PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE
SCHOOL OF ENGINEERING

PRICE DETERMINATION AND OPTIMAL COMPOSITION FOR A SET OF MULTIPLE BUNDLES THAT WILL BE INTRODUCED TO MULTIPLE MARKET SEGMENTS

ALEJANDRO ENRIQUE CATALDO CORNEJO

Thesis submitted to the Office of Graduate Studies in partial
fulfillment of the requirements for the degree of
Doctor in Engineering Sciences

Advisor
JUAN CARLOS FERRER

Santiago de Chile, Marzo 2018

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A mis queridos padres, hermana,
sobrinos, cuñado y tía, a quienes
amo y lucho por y con ellos...

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ABSTRACT

This doctoral thesis explores new aspects of the problem faced by a company that must determine the optimal price and composition of a number of bundles involving products and/or services that will be offered in one or more market segments. We first assume that consumers' decisions are driven toward utility maximization and that competing companies do not react to each other in the short term. We then incorporate the consumers' maximum willingness to pay for a bundle as an additional factor in our analysis. Under these considerations, we study three research problems involving multiple bundles: (1) a single market segment and consumers that base their purchase decision only on the utility that each alternative brings to them; (2) multiple market segments and consumers that base their purchase decision only on the utility that each alternative brings to them; and (3) a single market segment and consumers that base their purchase decision on their maximum willingness to pay.

The three research problems were formulated as nonlinear mixed integer programs. For each problem, we first tried to obtain a closed-form solution for the optimal price of each bundle assuming known bundle compositions. However, this was only possible for problem (1). This allowed us to solve this problem using a two-step approach. Problem (2) was solved via a Taboo Search algorithm while problem (3) was solved through exhaustive enumeration of all the possible solutions.

The main results obtained from the models developed as part of this thesis are: (i) if the bundles are created considering multiple market segments simultaneously, then the optimal bundle composition may not include the optimal bundle composition for each individual market separately; (ii) if the consumers' maximum willingness to pay is considered, then the expected benefit for the company decreases significantly with respect to the case without this additional factor, and the resulting bundle composition is different.

Keywords: Pricing; Composition of bundles; Products Selection; Dynamic Programming; Constrained Multinomial Logit.

RESUMEN

Esta tesis doctoral explora nuevas aristas al problema que enfrenta una compañía cuando debe determinar la composición y precio óptimo para un conjunto de paquetes de productos y/o servicios (bundles) que ofertará en uno o más segmentos de mercado. Se asume inicialmente que los consumidores basan su decisión en la maximización de su utilidad y que las compañías competidorañías no reaccionan en el corto plazo. Posteriormente, se incorpora el supuesto de que los consumidores tienen una disposición máxima a pagar por un bundle. Bajo estas consideraciones, se definieron tres investigaciones considerando siempre múltiples bundles y: (1) un único segmento de mercado y consumidores que basan su decisión de compra solo en la utilidad que les produce cada alternativa, (2) múltiples segmentos de mercado y consumidores que basan su decisión de compra solo en la utilidad que les produce cada alternativa, y (3) un único segmento de mercado y consumidores que incluyen en su decisión de compra su máxima disposición a pagar.

Las tres investigaciones fueron formulados como modelos de programación no lineal mixtos. En todos lo casos se vio si existía una expresión cerrada para determinar el precio óptimo de cada bundle cuando era conocida la composición de 'estos. Solamente en la investigación (1) esto sucedió, pudiendo resolverse el problema en dos fases. Para la investigación (2) se desarrolló un algoritmo basado en búsqueda tab'u y para la investigación (3) se resolvió por enumeración exhaustiva.

Los resultados más relevantes son: si los bundles son confeccionados considerando simultáneamente múltiples segmentos de mercado, la composición escogida para ellos puede no incluir la composición óptima para cada segmento de mercado de manera individual y al incluir la máxima disposición a pagar de los consumidores, el resultado obtenido disminuye significativamente el beneficio esperado de la compañía respecto a no considerar esta máxima disposición a pagar, dado que la composición escogida no es la misma.

Palabras Claves: Fijación de Precios; Composición de Paquetes de Productos; Selección de Productos; Programación Dinámica; Logit Multinomial.

1. INTRODUCTION

Companies, in the search for alternative ways to maximize their profit, have found in the design of their products/services and in determining their optimal price a new way of doing so. In its simplest form, the bundling strategy consists of grouping goods and/or services into a bundle and selling them at an overall price that is generally more attractive to the consumer than the prices the goods and services would sell at if they were sold separately (Guiltinan and Gordon, 1988). Dukart (2000) and Swartz (2000) have suggested that bundled services are of greater interest to business customers than private individuals because the former require more services and prefer paying them on a single bill.

The practice of bundling product and services is growing in importance in many industries and some service sector firms now base their business strategies on this tool. Its significance compared to other strategies has been studied empirically by Schoenherr and Mabert (2011), while the impacts of different bundling strategies (single, pure and mixed) for information products have been compared by Li et al. (2013).

An example of the application of bundling is a cable television provider that offers a package of three movie channels (HBO, Cinemax and Cinecanal), two sports channels (ESPN and FoxSport) and three cultural channels (NatGeo, Discovery and History). Alternatively, it could offer four movie channels (HBO, Cinemax, Sony and Warner), no sports channels and two cultural channels (NatGeo and History). Some other cases of product composition are holiday packages (return flight, hotel stay and car rental), restaurant menus (entre, main dish and dessert) and telecommunications packages (local calling, long distance calling, Internet access and cellular phone).

The mathematical structure of the problems that considering together the problem of bundling and of pricing are of the non-linear type in integer variables, and therefore are problems structurally difficult to solve (generally NP-Hard). The current literature shows that the efforts to solve problems of this type have allowed to solve the case in which it is desired to determine the composition and optimal price of a single bundle that will be introduced to a single market segment, seminal problem studied by Bitran and Ferrer (2007), where the price and attractiveness of each bundle allows to describe the buying behavior of consumers through a multinomial logit model. However, the real problem faced by companies is a variety of bundles to be offered in the market, considering that the market is divided into several market segments that are homogeneous with each other.

For example, if a provider distributed 15 movie channels, 10 sports channels and 5 cultural channels, the number of different bundles it could design would be $2^{30} > 10^{10}$. Clearly, coming up with a design for a good group of, say, five bundles and deciding how to price them for a set of given market segment is not an easy matter.

Generally speaking, firms that design more than one bundle do so because they intend to supply them to different market segments. In this article, however, we present the case of a business that seeks to market multiple product bundles even though it will supply them to a single market segment. This is a significant problem that arises often when firms face restrictions on production levels, supply of inputs or storage space that prevent them from producing the quantity required for the optimal bundle.

1.1. Summary of Contributions/Original Contributions

The present doctoral research attempts to determine the optimal composition and prices of a set of b bundles a firm intends to supply to its markets. The proposed analysis is a natural extension of the problem definition and methodology suggested by Bitran and Ferrer (2007) and contributes to the state of the art in that it develops a model and a solution approach for the multiple bundles and multiple segments case.

However, given the complex nature of this problem, it is convenient to progress gradually in the compression, modeling and resolution techniques of the subproblems that compose it. Consequently, this research proposes the resolution of three problems: (i) to develop an approach that allows determining the composition and optimal price of multiple bundles that will be offered to a single market segment, (ii) to develop an approach that allows to determine the composition and optimal price of multiple bundles that will be offered in multiple market segments and (iii) to develop an approach that allows to determine the composition and optimal price of multiple bundles when they are offered to a single segment of market and have explicit considerations Of the consumers' maximum willingness to pay in that market segment.

The first of these problems contributes to the state of the art by extending the scope of the work developed by Bitran and Ferrer (2007), extending decisions to multiple bundles in a non-forced manner, but motivated by the fact that many companies could present various types of logistical problems –for example shipping limits from some of their suppliers, restricted capacity in their warehouses or sales premises– and thus have limitations to produce the amount required by the market for the optimal bundle

The extension of the scope from the first to the second of the problems implies understanding how the inclusion of multiple market segments affect the joint dynamics of the decisions, verify if it is possible to maintain the solution approach developed in Bitran and Ferrer (2007) for the case of a single bundle and a single market segment and whose scope is extended to the realistic case in the development of the problem (i), and to determine if there is an significant benefit in solving the problem of multiple bundles and multiple segments instead of solving the quantity from multiple bundles to a single market segment. The definition of this second problem, the search for an adequate solution approach and the answer to the previous questions have not been treated in the literature of pricing and bundling, and therefore, to face this problem is a contribution to the state of the art of this kind of problems.

The third of the problems allows explicitly including the maximum willingness to pay a consumer for a bundle. This strongly modifies the assumption made by Bitran and Ferrer (2007) –and that was maintained when defining and facing problems (i) and (ii)– who assume that consumers make their choice of buying based exclusively on the utility that that election produces them, and therefore, regardless of the willingness to pay that consumers have. By including this provision to be paid as a probabilistic condition, and not as a condition that must be met if or for each consumer, the use of MNL models, which has typically been used for this purpose, should be replaced by other discrete choice models. The explicit inclusion of this maximum willingness to pay and the construction/choice of an appropriate discrete choice model has not been addressed so far in the pricing and bundling literature, and as such, are a contribution to the state of the art.

1.2. Thesis Outline

The remainder of this thesis document is organized as follows. Chapter 2 reviews the state of the art in pricing and composition of bundles; Chapter 3 defines the problem to be addressed and the research subproblems that were defined to be able to face it; Chapter 4 solves each of the subproblems by breaking it down into two consecutively solved subproblems consisting of setting optimal prices of the multiple bundles and then determining their optimal composition, and also provides someone managerial insight; and finally, Chapter 5 the conclusions obtained in this doctoral research are expressed and possible extensions are proposed for future research.

2. LITERARY REVIEW

The pricing and composition of bundles have been widely investigated in the contexts of consumer behavior. Studies such as Yadav and Monroe (1993) and Yadav (1994) focus on the way consumers evaluate different product packages. (Suri and Monroe, 2001) examine the effects of contextual factors on consumer intentions to buy product packages. Their research shows that giving price discounts on individual products may significantly reduce the attractiveness to consumers of bundles containing the same goods. Herrmann et al. (1997) found that discounts and complementarity of package components are factors that usually increase consumers' purchase intentions.

Price discrimination is, perhaps, the most important phenomenon studied in the bundling context; in fact, it was pointed out by the influential author Stigler (1963). Later, Adams and Yellen (1976) introduced a graphical framework to analyze bundling. Schmalense (1981) included reservation prices for consumers whose demands follow Gaussian functions, by means of numerical experimentation and with some assumptions, he shows that pure bundling is more convenient for the firm than separate pricing. In the tie-in sales case, Dansby and Conrad (1984) conclude that even non-dominant companies could have incentives to bundle. Another approach is to use price discrimination in a bundling framework as a tool for maximizing the profit of companies, which was introduced by Hanson and Martin (1990), and followed by Venkatesh and Mahajan (1993), Bakos and Brynjolfsson (1999), and Wu et al. (2008), among others.

From the point of view of markets, segmentation is one of the most useful strategies in marketing, as mentioned Kumar et al. (2009) in the article in which they suggest that by modeling the different needs and preferences of the clients, one can obtain a greater benefit by treating them differently. The relevance of a segmentation procedure is to divide the market into relatively homogeneous partitions in terms of its buying behavior (Blattberg et al., 2010). This idea is then rebutted by Allenby et al. (1998), who propose a model in which demand heterogeneity and uncertainty can be included within the same market components. In the context of *pricing*, it is important for a company to group its potential clients with respect to its willingness to pay, in order to generate strategies for a group of clients (Sedghi et al., 2017; Varella et al., 2017).

In the case of markets where customers have varying levels of knowledge of the bundle components, Basu and Vitharana (2009) demonstrate that those with more knowledge exhibit greater variability of reservation price. The authors use an analytic model to determine the conditions for obtaining the maximum benefits on each of three sale strategies, the first based on individual components (no bundling), the second using bundles only (pure bundling) and the third a mixed approach. (Chakravarty et al., 2013) compares the bundling set and bundling gain when the production and retail functions are integrated in a single firm with the bundling set and bundling gain of three supply chain scenarios with different levels of coordination. In the first-best scenario, bundle margins are determined so as to optimize the profit of the whole supply chain.

The inclusion of the consumers' behavior through the application of Multinomial Logit (MNL) functions is a standard practice in the literature related to pricing. For instance, in facility location and pricing problems is important to emphasize the papers of Lüer-Villagra and Marianov (2013) and Zhang (2015), where besides of modeling the consumers' behavior with Logit discrete choice model, they also decide the opening of new facilities. Given the non-linearity of the optimization models, in both cases metaheuristics methods are applied in order to find solutions. On the other side, Aydin and Porteus (2008) find by means of conditions of first order the optimal joint solution of prices and levels of inventory, dealing with the decision assorted in an exogenous way regarding the model. In a similar approach, Ghasemy Yaghin et al. (2014) consider the policies of products arrangement and pricing in a scheme of a supply chain of two levels. In order to solve that formulation, an algorithm of Swarm Optimization Particles solution is used. In Shao (2015), a product layout problem is considered with regard to vertical and horizontal dimensions, where two processes of sequenced choice from the consumers are supposed. Both theoretical as well as applied findings are presented over the optimal solution of the price given the problem layout dimensions. In another application area, in Chen et al. (2014) is analyzed an evolutionary approach of genetic algorithms for the layout of a range of products, and the changes connected to the mix of products and pricing based on the market demand and manufacturing cost. For this purpose, a Logit mixed and a costing model based on activities are used.

Other works in the literature attempt to develop and empirically test a general choice model for bundles that takes into account the interdependencies among the items composing them (Chung and Rao, 2003). Various studies conclude that incorrect modeling of reservation prices may result in large losses for the seller and that offering mid-season packages may be more effective than individual product price reductions (Gürler et al., 2009; Bulut et al., 2009). Martínez et al. (2009) leads to the constraints analysis on the consumers' problem through the Constrained Multinomial Logit (CMNL), which can present, for instance, bounds for consumers' maximum willingness to pay, that modify the classical Multinomial Logit via cutoff functions. This is particularly interesting when explicitly considering the consumers' maximum willingness to pay by means of application of the CMNL with cutoff functions.

Hui et al. (2012) extends the previous literature on bundling, which usually assumes consumer heterogeneity along a single consumer attribute, showing that an individual consumer's demand function can be expressed as the interaction of the demand function's intercept (indicating the consumer's initial willingness to pay, that is, to pay for the first unit of a product) and slope (representing the consumer's appetite, that is, the quantity consumed when the product is free). Using a combination of analytical and numerical methods, they demonstrate that appetite heterogeneity favors mixed bundling while initial willingness-to-pay heterogeneity may reduce its profitability relative to pure bundling. Banciu and Ødegaard (2016) analyzes the problem that arises when the valuation of a bundle's components are dependant on each other. By modeling the combined density of the reservation price, they are able to show under which circumstances it is more profitable to supply just the bundle or the entire line of products (individual products and bundles).

The impact of developing bundling strategies for retail products has also been analyzed in the literature. McCardle et al. (2007) consider a company that sells two kinds of retail products, basic and fashion, and develop models to calculate the optimal bundle prices, order quantities, and profits obtained under bundling. As part of their analysis, the authors establish the conditions under which a bundling strategy is profitable and confirm that bundling profitability depends on the demand for individual products, bundling costs, and the nature of the relationship between the demands of the products to be bundled. Gürler et al. (2009) and Bulut et al. (2009) study the case of a company that sells two types of perishable products in a single period. Products are sold individually and as a bundle. The authors model the problem stochastically, assuming that the arrival rate of customers follows a Poisson process with a price dependent rate. Customer reservation prices are assumed to have a joint distribution. The authors study the impact of the reservation price distributions, initial inventory levels, product prices, demand arrival rates and costs of bundling on the expected profit for the company.

Hitt and Chen (2005) describe the concept of customized bundling in which customers can choose a bundle composed of M out of a total of N products for a fixed price. They show that this approach outperforms pure bundling and no bundling strategies when there are positive marginal costs and consumer valuations are heterogeneous. More recently, Wu et al. (2008) and Yang and Ng (2010) find that customized bundling increases benefits to the firm compared to pure and no bundling approaches when consumers do not value all of the component goods positively. On the other hand, Armstrong (2013) shows that when consumer valuations of products in a bundle are non-additive and when the products are supplied by separate sellers, a seller has a unilateral incentive to offer a discount for the purchase of the bundle if its customers buy products from the competition.

Yet other studies suggest an optimization approach that simultaneously decides bundle design and pricing. Hanson and Martin (1990) propose a mathematical programming formula for determining the bundle configuration and price that maximize benefits without explicitly considering the entire range of feasible solutions. In a later article, Venkatesh and Mahajan (1993) consider two dimensions of the consumer decision-making process (time and money) in determining the optimal price for a given bundle under different bundling strategies. A pair of publications by Green et al. (1991); Green and Krieger (1992) present an algorithm that solves a stochastic mixed integer programming model to determine a bundle's price and configuration. In more recent works, Proano et al. (2012) builds a mixed integer non-linear programming model to identify the number of vaccines in bundles of antigens and the range of feasible prices that maximize the sum of producer benefits and consumer surplus, and Ferrer et al. (2010) addresses the problem facing a seller of bundles composed of a service and a related product offered for a monthly flat rate plus a subscription fee to a customer base that changes over time. In the latter article, pricing policy is determined via a dynamic programming approach that identifies the long-run optimal number of customers for each bundle. Meyer and Shankar (2016) and Ødegaard and Wilson (2016), on the other hand, tackle the problem of determining the optimal price for a bundle composed of assets and services (hybrid bundle). They show how the optimal price changes with variations in the quality of the services or the products in the bundle.

Other studies in the literature discuss the algorithms used to define static and dynamic pricing policies for products or services that are sold under bundling strategies (Xia and Dube, 2007). Grigoriev et al. (2008)consider an auction involving a set of bidders and an unlimited supply of items. The authors solve an optimization problem that seeks to maximize the total revenue collected from all the bidders considering that not all bids need to be granted. Ferrer et al. (2010) study the pricing problem faced by a seller of bundles composed of a service and an associated product. Bundles are offered to customers on a subscription basis using a two-part tariff scheme (subscription plus a fixed monthly fee). The authors assume that the customer base can change over time and obtain an optimal pricing policy that maximizes the profit of a firm using a dynamic programming approach. They also conclude that, in the long run, there is an optimal number of customers associated with each bundle.

Finally, the problem of how to determine the composition and price of a single bundle that maximize the total expected benefits to the firm in a competitive market is analyzed by Bitran and Ferrer (2007), who study formulates a mixed integer non-linear programming model and shows that an optimal policy for bundle composition and price determination can be identified. The optimal pricing problem is first solved with a closed-expression that depends on the composition. The bundle composition problem is then handled with an algorithm that builds the bundle component by component considering the contribution made by each possible component alternative to the objective function. Noteworthy in this approach is that the algorithm solves to optimality in the same number of iterations as there are components in the bundle. In this doctoral research work we will extend the scope of this problem to multiple bundles and multiple segments, considering also for the case of a single bundle and a single segment a modification in the model of choice used to describe the buying behavior of the consumers, passing from the traditional MNL to the CMNL.

