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# QUBIT GENERATION AND PHOTON COOLING IN A RAMSEY INTERFEROMETER

by

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*To my parents.*



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# Abstract

In this thesis, we propose a realizable experimental scheme to prepare a superposition of the vacuum and one-photon states using a typical cavity QED-setup. This is different from previous schemes, where the superposition state of the field is generated by resonant atom-field interaction and the cavity is initially empty. Here, we consider only dispersive atom-field interaction and the initial state of the cavity field is coherent. Then, we determine the parameters to prepare the desired state via atomic post-selection. We also include the effect of cavity losses and detection imperfections in our analysis against which this preparation of the optical qubit in a real Fabry-Perot superconducting cavity is robust. In addition, we show that this scheme can be used for the preparation of other photon number Fock state superpositions. In short, our task is achieved with high fidelity and a post-selection probability within experimental reach. With the same scheme, we propose to cool the cavity field to its ground state starting from a thermal distribution by a dispersive atom-field coupling followed by an atomic post-selection. We also analyze the effect of the cavity and atomic losses. The proposed experiment can be realized with real parameters with high fidelity.

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# Chapter 1

## Introduction

The generation and engineering of nonclassical states of light is central to quantum optics and quantum information. Over the years, various schemes for the preparation of Fock states [1–9] and their arbitrary finite superpositions [10–22] have been developed. Such states have been shown to be generated by nonlinear media or by conditional measurements. For example, the method proposed by Brune *et al.* [2,4] can generate Fock states of a cavity field. This method is based on a quantum nondemolition measurement (QND) of the photon number of a field stored in a high-Q cavity, where the information acquired by detecting a sequence of atoms modifies the field step by step, until it eventually collapses into a Fock state. This has been done experimentally by Guerlin *et al.* [23]. Although the collapse of the field into a Fock state is not predictable due to the randomness of the measurement, it is possible to prepare on demand photon number states (Fock states) using a quantum feedback scheme in the context of a QND measurement of the photon number of a cavity field [9,24]. Other interesting schemes proposed by Leoński [5,6] relies on the nonlinearity of the time evolution of the cavity field in a Kerr medium. In these methods, by adjusting the parameters of the Hamiltonian it is possible to generate a Fock state via unitary time evolution of a given initial cavity field.

In the case of the preparation of a finite superposition of the number state, most of the proposed schemes are based on a conditional measurement at the outports of beam splitters. For example, the method proposed by Dakna *et al.* [15] generates an arbitrary

(finite) superposition of Fock states by performing alternaly coherent displacement and single-photon adding in a well-defined succession via conditional measurements on beam splitters. However, one of the simplest methods is the optical truncation of coherent light, also referred to as *quantum scissors device* (QSD), proposed by Pegg, Phillips and Barnett (PPB) [13, 14]. Later, Resch *et al.* proposed and experimentally demonstrated a QSD-like state preparation technique based on conditional coherence [25]. In addition, due to its simplicity, the basic idea of the QSD has been modified and generalized [16]. It was shown that an optical qubit can be generated experimentally with high accuracy using the PPB scheme with commercially available detectors and single-photon sources [20, 21]. Moreover, optical qubit generation by nonlinear quantum scissors has been proposed [22]. In addition, various schemes have been presented in the field of the QED cavity for the generation of superpositions states of the radiation field by a conditional measurement of the atoms. Vogel *et al.* have basically employed resonant atom-field interaction to build an arbitrary field in an initial empty cavity [10]. Another proposal [11], based on [10], has been presented considering both resonant and dispersive atom-field interaction for preparing a reciprocal-binomial state of the radiation field from an initial empty cavity. Moreover, a different method considers an initial coherent state and two Ramsey zones for the generation of a general cavity field state by the conditional measurement of the atoms interacting resonantly with the field [18].

In addition, a variety of experiments developed for generating nonclassical states of light assume that the cavity field is initially in its vacuum state [1–4, 7–12, 18, 19, 23, 24, 26]. Nevertheless, in most situations, the system is unavoidable coupled to the environment, that produces detrimental effects on the ideal realization of an experiment. In this way, assuming that the environment is in thermal equilibrium with the system, a cavity mode contains thermal photons on average that have to be removed at the beginning of each experiment. One way to remove the residual thermal photons is sending across the cavity a number of atoms initially prepared in the lower atomic level  $|g\rangle$  and tuned in resonance with the cavity mode [12, 19, 23]. However, in order to prepare the cavity field in its vacuum state, a more efficient technique to absorb thermal photons is by the principle of the rapid adiabatic passage (RAP), in which the atom-field frequency is swept [27, 28]. These two

techniques employ a cooling sequence of atoms to reduce the effective field temperature by energy exchange between the cavity field and the atoms.

In this thesis, we present a theoretical scheme for generating specific quantum states of a cavity field by a postselective measurement. First, we prepare a superposition of the vacuum and one-photon state which is the simplest optical qubit state. Then, we employ the scheme to create other photon number Fock states [29]. Finally, we develop a protocol to cool-down an initial thermal field to the pure vacuum state. Similar approaches have been presented for creating vibrational states and cooling a nanomechanical oscillator by performing a postselective measurement [30, 31].

The main idea of our scheme is based on an initial preparation of atoms that enter in a typical cavity QED-setup used for Ramsey interferometry and whose final states are postselected by adjusting the various parameters in such a way that we can generate the desired states. In particular, we consider a dispersive atom-field coupling of each atom in the sequence and thus there is no energy exchange between the atoms and the field which is the main difference with previous schemes presented. Hence, the process is probabilistic due to the conditioned atomic measurements required to obtain the desired state. Additionally, we study the feasibility of our processes in an open quantum system under real experimental conditions.

## 1.1 Structure of the Thesis

In chapter 2 we present a general introduction of the main concepts and processes relevant to the study of Cavity Quantum Electrodynamics systems. In chapter 3, we present our cavity QED model based on Ramsey interferometry and calculate the generalized equation of the cavity field state generated after postselection. Numerical results for the preparation of the desired cavity field states are presented in chapter 4. In addition, in chapter 4, we discuss the experimental feasibility of our scheme. The work we present here is summarized in chapter 5 where conclusions and further work are discussed.

## Chapter 2

# Theoretical description of atoms and photons

Basic quantum optics experiments allow to test the most intimate behaviours of the quantum world. In addition, these experiments opened the door to new promising technologies. For example, quantum computing and quantum communications.

In this work, we focus on Cavity Quantum Electrodynamics which is the study of the interaction between light confined in a reflective cavity and atoms under conditions where the quantum nature of light photons is significant. Therefore, this chapter is devoted to the introduction of the two basic elements in a cavity QED system: photons and atoms. First, we will review the formalism of the quantization of the electromagnetic field where photons are its quanta. Then, we will describe some quantum states of light and their properties which are of interest for this work. We will recall the phase space representation for visualizing the non-classical aspect of quantum states. Afterwards, we will introduce the two-level atom and its interaction with the quantized electromagnetic field in the resonant and dispersive regimes. Finally, we will present the master equation for taking into account the decoherence phenomena.

## 2.1 Quantization of the Electromagnetic Field

The objective is to quantize the electromagnetic field in free space starting by the classical field equations. The free electromagnetic field obeys the source free Maxwell's equations

$$\nabla \cdot \mathbf{B} = 0, \quad (2.1a)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (2.1b)$$

$$\nabla \cdot \mathbf{D} = 0, \quad (2.1c)$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}, \quad (2.1d)$$

where  $\mathbf{B} = \mu_0 \mathbf{H}$ ,  $\mathbf{D} = \varepsilon_0 \mathbf{E}$ ,  $\mu_0$  and  $\varepsilon_0$  being the magnetic permeability and electric permeability of free space, obeying the relation  $\mu_0 \varepsilon_0 = c^{-2}$ . We introduce the electromagnetic vector and scalar potentials  $\mathbf{A}$  and  $\Phi$  defined as:

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad (2.2a)$$

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \Phi. \quad (2.2b)$$

These definitions automatically satisfy equations (2.1a) and (2.1b).

As Maxwell's equations are gauge invariant, we choose the Coulomb gauge condition

$$\nabla \cdot \mathbf{A} = 0, \quad (2.3)$$

$$\Phi = 0.$$

With the above gauge and substituting (2.2a) and (2.2b) into (2.1d), we find that  $\mathbf{A}$  satisfies the wave equation

$$\nabla^2 \mathbf{A} = \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} \quad (2.4)$$

It is convenient to expand the vector potential in terms of a discrete set of orthogonal mode

functions restricted to a certain volume of space

$$\mathbf{A}(\mathbf{r}, t) = \sum_k c_k \left[ a_k(t) \mathbf{u}_k(\mathbf{r}) + a_k^\dagger(t) \mathbf{u}_k^*(\mathbf{r}) \right], \quad (2.5)$$

where the coefficients  $c_k$  are constant for a free field. Substituting (2.5) into (2.4) gives a separate set of equations

$$\begin{aligned} \left( \nabla^2 + \frac{\omega_k^2}{c^2} \right) \mathbf{u}_k(\mathbf{r}) &= 0, \\ \left( \frac{\partial^2}{\partial t^2} + \omega_k^2 \right) a_k(t) &= 0. \end{aligned} \quad (2.6)$$

The mode functions  $\mathbf{u}_k(\mathbf{r})$ , in the Coulomb gauge, must satisfy the transversality condition

$$\nabla \cdot \mathbf{u}_k(\mathbf{r}) = 0. \quad (2.7)$$

Also, these functions form a complete orthonormal basis

$$\int_V \mathbf{u}_k^*(\mathbf{r}) \mathbf{u}_{k'}(\mathbf{r}) d\mathbf{r} = \delta_{kk'}. \quad (2.8)$$

Depending on the boundary conditions of the physical volume under consideration, the mode functions could be travelling waves (periodic boundary conditions) or standing waves (cavity boundary conditions). For example, planes waves may be written as

$$\mathbf{u}_k(\mathbf{r}) = \frac{\mathbf{e}_k \exp(i\mathbf{k} \cdot \mathbf{r})}{\sqrt{V}}, \quad (2.9)$$

where  $V$  is the volume and  $\mathbf{e}_k$  is the unit polarization vector, which is required to be perpendicular to  $\mathbf{k}$  by the transversality condition (2.7). Therefore, we have two possible and mutually orthogonal polarizations contained in a plane perpendicular to  $\mathbf{k}$ .

Next, solving equation (2.6) for the amplitudes  $a_k(t)$

$$\begin{aligned} a_k(t) &= a_k \exp(-i\omega_k t), \\ a_k^\dagger(t) &= a_k^\dagger \exp(i\omega_k t). \end{aligned} \quad (2.10)$$

The potential vector can be written in the form

$$\mathbf{A}(\mathbf{r}, t) = \sum_k \sqrt{\frac{\hbar}{2\omega_k \varepsilon_0 V}} \mathbf{e}_k \left[ a_k e^{i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)} + a_k^\dagger e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)} \right]. \quad (2.11)$$

Then from equations (2.2a) and (2.2b), the corresponding magnetic and electric field are

$$\mathbf{E}(\mathbf{r}, t) = i \sum_k \sqrt{\frac{\hbar \omega_k}{2\varepsilon_0 V}} \mathbf{e}_k \left[ a_k e^{i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)} - a_k^\dagger e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)} \right], \quad (2.12)$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{i}{c} \sum_k \sqrt{\frac{\hbar \omega_k}{2\varepsilon_0 V}} (\hat{k} \times \mathbf{e}_k) \left[ a_k e^{i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)} - a_k^\dagger e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)} \right], \quad (2.13)$$

where  $\hat{k}$  is a normalized vector using the relation  $\mathbf{k} = (\omega_k/c)\hat{k}$ .

In the Classical Electromagnetic theory, the amplitudes  $a_k$  and  $a_k^\dagger$  are a pair of conjugate complex numbers. Then, to quantize the electromagnetic field we choose  $a_k$  and  $a_k^\dagger$  to be mutually adjoint operators. Since photons are bosons, these operator obey the boson commutation relations

$$[a_k, a_{k'}^\dagger] = \delta_{kk'}, \quad [a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0. \quad (2.14)$$

These operators are dimensionless. Therefore, the normalization factor  $\sqrt{\frac{\hbar \omega_k}{2\varepsilon_0 V}}$  is the electric field amplitude of the vacuum in the mode  $k$  and depends only on the geometry and on the frequency of the mode.

The Hamiltonian for the electromagnetic field is

$$H = \int \frac{1}{2} (\epsilon_0 \mathbf{E}^2 + \mu_0 \mathbf{B}^2) d\mathbf{r}. \quad (2.15)$$

Replacing equations (2.12) and (2.13), the Hamiltonian may be reduced to the form

$$H = \sum_k \hbar \omega_k \left( a_k^\dagger a_k + \frac{1}{2} \right). \quad (2.16)$$

The first term of equation (2.16) represents an ensemble of independent quantum oscillators. The second term,  $\frac{1}{2}\hbar\omega_k$ , represents the energy of the vacuum fluctuations in each mode.

## 2.2 Fock or Number States

Each mode of the Hamiltonian (2.16) has the eigenvalues  $\hbar\omega_k(n_k + \frac{1}{2})$ , where  $n_k$  is an integer ( $n_k = 0, 1, 2, \dots, \infty$ ). We write the eigenstates as  $|n_k\rangle$  and are known as Fock or number states. They are eigenstates of the number operator  $N_k = a_k^\dagger a_k$

$$a_k^\dagger a_k |n_k\rangle = n_k |n_k\rangle. \quad (2.17)$$

The operators  $a_k^\dagger$  and  $a_k$  are the raising and lowering operators for the quantum harmonic oscillator. For this particular case, the application of the operator  $a_k$  ( $a_k^\dagger$ ) to a Fock state decreases (increases) the energy of  $k$ th mode of the radiation field. Hence the terminology, annihilation and creation photon operators. The action on the Fock states of the creation and annihilation operators is given by

$$a_k |n_k\rangle = \sqrt{n_k} |n_k - 1\rangle, \quad a_k^\dagger |n_k\rangle = \sqrt{n_k + 1} |n_k + 1\rangle \quad (2.18)$$

To generate any Fock state of the  $k$ th mode from the vacuum, we just have to apply the creation operator successively

$$|n_k\rangle = \frac{(a_k^\dagger)^{n_k}}{\sqrt{n_k!}} |0\rangle, \quad n_k = 0, 1, 2, \dots \quad (2.19)$$

The number states are orthogonal

$$\langle n_k | m_k \rangle = \delta_{nm}, \quad (2.20)$$

and complete

$$\sum_{k=0}^{\infty} |n_k\rangle \langle n_k| = 1. \quad (2.21)$$

We notice that the energy of the ground state is given by

$$\langle 0|H|0\rangle = \frac{1}{2} \sum_k \hbar\omega_k. \quad (2.22)$$

Since there is no upper bound over the sum of the electromagnetic modes, the energy of the ground state diverges, causing a conceptual difficulty of the quantization method. However, in practical experiments, one measures a change in the total energy. So, there is no divergence in practice caused by the infinite zero-point energy. Moreover, for the purpose of this work we consider single modes of the electromagnetic field confined in a cavity.

An important property of number states  $|n\rangle$  is that the expectation value of the electric field vanishes, i.e.,  $\langle n|E|n\rangle = 0$ . However, these states present field fluctuations because the expectation value of the intensity operator  $E^2$  is given by

$$\langle n|E^2|n\rangle = \hbar\omega(n + \frac{1}{2}) \quad (2.23)$$

This is a peculiar effect of Fock states, where the expectation value of the electric field vanishes no matter how many photons are contained in the state. Number states are highly nonclassical with a well-defined amplitude and completely random phase.

## 2.3 Coherent States

Classical electromagnetic fields consist of waves with a well-defined amplitude and phase. However, quantization of the electromagnetic field implies in an uncertainty relation for the conjugate field variables. Therefore, a state which minimizes the uncertainty relation for the field conjugate variables can be interpreted as the closest to a classical description. A coherent state, as introduced by Glauber [32], is a minimum uncertainty state defined as an eigenstate of the annihilation operator  $a$

$$a|\alpha\rangle = \alpha|\alpha\rangle, \quad (2.24)$$

where  $\alpha = |\alpha|e^{i\theta}$  is complex number. Coherent states can be expanded over the Fock basis

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \quad (2.25)$$

The probability distribution of photons  $P(n)$  in a coherent state is a Poisson distribution

$$P(n) = |\langle n|\alpha\rangle|^2 = \frac{e^{-|\alpha|^2} |\alpha|^{2n}}{n!}, \quad (2.26)$$

with a mean photon number  $\bar{n} = \langle \alpha | a^\dagger a | \alpha \rangle = |\alpha|^2$  and photon number mean square-root deviation  $\Delta N = |\alpha| = \sqrt{\bar{n}}$ .

As we mentioned earlier, a coherent state is a minimum uncertainty state (MUS). If we write the usual relation between  $a$ ,  $a^\dagger$  and  $q$ ,  $p$

$$\begin{aligned} q &= \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger), \\ p &= i\sqrt{\frac{\hbar m\omega}{2}} (a - a^\dagger). \end{aligned} \quad (2.27)$$

Then, taking the expectation over a coherent state we get

$$\langle q \rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle \alpha | (a + a^\dagger) | \alpha \rangle = \sqrt{\frac{\hbar}{2m\omega}} (\alpha + \alpha^*), \quad (2.28)$$

and

$$\langle q^2 \rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle \alpha | (a + a^\dagger)^2 | \alpha \rangle = \sqrt{\frac{\hbar}{2m\omega}} (1 + (\alpha + \alpha^*)^2). \quad (2.29)$$

Hence, calculating the variance of  $q$ , we find

$$\Delta q = \sqrt{\langle q^2 \rangle - \langle q \rangle^2} = \frac{\hbar}{2m\omega}. \quad (2.30)$$

Similarly for the variable  $p$ , we get

$$\Delta p = \frac{\hbar m\omega}{2}. \quad (2.31)$$

Finally, the coherent state is a *minimum uncertainty state* because the conjugate variables

$q$  and  $p$  satisfy

$$\Delta x \Delta p = \frac{\hbar}{2}. \quad (2.32)$$

Another property of coherent states is the time evolution of these states  $|\alpha, t\rangle$ , using (2.16) and the Fock representation basis (2.25)

$$\begin{aligned} |\alpha, t\rangle &= e^{-\frac{i}{\hbar} H t} |\alpha\rangle \\ &= e^{-|\alpha|^2/2} \sum_n e^{-i\omega t(n+1/2)} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \\ &= e^{-i\omega t/2} |\alpha e^{-i\omega t}\rangle. \end{aligned} \quad (2.33)$$

Within a global phase factor, the state remains coherent. The evolution of the state is a rotation in the phase space at frequency  $\omega$ .

It is convenient to define the displacement operator

$$D(\alpha) = e^{(\alpha a^\dagger - \alpha^* a)}, \quad (2.34)$$

which satisfies:

$$D^{-1}(\alpha) = D^\dagger(\alpha) = D(-\alpha); \quad D(0) = \mathbb{1}. \quad (2.35)$$

As we can see, the displacement operator is unitary ( $D^\dagger D = \mathbb{1}$ ). The application of the displacement operator on the vacuum state generates a coherent state

$$|\alpha\rangle = D(\alpha)|0\rangle. \quad (2.36)$$

Where we used the relation  $e^{(A+B)} = e^A e^B e^{-\frac{1}{2}[A,B]}$ , valid if  $[A[A, B]] = [B[A, B]] = 0$ , in (2.34) to prove (2.36). In the case of an electromagnetic field, the displacement operation can be done by the action of a classical field source into the cavity [33].

Also, the set of all coherent states define a complete basis

$$\frac{1}{\pi} \int d^2\alpha |\alpha\rangle \langle \alpha| = \mathbb{1}. \quad (2.37)$$

And, two coherent states are not orthogonal

$$|\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2}. \quad (2.38)$$

Further readings about the properties of coherent states can be found in [34].

