Topics in Large Deviations and Localization for Random Walks in Random Environment

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Abstract

In this thesis, we study the model of Random Walks in Random Environment (RWRE) on the integer lattice $\mathbb{Z}^d$. The first topic in consideration is about the equality and difference of the quenched and averaged large deviations rate functions in terms of the environment's disorder. We measure this quantity in terms of how close it is to its expected value. After the introduction, in Chapter 2 we consider the problem on the interior of the domain of the rate functions, namely, on the set $\mathrm{int}(\mathbb{D}) := \{x \in \mathbb{R}^d : |x|_1 < 1\}$ when $d \geq 4$. We show that on any compact set that does not contain the origin, the rate functions agree on that set if the disorder is low enough.

In Chapter 3, we look at the same problem on the set $\partial\mathbb{D} := \{x \in \mathbb{R}^d : |x|_1 = 1\}$. In addition to the results from the previous chapter, we also prove equality under a weaker condition we call (low) *imbalance*. Our results allow us to deduce an explicit formula for the quenched rate function on the boundary in the low disorder (or imbalance) regime. Moreover, we show a phase transition in terms of equality/inequality under a parametrized family of environments.

Finally, Chapter 4 is devoted to the study of localization at the boundary for RWRE. Roughly, we say the walk is localized at the boundary if, conditioned on the event that its path $(X_n)_{n\in\mathbb{N}}$ satisfies $|X_n|_1 = n$, there is a sequence of random points $(x_n)_{n\in\mathbb{N}}$ which are asymptotically more likely of being visited. We show that when $d = 2$ or 3, the walk is localized. In contrast, when $d \geq 4$, there is also a phase transition (as in Chapter 3) for the localization/delocalization phenomena.

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Contents

List of Symbols

- $\subseteq$: refers to the inclusion between sets, while $\subsetneq$ refers to the strict inclusion.
- $\bigsqcup$: for two disjoint sets A and B, its union is denoted by $A \bigsqcup B$.
- $\mathbb{N}$: the set of natural numbers including 0.
- $\mathbf{i}$: the imaginary number.
- e_i: the vector $(0, \cdots, \underbrace{1}_{\text{i-th position}}, \cdots 0)$.
- $\mathbb{V} = \mathbb{V}(d) := \{\pm e_1, \cdots, \pm e_d\}$.
- $\mathbb{V}^+ = \mathbb{V}^+(d) := \{e_1, \cdots, e_d\}$.
- $|\cdot|_p$: the ℓ^p norm $(p \geq 1)$.
- $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : |x|_2 = 1\}$.
- $\langle \cdot, \cdot \rangle$: the standard inner product in $\mathbb{R}^d$.
- $||\cdot||$: the operator 1-norm.
- $B_\delta(x)$: the ball of center x and radius δ. Most of the times will be with respect to $|\cdot|_1$, and occasionally with respect to $|\cdot|_2$.
- $\text{int}(G)$, $\overline{G}$: the interior and closure respectively of the set G.

- $\mathcal{B}(A)$: the Borel σ-algebra of the topological space A.
- $\mathcal{M}_1(A)$: the space of probability measures on A.

Chapter 1

Introduction

This thesis is concerned with the model of *random walks in random environment* (RWRE) on $\mathbb{Z}^d$, for $d \geq 1$. Under this framework, instead of considering random walks with fixed jump probabilities, these are now random. As a simple example, suppose that $d = 1$, and consider a sequence of i.i.d random variables $(p_i)_{i\in\mathbb{Z}}$ which takes values on $[0, 1]$, on some probability space. Then we can consider a random walk $\{X_n\}_{n\in\mathbb{N}}$ with $X_0 = 0$ and such that the probability of jumping from i to $i + 1$ is p_i, and the probability of jumping from i to $i - 1$ is $1 - p_i$. Once a realization of the random variables $(p_i)_{i\in\mathbb{Z}}$ is fixed, we can apply Markov chain techniques to study its behavior. One then would like to know if certain properties of the walk are satisfied for almost any realization of the p_is. Typical questions are related to the recurrence/transience phenomenon, the law of large numbers (LLN), and the central limit theorem (CLT). It is customary to refer to the probabilities for fixed p_is as *quenched* probabilities. On the other hand, one could consider an "averaged" random walk; as the jump probabilities are themselves random variables, one could think of the distribution of the walk's trajectory as a random variable. Thus, the *averaged* (or *annealed*) random walk is obtained by taking expectations over the quenched probabilities. Similarly, the same questions that arise in the quenched setting can be analyzed in the averaged one.