3. DEFINITION OF THE PROBLEM

Consider a hypothetical firm faced with the problem of determining the composition and prices of a single or a set of bundles to be supplied to a single or multiple segments of homogeneous customers. The company aims to define a set of bundles and find the optimal price for each one so as to maximize its benefits. The individual bundles each consist of a set of component products or services, and for each such set there is a group of known choice alternatives. Certain components must always be selected for inclusion in a given bundle while others may or may not be. The non-selection of a particular component in a given case is considered to be a valid option and can therefore be treated in a similar manner to the other alternatives.

The two main considerations that determine the composition of possible bundles are technological and competitive feasibility. Technological feasibility refers to certain technical specifications and requirements that bundles must satisfy such as containing certain components or containing them in certain quantities. Competitive feasibility denotes the requirement that the bundles offered by a firm be competitive with those supplied by its rivals. Though the firm's decision makers obviously do not control the composition or prices of competing bundles, they must take them into account.

A simpler version of this problem was posed and solved by Bitran and Ferrer (2007), who devised a method of finding the optimal composition and price of a single bundle. They showed that the optimal composition could be determined by building a space of all feasible bundles and then finding the optimal one within it. In the following subsections we will see three investigations that extend the development of this seminal problem.

3.1. Optimal Pricing and Composition of multiple bundles and single market segment

To specify our model we begin by assuming that the number of bundles to be supplied by our hypothetical firm to a single market segment is exogenously defined as b, where $b \geq$

1. As in Bitran and Ferrer (2007), a single consumer segment is considered. This means that no coupling constraint will be specified, thus dispensing with the need to determine the optimal bundle composition simultaneously over all market segments.

We also assume that a consumer's willingness to pay, known as the reservation price, varies from bundle to bundle given that the attractiveness of each bundle to the consumer is different. Following Bitran and Ferrer (2007) we call such valuations the *bundle attraction factor*. Note that it does not include price, which is treated as a separate attribute. This factor is measured by the weighted sum of the individual attraction factors of each bundle component. As with Green and Krieger (1992), we assume further that there are no interaction factors between the components.

The necessary notation for the model's sets, indexes, parameters and variables is set out below.

Sets:

$\mathcal{N}$: The set of bundles offered on the market by the competition. By abuse of notation, we say that $Card(\mathcal{N}) = \mathcal{N}$.

$\mathcal{M}$: Set of components in a bundle. By abuse of notation, we say that $Card(\mathcal{M}) = \mathcal{M}$.

$\mathcal{S}_j$: Set of alternatives for component j of a bundle.

Indexes:

k, l, t : indexes for the bundles offered by the firm.

n : Index for the bundles offered by the competition.

j : Index for the set of components of a bundle.

u : Index for the possible choice alternatives of a component.

i : Index for all bundles offered on the market.

Parameters:

b : Number of bundles to be constructed.

$\hat{p}_n$: Price at which bundle n is offered by the competition.

c_{uj} : Cost of alternative u of the set of components j.

I_{uj} : Attractiveness of alternative u of the set of components j.

a_j : Attractiveness weight of component j in the bundle composition.

γ : Utility of the bundles offered by the competition and of non-purchase.

g : Number of bundles offered by the competition.

β : Sensitivity of utility to bundle price.

Variables:

x_{ujk} : Binary variable indicating whether or not alternative u of the set of components j is chosen for the composition of bundle k.

X_k : Binary matrix representing the composition of bundle k.

p_k : Price of bundle k.

q_k : Probability that a consumer chooses bundle k.

c_{X_k} : Binary matrix representing the cost to the firm of bundle k.

I_{X_k} : Binary matrix representing the attractiveness of bundle k.

Decisions that are under the firm's control are the composition of the b bundles it offers to its customer segment and the prices it sets for them so as to maximize profits (recall that b is an integer greater than or equal to 1). Thus, these decisions are described by the attractiveness of the bundle $I_{X_k} = \sum_{j \in \mathcal{M}} a_j \sum_{u \in \mathcal{S}_j} I_{uj} x_{ujk}$, parameter $a_j > 0$ for all $j \in \mathcal{M}$, the bundle's cost to the firm $c_{X_k} = \sum_{j \in \mathcal{M}} \sum_{u \in \mathcal{S}_j} c_{uj} x_{ujk}$, parameter $c_{uj} \geq 0$ for all $u \in \mathcal{S}_j, j \in \mathcal{M}$ and the bundle's price p_k for $k = 1, \ldots, b$. Bundles not under the control of the firm are considered as given information and are characterized by attractiveness I_{X_n} where $n \in \mathcal{N}$ and price $\hat{p}_n$ where $n \in \mathcal{N}$. Since we assume that competitors do not react in the short run to the firm's decisions, the model is static rather than dynamic.

Consumers may choose any one of the bundles offered on the market, or decline to choose any. They are considered to be rational and random utility maximizers, the latter an increasingly common assumption in consumer choice models. This implies, first of all, that consumers will prefer the option giving them the greatest perceived utility. According to McFadden et al. (1973) they choose over a set of attributes, making an overall evaluation of every possible alternative on the basis of a random utility function and then picking the one that confers the highest value.

The random utility maximizer assumption also implies that utility is divided into a deterministic component and a stochastic one. Early work on this component-based approach in bundle choice models was developed by Hanson and Martin (1990) and then Venkatesh and Mahajan (1993). In our formulation, the perceived utility conferred by bundle i on a given consumer in the single consumer segment is specified as the sum of a deterministic component denoted $V(\cdot)$ that depends on the price and composition of the bundle, and a stochastic component expressed by an independent disturbance term ε_i that includes all factors preventing the consumer from determining a good's exact utility. The random utility model is $U_i = V(p_i, X_i) + \varepsilon_i$ for all $i = 1, \ldots, \mathcal{N} + b + 1$, where the deterministic utility is $V(p_i, X_i) = I_{X_i} + \beta p_i$ and parameter $\beta < 0$ is a scalar that expresses the sensitivity of utility to price.

Ben-Akiva et al. (1985) and McFadden et al. (1973) show that the probability of choosing a bundle from among a set of alternatives is given by the closed-form expression $q(p_i, X_i) = \frac{e^{V(p_i, X_i)}}{\xi + e^{V(p_i, X_i)}}$, where ξ is the utility of all the bundles offered to the consumer segment plus the non-purchase utility. Since the firm supplies b bundles to the market, the probability that the firm's bundle i is chosen is given by

$$q_i(p_1, \ldots, p_b, X_1, \ldots, X_b) = \frac{e^{V(p_i, X_i)}}{\gamma + \sum_{k=1}^{b} e^{V(p_k, X_k)}} \tag{3.1}$$

The probability of choosing a bundle is thus represented by a multinomial logit (MNL) model (Ben-Akiva et al., 1985; Zhang, 2015). If we let V_0 be the deterministic non-purchase utility, then $\gamma = e^{V_0} + \sum_{n\in\mathcal{N}} e^{V(\hat{p}_n, X_n)}$. Among the properties of (3.1), as explained in Bitran and Ferrer (2007), is that $lim_{p_i\to\infty} q_i(p_1,\ldots,p_b,X_1,\ldots,X_b) = 0$, $0 \leq q_i(p_1,\ldots,p_b,X_1,\ldots,X_b) \leq 1$ and $0 \leq \sum_{i=1}^{b} q_i(p_1,\ldots,p_b,X_1,\ldots,X_b) \leq 1$.

Given that we are using a logit model to simultaneously determine the composition of multiple bundles which will all be different from, and perfect substitutes for, each other, the solution must also satisfy the independence of irrelevant alternatives (IIA) property. This condition is handled by Equation (3,8), which makes pairwise comparisons of all the chosen bundles to check that there are no more than $\mathcal{M} - 1$ identical components, thus ensuring each of the b bundles differs from every other one in at least one component.

With the foregoing as our basis, we now set out our proposed mixed integer non-linear programming model for solving the *pricing and composition of multiple bundles problem* (MBP) model in order to determining the composition and pricing of the b bundles that will be supplied to a single market segment.

(MBP)
$$\max_{p_1,\ldots,p_b,X_1,\ldots,X_b} \Pi(p_1,\ldots,p_b,X_1,\ldots,X_b) = \sum_{k=1}^{b} q_k(p_1,\ldots,p_b,X_1,\ldots,X_b)(p_k - c_{X_k}) \tag{3.2}$$

$$s.t: \; q_k(p_1,\ldots,p_b,X_1,\ldots,X_b) = \frac{e^{I_{X_k}+\beta p_k}}{\gamma + \sum_{l=1}^{b} e^{I_{X_l}+\beta p_l}} \qquad \forall\, k = 1,\ldots,b. \tag{3.3}$$

$$\gamma = e^{V_0} + \sum_{n\in\mathcal{N}} e^{I_{X_n}+\beta\hat{p}_n} \tag{3.4}$$

$$c_{X_k} = \sum_{j\in\mathcal{M}} \sum_{u\in\mathcal{S}_j} c_{uj} x_{ujk} \qquad \forall\, k = 1,\ldots,b. \tag{3.5}$$

$$I_{X_k} = \sum_{j \in \mathcal{M}} a_j \sum_{u \in \mathcal{S}_j} I_{uj} x_{ujk} \qquad \forall\, k = 1, \ldots, b. \tag{3.6}$$

$$\sum_{u \in S_j} x_{ujk} = 1 \qquad \forall\, j \in \mathcal{M};\ \forall\, k = 1, \ldots, b. \tag{3.7}$$

$$\sum_{j \in \mathcal{M}} \sum_{u \in S_j} x_{ujk} x_{ujl} \leq \mathcal{M} - 1 \qquad \forall\, k = 1, \ldots, b;\ \forall\, l = k+1, \ldots, b. \tag{3.8}$$

$$x_{ujk} \in \{0, 1\} \qquad \forall\, j \in \mathcal{M};\ \forall\, u \in \mathcal{S}_j;\ \forall\, k = 1, \ldots, b. \tag{3.9}$$

$$p_k \geq 0 \qquad \forall\, k = 1, \ldots, b. \tag{3.10}$$

The objective function in (3.2) seeks to maximize the expected value of the benefit obtained by the company, building it as the sum of the individual benefits obtained for the sale of each bundle. The various constraints in the model may be described briefly as follows. Constraint (3.3) determines the probability an individual chooses one of the firm's bundles given its composition and price. Constraint (3.4) determines the consumer utility of the bundles offered by the competition and of non-purchase. Constraints (3.5) and (3.6) determine the cost and attractiveness of each bundle designed. Constraint (3.7) imposes that in the design of each bundle a single alternative is chosen for each component. Constraint (3.8), as already noted above, ensures that no two bundles have identical compositions. Finally, constraints (3.9) and (3.10) define the nature of the variables.

As can be appreciated, the model as a whole is a non-linear programming formulation with continuous and binary variables. Dobson and Kalish (1993) have shown that for both product selection and pricing problems, such models are usually NP-hard given that the product selection problem is a set covering problem. Nevertheless, Bitran and Ferrer

(2007) were able to solve the problem for the case where $b = 1$. In the next chapter, we develop an approach that solves the MBP to optimality in a reasonable amount of time.

3.2. Optimal Pricing and Composition of multiple bundles and multiple market segments

In this section we present the problem to be addressed is that of a firm which must determine the composition and prices of multiple bundles it intends to sell in multiple competitive market segments with the objective of maximizing profit. The number of bundles to be offered is known and equal to b for every market segment. Each of the b bundles is made up of a set of components, and for every component there is a group of alternative choices. Certain components must always be present in a given bundle while others may or may not be. Thus, not including certain components is a valid choice alternative.

A seminal version of this problem in which one bundle is supplied to a single market segment was solved by Bitran and Ferrer (2007). The problem was then extended in the case of multiple bundles but still for a single market segment. In the present paper we further extend these previous studies to the case of b bundles offered in W market segments, where $b > 1$ and $W > 1$. The price of any particular bundle is assumed to be the same in every segment.

In modeling the demand for the bundles in each of the W segments we assume that a consumer's willingness to pay varies as a function of the segment they belong to and is different for each bundle. This last point is justified on the grounds that the attractiveness to consumers of each bundle will differ from segment to segment. The value consumers place on a bundle is specified for present purposes by a "bundle attraction factor" as defined in Bitran and Ferrer (2007). It is equal to the weighted sum of the individual attraction factors of each bundle component. Following Green and Krieger (1992) we assume there is no interaction between components. Note also that the factor excludes the price attribute, which is treated separately.

We begin the formulation of our proposed model with the necessary notation for the various sets, indexes, parameters and variables.

The sets:

$\mathcal{W}$: The set of market segments in which the firm intends to offer product bundles. By abuse of notation we write $Card(\mathcal{W}) = \mathcal{W}$.

$\mathcal{M}$: The set of components in a bundle. By abuse of notation we write $Card(\mathcal{M}) = \mathcal{M}$.

$\mathcal{N}$: The set of bundles offered by the firm's competition. By abuse of notation we write $Card(\mathcal{N}) = \mathcal{N}$.

$\mathcal{S}_j$: The set of choice alternatives for component j of a bundle.

The indexes:

i : The index of the market segments.

k, l, t : The indexes of the bundles offered by the firm.

n : The index of the bundles offered by the competition.

j : The index of the set of bundle components.

u : Index of the possible choice alternatives for a bundle component.

The parameters:

b : The number of bundles to be designed and offered by the firm.

$\hat{p}_n$: Price at which the competition offers bundle n.

c_{ju} : Cost to the firm of choosing alternative u for component j.

I^i_{ju} : Attractiveness of component j if alternative u is chosen for sale in market segment i.

a^i_j : Weight of component j in the attractiveness of a bundle for market segment i.

H^i : Number of consumers in market segment i.

γ^i : Utility of a non-purchase and of the bundles offered by the competition in market segment i.

β^i : Sensitivity of utility to the price of a bundle in market segment i (does not depend on the bundle observed by the individual consumer in the segment).

The variables:

x_{juk} : Binary variable indicating whether or not is chosen for component j the alternative u in the composition of bundle k.

X_k : An $h \times m$ binary matrix representing the composition of bundle k.

p_k : Price of bundle k

q_k^i : The probability that a given consumer from market segment i chooses bundle k.

c_{X_k} : An $h \times m$ binary matrix that represents the cost to the firm of bundle k.

$I_{X_k}^i$: An $h \times m$ binary matrix that represents the attractiveness of bundle k in market segment i.

Information on the bundles not under the control of the firm (i.e., the competitors' bundles) is considered in the proposed model to be given. This information consists of the attractiveness of these bundles in each market segment (hereafter simply "segment"), which is denoted as $I_{X_n^i}$ where $n \in \mathcal{N}$ and $i \in \mathcal{W}$, and their respective prices $\hat{p}_n$ where $n \in \mathcal{N}$ (recall that the price of a given bundle is the same in all market segments). The decisions under the control of the firm are the composition and price of each of the bundles b it offers to the $\mathcal{W}$ segments. The information on these decisions therefore consists of the attractiveness of the firm's bundles in each market segment $I_{X_k}^i$, the cost to the firm of the bundles' composition c_{X_k} and the bundles' prices p_k. As with Cataldo and Ferrer (2017) we assume that the competition does not react in the short run to the firm's decisions, which implies that the proposed model is static.

The consumers in the model can purchase only one bundle among those offered in their segment or none at all. They are considered to be utility maximizers, an assumption shared by many other consumer choice models, and thus make their choices based on their evaluation of a set of attributes (McFadden et al., 1973). More specifically, they choose based on an overall evaluation of each possible choice as determined by their utility functions, selecting the option that gives them the most utility.

We therefore model the utility of each bundle for a given consumer in a given segment as the sum of a deterministic component and a stochastic component. The deterministic part, denoted $V(\cdot)$, depends on the price and the composition of the bundle while the stochastic part is an independent disturbance term ε_k that embraces all of the factors that prevent a consumer from accurately determining a product's utility. The utility perceived by consumers in segment i of choosing bundle k is written as $U_k^i = V^i(p_k, X_k) + \varepsilon_i$ for all $i = 1, ..., \mathcal{W}$ and $k = 1, ..., \mathcal{N} + b + 1$, (i.e., the competition's bundles, the firm's bundles to be designed and bundles not purchased) where the deterministic utility is given by $V^i(p_k, X_k) = I^i_{X_k} + \beta^i p_k$, the attractiveness of the composed bundle is $I^i_{X_k} = \sum_{j \in \mathcal{M}} a^i_j \sum_{u \in \mathcal{S}_j} I^i_{ju} x_{juk}$, the parameter $a^i_j > 0$ for all $j \in \mathcal{M}$ and $i \in \mathcal{W}$, and parameter $\beta^i < 0$ is a scalar that gives the sensitivity of utility to price for consumer segment $i \in \mathcal{W}$.

Thus, given that the firm will design b bundles, the probability a bundle b will be chosen by a consumer in segment i is given by

$$q^i_k := q^i_k\left(p_1, \dots, p_b, X_1, \dots, X_b\right) = \frac{e^{V^i(p_k, X_k)}}{\gamma^i + \sum\limits_{l=1}^{b} e^{V^i(p_l, X_l)}}, \tag{3.11}$$

where, given that V^i_0 is the deterministic utility of a consumer in segment i who decides not to purchase a bundle (whether of the firm or the competition), $\gamma^i = e^{V^i_0} + \sum\limits_{n \in \mathcal{N}} e^{V^i(\hat{p}_n, X_n)}$. As is explained in Bitran and Ferrer (2007), one of the properties of (3.1) is that:

$lim_{p_k \to \infty} q^i_k(p_1, \dots, p_b, X_1, \dots, X_b) = 0$, $0 \leq q^i_k(p_1, \dots, p_b, X_1, \dots, X_b) \leq 1$

and $0 \leq \sum_{k=1}^{b} q^i_k(p_1, \dots, p_b, X_1, \dots, X_b) \leq 1$.