## 2.4 Quantum states and density operator

Up to this point, we have used the state vector representation to refer to a quantum state. A more general mathematical representation of a quantum state is given by the density operator  $\rho$ . In this case, the state vector of a quantum system is not completely known. More precisely, we only know that a quantum system is in the state  $|\psi_i\rangle$  with a respective probability  $p_i$ . The density operator of a quantum system is defined as

$$\rho = \sum_{i=1}^D p_i |\psi_i\rangle\langle\psi_i|, \quad (2.39)$$

where  $0 \leq p_i \leq 1$  and  $\sum_i p_i = 1$ .

Next, we will introduce the conditions that an operator must satisfy to be called a density operator.

### 2.4.1 Properties of the density operator

Any operator satisfying the following conditions is a valid density operator.

1. Normalization condition:  $\text{Tr}(\rho) = 1$ .
2. Hermiticity condition:  $\rho = \rho^\dagger$ .
3. Positivity condition:  $\langle\psi|\rho|\psi\rangle \geq 0$  for all  $|\psi\rangle$ .

### 2.4.2 Pure and mixed states

A quantum system whose state  $|\psi\rangle$  is known exactly is said to be *pure state*. Then, the density operator is a projector  $\rho = |\psi\rangle\langle\psi|$ , which implies that  $\rho^2 = \rho$ . Therefore, due

to the normalization, the following condition is necessary and sufficient to identify pure states

$$\text{Tr}(\rho^2) = 1. \quad (2.40)$$

Fock and coherent states are examples of pure states.

In general, the purity of a state is defined by [35]

$$P(\rho) = \text{Tr}(\rho^2), \quad (2.41)$$

which satisfies  $P(\rho) \leq 1$ , with equality if and only if  $\rho$  is a pure state. Then, the states satisfying  $P(\rho) < 1$  are called *mixed states*, being  $P(\rho) = 1/D$  the value for a completely mixed state.

An example of a mixed state is a thermal field. The density operator for a one-mode thermal state is given by

$$\rho_{th} = \sum_n \frac{n_{th}^n}{(1 + n_{th})^{n+1}} |n\rangle\langle n|, \quad (2.42)$$

where the photon number distribution follows a Bose-Einstein statistics and the mean photon number is

$$n_{th} = \frac{1}{\exp\left(\frac{\hbar\omega}{k_B T}\right) - 1}. \quad (2.43)$$

### 2.4.3 Evolution

Differentiating the density operator of equation (2.39) and employing the Schrödinger equation  $i\hbar\partial_t|\psi_i\rangle = H|\psi_i\rangle$ , we can write down the equation of motion for the density operator:

$$\begin{aligned} \frac{\partial\rho}{\partial t} &= \sum_i p_i \left( \frac{\partial|\psi_i\rangle}{\partial t} \right) \langle\psi_i| + \sum_i p_i |\psi_i\rangle \left( \frac{\partial\langle\psi_i|}{\partial t} \right) \\ &= -\frac{i}{\hbar} H \sum_i p_i |\psi_i\rangle \langle\psi_i| + \frac{i}{\hbar} \sum_i p_i |\psi_i\rangle \langle\psi_i| H \\ &= -\frac{i}{\hbar} [H, \rho]. \end{aligned} \quad (2.44)$$

This is referred to as the **Schödinger-Von Neumann equation**.

## 2.5 Phase space representation

In classical mechanics a useful representation is the *phase space* representation and the concept of probability distributions in this space. In this representation, the state of a system with  $n$  degrees of freedom is represented entirely by a point in a  $2n$ -dimensional phase space. By convention, the first  $n$  are the generalized positions ( $q_i$ ) of the system's state, and the second the conjugate momenta ( $p_i$ ).

In the case of  $n = 1$ , we are considering the one dimensional motion of a particle, and thus the phase space is a plane, with the position  $q$  and conjugate momentum  $p$  as coordinates. Analogue to a one-dimensional motion, it is possible to represent one mode of a cavity field using two orthogonal field quadratures (equation (2.27)) as position and momentum. In the classical world it is possible to determine precisely and simultaneously  $q$  and  $p$  at a given point. Thus, their probability distributions are Dirac distributions. However, if we have statistical uncertainties or lack of knowledge about the system, we must replace the Dirac distributions associated to this point by a density of probability,  $f(x, p)$ , positive, normalized, and non-zero in a limited region of the phase space. The statistical average of any observable  $o(x, p)$  is given by

$$\bar{o} = \int f(x, p) o(x, p) dx dp. \quad (2.45)$$

The extension of this representation to quantum states was first discussed by Wigner in 1932. The generalization of this representation for quantum states is not trivial since the Heisenberg uncertainty principle adds a natural blurring even in the absence of any statistical uncertainty. However, it is possible to define a real phase space distribution for quantum systems which preserve the main features of a classical probability distribution. This distribution is called Wigner function ( $W$ ), and contains all the information required for determine the expectation value of every observable of the system. Therefore, the Wigner function representation is equivalent to the density operator representation.

There are two phase space distributions different from  $W$ , the Husimi- $Q$  distribution and the Glauber-Sudarshan  $P$  distribution. These distributions are the Fourier transform of different characteristic functions depending on the order of the creation and annihilation

operators. In this work, we focus on the Wigner distribution to evidence the quantumness of a state.

### 2.5.1 Characteristic functions

Previously, we mentioned that the phase space distributions  $W$ ,  $Q$  and  $P$  are the Fourier transforms of characteristic functions related to the expectation value of the displacement operator  $D$  (see equation (2.34)). It is important to mention that when we expand a function in a power series of operators, we must address the issue of their ordering. Particularly, there are three orderings when we deal with a function of the creation and annihilation operators  $a$  and  $a^\dagger$ . These orderings are:

- normal order: All creation operators are placed to the left to the left of the annihilation ones ( $a^\dagger \longleftrightarrow a$ ).
- anti-normal order: All annihilation operators are placed to the left of the creation ones ( $a \longleftrightarrow a^\dagger$ ).
- symmetric order: All products of creation and annihilation operators are symmetrized.

We notice that the expansion of the displacement operator is naturally in symmetric order. Then, the characteristic function in symmetric order is defined by

$$C_S^{[\rho]}(\lambda) = \langle D(\lambda) \rangle = \text{Tr} \left[ \rho e^{\lambda a^\dagger - \lambda^* a} \right], \quad (2.46)$$

with  $\lambda$  being a complex number. The expression of the symmetric characteristic function for a pure state  $|\Psi\rangle$

$$C_S^{[|\Psi\rangle\langle\Psi|]} = \langle \Psi | D(\lambda) | \Psi \rangle. \quad (2.47)$$

It is simply the overlap between the state  $|\Psi\rangle$  and its translation in the phase space by  $D(\lambda)$ . We say that it is an autocorrelation function.

The normal- and anti-normal-order characteristic functions are defined as

$$C_n^{[\rho]}(\lambda) = \text{Tr} \left[ \rho e^{\lambda a^\dagger} e^{-\lambda^* a} \right] \quad (2.48)$$

$$C_{an}^{[\rho]}(\lambda) = \text{Tr} \left[ \rho e^{-\lambda^* a} e^{\lambda a^\dagger} \right]. \quad (2.49)$$

These functions can be related using the Glauber identity  $e^A e^B = e^{A+B} e^{[A,B]/2}$ , resulting in the relations:

$$C_S^{[\rho]}(\lambda) = e^{-|\lambda|^2/2} C_n^{[\rho]}(\lambda) = e^{|\lambda|^2/2} C_{an}^{[\rho]}(\lambda). \quad (2.50)$$

### 2.5.2 Wigner function

The Wigner function is defined as the two-dimensional Fourier transform of the symmetric characteristic function

$$W(\alpha) = \frac{1}{\pi^2} \int d^2\lambda C_S(\lambda) e^{\alpha\lambda^* - \alpha^*\lambda}, \quad (2.51)$$

where  $\alpha$  and  $\lambda$  are complex numbers. This function is real and normalized which is an essential requirement for a probability distribution, but is not positive definite. Hence, the Wigner function cannot be thought as a genuine probability, and has been defined as a *quasi-probability distribution*.

### Two equivalent expressions of W

The Wigner function can be written in two equivalent expressions, one related to the elements of the density matrix in the position eigenstates basis and the other related to the expectation value of the parity operator  $P$ .

- **Position eigenstates basis**

This expression is the most widely used. It directly relates the quantum state of the system described by  $\rho$  with the Wigner function using the position eigenstates basis, by

$$W(x, p) = \frac{1}{\pi} \int du e^{-2ipu} \langle x + \frac{u}{2} | \rho | x - \frac{u}{2} \rangle. \quad (2.52)$$

Then,  $W$  is the Fourier transform of the off-diagonal terms of the density matrix  $\rho$  written in the position eigenstates basis. If we invert the Fourier transform we can

get the matrix elements of  $\rho$  in terms of  $W$

$$\langle x + \frac{u}{2} | \rho | x - \frac{u}{2} \rangle = \int dp e^{2ipu} W(x, p). \quad (2.53)$$

Also, this expression shows us that both representations, density matrix  $\rho$  and Wigner distribution  $W$ , are completely equivalent. Therefore,  $W$  contains all the information required to determine the expectation value of any observable of the quantum system.

• **Expectation value of the parity operator**

Another expression, which is of principal interest for this work, is the expression linking the Wigner with the parity operator  $P$  by

$$W(\alpha) = \frac{2}{\pi} \text{Tr} [D(-\alpha) \rho D(\alpha) P]. \quad (2.54)$$

The Wigner function is the expectation value of  $2P/\pi$  over the state obtained by displacing  $\rho$  in the phase space by  $-\alpha$ .

The hermitian parity operator  $P$  performs a symmetry in the phase space around the origin according to

$$P|x\rangle = |-x\rangle; \quad P|p\rangle = |-p\rangle \quad (2.55)$$

Moreover, the operator  $P$  is the photon number parity operator. This can be shown remembering that the wavefunctions of a Fock state  $|n\rangle$  in either the position or momentum representation are

$$\psi_n(x) = \langle x | n \rangle = (2/\pi)^{1/4} \frac{1}{\sqrt{2^n n!}} e^{-x^2} (-1)^n e^{2x^2} \frac{d^n}{du^n} e^{-u^2}; \quad \text{with } u = x\sqrt{2}. \quad (2.56)$$

We write these wavefunctions as  $\psi_n(x) = (-1)^n f_n(x)$ , where  $f_n(x)$  is an even function. So, reversing the sign of  $x$  or  $p$  in these functions produces the effect of a multiplication by  $(-1)^n$ :

$$\psi_n(-x) = (-1)^n f_n(x). \quad (2.57)$$

And thus, we can write the effect of  $P$  on the Fock basis as

$$P|n\rangle = (-1)^n|n\rangle. \quad (2.58)$$

At this stage, it is easy to conclude that the operator  $P$  is the photon number parity observable

$$P = e^{i\pi aa^\dagger}. \quad (2.59)$$

Since  $W$  is the expectation value of an observable,  $W$  is a measurable quantity. Hence,  $W$  can be determined experimentally using equation (2.54), allowing the reconstruction of quantum states and the study of their decoherence process [36]. Equation (2.54) imposes boundaries to the range of values that  $W$  can take owing to the fact that

$$-1 \leq P \leq 1 \quad \Rightarrow \quad \frac{-2}{\pi} \leq W \leq \frac{2}{\pi}. \quad (2.60)$$

### Properties of the Wigner distribution

- **Expectation value of operators:** The Wigner function can be used to compute expectation values of any operator written in symmetric ordering  $o_s$ :

$$\langle o_s(a, a^\dagger) \rangle = \text{Tr}[\rho o_s] = \int d\alpha^2 o_s(\alpha, \alpha^*) W(\alpha). \quad (2.61)$$

Equation (2.61) reminds us to equation (2.45), which is used for computing the expectation values of classical observables.  $W(\alpha)$  plays the role of  $f(x, p)$  in the quantum phase space.

- **Marginal distributions:** Using equation (2.53) with  $u = 0$  gives the probability distribution of the field quadrature  $X_0$ , also called position quadrature:

$$P(x) = \langle x|\rho|x \rangle = \int dp W(x, p). \quad (2.62)$$

Similarly, the probability distribution of the field quadrature  $X_{\pi/2}$  (momentum) can

be determined by

$$P(p) = \langle p | \rho | p \rangle = \int dx W(x, p). \quad (2.63)$$

Therefore, we obtain the marginal probability distributions of  $x$  and  $p$  by integrating the Wigner distribution over the conjugate variable.

In general, this result is valid for any couple of orthogonal field quadratures  $x_\phi$  and  $p_\phi$

$$P(p_\phi) = \int dx_\phi W(x_\phi, p_\phi). \quad (2.64)$$

The Wigner function is the only quantum phase space distribution satisfying this property.

For a lot of quantum states, the probability distributions  $P(x)$  or  $P(p)$  present nodes. Since  $P(x_0) = \int dp W(x_0, p) = 0$ , it follows that  $W$  cannot be positive everywhere. When it takes negative values in some regions of the phase space, it cannot be regarded a genuine probability distribution. We will see next, that negativities in  $W$  are indicators of the non-classical features of a state.

### 2.5.3 Examples of Wigner functions

In this section, we present examples of the Wigner distribution of some quantum states. We can divide these states into two categories: First, the group of those states whose Wigner distribution is positive everywhere, which we called *quasi-classical* states, and then the group whose Wigner functions present negativities, called *non-classical* states.

#### Coherent states

For a coherent state  $|\beta\rangle$ , the Wigner distribution is

$$W^{[|\beta\rangle\langle\beta|]}(\alpha) = \frac{2}{\pi} e^{-2|\beta-\alpha|^2}. \quad (2.65)$$

This function is a Gaussian centered at  $\beta$ , with a width of  $1/\sqrt{2}$ . It can be observed directly that the Wigner function of a coherent state is always positive, taking its maximum value  $2/\pi$  at  $\beta = \alpha$ . The coherent state is thus a quasi-classical state. Figure 2.1 (a) and (b)

present the Wigner distributions of the vacuum state ( $\beta = 0$ ) and a coherent state with a mean photon number  $\bar{n} = 5$  ( $\beta = \sqrt{5}$ ), where  $\alpha = x + ip$ .

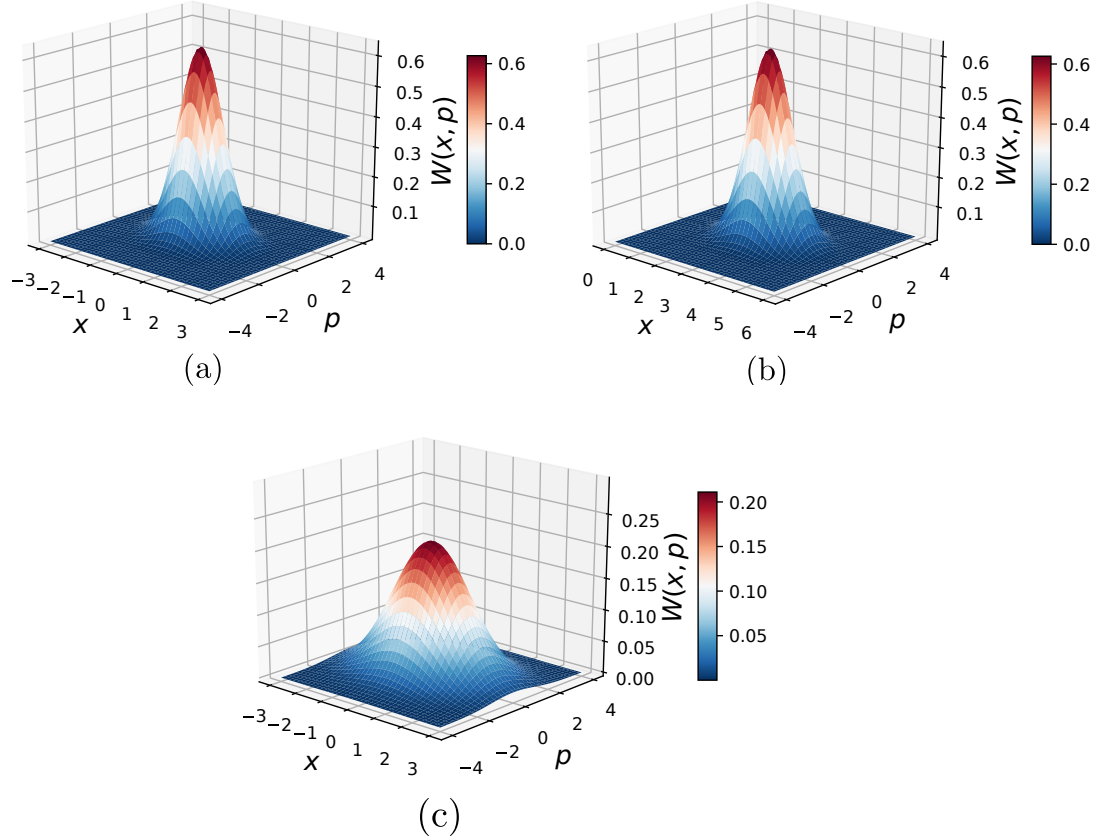


Figure 2.1: Quasi-classical states Wigner functions. (a) vacuum state. (b) Coherent state with  $\beta = \sqrt{5}$ . (c) Thermal state with  $n_{th} = 1$  mean photon number.

### Thermal state

The thermal state is the steady state of the field in cavity at a finite temperature. Its Wigner function is given by

$$W^{[\rho_{th}]}(\alpha) = \frac{2}{\pi} \frac{1}{(2n_{th} + 1)} e^{-2|\alpha|^2/(2n_{th}+1)}. \quad (2.66)$$

It is a Gaussian centered at the origin, with a width of  $\sqrt{n_{th} + 1/2}$ . It is positive everywhere, taking its maximum value  $1/\pi(n_{th} + 1/2)$  at the origin. Thus, the thermal state

is a quasi-classical state. Figure 2.1 (c) shows the Wigner distribution of a thermal state with a mean photon number  $n_{th} = 1$ .

For these states, the Wigner function is definite and positive. It has all the properties of a classical probability distribution in the phase space.

### Fock states

The Wigner function of a Fock state reads as

$$W^{[|n\rangle\langle n|]}(\alpha) = \frac{2}{\pi}(-1)^n e^{-2|\alpha|^2} L_n(4|\alpha|^2), \quad (2.67)$$

where  $L_n$  is the  $n$ th Laguerre polynomials. Since  $L_n(0) = 1$ :

$$W^{[|n\rangle\langle n|]}(0) = \frac{2}{\pi}(-1)^n. \quad (2.68)$$

The value of the Wigner function at the origin is  $\pm 2/\pi$ . It takes its minimum value  $-2/\pi$  for odd photon number states and maximum value for even photon number states. In fact, all Fock states present some regions with negativities in the phase space, and thus the Wigner cannot be regarded as a classical probability distribution. They evidence the quantumness of Fock states. For example, the Wigner distribution of the single-photon state is

$$W^{[|1\rangle\langle 1|]}(\alpha) = -\frac{2}{\pi}(1 - 4|\alpha|^2)e^{-2|\alpha|^2}, \quad (2.69)$$

which is shown in figure 2.2 (a), also in figure 2.2 (b), the five-photon number state is presented.