Before getting into more details, let us introduce first the model in more generality. As its name indicates, we need to define a random walk and a (random) environment. First, we define the notion of environment. Let d be a positive integer, which represents the dimension where the walk moves. We define the canonical vectors $e_1, \cdots, e_d$ as $e_i := (0, \cdots, \underbrace{1}_{\text{i-th position}}, \cdots 0)$. Set $\mathbb{V} := \{\pm e_1, \cdots, \pm e_d\}$ and $\mathbb{V}^+ := \{e_1, \cdots, e_d\}$. Next, we define the set of (nearest-neighbor) jump probabilities as $\mathcal{M}_1(\mathbb{V}) := \{p : \mathbb{V} \to [0,1] : \sum_{e \in \mathbb{V}} p(e) = 1\}$. Finally, the set of *environments* is $\Omega := \mathcal{M}_1(\mathbb{V})^{\mathbb{Z}^d}$. Thus, an element $\omega \in \Omega$ can be represented as $\omega = (\omega(x))_{x \in \mathbb{Z}^d}$, where $\omega(x) \in \mathcal{M}_1(\mathbb{V})$. By definition, for each $x \in \mathbb{Z}^d$, $\omega(x)$ is a jump probability, so it can be written as $\omega(x) = (\omega(x,e))_{e \in \mathbb{V}}$. In conclusion, ω is completely determined by the sequence $(\omega(x,e))_{x \in \mathbb{Z}^d, e \in \mathbb{V}}$. The quantity $\omega(x,e)$ represents the (quenched) probability of a random walk jumping from x to $x+e$. Indeed, if we fix $\omega \in \Omega$ and the initial point of the walk $z \in \mathbb{Z}^d$, then we say that the Markov chain $(X_n)_{n \in \mathbb{N}}$ is a *random walk in the environment* ω*, starting at* z, with law $P_{z,\omega}$, if it satisfies the conditions

$$P_{z,\omega}(X_0 = z) = 1$$

$$P_{z,\omega}(X_n + 1 = x + e | X_n = x) = \begin{cases} \omega(x,e), & \text{if } x \in \mathbb{Z}^d, e \in \mathbb{V} \text{ and } P_{z,\omega}(X_n = x) > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{1.0.1}$$

The measure $P_{z,\omega}$ is called the *quenched measure*, in contrast to the *averaged* (or *annealed*) measure we describe next. First, we need to consider some probability measure $\mathbb{P}$ on the space $(\Omega, \mathcal{B}(\Omega))$. Then we define the annealed measure starting at z as the semi-direct product on $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}$ given by the formula

$$P_z(A \times B) = \int_A P_{z,\omega}(B) \mathrm{d}\mathbb{P} \qquad \forall\, A \in \mathcal{B}(\Omega)\,,\, B \in \mathcal{B}((\mathbb{Z}^d)^{\mathbb{N}}).$$

In general, one needs to impose certain conditions on the measure $\mathbb{P}$ so that the model can be mathematically tractable. Two classes of such conditions are usually assumed. The first one deals with *ergodicity*. We define the canonical shifts $(t_z)_{z\in\mathbb{Z}^d}$ on Ω via the map $t_z\omega(x,e) := \omega(x+z,e)$, $e \in \mathbb{V}$. Then we define the condition **(ERG)** as

$$\textbf{(ERG)} \text{ the maps } (t_z)_{z\in\mathbb{Z}^d} \text{ form an ergodic family on the space } (\Omega, \mathcal{B}(\Omega), \mathbb{P}). \tag{1.0.2}$$

That is, if $A \in \mathcal{B}(\Omega)$ satisfies $A = t_x^{-1}(A)$ for each $x \in \mathbb{Z}^d$, then $\mathbb{P}(A) \in \{0,1\}$. A more restricted —but the one we will mostly use in this thesis —hypothesis is the **(IID)** condition:

$$\textbf{(IID)} \text{ the random variables } (\omega(x))_{x\in\mathbb{Z}^d} \text{ are i.i.d under } \mathbb{P}. \tag{1.0.3}$$

The second class of conditions deals with the likelihood of the environments being close to 0. We distinguish between two properties, **(E)** (for ellipticity) and **(UE)** (for uniform ellipticity), defined as follows:

- **(E)** for each $x \in \mathbb{Z}^d$, $\mathbb{P}(\omega(x,e) > 0) = 1$ for all $e \in \mathbb{V}$,
- **(UE)** there exists some $\kappa > 0$ such that, for all $x \in \mathbb{Z}^d$ and $e \in \mathbb{V}$, $\mathbb{P}(\omega(x,e) \geq \kappa) = 1$.

  The smallest such κ is called the *ellipticity constant.*

(1.0.4)

The essential difference between the elliptic and uniform elliptic case is that in the latter, the environments are uniformly bounded by below (hence its name). In contrast, in the former, the environments are positive, but they can be arbitrarily close to 0. The elliptic condition makes the model more difficult to handle compared to the uniform elliptic case. One of such difficulties is the presence of traps (see, for example, Example 2.18 in [DR]).

1.1 Known results

We proceed now to give a summary of some known results for the model of RWRE. We will mostly state results that are somewhat related to our work. By no means this will be a complete account of the literature. Some complementary references are the lectures notes of Bolthausen and Sznitman [BS2], Zeitouni [Zei] and Drewitz and Ramírez [DR].

1.1.1 One-dimensional case

The case $d = 1$ has been studied since the 70s, starting with the seminal works of Kozlov [Koz1] and Solomon [Sol][1]. In [Sol], the author gives a characterization of the recurrence/transience phenomenon for an RWRE that satisfies **(E)** and **(IID)**. When $d = 1$, we simplify the notation for the environments, writing for $x \in \mathbb{Z}$, $p_x := \omega(x, 1)$ and $q_x := 1 - p_x = \omega(x, -1)$. Also, define $\rho_x := \frac{q_x}{p_x}$. We now state the mentioned result of Solomon.

Theorem 1.1.1. *Let $\mathbb{P}$ satisfies **(E)** and **(IID)**. Assume that $\mathbb{E}[\log \rho] := \mathbb{E}[\log \rho_0] \in [-\infty, \infty]$ is well defined. Then we have the following:*

(i) If $\mathbb{E}[\log \rho] < 0$, then $\lim_{n\to\infty} X_n = \infty$ P_0-a.s.

