$

In particular, this means that all representations with $\alpha_{\{s\}} \neq 0$ will necessarily have a null supertrace for the unit supermatrix in the representation $\{s\}$.

(ii) From equations (21) and (23) we obtain

$$\int [dU] s\chi_{\{s\}}(U) s\chi^*_{\{t\}}(U) = 0, \tag{24}$$

even if $\{s\} = \{t\}$, because this relation involves again the supertrace of the corresponding unit supermatrix.

4.4 Determination of the $\alpha_{\{t\}}$ coefficient for ordinary representations

If we introduce only one supercharacter in the integration of Eq.(21), we are left with

$$\int [dU] s\chi_{\{s\}}(U)\mathcal{D}^{\{t\}*}_{KL}(U) = \alpha_{\{t\}}\delta^{\{s\},\{t\}}\delta^{\{t\}}_{KL}, \tag{25}$$

which plays the role of the standard orthogonality condition of the characters in the classical case.

The condition (25) implies de following useful

Lemma: The supercharacters $s\chi_{\{t\}}(U) \equiv \sum_I (-1)^{\epsilon_I^{\{t\}}}\mathcal{D}^{\{t\}}_{II}(U)$ of the representations $\mathcal{D}^{\{t\}}(U)$ for which $\alpha_{\{t\}} \neq 0$ constitute a linearly independent set.

The proof goes as follows: let us consider a null linear combination of supercharacters of representations with $\alpha_{\{s\}} \neq 0$: $\sum_{\{s\}} a_{\{s\}} s\chi_{\{s\}}(U) = 0$. Multiplying this equation by $\mathcal{D}^{\{t\}*}_{KL}(U)$, integrating over $[dU]$ and using Eq.(25) we have $a_{\{t\}}\alpha_{\{t\}}\delta^{\{t\}}_{kl} = 0$ for each representation $\{t\}$, which shows that $a_{\{t\}} = 0$ provided $\alpha_{\{t\}} \neq 0$.

The starting point that leads to the determination of the undotted representations $\{t\}$ which have non-zero values for $\alpha_{\{t\}}$ in (21) is the supersymmetric extension of the HCIZ integral given in Refs.[16, 17].

A convenient way of rewriting the standard HCIZ integral (defined in equation (98), Appendix 8.1) is in terms of its expansion in characters of the corresponding irreducible representations of the unitary group [3]

$$I(\lambda_1, \lambda_2; \beta) = \sum_{\{n\}} \frac{\beta^{|n|}}{|n|!}\frac{\sigma_{\{n\}}}{d_{\{n\}}}\chi_{\{n\}}(\lambda_1)\chi_{\{n\}}(\lambda_2), \tag{26}$$

where $d_{\{n\}}$ is the dimension of the representation $\{n\}$ and $\sigma_{\{n\}}$ and $|n|$ were already defined in (9) and (10).

It will prove convenient for our purposes, to obtain the analogous supercharacter expansion of the expression given in (97) for the SUSY HCIZ integral. This we do by using the orthogonality relations (21). The construction goes as follows: starting from the SUSY HCIZ integral

$$\tilde{I}(M_1, M_2; \beta) = \int [dU] e^{\beta str(M_1 U M_2 U^\dagger)} = \int [dU] \sum_{p=0}^{\infty} \frac{\beta^p}{p!} (str(M_1 U M_2 U^\dagger))^p, \quad (27)$$

and using the result in (12) we get

$$\tilde{I}(M_1, M_2; \beta) = \sum_{p=0}^{\infty} \frac{\beta^p}{p!} {\sum_{\{t\}}}' \sigma_{\{t\}} \int [dU] s\chi_{\{t\}}(M_1 U M_2 U^\dagger), \quad (28)$$

where the representations that contribute to the above primed sum are the ones for which $|t| = p$, for a given p.

Let us now calculate the integral

$$I_{\{t\}}(M_1, M_2) = \int [dU] s\chi_{\{t\}}(M_1 U M_2 U^\dagger). \quad (29)$$

Using the definition of the supercharacter together with the properties of a representation we have

$$\begin{aligned}
I_{\{t\}}(M_1, M_2) &= \int [dU] \sum_{a=1}^{sd_{\{t\}}} (-1)^{\epsilon_a} \mathcal{D}^{\{t\}}_{aa}(M_1 U M_2 U^\dagger) \\
&= \int dU \sum_{a,b,c,d=1}^{sd_{\{t\}}} (-1)^{\epsilon_a} \mathcal{D}^{\{t\}}_{ab}(M_1) \mathcal{D}^{\{t\}}_{bc}(U) \mathcal{D}^{\{t\}}_{cd}(M_2) \mathcal{D}^{\{t\}}_{da}(U^\dagger) \\
&= \int [dU] \sum_{a,b,c,d=1}^{sd_{\{t\}}} (-1)^{\epsilon_a} (-1)^{(\epsilon_b+\epsilon_c)(\epsilon_c+\epsilon_d)} \mathcal{D}^{\{t\}}_{ab}(M_1) \mathcal{D}^{\{t\}}_{cd}(M_2) \mathcal{D}^{\{t\}}_{bc}(U) \mathcal{D}^{\{t\}}_{da}(U^\dagger) \\
&= \sum_{a,b,c,d=1}^{sd_{\{t\}}} (-1)^{\epsilon_a} (-1)^{(\epsilon_b+\epsilon_c)(\epsilon_c+\epsilon_d)} \mathcal{D}^{\{t\}}_{ab}(M_1) \mathcal{D}^{\{t\}}_{cd}(M_2) \underbrace{\int dU \mathcal{D}^{\{t\}}_{bc}(U) \mathcal{D}^{\{t\}}_{da}(U^\dagger)}_{(-1)^{\epsilon_c} \delta_{ba} \delta_{cd} \alpha_{\{t\}}}
\end{aligned} \quad (30)$$

Finally we obtain

$$I_{\{t\}}(M_1, M_2) = \alpha_{\{t\}} \sum_{a,c=1}^{sd_{\{t\}}} (-1)^{\epsilon_a}(-1)^{\epsilon_c} \mathcal{D}_{aa}^{\{t\}} \mathcal{D}_{cc}^{\{t\}}$$
$$\Rightarrow I_{\{t\}}(M_1, M_2) = \alpha_{\{t\}} s\chi_{\{t\}}(M_1) s\chi_{\{t\}}(M_2) \qquad (31)$$

Substituting this last result in equation (28) we get the expansion in supercharacters for the SUSY HCIZ integral

$$\tilde{I}(M_1, M_2; \beta) = \sum_{\{t\}} \frac{\beta^{|t|}}{|t|!} \sigma_{\{t\}} \alpha_{\{t\}} s\chi_{\{t\}}(M_1) s\chi_{\{t\}}(M_2), \qquad (32)$$

which contains only undotted representations.

In virtue of the Lemma proved at the begining of this section, we see that the representations which contribute to Eq.(32) have supercharacters that form a linearly independent set.

Up to now, the $\alpha_{\{t\}}$'s are still unknowns. Next we identify the representations with non-zero $\alpha_{\{t\}}$. The basic expression we use is the character expansion in both sides of Eq.(101), which is

$$\sum_{\{t\}} \frac{\beta^{|t|}}{|t|!} \sigma_{\{t\}} \alpha_{\{t\}} s\chi_{\{t\}}(M_1) s\chi_{\{t\}}(M_2) = \sum_{\{p\}} \sum_{\{q\}} \frac{\beta^{|p|+|q|+mn}}{|p|!|q|!} \frac{\sigma_{\{p\}}\sigma_{\{q\}}}{d_{\{p\}}d_{\{q\}}} (-1)^{|q|} \times$$
$$\times \Sigma(\lambda_1, \bar{\lambda}_1) \chi_{\{p\}}(\lambda_1) \chi_{\{q\}}(\bar{\lambda}_1) \Sigma(\lambda_2, \bar{\lambda}_2) \chi_{\{p\}}(\lambda_2) \chi_{\{q\}}(\bar{\lambda}_2), \qquad (33)$$

where

$$\Sigma(\lambda, \bar{\lambda}) = \prod_{i=1}^{m} \prod_{\alpha=1}^{n} (\lambda_i - \bar{\lambda}_\alpha). \qquad (34)$$

Now we analize this equation by considering the following cases:

4.4.1 Case of $|t| < mn$

Before making any further analysis, from (33) we can immediately conclude that

$$\alpha_{\{t\}} = 0, \text{ for } |t| = 0, 1, \ldots, (mn-1). \qquad (35)$$

This is because in both sides of that equation we have a power series in β, and the right hand side (RHS) of it starts with β^{mn} while the left hand side (LHS) starts with β^0. The proof goes by assuming that some coefficients $\alpha_{\{t\}}$ are non-zero. The linear independence of the $s\chi_{\{t\}}(M)$'s associated to those representations imply that $\alpha_{\{t\}}$ must be zero.