### Fock states superpositions

Since this work focuses on the generation of some Fock states superpositions which are highly non-classical states, we present some Wigner functions of Fock states superpositions realized in [29]. The general equation of these states is given by

$$|\psi\rangle = \alpha|m\rangle + \beta|n\rangle, \quad (2.70)$$

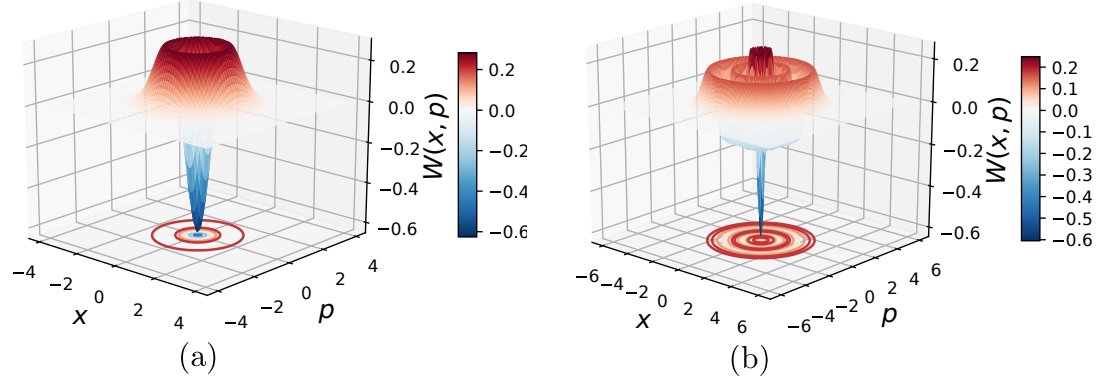


Figure 2.2: Fock states Wigner functions. (a) single-photon Fock state. (b) five-photon Fock state.

with  $m \neq n$ . One of these states of interest is an *optical qubit*, which is a superposition of the vacuum and the one-photon number state:

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle.$$

The Wigner function of the optical qubit and Fock state superpositions in general can be determined by using equations (2.51) and (2.67). Apart of the non-classical feature of Fock states, the quantum nature of superposition states arise from the interference terms ( $\langle m|D(\lambda)|n\rangle$ ) when equation (2.51) is applied. Particularly, for  $\cos(\theta/2) = \sin(\theta/2) = 1/\sqrt{2}$ , the Wigner function of the optical qubit has the form

$$W_{qubit}(x, p) = \frac{4}{\pi} e^{-2(x^2+p^2)} [x^2 + p^2 + x\cos(\phi) + p\sin(\phi)]. \quad (2.71)$$

In figure 2.3 (a) and (b), we show an optical qubit state and a superposition of states  $|1\rangle$  and  $|3\rangle$  states, respectively.

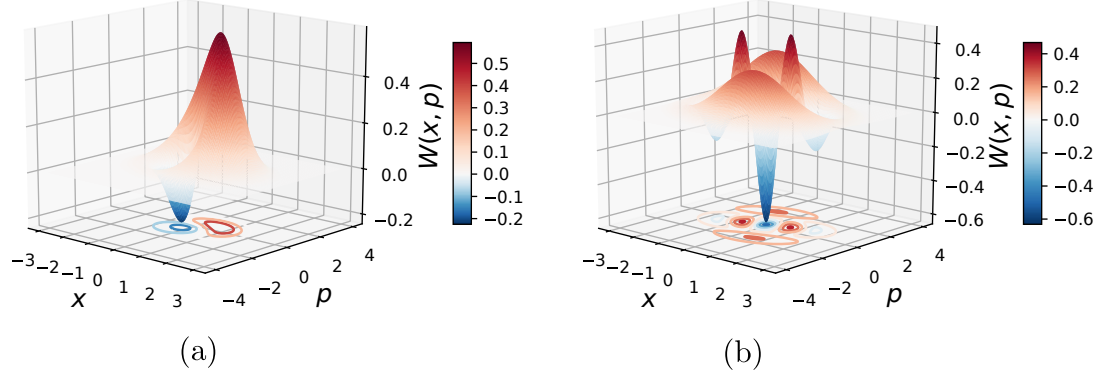


Figure 2.3: Fock states superpositions Wigner functions. (a) superposition of states  $|0\rangle$  and  $|1\rangle$ . (b) superposition of states  $|1\rangle$  and  $|3\rangle$ .

## 2.6 Two-level atoms

### 2.6.1 A two-level atom as a spin-1/2 system

In general, we can describe a two-level system as a spin-1/2 system like the electron. Then, we call this system a *pseudo-spin*  $\mathbf{S}$ , where  $\mathbf{S} = \hbar\boldsymbol{\sigma}/2$  and  $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ , is a vector formed by Pauli matrices. In the basis of the  $\sigma_z$  eigenstates, the Pauli matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.72)$$

These matrices obey the commutation rule

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k. \quad (2.73)$$

The eigenvalues of  $\sigma_i$  are  $\pm 1$ , so the pseudo-spin  $\mathbf{S}$  along one direction of the space can take only two possible values,  $\pm\hbar/2$ .

In the case of a two-level atom with an upper state  $|e\rangle$  and a ground state  $|g\rangle$ , we use the states  $|e\rangle$  and  $|g\rangle$ , as the eigenstates of the operator  $\sigma_z$  with eigenvalues  $+1$  and  $-1$ ,

respectively. The Hamiltonian of the atom with eigenstates  $|e\rangle$  and  $|g\rangle$  can be written as

$$H_{at} = \frac{\hbar\omega_{eg}}{2}\sigma_z, \quad (2.74)$$

where  $\omega_{eg}$  is the angular frequency of the transition  $|e\rangle \longleftrightarrow |g\rangle$ , and  $\sigma_z$  is the Pauli matrix from equation (2.72).

The most general pure *atomic pseudo-spin* state can be written as

$$|\psi_{at}\rangle = \cos(\theta/2)|e\rangle + e^{i\varphi}\sin(\theta/2)|g\rangle, \quad (2.75)$$

with the two angles constrained by  $0 \leq \theta \leq \pi$  and  $0 \leq \varphi \leq 2\pi$ .

As shown in figure 2.4, when the unit vector  $\hat{\mathbf{n}}$  rotates in space, the state  $|\psi_{at}\rangle$  explores the entire Hilbert space. This general state  $|\psi_{at}\rangle$  is an eigenstate with eigenvalue  $+1$  of the spin component along the direction  $\hat{\mathbf{n}}$  with polar angles  $\theta$  and  $\varphi$ . If we vary the parameters  $\theta$  and  $\varphi$ , the tip of the unit vector  $\hat{\mathbf{n}}$  form a spherical surface called *Bloch sphere*. Each point of the Bloch sphere represents a superposition of  $|e\rangle$  and  $|g\rangle$ .

Now lets us consider the time evolution of the state  $|\psi_{at}\rangle$  under the Hamiltonian (2.74). This state is given by

$$|\psi_{at}(t)\rangle = e^{-i\omega_{eg}t/2} \left( \cos(\theta/2)|e\rangle + e^{i(\varphi+\omega_{eg}t)}\sin(\theta/2)|g\rangle \right). \quad (2.76)$$

In the Bloch sphere picture, this evolution is a *Larmor precession* of the atomic pseudo-spin around the quantization axis  $z$ .

### 2.6.2 Manipulation of atomic states with a classical field

Preparation of arbitrary atomic states is essential in quantum optics experiments described below. This can be realized performing spin rotations on the Bloch sphere by coupling the atom with a resonant or quasi-resonant classical electric field.

We can write the usual Pauli matrices in terms of the energy eigenstates as

$$\sigma_x = |g\rangle\langle e| + |e\rangle\langle g|; \quad \sigma_y = i(|g\rangle\langle e| - |e\rangle\langle g|); \quad \sigma_z = |e\rangle\langle e| - |g\rangle\langle g|. \quad (2.77)$$

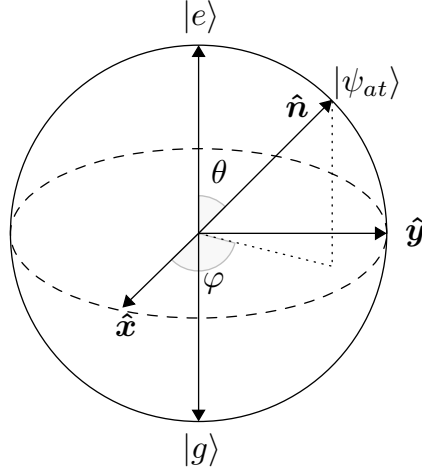


Figure 2.4: Representation of the states of a two-level system on the Bloch sphere.

And also, define the raising and lowering operators  $\sigma_{\pm}$

$$\sigma_+ = |e\rangle\langle g|; \quad \sigma_- = |g\rangle\langle e|. \quad (2.78)$$

The atom-field Hamiltonian is

$$H = H_a + H_r, \quad (2.79)$$

where  $H_a$  is the Hamiltonian from (2.74), and

$$H_r = -\mathbf{d} \cdot \mathbf{E}_r, \quad (2.80)$$

describes the coupling between the atom and the classical electric field in the dipole approximation. In the dipole approximation or long-wavelength approximation, we assume that the wavelength of the field is much longer than the size of the atom so that we can neglect any variations of the field over the extent of the atom. With this approximation, we consider a classical time-dependent electric field given by

$$\mathbf{E}_r = i\varepsilon_r \left[ \hat{\mathbf{e}}_r e^{-i\omega_r t} e^{-i\varphi_0} e^{-i\varphi} + \hat{\mathbf{e}}_r^* e^{i\omega_r t} e^{i\varphi_0} e^{i\varphi} \right]. \quad (2.81)$$

$\varepsilon_r$ ,  $\omega_r$  and  $\hat{\mathbf{e}}_r$  represent the real amplitude, angular frequency and the polarization of the field, respectively. Also,  $\varphi_0 + \varphi$  describe the phase.

In the dipole approximation, the atom couples to the field by the dipole operator,  $\mathbf{d} = q\mathbf{r}$ . In order to determine the form of the atomic dipole operator, we invoke the parity operator  $\Pi$  which flips the sign of the position operator  $\mathbf{r}$ , and thus is defined by the transformation

$$\Pi\mathbf{r}\Pi^\dagger = -\mathbf{r}. \quad (2.82)$$

We note that the action of the parity operator twice returns  $\mathbf{r}$  and thus the parity operator is unitary ( $\Pi^\dagger\Pi = \mathbb{1}$ ) and  $\Pi^2 = \mathbb{1}$ . So, the possible eigenvalues of  $\Pi$  are  $\pm 1$ . Operating the parity operator on the right of the transformation (2.82) gives  $\Pi\mathbf{r} = -\mathbf{r}\Pi$ , and thus the anticommutator of  $\Pi$  with the position operator  $\mathbf{r}$  vanishes:

$$[\Pi, \mathbf{r}]_+ = 0. \quad (2.83)$$

Then, using the basis of the atomic energy eigenstates, allows that the matrix elements of the anticommutator to vanish,

$$\langle i | [\Pi, \mathbf{r}]_+ | j \rangle = 0, \quad (2.84)$$

these matrix elements can be written in the same basis as

$$\langle i | [\Pi, \mathbf{r}]_+ | j \rangle = \langle i | (\Pi\mathbf{r} + \mathbf{r}\Pi) | j \rangle = (\pi_i + \pi_j) \langle i | \mathbf{r} | j \rangle, \quad (2.85)$$

where  $\pi_a$  and  $\pi_b$  are the eigenvalues of  $\Pi$ . This is valid because  $\Pi$  commutes with the atomic Hamiltonian which has the form  $\mathbf{p}^2/2m - \alpha/|\mathbf{r}|$ , and thus  $\Pi$  and  $H$  have simultaneous eigenstates. This is because  $\mathbf{p}^2 = -\hbar^2\nabla^2$  in the position representation, and thus is not affected by the parity operator. Also, the same happens to the modulus of  $\mathbf{r}$ . We have mentioned that the eigenvalues of the parity operator are  $\pm 1$ , then combining equations (2.84) and (2.85), we have that  $\pi_i + \pi_j = 0$  or  $\langle i | \mathbf{r} | j \rangle = 0$ . We can see that the diagonal matrix elements of  $\mathbf{d}$  vanish, since  $\pi_g$  and  $\pi_e$  are both nonzero,

$$\langle g | \mathbf{d} | g \rangle = \langle e | \mathbf{d} | e \rangle = 0. \quad (2.86)$$

The off-diagonal matrix elements  $\langle e | \mathbf{d} | g \rangle = \langle g | \mathbf{d} | e \rangle^*$  are nonvanishing, because  $\pi_g = -\pi_e$ .

Using the Completeness property of the atomic eigenstates, i.e.  $|g\rangle\langle g| + |e\rangle\langle e| = \mathbb{1}$ , the dipole operator can be written as

$$\begin{aligned}\mathbf{d} &= (|e\rangle\langle e| + |g\rangle\langle g|)\mathbf{d}(|e\rangle\langle e| + |g\rangle\langle g|) \\ &= \langle g|\mathbf{d}|e\rangle\sigma_- + \langle e|\mathbf{d}|g\rangle\sigma_+ \\ &= d(\hat{\epsilon}_{\mathbf{a}}\sigma_- + \hat{\epsilon}_{\mathbf{a}}^*\sigma_+),\end{aligned}\tag{2.87}$$

where  $d$  is the dipole matrix element (assumed to be real without loss of generality) and  $\hat{\epsilon}_{\mathbf{a}}$  the unit vector describing the atomic transition polarization.

With the matrix elements of the dipole operator, the atom-field Hamiltonian is

$$H = \frac{\hbar\Delta_r}{2}\sigma_z + \frac{\hbar\omega_r}{2}\sigma_z - d(\hat{\epsilon}_{\mathbf{a}}\sigma_- + \hat{\epsilon}_{\mathbf{a}}^*\sigma_+) \cdot \mathbf{E}_r,\tag{2.88}$$

where we have introduced the atom-field detuning  $\Delta_r = \omega_{eg} - \omega_r$ . Then, we apply an unitary transformation given by

$$U = e^{i\omega_r\sigma_z t/2}.\tag{2.89}$$

A Hamiltonian transforms under an unitary transformation  $U$  in the following way. Suppose an unitary transformation  $U$  which induces the transformation  $|\tilde{\psi}\rangle = U|\psi\rangle$ . Both, original and transformed states must satisfy the Schrödinger equation

$$i\hbar\frac{\partial|\psi\rangle}{\partial t} = H|\psi\rangle; \quad i\hbar\frac{\partial|\tilde{\psi}\rangle}{\partial t} = \tilde{H}|\tilde{\psi}\rangle,\tag{2.90}$$

where  $\tilde{H}$  is the transformed Hamiltonian. We can write the first equation here as

$$i\hbar\frac{\partial(U^\dagger|\tilde{\psi}\rangle)}{\partial t} = HU^\dagger|\tilde{\psi}\rangle.\tag{2.91}$$

By using the product rule for derivatives and operating on the left by  $U$ , one gets

$$i\hbar\frac{\partial|\tilde{\psi}\rangle}{\partial t} + i\hbar U\frac{\partial U^\dagger}{\partial t}|\tilde{\psi}\rangle = UHU^\dagger|\tilde{\psi}\rangle.\tag{2.92}$$

We can rewrite this as

$$i\hbar \frac{\partial |\tilde{\psi}\rangle}{\partial t} = \left[ U H U^\dagger - i\hbar U \frac{\partial U^\dagger}{\partial t} \right] |\tilde{\psi}\rangle. \quad (2.93)$$

Comparing this to the Schrödinger equation in the transformed variables, one can identify the transformation rule for the Hamiltonian

$$\tilde{H} = U H U^\dagger - i\hbar U \frac{\partial U^\dagger}{\partial t}. \quad (2.94)$$

Using this transformation rule for the Hamiltonian in (2.88) and the unitary transformation of (2.89), we have the Hamiltonian

$$\tilde{H} = \frac{\hbar\Delta_r}{2}\sigma_z - d(\hat{\epsilon}_a\sigma_- e^{-i\omega_r t} + \hat{\epsilon}_a^*\sigma_+ e^{i\omega_r t}) \cdot \mathbf{E}_r. \quad (2.95)$$

The expansion of the scalar product  $\mathbf{d} \cdot \mathbf{E}_r$  from above has two time-independent terms and two terms oscillating at angular frequencies  $\pm 2\omega_r$ . These two terms oscillate rapidly and we can make the **rotating-wave approximation** (RWA) to neglect them. With this approximation the Hamiltonian is

$$\tilde{H} = \frac{\hbar\Delta_r}{2}\sigma_z - i\hbar \frac{\Omega_r}{2} [e^{-i\varphi}\sigma_+ - e^{i\varphi}\sigma_-], \quad (2.96)$$

where

$$\Omega_r = \frac{2d}{\hbar} \epsilon_r \hat{\epsilon}_a^* \cdot \hat{\epsilon}_r e^{i\varphi_0}, \quad (2.97)$$

is the classical Rabi frequency. The phase  $\varphi_0$  is adjusted to make the Rabi frequency a positive real value. The lowering and raising also are defined by  $\sigma_\pm = (\sigma_x \pm i\sigma_y)/2$ , and thus the Hamiltonian in (2.96) can be written in a compact form

$$\tilde{H} = \frac{\hbar\Omega'_r}{2} \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}, \quad (2.98)$$

with  $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$  being the vector of Pauli matrices,  $\Omega'_r = \sqrt{\Omega_r^2 + \Delta_r^2}$  and

$$\hat{\mathbf{n}} = \frac{-\Omega_r \sin(\varphi) \hat{\mathbf{e}}_x + \Omega_r \cos(\varphi) \hat{\mathbf{e}}_y + \Delta_r \hat{\mathbf{e}}_z}{\Omega'_r}. \quad (2.99)$$

The Hamiltonian of equation (2.98) describes a rotation with angular frequency  $\Omega_r$  around the  $\hat{\mathbf{n}}$  axis.

Note that:

- if  $\Omega_r = 0$ , then  $\hat{\mathbf{n}} = \hat{\mathbf{e}}_z$ ,
- if  $\Delta_r = 0$ , then  $\hat{\mathbf{n}}$  is in the equatorial plane of the Bloch sphere, and  $\Omega'_r = \Omega_r$ .

The time evolution operator is given by

$$e^{-i\Omega'_r t \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}/2} = e^{-i\theta \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}/2} = \cos(\theta/2)\mathbb{1} - i \sin(\theta/2) \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}, \quad (2.100)$$

where  $\theta = \Omega'_r t$ , and the last identity resulting from a power series expansion of the exponential, with  $(\boldsymbol{\sigma} \cdot \hat{\mathbf{n}})^2 = \mathbb{1}$ . Then, the resonant case ( $\Delta_r = 0$ ) provides a simple way to prepare a state with arbitrary polar angles on the Bloch sphere. The interaction during a time  $t = \theta/\Omega_r$ , with a resonant field having a phase  $\varphi$  performs the rotation:

$$\cos(\theta/2)\mathbb{1} - i \sin(\theta/2) \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2)e^{-i\varphi} \\ \sin(\theta/2)e^{i\varphi} & \cos(\theta/2) \end{pmatrix}. \quad (2.101)$$

This rotation transforms  $|e\rangle$  into  $\cos(\theta/2)|e\rangle + \sin(\theta/2)e^{i\varphi}|g\rangle$ , the state whose Bloch vector points in the direction of polar angles  $\theta, \varphi$ .

If we consider an atom initially in the state  $|e\rangle$  or  $|g\rangle$  interacting with a resonant classical electric field during a time  $t$  which satisfies the condition  $\Omega_r t = \pi/2$ . Then, the corresponding state is a superposition of  $|e\rangle$  and  $|g\rangle$  with equal probabilities

$$|e\rangle \rightarrow \frac{1}{\sqrt{2}} (|e\rangle + e^{i\varphi}|g\rangle); \quad |g\rangle \rightarrow \frac{1}{\sqrt{2}} (-e^{-i\varphi}|e\rangle + |g\rangle). \quad (2.102)$$

This manipulation of the atomic state satisfying  $\Omega_r t = \pi/2$  is called as  $\pi/2$  pulse.

The resonant interaction with a classical field is a basic ingredient in CQED experiments. Particularly, we will see that the  $\pi/2$  pulse plays an important role in the atomic Ramsey interferometry.

