PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE
ESCUELA DE INGENIERÍA

TIME-VARYING OIL RISK PREMIUMS

PHILIP LIEDTKE

Thesis submitted to the Office of Research and Graduate Studies
in partial fulfillment of the requirements for the degree of
Master of Science in Engineering

Advisor:
GONZALO CORTÁZAR

Santiago de Chile, July 2018

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Gratefully to my parents

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ABSTRACT

This thesis proposes to extract time-varying commodity risk premiums from multi-factor models using futures prices and analyst's forecasts of future prices. The model is calibrated for oil using a 3-factor stochastic commodity-pricing model with an affine risk-premium specification. WTI futures price data is from NYMEX and analysts forecasts from Bloomberg and the U.S Energy Information Administration. Weekly estimations for short, medium and long-term risk premiums between 2010 and 2017 are obtained. Results from the model calibration show that risk premiums are clearly stochastic, that short-term risk premiums tend to be higher than long-term ones and that risk premium volatility is much higher for short maturities. An empirical analysis is performed to explore the macroeconomic and oil market variables that may explain the stochastic behavior oil risk-premiums.

Keywords: Risk premium, futures prices, oil prices

RESUMEN

Esta tesis propone extraer premios por riesgo variables en el tiempo de modelos de múltiples factores usando precios de futuros y pronósticos de precio de analistas. Se utiliza un modelo de valorización de commodities de 3 factores, con una estructura de premios por riesgo afines, el cual es calibrado para petróleo. Los precios de futuros de WTI son obtenidos de NYMEX y los pronósticos de analistas de *Bloomberg* y la *U.S Energy Information Administration*. Se obtienen estimaciones de premios por riesgo a corto, mediano y largo plazo entre 2010 y 2017. Los resultados de la calibración del modelo muestran que los premios por riesgo son claramente estocásticos, que los de corto plazo tienden a ser mayores que los de largo plazo y que su volatilidad es mayor para plazos menores. Un análisis empírico es realizado para explorar las variables macroeconómicas y las del mercado del petróleo que puedan explicar el comportamiento estocástico de los premios por riesgo de este commodity.

Palabras Claves: Premios por reisgo, contratos futuros, petróleo

1. GENERAL OVERVIEW

Commodity prices have become an important matter of interest in the last few years showing large volatilities and fast declining prices of some of them, rising the practitioners' and researchers' interest in understanding their behavior and main drivers. Most of the commodity producers and many goods manufacturers, whose operations strongly depend on commodity prices, are in need of credible price estimations to evaluate their projects and define their business strategy. However, as no method has shown to be a totally valid way of predicting future prices, most of them rely on the use of futures contracts' prices as the most likely price of a specific commodity in the future.

Futures contracts are one of the most basic financial contracts which essentially represent an agreement to buy or sell something at a certain price in a specific time-point in the future. Their main function is to allow producers and buyers of different assets to be able to trade it at a fixed price in the future to assure their continued business operations and to be protected from any possible price fluctuation. A futures contract differs from an option in the fact that it is an obligation for the participant to trade a certain asset at the agreed price, even if the market price is more convenient. Futures are two-way contracts, which means that a buyer and a seller need to agree on a certain price and time horizon for it to exist.

Futures prices should theoretically represent the expected prices of the underlying commodity, as producers and buyers are willing to trade it at the price they expect to get in the future. However, this only holds in a perfect and risk free world. In the real world there is always an imbalance between the amount of market players willing to buy and sell futures contracts to hedge their risk, and in order to compensate this imbalance speculators join the market asking for an additional premium, the risk premium, for entering the other side of the contract and taking on the commodity price risk (Keynes (1930), Hicks (1939)).

Risk premiums represent the price difference between the futures contracts' prices and the markets' expected prices. Therefore, the use of futures prices as the most likely price for a commodity in the future is only valid if the risk premium does not exist. If the risk premium does exist and is different from zero, the most likely price of a commodity in the future would be given by the risk premium added to the futures price, which is basically the market expectation of the commodity price. The existence of a risk premium in futures contracts is not a new finding and the fact that futures prices are not equal to expected prices is accepted in the literature, however, many practitioners still use futures prices as proxies for expected prices because of the difficulty to properly estimate risk premiums.

Futures prices are directly accessible at different exchanges across the globe and even though their prices are not available for all possible time horizons, different methods have been developed to estimate them for any given maturity (e.g. Cortazar and Naranjo (2006)). Instead, neither expected prices nor risk premiums are directly available in market prices of any kind. This issue represents a huge challenge for researchers and practitioners needing to use expected prices' estimates. Risk premiums probably depend on other risk related variables of the market, whose estimation could be very straight forward, nevertheless no proper risk premium estimates have been achieved and therefore no clear relation has been found. It is interesting to note that if futures prices are known, which generally is the case, the estimation of risk premiums is equivalent to the estimation of expected prices.

Some efforts have been done in order to try to estimate risk premiums (or expected prices) from different market prices, but most of them fail due to the lack of information contained on them. Market prices do not contain, or contain too few information on expectations, nevertheless surveys done to market participants could contain significant information if properly executed and validated. The problem of using survey forecasts is that they are very noisy and their accuracy depends on the incentives placed on the participants and their knowledge of the market. If survey participants are well chosen, then it could be argued that at least some of the information in the survey is credible and can be used to enhance risk premium estimations. One good example of that are *Bloomberg's* commodity survey forecats, which regularly ask experts on different commodities, whose income directly depends on the quality of their forecast, and generates a commodity forecast with their input.

Some recent researches try to link futures prices with different survey forecasts in order to obtain a better estimate of expected prices for interest rates and commodities (e.g. Chun (2011), Altavilla, Giacomini, and Ragusa (2016), Cortazar et al. (2018)), nevertheless there still is no consensus on which is the proper way to use survey forecasts and to what end they are accountable.

This paper has two main goals, first, to develop a methodology capable of obtaining significant and consistent risk premium estimates of a commodity for different maturities using only futures prices and survey forecasts as input. This methodology will be applied to WTI oil prices and could then potentially be used for any other commodity as long as enough information is available. The application of the developed methodology to other commodities and the performance comparison between different commodities, however, is out of the scope of this study. And, second, to understand the main drivers of WTI oil's risk premiums and their relation with other market variables. As there still is a controversy on the basic behavior of risk premiums, this last point is key to generate a deeper understanding of them and to guide future investigations.

2. INTRODUCTION

Even though commodity risk premium is an important topic in financial economics, there is no consensus on its magnitude, behavior and appropriate estimation procedure (Baumeister and Kilian (2016); Bianchi and Piana (2017); de Roon, Nijman, and Veld (2000); Melolinna (2011); Palazzo and Nobili (2010)). Moreover, the recent financialization of commodity markets has increased its relevance for investors and strengthened arguments on its time-varying behavior (Hamilton and Wu (2014); Baker and Routledge (2017); Ready (2016)).

Understanding the stochastic behavior of commodity risk premiums is important for several reasons. First, it provides valuable information on investment returns for agents who treat commodities as an asset class. Second, it helps to relate risk-adjusted expected prices, which are readily available in futures markets, with those of true expected prices, which are required for NPV calculations or risk management purposes. Third, it may shed light on some public policy implications by uncovering their macroeconomic determinants.

This paper provides a procedure for estimating the stochastic process of the term structure of commodity risk premiums by calibrating a multifactor model using analysts' forecasts of future spot prices and futures contracts oil price data. Once time-varying oil risk premiums are obtained, an empirical analysis is performed to explore the main macroeconomic and oil market specific variables that explain their behavior.

There have been various attempts in the literature to estimate commodity risk premiums. Many practitioners and researchers use futures prices as proxies for market expectations (see Baumeister and Kilian (2016), Bianchi and Piana (2017)), implicitly assuming risk premiums are zero. But Keynes (1930) and Hicks (1939) had proposed in their theory of normal backwardation, that if producers and other market participants wanted to hedge their risk by selling future contracts, buyers should get a compensation in the form of a risk premium for taking on that risk. Furthermore, there is already evidence on its time-varying nature (de Roon et al. (2000); Sadorsky (2002); Pagano and Pisani (2009); Achraya, Lochstoer, and Ramadorai (2013); Etula (2013); Hamilton and Wu (2014); Singleton (2014)). In addition, in recent years there has been some discussion on the impact of the post 2005 growth of commodity index fund traders on risk premiums (Hamilton and Wu (2014); Singleton (2014); Hong and Yogo (2012); Stoll and Whaley (2010); Irwin and Sanders (2011); Ready (2016)).

Regardless of its increasing importance, there is no current consensus on how to estimate risk premiums and their stochastic behavior. In the last few years, different methods have been developed to extract risk premiums, or equivalently to calculate expected spot prices, from the available data. Even though most of the literature addresses how to get the market's expected interest rates (e.g. Diebold and Li (2006); Altavilla, Giacomini, and Costantini (2014); Chun (2011)), some efforts have been oriented to commodities.

In what follows we present one way of characterizing existing methods for estimating risk premiums in commodity markets by classifying them into three approaches: Econometric, Economic, and Market.

In what we call the econometric approach we include Gorton, Hayashi, and Rouwenhorst (2013), Hong and Yogo (2012), Pagano and Pisani (2009) and Baumeister and Kilian (2016) among others. This approach regresses realized spot commodity prices, or a function of them, on different lagged market variables to infer the expected market's spot price. Then the resulting risk premium is obtained by comparing this expected spot price with the futures price for the same maturity. Baumeister and Kilian (2016) extract expected spot prices from the historical payoffs of future contracts. They first calculate the payoff from different futures contracts as the difference between the futures price for a given maturity and the realized spot price at that date. Then, regressing the above payoffs on different set of variables, expected prices are obtained. Their results show that none of the sets of regressors used is capable of getting a lower MSPE than the Hamilton and Wu (2014) model detailed below when performing an out-of-sample analysis. However, given that realized future spot prices and current futures prices with same maturity are compared, the required data-sample gets bigger as longer-term risk premium are estimated.

In what we call the economic approach we include Hamilton and Wu (2014), Bianchi and Piana (2017), and Cortazar, Kovacevic, and Schwartz (2015). These models use no-arbitrage or rational expectation models to infer expected spot prices from past and current market variables, typically futures and spot prices. For example, Hamilton and Wu (2014), following the normal backwardation theory of Keynes (1930), present a model in which hedgers sell futures contracts to hedge their risk and speculators and investors buy those futures contracts in order to maximize their utility function caring about the expected value and the variance of their future income. They find a change in behavior of commodity risk premiums before and after 2005 due to the financialization of commodity markets. Bianchi and Piana (2017) argue against using realized risk premiums as they do not represent the ex-ante premiums if the spot prices are biased from their expectations. To directly capture the ex-ante risk premiums they create a model with adaptive learning to calculate expected spot prices for every date in the sample using only the past spot prices and the aggregate demand as input. Their model is based on the belief that investors learn from their mistakes predicting spot prices and their next predictions are therefore going to be influenced by their past prediction errors. They analyze the behavior of oil, copper, silver and corn, showing strong evidence on risk premia being time-varying.

Cortazar et al. (2015) follows the extensive literature on no-arbitrage commodity pricing models that uses multifactor models to explain the time-series and cross section of futures prices (Gibson and Schwartz (1990); Heston (1993); E. S. Schwartz (1997); Duffie, Pan, and Singleton (2000); E. Schwartz and Smith (2000); Cortazar and Schwartz (2003); Casassus and Collin-Dufresne (2005); Cortazar and Naranjo (2006); Trolle and Schwartz (2009); Cortazar and Eterovic (2010); Bhar and Lee (2011); Chiang, Hughen, and Sagi (2015)). They argue that these models, being successful in fitting futures prices, provide very poor risk premium estimates. Therefore, they propose using an asset-pricing model instead of restricting some of their parameters. Asset-pricing models have been extensively applied to estimate commodity risk premiums, diverging on their approach and application, including the definition and number of risk factors, obtaining mixed results (Dusak (1973); Bodie and Rosansky (1980); Carter, Rausser, and Schmitz (1983); Chang, Chen, and Chen (1990); Bessembinder and Chan (1992); Bjornson and Carter (1997); Erb and Harvey (2006); Hong and Yogo (2012); Dhume (2010)).