4.4.2 Case of $|t| \geq mn$

As we just said before, Eq.(33) is a power series in β , so for a given power $|t|$ of β we obtain

$$\frac{1}{|t|!}\sum_{\{t\}}{}' \sigma_{\{t\}}\alpha_{\{t\}} s\chi_{\{t\}}(M_1) s\chi_{\{t\}}(M_2) = \sum_{\{p\}}\sum_{\{q\}} \frac{(-1)^{|q|}}{|p|!|q|!}\frac{\sigma_{\{p\}}\sigma_{\{q\}}}{d_{\{p\}}d_{\{q\}}} \times \\ \times \Sigma(\lambda_1,\bar{\lambda}_1)\chi_{\{p\}}(\lambda_1)\chi_{\{q\}}(\bar{\lambda}_1)\ \Sigma(\lambda_2,\bar{\lambda}_2)\chi_{\{p\}}(\lambda_2)\chi_{\{q\}}(\bar{\lambda}_2), \quad (36)$$

where the sum in the LHS is made for all tableaux having a fixed number of boxes $|t|$, while the sum over $\{p\}$ and $\{q\}$ in the RHS is restricted to

$$|p| + |q| = |t| - mn. \quad (37)$$

We now want to prove that Eq.(36) necessarily implies that

$$s\chi_{\{t\}}(M) = c^{\{t\}}_{\{p\},\{q\}}\Sigma(\lambda,\bar{\lambda})\chi_{\{p\}}(\lambda)\chi_{\{q\}}(\bar{\lambda}), \quad (38)$$

for some $\{p\}$ and $\{q\}$ satisfying (37) and for a certain representation $\{t\}$ that we will determine.

In order to extract more information from Eq.(36) let us consider an arbitrary supermatrix M_2, while we restrict the supermatrix $M_1 = \tilde{M}$ in such a way that one of its λ-eigenvalues be equal to one of its $\bar{\lambda}$- eigenvalues. Namely, let $\lambda_j = \bar{\lambda}_\beta$, for example. Then, in Eq.(36) we are left with

$$\frac{1}{|t|!}\sum_{\{t\}}\sigma_{\{t\}}\alpha_{\{t\}} s\chi_{\{t\}}(\tilde{M}) s\chi_{\{t\}}(M_2) = 0, \quad (39)$$

because $\Sigma(\lambda_1, \bar{\lambda}_1)$ becomes zero. If we look at this relation as a null linear combination of the supercharacters $s\chi_{\{t\}}(M_2)$ with coefficients

$$\gamma_{\{t\}} = \frac{1}{|t|!}\sigma_{\{t\}}\alpha_{\{t\}}s\chi_{\{t\}}(\tilde{M}), \tag{40}$$

we conclude that the coefficients $\gamma_{\{t\}}$ are all zero, because the supercharacters appearing in (39) constitute a linearly independent set. But $\sigma_{\{t\}}$ and $\alpha_{\{t\}}$ are different from zero, so that we are left with $s\chi_{\{t\}}(\tilde{M}) = 0$. Recalling that $s\chi_{\{t\}}(M)$ is a polynomial function of the λ_i's and the $\bar{\lambda}_\alpha$'s, we conclude from this relation that $s\chi_{\{t\}}(M)$ must be divisible by $(\lambda_j - \bar{\lambda}_\beta)$. That is to say

$$s\chi_{\{t\}}(M) = (\lambda_j - \bar{\lambda}_\beta)F_{j\beta}(\lambda, \bar{\lambda}), \tag{41}$$

where $F_{j\beta}(\lambda, \bar{\lambda})$ is another polynomial function of the eigenvalues. The same reasoning can be extended to every λ_i $(i = 1, ..., m)$ and $\bar{\lambda}_\alpha$ $(\alpha = 1, ..., n)$, and this implies that $s\chi_{\{t\}}(M)$ must have the form

$$s\chi_{\{t\}}(M) = \prod_{i=1}^{m}\prod_{\alpha=1}^{n}(\lambda_i - \bar{\lambda}_\alpha)\ P(\lambda, \bar{\lambda}) = \Sigma(\lambda, \bar{\lambda})\ P(\lambda, \bar{\lambda}). \tag{42}$$

In Eq.(42), $P(\lambda, \bar{\lambda})$ must be an homogeneous polynomial function of all the eigenvalues, because $s\chi_{\{t\}}(\tilde{M})$ and $\Sigma(\lambda, \bar{\lambda})$ are so. The degree of homogeneity of $s\chi_{\{t\}}(\tilde{M})$ and $\Sigma(\lambda, \bar{\lambda})$ is $|t|$ and mn, respectively. This means that the degree of homogeneity of $P(\lambda, \bar{\lambda})$ must be $|t| - mn$. Also, we know that $s\chi_{\{t\}}(\tilde{M})$ and $\Sigma(\lambda, \bar{\lambda})$ are symmetric functions in the eigenvalues λ_i, $\bar{\lambda}_\alpha$, separately, and so should be $P(\lambda, \bar{\lambda})$. Summing up then, $P(\lambda, \bar{\lambda})$ is: (i) an homogeneous polynomial function of degree $|t| - mn$ in all the eigenvalues and (ii) a symmetric function of the λ_i's and the $\bar{\lambda}_\alpha$'s, separately. Since the characters $\chi_{\{a\}}(\lambda)$ $\left(\chi_{\{b\}}(\bar{\lambda})\right)$ are polynomial homogeneous functions of degree $|a|$ $(|b|)$, which are symmetric in the eigenvalues λ_i $(\bar{\lambda}_\alpha)$ and constitute a complete linearly independent set, $P(\lambda, \bar{\lambda})$ can be written as

$$P(\lambda, \bar{\lambda}) = \sum_{\{a\},\{b\}} c^{\{t\}}_{\{a\},\{b\}} \chi_{\{a\}}(\lambda) \chi_{\{b\}}(\bar{\lambda}), \tag{43}$$

where the sum in $\{a\}$ and $\{b\}$ is rectricted by $|a|+|b| = |t|-mn$. Substituing this last relation in (42) we have

$$s\chi_{\{t\}}(M) = \Sigma(\lambda, \bar{\lambda}) \sum_{\{a\},\{b\}} c^{\{t\}}_{\{a\},\{b\}} \chi_{\{a\}}(\lambda) \chi_{\{b\}}(\bar{\lambda}). \tag{44}$$

Using the above expression in the LHS of (36) and comparing both sides of this equation, we conclude that the RHS of (44) should be saturated only with one coefficient, for a certain tableaux $\{t\}$, which precise form is yet to be determined. That is

$$s\chi_{\{t\}}(M) = c^{\{t\}}_{\{p\},\{q\}} \Sigma(\lambda, \bar{\lambda}) \chi_{\{p\}}(\lambda) \chi_{\{q\}}(\bar{\lambda}),$$

where $\{p\}$ and $\{q\}$ satisfy (37). Thus, we have proved our result in (38).

In order to identify the Young tableau corresponding to the representation $\{t\}$ we will make use of the fact that the tableaux structure is independent of whether we are dealing with a group or supergroup. Of course, the specific symmetrization (antisimmetrization) properties will be different in each case. In this way we will identify the tableaux by looking only at the known characters of the $U(m), U(n)$ subgroups of $U(m|n)$, in Eq.(38).

4.4.2.1 The case of $\{p\} = \{q\} = 0$

Here we have $|t| = mn$ and

$$s\chi_{\{t\}}(M) = c^{\{t\}}_{\{0\},\{0\}} \Sigma(\lambda, \bar{\lambda}). \tag{45}$$

In order to proceed with the required identifications, let us consider the particular case where the only non-zero block of the supermatrix M is the

$m \times m$ block, i.e.

$$M = \begin{pmatrix} M' & 0 \\ 0 & 0 \end{pmatrix}. \tag{46}$$

Then Eq.(45) reduces to

$$\chi_{\{t\}}(M') = c^{\{t\}}_{\{0\},\{0\}} (\prod_{i=1}^{m} \lambda_i)^n. \tag{47}$$

Using Weyl's formula for the character of the representations of the unitary group [23]

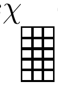
$$\chi_{\{r\}}(\lambda) = \frac{det(\lambda_i{}^{r_j+n-j})}{det(\lambda_i{}^{n-j})} \tag{48}$$

we conclude that the product of eigenvalues in (47) corresponds to the character of the representation $\{r\} = (r_1, r_2, \ldots, r_m)$ with $r_i = n$ of $U(m)$, which we denote by $\{r\} = \{mn\}$. So, pictorically, $\{r\}$ will look like

$$\{r\} = \{mn\} = \; m \overset{n}{\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}} \tag{49}$$

In this way we have that $\chi_{\{t\}}(M') = c^{\{t\}}_{\{0\},\{0\}} \chi_{(n,n,\ldots,n)}(M')$, which allows the identification of the representation $\{t\}$ as the one given by the tableau corresponding to $t_1 = t_2 = \ldots = t_m = n$, pictorically shown in (49), together with $c^{\{t\}}_{\{0\},\{0\}} = 1$. We will denote by $\{mn\}$ the representation just found. Besides, we identify $\Sigma(\lambda, \bar{\lambda})$ as the supercharacter of the representation referred to above:

$$s\chi_{\square}(M) = \Sigma(\lambda, \bar{\lambda}), \tag{50}$$



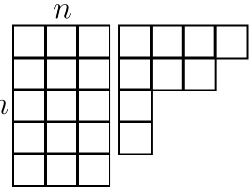
where $\Sigma(\lambda, \bar{\lambda})$ is given in (34).