Given all of the above, we now set out our proposed model for determining the optimal price and composition of each of the b bundles to be offered simultaneously in each segment W. We denote this the *multiple bundles and multiple segments problem* (MBMSP).

$$\text{(MBMSP)} \qquad \max_{p_1, \dots, p_b, X_1, \dots, X_b} \Pi(p_1, \dots, p_K, X_1, \dots, X_K) = \sum_{i=1}^{W} \sum_{k=1}^{b} H^i q^i_k(p_k, X_k)(p_k - c_{X_k}) \tag{3.12}$$

$$s.t:\ q_k^i = \frac{e^{I^i_{X_k} + \beta^i p_k}}{\gamma^i + \sum\limits_{l=1}^{b} e^{I^i_{X_l} + \beta^i p_l}} \qquad \forall\, i = 1, \ldots, W;\ k = 1, \ldots, b. \tag{3.13}$$

$$\gamma^i = e^{V_0^i} + \sum_{n \in \mathcal{N}} e^{I^i_{X_n} + \beta^i \hat{p}_n} \qquad \forall\, i = 1, \ldots, W. \tag{3.14}$$

$$c_{X_k} = \sum_{j \in \mathcal{M}} \sum_{u \in \mathcal{S}_j} c_{ju} x_{juk} \qquad \forall k = 1, \ldots, b. \tag{3.15}$$

$$I^i_{X_k} = \sum_{j \in \mathcal{M}} a^i_j \sum_{u \in \mathcal{S}_j} I_{ju} x_{juk} \qquad \forall\, i = 1, \ldots, W;\ k = 1, \ldots, b. \tag{3.16}$$

$$\sum_{u \in S_j} x_{juk} = 1 \qquad \forall\, j \in \mathcal{M}; k = 1, \ldots, b. \tag{3.17}$$

$$\sum_{j \in \mathcal{M}} \sum_{u \in S_j} x_{ujk} x_{ujl} \leq \mathcal{M} - 1 \qquad \forall\, k = 1, \ldots, b; l = k+1, \ldots, b. \tag{3.18}$$

$$x_{juk} \in \{0, 1\} \qquad \forall\, j \in \mathcal{M}; u \in \mathcal{S}_j; k = 1, \ldots, b. \tag{3.19}$$

$$p_k \geq 0 \qquad \forall k = 1, \ldots, b. \tag{3.20}$$

The objective function in (3.12) seeks to maximize the expected value of the benefit obtained by the company, building it as the sum of the individual benefits obtained in each market segment (then the size of each market segment is included in the expression). The various constraints in the model may be described briefly as follows. Constraint (3.13) determines the probability an individual consumer in segment i chooses one of the firm's bundles given its composition and price. Constraint (3.14) determines the consumer utility of the bundles offered by the competition and of non-purchase in each segment. Constraint (3.15) determines the cost and constraint (3.16) the attractiveness of each designed bundle in each segment. Constraint (3.17) imposes that in the design of each bundle a single alternative is chosen for each component. Constraint (3.18) ensures that no two bundles have identical compositions. Finally, constraints (3.19) and (3.20) define the nature of the variables.

As can be seen, this formulation is a non-linear programming model with both continuous and binary variables. We have already noted that in Bitran and Ferrer (2007) this problem was solved for the single bundle, single segment case while in Cataldo and Ferrer (2017) it was solved for the multiple bundle, single segment case.

3.3. Optimal Pricing and Composition of single bundle and single market segment with Constrained Multinomial Logit

In this section, we extend the scope of the seminal problem studied by Bitran and Ferrer (2007) and consider a company that must determine the price and optimal composition of a number of bundles that will be offered for sale in a single market segment. We assume that customer buying behavior can be described using a discrete choice model that considers a penalty associated with the consumers' maximum willingness to pay.

The consumers maximum willingness to pay must be interpreted as an amount of money that a consumer has budgeted to spend in the bundle, thus the customer adjusts his decision according to this amount. The fact that a price is greater than consumersmaximum willingness to pay does not imply that the customer will not buy the bundle, as it would be a hard constraint, but it will only exist a lesser probability that he buys the bundle, exceeding eventually the maximum willingness to pay.

The necessary notation for the model's sets, indexes, parameters and variables is set out below. Sets:

$\mathcal{N}$: The set of bundles offered on the market by the competition. By abuse of notation, we say that $Card(\mathcal{N}) = \mathcal{N}$.

$\mathcal{M}$: Set of components in a bundle. By abuse of notation, we say that $Card(\mathcal{M}) = \mathcal{M}$.

$\mathcal{S}_j$: Set of alternatives for component j of a bundle.

Indexes:

k, l : indexes for the bundles offered by the firm.

n : Index for the bundles offered by the competition.

j : Index for the set of components of a bundle.

u : Index for the possible choice alternatives of a component.

i : Index for all bundles offered on the market.

Parameters:

b : Number of bundles to be constructed.

$\hat{p}_n$: Price at which bundle n is offered by the competition.

c_{uj} : Cost of alternative u of the set of components j.

I_{uj} : Attractiveness of alternative u of the set of components j.

a_j : Attractiveness weight of component j in the bundle composition.

g : consumersmaximum willingness to pay.

γ : Utility of the bundles offered by the competition and of non-purchase.

g : Number of bundles offered by the competition.

β : Sensitivity of utility to bundle price.

Variables:

x_{ujk} : Binary variable indicating whether or not alternative u of the set of components j is chosen for the composition of bundle k.

X_k : Binary matrix representing the composition of bundle k.

p_k : Price of bundle k.

q_k : Probability that a consumer chooses bundle k.

c_{X_k} : Binary matrix representing the cost to the firm of bundle k.

I_{X_k} : Binary matrix representing the attractiveness of bundle k.

The models of discrete choice are a tool utilized to predict and analyze the decision that an agent makes (individual, home, firm, etc.) of a unique alternative based on the assumption of rationality, which the option that provides with the highest utility is chosen

Manski (1977). Mathematically speaking, we define V_{ri} as the profit that an agent that belongs to segment r obtains for the consumption of the goods i.

In the models of random utility (Block et al., 1960) the probability that r chooses the alternative i is:

$$P_{ri} = P(U_{ri} \geq U_{rj},\ \forall j \in A_r), \tag{3.21}$$

where A_r is the set of feasible options or choices for r.

In these models we consider that U_{ri} is known by the decisions maker, but not by the modeler, then this one represents it as the sum of two components:

$$U_{ri} = V_{ri} + \xi_{ri}, \tag{3.22}$$

a deterministic or systematic V_{ri}, known by the modeler being function of the characteristics vector Z_i of the alternative i, and a random component ξ_{ri}. If we work on the assumption that the errors ξ_{ri} allow a Gumbel distribution with a scale parameter (μ) independent and identical (iid), then the probability of choice has a functional form Logit multinomial (Chakravarty, 1999; Ortuzar and Willumsen, 2002; Kumar et al., 2009) is:

$$P_{ri} = \frac{e^{\mu V_{ri}}}{\sum\limits_{j \in A_r} e^{\mu V_{rj}}}. \tag{3.23}$$

The scale parameter μ is not identifiable, therefore its value is mostly set in 1. An assumption of equation (3.23) is that the decisions makers follow a compensatory behavior, i.e., they use a decision strategy that establishes the possibility of an interchange between attributes in order to keep a fixed utility level.

Then, because the compensatory assumption is not evident or true, it is necessary to reduce the number of alternatives to accomplish those constraints or include them in the profit of the agents. Then, we describe the *Constrained Multinomial Logit* model (*CMNL*) Martínez et al. (2009); due to its inclusion in the formulation of the mathematical pricing model. In order to analyze the constrained or semi-compensatory behavior by means of the constrained Logit bases on Swait (2001) and Cascetta and Papola (2001). That model assumes that the profit of each agent and the alternative is split in a compensatory term and the other non-compensatory part that points out the feasibility of that alternative for r:

$$U_{ri} = V_{ri} + \frac{1}{\mu}\ln(\phi_{ri}(z_i)) + \xi_{ri}, \tag{3.24}$$

where $\ln(\phi_{ri}(z_i))$ is a function *cutoff* or penalization imposed for r to the alternative i and μ is the scale parameter. That penalization with a logarithm function allows a smooth or flexible transition between the compensatory and non-compensatory space, permitting that the constraints can be subtly unfulfilled by the decisions maker.

Assuming that ξ_{ri} of equation 3.24 it is distributed (iid) Gumbel, the probability of choice of i is:

$$P_{ri} = \frac{\phi_{ri}(z_i)e^{\mu V_{ri}}}{\sum\limits_j \phi_{rj}(z_j)e^{\mu V_{rj}}}, \tag{3.25}$$

The model *CMNL* considers lower constraints ϕ^L_{rik} and greater than ϕ^U_{rik}, depending on the case. These constraints are defined by the binomial Logit obtaining that for each agent r that encloses the alternative i based on the characteristic j is defined as:

$$\phi^L_{rij} = \frac{1}{1 + e^{w_j(z_{ij} - g_{rj} + \rho_k)}} \tag{3.26}$$

$$\phi^U_{rij} = \frac{1}{1 + e^{w_j(a_{rj} - z_{ij} + \rho_k)}} \tag{3.27}$$

where g_{rj} y a_{rj} are the greater and lower benchmarks respectively that constrain the choice, $w_j > 0$ is the scale parameter of the binomial Logit, z_{ij} is the value of the attributes k and ρ_j is a parameter set by the following equation:

$$\rho_j = \frac{1}{w_j}\ln\left(\frac{1-\eta_j}{\eta_j}\right) \tag{3.28}$$

where η_j is the related value to the proportion of the population that violates the constraint of the characteristic j.

As in Martínez et al. (2009), the parameter w_j is positive because the negative values will simply switch upper constraints to lower and viceversa. Additionally, w_j is inversely proportional to the variance of the bounds. In order to incorporate several constraints, a cut-off ϕ_{ri} is defined as the multiplication of the cut-off functions enclosed to each one of the characteristics assuming that the constraints are independent.

With the foregoing as our basis, we now set out our proposed mixed integer non-linear programming model for solving the *pricing and composition of multiple bundles with constrained multinomial problem* (CMBP) model in order to determining the composition and pricing of the b bundles that will be supplied to a single market segment, making an explicit inclusion of the consumersmaximum willingness to pay, and considering a single market segment (i.e. $W = 1$).

$$\text{(CMBP)} \qquad \max_{p_1,\ldots,p_b,X_1,\ldots,X_b} \Pi(p_1,\ldots,p_b,X_1,\ldots,X_b) = \sum_{k=1}^{b} q_k(p_1,\ldots,p_b,X_1,\ldots,X_b)(p_k - c_{X_k}) \tag{3.29}$$

$$s.t: q_k(p_1,\ldots,p_b,X_1,\ldots,X_b) = \frac{\phi\left(p_k\right) e^{I_{X_k}+\beta p_k}}{\gamma + \sum_{l=1}^{b} \phi\left(p_l\right) e^{I_{X_l}+\beta p_l}} \qquad \forall\, k = 1,\ldots,b. \tag{3.30}$$

$$\phi\left(p_k\right) = \frac{1}{1 + e^{w(p_k - g + \rho)}} \tag{3.31}$$

$$\gamma = e^{V_0} + \sum_{n \in \mathcal{N}} e^{I_{X_n}+\beta \hat{p}_n} \tag{3.32}$$

$$c_{X_k} = \sum_{j \in \mathcal{M}} \sum_{u \in \mathcal{S}_j} c_{uj} x_{ujk} \qquad \forall\, k = 1,\ldots,b. \tag{3.33}$$

$$I_{X_k} = \sum_{j \in \mathcal{M}} a_j \sum_{u \in \mathcal{S}_j} I_{uj} x_{ujk} \qquad \forall\, k = 1,\ldots,b. \tag{3.34}$$

$$\sum_{u\in S_j} x_{ujk} = 1 \qquad \forall\, j \in \mathcal{M};\; k = 1,\ldots,b. \tag{3.35}$$

$$\sum_{j\in\mathcal{M}} \sum_{u\in S_j} x_{ujk}x_{ujl} \leq \mathcal{M} - 1 \qquad \forall\, k = 1,\ldots,b; l = k+1,\ldots,b. \tag{3.36}$$

$$x_{ujk} \in \{0,1\} \qquad \forall\, j \in \mathcal{M}; u \in \mathcal{S}_j; k = 1,\ldots,b. \tag{3.37}$$

$$p_k \geq 0 \qquad \forall\, k = 1,\ldots,b. \tag{3.38}$$

The objective function in (3.29) seeks to maximize the expected value of the benefit obtained by the company, building it as the sum of the individual benefits obtained for the sale of each bundle in each market segment. The various constraints in the model may be described briefly as follows. Constraint (3.30) determines the probability an individual chooses one of the firm's bundles given its composition and price. Constraint (3.31) allows you to calculate the value of the penalty (cutoff) as a function of the price. Constraint (3.32) determines the consumer utility of the bundles offered by the competition and of non-purchase. Constraints (3.33) and (3.34) determine the cost and attractiveness of each bundle designed. Constraint (3.35) imposes that in the design of each bundle a single alternative is chosen for each component. Constraint (3.36), as already noted above, ensures that no two bundles have identical compositions. Finally, constraints (3.37) and (3.38) define the nature of the variables. As can be seen, the CMBP differs from the MBP in the restriction (3.30), as it makes an explicit consideration of the consumers' maximum willingness to pay. In the next chapter, we develop an approach that solves the CMBP to optimality in a reasonable amount of time.

4. SOLUTION APPROACH

The problems described in the previous chapters have been approached in such a way as to take advantage of the structure of the specific situation being modeled in each case. Thus, although all these problems have a common root –that corresponds to the problem described and solved by Bitran and Ferrer (2007) regarding the composition of a single bundle offered in a single market segment– each situation has been approached and solved differently. The following subsections present the proposed solution approaches for each of the three problems defined in Chapter 3.

4.1. Optimal Pricing and Composition of multiple bundles and single market segment

The structure of the MBP is designed so that the solution process can be divided into two phases, each one solving a single subproblem. In the first phase, the optimal price (p_k) is determined for each of the b bundles on the assumption that their respective optimal compositions (X_k) are already known. Then, in the second phase, the optimal prices obtained in the first phase are used to generate the optimal compositions that maximize the benefits to the firm.

4.1.1. Phase 1: Multiple bundle optimal price subproblem

Given our assumption for this phase that the optimal composition of each bundle is already known, the decisions and constraints relating to composition are temporarily set aside. Substituting (3.3) above into the objective function (3.2), the unconstrained price optimization subproblem becomes

$$\Pi = \max_{p_1,\ldots,p_b \geq 0} \sum_{k=1}^{b} \frac{e^{I_{X_k}+\beta p_k}}{\gamma + \sum\limits_{l=1}^{b} e^{I_{X_l}+\beta p_l}} (p_k - c_{X_k}). \tag{4.1}$$

This leads to the following proposition:

PROPOSITION 4.1. *The optimal price for bundle k is given by the closed form expression*

$$p_k^* = c_{X_k} - \frac{1}{\beta}\left(1 + W\left(\frac{1}{\gamma}\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}\right)\right) \qquad \forall k = 1, \ldots, b, \tag{4.2}$$

where $W(\cdot)$ is the Lambert W-function. Thus, the optimal price of bundle k depends on the composition of all b bundles. This price will be greater than or equal to c_{X_k}, the cost to the firm of bundle k, whenever $\beta < 0$ and $W(\cdot) \geq 0$, the latter being the case if the W-function's argument is positive. In (4.2) all of these conditions are satisfied. The foregoing also implies that prices are always non-negative.

Proof: See Appendix 6.1.

Substituting the optimal price into the benefit function, we obtain the following corollaries:

Corollary 4.1. *The optimal expected benefit Π^* is*

$$\Pi^* = \frac{-1}{\beta} W\left(\frac{1}{\gamma}\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}\right). \tag{4.3}$$

Proof: See Appendix 6.2.

Corollary 4.2. *The probability of choosing bundle k when optimal price is p_k^* is*

$$q_k^* = \frac{W\left(\frac{1}{\gamma}\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}\right)}{\left(1 + W\left(\frac{1}{\gamma}\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}\right)\right)} \left(\frac{e^{I_{X_k}+\beta c_{X_k}-1}}{\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}}\right), \tag{4.4}$$

where the first factor is the probability a consumer chooses one of the b bundles offered by the firm and the second factor is the probability the bundle chosen is k.

Proof: See Appendix 6.3.

When $b = 1$, the expressions for price, probability of purchase and optimal benefit reduce to those given in Bitran and Ferrer (2007).

4.1.2. Phase 2: Multiple bundle optimal composition subproblem

Having just obtained a closed-form expression for the optimal prices of the multiple bundles we now address the second subproblem, which is the determination of the bundles' optimal composition and thus the maximization of total benefits.

Consider the definition of *Pareto bundles* posited by Bitran and Ferrer (2007). For any two bundles k and l, k is said to dominate l if $I_{X_k} \geq I_{X_l}$ and $c_{X_k} \leq c_{X_l}$. This implies that instead of having to check the entire feasible bundle space Ω, we can confine our search for the solution to the Pareto-efficient bundle frontier Ω^*, which is constructed with the non-dominated bundles.

Thus, to determine the composition of one of the b bundles we could explicitly construct Ω^* and then search for the best bundle. However, Garey and Johnson (2002) have shown that finding any point on a Pareto-efficient frontier is itself an NP-complete problem, and similar conclusions were drawn by Warburton (1987) and Beasley and Christofides (1989). It would appear, then, that the explicit construction of such a frontier is not in fact the best approach. In light of this, Bitran and Ferrer (2007) propose, for the single bundle case, an $O(\mathcal{HM})$-order pseudo-polynomial time algorithm, where $\mathcal{H}$ is the cardinality of the largest $\mathcal{S}_j$ set. This algorithm, which we will call BF, does not require explicit construction of the Pareto–efficient frontier.

But whereas the Pareto-efficient frontier for the single bundle problem is built of points corresponding to individual bundles, for the multiple bundle problem the points are groups of feasible bundles. It is possible, however, to identify the contribution of each of these bundles to the total objective function. The following proposition states a key result regarding this contribution:

PROPOSITION 4.2. *The partial derivatives of the objective function* Π^* *with respect to* I_{X_k} *and* c_{X_k} *are proportional to* q_k^* *for any arbitrary points* I_{X_k} *and* c_{X_k}*. More precisely,*

$$\nabla\Pi^* = \left(\frac{\partial\Pi^*}{\partial I_{X_k}}, \frac{\partial\Pi^*}{\partial c_{X_k}}\right) = \left(\frac{-1}{\beta}, -1\right) q_k^*, \tag{4.5}$$

where q_k^* *is the probability that bundle* k *is chosen when its price is* p_k^**.*

Proof: See Appendix 6.4.

The results in Proposition 4.2 are of fundamental importance, for they show that an increment in the objective function with respect to I_{X_k} is proportional to $-1/\beta$ while an increment with respect to c_{X_k} is proportional to -1. This implies that the point on the Pareto-efficient frontier for the multiple bundles problem that maximizes utility is the one whose bundles confer the largest utility increment. The determination of the optimal composition of multiple bundles therefore reduces to identifying the composition of those that maximize the contribution to the objective function. This conclusion is expressed as follows:

$$\max_{X_1, X_2, \ldots, X_b \in \Omega} \sum_{k=1}^{b} \frac{-I_{X_k}}{\beta} - c_{X_k}. \tag{4.6}$$

One of the bases of our solution approach is the separation of (4.6) into b stages, in each of which the optimal composition of only one bundle is determined. This means that in each stage k we must also consider the optimal composition of the bundles constructed in the previous stages ($l = 1, \ldots, k$). For this purpose we will need the following definition:

Definition 4.1. *(Inner adjacent frontier) Let* Ω_1 *be the set of all feasible bundles and* Ω_1^* *its Pareto-efficient frontier. Select a bundle* $\bar{b}$ *belonging to* Ω_1^* *and eliminate it from the set* Ω_1*. The result is a new set* Ω_2 *of all the feasible bundles, whose Pareto-efficient frontier is denoted* Ω_2^**. This new construction is called the inner adjacent frontier of* Ω_1^* *under* $\bar{b}$*.*

The significance of Definition 4.1 is the linkage it specifies between successive pairs of optimal bundle composition problems. Thus, if we want to obtain the optimal composition of any two bundles in a set of feasible bundles, we look for the first composition (X_1^*) on the Pareto-efficient frontier of the original problem and the second composition on that frontier's inner adjacent frontier under X_1^*.