(ii) If $\mathbb{E}[\log \rho] > 0$, then $\lim_{n\to\infty} X_n = -\infty$ P_0-a.s.

(iii) If $\mathbb{E}[\log \rho] = 0$, then $\liminf_{n\to\infty} X_n = -\infty$ P_0-a.s. and $\limsup_{n\to\infty} X_n = \infty$ P_0-a.s.

The result says that the walk is transient if and only if $\mathbb{E}[\log \rho] \neq 0$. Moreover, in the same article, Solomon shows a law of large numbers for the trajectory of the walk.

Theorem 1.1.2. *Let $\mathbb{P}$ satisfies **(E)** and **(IID)**. Then one and only one of the following cases holds:*

[1] Non-rigourous analysis were priorly made by Chernov [Che] for DNA chains and Temkin [Tem] for crystal growth.

(i) If $\mathbb{E}[\rho] < 1$, *then* P_0*-a.s.* $\lim_{n\to\infty} \frac{X_n}{n} = \frac{1-\mathbb{E}[\rho]}{1+\mathbb{E}[\rho]}$.

(ii) If $\mathbb{E}[\rho^{-1}] < 1$, *then* P_0*-a.s.* $\lim_{n\to\infty} \frac{X_n}{n} = -\frac{1-\mathbb{E}[\rho^{-1}]}{1+\mathbb{E}[\rho^{-1}]}$.

(iii) If $\mathbb{E}[\rho]^{-1} \leq 1 \leq \mathbb{E}[\rho^{-1}]$, *then* P_0*-a.s.* $\lim_{n\to\infty} \frac{X_n}{n} = 0$.

Remark 1.1.1.

1. Note that by Jensen's inequality, $\mathbb{E}[\rho^{-1}] \geq \mathbb{E}[\rho]^{-1}$, so the cases above are mutually exclusive and the only possible ones.

2. One of the interesting features of the one-dimensional case is that there are examples of walks that satisfy $\lim_{n\to\infty} X_n = \infty$ and $\lim_{n\to\infty} \frac{X_n}{n} = 0$, that is, walks that are transient to the right with zero velocity. Indeed, any measure $\mathbb{P}$ that fulfills $\mathbb{E}[\log \rho] < 0$ and $\mathbb{E}[\rho] \geq 1$ works, since by Jensen's inequality, $\mathbb{E}[\log \rho^{-1}] > 0$ implies $\mathbb{E}[\rho^{-1}] > 1$ (thus (iii) holds). One of the main conjectures in the multidimensional case is to show that this behavior cannot occur under the hypotheses **(UE)** and **(IID)**. More about this later (see Conjecture 1.1.1).

3. Another remarkable feature of one-dimensional RWREs was showed by Sinai [Sin]. In contrast to the one-dimensional simple random walk, which has fluctuations of order $\sqrt{n}$, the recurrent RWRE takes values of order $\log^2 n$, for large enough n. This case also has a localized behavior (more about localization in Section 1.2.2 and Chapter 4).

Later on, Alili in [Ali] extends the previous theorems assuming **(ERG)** instead of **(IID)**. We state the generalization of the law of large numbers to the stationary and ergodic case.

Theorem 1.1.3. *Let* $\mathbb{P}$ *satisfies **(E)** and **(ERG)**. Define* $S := 1 + \sum_{n=1}^{\infty} \prod_{i=1}^{n} \rho_i$ *and* $F := 1 + \sum_{n=1}^{\infty} \prod_{i=1}^{n} \rho_{-i}^{-1}$. *Then one and only one of the following cases holds:*

(i) If $\mathbb{E}[S] < \infty$, *then* P_0*-a.s.* $\lim_{n\to\infty} \frac{X_n}{n} = \frac{1}{\mathbb{E}[(1+\rho_0)S]}$.

(ii) If $\mathbb{E}[F] < \infty$*, then* P_0*-a.s.*$\lim_{n\to\infty} \frac{X_n}{n} = \frac{1}{\mathbb{E}[(1+\rho_0^{-1})F]}$.

(iii) If $\mathbb{E}[S] = \mathbb{E}[F] = \infty$*, then* P_0*-a.s.* $\lim_{n\to\infty} \frac{X_n}{n} = 0$.

Note that by the stationarity assumption and Jensen inequality, the cases above are mutually exclusive and the only possible ones. Furthermore, when the environments are i.i.d., we recover Theorem 1.1.2. As we will discuss later (cf. Section 1.1.2), the picture in the multidimensional case is significantly more complicated. This discrepancy is one of the characteristics of the model of RWRE.

Large deviations

The next topic we discuss is about large deviations. In general, large deviations deals with estimating atypical probabilities. If there is a law of large numbers, i.e. $\frac{Y_n}{n} \to v$ P-a.s., for some random sequence $(Y_n)_{n\in\mathbb{N}}$ with law P, the atypical probabilities are of the type $P(|\frac{Y_n}{n} - v|_1 > a)$, for $a > 0$. One then is interested in finding a function I, called rate function, that should satisfy $P(|\frac{Y_n}{n} - v|_1 > a) \approx \mathrm{e}^{-nI(a)}$. These vague notions can be formalized and generalized, as we do now.

