Dynamics of two interacting particles in a magnetic field in two dimensions

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The classical dynamics of two interacting particles of equal mass and equal or opposite charge, moving in a plane and a perpendicular magnetic field, is discussed. The simplest trajectories are similar to those of a single particle in the presence of crossed electric and magnetic fields (Hall configuration). Such motion occurs over a ribbon that may be straight (for opposite charges), or bent into a circle (for identical particles).

I. INTRODUCTION

Since the discovery of the quantum Hall effect there has been much interest in understanding the dynamics of electrons confined to move in two dimensions in the presence of a magnetic field perpendicular to the plane of motion.[1] The confinement is possible at the interface between two materials, typically a semiconductor and an insulator such as GaAs and AlGaAs, where a quantum well that traps the particles is formed, forbidding their motion in the direction perpendicular to the plane of the interface at low energies. The integral quantum Hall effect has been explained using a free electron model, while a proper treatment of the fractional effect requires that the electron–electron interaction be included.[2] The interacting quantum problem has been treated in the Hartree–Fock[3,4] and variational[5] approximations, as well as with numerical methods.[6] It is a difficult many-body problem for which a further understanding than that provided by the approximate treatments is needed. The simplest case, that of just two interacting particles in an additional confining parabolic potential has been treated by Taut.[7] He found exact analytical solutions for selected values of the magnetic field. Why other values do not lend themselves for such solutions is unclear.

In this paper we present a complete solution of the classical two-body problem ignoring radiation and relativistic effects. Our purpose is to provide information on the trajectories in order to guide further efforts in the understanding of the quantum effects. Also, there have been recent experiments involving interesting phenomena such as the Weiss oscillations,[8] in which electrons behave semiclassically. Although these effects may be explained using noninteracting electrons, it is possible that the interaction becomes relevant in the limit of very dilute electron systems, as is the case in the fractional quantum Hall effect (low-filling fraction).

In Sec. II we study the case of two identical particles. The problem is separable in center of mass and relative coordinates. The center of mass moves as a free particle in the magnetic field, of twice the charge and mass as each constituent of the pair. The Coulomb repulsion affects the relative motion. We find that this motion is similar to that of a single particle in crossed electric and magnetic fields (Hall configuration), only that the rectilinear strip in which this latter motion takes place is bent into a circle. In Sec. III we discuss the case of two particles of the same mass and opposite charge. The problem is nonseparable, yet becomes one dimensional in a special case for which we find solutions also similar to those in the Hall configuration over a rectilinear strip.

II. IDENTICAL PARTICLES

We consider two identical particles of mass m and charge e in a uniform magnetic field $\mathbf{B}$. We are interested in the motion in a plane perpendicular to the magnetic field. The particles interact with the field, and with each other through the Coulomb repulsion. The dynamics is derived from the Lagrangian (we use the Gaussian system of units):

$$L(\mathbf{r}_1,\mathbf{r}_2,\dot{\mathbf{r}}_1,\dot{\mathbf{r}}_2)=\tfrac{1}{2}m\dot{\mathbf{r}}_1^2+\tfrac{1}{2}m\dot{\mathbf{r}}_2^2+\frac{e}{c}\,\mathbf{A}(\mathbf{r}_1)\cdot\dot{\mathbf{r}}_1+\frac{e}{c}\,\mathbf{A}(\mathbf{r}_2)\cdot\dot{\mathbf{r}}_2-e^2/\kappa|\mathbf{r}_1-\mathbf{r}_2|. \tag{1}$$

Here, $\mathbf{r}_1$ ($\mathbf{r}_2$) is the position vector of particle 1 (2), $\mathbf{A}$ is the vector potential, and κ is the dielectric constant of the medium in which the particles move. The problem is separable if center of mass (CM) and relative coordinates are used. Let $\mathbf{R}=(\mathbf{r}_1+\mathbf{r}_2)/2$ denote the position of the CM and $\mathbf{r}=\mathbf{r}_2-\mathbf{r}_1$ the relative position vector. In the symmetric gauge $\mathbf{A}(\mathbf{r})=\frac{1}{2}\mathbf{B}\times\mathbf{r}$, we obtain $L=L_{cm}+L_{rel}$, where

$$L_{cm}(R,\dot{R};\theta_{cm},\dot{\theta}_{cm})=\frac{1}{2}\,M(\dot{R}^2+R^2\dot{\theta}_{cm}^2)+\frac{1}{2}\,\frac{Q}{c}\,BR^2\dot{\theta}_{cm} \tag{2}$$

describes the dynamics of the CM, and

$$L_{rel}(r,\dot{r};\theta_{rel},\dot{\theta}_{rel})=\frac{1}{2}\,\mu(\dot{r}^2+r^2\dot{\theta}_{rel}^2)+\frac{1}{2}\,\frac{q}{c}\,Br^2\dot{\theta}_{rel}-qQ/\kappa r \tag{3}$$

describes the relative motion, all in polar coordinates. From these Lagrangians we see that the CM motion is that of a single particle with charge $Q=2e$ and mass $M=2m$ in the presence of a magnetic field $\mathbf{B}$. It describes a circle with an angular frequency $\dot{\theta}=-\omega_c$, with $\omega_c=eB/mc$ the cyclotron frequency. The relative motion is, in turn, that of a particle with charge $q=e/2$ and mass $\mu=m/2$ in the presence of an external magnetic field, and the electric field produced by a charge $Q=2e$ fixed at the origin.