In what we call the market approach we include a recent paper by Cortazar et al. (2018) in which they propose extracting information on expected spot prices directly from market surveys and using them, in addition to spot and futures prices, to calibrate a term structure model. Thus, risk premiums are obtained directly from the model as the difference between the expected spot price and the futures price consensus curves. Including survey forecasts in economic models, even though it had not been previously applied to commodities, had been previously used in other contexts. For example, Chun (2011) shows that using GDP, inflation and other macroeconomic variables' survey forecasts adds important information, not fully incorporated in market prices, to interest rates prediction models and gives them a higher accuracy. Altavilla et al. (2016) develop a method in which interest rate predictions become more accurate using interest rate surveys.

This paper proposes to extract time-varying risk premium observations using the market approach by extending Cortazar et al. (2018) to allow for a stochastic specification of risk premiums. We propose a 3-factor model based on Cortazar and Naranjo (2006) and Dai and Singleton (2000), and consider an affine risk premium specification following Duffee (2002). The model is estimated with the Kalman Filter using WTI oil analysts' forecasts of spot prices and futures contracts price data between 2010 and 2017. Analysts' forecasts are provided by Bloomberg and the U.S. Energy Information Administration (EIA) for up to 25 years, and oil futures price data is obtained from the New York Mercantile Exchange (NYMEX) for maturities up to 10 years. This allows us to obtain weekly estimates for short, medium and long-term oil risk premiums and to analyze the market determinants of these premiums comparing them with previous findings in the literature.

This analysis requires having time-varying risk premium estimates provided by our procedure and which were not available in the previous literature.

Once the term structures for oil risk premiums between 2010 and 2017 are computed, we explore the market determinants of those premiums. Following Bhar and Lee (2011) among others, we perform several regressions on different market variables that have been previously proposed in the literature. In this way, we provide some light on the determinants of risk premium variations and propose an adjustment to futures prices as a new simple way to estimate market expected prices.

The remainder of this paper is organized as follows. Section 3 presents the model to estimate time-varying term structures of risk premiums. Section 4 describes the data used. Section 5 provides the risk premium results. Section 6 discusses the market determinants of risk premiums and Section 7 concludes.

3. THE MODEL TO ESTIMATE RISK PREMIUMS

3.1. Model Definition

We present an N-factor term structure model which is a non-stationary version of the canonical $A_0(N)$ Dai and Singleton (2000) model with stochastic risk premiums as in Duffee (2002). We propose calibrating this model using both futures prices and analyst's forecasts to obtain a time-varying term structure of risk premiums[1]. Let S_t be the spot price of the commodity at time t, then assume that:

$$lnS_t = Y_t = h'x_t \tag{3.1}$$

$$dx_t = (-Ax_t + \begin{bmatrix} b_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix})dt + dw_t \tag{3.2}$$

where h is an $n \times 1$ vector of constants, x_t is an $n \times 1$ vector of state variables, b_1 is a scalar, A is an $n \times n$ upper triangular matrix with its first diagonal element being zero and the other diagonal elements all different and strictly positive. Let dw_t be an $n \times 1$ vector of uncorrelated Brownian motions following

$$dw_t dw_t' = Idt \tag{3.3}$$

where I is an $n \times n$ identity matrix. Dai and Singleton (2000) show that their model has the maximum number of econometrically identifiable parameters and at the same time nests most of the models used in literature.

To specify a time-varying risk premium in our constant-volatility model we resort to Duffee (2002) who shows how to use affine risk premiums in all types of Dai and Singleton (2000) canonical models, including the ones with non-stochastic volatility. Let RP_t be the

[1]This paper builds on Cortazar et al. (2018) which also used futures and analysts' forecasts, but assumed constant risk premiums. That paper used the Cortazar and Naranjo (2006) N-factor model. In Appendix A we show that our proposed model is a rotated version of the Cortazar and Naranjo (2006) model.

commodity risk premium and assume that:

$$RP_t = \lambda + \Lambda x_t \tag{3.4}$$

and the risk adjusted version of the model shown in Equations 3.1 and 3.2, is

$$Y_t = h' x_t \tag{3.5}$$

$$dx_t = \left[-(A+\Lambda)x_t + \begin{bmatrix} b_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} - \lambda \right] dt + dw^Q \tag{3.6}$$

where λ is a $n \times 1$ vector and Λ is a $n \times n$ matrix which does not need to be diagonal nor triangular. No further restrictions are set for the elements[2] in λ and Λ.

Notice that in our model the risk-adjusted process differs from the true one not only by a constant risk premium, λ, but also by the Λ matrix. Thus, futures prices and expected prices depend on different processes for the state variables, the former with the $A + \Lambda$ matrix, while the latter only with matrix A. However, if the Λ matrix were set to zero, risk premiums would be a constant and not time-varying.

It is well known (Cox, Ingersoll, and Ross (1981)) that futures prices are the expected value of the spot price, S_t, under the risk-adjusted probability measure, Q. Given that the risk-adjusted spot price follows a log-normal distribution, futures prices are given by:

$$F_t(T) = E_t^Q(S_T) = e^{E_t^Q(Y_T) + \frac{1}{2}Var^Q(Y_T)} \tag{3.7}$$

where the risk-adjusted expected price and variance of Y_T can be obtained by replacing Equation 3.1 into 3.7:

$$F_t(T) = E_t^Q(S_T) = e^{h' E_t^Q(x_T) + \frac{1}{2} h' Cov^Q(x_T) h} \tag{3.8}$$

[2]An equivalent model definition is also used by Casassus and Collin-Dufresne (2005), Dai and Singleton (2002), Duarte (2004), Kim and Orphanides (2012), Palazzo and Nobili (2010) among others, however none of them use observations on analysts' forecasts as expected prices as we propose, having difficulties estimating significant risk premiums.

with[3]

$$E_t^Q(x_T) = e^{-(A+\Lambda)(T-t)}x_t + \left(\int_0^{T-t} e^{-(A+\Lambda)\tau}d\tau\right)(b-\lambda) \tag{3.9}$$

$$Cov_t^Q(x_T) = \int_0^{T-t} e^{-(A+\Lambda)\tau}(e^{-(A+\Lambda)\tau})'d\tau \tag{3.10}$$

Analogous to Equations 3.7, 3.8, 3.9 and 3.10, the expected price should satisfy the following equations:

$$E_t(S_T) = e^{E_t(Y_T)+\frac{1}{2}Var(Y_T)} \tag{3.11}$$

$$E_t(S_T) = e^{h'E_t(x_T)+\frac{1}{2}h'Cov(x_T)h} \tag{3.12}$$

$$E_t(x_T) = e^{-A(T-t)}x_t + \left(\int_0^{T-t} e^{-A\tau}d\tau\right)b \tag{3.13}$$

$$Cov_t(x_T) = \int_0^{T-t} e^{-A\tau}(e^{-A\tau})'d\tau \tag{3.14}$$

It can be shown[4] that Equations 3.9 and 3.10 have a closed form solution if matrix $A+\Lambda$ is diagonal. The same occurs for Equations 3.13 and 3.14, now considering matrix A. In a more general case, as in our model, futures prices and expected prices have to be obtained numerically[5].

Risk premiums may be defined as the return of the expected spot price over the future price. Let, $\pi_t(T-t)$ be the instantaneous risk premium at time t for $T-t$ years ahead:

$$\pi_t(T-t) = \frac{ln\left(\frac{E_t(S_T)}{F_t(T)}\right)}{T-t} \tag{3.15}$$

Then, replacing the expected spot price and the future price from Equations 3.8 and 3.12 we obtain

$$\pi_t(T-t) = \frac{h'(E_t(x_T) - E_t^Q(x_T)) + \frac{1}{2}h'(Cov_t(x_T) - Cov_t^Q(x_T))h}{T-t} \tag{3.16}$$

[3]See Appendix B

[4]See Appendix B

[5]To solve the equations efficiently we follow Pashke and Prokopczuk (2009) who develop a way of avoiding numerical integration, using a decomposition of matrix $A+\Lambda$ in eigenvalues and eigenvectors. See Appendix C.

Finally, implied model volatilities for expected spots, σ_e, and for futures prices, σ_f, may be computed using the following expressions[6]:

$$\sigma_E = \sqrt{h' e^{-A(T-t)} \left(e^{-A(T-t)}\right)' h} \tag{3.17}$$

and for the futures prices by,

$$\sigma_F = \sqrt{h' e^{-(A+\Lambda)(T-t)} \left(e^{-(A+\Lambda)(T-t)}\right)' h} \tag{3.18}$$

3.2. Model Estimation

The parameters of the model and the state variables are estimated using the Kalman Filter (Kalman (1960)), which computes the optimal value of each state variable for any given time taking all past information into account. The procedure can handle a large number of observations (in our case analysts' forecasts and futures prices) and allow for measurement errors.

At any given time-iteration (date), a variable number of observations is available, so we use the incomplete data panel specification of the Kalman filter previously used for Futures (Cortazar and Naranjo (2006)), Bonds (Cortazar, Naranjo, and Schwartz (2007)) and Analysts' forecasts (Cortazar et al. (2018)):

$$z_t = Hx_t + d + v_t \qquad v_t \sim N(0, R) \tag{3.19}$$

$$x_{t+1} = \bar{A}x_t + \bar{c} + w_t \qquad w_t \sim N(0, Q) \tag{3.20}$$

where z_t is an $m_t \times 1$ vector which contains the log-prices of each futures and analysts' forecast (in that order) observation at week t; H is an $m_t \times n$ matrix; d is an $m_t \times 1$ vector and v_t is an $m_t \times 1$ vector of measurement errors with zero mean and covariance given by R; x_t is the $n \times 1$ vector of the state variables from Equation 3.1; $\bar{A}$ and $\bar{c}$ are an $n \times n$ matrix and an $n \times 1$ vector, respectively, representing a discretization of the process described in Equation 3.2 and w_t is an $n \times n$ vector of random variables with mean zero

[6]See Appendix D

and covariance given by the $n \times n$ matrix Q. In this specification m_t varies depending on the number of available observations changing the size of z_t, H, d, v_t and R on every iteration. In contrast to Cortazar et al. (2018) we specify two error terms in Equation 3.19, with different variances to differentiate between futures prices and forecasts, since the latter include estimations from different analysts' and should be much noisier.

Thus, we define the $m_t \times m_t$ matrix R as follows:

$$R_t = \begin{bmatrix} \sigma_f & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_f & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \sigma_e & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \sigma_e \end{bmatrix} \tag{3.21}$$

To estimate the parameters of this model a maximum-likelihood approach is used.

4. DATA

To be able to estimate the risk premiums, futures prices and analyst's forecasts for different dates and maturities are required. This section describes the data used.

4.1. Futures Contracts

WTI crude oil futures prices are obtained from the *New York Mercantile Exchange*. We used weekly futures prices with expiration every 6 months, including the closest one to maturity. The longest traded contracts expire in approximately 9.2 years. Table 4.1 presents the futures price data, separated in one-year buckets.

Table 4.1. Futures price observations between January 2010 and June 2017 by yearly maturity buckets.

Maturity Bucket (years)	Mean Price ($/bbl.)	Price SD	Max Price ($/bbl.)	Min Price ($/bbl.)	Mean Maturity (years)	Number of Observations
0-1	77.8762	22.2808	113.7	26.55	0.4472	968
1-2	78.2296	19.4252	112.83	35.36	1.4795	795
2-3	77.5466	17.5891	109.33	38.66	2.4947	821
3-4	77.2896	16.4093	107.14	41.34	3.5103	783
4-5	77.39	15.7564	105.8	43.24	4.4861	786
5-6	77.4764	15.4024	105.56	44.42	5.4722	809
6-7	78.0038	15.2228	105.88	45.77	6.5043	767
7-8	78.1963	15.2019	106.3	46.5	7.4942	774
8-9	78.2701	15.7178	106.95	46.99	8.4316	635
9-10	77.1498	13.7851	95.16	55.08	9.0582	44

4.2. Survey Based Expected Prices

Since we assume that analysts' forecasts are noisy proxies for expected future spot prices, WTI's expected prices were collected from Bloomberg's analysts' predictions, a list of surveys done to professional analysts on the expected future commodity prices. The expectations are given quarterly for the next 8 quarters and yearly for the next 4 years. Data is available only when one of the many analysts does a prediction, and may be available any day of the week. Each prediction is grouped on the oncoming Wednesday resulting in weekly groups of observations. If predictions for the same maturity on the same date are available, their mean value is used. On average, there are 220 oil price predictions available every month for different maturities. In addition to Bloomberg analysts' expectations, EIA's oil price forecasts are also used. Data is available once a year since 2010. EIA's data includes yearly long-term predictions for up to 33 years ahead. Even though both Bloomberg's and EIA's predictions are for the average price of each quarter or year they were assumed to represent the price in the middle of their time period. Data of the current quarter and year were left out. Table 4.2 describes the forecast data used. The bucket size grows with maturity due to the fewer observations available for longer maturities.