Definition 1.1.1. (lower semicontinuous function) Let X be a topological space. We say a function $f : X \to [-\infty, \infty]$ is lower semicontinuous if the set $\{x \in X : f(x) \leq a\}$ is closed for all $a \in \mathbb{R}$.

Definition 1.1.2. (large deviations principle, LDP) Let X be a topological space and $I : X \to [0, \infty]$ be a lower semicontinuous function. If $(\mu_n)_{n\in\mathbb{N}}$ is a sequence of probability measures on $(X, \mathcal{B}(X))$, we say it satisfies a large deviation principle (LDP) with rate function I if the following holds:

$$\begin{aligned} \limsup_{n\to\infty} \mu_n(F) &\leq -\inf_{x\in F} I(x) \quad \text{for all closed sets } F \subseteq X, \\ \liminf_{n\to\infty} \mu_n(G) &\geq -\inf_{x\in G} I(x) \quad \text{for all open sets } G \subseteq X. \end{aligned} \tag{1.1.1}$$

We refer the reader to the references [DZ] and [RAS4] for a complete account of large deviations.

The first result about large deviations for RWRE was given by Greven and den Hollander [GdH], where they show a quenched large deviation principle with a deterministic rate function. More precisely, we have the following result:

Theorem 1.1.4. *Assume **(E)** and **(IID)**. Suppose that* $\mathbb{E}[\log \rho] \leq 0$*. Then the sequence* $(P_{0,\omega}(X_n/n \in \cdot))_{n\in\mathbb{N}}$ *satisfies a quenched large deviation principle, that is, there exists a convex, deterministic function* I_q *(i.e. does not depend on* ω*) such that,* $\mathbb{P}$*-a.s.* (1.1.1) *holds with* $\mu_n(\cdot) = P_{0,\omega}(X_n/n \in \cdot)$*,* $X = \mathbb{R}$ *and* $I = I_q$*. Moreover,* I *is continuous and*

$$\lim_{n\to\infty} P_{0,\omega}(X_n = \lfloor n\theta \rfloor) = -I_q(\theta) \ \mathbb{P}\text{-}a.s. \ for\ \theta \in [-1,1].$$

There is no loss of generality assuming that $\mathbb{E}[\log \rho] \leq 0$; if $\mathbb{E}[\log \rho] > 0$, replace ρ by ρ^{-1} and X_n by $-X_n$ in the previous theorem. Theorem 1.1.4 was generalized to stationary and ergodic environments by Comets, Gantert and Zeitouni in [CGZ]. They also show an annealed large deviation principle (i.e. choosing $\mu_n(\cdot) = P_0(X_n/n \in \cdot)$) with a rate function I_a for a class of measures that includes the case **(IID)**. Moreover, the annealed and quenched rate functions are related via the formula

$$I_a(x) = \inf_{\mathbb{Q}} \left[I_q^{\mathbb{Q}}(x) + x|h(\mathbb{Q}|\mathbb{P})| \right].$$

In the formula above, the infimum is over all the stationary and ergodic measures $\mathbb{Q}$, $I_q^{\mathbb{Q}}$ is the quenched rate function with respect to $\mathbb{Q}$, and $h(\cdot|\cdot)$ is the specific relative entropy. Another interesting feature of the one-dimensional rate functions is the presence of *linear pieces*

1.1.2 Multidimensional RWRE ($d \geq 2$)

While the understanding of one-dimensional RWREs is rather advanced, the multidimensional case is still under substantial research. Several fundamental questions have not been solved up to this day. We mention some advances during the last years in the topic. Throughout the sequel, the concepts of *transience in a given direction and ballisticity* are crucial to the development of the model, so we introduce them now.

Definition 1.1.3. (Transience in a given direction) Let $\ell \in \mathbb{S}^{d-1}$. Given an RWRE $(X_n)_{n\in\mathbb{N}}$, denote by $A_\ell := \{\lim_{n\to\infty} X_n \cdot \ell = \infty\}$. We say the RWRE is transient in the direction ℓ if $P_0(A_\ell) = 1$.

Definition 1.1.4. (Ballisticity) Let $\ell \in \mathbb{S}^{d-1}$. We say the RWRE $(X_n)_{n\in\mathbb{N}}$ is ballistic in the direction ℓ if

$$\liminf_{n\to\infty} \frac{X_n \cdot \ell}{n} > 0 \ P_0\text{-a.s.} \tag{1.1.2}$$

It is clear that if the walk is ballistic in a given direction, then it is transient. One of the main conjectures in the field is that (under certain conditions) the converse also holds:

Conjecture 1.1.1. Assume $\mathbb{P}$ satisfies **(IID)** and **(UE)**. If the walk is transient in a given direction, then it is ballistic in the same direction.

Much of the past development in the last two decades has been in trying to solve this conjecture. In order to close the gap, certain *ballisticity conditions*, which we will address later, have been introduced.

The work of Kalikow [Kal] was the first in studying the multidimensional case. He introduced the first ballisticity condition, the so-called *Kalikow's condition*, and showed that it implies transience and a 0-1 law. This criterion is quite involved, but for completeness, we define it below. A couple of preliminary definitions are needed.

Definition 1.1.5. (Exterior boundary) For any set $U \subseteq \mathbb{Z}^d$, its exterior boundary is given by $\partial U := \{z \in \mathbb{Z}^d \setminus U : \exists u \in U, |z-u|_1 = 1\}$.