With this definition we now describe the separation into stages, in each of which a single optimal bundle is determined. In stage 1 we obtain the optimal composition of a single bundle in the space Ω_1 containing all feasible bundles and therefore also on the Pareto-efficient frontier Ω_1^*. Let us call this problem $P(1)$. The solution of $P(1)$ is denoted X_1^* and indicates the composition of the first of the b optimal bundles. The information on this optimal composition X_1^* is passed on from stage 1 to stage 2 so that the second bundle is not given the same composition. This is ensured simply by eliminating X_1^* from the feasible bundle space Ω_1, thus obtaining a new Pareto-efficient frontier Ω_2^* which by construction is the inner adjacent frontier of Ω_1^* under X_1^*.

Now let us call $P(2)$ the problem of determining the optimal composition of a single bundle located in the feasible bundle space Ω_2, and therefore also on Ω_2^*. The solution to $P(2)$ will give the optimal composition X_2^*, the composition of the second bundle. This second bundle is then eliminated from the feasible bundle space Ω_2, resulting in the construction of a new Pareto-efficient frontier that will be the inner adjacent frontier of Ω_2^* under X_2^*. The optimal composition and the new Pareto-efficient frontier are then passed on to the next stage and the third bundle is determined in the same fashion as the previous two. Iterating this process b times will identify the optimal composition of all b bundles.

The solution approach just outlined can be considered an application of dynamic programming given that it divides the original problem into b sequential subproblems and uses the solution of each of them to obtain the solution of the next one. We can therefore reformulate (4.6) as the following dynamic programming problem:

$$\underbrace{\max_{X_b \in \Omega_b^*} \frac{-I_{X_b}}{\beta} - c_{X_b}}_{\text{stage } b} + \left\{ \underbrace{\max_{X_{b-1} \in \Omega_{b-1}^*} \frac{-I_{X_{b-1}}}{\beta} - c_{X_{b-1}}}_{\text{stage } b-1} + \ldots + \left\{ \underbrace{\max_{X_1 \in \Omega_1^*} \frac{-I_{X_1}}{\beta} - c_{X_1}}_{\text{stage } 1} \right\} \ldots \right\}, \tag{4.7}$$

where Ω_{k+1}^* is the inner adjacent frontier of Ω_k^* under X_k^* for $k = 1, \ldots, b-1$. This in turn can be rewritten as a formulation of the Bellman equation for each stage k:

$$F_k^*(\Omega_k^*) = \max_{X_k \in \Omega_k^*} \left\{ \frac{-I_{X_k}}{\beta} - c_{X_k} + F_{k-1}^*(\Omega_{k-1}^*) \right\}, \tag{4.8}$$

where $F_0^*(\cdot) = 0$.

Bellman's optimality principle (Bellman, 2013) ensures that the optimal compositions of the bundles, and then $X_1^*, X_2^*, \ldots, X_b^*$, each one obtained as the solution of its respective stage-level problem, constitute the optimal solution of the complete subproblem. To solve equation (4.8), the Pareto-efficient frontiers must be constructed for each of the stages $k = 1, \ldots, b$. But as has already been observed, their construction is highly complex. However, they can in fact be identified by making use of the inner adjacent dependency existing between the frontiers of two successive stages (k and $k+1$).

To understand this dependency notion, assume, recalling our example in Chapter 1, that a cable TV company wants to build two bundles, each containing a movie channel (component A), a sports channel (component B) and a cultural channel (component C). The choice alternatives for each component and their attractiveness and cost levels are as given in Table 4.1, with sensitivity of utility to price β equal to -0.007. The Pareto-efficient frontier for stage 1 and the bundles not on it (together constituting the set of all feasible bundles) are shown in Figure 4.1. The optimal composition of the first bundle, located on Ω_1^*, is X_1^*. The Pareto-efficient frontier for stage 2 includes two bundles that were dominated by X_1^* in stage 1 but are now on the frontier because X_1^* no longer belongs to the feasible bundle space and no other bundle still in the space dominates them. The other

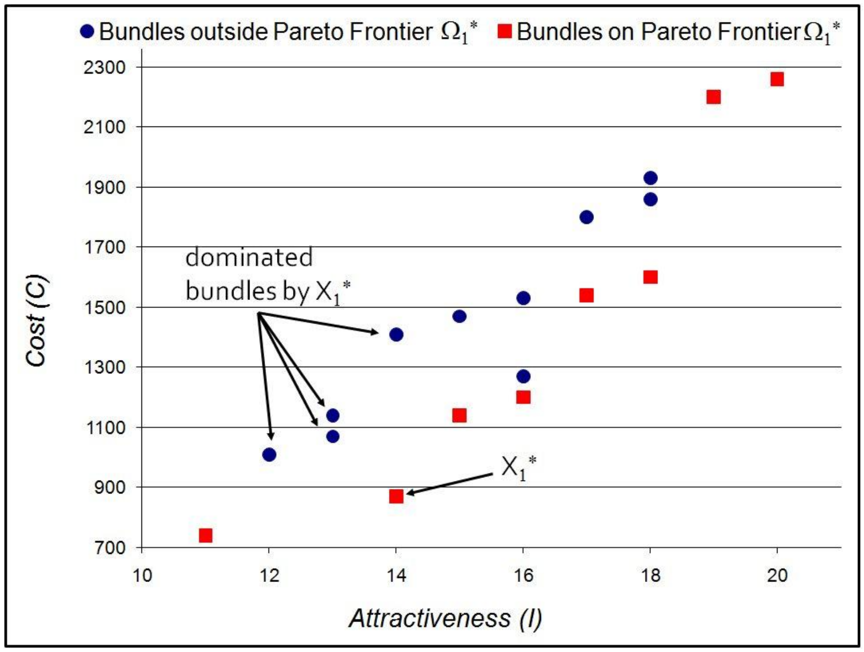

Figura 4.1. Efficient Pareto frontier Ω_1^*.

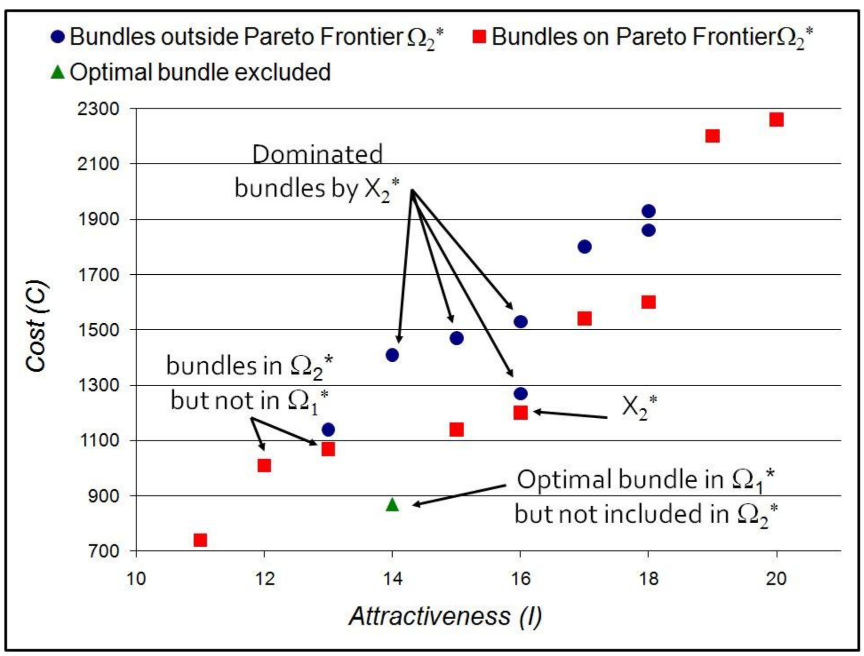

Figura 4.2. Efficient Pareto frontier Ω_2^*.

two of the four bundles that were dominated by X_1^* in stage 1 continue to be dominated in stage 2 even though X_1^* has been eliminated, as is shown in Figure 4.2.

Tabla 4.1. Data for cable TV example.

Component	Alternative	Attractiveness	Cost	Index BF
A: movie channels	1 (HBO)	2	110	175.71
	2 (Cinemax)	5	240	474.29
	3 (Cinacanal)	7	900	100.00
B: sport channels	1 (ESPN)	5	400	314.29
	2 (FoxSport)	7	800	200.00
C: cultural channels	1 (NatGeo)	4	230	341.43
	2 (Discovery)	5	500	214.29
	3 (History)	6	560	297.14

With this example in mind, we present the next proposition.

PROPOSITION 4.3. *Let Ω be the feasible bundle space. Also, let Ω_1^* be a Pareto-efficient frontier containing d bundles whose optimal bundle has the composition X_1^*. Finally, let Ω_2^* be the inner adjacent frontier of Ω_1^* under bundle X_1^*. Then Ω_2^* will contain the $d-1$ bundles other than X_1^* that were on Ω_1^* as well as some of the bundles in Ω that were not on Ω_1^* because they were dominated exclusively by X_1^*.*

Proof: See Appendix 6.5.

Proposition 4.3 leads us to conclude that constructing inner adjacent frontiers is also a complex task. Even though $d-1$ bundles on the Pareto-efficient frontier are known, it is not clear which bundles or how many of them will appear on it once the bundle that dominated them has been eliminated from the feasible bundle space.

In view of the above, we set out in what follows an alternative procedure for solving (4.7) without having to build either the Pareto-efficient or the inner adjacent frontier at each stage. We start with two useful definitions:

Definition 4.2. *(Ranked list of a component) If we calculate the contribution of each component alternative to the gradient of the objective function using the expression $f(I,c) = \frac{-I}{\beta} - c$, a descending ordered list of the alternatives can be constructed as a function of the value of $f(I,c)$. This list is called the ranked list of a component.*

Definition 4.3. *(Adjacent bundle) Let k and l be two bundles. Bundle l is said to be adjacent to bundle k if they are the same for all components except one and their respective chosen alternatives for that exception are in immediately adjacent places on the ranked list. In the special case where bundle k's chosen alternative is the last one on the list, bundle l's chosen alternative must be the first one on the list (boundary condition to properly define the algorithm and its correctness).*

It follows from Definition 4.3 that a bundle will have as many adjacent bundles as it has components. This leads to the definition of what we call a candidate bundle set:

Definition 4.4. *(Candidate bundle set) Let X_1^*, X_2^*,…,X_k^* be the optimal composition of the bundles obtained in stages 1 to k, respectively. Also, let $L(X_1^*)$ be the set of bundles adjacent to X_1^*, $L(X_2^*)$ the set of bundles adjacent to X_2^*, and so on up to $L(X_k^*)$, the set of bundles adjacent to X_k^*. Now construct a new set $T(X_{k+1})$ as the union of the k sets of adjacent bundles. All of its bundles should have an index value $f(I,c) = \frac{-I}{\beta} - c$ that is less than or equal to the optimal composition X_k^* obtained in the previous stage. Thus, $T(X_{k+1})$ is the set of candidate bundles for solving stage k+1.*

With Definitions 4.2, 4.3 and 4.4 we can now develop the following proposition:

PROPOSITION 4.4. *Let Ω_k be the feasible bundle space in stage k. Also, let Ω_k^* be the Pareto-efficient frontier of Ω_k and X_k^* the optimal composition of bundle k. Then the optimal composition of bundle k+1 bundle with $k = 1, \ldots, b-1$, denoted X_{k+1}^*, is the bundle in $T(X_{k+1})$ that has the highest index as given by $f(I,c) = \frac{-I}{\beta} - c$.*

Proof: See Appendix 6.6.

The importance of Proposition 4.4 lies in the fact that when we want to solve stage k+1 of (4.7), we need only identify the optimal composition of bundle k+1 in the set $T(X_{k+1})$; there is no need to determine the Pareto-efficient frontier Ω_{k+1}^*. Note that since the number of adjacent bundles for X_k^* is $\mathcal{M}$, the number of bundles in $T(X_{k+1})$ is bounded above by $k \cdot \mathcal{M}$.

Thus, Proposition 4.4 ensures that once we know the composition of the first bundle, we can obtain the optimal composition of the remaining $b-1$ bundles iteratively using the recurrence function (4.8). We therefore introduce the following proposition:

Tabla 4.2. All feasible bundles for cable TV example.

N	A	B	C	Attractiveness	Cost	N	A	B	C	Attractiveness	Cost
1	1	1	1	11	740	10	2	2	1	16	1,270
2	1	1	2	12	1,010	11	2	2	2	17	1,540
3	1	1	3	13	1,070	12	2	2	3	18	1,600
4	1	2	1	13	1,140	13	3	1	1	16	1,530
5	1	2	2	14	1,410	14	3	1	2	17	1,800
6	1	2	3	15	1,470	15	3	1	3	18	1,860
7	2	1	1	14	870	16	3	2	1	18	1,930
8	2	1	2	15	1,140	17	3	2	2	19	2,200
9	2	1	3	16	1,200	18	3	2	3	20	2,260

PROPOSITION 4.5. *The optimal composition of the stage 1 bundle is found by applying the BF algorithm to the optimal composition problem for a single bundle in a single market segment.*

Proof: See Appendix 6.7.

We are now able to set out our proposed solution approach, for which we will again use our cable TV case as an example (see Table 4.1) and add the following data: first, the utility of the bundles offered by the competition is $\gamma = 12{,}000$, and second, the number of cable TV bundles is $b = 3$.

To find the optimal composition of these 3 bundles, the first step is to build the ranked list for all of the components. The first bundle's composition is obtained by solving the stage 1 problem of (4.7) to get X_1^*. This solution is generated using the BF algorithm. In our example, X_1^* is the composition of bundle 7 (see Table 4.2), which consists of the alternatives 2 (Cinecanal), 1 (ESPN) and 1 (NatGeo) for components A, B and C, respectively. To determine the optimal composition of the second bundle X_2^*, we obtain the list of bundles adjacent to bundle 7 and build the candidate bundle set $T(X_2)$. According to

Table 4.2 these bundles could be 1, 8, 9, 10 and 13, but by using Definition 4.3 we find that they in fact are 1, 9 and 10, and since we are in the first stage, these three are the only ones containing $T(X_2)$ (see Figure 4.3). By Proposition 4.4, the optimal composition of the stage 2 bundle is the bundle in $T(X_2)$ with the highest index value for $f(I, c) = \frac{-I}{\beta} - c$. The optimal composition X_2^* in our example is alternatives 2 (Cinecanal), 1 (ESPN) and 3 (History) for components A, B and C, respectively, which is bundle 9. Finally, to determine the optimal composition of the third bundle X_3^* we build the candidate bundle set for stage 3, obtaining bundle 10 which consists of alternatives 2 (Cinecanal), 2 (FoxSport) and 1 (NatGeo) for components A, B and C, respectively.

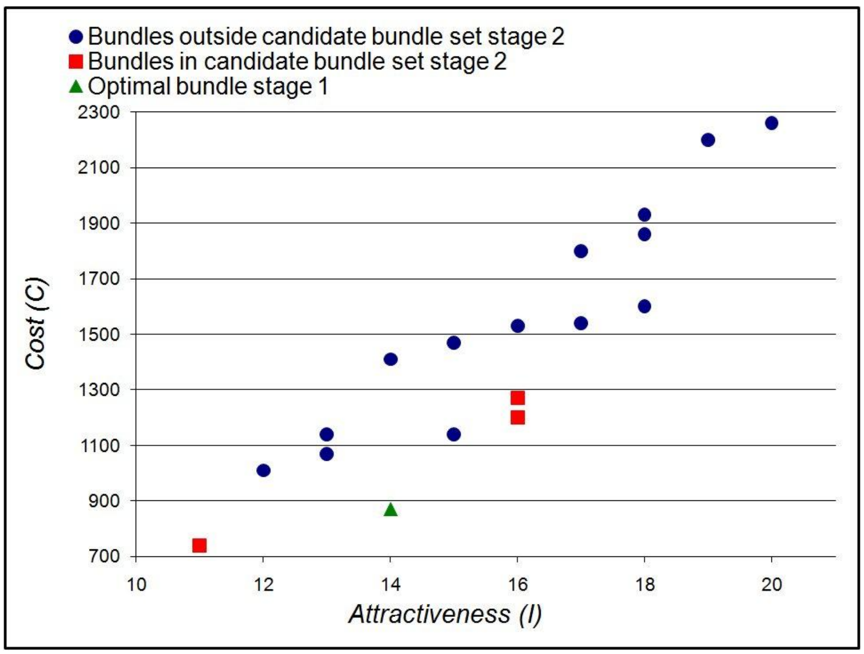

Figura 4.3. Results for stage 1 and sets for stage 2.

The change in the candidate bundle set between stage 2 and stage 3 is shown in Figure 4.4, which also indicates the optimal bundle for the latter stage. Note that the set $T(X_3)$ contains the $T(X_2)$ and $L(X_2^*)$ bundles.

The above-described procedure is captured in the following solution algorithm for the multiple bundle composition problem.

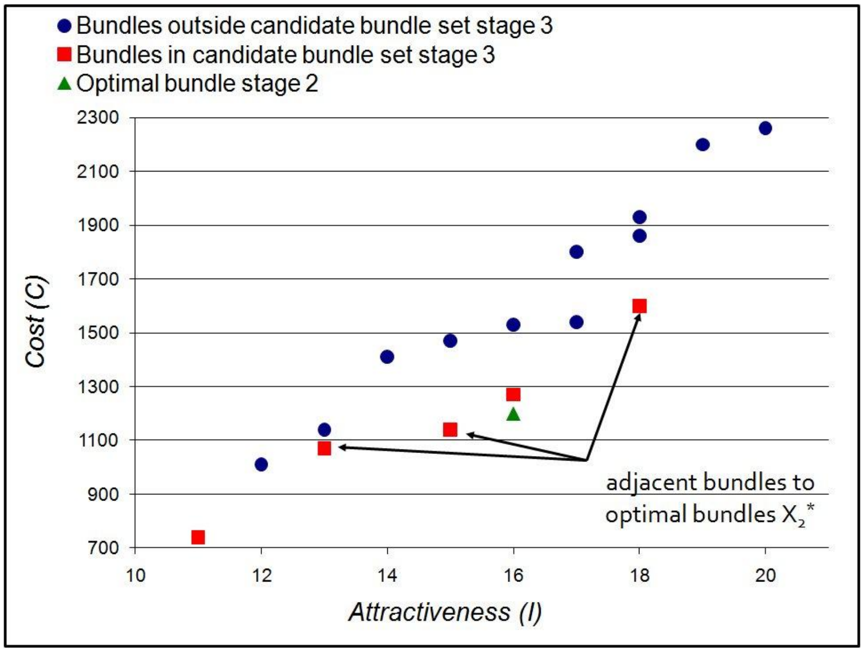

Figura 4.4. Results for stage 2 and sets for stage 3.