Table 4.2. Analysts' price forecasts between January 2010 and June 2017 separated by maturity bucket.

Maturity Bucket (years)	Mean Price ($/bbl.)	Price SD	Max Price ($/bbl.)	Min Price ($/bbl.)	Mean Maturity (years)	Number of Observations
0-1	81.0201	22.2566	122	35	0.5314	1118
1-2	85.619	21.2545	135	40	1.4296	808
2-3	89.0411	23.4254	189	44	2.4752	289
3-4	88.3383	23.0256	154	40	3.4448	239
4-5	86.2134	22.6815	150	38.5	4.4235	179
5-10	101.217	22.0268	152.96	60	6.2903	79
10-34	171.5592	34.2276	265.2	104.678	18.4838	134

5. RESULTS

This section presents the results of using WTI oil weekly data between January 2010 and June 2017 to calibrate the N-factor term structure model using a 3-factor specification.

Table 5.1 shows the model parameter estimates. It can be noted that half of the parameter estimates are statistically significant at a 1% and 3/4 of them at a 10% significance level.

Table 5.1. Parameter estimates for the 3-factor model. Data between January 2010 and June 2017. Significance levels are given by ***1%, **5% and *10%.

	Estimate	Deviation	tStat	pValue
A_{11}	0	-	-	-
A_{12}	0.728*	0.3676	1.9802	0.0564
A_{13}	1.4204	0.9677	1.4678	0.1358
A_{22}	1.4929***	0.185	8.0674	0
A_{23}	2.7146*	1.3379	2.0291	0.0512
A_{33}	0.163***	0.0238	6.8577	0
Λ_{11}	0.2267***	0.0076	29.7516	0
Λ_{12}	-0.7768*	0.3877	-2.0037	0.0539
Λ_{13}	-1.5684*	0.9397	-1.669	0.0992
Λ_{21}	-0.044	0.0423	-1.0404	0.2319
Λ_{22}	-1.3074***	0.2862	-4.5686	0
Λ_{23}	-2.2669	1.4022	-1.6166	0.108
Λ_{31}	-0.0306	0.0248	-1.2314	0.1867
Λ_{32}	0.2826***	0.0673	4.2015	0.0001
Λ_{33}	0.4187***	0.111	3.7708	0.0004
h_1	0.1521***	0.0184	8.2626	0
h_2	0.2146*	0.117	1.8333	0.0745
h_3	0.7469***	0.0431	17.3302	0
λ_1	-6.081***	0.5953	-10.2143	0
λ_2	1.2692	1.2894	0.9843	0.2454
λ_3	1.0407*	0.6039	1.7233	0.0905
b_1	0.1767***	0.0543	3.2549	0.0021
σ_f	0.0058***	0	302.9228	0
σ_e	0.1***	0.0005	195.163	0

Figure 5.1 shows the term structure (from 1 month to 10 years) of annualized risk premiums over the whole sample period (01/2010 to 06/2017). Three things are worth noting. First, risk premiums are clearly stochastic. Second, short-term risk premiums tend to be higher that long-term ones. Third, risk premium volatility is much higher for short maturities.

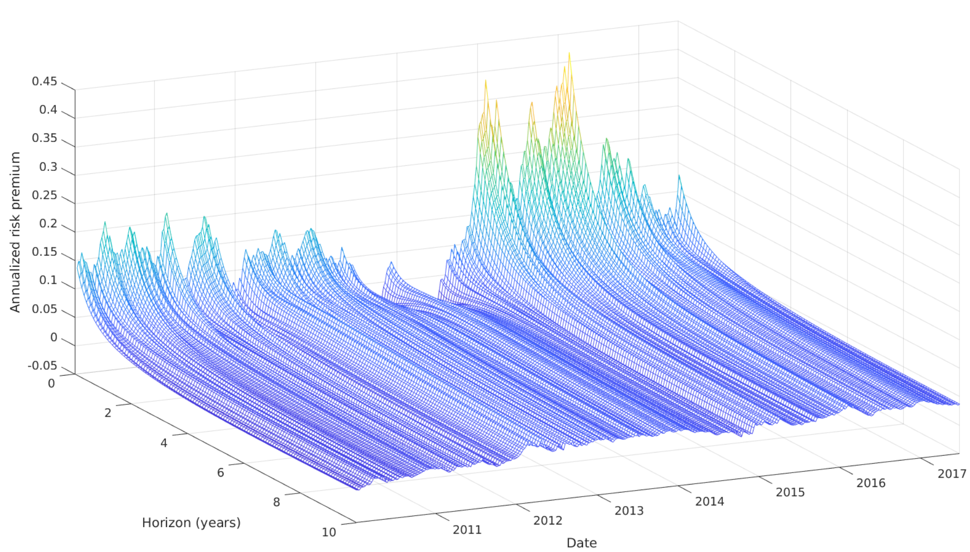
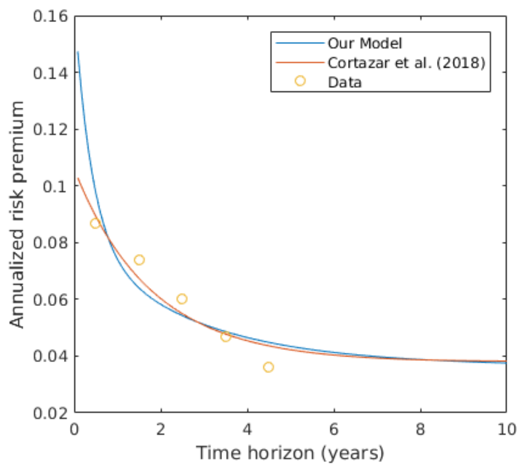


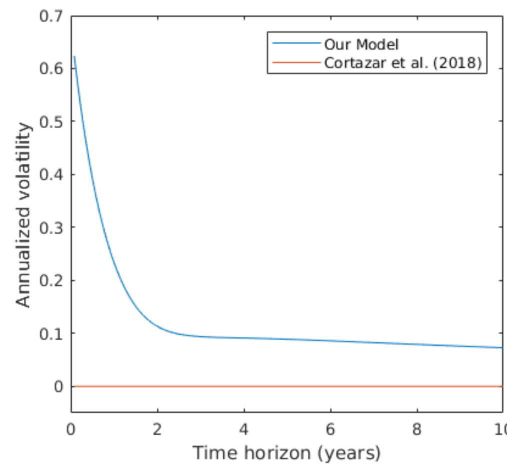
Figure 5.1. Annualized risk premium term structure from 1 month to 10 years. Data between January 2010 and June 2017.

Figure 5.2 analyzes the term structure mean and volatility of risk premiums. Figure 5.2a compares our model's mean risk premiums to those of Cortazar et al. (2018) constant risk premium model (with our same data) and to the data means. It can be noted that our model's mean risk premium level is similar to that of Cortazar et al. (2018) and both fit the data risk premiums well. Additionally, both premiums decrease with maturity.

Where both models diverge is Figure 5.2b that shows the volatility term structure because by construction Cortazar et al. (2018)'s assumes constant risk premiums while an essential element of our model are time-varying risk premiums. Finally, we analyze the goodness-of-fit of our model to futures and analysts' forecasts data. Table 5.2 presents the mean absolute percentage error (MAPE) of our model and shows that its fit for both data sets is better than for the constant risk premium model in Cortazar et al. (2018).



(a) Mean risk premiums



(b) Risk premium volatility

Figure 5.2. Mean risk premiums (a) and risk premium volatility (b) for our model and for the constant volatility model in Cortazar et al. (2018). Data mean risk premiums are also included in Figure (a). Data between January 2010 and June 2017.

Table 5.2. Mean Absolute Percentage Error (MAPE) for our time-varying risk premium model and for Cortazar et al. (2018) constant risk premium model. Data between January 2010 and June 2017.

	Our model	Cortazar et al. (2018)'s model
Futures prices	0.37%	0.39%
Expected prices	7.39%	8.00%

6. THE DETERMINANTS OF OIL RISK PREMIUMS

6.1. The Methodology

In this section, we explore the market determinants that may explain the variations of the estimated oil risk premiums. To do this we gather a set of market variables that have been previously reported in the literature as candidates for being related to risk premiums. We then perform a series of linear regressions in order to find which variables are the most significant in explaining the term structure of oil risk premiums.

There are few studies which analyze risk premiums directly (e.g. Bhar and Lee (2011); Bianchi and Piana (2017); Chen and Zhang (2011); Melolinna (2011)) as most investigations only calculate them as a side result from a price prediction model. However, there is some literature that discusses the impact of different market variables on risk premiums which we review below. The potential explanatory variables for the oil's risk premiums that we consider are: the S&P500 Index returns, the NASDAQ Emerging Markets Index returns (EMI), oil inventories percentage variation, oil futures open interest percentage variation, hedging pressure, the term premium, the default premium and the 5-year treasury bill rate. These variables have been shown to include most of the risk factors taking part in the oil market as we explain below.

The **S&P500 index returns** is used in some studies (de Roon et al. (2000); Bianchi and Piana (2017)) as a proxy for the state of the US' economy which could affect oil risk premiums. Daily data is available in Bloomberg since 1950.

The **NASDAQ Emerging Markets Index (EMI)** represents the state of the emerging markets' economy. It is known that many big emerging economies, such as Russia or China, are important oil market players, hence their economic performance could directly affect oil prices and premiums. EMI daily returns are available from the NASDAQ database since 2001.

Oil inventories percentage variation is a commonly used regressor in oil studies (Gorton et al. (2013); Melolinna (2011)) since it directly affects the supply of oil and therefore its price. The theoretical relationship between available stocks and risk premiums was first introduced by Kaldor (1939) in his Theory of Storage, in which he proposes the existence of a convenience yield to explain differences between current spot and futures prices. Gorton et al. (2013) develop a model, based on Kaldor (1939)'s Theory of Storage, which under a few assumptions implies that a rise in inventories should lead to a decrease in the overall risk premiums, and they find empirical results supporting their model. Weekly US WTI inventories starting at 1983 are available from the EIA and their percentage differences were calculated in order to obtain a stationary time series.

Open Interest (OI) and **Hedging Pressure (HP)** are the usual measures to represent the size and behavior of an instrument's market (in our case WTI futures). OI is measured as the total number of outstanding contracts, and therefore represents the market's size. It could be linked to the risk premiums as a larger amount of outstanding contracts could affect market's liquidity and therefore its premium. Kang, Rouwenhorst, and Tang (2017) propose that there exists a liquidity premium on commodity futures markets. OI is often used as an explanatory variable for commodity related studies (Bianchi and Piana (2017); Hong and Yogo (2012)).

HP is measured as the net positions of hedgers in a specific market, and represents the difference between hedgers' and speculators' positions, which according to Keynes (1930)'s and Hicks (1939)'s theories should have a strong correlation with risk premiums. According to them if hedgers want to hedge their risk by selling futures contracts, the buyers of those contracts should get a compensation for taking on that risk. As HP rises, risk premium will rise, because speculators will be willing to accept a greater amount of risk only if the premium is big enough. The relation between HP and prices or premiums has been empirically tested by different studies (Bianchi and Piana (2017); de Roon et al. (2000); Gorton et al. (2013); Kang et al. (2017); among others) generally supporting Keynes (1930). OI and HP weekly data was obtained from reports from the Commodity

Futures Trading Commission (CFTC), which is available since 2007. OI is directly available in the reports and their weekly percentage variations were used in the analysis. HP was computed as the short minus long commercial positions, divided by the total amount of outstanding contracts:

$$HP_t = \frac{CS_t - CL_t}{OI_t} \tag{6.1}$$

where CS_t and CL_t stand for short and long commercial positions, respectively.