Definition 1.1.6. (Exit time) Given any set $U \subseteq \mathbb{Z}^d$, the first exit time of an RWRE $(X_n)_{n\in\mathbb{N}}$ from U is $T_U := \inf\{n \in \mathbb{N} : X_n \notin U\}$.

Definition 1.1.7. (Kalikow's random walk) Let $U \subsetneq \mathbb{Z}^d$ be a connected set that contains 0. Denote the class of such U's as $\mathcal{C}$. Given an RWRE $(X_n)_{n\in\mathbb{N}}$ with environmental law $\mathbb{P}$ that satisfies **(E)** and **(IID)**, the Kalikow's random walk starting at $x \in U$ is the Markov chain $(\tilde{X}_n)_{n\in\mathbb{N}}$ with transition probability $\widehat{P}_U$ given by

$$\widehat{P}_U(y, y+e) = \begin{cases} \frac{E_0[\sum_{n=0}^{T_U} \mathbb{1}_{\{X_n=y\}}\omega(y,e)]}{E_0[\sum_{n=0}^{T_U} \mathbb{1}_{\{X_n=y\}}]}, & \text{if } y \in U, e \in \mathbb{V}, \\ 1, & \text{if } y \in \partial U, e = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{1.1.3}$$

The law of this chain starting at $x \in U \cup \partial U$ is denoted by $\widehat{P}_{x,U}$.

Now we can define the Kalikow's condition:

Definition 1.1.8. (Kalikow's condition) Let $\ell \in \mathbb{S}^{d-1}$. The Kalikow's condition in direction ℓ is satisfied if and only if

$$\inf_{U\in\mathcal{C}, x\in U} \sum_{e\in\mathbb{V}} (e\cdot\ell)\widehat{P}_U(x, x+e) > 0. \tag{1.1.4}$$

One of the main results in [Kal] is that Kalikow's condition implies transience:

Theorem 1.1.5. *Suppose $\mathbb{P}$ satisfies **(UE)** and **(IID)**. Let $\ell \in \mathbb{S}^{d-1}$. If Kalikow's condition is satisfied with respect to v, then the RWRE is transient in direction ℓ.*

Moreover, it can be deduced from the aforementioned article (as showed in Lemma 1.1 in [SZ]) the so-called Kalikow's 0-1 law:

Theorem 1.1.6. *Suppose $\mathbb{P}$ satisfies **(UE)** and **(IID)**. For each $\ell \in \mathbb{S}^{d-1}$, $P_0(A_\ell \cup A_{-\ell}) \in \{0,1\}$.*

The natural question is if one can deduce that $P_0(A_\ell) \in \{0,1\}$. Unfortunately, this is an open problem for $d \geq 3$. However, Zerner and Merkl in [ZM] showed that the 0-1 law holds for $d = 2$:

Theorem 1.1.7. *Suppose $\mathbb{P}$ satisfies **(E)** and **(IID)**. If $d = 2$, for each $\ell \in \mathbb{S}^{d-1}$, $P_0(A_\ell) \in \{0,1\}$.*

Moreover, Kalikow's condition was used by Sznitman and Zerner in [SZ] to improve Theorem 1.1.5, where they deduce ballisticity.

Theorem 1.1.8. *Suppose $\mathbb{P}$ satisfies **(UE)** and **(IID)**. Let $\ell \in \mathbb{S}^{d-1}$. If Kalikow's condition is satisfied with respect to ℓ, then the RWRE is ballistic in that direction. More precisely, there exists some $v \in \mathbb{R}^d$ such that $v \cdot \ell > 0$ and P_0-a.s.*

$$\lim_{n\to\infty} \frac{X_n}{n} = v. \tag{1.1.5}$$

Indeed, one can obtain more information about the LLN velocity, but we need to introduce first a *regeneration structure*, proposed in [SZ], as a multidimensional version of the renewal structure of Kesten, cf. [Kes].

Definition 1.1.9. (First regeneration time, τ_1) Let $\ell \in \mathbb{S}^{d-1}$. We define the first regeneration time in the direction ℓ as

$$\tau_1 = \tau_1(\ell) := \min\{n \geq 1 : \max_{0 \leq m \leq n-1} X_n \cdot \ell < X_n \cdot \ell \leq \inf_{m \geq n} X_n \cdot \ell\}. \tag{1.1.6}$$

Since to determine τ_1 we need to look at the future, this random time is not a stopping time. However, one of the main features of τ_1 is that one can construct a sequence of random times $(\tau_n)_{n\geq 0}$ (here $\tau_0 = 0$) with the property that, under $P_0(\cdot|A_\ell)$ (so that for all n, $\tau_n < \infty$ P_0-a.s.), $(X_{\tau_{n+1}} - X_{\tau_n}, \tau_{n+1})_{n\geq 0}$ form an independent sequence . Furthermore, if $D := \inf\{n \geq 0 : X_n \cdot \ell < X_0 \cdot \ell\}$, then under $P_0(\cdot|A_\ell)$, $(X_{\tau_{n+1}} - X_{\tau_n}, \tau_{n+1})_{n\geq 1}$ are distributed as (X_{τ_1}, τ_1) under $P_0(\cdot|D = \infty)$ (see Corollary 1.5. in [SZ]).