### 2.6.3 The Ramsey interferometer

The Ramsey interferometer is realized by combining two  $\pi/2$  pulses applied in zones  $R_1$  and  $R_2$  as shown in figure. 2.5.

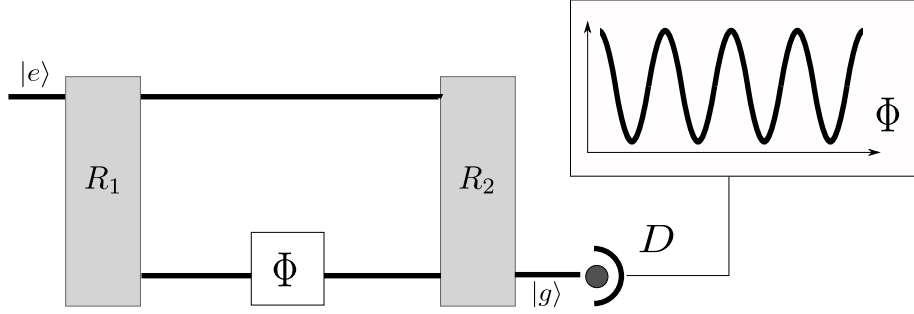


Figure 2.5: Scheme of the Ramsey interferometer.  $R_1$  and  $R_2$  are the zones where the  $\pi/2$  pulses are applied, while  $D$  is the state detector of the atom. The probability of finding the atom in  $|e\rangle$  or  $|g\rangle$  is an oscillating function depending on the phase shift  $\Phi$ .

First, the  $\pi/2$  pulse in  $R_1$  rotates the state  $|e\rangle$  to a superposition state

$$|e\rangle \rightarrow \frac{1}{\sqrt{2}} (|e\rangle + |g\rangle). \quad (2.103)$$

Then, suppose that on the path to  $R_2$  the atomic state acquires a phase shift  $\Phi$

$$\frac{1}{\sqrt{2}} (|e\rangle + e^{i\Phi} |g\rangle). \quad (2.104)$$

Another  $\pi/2$  pulse in  $R_2$  is applied to the atomic state. We consider that  $R_2$  has a relative phase  $\phi_r$  with  $R_1$ , and thus the transformation is the one from equation (2.102) with  $\varphi = \phi_r$ ,

$$\frac{1}{2} \left[ \left( 1 - e^{i(\Phi - \phi_r)} \right) |e\rangle + e^{i\Phi} \left( 1 + e^{i(\phi_r - \Phi)} \right) |g\rangle \right]. \quad (2.105)$$

The probabilities for detecting  $|e\rangle$  or  $|g\rangle$  are given by

$$\pi_e(\Phi, \phi_r) = \frac{1}{2} - \frac{1}{2} \cos(\Phi - \phi_r); \quad \pi_g(\Phi, \phi_r) = \frac{1}{2} + \frac{1}{2} \cos(\phi_r - \Phi). \quad (2.106)$$

We consider the case with  $\phi_r = 0$ . In this case, the probabilities  $\pi_e$  and  $\pi_g$  are oscillating functions depending only on  $\Phi$ . These oscillations in the detection probability can be

understood as an interference process. For example, the transition from the state  $|e\rangle$  to  $|g\rangle$  could take place either in  $R_1$  or  $R_2$ . There are two indistinguishable quantum paths leading the initial state to the final state so the final transition probability is obtained by the sum of the two corresponding amplitudes. The Ramsey fringes correspond to the interference between these two amplitudes.

The Ramsey interferometer is a very sensitive probe to phase disturbance acting on the atomic state while it crosses the set-up.

## 2.7 Atom-field coupling: Jaynes-Cummings model

In this section, we will describe the interaction of two quantum systems: one two-level atom (spin-1/2 system) interacting with a single-mode of a quantized electromagnetic field (quantum harmonic oscillator) confined in a cavity. As we mentioned in section 2.1, for cavity boundary conditions, the mode functions of the electromagnetic field are standing waves (sinusoidal functions). Thus, we write the electric cavity field as

$$\mathbf{E}_c = \varepsilon_0(\mathbf{r})(\hat{\mathbf{e}}_c a + \hat{\mathbf{e}}_c^* a^\dagger). \quad (2.107)$$

The complete Hamiltonian of the atom-cavity system is the quantum version of equation (2.79) and can be expressed as

$$H = H_a + H_c + H_{ac}, \quad (2.108)$$

where  $H_a$  and  $H_c = \hbar\omega_c N$  are the atom and cavity Hamiltonian. Note that we eliminated the constant  $\hbar\omega_c/2$  by redefining the zero energy level. The interaction Hamiltonian  $H_{ac}$  describing the coupling of the atom and the quantized electric field in the dipole approximation is

$$H_{ac} = -\mathbf{d} \cdot \mathbf{E}_c. \quad (2.109)$$

Then, the total Hamiltonian is explicitly written as

$$H = \frac{\hbar\omega_a}{2}\sigma_z + \hbar\omega_c a^\dagger a - d\varepsilon_0(\mathbf{r})(\hat{\boldsymbol{\varepsilon}}_{\mathbf{a}}\sigma_- + \hat{\boldsymbol{\varepsilon}}_{\mathbf{a}}^*\sigma_+) \cdot (\hat{\boldsymbol{\varepsilon}}_{\mathbf{c}}a + \hat{\boldsymbol{\varepsilon}}_{\mathbf{c}}^*a^\dagger). \quad (2.110)$$

Performing a similar procedure as the one in section 2.6.2, we apply an unitary transformation given by

$$U = e^{i\omega_c t a^\dagger a + i\omega_a t \sigma_z / 2}, \quad (2.111)$$

and thus the Hamiltonian transforms following the rule of equation (2.94) as

$$\tilde{H} = -d\varepsilon_0(\mathbf{r})e^{i\omega_a t \sigma_z / 2}(\hat{\boldsymbol{\varepsilon}}_{\mathbf{a}}\sigma_- + \hat{\boldsymbol{\varepsilon}}_{\mathbf{a}}^*\sigma_+)e^{-i\omega_a t \sigma_z / 2} \cdot e^{i\omega_c t a^\dagger a}(\hat{\boldsymbol{\varepsilon}}_{\mathbf{c}}a + \hat{\boldsymbol{\varepsilon}}_{\mathbf{c}}^*a^\dagger)e^{-i\omega_c t a^\dagger a}. \quad (2.112)$$

Making use of the property

$$e^{\alpha A} B e^{-\alpha A} = B + \alpha[A, B] + \frac{\alpha^2}{2}[A, [A, B]] + \frac{\alpha^3}{3!}[A, [A, [A, B]]] + \dots, \quad (2.113)$$

we have

$$e^{i\omega_c t a^\dagger a}(a + a^\dagger)e^{-i\omega_c t a^\dagger a} = ae^{-i\omega_c t} + a^\dagger e^{i\omega_c t}. \quad (2.114)$$

With the above, the atom-field Hamiltonian of equation (2.112) is

$$\tilde{H} = -d\varepsilon_0(\mathbf{r})(\hat{\boldsymbol{\varepsilon}}_{\mathbf{a}}\sigma_- e^{-i\omega_a t} + \hat{\boldsymbol{\varepsilon}}_{\mathbf{a}}^*\sigma_+ e^{i\omega_a t}) \cdot (\hat{\boldsymbol{\varepsilon}}_{\mathbf{c}}a e^{-i\omega_c t} + \hat{\boldsymbol{\varepsilon}}_{\mathbf{c}}^*a^\dagger e^{i\omega_c t}). \quad (2.115)$$

The expansion of the scalar product involves four possible processes. Two of them are proportional to  $a\sigma_-$  and  $a^\dagger\sigma_+$ . The first corresponds to an atomic transition to the lower level  $|g\rangle$  together with a photon annihilation. The second is a transition of the atom the upper level  $|e\rangle$  and the creation of a photon. From the above equation (2.115), we see that these two terms oscillate at angular frequencies  $\mp(\omega_a + \omega_c)$ , and thus when the cavity mode and the transition frequency are close,  $\omega_a \sim \omega_c$ , they can be neglected by performing the rotation-wave approximation. The other terms  $a\sigma_+$  and  $a^\dagger\sigma_-$  correspond to the usual processes of photon absorption or emission which dominates the evolution of the system.

Then, under the rotating-wave approximation the Hamiltonian is

$$\tilde{H} = \frac{\hbar\Omega_0(\mathbf{r})}{2}(a^\dagger\sigma_-e^{it(\omega_c-\omega_a)} + a\sigma_+e^{it(\omega_a-\omega_c)}), \quad (2.116)$$

where

$$\Omega_0(\mathbf{r}) = -\frac{2d\varepsilon_0\hat{\mathbf{e}}_a \cdot \hat{\mathbf{e}}_c}{\hbar}f(\mathbf{r}) = \Omega_0f(\mathbf{r}). \quad (2.117)$$

Finally, going back to the Schödinger picture by applying the inverse transformation of equation (2.111), we get the Hamiltonian in the dipole and rotating-wave approximation known as the Jaynes-Cummings model

$$H_{JC} = \frac{\hbar\omega_a}{2}\sigma_z + \hbar\omega_ca^\dagger a + \frac{\hbar\Omega_0}{2}(a\sigma_+ + a^\dagger\sigma_-)f(\mathbf{r}). \quad (2.118)$$

A natural basis representation is by the “uncoupled” eigenstates of  $H_a + H_c$ . They are the tensor product  $|e, n\rangle$  and  $|g, n\rangle$  of the atomic and cavity energy states and their eigenenergies are  $(\hbar\omega_a/2 + \hbar n\omega_c)$  and  $(-\hbar\omega_a/2 + \hbar n\omega_c)$ . These “uncoupled” states are eigenstates of the operator  $M = a^\dagger a + \sigma_+\sigma_-$ , representing the number of atomic and fields excitations, which commutes with  $H_{JC}$  and is a constant of motion. Since,  $M$  and  $H_{JC}$  are two commuting Hermitian operators, they possess a common eigenbasis. However, the pairs of eigenstates  $|e, n\rangle$  and  $|g, n+1\rangle$  of  $M$  are degenerate having the same number of excitations given by  $(n+1)$ , and thus generally  $H_{JC}$  is not diagonal in this basis representation due to the degeneracy. In fact, the structure of the Jaynes-Cummings Hamiltonian connects the pair of degenerate states corresponding to the excitation number  $(n+1)$  (the state  $|g, 0\rangle$  does not couple to any other state):

$$\begin{aligned} H_{JC}|e, n\rangle &= \left(\frac{\hbar\omega_a}{2} + \hbar\omega_cn\right)|e, n\rangle + \frac{\hbar\Omega_0f(\mathbf{r})}{2}\sqrt{n+1}|g, n+1\rangle, \\ H_{JC}|g, n+1\rangle &= \frac{\hbar\Omega_0f(\mathbf{r})}{2}\sqrt{n+1}|e, n\rangle + \left(-\frac{\hbar\omega_a}{2} + \hbar\omega_cn\right)|g, n+1\rangle. \end{aligned} \quad (2.119)$$

Then, the Hamiltonian is block diagonal, in 2x2 submatrices, making it simple to diagonalize because we have to solve separate two-level diagonalization problems.

Let us call  $H_n$  the restriction of the Jaynes-Cummings model to the subspace  $S_n =$

$\{|e, n\rangle, |g, n+1\rangle\}$  corresponding to the  $n$ th doublet with excitation number  $(n+1)$ . At the center of the cavity mode,  $f(\mathbf{r}) = 1$  and defining the atom-cavity detuning as  $\delta = \omega_a - \omega_c$ ,  $H_n$  can be expressed in the matrix form:

$$\begin{aligned} H_n &= \hbar \begin{pmatrix} \omega_c(n+1/2) + \delta/2 & \Omega_0\sqrt{n+1}/2 \\ \Omega_0\sqrt{n+1}/2 & \omega_c(n+1/2) - \delta/2 \end{pmatrix} \\ &= \hbar\omega_c(n+1/2)\mathbb{1} + V_n, \end{aligned} \quad (2.120)$$

with

$$\begin{aligned} V_n &= \frac{\hbar}{2} [\Omega_0\sqrt{n+1}\sigma_x + \delta\sigma_z] \\ &= \frac{\hbar}{2} \sqrt{\Omega_0^2(n+1) + \delta^2} \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}. \end{aligned} \quad (2.121)$$

The diagonalization of  $H_n$  is equivalent to the determination of the eigenstates of a spin placed in a magnetic field whose components along  $OZ$  and  $OX$  are proportional to  $\delta$  and  $\Omega_0\sqrt{n+1}$ . From the above, the unit vector  $\hat{\mathbf{n}} = \sin(\theta_n)\hat{\mathbf{e}}_x + \cos(\theta_n)\hat{\mathbf{e}}_z$  is pointing along the direction of this fictitious field which makes with  $Z$  the angle  $\theta_n$  defined by

$$\tan(\theta_n) = \Omega_0\sqrt{n+1}/\delta. \quad (2.122)$$

Also,  $(\boldsymbol{\sigma} \cdot \hat{\mathbf{n}})$  has  $\pm 1$  eigenvalues with the corresponding eigenvectors:

$$\begin{aligned} |+, n\rangle &= \cos(\theta_n/2)|e, n\rangle + \sin(\theta_n/2)|g, n+1\rangle; \\ |-, n\rangle &= \sin(\theta_n/2)|e, n\rangle - \cos(\theta_n/2)|g, n+1\rangle. \end{aligned} \quad (2.123)$$

These states are called the *dressed states* of the atom-field system, and the corresponding eigenenergies of  $H_n$  are

$$E_{\pm, n} = \hbar\omega_c(n+1/2) \pm \frac{\hbar}{2} \sqrt{\Omega_0^2(n+1) + \delta^2}. \quad (2.124)$$

Figure. 2.6 shows the eigenenergies,  $E_{\pm, n}$ , of dressed states as a function of the detuning for a given value of the ‘ $n$ -th Raby frequency’  $\Omega_n = \Omega_0\sqrt{n+1}$ . From the energy spectrum,

we can identify two regimes of the atom-field interaction. One is the dispersive regime or far detuned regime with  $|\delta| \gg \Omega_n$ , in which the uncoupled bare states remain good approximations of the dressed states. The other is the resonant regime with  $\delta = 0$ , in which the degeneracy of the uncoupled bare states is lifted by the coupling Hamiltonian. This results in an energy gap given by the Rabi frequency  $\Omega_n$  between the two dressed states. We discuss in more detail these two regimes in the following paragraphs.

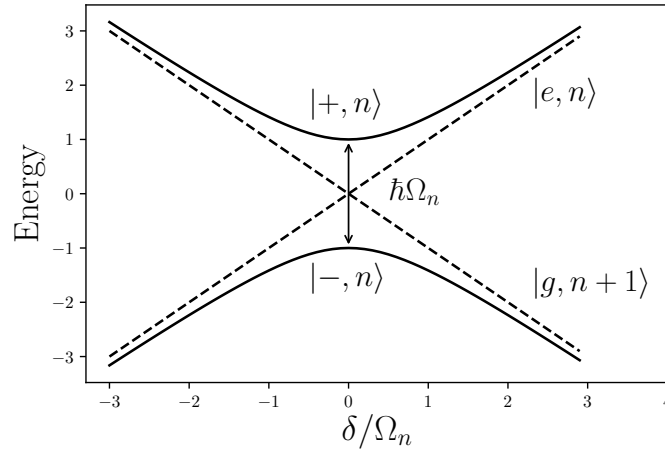


Figure 2.6: Energies of dressed states as functions of the ratio between detuning  $\delta$  and the ‘n-th photon Rabi frequency’  $\Omega_n = \Omega_0\sqrt{n+1}$ .

### 2.7.1 Resonant regime

In the resonant regime, when  $\delta = 0$ , we can see from equation (2.122) that  $\theta_n = \pi/2$  for all  $n$  values. Then, the dressed states are

$$|\pm, n\rangle = [|e, n\rangle \pm |g, n+1\rangle] / \sqrt{2}. \quad (2.125)$$

Consider an atom initially in state  $|e\rangle$ , inside a cavity with  $n$  photons. The initial state,  $|\psi_{ac}(0)\rangle = |e, n\rangle$ , expanded on the dressed basis is

$$|\psi_{ac}(0)\rangle = [|+, n\rangle + |-, n\rangle] / \sqrt{2}. \quad (2.126)$$

The time evolution of this state, neglecting a global phase factor given by the term  $(n + 1/2)\hbar\omega_c$  on the equation (2.124) of the eigenenergies, is

$$|\psi_{ac}(t)\rangle = \left[ |+, n\rangle e^{-i\Omega_n t/2} + |-, n\rangle e^{i\Omega_n t/2} \right]. \quad (2.127)$$

Returning to the uncoupled states basis, at a time  $t$ ,  $|\psi_{ac}(t)\rangle$  becomes

$$|\psi_{ac}(t)\rangle = \cos\left(\frac{\Omega_n t}{2}\right) |e, n\rangle - i \sin\left(\frac{\Omega_n t}{2}\right) |g, n+1\rangle, \quad (2.128)$$

with a probability of finding the atom in  $|e\rangle$  given by

$$P_e(t) = \frac{1 + \cos(\Omega_0 \sqrt{n+1}t)}{2}. \quad (2.129)$$

Equation (2.129) describes a reversible energy exchange between  $|e, n\rangle$  and  $|g, n+1\rangle$  at frequency  $\Omega_n$ . This phenomenon is called *quantum Rabi oscillations*.

Now, suppose that an atom initially in  $|e\rangle$ , interacting with a field in a state given by a superposition of Fock states,  $|\psi_c\rangle = \sum_n c_n |n\rangle$ . Using equation (2.128), we have the atom-field state at a time  $t$

$$|\psi_{ac}(t)\rangle = \sum_n c_n \left[ \cos\left(\frac{\Omega_n t}{2}\right) |e, n\rangle - i \sin\left(\frac{\Omega_n t}{2}\right) |g, n+1\rangle \right], \quad (2.130)$$

and a probability  $P_e(t)$  of finding the atom in  $|e\rangle$  given by

$$P_e(t) = \sum_n P(n) \frac{1 + \cos(\Omega_0 \sqrt{n+1}t)}{2}, \quad (2.131)$$

where  $P(n)$  is the photon number distribution ( $|c_n|^2$ ) of the field.

Now, the Rabi oscillation is a sum of oscillating terms at frequencies  $\Omega_0 \sqrt{n+1}$ , weighted by the corresponding probability of finding the photon number  $n$  in the initial field state. Figure. 2.7 shows  $P_e(t)$  for an initial coherent state with  $\bar{n} = 15$ , where we observe a collapse and a revival of the Rabi oscillations. These results are different for the oscillations predicted by the semiclassical description (two-level atom and a classical field). First of

all, in the quantum description, Rabi oscillations take place even for an initial field in the vacuum state. This oscillation occurs at a frequency  $\Omega_0$ .

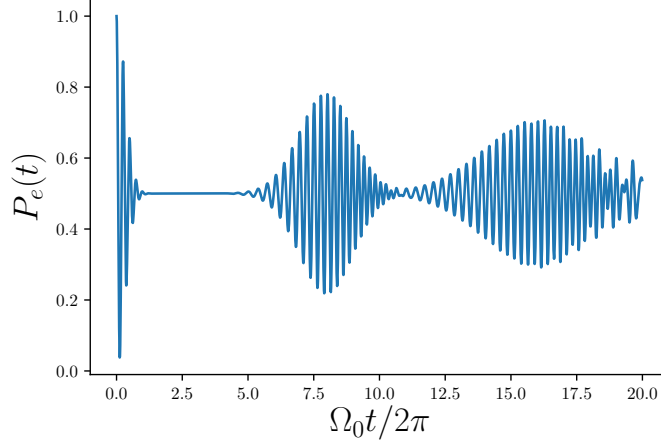


Figure 2.7: Collapses and revivals: Probability  $P_e(t)$  of finding the atom in the state  $|e\rangle$  versus time  $t$  in unit of  $2\pi/\Omega_0$ . The cavity is initially a coherent field with mean photon number  $\bar{n} = 15$ .