We will use the dimensionless notation $\xi = R/l_B$, $\rho = r/l_B$, $\dot{\rho} = (d\rho/dt)/\omega_c$, where $l_B = (mc^2/B^2)^{1/3}$ is the natural classical length scale. Time, energy, and angular momentum are expressed in units of $1/\omega_c$, e^2/l_B, and $m\omega_c l_B^2$, respectively. The Lagrangians (2) and (3) do not contain the azimuthal angle. The conjugate momenta

$$p_\theta^{cm} = 2\xi^2\left(\frac{\dot{\theta}_{cm}}{\omega_c} + \frac{1}{2}\right) \tag{4}$$

and

$$p_\theta^{rel} = \frac{1}{2}\rho^2\left(\frac{\dot{\theta}_{rel}}{\omega_c} + \frac{1}{2}\right) \tag{5}$$

are therefore constants of motion. The energies associated with the classical motion are

$$\epsilon_{cm} = \dot{\xi}^2 + \frac{1}{4}\left(\frac{p_\theta^{cm}}{\xi} - \xi\right)^2, \tag{6}$$

$$\epsilon_{rel} = \frac{1}{4}\dot{\rho}^2 + \left(\frac{p_\theta^{rel}}{\rho} - \frac{1}{4}\rho\right)^2 + \frac{1}{\kappa\rho}. \tag{7}$$

The form of ϵ_{cm} and ϵ_{rel} differ by the presence of a Coulomb term in ϵ_{rel}, making the relative and CM motion very different.

Integrating Eq. (6) with the aid of Eq. (4), we obtain for the radial coordinate of the CM, the equation

$$\xi^2 - 2\xi\sqrt{\epsilon_{cm} + p_\theta^{cm}}\cos(\theta_{cm} - \theta_0) + p_\theta^{cm} = 0, \tag{8}$$

where θ_0 is a constant of integration. Equation (8) represents a circle of radius $\sqrt{\epsilon_{cm}}$ centered at $\xi_0 = \sqrt{\epsilon_{cm} + p_\theta^{cm}}$, so that ϵ_{cm} defines the orbit radius, while p_θ^{cm} and ϵ_{cm} together fix the position of its center.

The integral of motion for the relative coordinate is obtained from Eqs. (5) and (7). We get,

$$\Delta\theta_{rel}(\rho) = 4\int_{\rho_<}^{\rho} \frac{d\rho'\left(\frac{p_\theta^{rel}}{\rho'} - \frac{1}{4}\rho'\right)}{\sqrt{-\rho'^4 + (16\epsilon + 8p_\theta^{rel})\rho'^2 - 16\rho'/\kappa - 16p_\theta^{rel^2}}}. \tag{9}$$

Here $\rho_< < \rho < \rho_>$, where $\rho_>$ and $\rho_<$ are the extreme values of the relative coordinate of the orbit. They are determined by the two real and non-negative solutions of Eq. (7) under the condition $\dot{\rho} = 0$. The other two solutions are $c_\pm$ (see the Appendix). In terms of these constants we can rewrite Eq. (9) in the form

$$\begin{aligned}\Delta\theta_{rel}(\rho) = {} & 2\frac{c_+ - \rho_<}{\sqrt{(\rho_> - c_+)(\rho_< - c_-)}} \\ & \times\left\{2p_\theta^{rel}\Pi\left[\lambda(\rho), \eta\frac{c_+}{\rho_<}, \sigma\right] + \Pi[\lambda(\rho), \eta, \sigma]\right\} \\ & + 2\frac{1}{\sqrt{(\rho_> - c_+)(\rho_< - c_-)}}\left(2\frac{p_\theta^{rel}}{c_+} - c_+\right) \\ & \times F[\lambda(\rho), \sigma],\end{aligned} \tag{10}$$

where $F(a,b)$ and $\Pi(a,b,c)$ are elliptic integrals of the first and third kind, respectively, and

$$\lambda(\rho) = \arcsin\sqrt{\frac{(\rho_> - c_+)(\rho - \rho_<)}{(\rho_> - \rho_<)(\rho - c_+)}}, \tag{11}$$

$$\eta = \frac{\rho_> - \rho_<}{\rho_> - c_+}, \tag{12}$$

$$\sigma = \sqrt{\frac{(\rho_> - \rho_<)(c_+ - c_-)}{(\rho_> - c_+)(\rho_< - c_-)}}. \tag{13}$$

The classical motion of the pair is the composition of the circular CM motion and the relative motion. The latter is in general noncircular and is not necessarily periodic. A simplifying property common to pairs of particles of equal mass is, however, that in the CM system both particles describe identical orbits with a phase difference of π.