4.4.2.2 The case of $\{p\} \neq 0$, $\{q\} = 0$

Here we have $|t| = |p| + mn$ and $s\chi_{\{t\}}(M) = c^{\{t\}}_{\{p\},\{0\}} \Sigma(\lambda, \bar{\lambda}) \chi_{\{p\}}(\lambda)$. Considering in this expression the same choice of M as in (46), we have $\chi_{\{t\}}(M') = c^{\{t\}}_{\{p\},\{0\}} (\prod_{i=1}^{m} \lambda_i)^n \chi_{\{p\}}(\lambda)$. Using again Weyl's formula we are able to make the identification $(\prod_{i=1}^{m} \lambda_i)^n \chi_{\{p\}}(\lambda) = \chi_{\{n+p\}}(\lambda)$, where by $\{n + p\}$ we mean the representation with Young tableau $(n + p_1, n + p_2, \ldots, n + p_m)$:

n

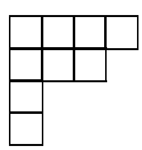
$$\{r\} = \{mn\}\{p\} = m$$



(51)

where we have generically drawn

$$\{p\} =$$



(52)

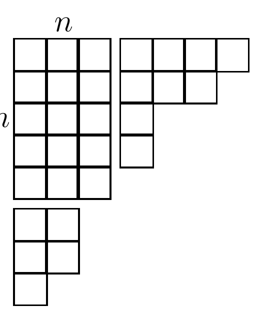
This leads to $\chi_{\{t\}}(M') = c^{\{t\}}_{\{p\},\{0\}}\chi_{(n+p_1,n+p_2,\ldots,n+p_m)}(\lambda)$ for this case and we conclude that $c^{\{t\}}_{\{p\},\{0\}} = 1$ with $\{t\}$ being the representation $(n+p_1, n+p_2, \ldots, n+p_m)$ of $U(m|n)$. Besides, we identify

$$s\chi_{\{t\}}(M) = \Sigma(\lambda, \bar{\lambda})\chi_{\{p\}}(\lambda) \tag{53}$$

4.4.2.3 The case of arbitrary $\{p\}$ and $\{q\}$

Now we discuss the main result of this section which states that the undotted representations of $U(m|n)$ with $\alpha_{\{t\}} \neq 0$ are characterized by the following Young tableaux:

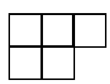
$$\{\tilde{t}\} = \{mn\}\{p\} = m$$
$$\{q\}^T$$



(54)

where $\{p\}$ is the same as in (52) and $\{q\}$ is pictorically identified with

$$\{q\} =$$



(55)

which, after interchanging its rows and columns, giving $\{q\}^T$, is put in the bottom left of $\{mn\}\{p\}$, producing (54). The representation $\{q\}^T$ is called the conjugate representation of $\{q\}$.

Besides identifying the particular representations involved, we are also able to calculate the corresponding non-zero normalization coefficient appearing in the orthogonality relations (21) for the representation $\{\tilde{t}\}$. It is given by

$$\alpha_{\{\tilde{t}\}} = (-1)^{|q|} \frac{|\tilde{t}|!}{|p|!|q|!} \frac{\sigma_{\{p\}}\sigma_{\{q\}}}{\sigma_{\{\tilde{t}\}}} \frac{1}{d_{\{p\}}d_{\{q\}}}. \tag{56}$$

Let us also remark that our expression (56) correctly reproduces the result

$$\alpha_{\{t\}} = \frac{1}{d_{\{t\}}} \tag{57}$$

for $U(N)$ (by making $m = N$ and $n = 0$). Note that in the $U(m|n)$ case, the $\alpha_{\{t\}}$ coefficient not only depends on the dimension of the representations involved, but also on the Clebsch-Gordan coefficients $\sigma_{\{t\}}$ and on the characteristic number $|t|$.

An important result that leads to the above conclusions is that

$$s\chi_{\{\tilde{t}\}}(M) = (-1)^{|q|}\Sigma(\lambda, \bar{\lambda})\chi_{\{p\}}(\lambda)\chi_{\{q\}}(\bar{\lambda}). \tag{58}$$

This relation is proved in Appendix 8.2, and after substituting it in equation (33), the result in (56) is obtained.

None of the results (50), (53) and (58) seem to be easily proved by standard methods like the supercharacter general formula (134) of Appendix 8.4 or the determinant formulas of Ref. [24]. This last formula consists in calculating the determinant of a matrix which components are supercharacters of completely symmetric representations. These supercharacters are expressed in terms of sums which, apparently, cannot be cast in closed form. Thus, our method provides an alternative derivation of the compact results already mentioned.

An immediate consequence of our relation (58) is that we can obtain the dimension $sd_{\{t\}}$ for the representations in $U(m|n)$ that arise in the supercharacter expansion, in terms of the dimension $d_{\{p\}}$ ($d_{\{q\}}$) of the $U(m)$ ($U(n)$) representations. Taking

$$M_0 = \begin{pmatrix} I_{m\times m} & 0 \\ 0 & -I_{n\times n} \end{pmatrix} \tag{59}$$

in (58) and observing that [24]

$$\begin{aligned} s\chi_{\{t\}}(M_0) &\to sd_{\{t\}} \\ \chi_{\{p\}}(\lambda) &\to d_{\{p\}} \\ \chi_{\{q\}}(-\bar{\lambda}) &\to (-1)^{|q|} d_{\{q\}} \\ \Sigma(\lambda, \bar{\lambda}) &\to 2^{mn}, \end{aligned}$$

we obtain the closed expression

$$sd_{\{\tilde{t}\}} = 2^{mn} d_{\{p\}} d_{\{q\}}, \tag{60}$$

for the dimensions of the representations of $U(m|n)$ characterized by the tableaux in (54).

Again, it should be possible to derive the general expression (60) for the dimension of the general tableaux (54) by using the formula developped by Balantekin and Bars [24] as a determinant of the supercharacters of completely symmetric representations. Nevertheless, we have not been able to reproduce the general result (60) in this way.

Before closing this section, let us illustrate the formula (58). Consider the representation

$$\{\tilde{t}\} = \ydiagram{3,2} \;, \tag{61}$$

whose supercharacter can be obtained with the aid of the character table of the symmetric group S_5 and the general expression (134), giving

$$s\chi_{\{\tilde{t}\}}(M) = \frac{1}{24}[(strM)^5 + 2(strM)^3 strM^2 - 4(strM)^2 strM^3 \\ -6strM strM^4 + 3strM(strM^2)^2 + 4strM^2 strM^3]. \tag{62}$$

Let us consider the tableau (61) as labeling a $U(1|2)$ representation. So according our notation (54) we have that

$$\{p\} = \square$$

$$\{q\} = \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \ .$$

Substituting the supermatrix M in its diagonal form,

$$M = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \bar{\lambda}_1 & 0 \\ 0 & 0 & \bar{\lambda}_2 \end{pmatrix}, \tag{63}$$

in the supercharacter expression (62) and after some algebra, we obtain

$$s\chi_{\{\tilde{t}\}}(M) = [(\lambda_1 - \bar{\lambda}_1)(\lambda_1 - \bar{\lambda}_2)]\ (\lambda_1)\ (\bar{\lambda}_1\bar{\lambda}_2), \tag{64}$$

which, for $U(1|2)$, can be equivalently written as

$$s\chi_{\{\tilde{t}\}}(M) = (-1)^2 \Sigma(\lambda, \bar{\lambda}) \chi_{\square}(\lambda) \chi_{\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}}(\bar{\lambda}). \tag{65}$$

Here $\chi_{\square}(\lambda)$ and $\chi_{\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}}(\bar{\lambda})$ are the $U(1)$ and the $U(2)$ characters of the corresponding representations, which can be taken from the character table in Appendix 8.4 and $\Sigma(\lambda, \bar{\lambda}) = (\lambda_1 - \bar{\lambda}_1)(\lambda_1 - \bar{\lambda}_2)$. We see, then, that (65) is in accordance with our general result (58).

5 Determination of $\alpha_{\{t\}}$ for complex conjugate and mixed representations

5.1 The case of complex conjugated (dotted) representations

We will need the following properties of $\bar{U}$

$$\left(str\bar{U}\right)^p = ((strU)^*)^p, \qquad str\left(\bar{U}^p\right) = \left(strU^p\right)^*, \tag{66}$$

which are just a consequence of the definition of $\bar{U}$ together with the group property of the $^-$operation. Since the supercharacter corresponding to the representation $\{\dot{p}\}$ has the same expression as the one corresponding to the representation $\{p\}$ except that U is replaced by $\bar{U}$, the properties (66) imply

$$s\chi_{\{\dot{p}\}}(U) = s\chi^*_{\{p\}}(U) = s\chi_{\{p\}}(U^\dagger), \tag{67}$$

where the Young tableau of the representation $\{\dot{p}\}$ is the same as that of the representation $\{p\}$ except that all boxes are dotted.

We will prove that

$$\alpha_{\{\dot{t}\}} = \alpha_{\{t\}}. \tag{68}$$

For this purpose we will look for two equivalent expressions for the integral

$$I_{\{n\}}(M_1, M_2) = \int [dU] s\chi_{\{n\}}(M_1 U M_2 U^\dagger)), \tag{69}$$

which we already presented in (29) and where M_1 and M_2 are hermitian supermatrices.

The first expression is equation (31), namely,

$$I_{\{n\}}(M_1, M_2) = \alpha_{\{n\}} s\chi_{\{n\}}(M_1) s\chi_{\{n\}}(M_2).$$

Before going to our second way of calculating (69) we observe that

$$s\chi_{\{n\}}((M_1UM_2U^\dagger)^\dagger) = s\chi_{\{n\}}(UM_2U^\dagger M_1) = s\chi_{\{n\}}(M_1UM_2U^\dagger),$$

which implies that for $B = M_1UM_2U^\dagger$ we have $s\chi_{\{n\}}(B^\dagger) = s\chi_{\{n\}}(B)$.

So, using (67) we have that $s\chi_{\{n\}}(M_1UM_2U^\dagger) = s\chi_{\{\dot{n}\}}(M_1UM_2U^\dagger)$ and therefore

$$I_{\{n\}}(M_1, M_2) = I_{\{\dot{n}\}}(M_1, M_2) = \alpha_{\{\dot{n}\}} s\chi_{\{\dot{n}\}}(M_1) s\chi_{\{\dot{n}\}}(M_2). \tag{70}$$

But for a hermitian supermatrix M we have that $s\chi_{\{\dot{n}\}}(M) = s\chi_{\{n\}}(M)$, and therefore

$$I_{\{n\}}(M_1, M_2) = \alpha_{\{\dot{n}\}} s\chi_{\{n\}}(M_1) s\chi_{\{n\}}(M_2). \tag{71}$$

Comparisson of (71) and (31) leads to our desired result in (68).