With the definition of τ_1 and D, we can now express the velocity v in Eq.(1.1.5) as

$$v = \frac{E_0[X_{\tau_1}|D = \infty]}{E_0[\tau_1|D = \infty]}. \tag{1.1.7}$$

As the results above demonstrate, Kalikow's condition has been very fruitful to understand the asymptotic behavior of RWREs. However, this condition is quite restrictive and difficult to check. One of the field's main objectives is to try to find more general conditions that assure ballistic behavior (being transience the ultimate goal), at least for i.i.d and uniform elliptic walks. The first of such ballisticity conditions, besides Kalikow's one, was discovered by Sznitman and published in [Szn2]. It is called condition (T).

Definition 1.1.10. (Condition (T)) Assume $\mathbb{P}$ satisfies **(UE)** and **(IID)**. Let $\ell \in \mathbb{S}^{d-1}$. We say condition (T) in the direction ℓ holds, denoted by (T)$|\ell$, if the following is satisfied:

(i) The walk is transient in the direction ℓ.

(ii) There exists $c > 0$ such that $E_0[\mathrm{e}^{c \max_{0\leq n\leq \tau_1} |X_n|}] < \infty$.

The direction ℓ is usually implicit, so we write (T) instead of (T)$|\ell$.

Remark 1.1.2. As shown in [Szn3], hypothesis (ii) in the above definition is equivalent to the following: for $\ell' \in \mathbb{S}^{d-1}$ and $L > 0$, let $U_{\ell',L} := \{x \in \mathbb{Z}^d : -L \leq x \cdot \ell' \leq L\}$. Then there exists a neighborhood of V of ℓ such that for all $\ell' \in V$,

$$\limsup_{L\to\infty} \frac{1}{L} \log P_0(X_{T_{U_{\ell',L}}} \cdot \ell' < 0) < 0. \tag{1.1.8}$$

Intuitively, this equation says that the probability of the walk exiting for the "bad" side (namely, contrary to direction ℓ) decays exponentially as the side of the slab $U_{\ell',L}$ goes to infinity.

One of the main results showed in [Szn2] is that condition (T) implies a ballistic law of large numbers, with the same velocity v as in (1.1.5), and a central limit theorem. Furthermore, he shows that Kalikow's condition implies condition (T).

For the sake of brevity, we refer the reader to the following references for more about ballisticity conditions: [Szn3] for the so-called *effective criterion* and the conditions (T′), $(\mathrm{T})_\gamma, \gamma \in (0,1]$[2], and [BDR] for the *polynomial condition*.

Large deviations

A substantial part of this thesis is devoted to the study of large deviations for RWRE, so we now proceed to give the context necessary to understand our work. We have already considered the one-dimensional case, so what remains is to summarize large deviations for $d \geq 2$.

The first important contribution came from Zerner in [Zer], where he showed a quenched LDP for the position of the walk. He assumes the RWRE is *nestling*. Let us define first this concept before stating the result.

Definition 1.1.11. (local drift, nestling and non-nestling walks) Define the local drift of the walk at zero as the random variable $d(\omega) := \sum_{e \in \mathbb{V}} \omega(0,e)e$. Let H be the convex hull of the support of the law of d. Then we say the RWRE is

(i) non-nestling if $0 \notin H$,

(ii) marginally nestling if $0 \in \partial H$, and

[2] In a series of papers, culminating with the work of Guerra and Ramírez in [GR], the authors show that the conditions $(\mathrm{T}), (\mathrm{T}')$ and $(\mathrm{T})_\gamma$ are all equivalent.

(iii) plain nestling if $0 \in \operatorname{int}(H)$.

In cases *(ii)* and *(iii)* we say the walk is nestling.

Theorem 1.1.9. *Suppose that $\mathbb{P}$ satisfies **(IID)** and also that for each $e \in \mathbb{V}$, $-\mathbb{E}\log\omega(0,e)^d < \infty$. Moreover assume that the RWRE is nestling. Then the sequence of measures $(\mu_n)_{n\in\mathbb{N}}$ with $\mu_n := P_{0,\omega}(X_n/n \in \cdot)$ satisfies a large deviation principle with deterministic rate function $I_q : \mathbb{R}^d \to [0,\infty]$. Moreover, $I_q(0) = 0$ and I_q is convex, continuous and finite if and only if $|x|_1 \leq 1$.*

Varadhan in [Var] proved both a quenched and annealed LDP; the former for stationary and ergodic environments (no nestling assumption is required), while the latter for i.i.d. environments. Here is the result.

Theorem 1.1.10. *Suppose $\mathbb{P}$ satisfies **(UE)** and **(ERG)**.*

(i) The sequence of measures $(\mu_n)_{n\in\mathbb{N}}$ with $\mu_n := P_{0,\omega}(X_n/n \in \cdot)$ satisfies a quenched large deviation principle with deterministic, convex rate function $I_q : \mathbb{R}^d \to [0,\infty]$, which is finite if and only if $|x|_1 \leq 1$.

*(ii) If instead of **(ERG)** we assume **(IID)**, then the sequence of measures $\mu_n := P_0(X_n/n \in \cdot)$ satisfies also an annealed large deviation principle with convex rate function $I_a : \mathbb{R}^d \to [0,\infty]$, which is finite if and only if $|x|_1 \leq 1$.*

Additionally, $I_a(x) \leq I_q(x)$ for all $x \in \mathbb{R}^d$ and $I_q(x) = 0$ if and only if $I_a(x) = 0$.