The simplest possible classical orbit one can obtain for this system is the circle. It corresponds to a situation in which the magnetic force and Coulomb repulsion combine to exactly produce the centripetal acceleration necessary to maintain a circular motion. The condition is

$$2\frac{v^2}{\rho} = v - \frac{1}{\kappa\rho^2}, \tag{14}$$

where $v = \rho|\dot{\theta}_{rel}|/2\omega_c$ is the dimensionless constant speed of each particle. If the CM is at rest the motion is truly circular in the laboratory frame, while only the relative motion is circular if the CM moves. The constant relative distance may be obtained from our formalism by noting that Eq. (7) is the sum of a kinetic energy term and the effective potential

$$V_{\text{eff}}(\rho) = \left(\frac{p_\theta^{rel}}{\rho} - \frac{1}{4}\rho\right)^2 + \frac{1}{\kappa\rho}. \tag{15}$$

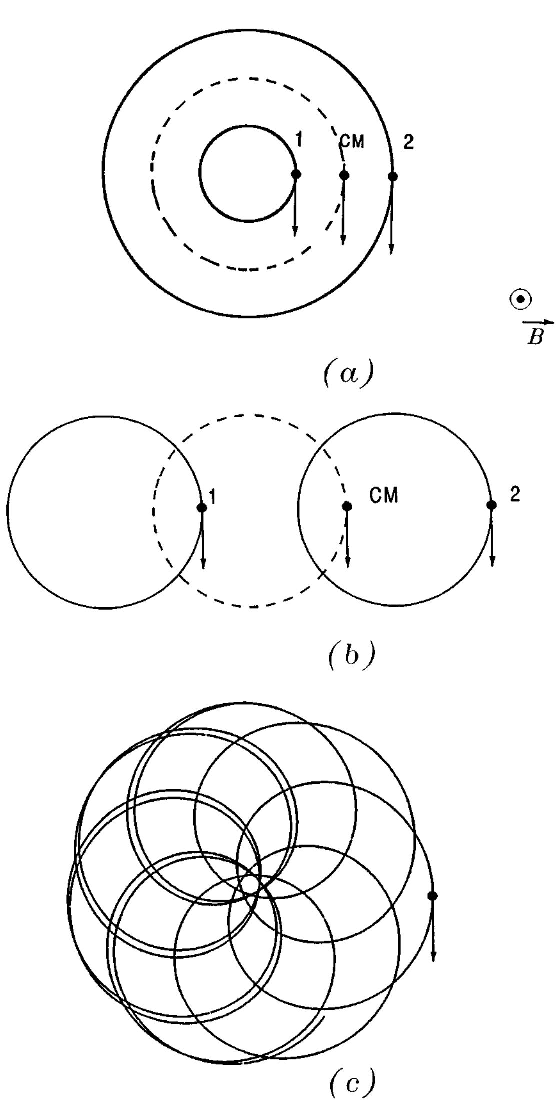
This potential has a minimum at which the relative motion is circular. Setting the derivative to zero one then obtains Eq. (14) with the aid of Eq. (5). One also obtains for the distance between the particles,

$$\rho_m = v + \sqrt{v^2 + \frac{1}{\kappa v}}. \tag{16}$$

In ordinary units the radius of the circle is then $R_m = \rho_m l_B/2$. With the aid of Eqs. (5) and (16) one obtains for the frequency of the circular motion in terms of the parameter ρ_m,

$$\frac{\dot{\theta}_{rel}}{\omega_c} = \begin{cases} -\frac{1}{2} + \frac{1}{2}\sqrt{1 - 8\frac{1}{\kappa\rho_m^3}}, & p_\theta^{rel} > 0, \\ -\frac{1}{2}, & p_\theta^{rel} = 0, \\ -\frac{1}{2} - \frac{1}{2}\sqrt{1 - 8\frac{1}{\kappa\rho_m^3}}, & p_\theta^{rel} < 0. \end{cases} \tag{17}$$

The role of p_θ^{rel} is clear in the noninteracting limit, for which there is relative motion between the particles if $p_\theta^{rel} < 0$ only. Then the center of mass and the particles themselves describe concentric circles with the same angular frequency $-\omega_c$, and radii $\sqrt{\epsilon_{cm}}$ and $|\sqrt{\epsilon_{cm}} \pm \rho_m/2|$, respec-