5.2 The case of mixed representations

We now prove the following expression for the α-coefficients for the mixed representations in the orthogonality relations:

$$\alpha_{\{\dot{p}\}|\{q\}} = \left[\frac{|p|!\ |q|!}{(|p|+|q|)!}\right]^2 \left[\frac{1}{\sigma_{\{p\}}\sigma_{\{q\}}}\right]^2 \sum_{\{t\}}{}' \ \rho_{\{t\}}^{\{p\},\{q\}} \ \sigma_{\{t\}}^2 \alpha_{\{t\}}. \tag{72}$$

Here the $\rho_{\{t\}}^{\{p\},\{q\}}$'s are the Clebsch-Gordan coefficients which appear in the decomposition of the tensor product of representations $\{p\}$ and $\{q\}$

$$\{p\} \otimes \{q\} = \oplus_{\{t\}}{}' \ \rho_{\{t\}}^{\{p\},\{q\}} \ \{t\}. \tag{73}$$

They are obtained by applying the Young tableaux rules for multiplying irreducible representations [25]. Our notation, $\sum'_{\{t\}}$ and $\oplus_{\{t\}}{}'$, means that the sums are carried only over the representations satisfying $|t| = |p| + |q|$.

Let us emphasize that all ingredients in our formula (72) are known : the $\alpha_{\{t\}}$'s are either null or given by (56), the $\rho_{\{t\}}^{\{p\},\{q\}}$'s are given by (73) and the $\sigma_{\{p\}}$'s are given by (10). Some examples of the relations (72) are given in Appendix 8.5.

Now, to prove (72) let us consider

$$I_{\{p\},\{q\}}(M_1, M_2) = \int [dU] s\chi_{\{\dot{p}\}|\{q\}}(M_1 U M_2 U^\dagger) \tag{74}$$

and, following the idea of the previous cases, we are going to calculate this expression in two different ways. The method that will be subsequently used consists basically in comparing these two expressions as polynomial expansions in $(strM_1^{k_1})^{l_1}(strM_2^{k_2})^{l_2}$. For our purposes it will be enough only to consider the highest power term

$$(str\ M_1)^{|p|+|q|}(str\ M_2)^{|p|+|q|}.$$

Since our argument is based only in the comparison of the highest power term $(str\ A)^{|p|+|q|}$ in the corresponding expressions, we next present the relevant approximations that will produce such terms. To begin with we consider the expansion

$$s\chi_{\{\dot{a}\}|\{b\}}(A) = s\chi_{\{\dot{a}\}}(A) s\chi_{\{b\}}(A) + \ldots, \tag{75}$$

which complete expression can be found in the Appendix 8.3. This is a function of supercharacters $s\chi_{\{\dot{r}\}}(A)$ and $s\chi_{\{s\}}(A)$ (with $|r| \leq |a|$, $|s| \leq |b|$). For our purposes it is enough only to consider the term written in (75). The remaining terms will contain the factor $(str(AA^\dagger)^i)^{r_i}$, thus lowering the power of $(strA)$. The next step is to express the corresponding supercharacters in terms of powers of supertraces. Again, what we need is to consider the highest power term

$$s\chi_{\{n\}}(A) = \frac{\sigma_{\{n\}}}{|n|!}(str\ A)^{|n|} + \dots. \tag{76}$$

of the full polynomial expresion (134).

In this way, using (75) and (76) for the case of a hermitian supermatrix M, we have that

$$s\chi_{\{\dot{a}\}|\{b\}}(M) = \frac{\sigma_{\{a\}}\sigma_{\{b\}}}{|a|!\ |b|!}(str\ M)^{|a|+|b|} + \dots, \tag{77}$$

where we have displayed only the highest power term in $(strM)$. We emphasize that the coefficient of $(str\ M)^{|a|+|b|}$ written in this last relation is exact.

With the above considerations we now proceed with the calculation. The direct integration over the supergroup in Eq.(74) gives

$$I_{\{p\},\{q\}}(M_1, M_2) = \alpha_{\{\dot{p}\}|\{q\}}\ s\chi_{\{\dot{p}\}|\{q\}}(M_1)\ s\chi_{\{\dot{p}\}|\{q\}}(M_2), \tag{78}$$

in analogy with (31). So, the first way of calculating (78) leads to

$$I_{\{p\},\{q\}}(M_1, M_2) = \left[\frac{\sigma_{\{p\}}\sigma_{\{q\}}}{|p|!\ |q|!}\right]^2 \alpha_{\{\dot{p}\}|\{q\}}(str\ M_1)^{|p|+|q|}(str\ M_2)^{|p|+|q|} + \dots \tag{79}$$

where only the term containing the highest power in $(str\ M_1)(str\ M_2)$ has been written.

Now, the second way of calculating $I_{\{p\},\{q\}}(M_1, M_2)$ consists in using the expansion (75) for the integrand in (74)

$$I_{\{p\},\{q\}}(M_1, M_2) = \int [dU]\, \underbrace{s\chi_{\{p\}}((M_1 U M_2 U^\dagger)^\dagger)}_{s\chi_{\{p\}}(M_1 U M_2 U^\dagger)}\ s\chi_{\{q\}}(M_1 U M_2 U^\dagger) + \dots. \tag{80}$$

and keeping only the highest power term. Next we combine the representations in the RHS of this relation

$$\int [dU] \; s\chi_{\{p\}}(M_1 U M_2 U^\dagger) \; s\chi_{\{q\}}(M_1 U M_2 U^\dagger) \\ = \int [dU] \quad s\chi_{\{p\}\otimes\{q\}}(M_1 U M_2 U^\dagger) \quad (81)$$

and subsequently we use the following Clebsh-Gordan expansion arising from (73)

$$s\chi_{\{p\}\otimes\{q\}}(A) = \sum_{\{t\}}{}' \; \rho_{\{t\}}^{\{p\},\{q\}} \; s\chi_{\{t\}}(A). \qquad (82)$$

So we have that

$$\int [dU] \; s\chi_{\{p\}}((M_1 U M_2 U^\dagger) \; s\chi_{\{q\}}(M_1 U M_2 U^\dagger) \\ = \sum_{\{t\}}{}' \; \rho_{\{t\}}^{\{p\},\{q\}} \quad \int [dU] s\chi_{\{t\}}(M_1 U M_2 U^\dagger)$$

$$= \sum_{\{t\}}{}' \; \rho_{\{t\}}^{\{p\},\{q\}} \; \alpha_{\{t\}} \; s\chi_{\{t\}}(M_1) \; s\chi_{\{t\}}(M_2). \qquad (83)$$

Therefore, substituting (83) in (80), we obtain

$$I_{\{p\},\{q\}}(M_1, M_2) = \sum_{\{t\}}{}' \rho_{\{t\}}^{\{p\},\{q\}} \; \alpha_{\{t\}} \; s\chi_{\{t\}}(M_1) \; s\chi_{\{t\}}(M_2) + \ldots, \qquad (84)$$

and using (76) we are left with

$$I_{\{p\},\{q\}}(M_1, M_2) = \frac{1}{[(|p| + |q|)!]^2} \sum_{\{t\}}{}' \rho_{\{t\}}^{\{p\},\{q\}} \alpha_{\{t\}} \; \sigma_{\{t\}}^2 (str \; M_1)^{|p|+|q|} \\ \times (str \; M_2)^{|p|+|q|} + \ldots. \qquad (85)$$

This is the result obtained by following the second method of calculation.

Finally we see that comparing the coefficient of the term $(str\ M_1)^{|p|+|q|}$ $\times(str\ M_2)^{|p|+|q|}$ of the two expressions for $I_{\{p\},\{q\}}(M_1, M_2)$, given in (79) and (85), we obtain the desired result stated in (72) for the α coefficients of the mixed representations.

As a consequence of our results (72) and (35) we derive the result

$$\alpha_{\{\dot{p}\}|\{q\}} = 0, \text{ for } |p| + |q| < mn, \tag{86}$$

which is similar to the one in (35).

Before closing this section we also observe that

$$\alpha_{\{\dot{p}\}|\{q\}} = \alpha_{\{\dot{q}\}|\{p\}}. \tag{87}$$

This property can be obtained from the relation (72) together with the fact that the tensor product of representations commutes.