Quantum revivals are also a pure quantum phenomenon directly related to the discrete energy spectrum of the electromagnetic field and, hence, to the field energy quantization itself. These revivals are not restricted to an initial coherent state field [33].

### 2.7.2 Dispersive regime

In the far detuned regime,  $|\delta| \gg \Omega_n = \Omega_0\sqrt{n+1}$ , the eigenenergies of the dressed states asymptotically approximate to the uncoupled levels (figure. 2.6). Hence, the bare states are good approximations at first order in  $\Omega_n/\delta$  of the dressed states:

$$|+, n\rangle \simeq |e, n\rangle \quad \text{and} \quad |-, n\rangle \simeq -|g, n+1\rangle \quad \text{if} \quad \delta > 0 \quad (2.132)$$

$$|+, n\rangle \simeq |g, n+1\rangle \quad \text{and} \quad |-, n\rangle \simeq |e, n\rangle \quad \text{if} \quad \delta < 0. \quad (2.133)$$

Then, the corresponding energies are modified at second order in  $\Omega_n/\delta$ :

$$E_{e,n} = E_{e,n}^0 + \frac{\hbar\Omega_0^2}{4\delta}(n+1) \quad (2.134)$$

$$E_{g,n+1} = E_{g,n+1}^0 - \frac{\hbar\Omega_0^2}{4\delta}(n+1), \quad (2.135)$$

where the energies with the superscript ‘0’ correspond to the energies of the bare states. The energy shift experienced by the system with respect to the unperturbed bare states is known as *light shift*. These shifts can be described by an effective Hamiltonian with the bare states as eigenstates:

$$H_{JC}^{\text{disp}} = H_c + \frac{\hbar}{2} \left[ \omega_a + \frac{\Omega_0^2}{2\delta} \left( a^\dagger a + \frac{1}{2} \right) \right] \sigma_z. \quad (2.136)$$

The second term of  $H_{JC}^{\text{disp}}$  can be considered either as an energy shift of the field state or of the atomic state. If we focus on the energy shift experienced by the atom, the atomic frequency is displaced by

$$\Delta\omega_a(n) = \frac{\Omega_0^2}{2\delta} \left( n + \frac{1}{2} \right), \quad (2.137)$$

with  $n$  being the photon number of the field, and the second contribution  $\Omega_0^2/2\delta$  is the *Lamb shift* due to the vacuum fluctuations in the cavity mode. Then, after an interaction time  $t_i$  of an atom with a field containing  $n$  photons, the atom will accumulate a relative phase shift between the states  $|e\rangle$  and  $|g\rangle$ ,

$$\Delta\omega_a(n)t_i = \varphi_0 \left( n + \frac{1}{2} \right) \equiv \varphi(n) + \frac{\varphi_0}{2}, \quad (2.138)$$

with

$$\varphi_0 = \frac{\Omega_0^2 t_i}{2\delta} \quad (2.139)$$

being the *phase shift per photon*. This phase shift can be experimentally tuned by adjusting the atom-field frequency detuning or the interaction time  $t_i$ . The Lamb shift  $\varphi_0/2$ , which is independent of the photon number can be compensated by redefining a phase origin for the atomic coherence.

If we absorb the Lamb shift into  $\varphi(n)$ , then an initial atomic state  $(|e\rangle + |g\rangle)/\sqrt{2}$  will

end up in a state

$$|+\rangle_n = \frac{1}{\sqrt{2}} \left( |e\rangle + e^{i\varphi(n)} |g\rangle \right), \quad (2.140)$$

after interacting with the cavity field during a time  $t_i$ . Figure 2.8 shows the Bloch sphere representation of the state  $|+\rangle_n$ , where the different directions in the equatorial plane depend on the photon number.

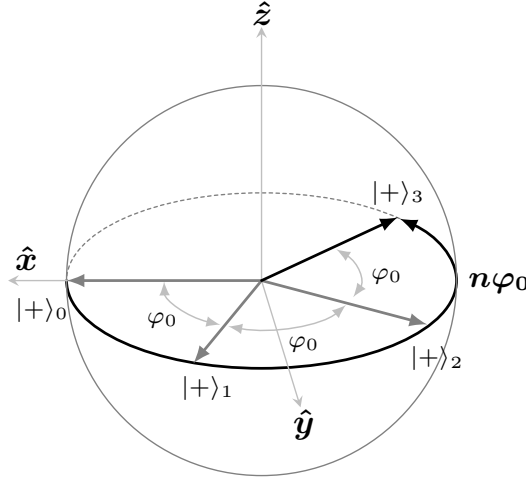


Figure 2.8: Evolution of the Bloch vector after a dispersive interaction with the cavity field in a Fock state  $|n\rangle$ . The global phase shift  $\varphi_0/2$  is not shown.

As we can see, the phase shift accumulated by the atomic state carries information about the number of photons inside the cavity. Combining this effect with the Ramsey interferometry technique explained in section 2.6.3, it is possible to realize Quantum Non-demolition measurement (QND) of the photon number which are going to be introduced in the next chapter [2–4].

## 2.8 General theory of measurement

### 2.8.1 Projective measurement

Consider a system  $S$  in a pure state represented by a vector  $|\Psi\rangle$  in the Hilbert space  $\mathcal{H}$ . The measured observable is described by a Hermitian operator  $O$  in  $\mathcal{H}$  with eigenvalues

$o_i$ , which are the possible measurement results. The observable  $O$  can be expressed as

$$O = \sum_i o_i P_i,$$

with  $P_i$  being the projective operator onto the eigenspace with eigenvalue  $o_i$ , and satisfies

$$P_i^\dagger = P_i = P_i^2 \qquad \sum_i P_i = \mathbb{1}.$$

The postulates in quantum mechanics tell that

1. The measurement results  $o_i$  are random with a probability given by

$$\pi_i = \langle \Psi | P_i | \Psi \rangle.$$

2. After obtaining the result  $o_i$ , the state is projected in

$$|\Psi\rangle \rightarrow \frac{P_i |\Psi\rangle}{\sqrt{\pi_i}}.$$

Those postulates can be generalized when the state of the system  $S$  is an statistical mixture described by the density matrix  $\rho$  to:

1. The measurement results  $o_i$  are random with a probability given by

$$\pi_i = \text{Tr}(\rho P_i). \tag{2.141}$$

2. After obtaining the result  $o_i$ , the state is projected in

$$\rho \rightarrow \rho_i = \frac{P_i \rho P_i}{\pi_i}. \tag{2.142}$$

If the measurement is unread, the state becomes

$$\rho \rightarrow \sum_i \pi_i \rho_i = \sum_i P_i \rho P_i. \tag{2.143}$$

These unread measurements preserve the diagonal terms of the density operator (spanned by the basis of eigenstates of the observable  $O$ ). The off diagonal elements are suppressed.

Now let us consider a more experimental situation applying the postulates mentioned earlier. Consider a system  $S$  interacting with a meter system  $A$  which performs a measurement on  $S$ . This measurement is possible due to the entanglement produced by a proper unitary operation  $U_M$  acting on both systems. Suppose that  $O_A$  is an observable of  $A$  with eigenvalues  $o_i^A$  and eigenstates  $|u_i^A\rangle$ , which are orthonormal. The unitary evolution  $U_M$  acting on the initial state  $|0^A\rangle$  of the meter and the state  $|\Psi^S\rangle$  of  $S$  leads to

$$\begin{aligned} U_M |\Psi^S\rangle \otimes |0^A\rangle &= \sum_i \langle u_i^A | U_M |0^A\rangle |\Psi^S\rangle \otimes |u_i^A\rangle \\ &= \sum_i M_i |\Psi^S\rangle \otimes |u_i^A\rangle. \end{aligned} \quad (2.144)$$

The unitary operator  $U_M$  transforms the initial *system-meter* state to a superposition of the eigenstates  $|u_i^A\rangle$  of  $A$  and a transformation described by the operators  $M_i$  to  $|\Psi^S\rangle$ .

If we perform a projective measurement of the meter  $A$  and obtain the outcome  $o_i^A$ , the state of  $A$  is then projected onto  $|u_i^A\rangle$ . Also, the state of  $S$  is transformed by

$$|\Psi^S\rangle \rightarrow \frac{M_i |\Psi^S\rangle}{\sqrt{\pi_i}}, \quad (2.145)$$

with

$$\pi_i = \langle \Psi^S | M_i^\dagger \hat{M}_i | \Psi^S \rangle \quad (2.146)$$

being the probability of obtaining the outcome result  $o_i^A$ .

Generally, if the system  $S$  is in a statistical mixture  $\rho^S = \sum_i p_i |\Psi_i^S\rangle \langle \Psi_i^S|$ , the unitary evolution of the coupled system is described by

$$U_M (\rho^S \otimes |0^A\rangle \langle 0^A|) U_M^\dagger = \sum_{i,j} M_i \rho^S M_j^\dagger \otimes |u_i^A\rangle \langle u_j^A|. \quad (2.147)$$

As before, if a projective measurement is performed on  $A$  obtaining the outcome  $o_i^A$ . Then,

the state of  $S$  is transformed to

$$\rho^S \rightarrow \frac{M_i \rho^S M_i^\dagger}{\pi_i}, \quad (2.148)$$

with

$$\pi_i = \text{Tr} \left( M_i \rho^S M_i^\dagger \right) \quad (2.149)$$

being the probability of obtaining  $o_i^A$ .

By comparing these results with the postulates given in (2.141) and (2.142), we see that the set of operator  $M_i$  also describes a measurement, this type of measurement is called *generalized measurement*.

### 2.8.2 Generalized measurement

This type of measurement does not require the operators  $M_i$  to be Hermitian. However, since the probabilities of all the possible outcomes must add up to one for any  $\rho^S$ , i.e.:

$$\sum_i \text{Tr} \left( \rho^S M_i^\dagger M_i \right) = 1 \quad \forall \rho^S, \quad (2.150)$$

they must satisfy

$$\sum_i M_i^\dagger M_i = \mathbb{1}. \quad (2.151)$$

If an unread measurement occurs, the density matrix of the system is given by

$$\rho^S \rightarrow \sum_i M_i \rho^S M_i^\dagger \quad (2.152)$$

Equation (2.152) is a linear process transforming a density matrix to another. In general, this type of transformation is known as a *quantum map* which is described by a *superoperator*. Moreover, any quantum map can be expressed as a finite sum of operators products called *Kraus sum representation* [33]. The expression in (2.152) is already in its Kraus representation, with the operators  $M_i$  being the *Kraus operators*.

Generalized measurements are widely used to formulate measurements in practical situations. In this work, they are useful for the formulation of QND measurements of the

photon number using a Ramsey interferometer, and thus we can apply them for generating quantum states of the field by a conditional measurement outcome which we call an atomic postselection.

## 2.9 Decoherence: Coupling with the environment

So far we have described the essential blocks of cavity electrodynamics experiments, which are the atoms, the quantized electromagnetic field, and their interaction. However, we have to add another ingredient to our description for an appropriate simulation of practical situations. In these situations, quantum systems are unavoidable coupled to the environment and suffer from decoherence effects which means that quantum states could have very short lifetimes due to the leakage of coherence towards the environment. In the following section, we describe the Lindblad master equation of the atom-field system, which allows us to model the decoherence process affecting our scheme for a more reliable simulation of an experimental situation.

### 2.9.1 Lindblad master equation

The Lindblad master equation will be thoroughly used in the next chapters to treat the realistic generation of quantum states of the field analyzed in this work.

As a general picture, if we consider the coupling between a system  $S$  and an environment  $E$ , the exact treatment is very complex because  $E$  is a large system with many degrees of freedom. However, when the environment  $E$  is a large system compared with the system  $S$ , it is possible to consider the state of the environment as a steady state which is not affected by its interaction with the system, and also uncorrelated with  $S$ . Precisely, this is possible when the environment  $E$  is a large system with a short memory time  $\tau_c$  compared to  $T_r$ , which is the characteristic time of evolution of  $S$ . Hence, if  $\tau_c \ll T_r$ , we can neglect the initial fluctuations of the environment and the system-environment correlations in an approximation known as *Markov approximation*. Then, if we choose a time slicing  $\tau$  satisfying  $\tau_c \ll \tau \ll T_r$ , we can obtain a coarse-grained description of the system  $S$  evolution in an equation known as *Lindblad master equation*. For further readings and

details see [33, 43].

The form of the master equation describing the evolution of the system affected by an environment is

$$\frac{d\rho_S}{dt} = \frac{1}{i\hbar} [H, \rho_S] + \sum_i \left[ L_i \rho_S L_i^\dagger - \frac{1}{2} (L_i^\dagger L_i \rho_S + \rho_S L_i^\dagger L_i) \right]. \quad (2.153)$$

where  $H$  is the total Hamiltonian of the system and the  $L_i$  Lindblad operators describe the decoherence process. In the case of an atom-field system in the presence of a thermal environment at a finite temperature  $T$ , the Lindblad operators are  $\sqrt{\Gamma(1+n_t)}\sigma_-$ ,  $\sqrt{\Gamma n_t}\sigma_+$ ,  $\sqrt{\kappa(1+n_t)}a$  and  $\sqrt{\kappa n_t}a^\dagger$ , where  $\Gamma$  is the spontaneous emission rate,  $\kappa$  is the cavity decay rate and  $n_t$  is the average thermal photon number. The first two Lindblad operators describe the emission and absorption thermal processes of the atom, while the other operators describe the photon loss in the environment or the gain of a thermal photon out of the thermal fluctuations of the environment.

Since our work is divided into two parts, we treat the master equation with different assumptions in each part. First, we consider a scheme for preparing an optical qubit and other Fock states superpositions in a typical microwave cavity QED system where the relaxation of the atoms is negligible when compared to the cavity damping time  $T_c = 1/\kappa$ . Therefore we will see that the dominant Lindblad operators are  $\sqrt{\kappa(1+n_t)}a$  and  $\sqrt{\kappa n_t}a^\dagger$  acting on the field in the cavity as a source of losses. In the second part, we apply the same model of the first part for cool-down the cavity field to its ground state, starting from a thermal distribution which is thermalized with the environment at a higher temperature compared with the first part of this work. Hence, the lifetime reduction of the atom is reduced by a factor  $(1+n_t)$  which can be relevant in this case, so we consider more appropriate to include the atomic and photon losses in this situation.

In the next chapter, we will present the scheme used in this thesis work for preparing an optical qubit, other Fock state superpositions and cooling-down the photons inside a cavity. The scheme is typically used for QND measurements of the photon number by Ramsey interferometry.

## Chapter 3

# The model: The Cavity QED Ramsey interferometer setup

In the previous chapters, we explained the basic elements of cavity QED studies. In this chapter, we will present a typical cavity QED setup which is based on Ramsey interferometry for measuring the phase of the atomic state and enables to perform QND measurements of the photon number of the cavity field. Moreover, this setup also enables us to generate specific quantum states of the cavity field by tuning the interaction parameters between the atoms and the electromagnetic field inside the cavity in addition to of the number of atoms measured and finally postselecting the atomic states that we measured. In what follows, we show the general equations for generating quantum states of light in this setup.

### 3.1 The model

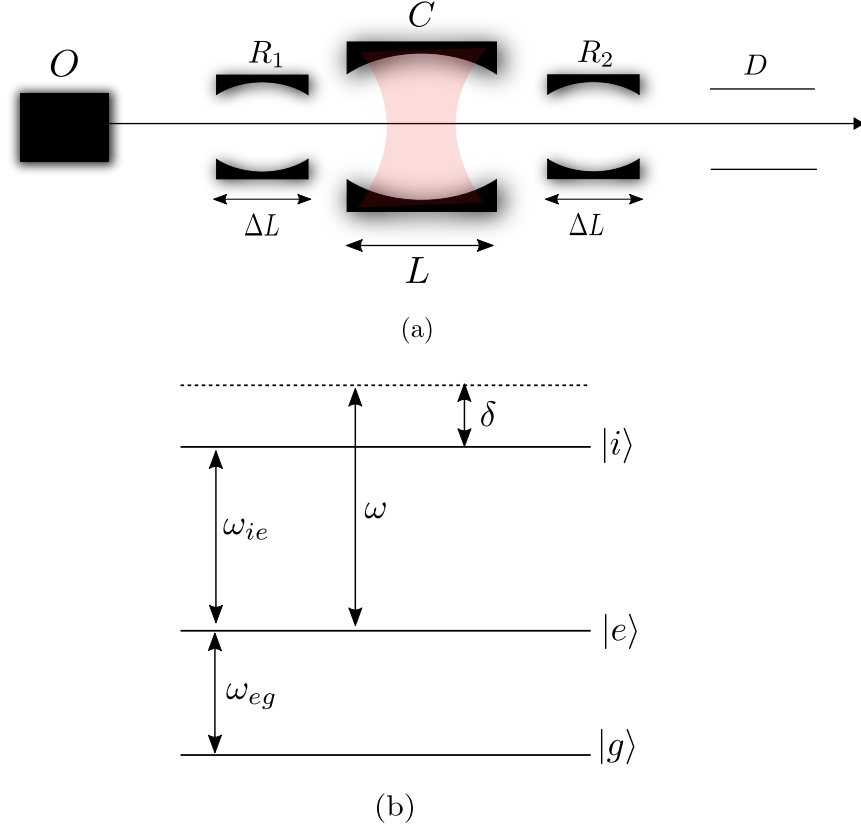


Figure 3.1: (a) Cavity QED-setup used for Ramsey interferometry. The field is initially prepared in the high-Q cavity  $C$ . The atoms are prepared and velocity-selected in the box  $O$ . Then, each atom interacts with three cavities:  $R_1$ ,  $C$  and  $R_2$ . In each of the zones  $R_1$  and  $R_2$ , the atom interacts with a classical microwave field. This interaction makes it possible to manipulate the atomic state before and after the interaction with  $C$ . Finally, after passing zone  $R_2$ , the atom is detected in the state  $|e\rangle$  or  $|g\rangle$  by the field ionization counter  $D$ . (b) Three-level atomic system for the experiment in the dispersive atom-field coupling. Here,  $\omega$  is the frequency of the field in cavity  $C$  which has a large detuning  $\delta$  from the atomic transition frequency  $\omega_{ie}$ .

As we mentioned in chapter 2, in the dispersive regime the effective interaction between the atoms and the field produces an energy shift to the atomic state. This energy shift leads to a phase shift on the atomic state which can be measured by a Ramsey interferometer setup as shown in figure 3.1 a [2]. In zone  $C$ , between the classical microwave zones  $R_1$  and  $R_2$  used for Ramsey interferometry, we have a superconducting cavity with an initial field

given by  $\rho_c(0) = \sum_{nn'} \rho_{nn'} |n\rangle\langle n'|$  written as an expansion using a Fock basis. A sequence of  $N$  three-level atoms, with the energy diagram shown in figure 3.1 b, is preselected in the state  $\rho_a(0) = \bigotimes_{k=1}^N |e_k\rangle\langle e_k|$  in box  $O$  and injected into the setup with a controlled velocity that allows us to assume that there is only one atom flying in the setup at a given time. Then, the initial state of the multipartite system (cavity - atoms) reads as

$$\rho_{ca}(0) = \rho_c(0) \otimes \rho_a(0). \quad (3.1)$$

For simplicity, we are going to consider only one atom flying through the setup and then extend that result for a sequence of  $N$  atoms. Then, we denote the initial atom-field state as  $\rho_{ca} = \rho_c \otimes |e_k\rangle\langle e_k|$ , maintaining the subscript  $k$  for the  $k$ th atom which is going to be useful for our description a sequence of atoms later.