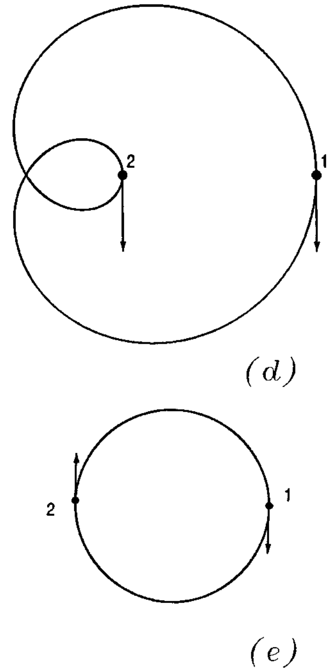


Fig. 1. Various orbits for which the separation between the particles is unchanged. (a) and (b) are for the noninteracting case, whereas in (c)–(e) the particles interact. In (d) they are given the same initial velocity, while in (e) the speed is the same but the motion is in opposite directions.

tively. This is shown in Fig. 1(a). When $p_\theta^{rel}>0$ in the same limit, there is no relative motion and the particles and center of mass all describe circles of the same radius. Their centers are then aligned and a distance $\rho_m/2$ apart, as shown in Fig. 1(b). When the interaction is turned on, however, the motion is more complex. An example is shown in Fig. 1(c), drawn for $p_\theta^{rel}=1$. The trajectory of only one particle is exhibited, enough to illustrate the complexity of motion in general.

The equations of motion in the interacting case are greatly simplified when $p_\theta^{rel}=0$. The trajectory in the CM frame is a circular orbit of radius $\rho_m=2/\kappa^{1/3}$ and frequency $\dot\theta_{rel}=-\omega_c/2$. Since the CM moves with angular frequency $-\omega_c$ then the frequencies are commensurate and the orbits are closed, as shown in Fig. 1(d), for the case in which both particles have the same initial velocity. If the CM is at rest then, as mentioned above, the particles are always at diametrically opposite points of a circle. This is shown in Fig. 1(e). The frequency of the small oscillations about the circular motion is easily obtained and we get $\omega_\rho=\sqrt{6/\kappa\rho_m^3}$.

When the motion is such that the distance between the particles is not constant one has, always for $p_\theta^{rel}=0$, that the general integral of motion Eq. (9) is reduced to

$$\Delta\theta_{rel}=-\frac{2\rho_<}{\sqrt{\rho_>(2\rho_>+\rho_<)}}\times\Pi\left(\arcsin\sqrt{\frac{\rho_>(\rho-\rho_<)}{\rho(\rho_>-\rho_<)}},\ \frac{\rho_>-\rho_<}{\rho_>},\ \sqrt{\frac{\rho_>^2-\rho_<^2}{\rho_>(2\rho_>+\rho_<)}}\right), \tag{18}$$

where $\Pi(a,b,c)$ is the elliptic integral of the third kind,

$$\rho_<=4\sqrt{\frac{2\epsilon_{rel}}{3}}\left(\sqrt{3}\,\sin\frac{\alpha}{3}-\cos\frac{\alpha}{3}\right), \tag{19}$$

$$\rho_>=8\sqrt{\frac{2\epsilon_{rel}}{3}}\cos\frac{\alpha}{3}, \tag{20}$$

and $\cos\alpha=-(2\sqrt{3/2\epsilon_{rel}})/\kappa$, with $\pi/2<\alpha\leqslant\pi$.

A general statement about the motion is that it is confined in spite of the Coulomb repulsion. This may be seen by

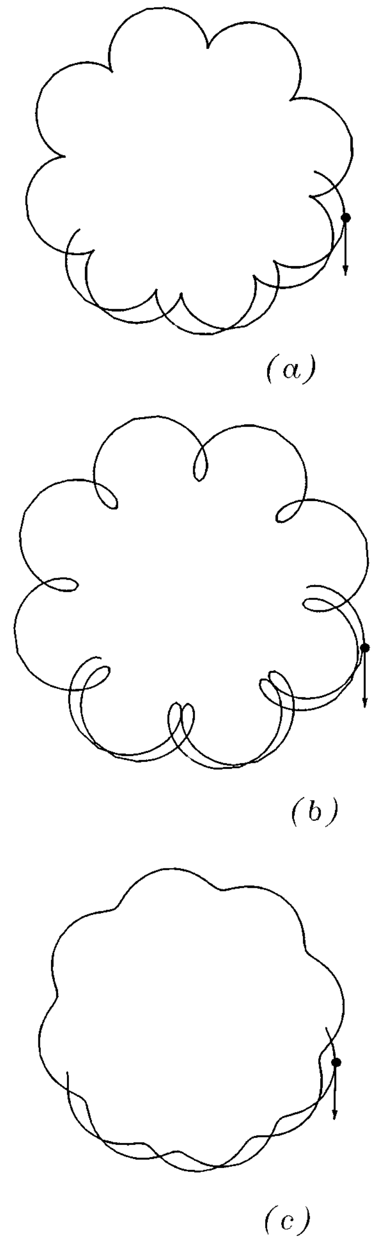


Fig. 2. Orbits for the energies (a) $\epsilon^{rel}=1/\kappa\rho_0$, (b) $\epsilon^{rel}>1/\kappa\rho_0$, (c) $\epsilon^{rel}<1/\kappa\rho_0$. The CM is at rest.

noting that the effective potential (15) diverges both at the origin and in the limit $\rho\rightarrow\infty$, at the separation (16). Confinement is provided by the magnetic field.