6 Identities for the dimensions of the $U(N)$ representations

The complete procedure already followed for the determination of the $\alpha_{\{\dot{p}\}|\{q\}}$ coefficients may be repeated step by step for the case of $U(N)$, obtaining exactly the same relation (72), but with the substitution $\alpha_{\{t\}} \to \frac{1}{d_{\{t\}}}$ everywhere (see Eq. (57)) and also with the replacement $str \to tr$. In this way we obtain the remarkable result

$$\frac{1}{d_{\{\dot{p}\}|\{q\}}} = \left[\frac{|p|!\ |q|!}{(|p|+|q|)!}\right]^2 \left[\frac{1}{\sigma_{\{p\}}\sigma_{\{q\}}}\right]^2 \sum_{\{t\}}{}' \rho_{\{t\}}^{\{p\},\{q\}} \frac{\sigma_{\{t\}}^2}{d_{\{t\}}}. \tag{88}$$

for the dimensions of the irreducible representations of $U(N)$. In fact, since these dimensions are all well known from an independent calculation ($d_{\{t\}} = \chi_{\{t\}}(I_{N\times N})$), Eq.(88) provides an identity relating the dimensions of mixed and undotted representations of this group. Many examples of the identity (88) are shown explicitly, mutatis mutandis, in the Appendix 8.5. Let us illustrate this, for example, in the case of the representation $\boxdot\square\square$. According to Appendix 8.5 (second row) we have that

$$\frac{1}{d_{\boxdot\square\square}} = \frac{1}{9}\frac{1}{d_{\square\square\square}} + \frac{4}{9}\frac{1}{d_{\substack{\square\square\\ \square}}}, \tag{89}$$

where

$$d_{\boxdot\square\square} = \tfrac{1}{2}N(N+2)(N-1), \quad d_{\square\square\square} = \tfrac{1}{6}N(N+1)(N+2),$$
$$d_{\substack{\square\square\\ \square}} = \tfrac{1}{3}N(N+1)(N-1), \tag{90}$$

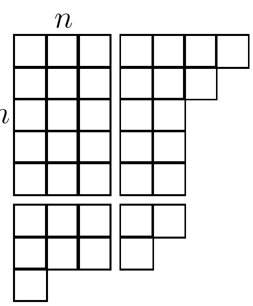
according to the formulas in Table I of the Appendix 8.4. The reader may verify that the identity in (89) is indeed fulfilled by expressions (90).

7 Relabeling of the $U(m|n)$ representations

The irreducible representations of $SU(m|n)$ have been characterized by Bars and Balantekin in terms of the Young tableaux notation [22]. We referred to this classification in section 3, when applying it to the $U(m|n)$ case. In this notation, to completely specify each representation, a set of numbers $(t_1, \ldots, t_k)$ is required, counting the number of boxes in the corresponding rows of the $\{t\}$ tableau. Contrary to what happens in the $U(N)$ $(SU(N))$ case, where the number of rows of the undotted tableau should not exceed N $((N-1))$, in the $U(m|n)$ $(SU(m|n))$ case there is no restriction for this number which, in principle, may be as large as wanted [22]. So, using the number of boxes on each row as a labelling of the $U(m|n)$ representations requires a non definite number of parameters.

Using our formula (58) we will show that it is possible to choose, at most, $(m+n)$ parameters in order to completly specify the undotted representations of $U(m|n)$. This is because representations of the type

$$\begin{array}{l} \{t\}_E = \{mn\}\{p\} = m \\ \qquad\quad \{q\}^T\{r\} \end{array}$$



(91)

do not exist, whenever representation $\{r\}$ is allowed to be placed there, that is, when $(\{q\}^T)_i = n$ for every $r_i \neq 0$ and $r_1 \leq p_m$.

To understand this property from our point of view, let us observe that formula (58) can be extended for representations $\{q\}^T \to \{q\}^T\{r\}$. In fact, in Appendix 8.2 we deal with this formula and validate it for a $\{q\}^T$ tableau having any number of rows and columns, as long as the Young tableaux rules are kept obeyed. Then for the representation $\{t\}_E$ (for which we mean 'Extended' $\{t\}$) we have that

$$s\chi_{\{t\}_E}(M) = (-1)^{|q|+|r|}\ \Sigma(\lambda,\bar{\lambda})\ \chi_{\{p\}}(\lambda)\ \chi_{\substack{\{q\}\\ \{r\}^T}}(\bar{\lambda}). \tag{92}$$

But any $U(n)$ tableau having more than n rows is forbidden, i.e.

$$\chi_{\substack{\{q\}\\ \{r\}^T}}(\bar{\lambda}) = 0,$$

so that

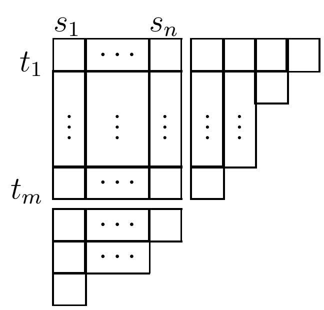
$$s\chi_{\{t\}_E}(M) = 0. \tag{93}$$

Given that the dimension of a supergroup representation can be calculated as $sd_{\{t\}} = s\chi_{\{t\}}(M_0)$, where M_0 is given in (59), we have that the dimension for $U(m|n)$ representations of the type $\{t\}_E$ is 0 and therefore they do not exist. This fact was already known in the literature [26], but it appears naturally in our calculations.

The above observation leads us to propose that any legal $U(m|n)$ representation can be completely characterized by

$$\{t\|s\} \equiv (t_1, \ldots, t_m\|s_1, \ldots, s_n), \tag{94}$$

in such a way that



(95)

where $(t_1, \ldots, t_m)$ is a $U(m)$ tableau denoting the number of boxes in the first m rows of $\{t\|s\}$, while $(s_1, \ldots, s_n)$ is a $U(n)$ tableau denoting the number of boxes of the first n columns of $\{t\|s\}$. These set of numbers completely specifies the existing undotted representations of $U(m|n)$.

If the t_i's and the s_j's in (95) satisfy respectively

$$t_i \geq n, \ s_j \geq m, \ (i = 1, \ldots, m; \ j = 1, \ldots, n), \tag{96}$$

then these numbers are completly independent. In this case $\{t\|s\}$ is a tableau of the type $\{\tilde{t}\}$ in (54). But if the t_i's and the s_j's do not all obey (96) then they will not be all independent. In fact, if $\{t\|s\}$ is such that every box of the tableaux is contained in the $\{mn\}$ tableau, then knowing all the t_i's is completely equivalent to knowing all the s_j's. Anyway, it is still true that knowing the $m+n$ numbers $(t_1,\ldots,t_m)$ and $(s_1,\ldots,s_n)$ (assumed to be given unambiguously and consistently) is enough to specify any $U(m|n)$ undotted representation.

Now, the analogue happens when considering purely dotted representations. Equation (58) is also valid for dotted representations since the corresponding derivation can be completely repeated for this case (the character and supercharacter expansions of the ordinary and supersymmetric HCIZ integral may be directly obtained for purely dotted representations). So following exactly the same arguments we are led to state that every $U(m|n)$ dotted representation can be completely specified by the notation $\{\dot{t}\|\dot{s}\} \equiv (\dot{t}_1,\ldots,\dot{t}_m\|\dot{s}_1,\ldots,\dot{s}_n)$. The pictorical tableau would be the same as in (95) but with all boxes dotted.

In the case of mixed representations, the undotted and the dotted parts will follow separately the previously established rules and the tableau will be abbreviated as $\{\dot{p}\|\dot{q}\}|\{u\|v\}$.

8 Appendices

8.1 The supersymmetric HCIZ integral

The basic tool we have used to determine the integration properties of the supergroup $U(m|n)$ in this work is the supersymmetric extension of the Harish-Chandra-Itzykson-Zuber (SUSY HCIZ) integral defined by [16, 17]

$$\tilde{I}(M_1, M_2; \beta) = \int [dU] e^{\beta str(M_1 U M_2 U^\dagger)}, \tag{97}$$

where M_1 and M_2 are hermitian $(m+n)\times(m+n)$ supermatrices, the integration is carried over the supergroup $U(m|n)$ and 'str' means the supertrace operation. This extension is made in complete analogy with the ordinary HCIZ integral which is [3]

$$I(N_1, N_2; \beta) = \int [dU] e^{\beta tr(N_1 U N_2 U^\dagger)}, \tag{98}$$

where N_1 and N_2 are $N \times N$ hermitian matrices, and the integration is carried over the group $U(N)$.

The calculation of the SUSY HCIZ integral has been made by following analogous steps to those taken by Itzykson and Zuber in the ordinary $U(N)$ case [16, 17]. In this approach, the integral is not calculated directly, but it is found as the solution of a differential equation. In the ordinary case this procedure is known as 'the difussion equation method', but in our case it was transformed to 'the Schrödinger equation method', in which, for convergence reasons we incorporated an imaginary factor 'i' to the difussion equation [16].

The result for the calculation of the SUSY HCIZ integral is

$$\begin{aligned}\tilde{I}(M_1, M_2; \beta) = \Sigma(\lambda_1, \bar{\lambda}_1)\Sigma(\lambda_2, \bar{\lambda}_2)\ \beta^{mn} \times (\beta)^{-\frac{m(m-1)}{2}} (-\beta)^{-\frac{n(n-1)}{2}} \times \\ \times \prod_{p=1}^{m-1} p! \prod_{q=1}^{n-1} q! \frac{det(e^{\beta \lambda_{1i} \lambda_{2j}})}{\Delta(\lambda_1)\Delta(\lambda_2)} \frac{det(e^{-\beta \bar{\lambda}_{1\alpha} \bar{\lambda}_{2\beta}})}{\Delta(\bar{\lambda}_1)\Delta(\bar{\lambda}_2)},\end{aligned} \tag{99}$$

where the diagonal supermatrices Λ_i $(i = 1, 2)$ contain the eigenvalues of the respective $(m + n) \times (m + n)$ hermitian supermatrices M_i $(i = 1, 2)$ (see (3) for the conventions).

Here, Δ is the usual Vandermonde determinant

$$\Delta(\lambda) = \prod_{i>j}(\lambda_i - \lambda_j), \quad \Delta(\bar{\lambda}) = \prod_{\alpha>\beta}(\bar{\lambda}_\alpha - \bar{\lambda}_\beta) \tag{100}$$

and the new function that appears is

$$\Sigma(\lambda, \bar{\lambda}) = \prod_{i=1}^{m}\prod_{\alpha=1}^{n}(\lambda_i - \bar{\lambda}_\alpha).$$

We observe that the polynomial $\Sigma(\lambda, \bar{\lambda})$ is completely symmetric under independent permutations of the λ_i's and the $\bar{\lambda}_\alpha$'s.