The **term premium (TRM)** and the **default premium (DEF)** have shown to predict market excess returns in stocks and bonds (Fama and French (1989); Keim and Stambaugh (1986)), and could, therefore, affect oil risk premiums. TRM is defined as the difference between the 10-year treasury bill rate and the 3-month treasury bond yield, and DEF as the difference between the BAA-rated and the AAA-rated corporate bond yield. Daily treasury bill rates are available at the Federal Reserve while corporate bond yields were obtained from the Federal Reserve Bank of St. Louis.

The **5-year treasury bill rate (5Y T-Bill)** was used directly as it represents a good approach for a medium-term interest rate. Daily rates are available at the Federal Reserve.

Once the potential independent variables were chosen a set of multivariate OLS regressions were conducted:

$$RP_{it} = \beta_{0i} + \beta_{1i} X_t + \epsilon_{it} \tag{6.2}$$

where RP_{it} is the risk premium for maturity i and date t, X_t is the set of regressors described previously which are independent of the maturity, β_{0i} and β_{1i} are the estimators for each maturity i, and ϵ_{it} is the regression error for maturity i and date t.

We conduct our analysis in two steps. In the first step, a univariate regression is done for each independent regressor to check whether it is able to explain risk premiums in a statistically significant way. Then a multivariate regression analysis is performed using only the variables that were significant[1] in the univariate regressions. We run risk premiums regressions for 3, 6, 12, 18 months and 2, 5 and 10 years maturities. An independent

[1]Meaning the variables that showed p-values under 5% or R squared of over 30% for most maturities.

regression is performed for every different time horizon, both in the univariate and multivariate regressions. Robust standard errors were used in order to account for possible heteroscedasticity.

6.2. The Results

Table 6.1 shows the results of the univariate regressions for each of the independent variables and maturities chosen. Inventories, HP, TRM, DEF and 5Y T-Bill have reasonable significance (p-value) to explain changes in oil risk premiums and are candidates for inclusion in the multivariate analysis, while the others are not.

Table 6.2 shows the results of multivariate regressions for each maturity using only the above variables. It can be noted that the R-Squared of the regressions vary between 47.61% and 60.10% , and all variables are significant for most of the maturities.

From the above tables several results are worth discussing. First, we find a statistically significant and maturity-independent positive relation between inventories and risk premiums, similar to Dincerler, Khokher, and Simin (2005) and Khan, Khokher, and Simin (2008). Our results are, however, contrary to Gorton et al. (2013)'s model which could be due to their assumptions not holding for our sample period.

Second, our statistical significance and positive value of the HP estimator over all studied maturities is backed up by Keynes (1930)' theory of normal backwardation, as a larger number of hedgers wanting to hedge their risk produces a greater HP which should by related to speculators demanding a larger premium to take on that risk. Basu and Miffre (2013), de Roon et al. (2000) and Bianchi and Piana (2017), among others, obtain similar results.

Third, TRM is negative and significant for all maturities. These results support the belief that a negative slope of the yield curve predicts a decrease in the GDP (Estrella and

Hardouvelis (1991); Harvey (1988)) which could lead to an inverse relation with premiums.

Finally, the 5Y T-Bill and DEF have a positive effect on risk premiums, however only for maturities up to two years. The relation between DEF and risk premiums was expected to be positive as the first one is highly correlated with the short-term market uncertainty, and should therefore affect risk premiums in a positive way. Higher short-term uncertainty should induce the average investor to demand a larger premium specially for short term investments which is consistent with DEF affecting only short term premiums in a significant way. If the treasury bill yield serves as a proxy for the current state of the economy, being higher when the economy grows and lower on slow economic periods, we would expect to get a negative effect of it on risk premiums, such as in Bhar and Lee (2011). However, interest rates were unusually and constantly low during our sample period, which might alter the way in which treasury bill yields represent the state of economy. These results suggest that these 5 market variables are able to explain half of the variation of oil risk premiums in our model for all studied maturities. In addition to the economic insight the regression results provide, they could also be used to obtain estimates of risk premiums and therefore expectations of future spot prices. For example, many practitioners who currently use futures prices as a proxy for the market's spot price expectations could infer them directly from our market variables.

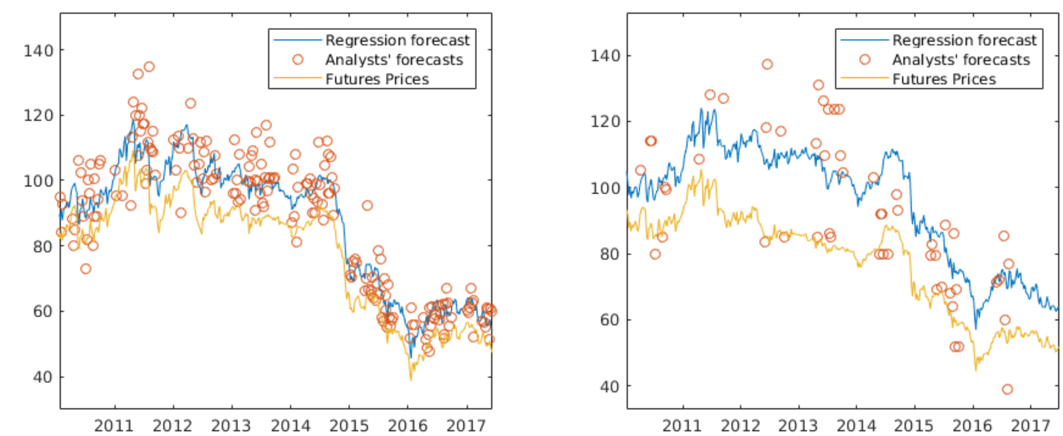
Figure 6.1 shows expected spot price estimations for two different maturities obtained by adding the expected risk premium from our regression analysis to the observed futures prices, along with analysts' forecasts and futures prices observations. The figure shows that by adding the risk premium to futures prices a less volatile estimate of expected prices is obtained. In addition, as Table 6.3 shows, this also increases its fit to analysts' forecasts, reducing estimation errors.

Table 6.1. Univariate regression analysis for each of the chosen independent variables and for each different maturity. Monthly maturities are written as “Mn” and yearly maturities as “Yn”. Data between January 2010 and June 2017.

		M3	M6	Y1	M18	Y2	Y5	Y10
S&P500	Estimate	-0.3639	-0.2689	-0.1437	-0.0723	-0.0312	0.0224	0.0175
	p-value	0.0518	0.0586	0.0893	0.1859	0.4446	0.4826	0.5018
	r2	0.0072	0.0066	0.0049	0.0019	-0.0011	-0.0013	-0.0014
EMI	Estimate	-0.2225	-0.1641	-0.0877	-0.0446	-0.0201	0.0109	0.0095
	p-value	0.1315	0.1433	0.1888	0.3009	0.5319	0.6628	0.6434
	r2	0.0033	0.0029	0.0019	0.0002	-0.0016	-0.0021	-0.002
Inventories	Estimate	4.5019	3.5144	2.225	1.4986	1.0845	0.5104	0.4163
	p-value	0	0	0	0	0	0.0049	0.0051
	r2	0.043	0.0456	0.052	0.0568	0.0533	0.0177	0.0175
OI	Estimate	-0.051	-0.041	-0.0283	-0.0213	-0.0175	-0.0122	-0.0096
	p-value	0.7157	0.6998	0.6549	0.6013	0.5651	0.608	0.6242
	r2	-0.0022	-0.0022	-0.0021	-0.0019	-0.0017	-0.0019	-0.002
HP	Estimate	-0.0269	-0.0025	0.0299	0.0485	0.0592	0.0689	0.0567
	p-value	0.3916	0.9149	0.0347	0	0	0	0
	r2	-0.0007	-0.0025	0.0089	0.0701	0.1917	0.4299	0.4325
TRM	Estimate	-0.0363	-0.0305	-0.0228	-0.0182	-0.0156	-0.011	-0.0094
	p-value	0	0	0	0	0	0	0
	r2	0.0984	0.1208	0.1919	0.2977	0.3895	0.3174	0.3463
5Y T-Bill	Estimate	-0.0005	-0.003	-0.0066	-0.009	-0.0105	-0.0122	-0.0096
	p-value	0.9503	0.6117	0.0578	0.0001	0	0	0
	r2	-0.0026	-0.0019	0.0067	0.0383	0.0977	0.2201	0.1993
DEF	Estimate	0.1332	0.0997	0.0559	0.0313	0.0174	-0.0006	-0.0009
	p-value	0	0	0	0	0	0.8105	0.6747
	r2	0.2051	0.1989	0.1768	0.1324	0.0721	-0.0024	-0.0021

Table 6.2. Multivariate regression coefficients for each maturity. Data between January 2010 and June 2017. Significance levels are given by ***1%, **5% and *10%.

	M3	M6	Y1	M18	Y2	Y5	Y10
Intercept	-0.0774***	-0.0506***	-0.0065	0.0212***	0.0358***	0.0461***	0.0395***
Inventories	2.6957***	2.1408***	1.4595***	1.0931***	0.8667***	0.4687***	0.3731***
HP	0.0784***	0.072***	0.0641***	0.0608***	0.0606***	0.0568***	0.0473***
TRM	-0.0693***	-0.0549***	-0.0355***	-0.0238***	-0.0167***	-0.0059***	-0.006***
5Y T-Bill	0.1083***	0.0828***	0.0485***	0.0284***	0.0166***	0.0002	0.0019
DEF	0.1827***	0.1377***	0.0784***	0.0448***	0.0259***	0.0008	0.001
R2	0.4761	0.49	0.5287	0.5727	0.601	0.534	0.5541



(a) 2 year ahead expected prices

(b) 5 year ahead expected prices

Figure 6.1. Expected prices obtained adding the regression estimated risk premiums to the observed futures prices (blue line) in comparison with analysts' forecasts (red dots) and futures prices (yellow line)

Table 6.3. MAPE between analysts' forecasts and two different expected price approaches: Futures and Futures plus Regression Market Risk Premium. Data between January 2010 and June 2017.

	M3	M6	M9	Y1	M18	Y2	Y5
Regression implied expectations	5.61%	5.65%	6.08%	6.11%	6.84%	7.7%	16.16%
Futures prices	6.01%	6.31%	7.71%	8.79%	11.24%	12.04%	17.72%

7. CONCLUSIONS

This paper proposes to extract time-varying commodity risk premiums from multi-factor models using futures prices and analyst's forecasts of future spot prices. The model is calibrated for oil between 2010 and 2017 using a 3-factor stochastic commodity-pricing model with an affine risk-premium specification, weekly WTI futures data from NYMEX and analyst's forecasts from Bloomberg and the U.S Energy Information Administration.

Results from the model calibration show that risk premiums are clearly stochastic, that short-term risk premiums tend to be higher than long-term ones and that risk premium volatility is much higher for short maturities.

Once weekly term structures of oil risk premiums are obtained an empirical analysis to explore the macroeconomic and oil market specific variables that may explain their stochastic behavior is performed. We find that inventories, hedging pressure, term premium, default premium and the level of interest rates all play a significant role in explaining the risk premium and thus could be used also for estimating expected commodity prices when reliable analyst's forecasts are not available.

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APPENDIX

A. ROTATION OF CORTAZAR AND NARANJO (2006)'S MODEL TO OURS AND BACKWARDS

From Cortazar and Naranjo (2006)'s to our model

Given the state space model of the form

$$Y_t = 1' x_t \tag{A.1}$$

$$dx_t = (-Ax_t + b)dt + \Sigma dw_t \tag{A.2}$$

where A and Σ are $n \times n$ diagonal matrices, b is a $n \times 1$ vector whose elements are zero excepting its first one and dw_t is an $n \times 1$ vector of correlated brownian motions such that $dw_t dw_t' = \Theta dt$. The covariance matrix $\Sigma\Theta\Sigma'$ is positive definite and therefore admits a Cholesky decomposition. Let define the matrix M as

$$M = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix} \tag{A.3}$$

where $M^{-1} = M$, then the matrix $M\Sigma\Theta\Sigma' M$ is still positive definite and still admits a Cholesky decomposition (L) so that

$$M\Sigma\Theta\Sigma' M = LL' \tag{A.4}$$

then applying the transformation $\xi_t = ML^{-1}Mx_t$ where $ML^{-1}M$ is an upper triangular matrix

$$Y_t = (\vec{1}' MLM)(ML^{-1}Mx_t) = h'\xi_t \tag{A.5}$$

$$d\xi_t = (-(ML^{-1}MAMLM)(ML^{-1}Mx_t) + ML^{-1}Mb)dt + ML^{-1}M\Sigma dw_t = (-\hat{A}\xi_t + \hat{b})dt + d\hat{w}_t \tag{A.6}$$

where h is an $n \times 1$ vector, $\hat{A}$ is an $n \times n$ upper triangular matrix whose first eigenvalue is zero, $\hat{b}$ is an $n \times 1$ vector with zeros in all its entries excepting the first one and $d\hat{w}_t$ is an

$n \times 1$ vector of uncorrelated brownian motions. This formulation is the one used by Dai and Singleton (2000) modified to hold for a matrix A with one zero valued eigenvalue by adding the $\hat{b}$ vector.