Generally, in cavity QED, the atom-field interaction is described by the Jaynes-Cummings model (equation (2.118))

$$H^{(k)} = \frac{\hbar}{2} \omega_{ie} \sigma_z^{(k)} + \hbar \omega a^\dagger a + \hbar g (a \sigma_+^{(k)} + a^\dagger \sigma_-^{(k)}), \quad (3.2)$$

where  $a$  ( $a^\dagger$ ) is the cavity photon annihilation (creation) operator, whereas for the  $k$ th atom the operators are  $\sigma_-^{(k)} = |e_k\rangle\langle i_k|$ ,  $\sigma_+^{(k)} = |i_k\rangle\langle e_k|$ ,  $\sigma_z^{(k)} = |i_k\rangle\langle i_k| - |e_k\rangle\langle e_k|$  and  $g = \Omega_0/2$  corresponds to the atom-field coupling constant, taken to be equal for all the atoms. From the atomic operators it can be seen that only levels  $|i_k\rangle$  and  $|e_k\rangle$  are affected by the atom-field interaction, whereas level  $|g_k\rangle$  is not involved in the dynamics. Particularly, in our scheme, we are interested in considering a nonresonant interaction by taking a large frequency detuning  $\delta = \omega_{ie} - \omega \gg g\sqrt{n}$  between the cavity field frequency  $\omega$  and the atomic transition frequency  $\omega_{ie}$ . Therefore, following the analysis of section 2.7.2 but restricting to the atomic Hamiltonian spanned by the states  $|e_k\rangle$  and  $|g_k\rangle$ , the effective interaction becomes a dispersive coupling (equation (2.136)) given by

$$V^{(k)} = -\frac{\hbar g^2}{\delta} a^\dagger a |e_k\rangle\langle e_k|. \quad (3.3)$$

For obtaining this result, we have performed a unitary transformation of

$U = \exp \left[ -i\omega_c t(a^\dagger a + 1/2) - i\omega_a t\sigma_z^{(k)}/2 \right]$  on equation (2.136), neglected the Lamb shift contribution and the projector  $|i_k\rangle\langle i_k|$  in  $\sigma_z^{(k)}$  because we are focusing only on the states  $|e_k\rangle$  and  $|g_k\rangle$ .

Then, after an interaction time  $\tau_k = L/v_k$ , the evolution operator reads as

$$U_I^{(k)} = \exp(-iV^{(k)}\tau_k/\hbar) = \exp(i\varphi_k a^\dagger a |e_k\rangle\langle e_k|). \quad (3.4)$$

In equation (3.4),  $L$  is the length of the cavity  $C$ ,  $v_k$  is the velocity of the  $k$ th atom passing through the cavity and  $\varphi_k = g^2\tau_k/\delta$  is the one photon phase shift, which characterizes the coupling strength between the  $k$ th atom and the cavity field. This interaction causes a dispersive phase shift to the  $|e_k\rangle$  level which is proportional to the photon number.

On the other hand, in each of the  $R_1$  and  $R_2$  zones, the  $k$ th atom interacts with a classical microwave field tuned at a frequency  $\nu_r$ , resonant with the atomic transition frequency  $\omega_{eg}$ . This interaction leads to a superposition of the  $|e_k\rangle$  and  $|g_k\rangle$  levels of the atomic state [34]. After an interaction time  $\Delta\tau_k = \Delta L/v_k$ , which satisfies  $\Omega_R\Delta\tau_k = \pi/2$ , the atom undergoes a  $U_{\pi/2}^{(k)}$  transformation given by

$$U_{\pi/2}^{(k)} = \frac{1}{\sqrt{2}}(|e_k\rangle\langle e_k| + |g_k\rangle\langle g_k| + i|e_k\rangle\langle g_k| + i|g_k\rangle\langle e_k|), \quad (3.5)$$

where  $\Delta L$  is the length of the zones  $R_1$  and  $R_2$ ,  $v_k$  is the velocity of the atom and  $\Omega_R$  is the Rabi frequency.

Finally, in the field ionization counter  $D$  we perform a projective measurement of the atom represented by the operators:

$$P_m^{(k)} = |m_k\rangle\langle m_k|, \quad m = e, g. \quad (3.6)$$

Given the outcome  $m$ , the complete operation made by one atom crossing the setup is described by

$$U_m^{(k)} = P_m^{(k)} U_{\pi/2}^{(k)} U_I^{(k)} U_{\pi/2}^{(k)}. \quad (3.7)$$

After this operation, the state evolves as

$$\rho_{ac,m} = U_m^{(k)} \rho_{ac} U_m^{\dagger(k)}. \quad (3.8)$$

Since we are interested in the state of the cavity field, we trace  $\rho_{ac,m}$  over the atomic states and normalize, resulting in the field of the system after the operation (3.7)

$$\rho_{c,m} = \frac{\text{Tr}_a \left( U_m^{(k)} \rho_{ac} U_m^{\dagger(k)} \right)}{\text{Tr} \left( U_m^{(k)} \rho_{ac} U_m^{\dagger(k)} \right)} = \frac{M_m^{(k)} \rho_c M_m^{\dagger(k)}}{\text{Tr}_c \left( M_m^{(k)} \rho_c M_m^{\dagger(k)} \right)}, \quad (3.9)$$

with

$$M_e^{(k)} = \langle e_k | U_{\pi/2}^{(k)} U_I^{(k)} U_{\pi/2}^{(k)} | e_k \rangle = \frac{1}{2} \left[ \exp(i\varphi_k a^\dagger a) - 1 \right] \quad (3.10)$$

$$M_g^{(k)} = \langle g_k | U_{\pi/2}^{(k)} U_I^{(k)} U_{\pi/2}^{(k)} | e_k \rangle = \frac{1}{2} \left[ \exp(i\varphi_k a^\dagger a) + 1 \right]. \quad (3.11)$$

These operators can be written in the Fock basis as

$$M_e^{(k)} = \sum_n \frac{[\exp(i\varphi_k n) - 1]}{2} |n\rangle \langle n|, \quad M_g^{(k)} = \sum_n \frac{[\exp(i\varphi_k n) + 1]}{2} |n\rangle \langle n| \quad (3.12)$$

Also, these operators satisfy

$$\sum_{i=e,g} M_i^{\dagger(k)} M_i^{(k)} = \mathbb{1}. \quad (3.13)$$

Therefore, equation (3.13) is a generalized measurement of the cavity field.

Now, if we consider the sequence of  $N$  atoms interacting with the same  $\varphi_k$  with the cavity, the measurement will result in  $N_e$  atoms measured in the  $|e\rangle$  state and  $(N - N_e)$  atoms measured in the  $|g\rangle$  state, generating a certain cavity field state regardless of the order of the atoms. This situation is represented by the cavity operator

$$M_{N_e} = \sqrt{C_N^{N_e}} M_e^{N_e} M_f^{N-N_e}, \quad (3.14)$$

where  $C_N^{N_e} = \frac{N!}{N_e!(N-N_e)!}$  is the number of combinations of having  $N_e$  atoms on the  $|e\rangle$  level of a set of  $N$  atoms.

As previously described in equation (3.13), these operators satisfy the generalization of the measurement shown in equation (3.15)

$$\sum_{N_e=0}^N M_{N_e}^\dagger M_{N_e} = \mathbb{1}. \quad (3.15)$$

Generalizing equation (3.9) to  $N$  atoms gives the following result:

$$\rho_{c,N_e} = \frac{M_{N_e} \rho_c M_{N_e}^\dagger}{\text{Tr}_c \left( M_{N_e} \rho_c M_{N_e}^\dagger \right)}. \quad (3.16)$$

The result of equation (3.16) can be derived by the action of the total evolution operator which is given by

$$U = U^{(N)} \dots U^{(1)}, \quad (3.17)$$

being  $U^{(k)} = U_{\pi/2}^{(k)} U_I^{(k)} U_{\pi/2}^{(k)}$  the evolution of the  $k$ th atom passing through the cavities ( $R_1$ ,  $C$  and  $R_2$ ).

After the interaction of the  $N$  atoms with the cavities, the state of the whole system evolves to

$$\tilde{\rho}_{ca} = U \rho_{ca} U^\dagger \quad (3.18)$$

$$= \sum_{nn'} \rho_{nn'} |\psi_n^{(1)}\rangle \langle \psi_{n'}^{(1)}| \otimes \dots \otimes |\psi_n^{(N)}\rangle \langle \psi_{n'}^{(N)}| \otimes |n\rangle \langle n'|, \quad (3.19)$$

with  $U^{(k)}|e_k\rangle = |\psi_n^{(k)}\rangle = \frac{1}{2} (e^{-i\varphi_k n}|e_k\rangle + ie^{-i\varphi_k n}|g_k\rangle + i|g_k\rangle - |e_k\rangle)$ . For simplicity, we assume the same coupling  $\varphi$  for all the atoms.

Subsequently, we perform the postselection of a symmetric state of the atomic levels on the  $\{|m_{(1)}, \dots, m_{(N)}\rangle\}$  basis, with  $m = \{e, g\}$

$$\begin{aligned} |\phi_{post}\rangle &= |e_1, \dots, e_{N_e}, g_{N_e+1}, \dots, g_N; S\rangle \\ &= \left( \frac{N_e!(N - N_e)!}{N!} \right)^{1/2} \sum_p |m_1, \dots, m_N\rangle. \end{aligned} \quad (3.20)$$

Here,  $S$  stands for a symmetric state. Therefore, the sum is taken over all the possible

combinations of  $N_e$  atoms on the  $|e\rangle$  level and  $N - N_e$  on the  $|g\rangle$  level.

Hence, the normalized state of the cavity after postselection is

$$\rho_c = C_N^{N_e} \sum_{nn'} \rho_{nn'} e^{\frac{i}{2} \varphi N(n'-n)} c_n^{N-N_e} c_{n'}^{N-N_e} d_n^{N_e} d_{n'}^{N_e} |n\rangle \langle n'|, \quad (3.21)$$

where the coefficients are  $c_n = \cos(\varphi n/2)$  and  $d_n = \sin(\varphi n/2)$ . Finally, because our work relies on a postselection process the desired field state is generated with a success probability given by

$$P_{post} = C_N^{N_e} \sum_n \rho_{nn} c_n^{2(N-N_e)} d_n^{2N_e}. \quad (3.22)$$

Following the same idea as before, we now consider the whole sequence of  $N$  atoms divided into  $K$  groups, each group with a total of  $N_k$  atoms interacting with the same  $\varphi_k$ . For each group denoted by the subscript  $k$ , the order of the measurements outcomes is not important, only the number of atoms  $N_{e,k}$  detected in  $|e\rangle$ . The equations for the cavity state generated and the postselection probability are basically the same as (3.21) and (3.22) but considering the possible combinations of measurement outcomes for each group of atoms in the sequence

$$\rho_c = \sum_{nn'} \rho_{nn'} \prod_{k=1}^K C_{N_k}^{N_{e,k}} e^{\frac{i}{2} \varphi_k N_k(n'-n)} c_{n,k}^{N_k-N_{e,k}} c_{n',k}^{N_k-N_{e,k}} d_{n,k}^{N_{e,k}} d_{n',k}^{N_{e,k}} |n\rangle \langle n'| \quad (3.23)$$

$$P_{post} = \sum_n \rho_{nn} \prod_{k=1}^K C_{N_k}^{N_{e,k}} c_{n,k}^{2(N_k-N_{e,k})} d_{n,k}^{2N_{e,k}}. \quad (3.24)$$

In what follows, we show that it is possible to generate an optical qubit and cooling down the cavity to the vacuum state with an appropriate atomic postselection.

## Chapter 4

# Results and discussion

In this chapter, we will show our results on generating specific quantum cavity states such as qubit states, Fock states superpositions and the vacuum state (zero photon state). We present the successful probability and fidelity of our protocol. Finally, we study a more realistic scenario taking into account the interaction of the system with the environment (cavity losses) and imperfect measurements of the atoms.

## 4.1 Preparing an optical qubit in dispersive cavity-QED

In this section, our task is to prepare a superposition of the vacuum and the one-photon states. First, we assume that the initial state of the field is a coherent state  $|\alpha\rangle = \sum_n b_n |n\rangle$ , where  $b_n = \alpha^n e^{-|\alpha|^2/2} / \sqrt{n!}$  and  $\alpha$  being a real value. Thus, equation (3.21) reduces straightforwardly for the case of an initial pure state for the cavity field, and for  $N_e = 0$  (because for  $N_e \neq 0$  the ket  $|0\rangle$  is eliminated). The state of the field after the postselection of  $N$  atoms in the  $|g\rangle$  level is

$$|\psi_f\rangle = \frac{\sum_{n=0}^{\infty} b_n e^{-\frac{iN\varphi}{2}} \cos^N\left(\frac{\varphi n}{2}\right) |n\rangle}{\left[\sum_{n=0}^{\infty} |b_n|^2 \cos^{2N}\left(\frac{\varphi n}{2}\right)\right]^{1/2}}, \quad (4.1)$$

with postselection probability

$$P_{post} = \sum_{n=0}^{\infty} |b_n|^2 \cos^{2N}\left(\frac{\varphi n}{2}\right). \quad (4.2)$$

As we can see from the numerator of equation (4.19), the parameters  $\alpha$ ,  $\varphi$  and  $N$  have to be adequate to ensure that only kets  $|0\rangle$  and  $|1\rangle$  survive.

We assume a target state of the form:

$$|\psi_t\rangle = \frac{|0\rangle + \alpha e^{-\frac{iN\varphi}{2}} \cos^N\left(\frac{\varphi}{2}\right) |1\rangle}{\sqrt{1 + \alpha^2 \cos^{2N}\left(\frac{\varphi}{2}\right)}}, \quad (4.3)$$

which can be written simply as  $|\psi_t\rangle = \frac{|0\rangle + \beta |1\rangle}{\sqrt{1 + |\beta|^2}}$ , with  $\beta = \alpha e^{-\frac{i\varphi N}{2}} \cos^N\left(\frac{\varphi}{2}\right)$ .

In order to estimate how far our final state  $|\psi_f\rangle$  is from the target state  $|\psi_t\rangle$ , we define a fidelity  $F = |\langle\psi_t|\psi_f\rangle|^2$  [37]. Using the equations (4.19) and (4.3), the fidelity reads as

$$F = \frac{1 + \alpha^2 \cos^{2N}\left(\frac{\varphi}{2}\right)}{\sum_{n=0}^{\infty} \frac{\alpha^{2n}}{n!} \cos^{2N}\left(\frac{\varphi n}{2}\right)}. \quad (4.4)$$

It can be easily seen from the above equation that a combination of  $\alpha$ ,  $\varphi$  and  $N$  can lead to a specific optical qubit. Particularly, if we want to prepare an equiprobable

superposition, e.g,  $|\langle 0|\psi_t\rangle|^2 = |\langle 1|\psi_t\rangle|^2 = 1/2$ , we require

$$\alpha \cos^N \left( \frac{\varphi}{2} \right) = 1. \quad (4.5)$$

Solving this condition for the variable  $\varphi$  gives

$$\varphi(N) = 2 \arccos \left( \frac{1}{\sqrt[N]{\alpha}} \right), \quad (4.6)$$

where  $\alpha$  is given. Replacing this condition in the expression of the fidelity in equation (4.4), we finally obtain a fidelity depending only on the number of atoms postselected in the  $|g\rangle$  level given by

$$F(N) = \frac{2}{2 + \sum_{n=2}^{\infty} \frac{\alpha^{2n}}{n!} \cos^{2N} \left( \frac{\varphi(N)n}{2} \right)}. \quad (4.7)$$

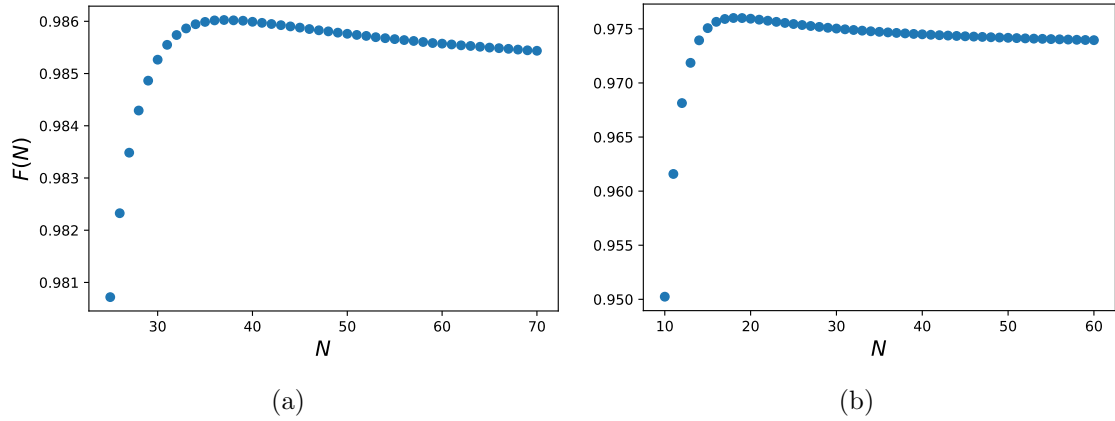


Figure 4.1: Fidelity given by equation (4.7) for different values of  $\alpha^2$ . Here, (a)  $\alpha^2 = 4.0$  and (b)  $\alpha^2 = 3.0$ . Fidelity is used to quantify the closeness between the final state in equation (4.19) and the target state given by equation (4.3).

In the above equation, the number of atoms ( $N$ ) has to be larger as  $\alpha$  grows in order to maximize this fidelity. We plot the expression from equation (4.7) for different values of  $\alpha^2$  in figure 4.1. As we can see, for each  $\alpha^2$  there is a number of atoms and a coupling value given by condition (4.6) for which the fidelity is optimal (close to 1.0). In figure 4.2a we determine the optimal fidelity  $F_{opt}$  for each value of  $\alpha^2$  and we also plot in figure

4.2b the postselection probability given by equation (4.20) for the optimal parameters. We found that for a range of  $3.0 < \alpha^2 < 5.0$ , the optimal qubit is generated with a fidelity and a postselection probability of  $0.976 < F_{opt} < 0.99$  and  $10.2\% > P_{post} > 1.36\%$ . In figure

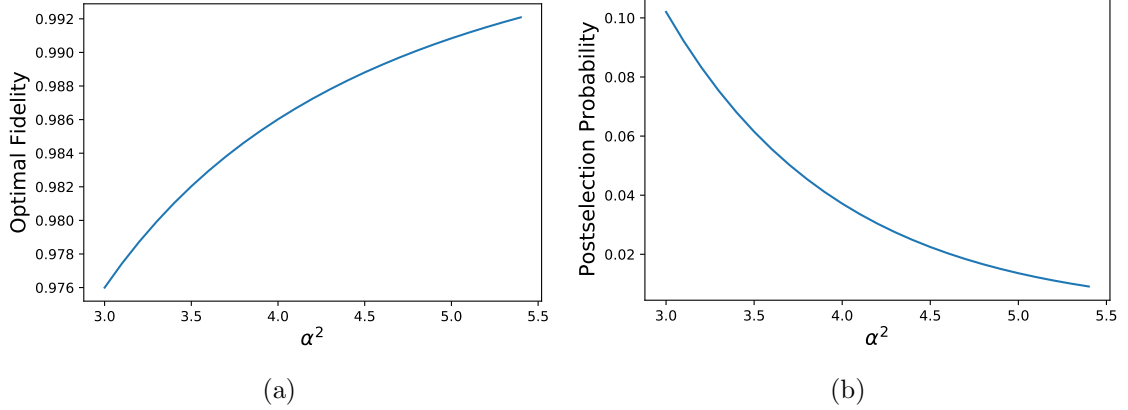


Figure 4.2: (a) Optimal fidelity from equation (4.7) versus  $\alpha^2$ . (b) Postselection probability for a set of parameters for which the fidelity is maximum (close to 1.0).