Varadhan's result does not provide explicit formulas for the quenched rate function (its proof is based upon the subadditive ergodic theorem). In contrast, for the annealed rate function, there is a variational formula that we do not reproduce here since it goes beyond this thesis's scope. On the other hand, Rosenbluth in [Ros] showed a quenched LDP for the walk's position, with rate function given by a variational formula. Before we present the result, we need some notation. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space —for now we simply forget about our space of environments Ω. Let $\mathcal{Q}$ be the set of measurable functions $q : \Omega \times \mathbb{V} \to [0,1]$ such that $\sum_{e \in \mathbb{V}} q(\omega, e) = 1$ for all $\omega \in \Omega$ (one can think of $\mathcal{Q}$ as the set of probability vectors $\mathcal{M}_1(\mathbb{V})$) and we fix some map $p \in \mathcal{Q}$. We also denote by $\mathcal{D}$ to the space of measurable maps $\phi : \Omega \to [0, \infty)$ for which $\int \phi d\mathbb{P} = 1$ and by $\mathcal{B}$ to the space of bounded and measurable functions $h : \Omega \to \mathbb{R}$.

Theorem 1.1.11. *Assume $\mathbb{P}$ satisfies **(ERG)** and there exists some $\alpha > 0$ such that for all $e \in \mathbb{V}$*

$$\int |\log p(\omega, e)|^{d+\alpha} \mathbb{P}(d\omega) < \infty. \tag{1.1.9}$$

Define $\Lambda : \mathbb{R}^d \to \mathbb{R}$ as

$$\Lambda(\lambda) := \sup_{q \in \mathcal{Q}, \phi \in \mathcal{D}} \inf_{h \in \mathcal{B}} \sum_{e \in \mathbb{V}} \int \left(\lambda \cdot e - \log \frac{q(\omega, e)}{p(\omega, e)} + h(\omega) - h(T_e \omega) \right) q(\omega, e) \phi(\omega) \mathbb{P}(d\omega). \tag{1.1.10}$$

Then the sequence of measures $(\mu_n)_{n \in \mathbb{N}}$ with $\mu_n := P_{0,\omega}(X_n/n \in \cdot)$ satisfies a quenched large deviation principle with deterministic, convex rate function $I_q : \mathbb{R}^d \to [0, \infty]$ given by

$$I_q(x) = \sup_{\lambda \in \mathbb{R}^d} [\lambda \cdot x - \Lambda(\lambda)].$$

Indeed, if we consider the RWRE $(X_n)_{n \in \mathbb{N}}$ with transition probabilities induced by p, then Λ satisfies $\Lambda(\lambda) = \lim_{n \to \infty} \frac{1}{n} \log E_{0,\omega}[\mathrm{e}^{\lambda \cdot X_n}]$ for all $\lambda \in \mathbb{R}^d$. From here, one can apply the Gärtner-Ellis theorem to conclude the proof of the LDP.

Once we know the quenched and annealed rate functions existence, a natural question is to ask about the relation between them. Theorem 1.1.10 says that $I_a \le I_q$ and that the zero-sets are the same for both functions. Also, $I_q(0) = I_a(0)$. Furthermore, both functions are finite on the ℓ^1-unit ball. In terms of the regularity of the rate functions, substantial progress has been made in the case of the annealed rate function, especially by Peterson and Zeitouni in [PZ] and Yilmaz in [Yil3]. In both articles, it is assumed some form of ballisticity, either condition (T), or the walk being non-nestling (which implies condition (T), see, for example, Proposition 2.4 in [SZ]). In particular, the LLN velocity v in (1.1.5) exists. We summarize in the next theorem the main results from both articles.

Theorem 1.1.12. *Suppose $\mathbb{P}$ satisfies **(UE)** and **(IID)**.*

(i) If the walk is non-nestling, then I_a is analytic on an open set $\mathcal{A}_{nn}$ that contains v.

(ii) If the walk is nestling and condition (T) holds, then there exists an open set $\mathcal{A}_n$ with the following properties:

(a) $(0, v] \subseteq \mathcal{A}_n$.

(b) $\mathcal{A}_n = \mathcal{A}_n^1 \sqcup \mathcal{A}_n^2 \sqcup \mathcal{A}_n^3$, where $\mathcal{A}_n^1$ is open, $\mathcal{A}_n^2 \subseteq \partial\mathcal{A}_n^1$ is a $d-1$ dimensional set such that its relative interior contains v, and $\mathcal{A}_n^3 = \{cu : c \in (0,1), u \in \mathcal{A}_n^1\}$.

(c) I_a is strictly convex and analytic on $\mathcal{A}_n^1$.

(d) I_a is strictly convex and 1-homogeneous on $\mathcal{A}_n^3$.

(e) I_a is continuously differentiable on $\mathcal{A}_n$.

Analogous results for the non-ballistic case are, to the best of the author's knowledge, unknown.