A special point in the relative motion is the separation at which the parentheses in Eq. (15) vanishes, that is, at $\rho_0 = 2\sqrt{p_\theta^{rel}}$. Then, the potential is just the Coulomb repulsion. Figure 2 shows the possible orbits in the relative coordinates, which we shall describe assuming the CM to be at rest. Figure 2(a) is for $\epsilon_{rel}=1/\kappa\rho_0$ and is obtained when the particles are released from rest at an initial separation ρ_0. They move instantaneously apart and then the Lorentz force curves the trajectories. The orbit is closed or open depending on whether or not Eq. (9) is an integral multiple of 2π. Figure 2(b) is for $\epsilon_{rel}>1/\kappa\rho_0$ and corresponds to equal and opposite initial velocities in the direction perpendicular to the line joining the particles and directed such that the initial motion is counterclockwise. Note that the sense of rotation is changed whenever $\rho=\rho_0$. Finally, Fig. 2(c) is for $\epsilon_{rel}<1/\kappa\rho_0$ and is obtained, always with the CM at rest, when the initial motion is as before, but the initial sense of rotation is clockwise. In this case $\dot{\theta}_{rel}$ does not change sign. When the CM is moving, the above orbits are superimposed on the uniform rotation of the CM. Also, while the three kinds of motion are possible for $p_\theta^{rel}>0$, in the case $p_\theta^{rel}<0$ they are all, in their relative motion, of the type shown in Fig. 2(b).

The curves in Fig. 2 correspond qualitatively to the possible trajectories of a particle in crossed magnetic and electric fields in the usual Hall configuration.[9] While in the standard Hall effect the field sources are fixed and the motion is over a straight strip, in our case they move over a circular strip. The observed curvature is produced by the motion of the field sources.

III. PARTICLES WITH OPPOSITE CHARGE

Another simple case is that of two particles that differ only in the sign of the charge, such as a particle and its antiparticle. Consider two particles of mass m and charge e and $-e$, respectively. As before the motion is limited to a planar surface and a magnetic field $\mathbf{B}$ perpendicular to this plane is present. The particles interact with the magnetic field and with each other through the Coulomb attraction $V(|\mathbf{r}_1-\mathbf{r}_2|)=-e^2/\kappa|\mathbf{r}_1-\mathbf{r}_2|$. As before we use CM and relative coordinates. In the symmetric gauge the Lagrangian is

$$L(\mathbf{R},\mathbf{r};\dot{\mathbf{R}},\dot{\mathbf{r}})=\frac{1}{2}M\dot{\mathbf{R}}^2+\frac{1}{2}\mu\dot{\mathbf{r}}^2 + \frac{1}{2}\frac{e}{c}(\mathbf{B}\times\mathbf{R}\cdot\dot{\mathbf{r}}+\mathbf{B}\times\mathbf{r}\cdot\dot{\mathbf{R}})+e^2/\kappa r. \quad (21)$$

As before, $\mathbf{r}=\mathbf{r}_2-\mathbf{r}_1$ and we have assumed particle 1 (2) to carry negative (positive) charge. In terms of the dimensionless units defined in Sec. II a first constant of motion is

$$-2\hat{z}\times\dot{\boldsymbol{\xi}}+\boldsymbol{\rho}=\boldsymbol{\rho}_0. \quad (22)$$

The CM position $\boldsymbol{\xi}$ and the relative position vector $\boldsymbol{\rho}$ are coupled in this equation. In fact, in contrast to the case of identical particles, now the Lagrangian (21) is not separable and the CM and relative motions are coupled.

A second constant of motion is the energy, which may be written in terms of relative coordinates only in the form,

$$\epsilon=\frac{1}{4}\dot{\boldsymbol{\rho}}^2+\frac{1}{4}(\boldsymbol{\rho}-\boldsymbol{\rho}_0)^2-\frac{1}{\kappa\rho}. \quad (23)$$

Note that information about motion of the CM is contained in this expression through the constant $\boldsymbol{\rho}_0$.