From our model back to Cortazar and Naranjo (2006)'s

Starting from equations A.5 and A.6 it is possible to get back to the model stated in equations A.1 and A.2 by finding a transformation matrix T which satisfies both following equations,

$$h'T^{-1} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \tag{A.7}$$

$$T\hat{A}T^{-1} = A \tag{A.8}$$

where A is the original diagonal matrix from equation A.2 which contains all $\hat{A}$'s eigenvalues. Therefore, if $\hat{A} = \hat{U}A\hat{U}^{-1}$, where $\hat{U}$ is a matrix containing $\hat{A}$'s eigenvectors, T^{-1} has to contain the columns of $\hat{U}$ scaled by a any chosen diagonal matrix G so that

$$T^{-1} = \hat{U}G \tag{A.9}$$

Equation A.9 assures that A.8 hold, thus to find T the system from equations A.7 and A.9 has to be solved for G obtaining for each of its i-th diagonal element

$$G_i = \left[h'\hat{U}\right]_i \tag{A.10}$$

having finally that $T = \left(\hat{U}G\right)^{-1}$. Applying the transformation $x_t = T\xi_t$ to equations A.5 and A.6,

$$Y_t = h'T^{-1}T\xi_t = 1'x_t \tag{A.11}$$

$$dx_t = (-T\hat{A}T^{-1}T\xi_t + T\hat{b})dt + Td\hat{w}_t = (-Ax_t + b)dt + \Sigma dw_t \tag{A.12}$$

where this last representation is the exact one shown in equations A.1 and A.2.

B. EXPECTED VALUE AND COVARIANCES OF STATE VARIABLES

In this section we show how to get the expected value an covariances of the state variables of any model of the type:

$$dx_t = (-Ax_t + b)dt + \Sigma dw_t \tag{B.1}$$

$$dw_t dw_t' = \Theta dt \tag{B.2}$$

Where dw_t are correlated brownian motions with a correlation matrix given by $dw_t dw_t' = \Theta dt$. First we define the following state space vector:

$$y_t = e^{At} x_t \tag{B.3}$$

and applying Itô's lemma

$$dy_t = e^{At} dx_t + Ae^{At} x_t dt \tag{B.4}$$

$$= e^{At} ((-Ax_t + b)dt + \Sigma dw_t) + Ae^{At} x_t dt \tag{B.5}$$

$$= e^{At} bdt + e^{At} \Sigma dw_t \tag{B.6}$$

This last equation can be integrated as follows:

$$\int_t^T y_s = \int_t^T e^{As} bds + \int_t^T e^{As} \Sigma dw_s \tag{B.7}$$

$$y_T - y_t = \left(\int_t^T e^{As} ds \right) b + \int_t^T e^{As} \Sigma dw_s \tag{B.8}$$

$$x_T = e^{-A(T-t)} x_t + e^{-AT} \left(\int_t^T e^{As} ds \right) b + e^{-AT} \int_t^T e^{As} \Sigma dw_s \tag{B.9}$$

Now it is straight forward to obtain the expected value and the variance of the state space variables:

$$E_t(x_T) = e^{-A(T-t)} x_t + \left(\int_0^{T-t} e^{-A\tau} d\tau \right) b \tag{B.10}$$

$$Cov_t(x_T) = \int_0^{T-t} e^{-A\tau} \Sigma \Theta \Sigma' (e^{-A\tau})' d\tau \tag{B.11}$$

C. METHOD TO AVOID NUMERICAL INTEGRATION

To get the expected values and covariances of the state variables as shown in equations B.10 and B.11 numerical integration seems to be necessary. Nevertheless, there is an alternative shown by Pashke and Prokopczuk (2009) which does not need numerical integration, but uses eigenvalues and eigenvectors of some matrices.

To solve for the expected value of the state variables like in equation B.10 first we decompose $A = UVU^{-1}$ where V is a matrix containing all A‘s eigenvalues in its diagonal and U is a matrix containing all its eigenvectors. It can be shown that $e^{-A\tau} = Ue^{-V\tau}U^{-1}$, where $e^{V(T-t)}$ is a diagonal matrix with $e^{v_i(T-t)}$ (where v_i is the ith eigenvalue of matrix A) in its ith position. It can be shown that

$$\int_0^{T-t} e^{-V\tau} d\tau = \begin{bmatrix} \frac{1-e^{v_1(T-t)}}{v_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \frac{1-e^{v_n(T-t)}}{v_n} \end{bmatrix} = \phi \tag{C.1}$$

thus the expected value of the state variables can be written as

$$E_t(x_T) = Ue^{V(T-t)}U^{-1}x_t + U\phi U^{-1}b \tag{C.2}$$

The variance shown in equation B.11 can be calculated using the same properties as the expected value, so that

$$Cov_t(x_T) = U\int_0^{T-t} e^{-V\tau}U^{-1}U'^{-1}(e^{-V\tau})'d\tau U' = UHU' \tag{C.3}$$

where H represents the integral just for ease of notation. As $e^{-V\tau}$ is a diagonal matrix containing $e^{-v_i\tau}$ in each of its diagonal elements, a closed form solution for the integral H can be obtained element-wise. To obtain the element in the i-th row an the j-th column of the matrix the next expression has to be evaluated

$$H_{i,j} = \int_0^{T-t} e^{-v_i\tau}\left[U^{-1}U'^{-1}\right]_{ij} e^{-v_j\tau}d\tau = \left[U^{-1}U'^{-1}\right]_{ij}\int_0^{T-t} e^{-(v_i+v_j)\tau}d\tau \tag{C.4}$$

$$= \left[U^{-1}U'^{-1}\right]_{ij} \frac{1 - e^{-(v_i+v_j)(T-t)}}{v_i + v_j} \tag{C.5}$$

D. MODEL IMPLIED VOLATILITIES

First, let D be a function of the state variables and time. Its returns can then be modeled as

$$\frac{dD}{D} = \mu_D dt + \sigma_D dw_D \tag{D.1}$$

Applying Itô's lemma we find that

$$\frac{dD}{D} = \frac{1}{D}\nabla D dx + \frac{1}{2}\frac{1}{D}\nabla D dx dx' \nabla D' + \frac{1}{D}\frac{dD}{dt}dt \tag{D.2}$$

where ∇ represents the jacobian operator. Replacing dx from equation 3.2,

$$\frac{dD}{D} = \frac{\nabla D(-Ax+c) + \frac{1}{2}\nabla D \nabla D' + \frac{dD}{dt}}{D}dt + \frac{\nabla D}{D}dw_x \tag{D.3}$$

Additionally it can be found that,

$$\left(\frac{dD}{D}\right)\left(\frac{dD}{D}\right) = \sigma_D^2 dt \tag{D.4}$$

and replacing equation D.3,

$$\sigma_D^2 = \frac{\nabla D \nabla D'}{D^2} \tag{D.5}$$

Now replacing D by the expected spot prices $E_t(S_T)$ calculated in section 3 the jacobian results in

$$\nabla E_t(S_T) = h' e^{-A(T-t)} E_t(S_T) \tag{D.6}$$

and replacing in D.5 we can get the following structure for the expected spot's implied volatility

$$\sigma_{E(S)}^2 = h' e^{-A(T-t)} \left(e^{-A(T-t)}\right)' h \tag{D.7}$$

Following the same procedure for futures prices and using the structure derived in section 3 the jacobian and the futures prices' implied volatility respectively result in

$$\nabla F_t(T) = h' e^{-(A+\Lambda)(T-t)} F_t(T) \tag{D.8}$$

$$\sigma_F^2 = h' e^{-(A+\Lambda)(T-t)} \left(e^{-(A+\Lambda)(T-t)}\right)' h \tag{D.9}$$

E. DUFFEE (2002)'S RISK PREMIUMS

In his investigation, Duffee (2002) proposes a new specification of affine risk premiums which differ from completely affine risk premiums, like the ones used by Dai and Singleton (2000), in their independence from stochastic volatility. In order to define the risk adjustment the next stochastic process for the state variables is defined,

$$dX_t = [(K\theta) - KX_t]\, dt + \Sigma S_t dW_t \tag{E.1}$$

where X_t is the $n \times 1$ vector of state variables, K and Σ are $n \times n$ matrices and $(K\theta)$ is an $n \times 1$ vector. S_t is an $n \times n$ diagonal matrix with the following in each of its diagonal elements,

$$S_t(ii) = \sqrt{\alpha_i + \beta_i' X_t} \tag{E.2}$$

being α_i the i-th element of vector a and β_i the i-th row of matrix β. Thus the model's degree of stochastic volatility depends solely on the composition of matrix β. The new premiums are then defined as,

$$\pi_t = S_t \lambda_t + S_t^- \Lambda_t X_t \tag{E.3}$$

where λ_t is an $n \times 1$ vector, Λ_t is an $n \times n$ matrix. S_t^- is an $n \times n$ diagonal matrix whose elements are defined as

$$S_t^-(ii) = \begin{cases} \sqrt{\alpha_i + \beta_i' X_t}, & \text{if } inf(\alpha_i + \beta_i' X_t) > 0 \\ 0, & \text{otherwise} \end{cases} \tag{E.4}$$

The previous specification can be rewritten in its canonical form according to Dai and Singleton (2000). In our case we are using a model with non-stochastic volatility, which taken to the canonical form implies that α becomes a vector of ones and β a matrix of zeros. This leads S_t and S_t^- into becoming identity matrices, so that the correct risk premium specification for us would be given by

$$\pi_t = \lambda_t + \Lambda_t X_t \tag{E.5}$$

F. ESTIMATION METHODOLOGY

To estimate the parameters of this model a maximum-likelihood approach is used. The number of parameters to be estimated[1] is given by $\frac{3N^2}{2} + \frac{5N}{2} + 3$, growing quadratically as the number of latent factors N increases. Thus, each additional factor enlarges the dimensionality of the optimization problem and toughens its numerical resolution significantly. Duffee (2002) uses time varying risk premiums with a stationary version of our model and concludes that quasi maximum likelihood functions, which should have exactly the same convergence properties as maximum likelihood functions, *"...have a large number of local maxima. The most important reason for this is the lack of structure placed on the feedback matrix K. ... . Another reason is that the feasible parameter space is not convex for any model with nonconstant volatility"*, where K is our matrix A. Nevertheless, we find that at least the first statement and probably the second one appear not to be true. The lack of structure placed in the matrix K has no influence in the non-convexity of the log-likelihood function as long as the model has the maximum number of statistically identifiable parameters. Moreover, the K (or A) matrix can be written in diagonal form by applying an invariant transformation to the model as shown in appendix A. Additionally we find that 2-factor models with time-changing risk premiums do not tend to show local maxima, while 3-factor models do. The only reasonable explanation of a statistically identified model showing local maximums as its dimension grows is the existence of numerical issues in the optimization process. When speaking of the dimension of the problem not only the number of parameters matter, but also the degrees of freedom of each model. For example when estimating the parameters of the 3-factor Cortazar et al. (2018) model which has constant volatility and 14 parameters to be estimated no important convergence issue arises, however if any of them is restricted to a specified value and one parameter for stochastic risk premiums (any element in our matrix Λ) is added the model presents serious convergence issues and gets stuck in multiple local maxima. We address those convergence problems to numerical issues of the additional degree of freedom the model

[1] The number of parameters to estimate includes the model's parameters and the two error terms from the Kalman filter

incorporated when their risk premiums were allowed to vary over time, because the number of parameters remained unchanged. Probably the multiple local maxima documented in several stochastic volatility models in literature is a numerical issue appearing due to the high dimensionality of them as well.