4.3a, we show the probability distribution ( $\Pr(n) = |\langle n | \psi_f \rangle|^2$ ) of the final state given by equation (4.19)

$$Pr(n) = \frac{|b_n|^2 \cos^{2N} \left( \frac{\varphi n}{2} \right)}{\sum_{n=0}^{\infty} |b_n|^2 \cos^{2N} \left( \frac{\varphi n}{2} \right)}. \quad (4.8)$$

We consider a set of parameters  $\alpha$ ,  $N$  and  $\varphi$  that satisfy the condition (4.5) for an equiprobable superposition ( $\Pr(n=0) = \Pr(n=1) = 1/2$ ) and for which the fidelity is maximum. However, the Hilbert space is not properly truncated up to just one photon, having  $\Pr(n=2) \approx 1.38\%$  in the case shown in figure 4.3a. To evidence the quantumness of the state, in figure 4.3b, we have numerically computed the Wigner quasi-probability distribution defined as  $W(x, p) = \frac{1}{\pi} \int_{-\infty}^{\infty} \langle x + x' | \psi_f \rangle \langle \psi_f | x - x' \rangle e^{-2ipx'} dx'$  [38]. We observe that the true quantum nature arises as a consequence of the considerable negative part of  $W(x, p)$ . To improve our results, we consider three groups of atoms crossing the cavities with three different couplings ( $\varphi_1$ ,  $\varphi_2$  and  $\varphi_3$ ) between the atoms and the cavity field. All atoms are postselected in the ground state  $|g\rangle$ , giving a final state

$$|\psi_f\rangle = \frac{\sum_{n=0}^{\infty} b_n e^{-\frac{in}{2}(\varphi_1 N_1 + \varphi_2 N_2 + \varphi_3 N_3)} \cos^{N_1} \left( \frac{\varphi_1 n}{2} \right) \cos^{N_2} \left( \frac{\varphi_2 n}{2} \right) \cos^{N_3} \left( \frac{\varphi_3 n}{2} \right) |n\rangle}{\left[ \sum_{n=0}^{\infty} |b_n|^2 \cos^{2N_1} \left( \frac{\varphi_1 n}{2} \right) \cos^{2N_2} \left( \frac{\varphi_2 n}{2} \right) \cos^{2N_3} \left( \frac{\varphi_3 n}{2} \right) \right]^{1/2}} \quad (4.9)$$

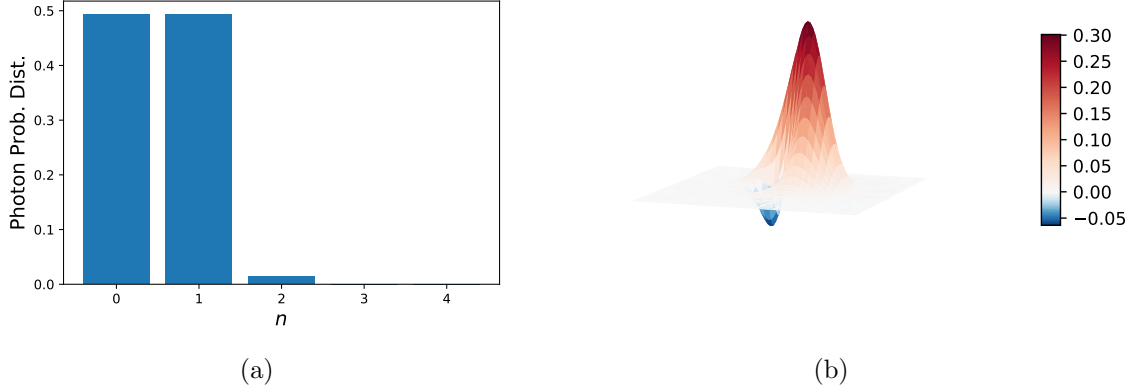


Figure 4.3: Generation of an optical qubit with  $\alpha^2 = 4.0$ ,  $N = 37$  and  $\varphi \approx 0.386$ . (a) Photon probability distribution as in equation (4.8). In (b) we show the Wigner function for the case described in (a). This optical qubit is generated with an optimal fidelity of  $F_{opt} \approx 0.986$  and a postselection probability of  $P_{post} \approx 3.72\%$ .

for the field after postselection, with a probability

$$P_{post} = \sum_{n=0}^{\infty} |b_n|^2 \cos^{2N_1} \left( \frac{\varphi_1 n}{2} \right) \cos^{2N_2} \left( \frac{\varphi_2 n}{2} \right) \cos^{2N_3} \left( \frac{\varphi_3 n}{2} \right), \quad (4.10)$$

where  $N_1$ ,  $N_2$  and  $N_3$  are the number of atoms postselected with couplings  $\varphi_1$ ,  $\varphi_2$  and  $\varphi_3$ , respectively. Similarly to equation (4.3), our target state is

$$|\psi_t\rangle = \frac{|0\rangle + \alpha e^{-\frac{i}{2}(N_1\varphi_1 + N_2\varphi_2 + N_3\varphi_3)} \cos^{N_1} \left( \frac{\varphi_1}{2} \right) \cos^{N_2} \left( \frac{\varphi_2}{2} \right) \cos^{N_3} \left( \frac{\varphi_3}{2} \right) |1\rangle}{\sqrt{1 + \alpha^2 \cos^{2N_1} \left( \frac{\varphi_1}{2} \right) \cos^{2N_2} \left( \frac{\varphi_2}{2} \right) \cos^{2N_3} \left( \frac{\varphi_3}{2} \right)}}. \quad (4.11)$$

Next, we consider  $N_1 = 1$ ,  $\varphi_1 = \pi/2$ ,  $N_2 = 1$  and  $\varphi_2 = \pi/3$  in order to kill the  $n = 2$  and  $n = 3$  components. As we can see from equation (4.9) these values,  $\varphi_1$  and  $\varphi_2$ , eliminated the kets  $|2\rangle$  and  $|3\rangle$ , respectively. Using the same method applied before to prepare an optical qubit with an equiprobable superposition, we require

$$\alpha \cos^{N_3} \left( \frac{\varphi_3}{2} \right) = 4/\sqrt{6}. \quad (4.12)$$

Solving this condition for the variable  $\varphi_3$ , we have a fidelity depending on the number

of atoms postselected in the  $|g\rangle$  level given by

$$F(N_3) = \frac{2}{2 + \sum_{n=2}^{\infty} \frac{\alpha^{2n}}{n!} \cos^2\left(\frac{\pi n}{4}\right) \cos^2\left(\frac{\pi n}{6}\right) \cos^{2N_3}\left(\frac{\varphi_3(N_3)n}{2}\right)}. \quad (4.13)$$

As before, the number of atoms ( $N_3$ ) has to increase when  $\alpha$  is larger in order to maximize the fidelity (equation (4.13)). Furthermore, the equation (4.13) is maximized with less atoms than the previous case for a given value of  $\alpha$ . In figure 4.4a we determine the optimal fidelity  $F_{opt}$  for each value of  $\alpha^2$  given that the other parameters satisfy condition (4.12). Also, we plot in figure 4.4b the postselection probability from equation (4.10) for the parameters for which the fidelity is maximum. In this case, we found that in the  $3.0 < \alpha^2 < 5.0$  range, the optical qubit is prepared with a fidelity and a postselection probability of  $0.95 < F_{opt} < 0.999$  and  $10.5\% > P_{post} > 1.35\%$ . Hence, we have improved the fidelity for the generation of the optical qubit keeping almost the same postselection probability. We show in figure 4.5a the photon probability distribution ( $\text{Pr}(n) = |\langle n|\psi_f\rangle|^2$ ) calculated using the final state in equation (4.9). We consider the same initial cavity field with  $\alpha^2 = 4.0$  to show that we have improved the preparation of the equiprobable qubit having a very similar postselection probability to the previous case presented in figure 4.3a. In figure 4.5b, we have displayed the Wigner function to evidence the quantumness of the state presented in figure 4.5a. It is also important to emphasize from the examples depicted in figure 4.3 and figure 4.5, that we require less atoms to prepare the improved qubit, thus imply in a shorter interaction time and less dissipation. The physical process that generates the superposition of the vacuum and one-photon states is the same involved in the reduction of the field into a Fock state in the QND procedure [2, 4, 23]. Essentially, after the continuous detection of the atoms, the field collapses into a coherent superposition of Fock states with amplitudes given by

$b_n e^{-\frac{in}{2}(\varphi_1 N_1 + \varphi_2 N_2 + \varphi_3 N_3)} \cos^{N_1}\left(\frac{\varphi_1 n}{2}\right) \cos^{N_2}\left(\frac{\varphi_2 n}{2}\right) \cos^{N_3}\left(\frac{\varphi_3 n}{2}\right)$  (within a normalization factor). Thus, the photon probability distribution is multiplied by an oscillating function of  $n$ . Consequently, the photon numbers for which this function is close to zero are efficiently decimated. However, in our work the decimation process is not random because we determine the parameters that efficiently decimate all the photon numbers except  $n = 0$

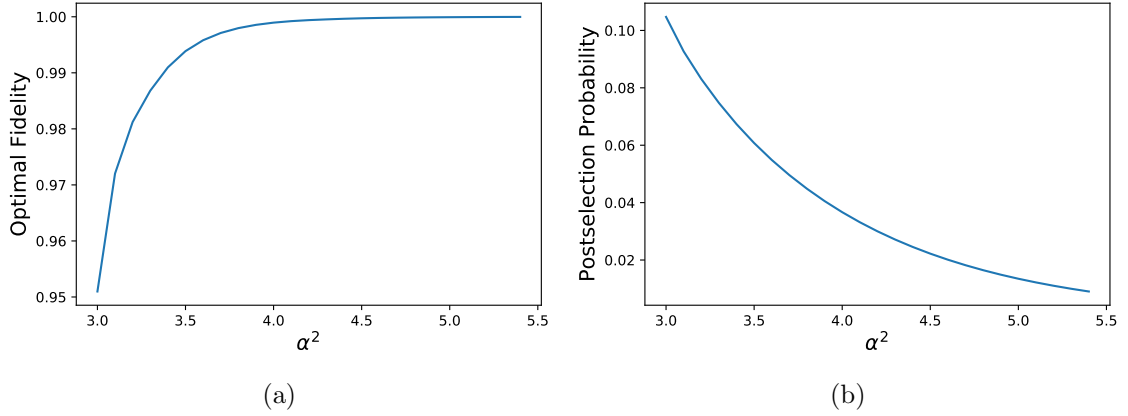


Figure 4.4: (a) Optimal fidelity from equation (4.13) versus  $\alpha^2$ . (b) Postselection probability for a set of parameters  $\alpha$ ,  $N_3$  and  $\varphi_3$  for which the fidelity is maximum.

and  $n = 1$ . Therefore, the most important ingredient is postselection.

#### 4.1.1 A more realistic scenario

The analysis described above is most valid in an idealized experiment. In the present section we describe a more realistic scenario and include the effects of experimental constraints such as cavity losses and imperfect detection of the atoms due to detection efficiency and error detection. Additionally, we present some examples of the generation of superpositions of higher-photon-number Fock states using this scheme.

This experiment can be performed in a typical microwave QED system. Here the atoms sent across the cavity are circular Rydberg atoms. This kind of atoms have a very long lifetime, in the order of tens of milliseconds, comparable to the lifetime of the photon in the superconducting cavity (130 milliseconds) with a fully open structure needed for passing the atoms through. So, we can neglect the atomic decay process during the interaction time between the atoms and the cavity field, and also consider that the atoms fly coherently through the cavities due to the short interaction time ( $\sim 0.4$  milliseconds). All of the parameters of the atomic samples are controllable (velocity, preparation time, interaction time, etc). Therefore, the different couplings  $\varphi_k$  needed in our scheme to prepare the qubit with high fidelity can be realized by controlling individually the velocity of the atoms by laser techniques. Previously, we mentioned that we need only one atom in the setup at a

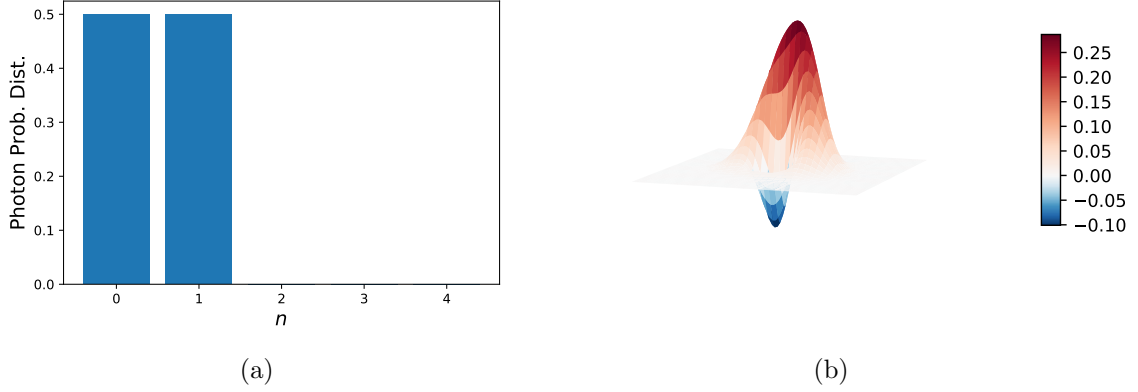


Figure 4.5: Generation of an optical qubit with  $\alpha^2 = 4.0$ ,  $N_3 = 11$  and  $\varphi_3 \approx 0.383$ . (a) Photon probability distribution of the postselected final state in equation (4.9). In (b) we show the Wigner function for the case depicted in (a). This qubit is prepared with an optimal fidelity of  $F_{opt} \approx 0.999$  and a postselection probability of  $P_{post} \approx 3.67\%$ .

given time. However, in real experiments, it is not readily possible to handle a deterministic single-atom preparation. One way to simulate single-atom experiments is preparing the atoms by weak laser excitation, producing a Poissonian statistics for the atoms (with a mean number of atoms per sample much less than one). Then, a postselection process takes place in which we retain only the data corresponding to the desired state. The single-atoms events are obtained with an increase in the time of the data acquisition [33]. Nevertheless, we assume in our study a deterministic single atom preparation and there are some proposals to achieve this preparation of Rydberg atoms making use of the called *dipole blockade* effect [39, 40].

In a typical experiment, the field is stored in a superconducting cavity  $C$  (cavity damping time  $T_c = 65$  ms) cooled down to a temperature  $T = 0.8$  K [41], and its dynamics is described by the master equation for a reservoir at temperature  $T$  [34]

$$\begin{aligned} \frac{d\rho}{dt} = \mathbf{L}\rho = & -\frac{\kappa}{2}(1 + n_{th})(a^\dagger a \rho + \rho a a^\dagger - 2a \rho a^\dagger) \\ & - \frac{\kappa}{2}n_{th}(a a^\dagger \rho + \rho a^\dagger a - 2a^\dagger \rho a), \end{aligned} \quad (4.14)$$

where  $\kappa = 1/T_{cav}$  is the cavity decay rate and  $n_{th} = 0.05$  is the equilibrium thermal photon

number. The atoms are sent at a  $T_a = 82 \mu\text{s}$  time interval [24]. Within the approximation of small time interval,  $T_a \ll T_c$ , we can describe the evolution of the field due to the cavity field relaxation during the time interval  $T_a$  between two atoms by the action of the superoperator  $\mathbf{T}$  [9]:

$$\mathbf{T}\rho = (\mathbb{1} + T_a \mathbf{L})\rho. \quad (4.15)$$

As we mentioned before the atomic detection is not perfect. The detector  $D$  has a finite detection efficiency  $\eta_d$  (probability of detecting an atom). Moreover, the limited state resolution of the Ramsey interferometer introduces a detection error probability of  $\eta_f$ . In our calculations we use  $\eta_d = 0.87$  and  $\eta_f = 0.05$ .

Because of nonideal detection efficiency and nonzero effective detection errors, a measurement outcome  $m' = e$  or  $g$  corresponds to a statistical mixture of different ideal measurement outcomes  $m$ . The conditional probabilities  $P(m'|m)$  and the ideal detection operators  $M_{m'}$  are given in [42]. We now give the explicit expression of a superoperator  $\mathbf{P}_{m'}$  acting on  $\rho$  describing the imperfect detection of an atom

$$\mathbf{P}_{m'}\rho = \frac{\sum_m P(m'|m) M_{m'} \rho M_{m'}^\dagger}{\text{Tr} \left( \sum_m P(m'|m) M_{m'} \rho M_{m'}^\dagger \right)}. \quad (4.16)$$

In our study to generate the optical qubit, we selected the measurement outcome  $m' = g$  using equation (4.16) the detection of this outcome is

$$\mathbf{P}_g\rho = \frac{\eta_d(1 - \eta_f)M_g\rho M_g^\dagger + \eta_d\eta_f M_e\rho M_e^\dagger}{\eta_d(1 - \eta_f)\text{Tr}(M_g\rho M_g^\dagger) + \eta_d\eta_f\text{Tr}(M_e\rho M_e^\dagger)} \quad (4.17)$$

As previously mentioned, our initial state is coherent and the target is the pure state given by equation (4.11). Requiring that all the atoms are detected in  $|g\rangle$ , we include the effect of the cavity relaxation between each detection using equation (4.15) and (4.17). We optimize the fidelity defined as  $F = \langle \psi_t | \rho_f | \psi_t \rangle$  [37], where the final state after the postselection is represented by the density matrix  $\rho_f$  due to the effect of the cavity losses and imperfect detections. In figure 4.6a we show the optimal fidelity versus  $\alpha^2$  with the parameters  $\varphi_3$  and  $N_3$  satisfying condition (4.12). Also, in figure 4.6b the postselection probability is shown. In figure 4.6a we observe that as  $\alpha^2$  increases, the fidelity becomes smaller as compared

to the result without photon losses (figure 4.4a). The reason behind this is that larger  $\alpha^2$  is also translated into more atomic postselection steps ( $N$ ) required to generate the optical qubit, i.e. longer interaction times are needed for our scheme to work. Therefore, the effects of photons leaking from the cavity are more probable for larger  $\alpha^2$ . As we can see from figure 4.6b the postselection probability is significantly reduced. However, we have a preparation of the optical qubit with a fidelity and postselection probability  $0.9 < F_{opt} < 0.94$  and  $4.28\% > P_{post} > 1.72\%$ , which is still within experimental reach. At

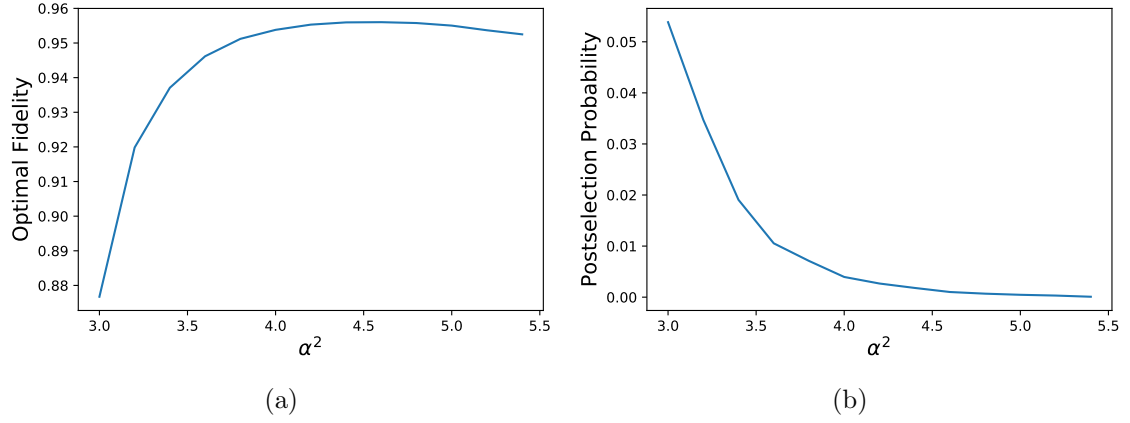


Figure 4.6: (a) Optimal fidelity including the effect of the cavity relaxation for each  $\alpha^2$ . (b) Post-selection probability for a set of parameters  $\alpha$ ,  $N_3$  and  $\varphi_3$  for which the fidelity is maximum.

this stage, we considered a more realistic scenario including the effect of the cavity losses, detection efficiency ( $\eta_d < 1$ ) and error detection ( $\eta_f > 0$ ), obtaining a robust preparation of the optical qubit in a real Fabry-Pérot superconducting cavity.