Motion is, in general, quite complex in this case. Simple trajectories are however obtained when $\dot{\theta}_{rel}=0$ and $p_\theta^{rel}=0$ in polar coordinates, since then the problem becomes one dimensional. This means $|\theta_{cm}-\theta_{rel}|=\pi/2$, so the relative position vector $\boldsymbol{\rho}$ does not change direction in time, though in general its length changes. The integral of motion is given by

$$\Delta t(\rho)=\frac{2}{\omega_c}\frac{\rho_>}{\sqrt{(\rho_>-c_-)\rho_<}}\Pi\left(\lambda(\rho),1-\frac{\rho_>}{\rho_<},\rho\right), \quad (24)$$

where $\rho_<\leqslant\rho\leqslant\rho_>$. Here $\rho_<$ and $\rho_>$ are the extremes of the orbit in the relative coordinate, c_- is a constant (see the

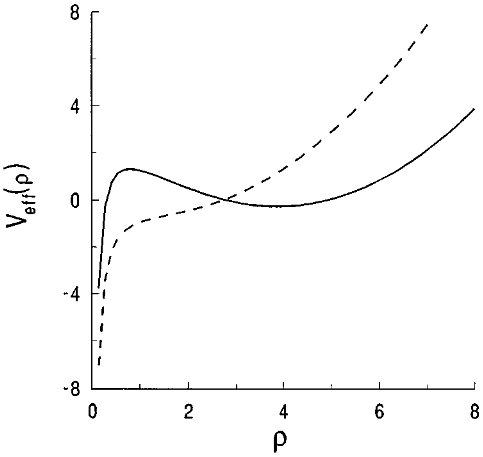


Fig. 3. Effective potential when $\rho_0 \leqslant 3(1/2\kappa)^{1/3}$ (dashed line), and $\rho_0 > 3(1/2\kappa)^{1/3}$ (full line).

Appendix) and $\lambda(\rho)$ is given by Eq. (11) with $c_+ = 0$. $\Pi(a,b,c)$ is the elliptic integral of the third kind.

As is apparent from Eq. (22) the family of solutions just discussed corresponds to a situation in which the CM velocity $\dot{\boldsymbol{\xi}}$ does not change direction in time and so the CM moves in a straight line. We also note that since all vectors in Eq. (23) keep their direction fixed one may treat this expression as the energy function of a one-dimensional problem with an effective potential

$$v_{\mathrm{eff}}(\rho) = \frac{1}{4}\,(\rho - \rho_0)^2 - \frac{1}{\kappa\rho}. \tag{25}$$

The simplest case occurs at the minimum of this function. The particles then move along parallel straight lines, with the Coulomb attraction perfectly balanced by the outward magnetic force. The constant separation between the particles is then

$$\rho_m = \frac{\rho_0}{3}\left(1 + 2\cos\frac{\alpha}{3}\right), \tag{26}$$

and they move with the CM speed $\dot{\xi} = (\rho_m - \rho_0)/2$. Here α is given by the relation $\cos\alpha = 1 - 27/\kappa\rho_0^3$. This parallel motion is not always possible, however. In fact, for Eq. (25) to have extrema the condition $\rho_0 > 3/(2\kappa)^{1/3}$ must be fulfilled. The dashed line in Fig. 3 shows v_{eff} when this condition is not met, while the full line is for the case when both a relative minimum and a relative maximum are present. The distance to the origin in this latter case is given by

$$\rho_M = \left(1 + \sqrt{3}\,\sin\frac{\alpha}{3} - \cos\frac{\alpha}{3}\right)\frac{\rho_0}{3}. \tag{27}$$

In Fig. 4 we show trajectories when $\rho_0 \leqslant 3/(2\kappa)^{1/3}$ and there are no extrema in the effective potential (the dashed line in Fig. 3). In this case only $\rho_>$ is real and positive, while the other extremum of the orbit is $\rho_< = 0$. The magnetic field is assumed to come out of the paper toward the reader. In Fig. 4 the motion of only one particle is shown, since the second particle follows a trajectory that is the mirror image about a vertical line passing through the center of the trajectory of the first. The particles start their motion a distance $\rho_>$ apart and oscillate about this vertical line with an average drift in the upward direction. The shoulders that appear at the center of Fig. 4 are regions where the Coulomb attraction dominates and the particles speed up toward each other in a collision course. The potential is divergent upon touching and in the immediate neighborhood of this point our solution is an extrapolation. Three different cases may be distinguished.