To solve this problem Duffee (2002) optimizes the model 1000 times from different starting points and restricts all parameters that show t-statistics lower than one. Alternatively Dai and Singleton (2002) propose to make a first optimization of the model and then setting the parameters with the largest relative standard errors to zero and reestimating the model. This would reduce the number of parameters to be estimated and therefore could eventually find a global maxima, nevertheless it imposes undesired restrictions to the model which do not necessarily represent the best possible estimation of the parameters. Furthermore, standard errors are not representative if calculated in a local maxima which exists because of numerical issues, so that the process of constraining parameters with the longest errors is probably mistaken. Additionally, when imposing restrictions to specific parameters the sense of a ”canonical representation” as Dai and Singleton (2000) propose is killed: Restricting one specific parameter to zero in their representation does not mean that all of the infinite alternative representations will also have a parameter restricted to zero. Thus if one parameter is set to zero this representation becomes the only one with the maximum number of identifiable parameters and is therefore unique and not ”canonical”.

More recently Chernov and Mueller (2012) evaluate the log-likelihood function in two billion starting points and select the best 20 thousand of them. Then they optimize alternating between simplex and sequential quadratic programming algorithms, eliminating half of the likelihoods at each stage. Other researches propose their own way to estimating them but most of them rely on trying a big amount of starting points and choosing the best possible solution.

We propose a slightly different approach, based on the belief that our log-likelihood function is actually convex, but presents numerical non-convexities due to its large dimensionality. If this would hold then different rotations in the sense of Dai and Singleton (2000)'s appendix A which maintain the number of parameters unchanged should have identical convexity properties, but not necessarily the same numerical issues. Therefore, when stuck in a numeric local maximum a rotation of the model will produce the same log-likelihood value, but probably not the same numerical issue leading out of the numeric local maxima. We implement this idea by rotating between the model described in section 3 and the one proposed by Cortazar and Naranjo (2006) as shown in appendix A. The new algorithm (we call it *rotating algorithm*) tends to show less local maxima and is able to find the global optimum much quicker, however multiple starting points are still needed in order to find a possible global maximum. This way we implement the following steps in order to find the optimal parameter estimates:

(i) Generate 70 random starting points and optimize them with the *fmincon* algorithm from MATLAB.

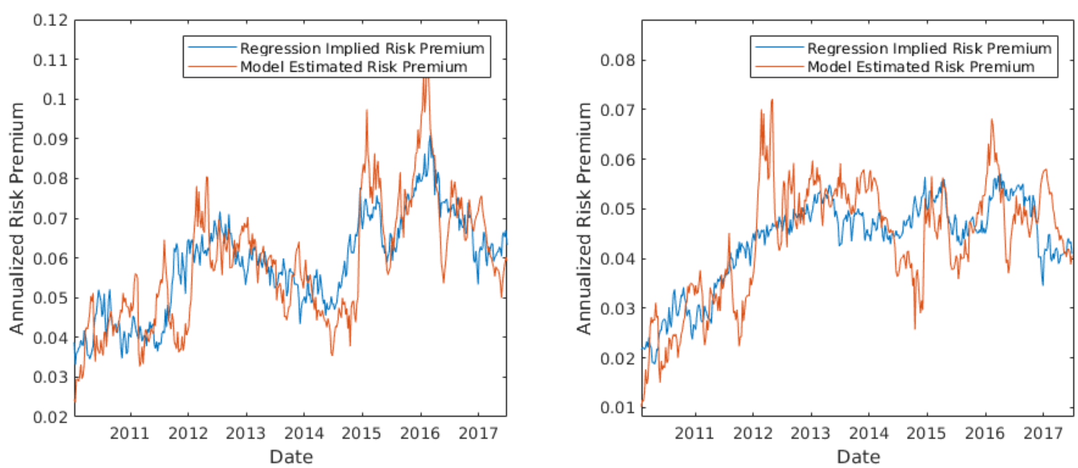
(ii) Choose the 2 best log-like values from 1 and optimize using the *rotating algorithm* implemented with MATLAB's *fminsearch*.

(iii) Compare both results and if their parameters differ significantly repeat step 2 with the next best log-like value.

It is not possible to assure that the found value is indeed the global optimum, nevertheless we argue that if the model lands at least two times in similar points with a higher log-likelihood value than any other optimization it is probably due to the presence of the global maxima.

G. RISK PREMIUM PREDICTION USING ONLY MARKET VARIABLES

Even though our model does not intend to create the best possible risk premium predicting regression it would be interesting to analyze how well are the market variables able to fit risk premium forecasts on their own. First we compare the model's risk premiums from section 5 with the regression implied risk premiums, which are obtained replacing all known market values and estimated coefficients from section 6 in equation 6.2 and assume a zero valued error. The results for a 2 and 5 year time horizon are shown in figure G.1. It can be noted that the market variables are able to replicate the risk premiums' behavior in an accurate way but showing less volatility



(a) 2 year risk premiums

(b) 5 year risk premiums

Figure G.1. Risk premiums created directly by the regression results of section 6 (blue line) in comparison with the models risk premiums of section 5

However the replication of risk premiums is more straight forward and of less importance than the replication of market's expectations themselves. Expected prices' estimates can be obtained by adding the corresponding risk premiums to the futures prices observations. The results for maturities of 2 and 5 years are shown in figure 6.1 compared to the use of futures prices only as predictors of expectations. The errors between the estimated expected prices and the analysts' forecasts are shown in table 6.3. It can be noted that the addition of this easily obtainable risk premium enhances the futures prices ability to represent the market expectations consistently across all studied maturities. These results are of special interest to practitioners currently using futures prices as a proxy for the market's price expectations.

H. TRANSFORMATION OF ANY LINEAR MODEL INTO AN EQUIVALENT CORTAZAR AND NARANJO (2006)'S MODEL

To transform any linear model into an equivalent one, an invariant transformation T has to be applied. In this section we will use Cortazar and Naranjo (2006)' definition of an invariant transformation to rewrite any general model as one following Cortazar and Naranjo (2006)'s structure.

PROPOSITION H.1. *Any given state space model with the next structure:*

$$Y_t = h' x_t + d \tag{H.1}$$

$$dx_t = (-Ax_t + b)dt + \Sigma dw_t \tag{H.2}$$

where dw_t are correlated Brownian motions such that $dw_t dw_t' = \Theta dt$ and A has all its eigenvalues positive and different excepting the first one which is set to zero, can be transformed into the model proposed in section 3.

Starting from the model shown in equations H.1 and H.2, it can be shown that $A = UVU^{-1}$ if A has all its eigenvalues different, where U is a matrix containing all A's eigenvectors and V is a diagonal matrix with A's eigenvalues in its diagonal. It is important to note that the matrix U can be rescaled by any diagonal matrix M whose elements can be chosen arbitrarily and the equation $A = UMV(UM)^{-1}$ would still hold. Thus, the matrix M can be chosen conveniently putting the inverse of the i-th element of $h'U$ in its i-th diagonal element which results in

$$h'UM = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \tag{H.3}$$

Then, applying the change of variables $\xi_t = (UM)^{-1} x_t$ the model can be rewritten as

$$Y_t = (h'UM)((UM)^{-1} x_t) + d = \mathbf{1}' \xi_\mathbf{t} + \mathbf{d} \tag{H.4}$$

$$d\xi_t = \left[-(UM^{-1}AUM)(UM^{-1}x_t) + UM^{-1}b \right] dt + UM^{-1}\Sigma dw = (-\mathbf{V}\xi_\mathbf{t} + \bar{\mathbf{b}})\mathbf{dt} + \mathbf{S}\mathbf{dw_t} \tag{H.5}$$

Even though the matrix S needs not to be diagonal, the covariance matrix $S\Theta S'$ is still symmetric and positive definite and can therefore be rewritten as $\bar{\Sigma}\bar{\Theta}\bar{\Sigma}'$, where $\bar{\Sigma}$ is a diagonal matrix and $\bar{\Theta}$ is a correlation matrix. Equation H.5 is equivalent to:

$$d\xi_t = (-V\xi_t + \bar{b})dt + \bar{\Sigma}d\bar{w}_t \tag{H.6}$$

$$d\bar{w}_t d\bar{w}_t{}' = \bar{\Theta}dt \tag{H.7}$$

Assuming now that the first element of V is the zero valued eigenvalue, if b_i represents the i-th element of $\bar{b}$ and u_i is the ith diagonal element from the matrix V, then applying the change of variables

$$\zeta_t = \xi_t + \begin{bmatrix} \sum_{i=2}^{N} b_i/u_i + d \\ -b_2/u_2 \\ \vdots \\ -b_N/u_N \end{bmatrix} \tag{H.8}$$

the next equivalent model can be obtained,

$$Y_t = \vec{1}'\zeta_t \tag{H.9}$$

$$d\zeta_t = (-V\zeta_t + \begin{bmatrix} b_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix})dt + \bar{\Sigma}d\bar{w}_t \tag{H.10}$$

Where ζ_t is a $n \times 1$ vector of state variables, V is a $n \times n$ diagonal matrix with its first diagonal element being zero and all others strictly positive, b_1 is a scalar, $\bar{\Sigma}$ is a $n \times n$ diagonal matrix and $d\bar{w}_t$ is a $n \times 1$ vector of correlated brownian motions following $d\bar{w}_t d\bar{w}_t{}' = \bar{\Theta}dt$.

I. TRANSFORMATION OF ANY LINEAR MODEL INTO OURS

To transform any linear model into an equivalent one, an invariant transformation T has to be applied. In this section we will use Cortazar and Naranjo (2006)' definition of an invariant transformation to rewrite any general model as one following our proposed structure.

PROPOSITION I.1. *Any given state space model with the next structure:*

$$Y_t = h' x_t + d \tag{I.1}$$

$$dx_t = (-Ax_t + b)dt + \Sigma dw_t \tag{I.2}$$

where dw_t are correlated Brownian motions such that $dw_t dw_t' = \Theta dt$ and A has all its eigenvalues positive and different excepting the first one which is set to zero, can be transformed into the model proposed in section 3.

As shown in appendix H any linear model of the form shown in proposition I.1 can be rewritten into

$$Y_t = \vec{1}' \zeta_t \tag{I.3}$$

$$d\zeta_t = (-V\zeta_t + \begin{bmatrix} b_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix})dt + \bar{\Sigma} d\bar{w}_t \tag{I.4}$$

where V and $\bar{\Sigma}$ are diagonal matrices and $d\bar{w}_t$ is a vector of correlated brownian motions such that

$$d\bar{w}_t d\bar{w}_t{}' = \bar{\Theta} dt \tag{I.5}$$

The covariance matrix is given by $\bar{\Sigma}\bar{\Theta}\bar{\Sigma}'$ and is symmetric and positive definite. Let define M as

$$M = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix} \tag{I.6}$$

where $M^{-1} = M$, then the matrix $M\Sigma\Theta\Sigma' M'$ is still positive definite and still admits a Cholesky decomposition such that

$$LL' = M\Sigma\Theta\Sigma' M' \tag{I.7}$$

where L is a lower triangular matrix then applying the transformation $\eta_t = ML^{-1}M\zeta_t$ where $ML^{-1}M$ is an upper triangular matrix

$$Y_t = (\vec{1}' MLM)(ML^{-1}M\zeta_t) = h'\eta_t \tag{I.8}$$

$$d\eta_t = (-(ML^{-1}MVMLM)(ML^{-1}M\zeta_t) + ML^{-1}M \begin{bmatrix} b_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix})dt + ML^{-1}M\bar{\Sigma}d\bar{w}_t = (-\hat{A}\eta_t + \hat{b})dt + d\hat{w}_t \tag{I.9}$$

where $\hat{A}$ is an upper triangular matrix whose first eigenvalue is zero and $d\hat{w}_t$ is a vector of uncorrelated brownian motions.