#### 4.1.2 Other superpositions of photon number Fock states

Finally, if we postselect not only atoms in the  $|g\rangle$  level, it is possible to generate other superpositions of photon number Fock states using this scheme. A more general expression for a pure state is derived from equation (3.23), giving us

$$|\psi_f\rangle = \frac{\sum_{n=0}^{\infty} b_n \prod_{k=1}^K C_{N_k}^{N_{e,k}} e^{-\frac{in}{2} \varphi_k N_k} \cos^{N_k - N_{e,k}} \left( \frac{\varphi_k n}{2} \right) \sin^{N_{e,k}} \left( \frac{\varphi_k n}{2} \right) |n\rangle}{\left[ \sum_{n=0}^{\infty} |b_n|^2 \prod_{k=1}^K C_{N_k}^{N_{e,k}} \cos^{2(N_k - N_{e,k})} \left( \frac{\varphi_k n}{2} \right) \sin^{2N_{e,k}} \left( \frac{\varphi_k n}{2} \right) \right]^{1/2}}. \quad (4.18)$$

Therefore, adjusting for each group of atoms denoted by  $k$  the number of atoms detected in each level ( $N_{e,k}$  in  $|e\rangle$  and  $N_k - N_{e,k}$  in  $|g\rangle$ ) and their interactions ( $\varphi_k$ ), we determine the parameters that decimate other photon number states to properly generate higher photon number Fock state superpositions. In figure 4.7a and 4.7b, we show a superposition of  $|0\rangle$  and  $|2\rangle$  states. First, one atom interacting with  $\varphi_1 = \pi$  is detected in  $|g\rangle$ , then five atoms interacting with  $\varphi_2 = 0.535$  are detected in  $|g\rangle$ . Also, we include the superposition of  $|1\rangle$  and  $|3\rangle$  states in figure 4.7c and 4.7d. First, two atoms interacting with  $\varphi_1 = \pi$  and  $\varphi_2 = \pi/5$  are detected in  $|e\rangle$  and  $|g\rangle$ , respectively. Then, with  $\varphi_3 = 0.372$  we detected one atom in  $|e\rangle$  ( $N_{e,3} = 1$ ) and four atoms in  $|g\rangle$  ( $N_3 - N_{e,3} = 4$ ).

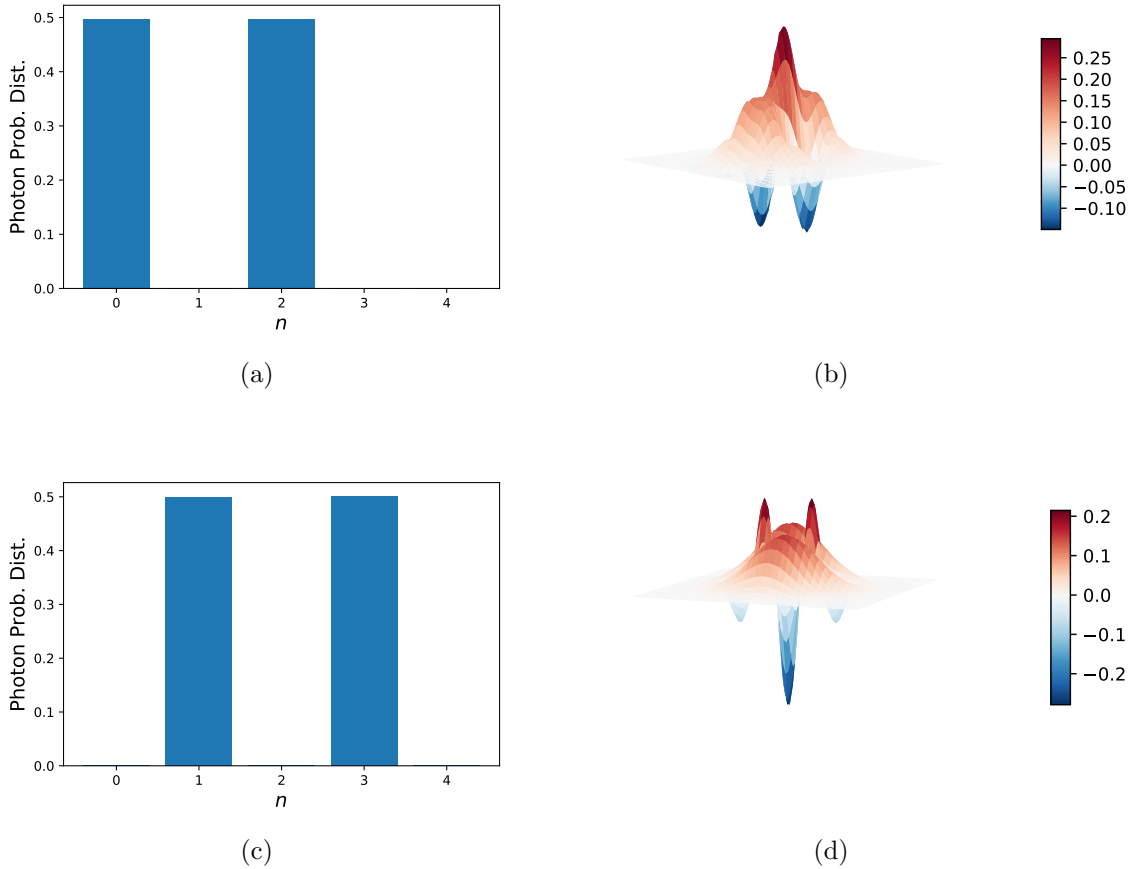


Figure 4.7: Other superpositions of photon number Fock states. In (a) and (b) we show a superposition of states  $|0\rangle$  and  $|2\rangle$ . This state is prepared with a fidelity of 0.99 and a probability of 10%. Figures (b) and (c) show a superposition of states  $|1\rangle$  and  $|3\rangle$ . This state is prepared with a fidelity of 0.97 and a probability of 5.5%.

## 4.2 Photon cooling

In the following, we show how to generate the vacuum state  $|0\rangle$  of the cavity field by an appropriate atomic postselection starting from a thermal state of the field  $\rho_c = \sum_n \rho_{nn} |n\rangle\langle n|$ , where  $\rho_{nn} = n_t^n / [(1 + n_t)^{n+1}]$  and  $n_t$  being the average photon number. As we can see from equation (3.23), we should have  $N_e = 0$  to keep the  $|0\rangle$  state. Thus, the normalized state of the field for an initial thermal state after the postselection of  $N$  atoms in the  $|g\rangle$  level ( $N_e = 0$ ) is

$$\rho_f = \frac{\sum_n \rho_{nn} \prod_{k=1}^N \cos^2(\varphi_k n/2) |n\rangle\langle n|}{\sum_n \rho_{nn} \prod_{k=1}^N \cos^2(\varphi_k n/2)}, \quad (4.19)$$

with postselection probability

$$P_{post} = \sum_{n=0}^{\infty} \rho_{nn} \prod_{k=1}^N \cos^2\left(\frac{\varphi_k n}{2}\right). \quad (4.20)$$

As shown in equation (4.19), the parameters  $\varphi_k$  have to be adequate to ensure that the oscillatory function  $\prod_{k=1}^N \cos^2(\varphi_k n/2)$  multiplying the projectors  $|n\rangle\langle n|$  is close to zero for all the photon numbers except  $n = 0$ . We propose a sequence of atoms where the  $k$ th atom crosses with  $\varphi_k = \pi/2^{k-1}$  in order to eliminate the photon numbers  $n = (2m - 1)2^k$ . Once the process has finished, the postselection probability is  $P_{post} \rightarrow \rho_{00} = 1/(1 + n_t)$ . Figure 4.8 shows the fidelity between the vacuum state and the final state of the cavity field after the postselection of a sequence of  $N$  atoms in  $|g\rangle$ . For an initial state with a mean photon number  $n_t = 100$ , the cooling process converges after the detection of a sequence of about 10 atoms.

In figure 4.9, we illustrate the convergence of the sequence with  $\varphi_k = \pi/2^{k-1}$ , considering that all the atoms are postselected in the state  $|g\rangle$ . In the top panel (a), we plotted an initial thermal state with  $n_t = 3.6$ , whereas in the bottom panel (b), we show the final field state after the postselection of 5 atoms in the state  $|g\rangle$ . As seen, the final state is the vacuum photon state  $|0\rangle$ , where the Wigner function  $W^{[\rho_t]}(\alpha) = \frac{2}{\pi} \frac{1}{2n_t+1} e^{-2|\alpha|^2/(2n_t+1)}$  becomes sharper than the initial thermal state.

In a realistic scenario, the quantum system is coupled to the environment and suffers

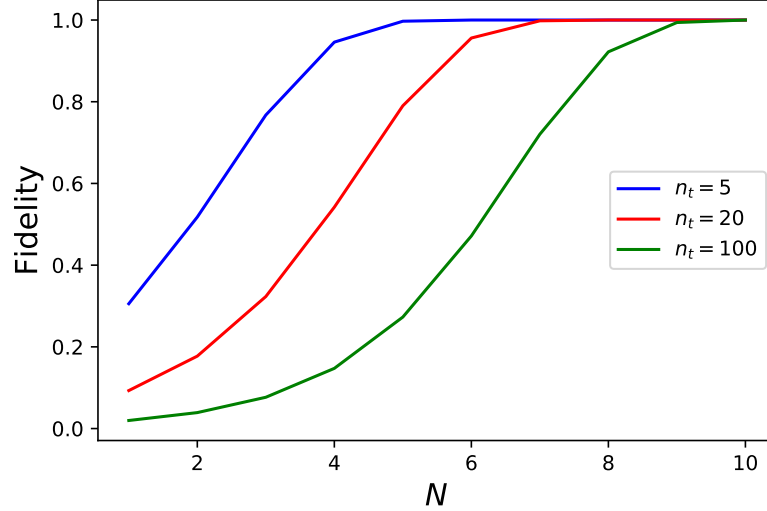


Figure 4.8: Fidelity between the vacuum and the final state of the cavity field given by Eq. (4.19) using a sequence of  $N$  atoms interacting with  $\varphi_k = \pi/2^{k-1}$ .

from decoherence effects. Since we are cooling photons, we are fighting against the thermalization effect of the reservoir. To simulate this scenario, we consider that the cavity field is initially in thermal equilibrium with an average photon number  $n_t$ , and that the atom-field system evolves with the following master equation:

$$\frac{d\rho_S}{dt} = \frac{1}{i\hbar}[H_{eff}, \rho_S] + \sum_i \left[ L_i \rho_S L_i^\dagger - \frac{1}{2}(L_i^\dagger L_i \rho_S + \rho_S L_i^\dagger L_i) \right], \quad (4.21)$$

where  $H_{eff}$  is the dispersive coupling Hamiltonian of equation (3.3) for each atom in the sequence and the  $L_i$  Lindblad operators are  $\sqrt{\Gamma(1+n_t)}\sigma_-$ ,  $\sqrt{\Gamma n_t}\sigma_+$ ,  $\sqrt{\kappa(1+n_t)}a$  and  $\sqrt{\kappa n_t}a^\dagger$ .

Typically, in microwave experiments with circular Rydberg atoms, the relaxation of these atoms is negligible, when compared to the cavity damping time  $T_c$  (equation (4.14)). However, we did consider the lifetime reduction by a factor  $(1+n_t)$ , as well as the field losses, but neglected the losses during the transit time through the Ramsey zones. The master equation for the period without atoms ( $t \sim 82 \mu s$ ), is given by:

$$\frac{d\rho_c}{dt} = \sum_i \left[ L_i \rho_S L_i^\dagger - \frac{1}{2}(L_i^\dagger L_i \rho_S + \rho_S L_i^\dagger L_i) \right], \quad (4.22)$$

where the Lindblad operators are  $\sqrt{\kappa(1+n_t)}a$  and  $\sqrt{\kappa n_t}a^\dagger$ . In our calculations, we used a cavity damping time  $T_c = 1/\kappa = 130$  ms [41]. Also, the atomic lifetime is  $T_a = 1/\Gamma = 30$  ms for circular Rydberg atoms of rubidium with principal quantum numbers 51 or 50 [33]. We consider the cavity tuned at a frequency  $\omega/2\pi = 51.1$  GHz and an atom-cavity detuning  $\delta/2\pi = 245$  kHz. The vacuum Rabi frequency is  $g/2\pi = 49$  kHz. All of these parameters are consistent with real experimental realizations [24].

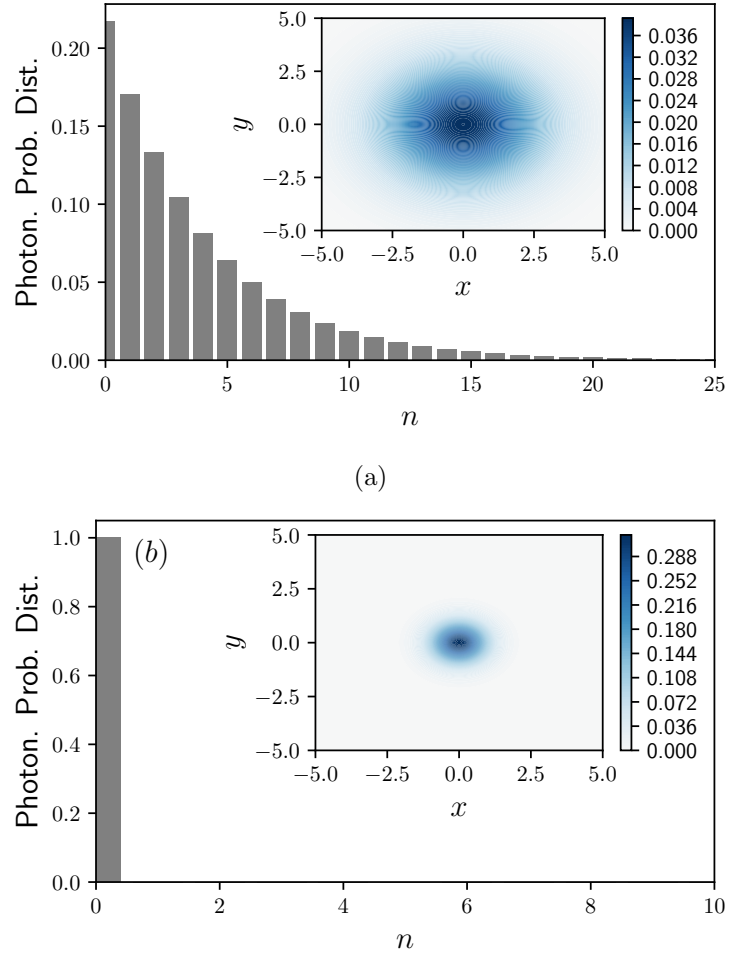


Figure 4.9: Cooling process of the thermal field. In (a) we show the initial thermal state ( $n_t = 3.6$ ), whereas in the bottom panel (b), we show the final state of the field after the postselection of a sequence of 5 atoms in  $|g\rangle$ . In both figures the photon number distribution and Wigner function are shown. The vacuum photon state is generated with a probability of  $P_{post} \approx 21.7\%$ .

The temperature of a thermal state can be determined by the relation  $n_t = 1/[\exp(\hbar\omega/k_bT) - 1]$ . Hence, for an initial thermal state with  $n_t = 3.6$  and the above given frequency, the corresponding bath and photon temperature is  $T = 10$  K (figure 4.9a). After the cooling process and considering the effect of the reservoir, the fidelity between the final state and vacuum goes down to 98.3% (figure 4.10). In this case, the final state has a 99.7% fidelity with respect to a thermal state with  $n_t = 0.017$  corresponding to a temperature of  $T = 0.6$  K. This result shows that the cooling process is robust even in the presence of decoherence, considered by the master equation in equation (4.21). As we mentioned in subsection 4.1.1,

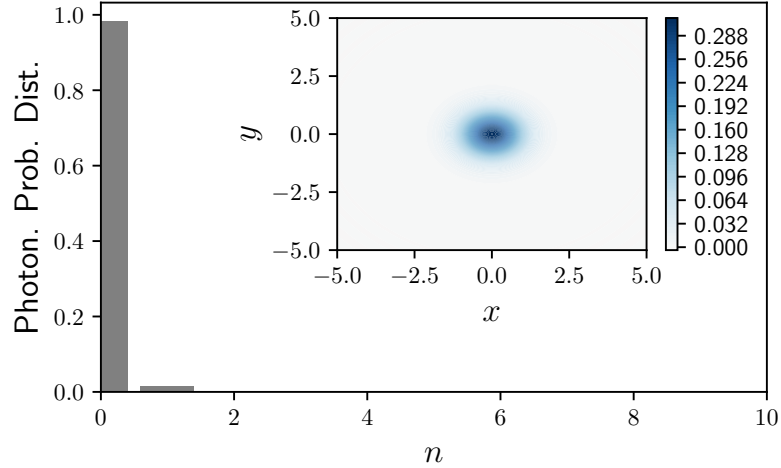


Figure 4.10: Cooling process of figure 4.9 considering thermal effects of the environment at a temperature  $T = 10$  K (mean photon number  $n_t = 3.6$ ). The final state is close to a thermal state with a mean photon number  $n_t = 0.017$ , corresponding to a temperature  $T = 0.6$  K.

this process can be done using circular Rydberg atoms. Hence, the couplings  $\varphi_k$  proposed in our atomic sequence to cool down the thermal photons in the cavity can be achieved by laser techniques. Also, we assume a deterministic preparation of single atoms.

## Chapter 5

# Conclusions and further work

### 5.1 Conclusions

In short, we suggest for the first time, a scheme to generate an optical qubit from an initial coherent state of the field in a typical cavity QED setup using a dispersive atom-field interaction and post-selection of atoms. Particularly, we study the case of an equiprobable superposition of the vacuum and one-photon states. First, the general scheme for the generation of a cavity field state from an initial state via atomic post-selection is presented. Then we focus on the preparation of the optical qubit by setting the parameters which optimize the fidelity between the final and our desired state. As seen in the previous sections, we can achieve this goal with a high fidelity and a post-selection probability within experimental reach. Then, we extend our study showing that this scheme can generate other superpositions of photon number Fock states.

Finally, we suggest a protocol to cool-down a thermal field to its vacuum state. In order to accomplish our task with a minimum number of atoms, we propose a sequence interacting with  $\varphi_k = \pi/2^{k-1}$  which rapidly eliminate the nonzero photon components. The reduction of the number of atoms needed in the process is important when the relevant system is coupled to a thermal reservoir with  $T \neq 0$ , since the whole process takes less time. We model this situation using the general master equation in (4.21) where we considered atomic and field losses by taking real experimental parameters. We conclude that even in the presence of decoherence, our protocol can be done with high fidelity.

## **5.2 Further work**

This work relies on postselection measurements. This means that if we obtain a different result than the expected in an atomic detection, we have to fully reinitialize the scheme to achieve our goal with a certain success probability. Hence, the first step in order to improve the presented results, is to study the feasibility of a quantum feedback protocol for on-demand preparation of the cavity field states analyzed in this work. A quantum feedback scheme for CQED is introduced in [24]. Furthermore, we considered an interaction between the cavity field and a single atom. In this way, there is evidence of a more efficient generation of mesoscopic field states in CQED considering the Dicke Model, particularly a simultaneous interaction between two atoms and the field [44]. A further work following this advice could be beneficial for a more efficient generation of cavity field states presented, i.e., successful results in a shorter time and with higher postselection probabilities.

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