The expression (99) is completely determined up to a normalization factor related to that of the measure $[dU]$ of the supergroup. This situation is analogous to the standard IZ case where the required factor can be fixed directly from the corresponding expression by taking the limit $\Lambda_1, \Lambda_2 \to 0$ in a convenient way and demanding $\int[dU] = 1$, for example. This procedure leads to the correct factors in Eq.(3.4) of Ref.[3]. In our case, a similar limiting procedure leads to the conclusion that $\int[dU] \equiv 0$, precisely due to the appearance of the $\Sigma(\lambda, \bar{\lambda})$ functions in the numerator. This is not an unexpected result since we are dealing with odd Grassmann numbers. For this reason we have chosen the normalization factor in such a way that

$$\tilde{I}(M_1, M_2; \beta) = \Sigma(\lambda_1, \bar{\lambda}_1)\Sigma(\lambda_2, \bar{\lambda}_2)\beta^{mn} I(\lambda_1, \lambda_2; \beta) I(\bar{\lambda}_1, \bar{\lambda}_2; -\beta), \tag{101}$$

where

$$\begin{aligned} \tilde{I}(\Lambda_1, \Lambda_2; \beta) &: \quad \text{HCIZ integral over } U(m|n) \\ I(\lambda_1, \lambda_2; \beta) &: \quad \text{HCIZ integral over } U(m) \\ I(\bar{\lambda}_1, \bar{\lambda}_2; -\beta) &: \quad \text{HCIZ integral over } U(n), \end{aligned}$$

and the HCIZ integral in (98) is given by

$$I(N_1, N_2; \beta) = \int [dU] e^{\beta tr(N_1 U N_2 U^\dagger)} = \beta^{-N(N-1)/2} \prod_{p=1}^{N-1} p! \frac{det(e^{\beta \lambda_{1i} \lambda_{2,j}})}{\Delta(\lambda_1)\Delta(\lambda_2)}. \quad (102)$$

We comment that the derivation of the expression (99) has been performed for purely imaginary β in order to guarantee the convergence of the method. Since both sides of Eq.(99) exist for every complex β we have made an analytic continuation of the result to all the β complex plane.

8.2 Expression for $s\chi_{\{\tilde{t}\}}(M)$

Here we show that the supercharacter of the particular representation

$$\{\tilde{t}\} = \begin{matrix} \{mn\}\{p\} \\ \{q\}^T \end{matrix} \quad , \qquad (103)$$

which is pictorically shown in (54), has the compact expression

$$s\chi_{\{\tilde{t}\}}(M) = (-1)^{|q|} \Sigma(\lambda, \bar{\lambda}) \chi_{\{p\}}(\lambda) \chi_{\{q\}}(\bar{\lambda}). \qquad (104)$$

This formula was previously stated in Ref.[18] and we now present the complete proof of it.

The basic idea of the proof is to start from Eq.(53)

$$s\chi_{\{mn\}\{p\}}(M) = \Sigma(\lambda, \bar{\lambda}) \chi_{\{p\}}(\lambda), \qquad (105)$$

valid for every representation $\{p\} \in U(m)$, and subsequently to perform an induction process in the number of boxes of the representation $\{q\} \in U(n)$. We will work with the simplified notation (103) instead of the one in (54).

Let $\{q\}^T = \{v\} = (v_1, \ldots, v_a)$ be the tableau which is placed in the bottom left of $\{mn\}\{p\}$ in (103). Our proof will go in two steps. The first one consists in making induction in the number of boxes of the last row of

$\{v\}$, that is, v_a. The second step consists in assuming (104) to be valid for $\{v\}$ and showing that is also valid for $\{v\}'$, which is constructed from $\{v\}$ by adding an extra row consisting in only one box, that is, $\{v\}' = (v_1, \ldots, v_a, 1)$. Thus, both proofs imply that the $\{q\}^T = \{v\}$ tableau may be as wide and as long as the Young tableaux rules allow.

(**i**) Here we perform the induction process in the number of boxes of the last row of $\{v_a\}$. We assume that (104) is valid for a tableau $\{q\}^T = \{v\}$, with

$$\begin{aligned} v_i \leq n \ (i = 1, \ldots, a-1) \\ \text{and for } v_a = 0, \ldots, V_a < v_{a-1}. \end{aligned} \tag{106}$$

Our task consists in showing that it is also valid for $v_a = V_a + 1$. Let $\{v_o\} = (v_1, \ldots, v_{a-1})$. We start by multiplying

$$s\chi_{\substack{\{mn\}\{p\} \\ \{v_0\}}}(M) = (-1)^{|v_0|}\Sigma(\lambda, \bar{\lambda})\chi_{\{p\}}(\lambda)\chi_{\{v_0\}^T}(\bar{\lambda}), \tag{107}$$

by the expression [24],

$$s\chi_{(V_a+1)}(M) = \sum_{k=0}^{V_a+1} (-1)^k \chi_{(V_a+1-k)}(\lambda)\chi_{(k)^T}(\bar{\lambda}), \tag{108}$$

where (s) and $(s)^T$ denote the completely symmetric and the completely antisymmetric tableau, respectively, both with s boxes. Using the Young tableaux rules for multiplying representations we have

$$\begin{aligned} s\chi_{\left(\substack{\{mn\}\{p\} \\ \{v_0\}}\right)\otimes(V_a+1)}(M) = (-1)^{|v_0|}\Sigma(\lambda, \bar{\lambda}) \times \\ \times \sum_{k=0}^{V_a+1} (-1)^k \chi_{\{p\}\otimes(V_a+1-k)}(\lambda)\chi_{\{v_0\}^T\otimes(k)^T}(\bar{\lambda}) \end{aligned} \tag{109}$$

$$\Rightarrow \sum_{k=0}^{V_a+1} s\chi_{\substack{\{mn\}(\{p\}\otimes(V_a+1-k)) \\ (\{v_0\}\otimes(k))}}(M) = (-1)^{|v_0|}\Sigma(\lambda,\bar{\lambda}) \times$$
$$\times \sum_{k=0}^{V_a+1}(-1)^k \chi_{\{p\}\otimes(V_a+1-k)}(\lambda)\chi_{\{v_0\}^T\otimes(k)^T}(\bar{\lambda}). \quad (110)$$

Now, we separate the (V_a+1)th term in both sides

$$\sum_{k=0}^{V_a} s\chi_{\substack{\{mn\}(\{p\}\otimes(V_a+1-k)) \\ (\{v_0\}\otimes(k))}}(M) + s\chi_{\substack{\{mn\}\{p\} \\ (\{v_0\}\otimes(V_a+1))}}(M) =$$
$$(-1)^{|v_0|}\Sigma(\lambda,\bar{\lambda})\sum_{k=0}^{V_a}(-1)^k \chi_{\{p\}\otimes(V_a+1-k)}(\lambda)\chi_{\{v_0\}^T\otimes(k)^T}(\bar{\lambda})$$
$$+(-1)^{|v_0|+V_a+1}\Sigma(\lambda,\bar{\lambda})\chi_{\{p\}}(\lambda)\chi_{\{v_0\}^T\otimes(V_a+1)^T}(\bar{\lambda}). \quad (111)$$

Using the property $(\{a\}\otimes\{b\})^T = \{a\}^T \otimes \{b\}^T$ and the fact that the representations $(\{v_0\}\otimes(k))$ are all of the type $\{v\}$ (for which the hypothesis of induction (106) is valid), the sums in both sides of (111) are cancelled, leading to

$$s\chi_{\substack{\{mn\}(\{p\}) \\ (\{v_0\}\otimes(V_a+1))}}(M) = (-1)^{|v_0|+V_a+1}\,\Sigma(\lambda,\bar{\lambda}) \times$$
$$\times\chi_{\{p\}}(\lambda)\chi_{(\{v_0\}\otimes(V_a+1))^T}(\bar{\lambda}). \quad (112)$$

Using the Young tableaux rules we have that

$$\{v_0\}\otimes(V_a+1) = \begin{array}{c}\{v_0\}\\(V_a+1)\end{array} \oplus \sum_{l_1,\ldots,l_a}{}'(v_1+l_1,\ldots,v_{a-1}+l_{a-1},l_a), \quad (113)$$

where $\begin{array}{c}\{v_0\}\\(V_a+1)\end{array} \equiv (v_1,\ldots,v_{a-1},V_a+1)$ and the prime means that the sum is restricted to *i*) $l_1+\ldots+l_a = V_a+1$, where all the l_i's are non negative integers; *ii*) $l_a \leq V_a$ and *iii*) $v_i \geq v_{i+1}+l_{i+1}$, for $i=1,\ldots a-1$.

Therefore, using (113) in (112) we have that

$$\begin{aligned}\sum_{l_1,\ldots,l_a}{}'' s\chi_{\begin{array}{l}\{mn\}\{p\}\\ (v_1+l_1,\ldots,v_{a-1}+l_{a-1},l_a)\end{array}}(M) + s\chi_{\begin{array}{l}\{mn\}\{p\}\\ \{v_0\}\\ (V_a+1)\end{array}}(M) &= \\ \sum_{l_1,\ldots,l_a}{}''(-1)^{|v_0|+V_a+1}\Sigma(\lambda,\bar{\lambda})\chi_{\{p\}}(\lambda)\chi_{(v_1+l_1,\ldots,v_{a-1}+l_{a-1},l_a)^T}(\bar{\lambda}) & \\ +(-1)^{|v_0|+V_a+1}\Sigma(\lambda,\bar{\lambda})\chi_{\{p\}}(\lambda)\chi_{\left(\begin{array}{l}\{v_0\}\\ (V_a+1)\end{array}\right)^T}(\bar{\lambda}) & .\end{aligned} \tag{114}$$

Here the double prime means that the summation is further restricted to $l_a < V_a + 1$. In virtue of the hipothesis of induction the sums that appear in both sides of this equation are equal and we are left with

$$s\chi_{\begin{array}{l}\{mn\}\{p\}\\ \left(\begin{array}{l}\{v_0\}\\ (V_a+1)\end{array}\right)\end{array}}(M) = (-1)^{|v_0|+V_a+1}\Sigma(\lambda,\bar{\lambda})\chi_{\{p\}}(\lambda)\chi_{\left(\begin{array}{l}\{v_0\}\\ (V_a+1)\end{array}\right)^T}(\bar{\lambda}), \tag{115}$$

which ends this part of the proof.