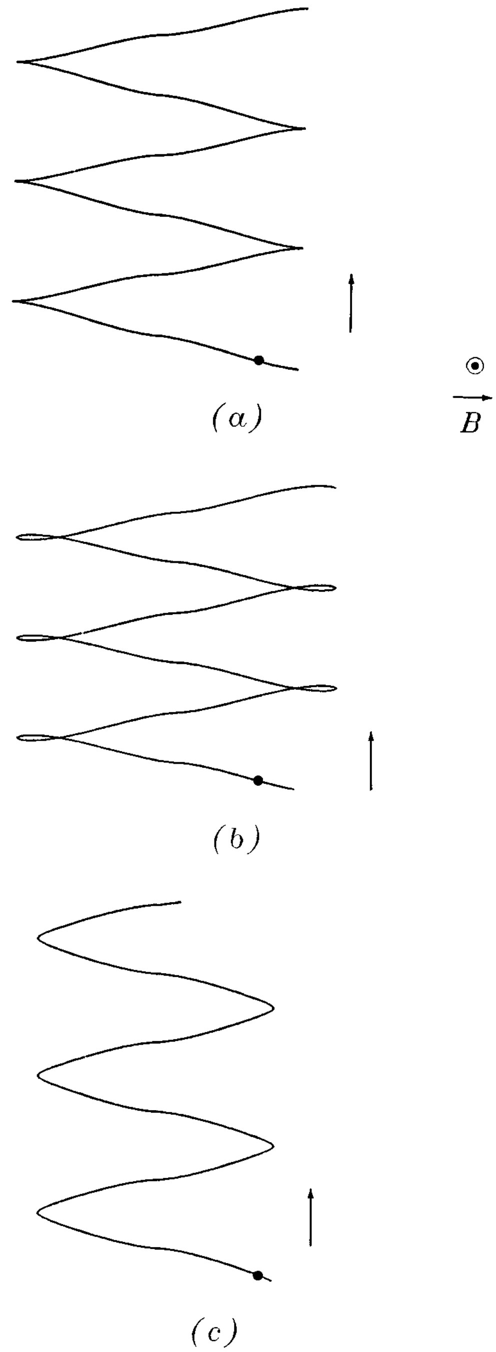


Fig. 4. Trajectories for the potential of Fig. 3, dashed line, and (a) $\epsilon^{rel} = -1/\kappa\rho_0$, (b) $\epsilon^{rel} > -1/\kappa\rho_0$, (c) $\epsilon^{rel} < -1/\kappa\rho_0$. $p_\theta^{rel} = 0$ in each case. The trajectory of only one particle is shown (see the text).

(i) For $\epsilon = -1/\kappa\rho_0$ motion starts from rest at an initial maximum separation $\rho_> = \rho_0$. The initial radial motion due to the Coulomb attraction is deflected by the magnetic field [Fig. 4(a)].

(ii) For $\epsilon > -1/\kappa\rho_0$, the maximum separation is always greater than ρ_0. The particles have equal initial velocities in the downward direction [Fig. 4(b)].

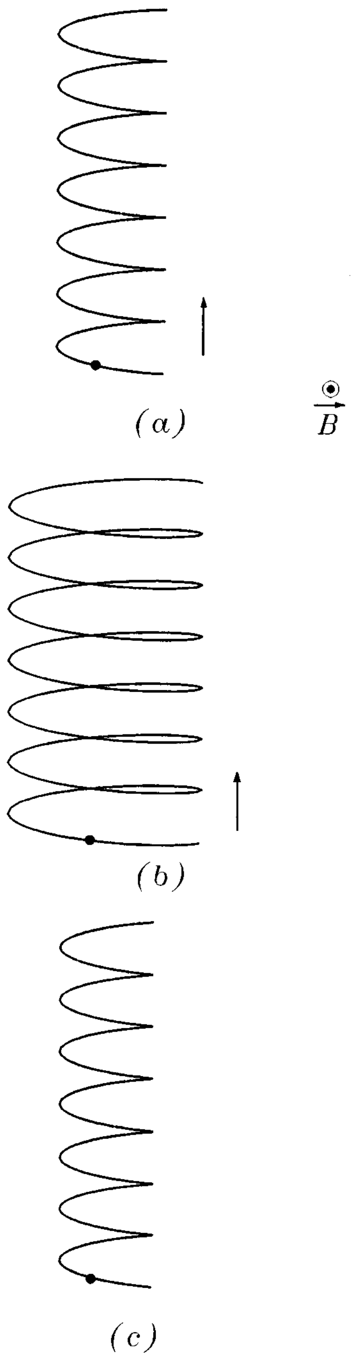


Fig. 5. Trajectories for the potential of Fig. 3, full line, and (a) $\epsilon^{rel}=-1/\kappa\rho_0$, (b) $-1/\kappa\rho_0<\epsilon^{rel}<v_{\text{eff}}(\rho_{\rho M})$, and (c) $v_{\text{eff}}(\rho_m)<\epsilon^{rel}<1/\kappa\rho_0$, for $\rho>3(1/2\kappa)^{1/3}$. As in Fig. 4 the trajectory of only one particle is shown.

(iii) For $\epsilon<-1/\kappa\rho_0$, one has $\rho_>< \rho_0$. It corresponds to particles with equal initial velocities in the upward direction [Fig. 4(c)].

When $\rho_0>3/(2\kappa)^{1/3}$ and the potential has a minimum (the full line in Fig. 3), motion is still as described above. A special case is when motion is bounded about the relative minimum of the potential, however, and the particles move in nonoverlapping parallel strips (Fig. 5).

IV. CONCLUSIONS

In summary, we have shown that the planar motion of two interacting charged particles in a uniform perpendicular magnetic field is bounded even when the Coulomb force is repulsive. We have ignored radiation and relativistic effects. When the particles are identical, the center of mass describes a circle with the single particle cyclotron frequency, while the pair move over a bounded circular ribbon in the relative coordinates. The simplest trajectory is a circle with the particles always in diametrally opposite points. When the particles have opposite charges the simplest trajectory is straight parallel motion with constant velocity. More complex trajectories include a family in which there is a parallel drift with periodic oscillations about the average direction of motion.