J. WHY WE CAN'T USE CORTAZAR AND NARANJO (2006)'S FORMULATION DIRECTLY

In this section the inconveniences of directly using the model proposed by Cortazar and Naranjo (2006) are shown. Although their model nests any other with the properties described in proposition H.1, it does not behave well for an affine risk premium structure. Their model can be derived following the same transformations done in appendix H, but replacing equation H.8 with

$$\zeta_t = \xi_t + \begin{bmatrix} \sum_{i=2}^{N} b_i/u_i + d - b_1 t \\ -b_2/u_2 \\ \vdots \\ -b_N/u_N \end{bmatrix} \tag{J.1}$$

resulting this time in

$$Y_t = \zeta_t + b_1 t \tag{J.2}$$

$$d\zeta_t = -V\zeta_t dt + \bar{\Sigma} d\bar{w}_t \tag{J.3}$$

which is very similar to equations H.9 and H.10. The problems appear when trying to apply the same transformations to the risk adjusted version of the model. Using an affine risk adjustment as defined in section 3 the risk adjusted process of a general model results in

$$dx_t = (-(A+\Lambda)x_t + b - \lambda)dt + \Sigma dw_t^Q \tag{J.4}$$

Then, applying the transformation $\xi_t = (UM)^{-1}x_t$ we obtain

$$d\xi_t = \left[-(V + (UM)^{-1}\Lambda UM)\xi_t + \bar{b} - (UM)^{-1}\lambda\right] dt + S dw_t^Q \tag{J.5}$$

$$d\xi_t = \left[-(V + \bar{\Lambda})\xi_t + \bar{b} - \bar{\lambda}\right] dt + \bar{\Sigma} d\bar{w}_t^Q \tag{J.6}$$

Using the transformation from equation H.8,

$$d\zeta_t^Q = \left[-(V+\bar{\Lambda}) \left(\zeta_t - \begin{bmatrix} \sum_{i=2}^{N} b_i/u_i + d \\ -b_2/u_2 \\ \vdots \\ -b_N/u_N \end{bmatrix} \right) + \bar{b} - \lambda \right] dt + \bar{\Sigma} d\bar{w}_t^Q \tag{J.7}$$

$$d\zeta_t^Q = \left[-(V+\bar{\Lambda})\zeta_t + \begin{bmatrix} b_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \bar{\Lambda} \begin{bmatrix} \sum_{i=2}^{N} b_i/u_i + d \\ -b_2/u_2 \\ \vdots \\ -b_N/u_N \end{bmatrix} - \lambda \right] dt + \bar{\Sigma} d\bar{w}_t^Q \tag{J.8}$$

$$d\zeta_t^Q = \left[-(V+\bar{\Lambda})\zeta_t + \begin{bmatrix} b_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} - \bar{\lambda} \right] dt + \bar{\Sigma} d\bar{w}_t^Q \tag{J.9}$$

Which means applying a Duffee (2002) risk premium adjustment to the original model or to its transformation is equivalent. On the other hand, when the transformation of equation J.1 is used the resulting model is

$$d\zeta_t^Q = \left[-(V+\bar{\Lambda}) \left(\zeta_t - \begin{bmatrix} \sum_{i=2}^{N} b_i/u_i + d - b_1 t \\ -b_2/u_2 \\ \vdots \\ -b_N/u_N \end{bmatrix} \right) + \bar{b} - \lambda \right] dt + \bar{\Sigma} d\bar{w}_t^Q + \begin{bmatrix} -b_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} dt \tag{J.10}$$

$$d\zeta_t^Q = \left[-(V+\bar{\Lambda})\zeta_t + \bar{\Lambda} \begin{bmatrix} \sum_{i=2}^{N} b_i/u_i + d - b_1 t \\ -b_2/u_2 \\ \vdots \\ -b_N/u_N \end{bmatrix} - \lambda \right] dt + \bar{\Sigma} d\bar{w}_t^Q \tag{J.11}$$

$$d\zeta_t^Q = \left[-(V + \bar{\Lambda})\zeta_t - \bar{\lambda}(t)\right] dt + \bar{\Sigma} d\bar{w}_t^Q \tag{J.12}$$

in which the part $\bar{\lambda}(t) = \bar{\lambda}_1 + \bar{\lambda}_2 t$ of the risk adjustment is a linear function of time t. The obtained risk premiums differ from the ones proposed by Duffee (2002) and have a greater number of parameters which is necessarily bigger than the maximum number of econometrically identifiable parameters. This issue makes the direct use of the Cortazar and Naranjo (2006) model inconvenient as the risk adjustment of the original model is no longer equivalent to the same adjustment in the transformed one.

K. 2-FACTOR MODEL RESULTS

This section presents the 2-factor model implementation results. Even though this models' estimation process was much simpler than the 3-factor models', the obtained results were not able to account for rapid changes in risk premiums as desired. The calibration was done for a weekly sample period between January 2010 and June 2017 and the estimated parameters can be observed in table K.1. 12 of the 13 estimated parameters appear to be statistically significant at the 1% level meaning that the model actually captures information about risk premiums just as the 3-factor model does, and hinting that a greater number of parameters would model the information contained by the data better.

Table K.1. Parameter estimates for the 2-factor model calibrated between January 2010 and June 2017

	Estimate	Deviation	tStat	pValue
A_{11}	0	-	-	-
A_{12}	-0.1644	0.057	-2.8817	0.0065
A_{22}	0.5488	0.0168	32.7558	0
Λ_{11}	0.0079	0.0025	3.1837	0.0027
Λ_{12}	0.1262	0.0346	3.6482	0.0006
Λ_{21}	-0.0299	0.0016	-18.2425	0
Λ_{22}	-0.1144	0.017	-6.7389	0
δ_1	0.131	0.0039	33.5316	0
δ_2	0.5087	0.0137	37.0754	0
λ_1	-0.0042	0.0895	-0.0472	0.3982
λ_2	1.0559	0.0512	20.607	0
b_1	0.2411	0.0105	22.9318	0
σ_f	0.0113	0	299.5937	0
σ_e	0.1061	0.0006	181.7876	0

Using the estimated parameters we are able to get the risk premium structure for every date as shown in figure K.1. The mean risk premiums over the whole sample period are shown in figure K.2a compared to our 3-factor model premiums. The risk premium volatilities of the 2-factor model and the 3-factor model are shown in figure K.2b for comparison. It can be noted that the levels of both mean risk premium curves differ for the short horizons but are very similar for longer ones. The 3-factor model risk premiums have a bigger time varying ability than the 2-factor model ones for all maturities.

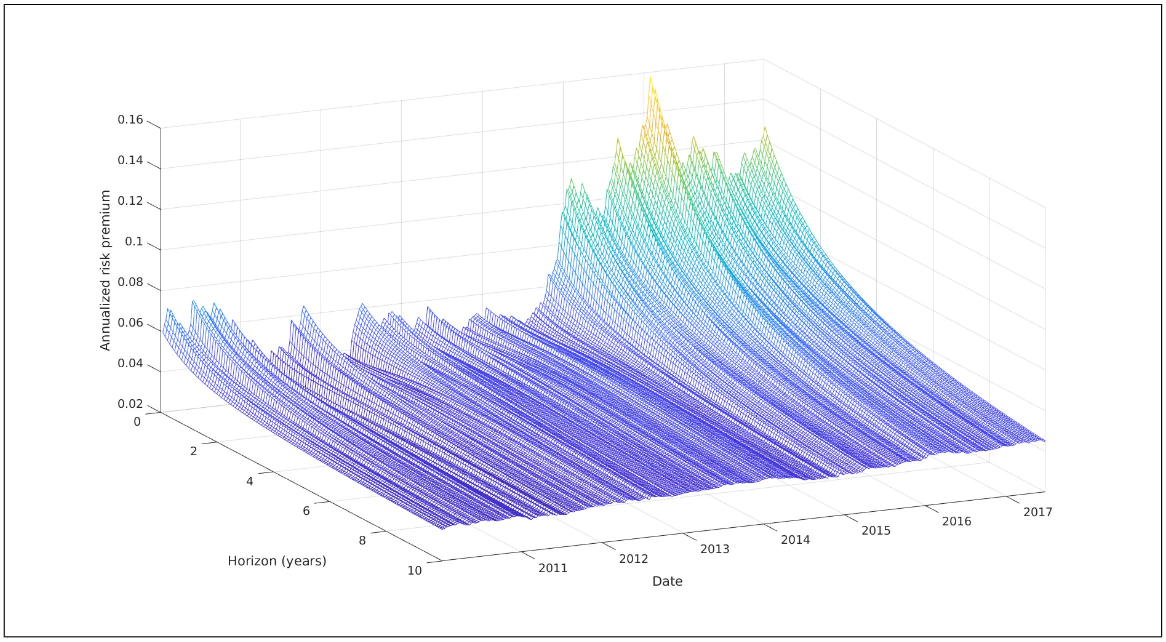
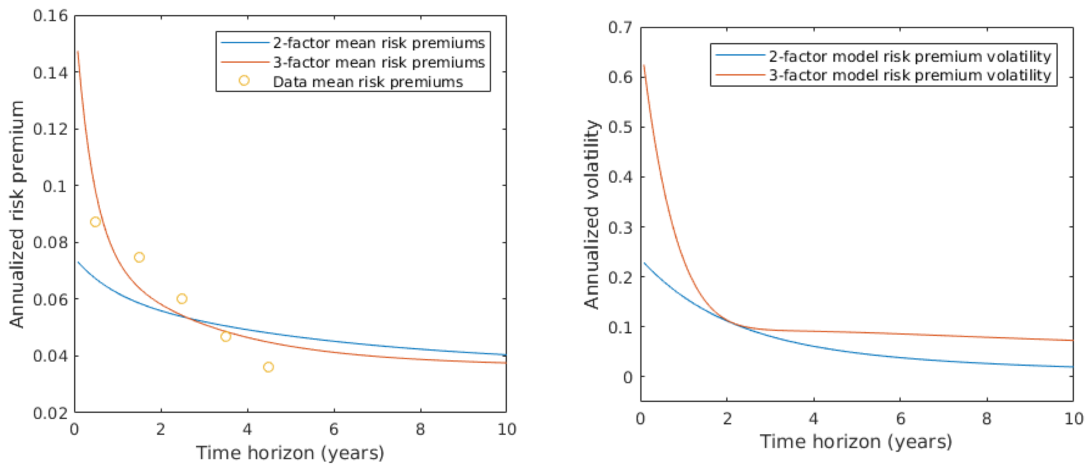


Figure K.1. Risk premium structure estimated using the 2-factor model with parameters calibrated between January 2010 and June 2017.

Table K.2 shows the MAPEs of our 2-factor and our 3-factor model for comparison. Our 3-factor model clearly outperforms the 2-factor model in both, futures and expected prices. These results are not surprising as a smaller number of factors is expected to fit the data in a worse way.

Table K.2. MAPEs of our 2- and 3- factor models for comparison. The sample time period starts in January 2010 and ends in June 2017

	2-factor model	3-factor model
Futures prices	0.78%	0.37%
Expected prices	7.98%	7.39%



(a) Mean risk premiums

(b) Risk premium volatility

Figure K.2. Mean risk premium and volatilities estimated using the 2-factor model for maturities between 0 and 10 years over the sample period from January 2010 to June 2017. Mean risk premiums are compared against the 3-factor models', and theoretical and curve volatilities are shown along with empirical data volatilities.

L. PARTIAL SAMPLE ESTIMATION

In order to test the model's behavior out-of-sample a different parameter calibration was done optimizing the parameters for data between January 2010 and June 2016 only. The parameter estimates are available in table L.1 and only $9/24$ of the parameters are significant at a 1% and $16/24$ at a 10% significance level this time.

Table L.1. Parameter estimates for the 3-factor model calibrated between January 2010 and June 2016

	Estimate	Deviation	tStat	pValue
A_{11}	0	-	-	-
A_{12}	2.182	0.9245	2.3602	0.025
A_{13}	1.6557	1.646	1.0059	0.2402
A_{22}	1.4818	0.1405	10.5446	0
A_{23}	1.104	0.8029	1.375	0.1549
A_{33}	0.0646	0.0296	2.1817	0.0373
Λ_{11}	0.1813	0.0437	4.1486	0.0001
Λ_{12}	-2.4181	0.9143	-2.6448	0.0124
Λ_{13}	-1.6518	1.7261	-0.9569	0.252
Λ_{21}	-0.1378	0.0345	-4.0007	0.0002
Λ_{22}	-0.6445	0.3394	-1.899	0.066
Λ_{23}	-0.5673	0.5961	-0.9518	0.2533
Λ_{31}	0.0969	0.052	1.8631	0.0705
Λ_{32}	0.267	0.2025	1.3186	0.1671
Λ_{33}	0.1855	0.2587	0.7171	0.3081
δ_1	0.0726	0.0214	3.3865	0.0014
δ_2	0.4915	0.1995	2.4639	0.0195
δ_3	0.619	0.1709	3.6227	0.0006
λ_1	-8.8957	1.0353	-8.5922	0
λ_2	7.3439	2.384	3.0805	0.0036
λ_3	-4.7814	3.7047	-1.2906	0.1733
b_1	0.6594	0.26	2.5362	0.0163
σ_f	0.0055	0	300.6888	0
σ_e	0.0983	0.0006	171.2085	0

It is possible to get in-sample and out-of-sample risk premium estimates using the latter parameters. Figure L.1 shows the risk premium structure for the whole sample, showing the in-sample and out-of-sample estimations together in order to compare them properly. Finally the mean risk premiums over the in-sample and out-of-sample periods are shown in figure L.2a and L.2b for the partial sample calibration and the full-sample calibration of section 5 for comparison.