(**ii**) We will now prove that if (104) is valid for $v_i \leq n$ $(i = 1, \ldots a)$, then it is also true that

$$s\chi_{\begin{array}{l}\{mn\}\{p\}\\ \{v\}\\ \Box\end{array}}(M) = (-1)^{|v|+1}\Sigma(\lambda,\bar{\lambda})\chi_{\{p\}}(\lambda)\chi_{\left(\begin{array}{l}\{r\}\\ \Box\end{array}\right)^T}(\bar{\lambda}). \tag{116}$$

We will follow very similar steps to those in (**i**). We multiply (104) by $s\chi_\Box(M) = \chi_\Box(\lambda) - \chi_\Box(\bar{\lambda})$, obtaining

$$s\chi_{\left(\begin{array}{l}\{mn\}\{p\}\\ \{v\}\end{array}\right)\otimes\Box}(M) = (-1)^{|v|}\Sigma(\lambda,\bar{\lambda})\chi_{\{p\}\otimes\Box}(\lambda)\chi_{\{v\}^T}(\bar{\lambda}) +$$

$$+(-1)^{|v|+1}\Sigma(\lambda,\bar{\lambda})\chi_{\{p\}}(\lambda)\chi_{\{v\}^T\otimes\square}(\bar{\lambda}) \qquad (117)$$

$$\Rightarrow s\chi_{\substack{\{mn\}(\{p\}\otimes\square)\\ \{v\}}}(M) + s\chi_{\substack{\{mn\}\{p\}\\ (\{v\}\otimes\square)}}(M) =$$
$$(-1)^{|v|}\Sigma(\lambda,\bar{\lambda})\chi_{\{p\}\otimes\square}(\lambda)\chi_{\{v\}^T}(\bar{\lambda}) + (-1)^{|v|+1}\Sigma(\lambda,\bar{\lambda})\chi_{\{p\}}(\lambda)\chi_{(\{v\}\otimes\square)^T}(\bar{\lambda}) \qquad (118)$$

Considering (104) for the case $\{p\}\to\{p\}\otimes\square$, the first term in both sides is the same and after cancelling it we have

$$s\chi_{\substack{\{mn\}\{p\}\\ (\{v\}\otimes\square)}}(M) = (-1)^{|v|+1}\Sigma(\lambda,\bar{\lambda})\chi_{\{p\}}(\lambda)\chi_{(\{v\}\otimes\square)^T}(\bar{\lambda}). \qquad (119)$$

Next we use the analogue formula to (113) which is

$$\{v\}\otimes\square = \substack{\{v\}\\ \square} \;\oplus \sum_{j_1,\ldots,j_a}{}^{'''}(v_1+j_1,\ldots,v_a+j_a), \qquad (120)$$

where the triple prime indicates the restrictions that the j_i's are non negative integers satisfying $j_1+\ldots+j_a = 1$ together with $v_i \geq v_{i+1}+j_{i+1}$, $(i = 1,\ldots,a-1)$. Using (120) in (119) we have that

$$\sum_{j_1,\ldots,j_a}{}^{'''} s\chi_{\substack{\{mn\}\{p\}\\ (v_1+j_1,\ldots,v_a+j_a)}}(M) + s\chi_{\substack{\{mn\}\{p\}\\ \{v\}\\ \square}}(M) =$$
$$\sum_{j_1,\ldots,j_a}{}^{'''}(-1)^{|v|+1}\Sigma(\lambda,\bar{\lambda})\chi_{\{p\}}(\lambda)\chi_{(v_1+j_1,\ldots,v_a+j_a)^T}(\bar{\lambda}) +$$
$$+(-1)^{|v|+1}\Sigma(\lambda,\bar{\lambda})\chi_{\{p\}}(\lambda)\chi_{\left(\substack{\{v\}\\ \square}\right)^T}(\bar{\lambda}). \qquad (121)$$

In virtue of the hipothesis of induction the sums in both sides are the same and we are left with the desired result.

8.3 Supercharacter of mixed representations

The general expression for the supercharacter of a mixed representation of the supergroup $GL(m|n)$ is the complicated expression given by

$$
s\chi_{\{\dot{a}\}|\{b\}}(A) = \sum_{l=0}^{k_{\{a\},\{b\}}} \sum_{\{m\}}{}'' \sum_{\{n\}}{}'' \delta(r_1 + 2r_2 + \ldots + lr_l - l)
$$

$$
\times \sum_{r_1,\ldots,r_l} \phi^{\{a\},\{b\}}_{l,\{m\},\{n\},\{r\}} \prod_{i=1}^{l} [(str(AA^\dagger)^i]^{r_i} \; s\chi_{\{\dot{m}\}}(A) s\chi_{\{n\}}(A), \tag{122}
$$

where A is an arbitrary $(m{+}n){\times}(m{+}n)$ supermatrix, $|m| = |a|{-}l$, $|n| = |b|{-}l$ and $k_{\{a\},\{b\}} = min\{|a|, |b|\}$. The coefficients $\phi^{\{a\},\{b\}}_{l,\{m\},\{n\},\{r\}}$ are known for all representations $\{a\}$ and $\{b\}$ of $GL(m)$ and $GL(n)$, respectively. Again, the double prime on each summation is to remind the reader of the constraints over which the sumations are performed.

In particular

$$
\phi^{\{a\},\{b\}}_{0,\{a\},\{b\},\{0\}} = 1, \tag{123}
$$

which corresponds to the terms in (122) which do not contain any factor $[str\ (AA^\dagger)^j]^{r_j}$. This term is precisely the one that we consider in Eq.(75).

The formula (122) is a generalization of the expression appearing in Ref. [15], which correspond to the superunitary case where $AA^\dagger = 1$

Simple examples of the formula (122) are

$$
s\chi_{\boxdot\square}(A) = s\chi_{\boxdot}(A) s\chi_{\square}(A) - \frac{1}{m-n} str(AA^\dagger), \tag{124}
$$

$$
s\chi_{\boxdot\square\square}(A) = s\chi_{\boxdot}(A) s\chi_{\square\square}(A) - \frac{1}{m-n} str(AA^\dagger) s\chi_{\square}(A), \tag{125}
$$

$$
s\chi_{\substack{\boxdot\square\\ \;\;\square}}(A) = s\chi_{\boxdot}(A) s\chi_{\substack{\square\\ \square}}(A) - \frac{1}{m-n} str(AA^\dagger) s\chi_{\square}(A). \tag{126}
$$

The reader may verify that these expressions coincide with the ones of Ref. [15] when A is a unitary supermatrix.

We are not going to perfom here the derivation of (122). Instead, we will present a simple example which illuminates the general procedure. Let us take the case $\{\dot{a}\}|\{b\} = \boxdot\!\square$. In order to construct the result for $s\chi_{\boxdot\!\square}(A)$ given in (124) we consider

$$\mathcal{D}^{\boxdot\!\square}_{ac,bd}(A) = \mathcal{D}^{\boxdot\times\square}_{ac,bd}(A) - \frac{1}{m-n}\delta_{bd}(-1)^{\epsilon_e}\mathcal{D}^{\boxdot\times\square}_{ac,ee}(A). \tag{127}$$

The above expression is obtained starting from the fundamental representations

$$\mathcal{D}^{\square}_{ij}(A) = A_{ij} \qquad \mathcal{D}^{\boxdot}_{ij}(A) = (-1)^{\epsilon_i(\epsilon_i+\epsilon_j)}A^*_{ij}, \tag{128}$$

and imposing the representation $\boxdot\!\square$ to be irreducible. The representation $\boxdot \times \square$ is given by

$$\mathcal{D}^{\boxdot\times\square}_{ac,bd}(A) = (-1)^{(\epsilon_a+\epsilon_c)(\epsilon_a+\epsilon_b)}A^\dagger_{ba}A_{cd}, \tag{129}$$

according to the general rule described in section 3.

The construction of (127) leads to

$$\mathcal{D}^{\boxdot\!\square}_{ac,bd}(A) = (-1)^{\epsilon_a+\epsilon_c)(\epsilon_a+\epsilon_b)}A^\dagger_{ba}A_{cd} - \frac{1}{m-n}\delta_{bd}(-1)^{\epsilon_e}(AA^\dagger)_{ca}. \tag{130}$$

Calculating the supercharacter $s\chi_{\boxdot\!\square}(A) = \sum(-1)^{\epsilon_a+\epsilon_c}\mathcal{D}^{\boxdot\!\square}_{ac,ac}(A)$,we obtain (124).

Let us observe that in order to get (124) we have begun from (127), which is the product of the representations $\boxdot$ and $\square$ to which we have substracted a similar term with a repeated index e. This index contraction produces the term $AA^\dagger$ in (130) and subsequently it becomes $str(AA^\dagger)$, after calculating the supercharacter.