Multiple Bundle Composition Algorithm

Begin

Do $Rank[j, u] := 0$ for all $j \in \mathcal{M}$ and $u \in \mathcal{S}_j$; (Ranked list building)

$k := 1$;

While $k \leq b$ do

If $k = 1$ then

Solve stage 1 with algorithm $BF \mapsto X_1^*$

End If

If $k > 1$ then

Determine $L(X_{k-1}^*)$

Build $T(X_k)$

Solve stage k:

$X_k^* := \arg \max_{X_k \in T(X_k)} f(I_{X_k}, c_{X_k}) = \frac{-I_{X_k}}{\beta} - c_{X_k}$

End If

$k = k + 1$

While End

End

The iterated application of this algorithm ensures that the b bundles will be obtained in exactly b stages. This procedure has the notable advantage of obviating the need to explicitly construct Pareto-efficient frontiers in each stage. Thus, on the basis of equations (4.2), (4.3) and (4.4) associated respectively with Proposition 4.1, Corollary 4.1 and Corollary 4.2, all developed in Phase 1, the algorithm built in Phase 2 ensures the optimal composition and price will be obtained for the three bundles supplied by a firm to a single market segment. The results generated by the algorithm for the cable TV example are displayed in Table 4.3. As can be seen, the expected utility was 22.27 and the market share attained was 13.09 %.

Tabla 4.3. Cable TV example results ($b = 3$).

N	A	B	C	Attractiveness	Cost	Price	q_i [%]	Π_i
7	2	1	1	14	870	1,035.1	6.18	10.20
9	2	1	3	16	1,200	1,365.1	4.53	7.48
10	2	2	1	16	1,270	1,435.1	2.78	4.59

The proposed algorithm will function correctly as long as all of the component sets $\mathcal{S}_j$ are well-defined in the sense that all of the components have non-negative cost and attractiveness factor values. The computational complexity of the algorithm is of order $O(\mathcal{HM}+(b-1)\mathcal{M})$, where b is the number of bundles to be composed, $\mathcal{H}$ is the cardinality of the largest set $\mathcal{S}_j$ and $\mathcal{M}$ is the number of components in a bundle.

4.1.3. Evaluation of algorithm performance: comparison with an optimization software

To evaluate the proposed algorithm we compared its performance to that of an optimization software. For this purpose we constructed six test cases, each with three components, based on the cable TV data in Table 4.1. The only change was the number of bundles to be designed. Two more complex cases having many more possible solutions were also tested. For these two, the values for cost and attractiveness were chosen randomly, the former ranging between 1 and 1,000 and the latter between 10 and 100. The first one had four components with 5, 4, 3, and 5 choices, respectively while the second one had five components with 4, 4, 5, 6, and 4 choices, respectively. For all eight test cases, $\gamma = 12.000$ and $\beta = -0{,}007$.

The eight cases were then solved by both the algorithm and the GAMS IDE version 23.8.1 optimization software with the Bonmin solver, which is designed for integer non-linear programming problems. The results are given in Table 4.4. The software solutions are the best ones found within 90 minutes of running time.

Tabla 4.4. Results of proposed algorithm and optimization software for 8 test cases.

Components	Bundles to compose	Possible solutions	Algorithm utility	Algorithm time [sec]	Software utility	Software time [sec]
3	2	153	18.21	0.00	18.21	0.51
3	3	816	22.27	0.00	22.27	1.53
3	4	3,060	25.82	0.00	25.82	7.03
3	5	8,568	28.56	0.00	25.56	22.21
3	6	18,564	30.04	0.03	30.04	2,042.20
3	7	31,824	31.03	0.04	31.03	1,913.02
4	3	4,455,100	46,055.43	42.90	44,679.20	5,400
5	2	1,842,240	58,693.40	12.90	58,656.41	5,400

As can be seen, the more possible solutions there were, the greater was the algorithm's advantage in processing time over the optimization software. In the most complex case, set out in the second-to-bottom row of the table, the algorithm took only 42 seconds to find the solution whereas the software needed 5,400 seconds. Also, the utility of the algorithm solution was always greater than that of the software. Specifically, in the most complex case 3 % higher (46,055.43 versus 44,679.20).

4.1.4. Relationship between optimal solution for b y $b + 1$ bundles

Based on the analysis of Phase 2, we are able to establish the following proposition:

PROPOSITION 4.6. *In the absence of administration costs, the optimal composition for $b+1$ bundles contains the optimal solution for b bundles.*

Proof: The demonstration is straightforward, following directly from equations (4.7) and (4.8). Clearly, when solving the problem of $b+1$ bundles the same b first stages are consecutively solved as for the problem of b bundles (equation (4.7)). Only the solution of stage $b+1$ will be different, and it is found by solving over the set Ω^*_{b+1} as shown in (4.8).

Consequently, the bundles that comprise the solution of the b bundles case must necessarily include all those in the solution of any $\tilde{b}$ bundles case where $1 \leq \tilde{b} < b$. However, enlarging the problem from b bundles to $b+1$ may change the b bundles' optimal prices given that the optimal price of each bundle is dependent on the composition of all the others (Proposition 4.1).

4.1.5. Optimal number of bundles

From our analysis so far we are now in a position to answer the question whether there exists an optimal number of bundles to be marketed. Assuming there are no administration costs, we propose the following:

PROPOSITION 4.7. *The expected utility is increasing with respect to the number of bundles to be marketed when there are no administration costs depending on the number of bundles to be designed.*

Proof: See Appendix 6.8.

The variation in utility as the number of bundles to be supplied increases from 1 to all 18 that can possibly be composed in our cable TV example (see Table 4.1) is shown as a continuous line in Figure 4.5. The dashed line in the figure is the trend in the marginal utility of adding a new bundle. The two curves are consistent with Proposition 4.5 according to which utility (Π) grows as the number of bundles (b) to be composed increases but at a decreasing marginal rate.

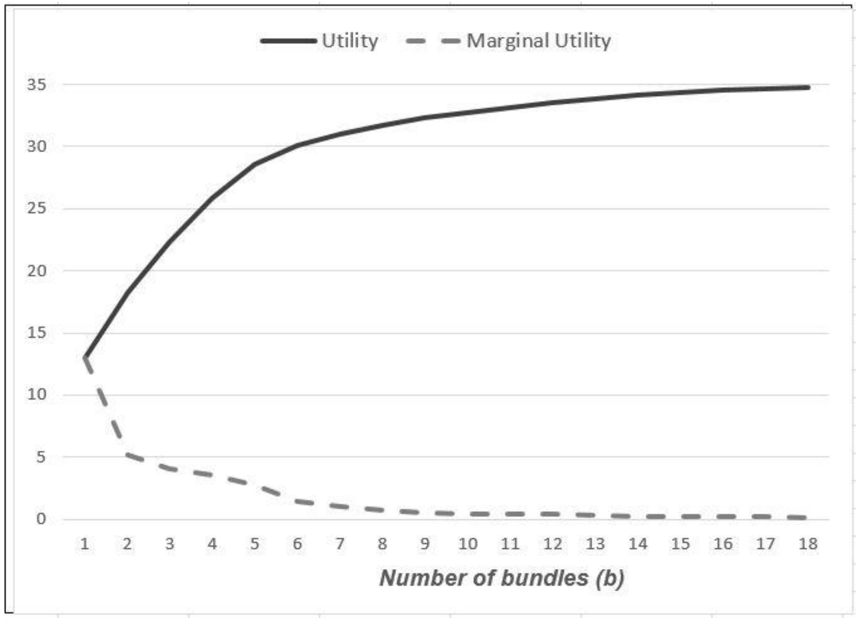

Figura 4.5. Variation in utility as the number of bundles to be marketed varies.

If we include bundle administration costs, the optimal number of bundles to be put on the market will be determined by the interaction of the marginal cost function with the marginal utility function. For example, if the cost function is linear –say, 5 per bundle– the optimal number will be 2 given that the marginal utility of marketing the second bundle is 5.21 whereas that of the third is only 4.06.

4.2. Optimal Pricing and Composition of multiple bundles and multiple market segments

Next, we show that the structure of the MBMSP does not allow us to solve the problem in two phases as in the case of a single market segment discussed in the previous subsection. Consequently, we propose a heuristic approach to determine the price and bundle compositions under the conditions described for the MBMSP.

4.2.1. Bundle price optimization problem

If we assume that we know the composition of each of the b bundles, the decision variables and constraints associated with their composition (i.e., constraints (3.14) through (3.19)) drop out, leaving only constraints (3.13) and (3.20). Then, if we substitute constraint (3.13) into objective function (3.12), we obtain the following optimization problem in which the decision variables are the prices of the b bundles and the only remaining condition is the non-negativity of the bundle prices. If we further assume that the price non-negativity condition is always satisfied for values strictly greater than zero, this problem can be said to be unconstrained. We then have

$$\Pi = \max_{p_1,\dots,p_K} \sum_{k=1}^{b} \sum_{i=1}^{W} H^i \frac{e^{I^i_{X_k}+\beta^i p_k}}{\gamma^i + \sum_{l=1}^{b} e^{I^i_{X_l}+\beta^i p_l}} (p_k - c_{X_k}). \tag{4.9}$$

Applying the first-order optimization conditions to (4.9) we obtain the following proposition:

PROPOSITION 4.8. *The optimal price of the kth bundle cannot be obtained in closed form and is given by the following expression:*

$$p_k^* = c_{X_k} + \frac{\left(\sum_{i=1}^{W} \sum_{t=1:t\neq k}^{b} H^i \beta^i q^i_t q^i_k \left(p^*_t - c_{X_t}\right)\right) - \left(\sum_{i=1}^{W} H^i q^i_k\right)}{\sum_{i=1}^{W} H^i \beta^i q^i_k \left(1 - q^i_k\right)} \tag{4.10}$$

where all q^i_k depend on p_k.

Proof: See Appendix 6.9.

As may be observed, in order for the optimal price p_k to be greater than or equal to the cost (c_{X_k}), it must be the case that $p_t - c_{X_t} \geq 0$ for every designed bundle given that $H^i > 0$, $q^i_t, q^i_k \geq 0$, $\beta^i < 0$, and $(1 - q^i_k) \geq 0$ for every segment. It can be shown by (4.10) that price is always greater than cost. If the price of any bundle – say, bundle k – is less than its cost, the term $p_k - c_{X_k} < 0$, and the bundle's contribution to expected utility will be negative or zero if the remaining terms are non-negative. If this is so, it will be the case that $H^i > 0$ for all segments and $q_k^i \geq 0$ for every bundle and segment.

Further analysis of (4.9) brings us to Proposition 4.9:

PROPOSITION 4.9. *The objective function (4.9) is quasi-concave for prices such that $p_k \geq c_{X_k}$ for all $k = 1, \ldots, b$.*

Proof: See Appendix 6.10.

The implication of 4.9 is that if we know the composition of the b bundles, then every local optimum is also a global optimum as long as the search space is a convex set. This being the case, the optimization problem (4.9) must satisfy the single condition $p_k \geq c_{X_k}, \;\; \forall k = 1, \ldots, b$ that defines a convex space. Recall that we assume the composition of the bundles to be known, so c_{X_k} $\forall k = 1, \ldots, b$ is a known parameter for this problem.

It follows from the foregoing that given the composition of each of the b bundles, we can determine the optimal price for each one using any of the existing numerical methods for finding the maximum of a non-linear function with linear constraints, and any local optimum so found will also be a global optimum.

4.2.2. Solution approach for MBMSP

We have already seen in the previous subsection that it is not possible to obtain a closed expression for the determination of the optimal price of the b bundles when the composition of these bundles is known. This prevents facing the composition problem as it was done in Bitran and Ferrer (2007) and in the Chapter 4.1.

We therefore propose a solution metaheuristic that proceeds in three stages as shown in Figure 4.6: (0) initial composition, (1) optimal prices, and (2) final composition.

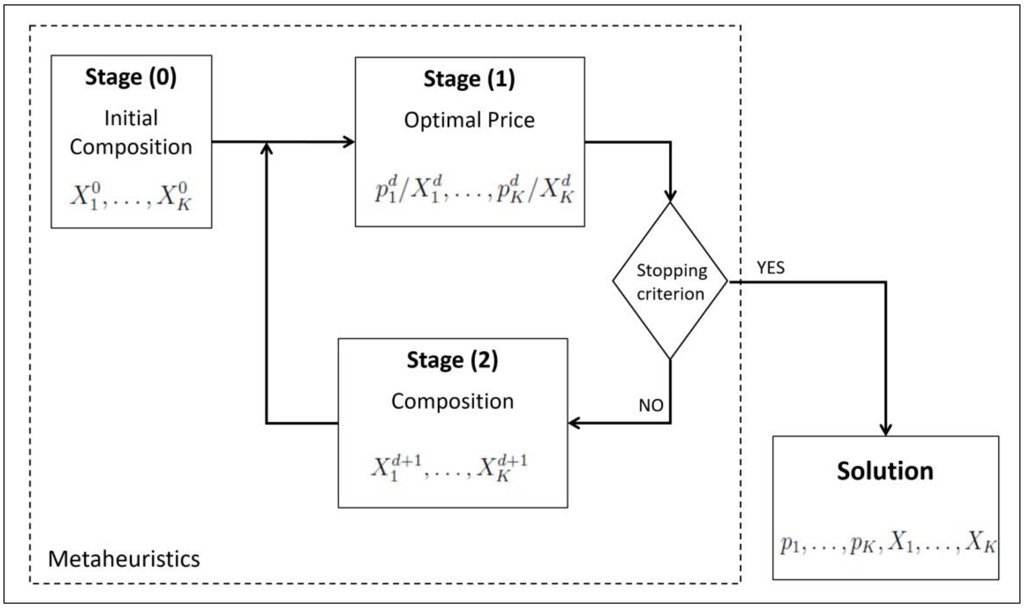

Figura 4.6. Flow diagram of proposed solution approach.

In the following subsections the three stages are described in turn.

4.2.2.1. Composicin Inicial

To determine the initial composition of the b bundles we use an algorithm based on the Multiple Bundle Composition Algorithm developed in Cataldo and Ferrer (2017), which identifies the price and optimal composition for multiple bundles offered in a single segment. The steps in the algorithm are as follows:

One of the steps calls the Multiple Bundle Composition Algorithm to determine the optimal composition of the b bundles for each segment separately. The indicator $Y(k,i)$ is used to signal whether the composition of bundle k in segment i already appeared in a previously visited segment. We define the index CFI as the weighted attractiveness level of a bundle across all segments, which is a function of the number of consumers in each segment (H^i), and that is constructed with the following expression: $\sum_{i=1}^{W} H^i \left(\frac{-I_{\hat{X}_k^i}}{\beta^i} - c_{\hat{X}_k^i} \right)$.

Initial Composition Algorithm

Begin

$i := 1$;

While $i \leq W$ do

Use Multiple Bundle Composition Algorithm $\mapsto \hat{X}_k^i \;\; \forall k = 1, \ldots, b$

Set indicator of valid combination $\mapsto Y(k,i) = 0 \;\; \forall k = 1, \ldots, b$

$i = i + 1$

While End

i:=1

While $i \leq W$ do

k:=1

While $k \leq b$ do

Procedure to indicate if combination (bundle,segment) already appeared $\mapsto Y(k,i) = 1$

$k = k + 1$;

While End

$i = i + 1$;

While End

While $k \leq b$ do

Calculate CFI index for each bundle such that $Y(k,i) = 0$ as $\sum_{i=1}^{W} H^i \left(\frac{-I_{\hat{X}_k^i}}{\beta^i} - c_{\hat{X}_k^i} \right)$

$k = k + 1$

While End

Select b bundles with the highest CFI index where $Y(k,i) = 0 \mapsto X_k^0 \;\; \forall k = 1, \ldots, b$

End

Thus, the Initial Composition Algorithm specifies the composition of the b bundles that will be used as the initial composition (X_k^0) for the metaheuristic. Note that the b bundles created by this algorithm all differ from each other by construction because the Multiple Bundle Composition Algorithm ensures this is so separately for each segment.

4.2.2.2. Optimal prices

Proposition 4.2 states that for a given composition there exists a global optimum for problem (4.1), which generates the optimal prices associated with the expected maximum utility for the problem. It further says that this global optimum is obtained at any local optimum.

We therefore use the conjugate gradient method to determine the combination of prices that maximizes the value of the objective function in (4.1) for b bundles of known composition.

4.2.2.3. Final composition

To determine the final composition for the bundles to be designed, we use a Tabu search in three phases as shown in Figure 4.7.

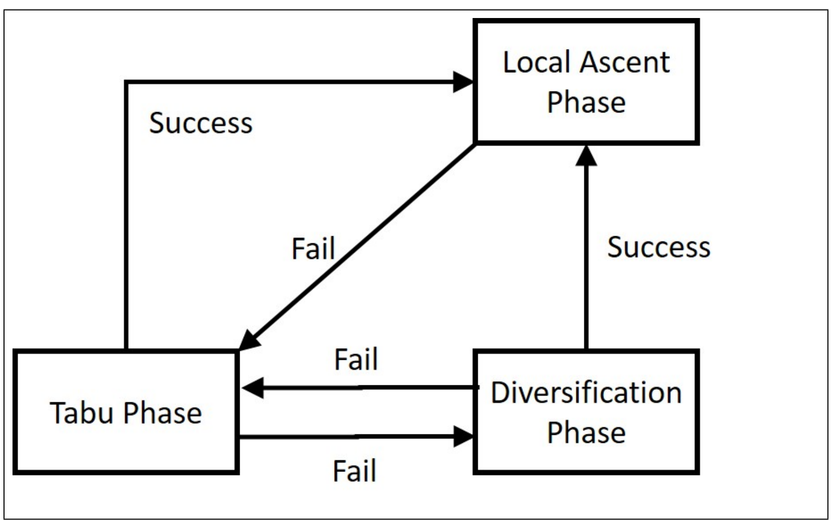

Figura 4.7. Tabu search algorithm framework.

The local ascent phase determines the best solution within the neighborhood of the current solution. If no neighboring solutions better than the current one are found, the phase terminates and the algorithm switches to the Tabu phase. In this phase there are movements that do not improve the current solution. A Tabu list maintains a recent search history to prevent the algorithm from cycling between a set of solutions. If a solution better than the current one is found, the Tabu phase terminates and the algorithm returns to the local ascent phase. If, on the other hand, a better solution is not found before a predetermined number of iterations has been reached, the algorithm goes from the Tabu phase to the diversification phase. A previously unexplored portion of the solution space is then explored over a set number of iterations. If a better solution is found, the algorithm goes back to the local ascent phase, otherwise it returns to the Tabu phase with the best solution found up to that point.

For our problem in which b bundles must be designed, we define a neighboring solution as one that is identical to the current one in the design of $b - 1$ bundles, that is, a solution in which only one bundle is different, and is so only in one component.