The dynamics of interacting particles in a magnetic field is a fundamental problem both in classical and quantum physics. It is hoped that the insight gained by our classical solution for the interacting pair will be helpful in the search for a better understanding of its quantum counterpart.

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APPENDIX: ROOTS OF THE ORBIT EQUATION

1. Identical particles

The equation for the extremes of the orbit is

$$\rho^4-8(2\epsilon+p_\theta^{rel})\rho^2+16\frac{1}{\kappa}\rho+16p_\theta^{rel^2}=0 \tag{28}$$

with roots[10] $\rho_>=a+b$, $\rho_<=a-b$, $c_+=-a+b$, and $c_-=-a-b$, where

$$a=\tfrac{1}{2}\sqrt{8(2\epsilon+p_\theta^{rel})+u_1}, \tag{29}$$

$$b=\frac{1}{2}\sqrt{8(2\epsilon+p_\theta^{rel})-u_1+4\sqrt{\frac{u_1^2}{4}-16p_\theta^{rel^2}}}. \tag{30}$$

Here u_1 is given by

$$u_1=(r+\sqrt{q^3+r^2})^{1/3}+(r-\sqrt{q^3+r^2})^{1/3}+\tfrac{8}{3}(2\epsilon+p_\theta^{rel}), \tag{31}$$

where

$$r=\frac{2048}{9}\left(p_\theta^{rel^2}\epsilon+\frac{2}{3}p_\theta^{rel^3}-\frac{2}{3}\epsilon^3-\epsilon^2p_\theta^{rel}\right)+128/\epsilon_m^2, \tag{32}$$

$$q=-\frac{256}{9}(p_\theta^{rel^2}+\epsilon^2+\epsilon p_\theta^{rel}). \tag{33}$$

2. Particles of opposite charge

The equation for the extremes of the orbit is

$$\rho^3-2\rho_0\rho^2+(\rho_0^2-4\epsilon)\rho-4\epsilon_c=0. \tag{34}$$

The roots are[10]

$$\rho_>=f_++f_-+\tfrac{2}{3}\rho_0, \tag{35}$$

$$\rho_<=-\frac{1}{2}(f_++f_-)+\frac{2}{3}\rho_0+\frac{\sqrt{3}i}{2}(f_+-f_-), \tag{36}$$

$$c_-=-\frac{1}{2}(f_++f_-)+\frac{2}{3}\rho_0-\frac{\sqrt{3}i}{2}(f_+-f_-), \tag{37}$$

where f_+ and f_- are given by

$$f_\pm=\left(\frac{2}{\epsilon_m}+\frac{4}{3}\rho_0\epsilon-\frac{1}{27}\rho_0^3 \pm\sqrt{\left(\frac{2}{\epsilon_m}+\frac{4}{3}\rho_0\epsilon-\frac{1}{27}\rho_0^3\right)^2-\left(\frac{4}{3}\epsilon+\frac{1}{9}\rho_0^2\right)^3}\right)^{1/3}. \tag{38}$$

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THE FUTURE OF TRANSISTOR ELECTRONICS

It may be appropriate to speculate at this point about the future of transistor electronics. Those who have worked intensively in the field share the author's feeling of great optimism regarding the ultimate potentialities. It appears to most of the workers that an area has been opened up comparable to the entire area of vacuum and gas-discharge electronics. Already several transistor structures have been developed and many others have been explored to the extent of demonstrating their ultimate practicality, and still other ideas have been produced which have yet to be subjected to adequate experimental tests. It seems likely that many inventions unforeseen at present will be made based on the principles of carrier injection, the field effect, the Suhl effect, and the properties of rectifying junctions. It is quite probable that other new physical principles will also be utilized to practical ends as the art develops.

William Shockley, *Electrons and Holes in Semiconductors* (D. Van Nostrand Co., Inc., New York, 1950), p. 349.

NOHOW

How, finally, do we teach why? How do we teach logic and mathematics, how do we teach abstract concepts and the relations among them, how do we teach intuition, recognition, understanding? How do we teach these things so that when we are done our ex-student can not only pass an examination by naming the concepts and listing the relations, but he can also get pleasure from his insight, and, if he is talented and lucky, be vouchsafed the discovery of a new one? The only possible answer that I can see is: nohow. Don't do nuttin'; just wait. The only way I know of for an individual to share in humanity's slowly acquired understanding is to retrace the steps. Some old ideas were in error, of course, and some might have become irrelevant to the world of today, and therefore no longer fashionable, but on balance every student must repeat all the steps—ontogeny must recapitulate philogeny every time.

Paul R. Halmos, ''What is Teaching?,'' The American Mathematical Monthly **101** (9), 848–854 (1994).