When the same procedure is applied to more complicated cases like that of the representation $\boxdot\boxdot\boxdot\square\square$, we will obtain an expression of the type

$$s\chi_{\boxdot\boxdot\boxdot\square\square}(A) = s\chi_{\boxdot\boxdot\boxdot}(A)s\chi_{\square\square}(A) + a\ str(AA^{\dagger})s\chi_{\boxdot\boxdot}(A)s\chi_{\square}(A) + \\ +(b[str(AA^{\dagger})]^2 + c\ str(AA^{\dagger})^2)\ s\chi_{\boxdot}(A), \tag{131}$$

where the coefficients a, b y c take known numerical values. When the same procedure is extended to the general case, one obtains the formula (122).

8.4 Character and Supercharacter tables of $GL(N)$ and $GL(m|n)$

The character of any $U(N)$ representation may be written in terms of traces of powers of the fundamental ordinary and complex representations, U and $\bar{U}$.

A general formula for the character of an undotted representation $\{t\}$ is [27]

$$\chi_{\{t\}}(U) = \frac{1}{|t|!} \sum_{a_1,\ldots,a_{|t|}=0}^{|t|} \delta(a_1 + 2a_2 + \ldots + |t|a_{|t|} - |t|) h_{(a)} \chi_{(a)}^{\{t\}} \prod_{i=1}^{|t|} (trU^i)^{a_i}, \tag{132}$$

where the $\chi_{(a)}^{\{t\}}$ coefficients are the characters of the symmetric group of degree $|t|$, $S_{|t|}$, and

$$h_{(a)} = \frac{|t|!}{1^{a_1} a_1!\ 2^{a_2} a_2! \ldots |t|^{a_{|t|}} a_{|t|}!} \tag{133}$$

is the order of class $\{a\}$ of $S_{|t|}$.

The character of a dotted representation has exactly the same expression (132), but replacing $U \to U^{\dagger}$.

For the character of a mixed representation see Ref. [15] and our Appendix 8.3, replacing supertrace for trace, whenever it is necessary. Table I is constructed with these ingredients.

Replacing trace by supertrace [24], we obtain the analogue of formula (132) for the supercharacter of the representation $\{t\}$ of $GL(m|n)$:

$$s\chi_{\{t\}}(U) = \frac{1}{|t|!} \sum_{a_1,\ldots,a_{|t|}=0}^{|t|} \delta(a_1 + 2a_2 + \ldots + |t|a_{|t|} - |t|) h_{(a)} \chi_{(a)}^{\{t\}} \prod_{i=1}^{|t|} (strU^i)^{a_i}. \tag{134}$$

Some examples of this formula appear in Table II.

Table I. Characters and dimensions for some representations of the linear group $GL(N)$ (modified from [3])

Young Tableau	$\sigma_{\{n\}}$	$\chi_{\{n\}}(A)$	$d_{\{n\}}$
□	1	trA	N
□□	1	$\frac{1}{2}[(trA)^2 \ + \ trA^2]$	$\frac{1}{2}N(N+1)$
□ □	1	$\frac{1}{2}[(trA)^2 - trA^2]$	$\frac{1}{2}N(N-1)$
□□□	1	$\frac{1}{6}[(trA)^3 + 2trA^3 + 3trA\ trA^2]$	$\frac{1}{6}N(N+1)(N+2)$
□□ □	2	$\frac{1}{3}[(trA)^3 - trA^3]$	$\frac{1}{3}N(N+1)(N-1)$
□ □ □	1	$\frac{1}{6}[(trA)^3 + 2trA^3 - 3trA\ trA^2]$	$\frac{1}{6}N(N-1)(N-2)$
⊡	1	$trA^\dagger$	N
⊡□	-	$trA^\dagger\ trA - \frac{1}{N}trA\ A^\dagger$	$(N+1)(N-1)$
⊡□□	-	$\frac{1}{2}trA^\dagger\ [(trA)^2 \ + \ trA^2] - \frac{1}{N}trA\ tr(AA^\dagger)$	$\frac{1}{2}N(N+2)(N-1)$
⊡□ □	-	$\frac{1}{2}trA^\dagger\ [(trA)^2 \ - \ trA^2] - \frac{1}{N}trA\ tr(AA^\dagger)$	$\frac{1}{2}N(N+1)(N-2)$

Table II. Supercharacters for some representations of the linear supergroup $GL(m|n)$ (constructed from [3] and [24])

Young Tableau	$\sigma_{\{n\}}$	$s\chi_{\{n\}}(B)$	$sd_{\{n\}}$
$\square$	1	$strB$	$m+n$
$\square\square$	1	$\frac{1}{2}[(strB)^2+strB^2]$	$\frac{1}{2}((m+n)^2+(m-n))$
$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	1	$\frac{1}{2}[(strB)^2-strB^2]$	$\frac{1}{2}((m+n)^2-(m-n))$
$\square\square\square$	1	$\frac{1}{6}[(strB)^3+2strB^3+$ $+3strB\ strB^2]$	$\frac{1}{6}(m+n)((m+n)^2+3(m-n)+2)$
$\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}$	2	$\frac{1}{3}[(strB)^3-strB^3]$	$\frac{1}{3}(m+n)(m+n+1)(m+n-1)$
$\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$	1	$\frac{1}{6}[(strB)^3+2strB^3-$ $-3strB\ strB^2]$	$\frac{1}{6}(m+n)((m+n)^2-3(m-n)+2)$
$\boxdot$	1	$strB^\dagger$	$m+n$
$\boxdot\square$	-	$strB^\dagger\ strB-\frac{1}{m-n}str(B\ B^\dagger)$	$(m+n+1)(m+n-1)$
$\boxdot\square\square$	-	$\frac{1}{2}strB^\dagger[(strB)^2+strB^2]-$ $-\frac{1}{m-n}strB\ str(BB^\dagger)$	$\frac{1}{2}(m+n)((m+n)^2+(m-n)-2)$
$\begin{smallmatrix}\boxdot&\square\\&\square\end{smallmatrix}$	-	$\frac{1}{2}strB^\dagger[(strB)^2-strB^2]-$ $-\frac{1}{m-n}strB\ str(BB^\dagger)$	$\frac{1}{2}(m+n)((m+n)^2-(m-n)-2)$

8.5 $\alpha_{\{\dot{p}\}|\{q\}}$ coefficient table

Some examples of the formula (72) are given in the following table

$\alpha_{\{\dot{p}\}\|\{q\}}$	$= \left[\frac{\|p\|!\ \|q\|!}{(\|p\|+\|q\|)!}\right]^2 \left[\frac{1}{\sigma_{\{p\}}\sigma_{\{q\}}}\right]^2 \sum_{\{t\}} \rho^{\{p\},\{q\}}_{\{t\}} \sigma^2_{\{t\}} \alpha_{\{t\}}$ with $\|t\| = \|p\| + \|q\|$
$\alpha_{\begin{ytableau} \cdot & {} \end{ytableau}}$	$\frac{1}{4}\alpha_{\ydiagram{2}} + \frac{1}{4}\alpha_{\ydiagram{1,1}}$
$\alpha_{\begin{ytableau} \cdot & {} & {} \end{ytableau}} = \alpha_{\begin{ytableau} \cdot & \cdot & {} \end{ytableau}}$	$\frac{1}{9}\alpha_{\ydiagram{3}} + \frac{4}{9}\alpha_{\ydiagram{2,1}}$
$\alpha_{\begin{ytableau} \cdot & {} \\ \none & {} \end{ytableau}} = \alpha_{\begin{ytableau} \cdot & {} \\ \cdot \end{ytableau}}$	$\frac{4}{9}\alpha_{\ydiagram{2,1}} + \frac{1}{9}\alpha_{\ydiagram{1,1,1}}$
$\alpha_{\begin{ytableau} \cdot & \cdot & \cdot & {} \end{ytableau}} = \alpha_{\begin{ytableau} \cdot & {} & {} & {} \end{ytableau}}$	$\frac{1}{16}\alpha_{\ydiagram{4}} + \frac{9}{16}\alpha_{\ydiagram{3,1}}$
$\alpha_{\begin{ytableau} \cdot & \cdot & {} \\ \none & \cdot \end{ytableau}} = \alpha_{\begin{ytableau} \cdot & {} & {} \\ \none & {} \end{ytableau}}$	$\frac{9}{64}\alpha_{\ydiagram{3,1}} + \frac{1}{16}\alpha_{\ydiagram{2,2}} + \frac{9}{64}\alpha_{\ydiagram{2,1,1}}$
$\alpha_{\begin{ytableau} \cdot & {} \\ \cdot \\ \cdot \end{ytableau}} = \alpha_{\begin{ytableau} \cdot & {} \\ \none & {} \\ \none & {} \end{ytableau}}$	$\frac{9}{16}\alpha_{\ydiagram{2,1,1}} + \frac{1}{16}\alpha_{\ydiagram{1,1,1,1}}$
$\alpha_{\begin{ytableau} \cdot & \cdot & {} \\ \none & \none & {} \end{ytableau}} = \alpha_{\begin{ytableau} \cdot & {} & {} \\ \cdot \end{ytableau}}$	$\frac{1}{4}\alpha_{\ydiagram{3,1}} + \frac{1}{4}\alpha_{\ydiagram{2,1,1}}$
$\alpha_{\begin{ytableau} \cdot & {} \\ \cdot & {} \end{ytableau}}$	$\frac{1}{9}\alpha_{\ydiagram{2,2}} + \frac{1}{4}\alpha_{\ydiagram{2,1,1}} + \frac{1}{36}\alpha_{\ydiagram{1,1,1,1}}$
$\alpha_{\begin{ytableau} \cdot & \cdot & {} & {} \end{ytableau}}$	$\frac{1}{9}\alpha_{\ydiagram{2,2}} + \frac{1}{4}\alpha_{\ydiagram{3,1}} + \frac{1}{36}\alpha_{\ydiagram{4}}$

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