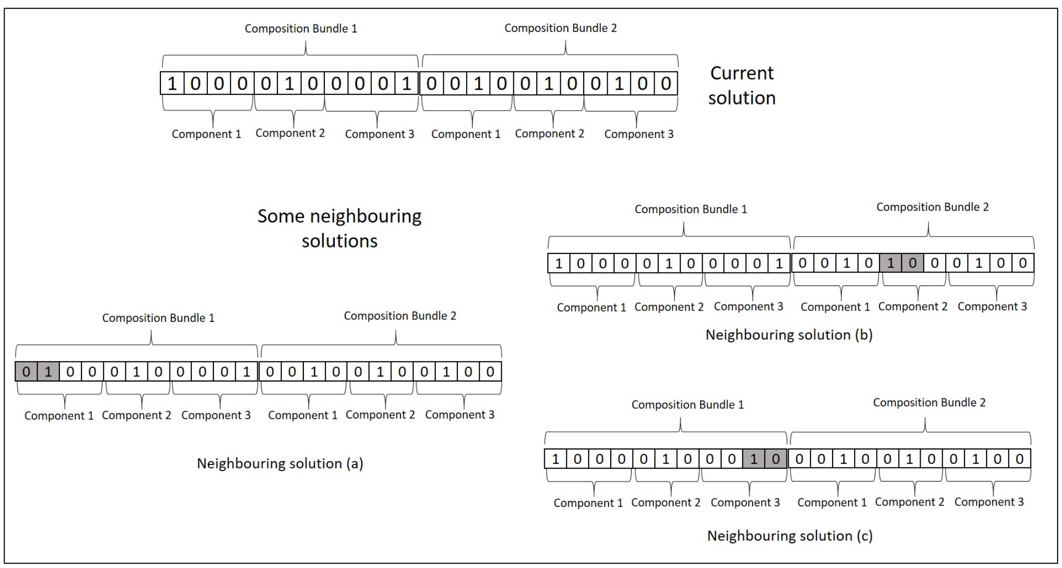

Figura 4.8. Example of the definition of a neighborhood.

As an example, consider the case shown in Figure 4.8. Two bundles must be designed with three components each and there are four choice alternatives for component 1, three for alternative 2 and four for alternative 3. The current and neighboring solutions for the two bundles are each illustrated by an ordered set of squares, each one representing a choice alternative for one of the bundle components as marked. A square containing a 1 in, say, the third square from left to right for a given component indicates that in the solution in question, the component is the third alternative; a 0 indicates it is not. In the current solution, shown by the top set of squares, the three alternatives for the bundle 1 components in numerical order are 1, 2 and 4, respectively, while for bundle 2 the corresponding alternatives are 3, 2 and 2. In the three neighboring solutions, the differences between them and the current solution are marked in grey, indicating which components in which bundle have changed. For instance, in neighboring solution (a), bundle 1 differs from the current solution in that component 1 is the second alternative rather than the first. In neighboring solution (b) it is bundle 2 that has changed, component 2 being the first alternative instead of the second.

4.2.3. Case Study

Our case study is the cable TV example previously used in (Cataldo and Ferrer, 2017), in which a cable television company designs two bundles, each one containing a movie channel (component A), a sports channel (component B) and a cultural channel (component C). This time, however, the two bundles are to be offered in two market segments instead of just one. Three different scenarios will be considered. In the first scenario, although both bundles are sold in both segments, the cable TV company optimizes the composition and price of each bundle for only one (different) segment. Thus, two independent problems are solved. In the second scenario, the same bundle compositions are used but the company optimizes the bundle prices over both segments. The prices are therefore obtained solving a joint problem in which the two segments are mutually dependent. Finally, the third scenario is our proposed problem in which the company optimizes both composition and price jointly over the two segments, which are thus fully mutually dependent.

4.2.4. Scenario data and results

For all three scenarios, the choice alternatives for each component and their respective attractiveness and cost levels in each segment are given in Table 4.5. All 18 feasible bundle

Tabla 4.5. Component alternatives and their attractiveness and cost for case study.

Component	Alternative	Attractiveness Segment 1	Attractiveness Segment 2	Cost
A: movie channel	1 (HBO)	2	3	110
	2 (Cinemax)	5	4	240
	3 (Cinacanal)	7	9	900
B: sport channel	1 (ESPN)	5	4	400
	2 (FoxSport)	7	8	800
C: cultural channel	1 (NatGeo)	4	2	230
	2 (Discovery)	5	5	500
	3 (History)	6	7	560

Tabla 4.6. All feasible bundles and their attractiveness and cost for case study.

N	A	B	C	Attr. Seg. 1	Attr. Seg. 2	Cost	N	A	B	C	Attr. Seg. 1	Attr. Seg. 2	Cost
1	1	1	1	11	9	740	10	2	2	1	16	14	1,270
2	1	1	2	12	12	1,010	11	2	2	2	17	17	1,540
3	1	1	3	13	14	1,070	12	2	2	3	18	19	1,600
4	1	2	1	13	13	1,140	13	3	1	1	16	15	1,530
5	1	2	2	14	16	1,410	14	3	1	2	17	18	1,800
6	1	2	3	15	18	1,470	15	3	1	3	18	20	1,860
7	2	1	1	14	10	870	16	3	2	1	18	19	1,930
8	2	1	2	15	13	1,140	17	3	2	2	19	22	2,200
9	2	1	3	16	15	1,200	18	3	2	3	20	24	2,260

designs and their individual attractiveness and cost levels in each segment are shown in Table 4.6. The parameters are $H^1 = 20,000$; $H^2 = 35,000$; $\gamma^1 = 6,520$; $\gamma^2 = 8,730$; $\beta^1 = -0.0064$ and $\beta^2 = -0.0071$.

The solution method for the first scenario, in which each bundle is optimized independently for a single different segment, follows the approach presented in Bitran and Ferrer (2007). The results are summarized in Table 4.7.

As can be seen, the optimal composition for a single bundle to be sold in segment 1 is alternatives 2, 1 and 1 for components A, B and C, respectively. This corresponds to bundle 7 in Table 4.6. The optimal composition for the same components in segment 2 is

Tabla 4.7. Summary of results for first scenario, optimizing for each segment independently.

Bundle	A	B	C	Price	Cost	q_1^1	q_1^2	q_2^1	q_2^2	Utility
7	2	1	1	1,060.2	870	17.0 %	0.1 %	-	-	656,696
18	3	2	3	2,415.9	2,260	-	-	1.2 %	9.7 %	566,930

alternatives 3, 2 and 3, corresponding to bundle 18 in Table 4.6. The optimal prices determined specifically for each segment are 1,060.2 for segment 1 and 2,415.9 for segment. Note that although the probability that bundle 7 is bought by a segment 1 consumer is 17.0 % while the probability it is bought by a segment 2 consumer is only 0.1 %, the latter figure is nevertheless greater than zero. In the case of bundle 18, the situation is similar: the probability it is bought by a segment 2 consumer, for whom it was designed, is 9.7 % whereas the probability a segment 1 consumer buys it is 1.2 %, that is, much lower than segment 2 but still a positive number. The expected utility in the two segments is 656,696 for bundle 7 and 566,930 for bundle 18, totaling 1,223,626 for the two bundles combined. Taken separately, segment 1 utility is 685,123 while segment 2 utility is 538,503.

The solution for the second scenario, in which the two bundles maintain their respective first scenario compositions but price policy is optimized jointly over the two segments, is determined using equation (4.1). The results are set forth in Table 4.8.

Tabla 4.8. Summary of results for second scenario, optimizing price jointly over both segments.

Bundle	A	B	C	Price	Cost	q_1^1	q_1^2	q_2^1	q_2^2	Utility
7	2	1	1	1,060.1	870	17.1 %	0.1 %	-	-	656,825
18	3	2	3	2,418.2	2,260	-	-	1.2 %	9.6 %	566,871

As the table shows, total expected utility is 1,223,696, an increase of only 70 (0.006 %) over the total figure for the two segments when solved separately.

Finally, the third scenario representing our proposed problem is solved using the approach proposed in Subsection 4.2.2 in which both composition and price are optimized jointly over the two segments. The results are given in Table 4.9.

Tabla 4.9. Summary of results for third scenario, optimizing composition and price jointly over both segments.

Bundle	A	B	C	Price	Cost	q_1^1	q_1^2	q_2^1	q_2^2	Utility
9	2	1	3	1,390.6	1,200	14.3 %	1.8 %	-	-	661,667
12	2	2	3	1,773.8	1,600	-	-	9.1 %	6.4 %	702,711

The optimal composition in this case is alternatives 2, 1 and 3 respectively for components A, B and C in segment 1, corresponding to bundle 9 in Table 4.6, and alternatives 2, 2 and 3 for the same components in segment 2, which is bundle 12 in Table 4.6. The respective optimal prices for the two bundles are 1,390.6 and 1,773.8. The expected market share for segment 1 falls from 17.2 % to 16.1 % while for segment 2 it rises from 10.8 % to 15.5 %. The net effect of these changes is that total expected utility for the two bundles increases from 1,223,696 to 1,364,378, or 11.5 %.

In light of the foregoing results, the following observations are in order. First, the optimal compositions of the bundles are not necessarily the compositions obtained when each segment is considered independently. Second, when the two segments are considered jointly, the optimal compositions of the bundles are not necessarily those that are optimal for either one of the segments considered independently. And third, optimizing price jointly over both segments with predetermined (i.e., not jointly optimized) bundle designs increases the expected profits, but not to the same extent as jointly optimizing both price and bundle composition over the two segments.

4.2.4.1. Sensitivity Analysis

We now turn to the analysis of how the solution of the problem changes in price and composition when structural aspects of the market are modified. Two such modifications are considered: market size (H^i) and level of competition (γ^i). The analysis maintains the case study characteristics set out in Table 4.5.

4.2.4.2. Relative size of market segments

We assume that there are 55,000 consumers and that the structure of the segments remains the same except for their size, which is modified by varying the ratio of their respective numbers of consumers H^1 and H^2 from 0 to 1 in equal increments of $\delta = 0{,}001$. The results are summarized in Table 4.10, which displays only those pairs of bundles that are optimal, namely, 12-18, 9-12 and 7-9. Note that none of the pairs is 7-18, the bundles that were optimal for segments 1 and 2 considered independently.

Tabla 4.10. Results when relative segment size (H^1/H^2) is varied.

Bundle 1	Price Bundle 1	Bundle 2	Price Bundle 2	Utility	H^1/H^2
12	1,768.5	18	2,425.7	1,301,857	0.2
9	1,389.6	12	1,772.4	1,399,581	0.4
7	1,079.8	9	1,401.7	1,593,399	0.5
7	1,080.0	9	1,404.4	1,870,678	0.6
7	1,080.2	9	1,408.0	2,425,659	0.8
7	1,080.4	9	1,410.5	2,980,958	1.0

Also, when the ratio is varied from one instance to the next (for example, from 0.2 to 0.2+δ), one of two things occur: (i) the optimal pair of bundles remains the same but the optimal prices change, or (ii) just one of the bundles changes and the optimal prices change accordingly.

4.2.4.3. Relative level of competition

We now modify the level of competition in segment 2 while maintaining all other characteristics of the segments' structure. This is done by increasing the value of γ^2) from 652 to 16,300, which in turn increases the ratio γ^2/γ^1. These variations are made in equal increments of $\delta = 1$. The principal results are shown in Table 4.11.

In the case where the ratio is low, competition in segment 2 is weak and its attractiveness is therefore high relative to segment 1, but as the ratio increases, competition in segment 2 intensifies and the relationship reverses. Thus, in the case where the ratio is high, segment 1 is more attractive relative to segment 2. The effect is evident in the bundles found to be optimal. In the first case the optimal bundles are 12 and 18, which are focused on segment 2 (recall from Table 4.7 that in segment 2 it was bundle 18 that had the optimal composition) whereas in the second case, the optimal bundles are 7 and 9, which are focused on segment 2 (recall from Table 4.9 that in segment 1 it was bundle 7 that had the optimal composition).

Tabla 4.11. Results when relative competition level (γ^2/γ^1) is varied.

Bundle 1	Price Bundle 1	Bundle 2	Price Bundle 2	Utility	γ^2	γ^2/γ^1
12	1,870.1	18	2,542.0	1,350,876	652	0.1
12	1,795.5	18	2,461.6	1,301,857	2,608	0.4
12	1,777.6	18	2,438.3	1,399,581	5,216	0.8
9	1,389.6	12	1,774.0	1,593,399	6,520	1.0
9	1,390.5	12	1,773.8	1,870,678	8,476	1.3
9	1,391.1	12	1,784.2	2,425,659	9,780	1.5
7	1,079.8	9	1,401.9	2,980,958	16,300	2.5

Note also that the optimal bundles in Table 4.11 are the same ones found in the market size sensitivity analysis. Bundles 7 and 18 never appear as a pair, being the bundles that were optimal for segments 1 and 2, respectively, when the latter were solved for independently.

Once again, as the sensitivity analysis proceeds through the various instances, one of two things occur: (i) the optimal pair of bundles remains the same but the optimal prices change, or (ii) just one of the bundles changes and the optimal prices change accordingly.

4.2.5. Evaluation of the proposed approach

To test the performance of the proposed approach, 1,000 experimental problems were run using the heuristic. The problems involved 2 or 3 market segments and the design of 2 to 4 bundles. The bundles were composed of either 4 or 5 components, each with at least 3 and at most 6 choice alternatives. With these characteristics the largest problem involved 7,776 different possible bundle designs and more than 30 million solutions. To limit the possibilities to a more manageable number, the problems were set up in such a way that there could be no more than 100,000 solutions. The parameter values set for H^i ranged from 2 to 90, for β^i from -0.006 to -0.009 and for γ^i from 100,000 to 500,000. Computer runs that did not terminate on one of the heuristic stopping criteria were cut off at 5 minutes. In each case, the problem was first solved to optimality (exhaustive enumeration) and the solution found was used as the benchmark for comparison with the solution obtained using the proposed approach.

Tabla 4.12. Summary of results for 1,000 experiments.

Indicator	Tabu Search
Percentage of cases where optimal solution found (%)	96.10
Average GAP (%)	0.19
Average GAP without cases where optimal solution found (%)	1.71
Maximum GAP (%)	5.35
Standard Deviation GAP (%)	0.71

The main results obtained are summarized in Table 4.12. The indicator “percentage of cases where the optimal solution was found” refers to the proportion of problems solved for which the heuristic found the optimal solution(for both composition and price). “Average GAP” indicates the average percentage by which the heuristic solution fell short of the optimal one, and was calculated including the 0 % gaps where the heuristic identified the optimum. “Average GAP” without cases where an optimal solution was found is the average percentage by which the heuristic solution fell short of the optimal one for cases where the heuristic did not find an optimal solution. “Maximum GAP” is the largest percentage difference between the heuristic and optimal solutions for any case. Finally, “standard deviation GAP” is the standard deviation for all of the gap values.

As these results indicate, our proposed approach found the optimal composition and price in 96.1 % of cases, falling short of the optimum in the remaining cases by an average of only 1.71 %. In the worst case, the solution found by our approach was only 5.35 % below the optimum.

4.3. Optimal Pricing and Composition of single bundle and single market segment with Constrained Multinomial Logit

As in Bitran and Ferrer (2007), the non-linear mixed integer programming problem is solved in two phases. The first one is aimed at pricing for a generic composition. In the second stage, we solve the problem of defining the optimal composition, given the price defined in the first step.

4.3.1. Phase 1: Multiple bundle optimal price subproblem

Given our assumption for this phase that the optimal composition of each bundle is already known, the decisions and constraints relating to composition are temporarily set aside. Substituting (3.30) above into the objective function (3.29), the unconstrained price optimization subproblem becomes:

$$\Pi = \max_{p_1,\ldots,p_b \geq 0} \sum_{k=1}^{b} \frac{\phi\left(p_k\right) e^{I_{X_k}+\beta p_k}}{\gamma + \sum\limits_{l=1}^{b} \phi\left(p_l\right) e^{I_{X_l}+\beta p_l}} \left(p_k - c_{X_k}\right). \tag{4.11}$$

This leads to the following proposition:

PROPOSITION 4.10. *The optimal price of the bundle k can not be obtained in a closed form, and is determined by the following expression, which corresponds to an expression for a fixed-point system:*

$$p_k^* = \underbrace{c_{X_k} + \frac{\gamma_k + \phi\left(p_k^*\right) e^{I_{X_k}+\beta p_k^*}}{\gamma_k\left(w\left(1-\phi\left(p_k^*\right)\right)-\beta\right)}}_{b=1 \text{ term}} + \underbrace{\frac{\sum\limits_{l=1:l\neq k}^{b} \phi\left(p_l^*\right) e^{I_{X_l}+\beta p_l^*}\left(p_l^* - c_{X_l}\right)}{\gamma_k}}_{b>1 \text{ term}} \qquad \forall k = 1,\ldots,b, \tag{4.12}$$

where $\gamma_k = \gamma + \sum_{l=1:l\neq k}^{b} \phi\left(p_l^*\right) e^{I_{X_l}+\beta p_l^*}$. Thus, the optimal price of bundle k depends on the composition of all b bundles. This price will be greater than or equal to c_{X_k}, the cost *to the firm of bundle k, whenever $\beta < 0$, $\phi(p_k) \geq 0$ and $\gamma \geq 0$. In (4.12) all of these conditions are satisfied. The foregoing also implies that prices are always non-negative.*

Proof: See Appendix 6.11.

It is important to notice that when the maximum willingness to pay is not modelled, i.e., the customers' decision is independent of g, then $\phi(p) = 1$ and the same result by Bitran and Ferrer (2007) is reproduced for the case in which a price and a composition for a given unique customers' segment is analyzed. Both in Bitran and Ferrer (2007) and in the problem of the Chapter 4.1 was derived a closed expression for the price, which corresponds to a Lambert function. For the current case, in which we include the cutoff functions regarding the maximum willingness to pay, we have proceeded in numerical way to establish the solution of the fixed point equation.

From the equation (4.12), it is possible to see that for a given composition the optimal price found for the case with CMNL on the consumers' maximum willingness to pay is greater than or equal to the cost c_{X_k}, and less than or equal to the price found according to Bitran and Ferrer (2007). Namely, $c_{X_k} \leq p_k^* \leq p_{BF}^*$, which is possible to see graphically in the Figures 4.9 and 4.10.

In Figure 4.9, it is also possible to see that the effect of the maximum willingness to pay variations on the optimal price p_k^*. We can observe that as much as the willingness g grows, then p_k^* tends to a p_{BF}^*; whereas when the g decreases p_k^* tends to a c_{X_k}. The above has merits with reality by extending Bitran and Ferrer work; in which was assumed implicitly that the consumers' maximum willingness to pay was not bounded. On the other hand, if the maximum willingness to pay is less than the cost, then the firm will not charge less than the cost for it would incur in losses.