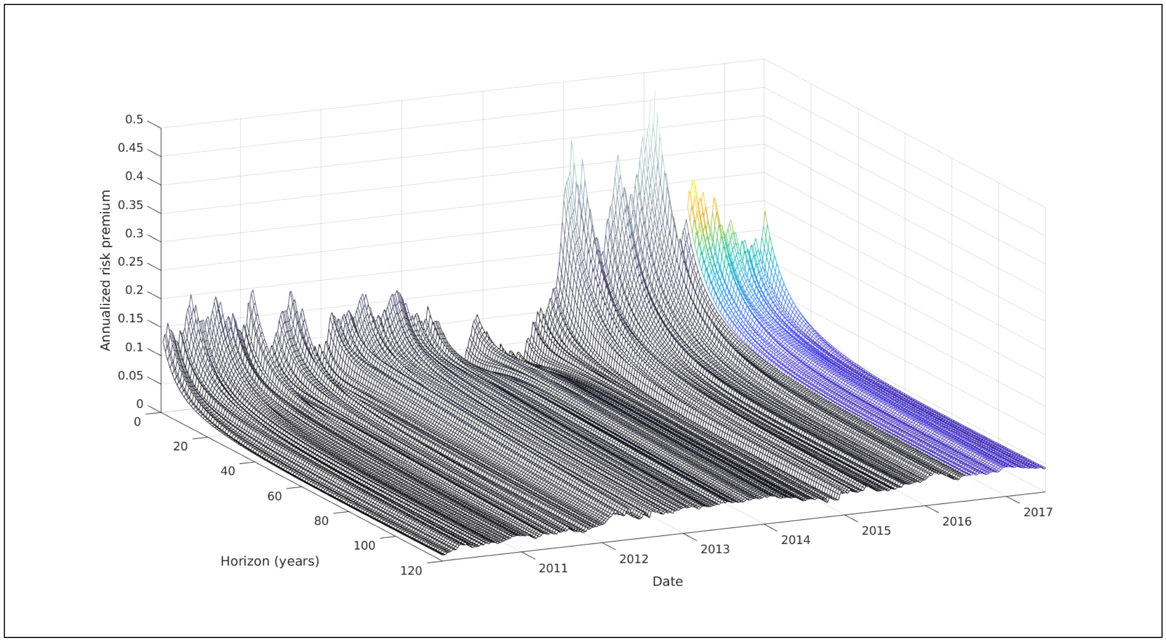
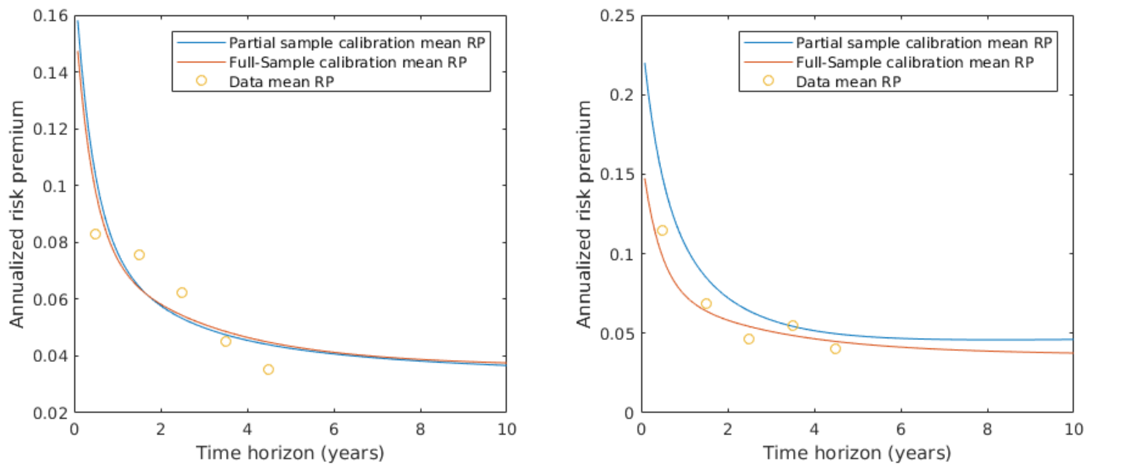


Figure L.1. Risk premium structure estimated using the 3-factor model with parameters calibrated between January 2010 and June 2016 (grey). Out-of-sample premiums start in July 2016 and end in June 2017 (in colour).

The adjustment errors to the data are shown in table L.2. Both calibrations have similar errors during the period between January 2010 and June 2016, however the full-sample calibrated model beats the partial-sample calibration in the July 2016 - June 2017 time period.



(a) Mean risk premiums between January 2010 and June 2016

(b) Mean risk premiums between July 2016 and June 2017

Figure L.2. Mean risk premium for the in-sample period from January 2010 to June 2016 and out-of-sample period from July 2016 to June 2017.

Table L.2. MAPEs of our model in comparison with Cortazar et al. (2018)'s model. The in-sample time period starts in January 2010 and ends in June 2016, the out-of-sample period starts in July 2016 and ends in June 2017. The same data and Kalman filter specification was used for both models.

		Partial-Sample calibration	Full-Sample calibration
In sample period	Futures prices	0.36%	0.38%
	Expected prices	7.25%	7.23%
Out of sample period	Futures prices	0.72%	0.39%
	Expected prices	8.81%	8.35%

M. ANALYSIS OF RISK PREMIUM PRINCIPAL COMPONENTS

Our risk premium estimates from section 5 are calculated from only 3 different latent factors, hence our premiums should only have 3 principal components which explain 100% of their volatility. In order to understand the relation of the market variables of section 6 and our risk premium estimates we try to find out how the 3 principal components relate to those variables. We start by computing the 3 principal components of our monthly risk premium panel, that is 1 to 120 months. The volatilities explained by each of the 3 components is given in table M.1. It can be noted that the first component explains most of the variance and together with the second they are able to explain more than 99.9% of it.

Table M.1. Percentage of risk premium volatility explained by each principal component. A risk premium panel from 1 to 120 months was used and only the 3 first principal components are shown as they explain 100% of the volatility.

Principal Component	Variance Explained
First	77.74%
Second	22.24%
Third	0.03%

As only one component explains most of the risk premiums variance the risk premiums probably depend on the same market variables for all maturities which is consistent with the results found in section 6. Furthermore, probably the same variables used that section are able to predict the first principal component in a very significant way. Repeating the multivariate regression of section 6 replacing the different risk premium maturities with the three components we obtain the results shown in table M.2. It can be noted that the three principal components are closely related with the selected market variables. Analyzing the first one it is possible to note that even the signs of the estimated coefficients are consistent with the results from section 6, probably because of it being the main source of variability in risk premiums.

Table M.2. Regression analysis of the three principal components of the risk premium structure. f_1, f_2 and f_3 represent the first, second and third principal component respectively.

	f_1	f_2	f_3
Intercept	-0.5512***	-0.1026**	-0.0025**
Inventories	9.3405***	-3.0992**	0.1397***
HP	0.3752***	-0.4924***	-0.0029**
TRM	-0.2238***	0.0335***	0.0061***
5Y T-Bill	0.3158***	0.0322**	-0.0086***
DEF	0.5272***	0.0635***	0.0016**
R2	0.5099	0.4882	0.625

N. PREDICTIVE ABILITY

The prediction of oil prices has shown to be very difficult and futures prices are known as the best predictors of future spot prices. Even the no-change forecast appears to achieve better results than any model over long periods of time (Baumeister and Kilian (2016), Bernard, Khalaf, Kichian, and Yelou (2015)), hence we choose to compare our model's price forecasts with futures prices and the no-change forecast. As only 7.5 years of data are available we can only evaluate predictions for horizons up to 90 months. However we decided to evaluate predicting errors only up to 60 months in order to have a long enough data window.

We calculate the mean absolute percentage error (MAPE) as the average absolute values of all percentage deviations of the forecasts from the actual values. We do not analyze forecast done the first 10 weeks of the model as it had too little information until then. Figure N.1 compares the MAPEs of our 3-factor model's predictions with the no-change and the future-based forecasts. It can be concluded that our model is equivalent to the other two when doing short term forecasts, but clearly gets outperformed on longer time horizons. A few comments are in order, first non of the three models is able to achieve decent forecasts for long maturities having all of them errors of over 80% for 5 year forecasts. Second, the no-change forecast is not significantly worse than the futures forecasts and they are equivalent for longer maturities which raises doubts about the existence of any predictive ability in futures and our model. Third, the high value of the MAPEs (some bigger than 100%) are probably due to the 2014's worldwide drop in commodity and specifically oil prices. Figure N.2a shows the spot price of WTI oil between 2010 and 2017, while N.2b gives the errors done by the three forecasting alternatives for a one year time horizon. It is evident that the big drop in prices occurred during 2014 drove the forecasting errors to a much higher level for the three models simultaneously, indicating that neither our model nor the benchmarks were able to predict the fall in prices. This is an important finding because it means neither futures prices nor analysts' expectations contained significant information about it. Because prices have remained on a lower level ever since, long horizon forecasts

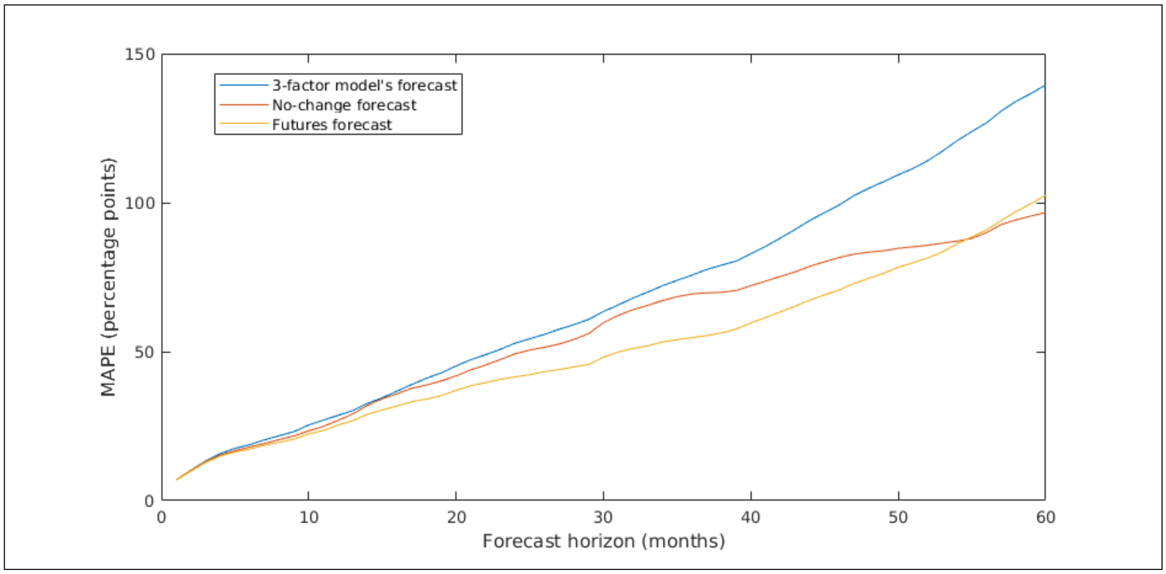


Figure N.1. MAPEs from the 3-factor model (blue) compared to the ones from the no-change forecast (red) and the futures forecast (green). Errors are shown for every forecast horizon up to 60 months. The comparison was made for forecasts done between April 2010 and June 2017.

are obviously more affected and their accuracy is even worse. Given the bad results to forecast futures prices it does not appear necessary to make an out-of-sample analysis as it probably will give similar or worse results.

(a) WTI spot price

(b) Prediction error for the 12 month forecast

Figure N.2. WTI spot prices and prediction errors done by the one year forecast of our model (blue), the no change forecast (red) and futures prices (green).