In Figure 4.10 it is possible to see the effect of the variations of value w on the optimal price p_k^*. The parameter w has a relation to the dispersion of g in terms of the cutoff function ϕ. The gradient of the dependent function of the price in the fixed point system

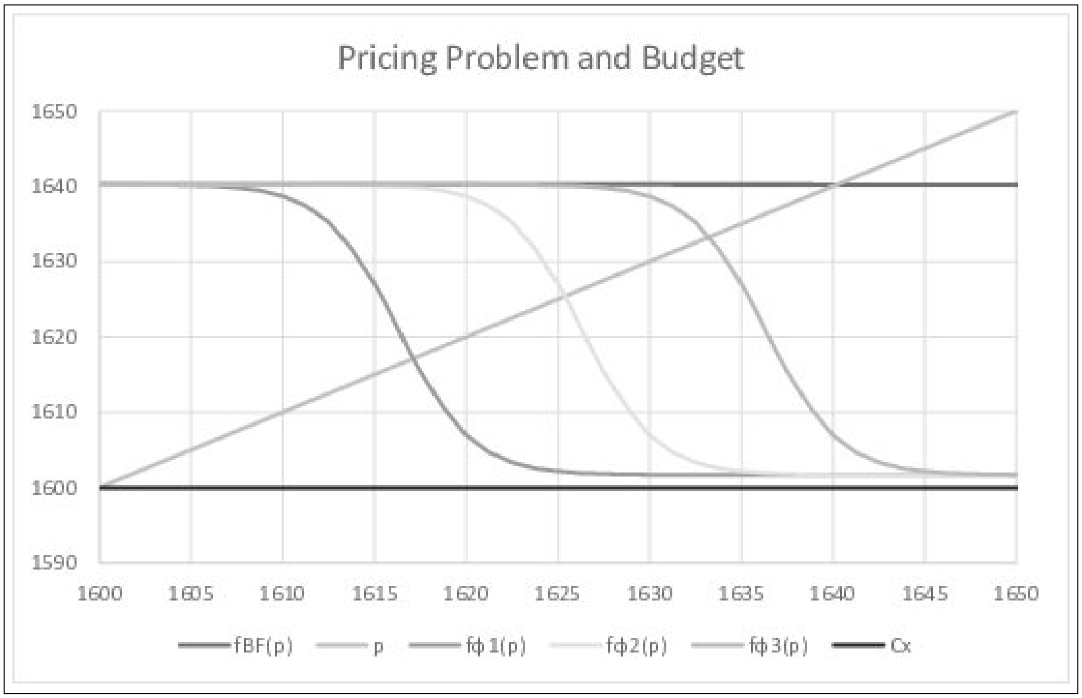

Figura 4.9. Numerical examples of optimal prices with CMNL with respect to g

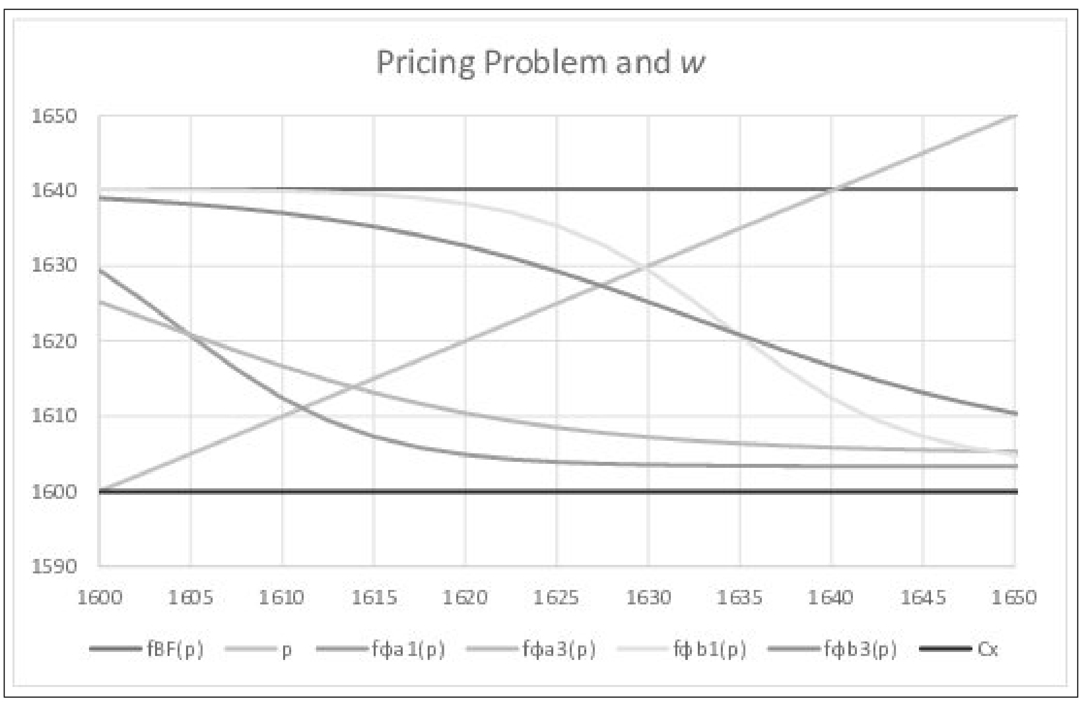

Figura 4.10. Numerical examples of optimal price with CMNL with respect to w (for two different g)

will change, and with this change into account, the optimal price will do so. In this environment, it is possible to see that when $g \leq c_{X_l}$ and w increases, then p_k^* decreases. On the other hand, when $g \geq p_{BF}^*$ and w increases, then p_k^* does the same.

4.3.2. Phase 2: Multiple bundle optimal composition subproblem

Thus, having analyzed the fixed point equation that permits to calculate the optimal price for the bundle k, i.e. p_k^*, for a given bundle composition, it is possible to identify this one through, for instance, the enumeration of all possible bundles. Then, it is possible to define the profit obtained for each composition evaluating in the optimal price, and we can set which is the price and the optimal composition for every possible bundle. We must notice that the optimal solution found with this methodology is a global optimal of the problem (BP), since we have proceeded by enumeration. Given that $p_k^* \leq p_{BF}^*$, then the composition $\mathbf{X}$ neither it is necessarily equal to the one of Bitran and Ferrer (2007).

Let us take the numerical example set out in Chapter 4.1, taking into account the information given in the Tables 4.1 and 4.2, which correspond to information on cost and attractiveness per component and the enumeration of all bundles feasible to compose for this example.

Let us consider as an example (see Table 4.13) a case in which, on the one hand, maximum willingness to pay is not considered, Bitran and Ferrer type, and on the other, cutoff functions on the maximum willingness to pay such that $w = 0{,}0015$, $g = 850$ y $\rho = 0{,}1$.

In Table 4.13, it is possible to see that, when the consumers' maximal willingness to pay is taken into account, the optimal composition is the **bundle 1**, whilst when the consumers' maximal willingness to pay is not considered (Bitran and Ferrer base case), then the optimal composition is the **bundle 7**. The above proves the fact that not only the price is different when considering the consumers' maximum willingness to pay, but also the composition of the bundle is different with respect to the Bitran and Ferrer base

Tabla 4.13. p^* and objective function, Bitrn y Ferrer case versus CMNL

Bundle	p^*_{BF}	$\Pi\left(p^*_{BF}\right)$	p^*	$\Pi\left(p^*\right)$
1	883.168	0.311	**869.006**	**0.1526**
2	1,152.985	0.128	1,136.492	0.0501
3	1,213.085	0.228	1,196.022	0.0845
4	1,282.997	0.140	1,265.418	0.0484
5	1,552.915	0.057	1,533.411	0.0150
6	1,612.960	0.103	1,593.034	0.0251
7	**1,013.197**	**0.340**	997.818	0.1505
8	1,282.997	0.140	1,265.418	0.0484
9	1,343.107	0.250	1,324.981	0.0815
10	1,413.010	0.153	1,394.420	0.0465
11	1,682.920	0.063	1,662.596	0.0142
12	1,742.969	0.112	1,722.260	0.0236
13	1,672.882	0.025	1,652.647	0.0057
14	1,942.867	0.010	1,921.221	0.0017
15	2,002.875	0.018	1,980.956	0.0028
16	2,072.868	0.011	2,050.666	0.0016
17	2,342.862	0.005	2,319.746	0.0004
18	2,402.865	0.008	2,379.580	0.0007

case. This sets forth explicitly the contribution of this research, since it is concrete that by means of inclusion of the consumers' maximum willingness to pay, the original model result changes completely, in both price and composition. The latter not only makes a theoretical finding in terms of modelling, but it also has applied implications in terms of the decision makers (managerial insights).

Additionally, in Table 4.14, we analyze the optimal composition and profit when considering multiple values for the maximum willingness to pay b for w, noting that for some combinations of g and w the bundle composition is the same as in Bitran and Ferrer's case, whereas for other combinations they are not.

Particularly in Table 4.14 is appreciated that for most of proposed scenarios, the profit function value varies considerably with respect to w and g. It is observed that when the value of w is 0.0015 and the maximum willingness to pay goes from 500 to 1500; here is

Tabla 4.14. Optimal price, profit and composition varying g y w

		g					
		500	700	900	1,100	1,300	1,500
w = 0.001	Π^*	0.128	0.144	0.161	0.178	0.195	0.211
	X	7	7	7	7	7	7
	p	1,001,314	1,002,158	1,003,038	1,003,939	1,004,843	1,005,732
w=0.0015	Π^*	0.113	0.135	0.158	0.182	0.207	0.23
	X	1	1	1	7	7	7
	p	865.886	867.624	869.476	1.000.154	1.002.043	1.003.863
w=0.002	Π^*	0.1	0.129	0.16	0.19	0.219	0.248
	X	1	1	1	1	7	7
	p	859,929	862,658	865,696	868,841	999,942	1,002,870
w=0.003	Π^*	0.078	0.117	0.163	0.206	0.243	0.277
	X	1	1	1	1	1	7
	p	848,538	853,251	859,093	865,317	870,975	1,002,709
w=0.004	Π^*	0.06	0.107	0.166	0.222	0.263	0.298
	X	1	1	1	1	1	7
	p	838,303	844,672	853,604	863,394	871,650	1,003,924
w=0.005	Π^*	0.046	0.098	0.169	0.236	0.277	0.313
	X	1	1	1	1	1	7
	p	829,389	836,954	849,055	862,640	873,094	1,005,641
w=0.09	Π^*	0	0.001	0.299	0.34	0.34	0.34
	X	1	1	1	1	1	1
	p	750,309	750,411	855,751	1,012,497	1,013,197	1,013,197

possible to see that the optimal composition is the **bundle 1** for $g \in \{500, 700, 900\}$, and for $g \in \{1100, 1300, 1500\}$ the optimal composition is **bundle 7**.

In the analyze the optimal profit value for a particular composition, in this case **bundle 1**, regarding the change of value for the maximum willingness to pay g, and for different values of w. In Figure 4.11, the optimal profit behavior for a given composition follows a similar pattern to that observed regarding the optimal price, i.e., for low willingness to pay, when the price is similar to the cost, have to obtain that the profit is inclined to zero; on the other hand, when the willingness to pay increases, then the price is similar to Bitran and Ferrer's one, and the same goes for the profit. The parameter w standardizes the gradient with which this change is produced, whereas w is smaller, then the approach to Bitran and Ferrer values is given in values of higher g.

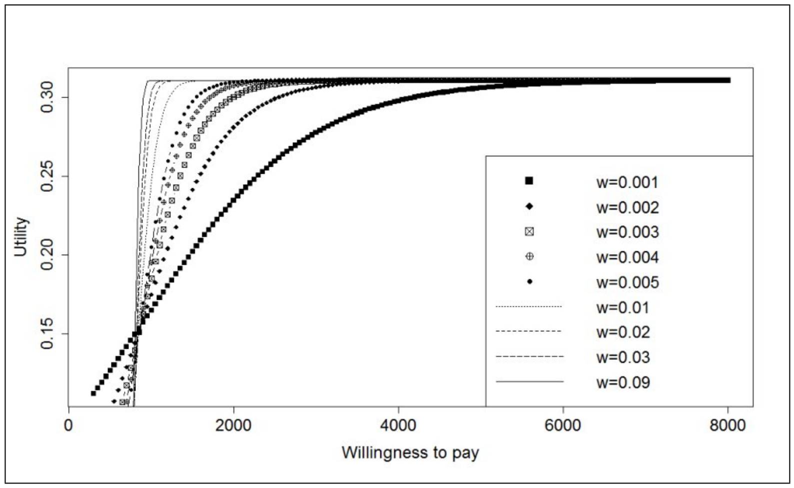

Figura 4.11. Optimal profit for bundle composition 1

5. CONCLUSION AND FUTURE RESEARCH

5.1. Review of the Results and General Remarks

This doctoral research addresses the problem of simultaneously determining the optimal composition and pricing of a set b of bundles marketed by a firm whose objective is to ensure its composition and pricing decisions maximize total benefits, extending in this way the scope of the previous research carried out by Bitran and Ferrer (2007), in which it was considered a single bundle and a single market segment.

In the first problem (3.1), a single market segment and a set of b bundles were considered to be designed and priced. In the second problem 3.2 extends the reach to multiple market segments, considering that the b bundles are offered in all market segments. In both problems the focus of the research was on developing ways to solve them, and through the solution obtained mention some managerial insight. In the third problem 3.3, a single market segment was once again incorporated, incorporating the consumers' maximum willingness to pay. The focus on this problem was to prove that, using the model and approach proposed by Bitran and Ferrer, a non-optimal bundle (or a combination of bundles) could be designed.

For the first and second problem faced in this work of doctoral thesis, these consumer preferences are assumed to maximize consumer utility as defined by a random utility model. It is also assumed that demand can be described in terms of the price and attributes of all of the firm's bundles. The random utility model is a multinomial logit formulation. For the third problem, a constraint multinomial logit formulation is used to model the consumers' behavior when included their maximum willingness to pay. In this sense, this consumers' maximum willingness to pay does not represent a fixed threshold, and then under that threshold buys and on that threshold buys the bundle if it reports the maximum benefit among the alternatives evaluated, but represents a smooth curve described probabilistically through the cutoff functions. In this sense, this maximum willingness to pay by the consumer does not represent a fixed threshold (and therefore does not indicate that below that threshold the consumer buys the bundle and over that threshold the consumer does not buy the bundle, even if that bundle gives him the maximum benefit among the evaluated alternatives), but represents a smooth curve described probabilistically through the cutoff functions.

The three problems described were formulated using non–linear optimization models in mixed variables. Though such models are normally difficult to solve, in all cases studied the problem's mathematical structure is such that it can be addressed in two phases. In the first phase it is assumed the optimal composition of the b bundles offered by the firm is known, and an expression is obtained that determines the optimal price for each bundle. In the second phase, the composition of all b bundles is obtained.

In the first problem, a closed expression is obtained to determine the optimal price for each bundle (first phase), and the optimal composition of all b bundles is derived by substituting the closed price expression into the original problem formulation, thus obtaining a new optimization model that is pre-optimized for price (second phase). This new model is rewritten as a dynamic programming problem that in each stage (or subproblem) determines the optimal composition of one of the b bundles. The optimal solution of the global problem is generated by an algorithm constructed around the idea of a Pareto-efficient frontier, first described by Bitran y Ferrer, in combination with the novel concepts of inner adjacent frontier, ranked list of a component, adjacent bundle and candidate bundle set. The algorithm is a pseudo-polynomial of order $O(\mathcal{HM} + (b-1)\mathcal{M})$. It requires an initial bundle, designed following the approach also developed by Bitran and Ferrer for the single bundle composition problem in a single market segment. It is demonstrated that this bundle will always be one of the b bundles chosen.

Two aspects of the solution are particularly worthy of note. First, the optimal price of each bundle designed by a firm depends on the composition of all of its other bundles, but not on their prices; and second, the bundles designed are very similar in their composition. The fact that any one bundle's price does not depend directly on the prices of the others

–though it does indirectly through its composition– ensures that there exists a closed expression to calculate it. The similarity of the bundles' composition is due to the fact that an multinomial logit consumer choice model was used.

In the second and third problem, was possible to determine the optimal price as a function of a given composition, obtaining a fixed point type function which could have more than one solution, and consequently, was developed a different solution approach than the proposed to face the first problem.

In the second problem, the structure of the objective function is quasi-concave in function of the prices, if it is known the composition of the bundles and the search space is convex, which guarantees that any local minimum of the problem is global minimum. This allows us, by numerical techniques, to determine the optimal price for a given bundle composition. Then, a solution approach corresponding to a three-stage heuristic algorithm was developed: Initial Composition; Optimal Price, and Composition. In the Initial Composition stage, the Multiple Bundle Composition Algorithm described in 4.1.2 is used. The other two stages are based on a Tabu Search heuristic that uses a conjugate gradient method to determine the utility and optimal price for each solution analyzed. We have tested the performance of the proposed approach for 1,000 cases, and we have reached the optimum in 96.1 % of them, being interesting to note that the average gap reached in cases where the optimum is not reached is only 1.71 %, being Of 5.35 % the greatest gap obtained.

Given the focus on the third problem, it was sufficient to make an explicit enumeration of all bundles (or groups of bundles that can be formed) and to use numerical techniques to determine their optimal price.

In the light of the results obtained, we have the following managerial insights: (i) composition of the optimal bundles does not need to correspond to the composition which is obtained when the design decision is made in each segment independently; (ii) when bundles are designed considering both market segments simultaneously, the optimal composition may not include the composition of the optimal bundles for any of the specific market segments; (iii) a pricing policy considering both market segments over a pre-established bundle design increases the expected profit, but not to the level that is achieved with an optimal design of these bundles, and (iv) by including the maximum willingness to pay by consumers, the bundle obtained is not always the one obtained when not considering it, in such a case, the bundle obtained not considering consumers willingness to pay does not only is not optimal with respect the expected profit of the company, but also overestimates this expected profit.

5.2. Future Research Topics

The natural extension of the research is to include the consumers' maximum willingness to pay for the problem of multiple bundles and multiple market segments. In this case, it is very likely that there is no closed expression to determine the price of bundles based on a given composition for them. A possible way to deal with this variant of the problem would be to see how to adapt the proposed methodology to face the second problem presented in this doctoral thesis.

Another line of research is to explicitly incorporate a benefit by diversifying the design chosen for bundles. For example, in the case of two bundles, the benefit could be linear or non-linear depending on the number of components in which a different alternative has been chosen. Other avenues for further research including competitors' possible reactions to the firm's choices of bundle compositions and prices.

6. PROOFS

6.1. Proof of Proposition 4.1

Recalling (4.1) in the main text, we differentiate Π with respect to the price of bundle k and set the derivative to 0 to obtain the first-order conditions. Thus, for all $k = 1, \ldots, b$,

$$\frac{\partial \Pi}{\partial p_k} = \frac{e^{I_{X_k}+\beta p_k}\left[(1+\beta p_k - \beta c_{X_k})\left(\gamma + \sum_{l=1}^{b} e^{I_{X_l}+\beta p_l}\right) - \beta \sum_{l=1}^{b}(p_l - c_{X_l})e^{I_{X_l}+\beta p_l}\right]}{\left(\gamma + \sum_{l=1}^{b} e^{I_{X_l}+\beta p_l}\right)^2} = 0 \tag{6.1}$$

Since $\left(\gamma + \sum_{l=1}^{b} e^{I_{X_l}+\beta p_l}\right) > 0$ and $e^{I_{X_k}+\beta p_k} > 0$, then

$$(1+\beta p_k - \beta c_{X_k})\left(\gamma + \sum_{l=1}^{b} e^{I_{X_l}+\beta p_l}\right) - \beta \sum_{l=1}^{b}(p_l - c_{X_l})e^{I_{X_l}+\beta p_l} = 0 \tag{6.2}$$

If we rewrite (6.2) in terms of bundle w, subtract it from (6.2) and then divide by β, we are left with

$$(p_k - c_{X_k} - (p_w - c_{X_w}))\left(\gamma + \sum_{l=1}^{b} e^{I_{X_l}+\beta p_l}\right) = 0 \tag{6.3}$$

Since $\gamma + \sum_{l=1}^{b} e^{I_{X_l}+\beta p_l} > 0$, then $p_k - c_{X_k} = p_w - c_{X_w}$. Now define $r = p_k - c_{X_k}$ for all $k = 1, \ldots, b$. Equation (6.2) then becomes $(1+\beta r)\left(\gamma + \sum_{l=1}^{b} e^{I_{X_l}+\beta r+\beta c_{X_l}}\right) - \beta \sum_{l=1}^{b} r e^{I_{X_l}+\beta r+\beta c_{X_l}} = 0$. Grouping terms, this reduces to $(1+\beta r)\,\gamma + \sum_{l=1}^{b} e^{I_{X_l}+\beta r+\beta c_{X_l}} = 0$, and if we let $Q = \sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}}$, which is constant for a given set of bundles, we obtain $(1+\beta r)\,\gamma + e^{\beta r}Q = 0$. Collecting terms, we have $(-1-\beta r)\,e^{(-1-\beta r)} = \frac{Qe^{-1}}{\gamma}$. If we then apply the Lambert W-function and define $W(z) = (-1-\beta r)$ and $z = \frac{Qe^{-1}}{\gamma}$, we get $(-1-\beta r) = W\left(\frac{1}{\gamma}Qe^{-1}\right)$, where $r = \frac{-1}{\beta}\left(1 + W\left(\frac{1}{\gamma}Qe^{-1}\right)\right)$. Finally, since $Q =$

$\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}}$ we arrive at the following closed-form expression for the optimal price:

$$p_k^* = c_{X_k} - \frac{1}{\beta}\left(1 + W\left(\frac{1}{\gamma}\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}\right)\right) \qquad \forall k = 1, \ldots, b.$$

6.2. Proof of Corollary 4.1

If (4.1) is expressed as a function of p_k^*, we have

$$\Pi^* = \sum_{k=1}^{b} \frac{e^{I_{X_k}+\beta p_k^*}}{\gamma + \sum_{l=1}^{b} e^{I_{X_l}+\beta p_l^*}}(p_k^* - c_{X_k}) \tag{6.4}$$

Substituting the result of (4.2) into (6.4), $\Pi^* = \sum_{k=1}^{b} \frac{e^{I_{X_k}+\beta G_k}}{\gamma+\sum_{l=1}^{b} e^{I_{X_l}+\beta G_l}} (G_k - c_{X_k})$, where $G_k = c_{X_k} - \frac{1}{\beta}\left(1 + W\left(\frac{1}{\gamma}\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}\right)\right)$. Again letting $Q = \sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}$ and $z = \frac{Q}{\gamma}$ (all of which are constants), we have

$$\Pi^* = \frac{-1}{\beta}\left(W(z)+1\right)\sum_{k=1}^{b} \frac{e^{I_{X_k}+\beta c_{X_k}-1-W(z)}}{\left(\gamma + \sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1-W(z)}\right)} \tag{6.5}$$

Since $\left(\gamma + \sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1-W(z)}\right)$ is independent of k,

$$\Pi^* = \frac{-1}{\beta}\left(W(z)+1\right)\frac{\sum_{k=1}^{b} e^{I_{X_k}+\beta c_{X_k}-1-W(z)}}{\left(\gamma + \sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1-W(z)}\right)}$$

Recalling that $Q = \sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}$, we have $\Pi^* = \frac{-1}{\beta}\left(W(z)+1\right)\left(\frac{Qe^{-W(z)}}{\gamma+Qe^{-W(z)}}\right)$, and since $z = \frac{Q}{\gamma}$, then $\Pi^* = \frac{-1}{\beta}\left(W(z)+1\right)\left(\frac{ze^{-W(z)}}{1+ze^{-W(z)}}\right)$.

Finally, by definition of the Lambert W-function we know that $\left(\frac{ze^{-W(z)}}{1+ze^{-W(z)}}\right) = \frac{W(z)}{W(z)+1}$, which leaves

$$\Pi^* = \frac{-1}{\beta}\left(W(z)+1\right)\frac{W(z)}{W(z)+1} \tag{6.6}$$

thus proving Corollary 4.1

$$\Pi^* = \frac{-1}{\beta} W\left(\frac{1}{\gamma}\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}\right)$$

6.3. Proof of Corollary 4.2

By virtue of (6.5) and (6.6) we have $\frac{W(z)}{W(z)+1} = q$, where q is the probability that one of the b bundles offered by the firm will be chosen. We therefore have

$$q = \frac{W(z)}{W(z)+1}\left(\frac{\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}}{\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}}\right) = \frac{W(z)}{W(z)+1}\left(\frac{e^{I_{X_1}+\beta c_{X_1}-1}}{\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}} + \ldots + \frac{e^{I_{X_b}+\beta c_{X_b}-1}}{\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}}\right)$$

where $\frac{e^{I_{X_k}+\beta c_{X_k}-1}}{\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}}$ is defined as the proportion of q contributed by bundle k. Finally, again letting $Q = \sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}$ and $z = \frac{Q}{\gamma}$, we have the proof of Corollary 4.2.

$$q_k^* = \frac{W\left(\frac{1}{\gamma}\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}\right)}{\left(1+W\left(\frac{1}{\gamma}\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}\right)\right)}\left(\frac{e^{I_{X_k}+\beta c_{X_k}-1}}{\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}}\right)$$

6.4. Proof of Proposition 4.2

The derivative of Π^* with respect to the composition of bundle k is given by

$$\frac{\partial \Pi^*}{\partial I_{X_k}} = \frac{-1}{\beta}\left(\frac{\partial W(z)}{\partial I_{X_k}}\right) = \frac{-1}{\beta}\left(\frac{\partial W(z)}{\partial z}\frac{\partial z}{\partial I_{X_k}}\right) = \frac{-1}{\beta}\frac{W(z)}{z\left(1+W(z)\right)}\frac{1}{\gamma}e^{I_{X_k}+\beta c_{X_k}-1}$$

Recalling that $Q = \sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}$ and $z = \frac{Q}{\gamma}$, we have

$$\frac{\partial \Pi^*}{\partial I_{X_k}} = \frac{-1}{\beta} \frac{W\left(\frac{1}{\gamma}\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}\right)}{\left(1+W\left(\frac{1}{\gamma}\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}\right)\right)} \left(\frac{e^{I_{X_k}+\beta c_{X_k}-1}}{\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}}\right)$$

The derivative of Π^* with respect to the cost c_{X_k} of bundle k is

$$\frac{\partial \Pi^*}{\partial c_{X_k}} = \frac{-1}{\beta}\left(\frac{\partial W(z)}{\partial c_{X_k}}\right) = \frac{-1}{\beta}\left(\frac{\partial W(z)}{\partial z}\frac{\partial z}{\partial c_{X_k}}\right) = \frac{-1}{\beta}\frac{W(z)}{z\left(1+W(z)\right)}\frac{\beta}{\gamma}e^{I_{X_k}+\beta c_{X_k}-1}$$

since $Q = \sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}$ and $z = \frac{Q}{\gamma}$, we have

$$\frac{\partial \Pi^*}{\partial c_{X_k}} = -\frac{W\left(\frac{1}{\gamma}\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}\right)}{\left(1+W\left(\frac{1}{\gamma}\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}\right)\right)} \left(\frac{e^{I_{X_k}+\beta c_{X_k}-1}}{\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}}\right)$$

Finally, using the expression for q_k^* from Corollary 4.2, we obtain

$$\nabla \Pi^* = \left(\frac{\partial \Pi^*}{\partial I_{X_k}}, \frac{\partial \Pi^*}{\partial c_{X_k}}\right) = \left(\frac{-1}{\beta}, -1\right) q_k^*$$

6.5. Proof of Proposition 4.3

We may state without loss of generality that the feasible bundle space Ω contains the following f+1 bundles: X_{b_1}, X_{b_2},. . .,X_{b_d},. . ., X_{b_f} and X_1^*. We also assume that the Pareto-efficient frontier Ω_1^* is composed of the d bundles X_{b_1}, X_{b_2},. . ., $X_{b_{d-1}}$ and X_1^*. Then for all $j = b_d, \ldots, b_f$, it must be the case that $I_{X_j} \leq I_{X_i}$ and $c_{X_j} \geq c_{X_i}$ for at least one i in $\{b_1, \ldots, b_{d-1}, X_1^*\}$.

The construction Ω_2^*, which is the inner adjacent frontier of Ω_1^* under X_1^*, will contain the d-1 bundles b_1,. . .,b_{d-1} since the latter are not dominated by any bundle and will not be in any way affected by the extraction of bundle X_1^*. In addition, a bundle j in $\{b_d, \ldots, b_f\}$

will be incorporated into Ω_2^* provided $I_{X_j} \leq I_{X_i}$ and $c_{X_j} \geq c_{X_i}$ are not satisfied for all i in $\{b_1, \ldots, b_{d-1}\}$. This ensures that any bundle that was originally dominated only by X_1^* will be on the new Pareto-efficient frontier when the latter is eliminated from the feasible bundle set. Therefore, Ω_2^* will contain the d-1 bundles inherited from the previous frontier Ω_1^* and the bundles in $\{b_1, \ldots, b_{d-1}\}$ to be incorporated.

Proposition 4.3 is therefore proved.

6.6. Proof of Proposition 4.4

Proposition 4.2 states that the contribution of any given bundle to the objective function is independent of the contribution of every other bundle. Therefore, the optimal composition of the stage k+1 bundle is independent of the term $F_k^*(\Omega_k^*)$ given by (4.8), which is the contribution made by all of the already formed bundles to optimal utility.

It follows from the above that determining the composition of the optimal bundle in subproblem k+1 requires only the state information of the candidate bundle set $T(X_{k+1})$ for that stage. As was explained in Definition 4.4, building the set $T(X_{k+1})$ only requires information on the optimal bundles of the previous stages, that is, X_1^*, X_2^*,..., X_k^*, which is always present by virtue of the method used.

Therefore, to determine the optimal composition of the stage k+1 bundle we must solve

$$F_{k+1}^*(\Omega_{k+1}^*) = \max_{X_{k+1} \in T(X_{k+1})} \left\{ \frac{-I_{X_{k+1}}}{\beta} - c_{X_{k+1}} \right\} \tag{6.7}$$

thus proving Proposition 4.4.

6.7. Proof of Proposition 4.5

We define C as the total number of bundles that can be formed. Thus, $C = Card(\Omega)$. Rewriting (4.6) so as to choose the best b bundles, we obtain the following knapsack problem:

$$\max_{Y_1,Y_2,\ldots,Y_C} \sum_{k=1}^{C} \left(\frac{-I_k}{\beta} - c_k\right) Y_k$$
$$s.t: \sum_{k=1}^{C} Y_k \leq b$$
$$Y_1, Y_2, \ldots, Y_C \in \{0,1\}$$

where Y_k is a binary variable that indicates whether the kth bundle should be chosen and $(-I_k/\beta) - c_k = d_k$ is the benefit obtained by choosing bundle k. The number of bundles to be chosen is limited by the budget constraint, the cost a_k in resources of choosing any given bundle being set at $1\ \forall k$. We then construct the quotient $v_k = d_k/a_k$ and order the C bundles so that $v_1 \geq v_2 \geq v_3 \geq \ldots \geq v_C$. Given this ordering and the fact that all $a_k = 1$, the bundle designed by the BF algorithm Bitran and Ferrer (2007) will be the one that is associated with v_1. The second turnpike theorem, described in detail by Garfinkel and Nemhauser (1972), states that if $v_1 > v_2$ and there exists an $h = (a_1 - 1)\cdot \max_{k\geq 2}\{a_k\}$ such that $b > h$ (where b is the knapsack-type constrained resource), then the article associated with v_1 is the optimal choice among all the choosable articles. In our case, $h = 0$ given that $a_1 = 1$, and $b \geq 1$ is the number of bundles to be formed. The conditions set by the theorem are thus satisfied and the first bundle to be chosen must be the one constructed by the BF algorithm for the single-bundle optimal composition problem.

6.8. Proof of Proposition 4.7

By (4.3) we have that: $\Pi^* = \frac{-1}{\beta} W\left(\frac{1}{\gamma}\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}\right)$. If an additional bundle is composed, the benefit can be written as

$$\Pi^* = \frac{-1}{\beta} W\left(\frac{1}{\gamma}\sum_{l=1}^{b+1} e^{I_{X_l}+\beta c_{X_l}-1}\right) = \frac{-1}{\beta} W\left(\frac{1}{\gamma}\left(e^{I_{X_{b+1}}+\beta c_{X_{b+1}}-1} + \sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}\right)\right),$$

where $I_{X_{b+1}} + \beta c_{X_{b+1}} - 1 > -1$ given that $I_{X_{b+1}} + \beta c_{X_{b+1}} > 0$ for every case. Then $e^{I_{X_{b+1}}+\beta c_{X_{b+1}}-1} > e^{-1} > 0$, implying in turn that $e^{I_{X_{b+1}}+\beta c_{X_{b+1}}-1} + \sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1} >$

$\sum_{l=1}^{b} e^{I_{X_l}+\beta c_{X_l}-1}$. Since the Lambert W function is strictly increasing, the utility of b bundles is always less than the utility of $b+1$ bundles provided there are no administration costs.

6.9. Proof of Proposition 4.8

Rewrite (4.9) as follows:

$$\Pi = \left(\sum_{i=1}^{W} \sum_{t=1:t\neq k}^{K} H^i \frac{e^{I^i_{X_t}+\beta^i p_t}}{\gamma^i + \sum_{l=1}^{K} e^{I^i_{X_l}+\beta^i p_l}} \left(p_t - c_{X_t}\right) \right) + \left(\sum_{i=1}^{W} H^i \frac{e^{I^i_{X_k}+\beta^i p_k}}{\gamma^i + \sum_{l=1}^{K} e^{I^i_{X_l}+\beta^i p_l}} \left(p_k - c_{X_k}\right) \right).$$

Denote the two terms on the right-hand side A and B, respectively.

Taking the partial derivatives of A and B with respect to the price of bundle k, we obtain

$$\frac{\partial A}{\partial p_k} = -\left(\sum_{i=1}^{W} \sum_{t=1:t\neq k}^{K} H^i \frac{e^{I^i_{X_t}+\beta^i p_t}}{\left(\gamma^i + \sum_{l=1}^{K} e^{I^i_{X_l}+\beta^i p_l}\right)^2} \left(p_t - c_{X_t}\right) \beta^i e^{I^i_{X_k}+\beta^i p_k} \right)$$

and

$$\frac{\partial B}{\partial p_k} = \sum_{i=1}^{W} H^i \frac{e^{I^i_{X_k}+\beta^i p_k}}{\gamma^i + \sum_{l=1}^{K} e^{I^i_{X_l}+\beta^i p_l}} + \sum_{i=1}^{W} H^i \left(p_k - c_{X_k}\right) \frac{\beta^i e^{I^i_{X_k}+\beta^i p_k} \left(\gamma^i + \sum_{l=1}^{K} e^{I^i_{X_l}+\beta^i p_l} - e^{I^i_{X_k}+\beta^i p_k}\right)}{\left(\gamma^i + \sum_{l=1}^{K} e^{I^i_{X_l}+\beta^i p_l}\right)^2}.$$

The first-order conditions are then

$$\frac{\partial \Pi}{\partial p_k} = -\left(\sum_{i=1}^{W} \sum_{t=1:t\neq k}^{K} H^i \frac{e^{I^i_{X_t}+\beta^i p_t}}{\left(\gamma^i + \sum_{l=1}^{K} e^{I^i_{X_l}+\beta^i p_l}\right)^2} \left(p_t - c_{X_t}\right) \beta^i e^{I^i_{X_k}+\beta^i p_k} \right)$$

$$\left(\sum_{i=1}^{W} H^i \frac{e^{I^i_{X_k}+\beta^i p_k}}{\gamma^i + \sum_{l=1}^{K} e^{I^i_{X_l}+\beta^i p_l}} \right) + \left(\sum_{i=1}^{W} H^i \beta^i \left(p_k - c_{X_k}\right) e^{I^i_{X_k}+\beta^i p_k} \frac{\gamma^i + \sum_{l=1}^{K} e^{I^i_{X_l}+\beta^i p_l} - e^{I^i_{X_k}+\beta^i p_k}}{\left(\gamma^i + \sum_{l=1}^{K} e^{I^i_{X_l}+\beta^i p_l}\right)^2} \right) = 0.$$

By equation (3.13) we can rewrite this last expression as

$$\frac{\partial \Pi}{\partial p_k} = -\left(\sum_{i=1}^{W} \sum_{t=1:t\neq k}^{K} H^i \beta^i q^i_t q^i_k \left(p_t - c_{X_t}\right) \right) + \left(\sum_{i=1}^{W} H^i q^i_k \right) + \left(\sum_{i=1}^{W} H^i \beta^i q^i_k \left(1 - q^i_k\right) \left(p_k - c_{X_k}\right) \right) = 0$$

Rearranging terms, we have

$$p_k = c_{X_k} + \frac{\left(\sum_{i=1}^{W} \sum_{t=1:t\neq k}^{K} H^i \beta^i q^i_t q^i_k \left(p_t - c_{X_t}\right) \right) - \left(\sum_{i=1}^{W} H^i q^i_k \right)}{\sum_{i=1}^{W} H^i \beta^i q^i_k \left(1 - q^i_k\right)}.$$

6.10. Proof of Proposition 4.9

We know from Caplin and Nalebuff (1991) that if marginal costs are constant, then as long as $D(p_k > 0)$, a sufficient condition for $\Pi_k = D(p_k)(p_k - c_k)$ to be quasiconcave in p_k is that $1/D(p_k)$ $1/D(p_k)$ be convex p_k. This has been proven for logit functions by Torrents (2013).

6.11. Proof of Proposition 4.10

If the composition in set in X_l, then the optimal price for it can be defined solving the problema is:

$$\max_{p_1,\ldots,p_b\geq 0}\sum_{k=1}^{b}\frac{\phi\left(p_k\right)e^{I_{X_k}+\beta p_k}}{\gamma+\sum\limits_{l=1}^{b}\phi\left(p_l\right)e^{I_{X_l}+\beta p_l}}\left(p_k-c_{X_k}\right)$$

We proceed by deriving the expression of profit of the firm regarding the price, and applying the optimal condition of first order:

$$\frac{\partial\Pi_{X_k}}{\partial p_k}=0 \qquad \forall k=1,\ldots,b.$$

We find the fixed point equations system that allows us to set the optimal prices p_k with $k=1,\ldots,b$, which is:

$$p_k^*=\underbrace{c_{X_k}+\frac{\gamma_k+\phi\left(p_k^*\right)e^{I_{X_k}+\beta p_k^*}}{\gamma_k\left(w\left(1-\phi\left(p_k^*\right)\right)-\beta\right)}}_{b=1\text{ term}}+\underbrace{\frac{\sum\limits_{l=1:l\neq k}^{b}\phi\left(p_l^*\right)e^{I_{X_l}+\beta p_l^*}\left(p_l^*-c_{X_l}\right)}{\gamma_k}}_{b>1\text{ term}} \qquad \forall k=1,\ldots,b.$$

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