Another important question that we will address now—and which is directly related to this thesis—is the relation between I_a and I_q. Recall that $I_a(x) \leq I_q(x)$ for all $x \in \mathbb{R}^d$. This inequality leads to asking when the rate functions are equal and when they are different. Yilmaz derived the first result in this vein in [Yil1] for random walks in *space-time* product random environment. These walks have the nice property that the environments are independent for different times. In particular, the LLN velocity v exists. More precisely, one can consider an RWRE on $\mathbb{Z}^{d+1}$ of the type $Y_n = (X_n, n)_{n\in\mathbb{N}}$, where $(X_n)_{n\in\mathbb{N}}$ is a "standard" RWRE on $\mathbb{Z}^d$, and the last (deterministic) coordinate represents time. To avoid ambiguities, we usually say that these types of walks have dimension $1 + d$, with d being the dimension of the walk $(X_n)_{n\in\mathbb{N}}$. The result is the following:

Theorem 1.1.13. *Consider an i.i.d. and uniform elliptic space-time RWRE in dimension $1 + d$, with $d \geq 3$. Then there exists a neighborhood of v for which $I_q = I_a$.*

One of the advantages of working in the space-time regime is that the regeneration times are $\tau_n = n$, since the walk never visits the same point twice. The proof of Theorem 1.1.13 uses the so-called martingale method, where one shows first that the quenched and averaged logarithmic moment generating functions coincide in a neighborhood of 0. Using standard methods, one can then conclude that the rate functions coincide in a neighborhood of the velocity. Trying to adapt this proof to conventional RWREs is not trivial since one needs to deal with random regeneration times. This forces to require some sort of ballisticity. Nevertheless, if we restrict the study of the rate functions to the boundary of their domains (i.e., to $\partial\mathbb{D} = \{x \in \mathbb{R}^d : |x|_1 = 1\}$), then the walks at these points behave very similarly to a space-time RWRE. Exploiting this fact is one of the main results of this thesis (see Theorem 1.2.2).

Coming back to "static" RWREs (in contrast to "dynamic" ones, that is, space-time walks), Yilmaz was able to establish equality between quenched and annealed rate functions in a neighborhood of the velocity, when $d \geq 4$. The theorem is stated below.

Theorem 1.1.14. *Suppose $\mathbb{P}$ satisfies **(UE)** and **(IID)**.*

(i) If the walk is non-nestling, then there exists an open set $\mathcal{A}_{eqnn}$ which contains the LLN velocity such that $I_q(x) = I_a(x)$ for each $x \in \mathcal{A}_{eqnn}$.

(ii) If the walk is nestling and condition (T) holds, then I_q and I_a are equal on an open set $\mathcal{A}_{eqn}$. The LLN velocity v satisfies $v \in \partial\mathcal{A}_{eqn}$.

Remark 1.1.3. Some comments about the last theorem:

(i) The result assumes the existence of the velocity, thus some kind of ballisticity is needed. That is why in the nestling case the hypothesis of condition (T) is added (it is also heavily used in the proof).

(ii) Both Theorem 1.1.12 and Theorem 1.1.14 do not give information far away of the velocity.

Combining both points in the last remark, natural questions that arise are the following:

Question 1.1.1. Can we say something about equality of the rate functions in neighborhoods that may not necessarily contain the velocity?

If the answer to the previous question is affirmative, a follow-up question is

Question 1.1.2. Do we need to assume that the walk is ballistic?

In this thesis, we will provide a positive answer to the first question, and a negative answer to the second (see Section 1.2).

The reader may ask what happens in dimensions two and three. It turns out that the conclusion of Theorem 1.1.14 cannot be extended to these cases. The result for the space-time case is given first.

Theorem 1.1.15. *Consider an i.i.d and uniform elliptic space-time RWRE in dimension $1+d$.*

(i) If $d=1$, then $I_q(x) = I_a(x) < \infty$ if and only if x is the LLN velocity.

(ii) If $d=2$, there exists a punctured neighborhood of the velocity for which $I_a < I_q$.

The corresponding result for static RWREs is given next.

Theorem 1.1.16. *Let $d = 2$ or 3. Assume $\mathbb{P}$ satisfies **(UE)** and **(IID)**. Then there exists a class of non-nestling walks such that for any open set that contains the velocity, I_q and I_a are not identically equal in such set.*

The proof of Theorems 1.1.15 and 1.1.16 uses the *fractional moment* method developed by Lacoin in [Lac] to study the difference between the quenched and averaged free energies for directed polymers in random environment.

1.1.3 Closing remarks

The LDP we have discussed so far is sometimes called *level 1* large deviations, in contrast to the level 2 (which deal with the empirical measure) and level 3 (also called *process level*) large deviations[3]. In this thesis we will be focused only in level 1 large deviations. For the sake of completeness, we mention some results from higher levels, and generalizations of previous contributions.

Level 2 large deviations for RWRE were studied by Yilmaz in [Yil2]. Rather than considering the position, he deals with the random variable $\nu_{n,X} := \frac{1}{n}\sum_{k=0}^{n-1} \delta_{T_{X_k}\omega, X_{k+1}-X_k} \in \mathcal{M}_1(\Omega \times \mathbb{V})$. Then Yilmaz provided a LDP for the sequence of measures $\mu_n := P_{0,\omega}(\nu_{n,X} \in \cdot)$, assuming (1.1.9). Rassoul-Agha and Seppäläinen in [RAS2] showed a level 3 large deviation principle, where instead of $\nu_{n,X}$ they study $R^{1,\infty}_{n,X} := \frac{1}{n}\sum_{k=0}^{n-1} \delta_{T_{X_k}\omega,(X_{i+1}-X_i)_{i\geq k}}$.

Generalization of some of the large deviation results mentioned so far to the so-called *random walks in random potential* (RWRP) can be found in [RASY1], [RAS3], [RASY3], [RASY2] and [JNRA].

[3] The nomenclature of levels is used since one can go from higher levels to lower levels via the contraction principle. See the references [DZ] and [RAS4] for more details.

1.2 Our results

The results of this thesis fall into two categories: large deviations and localization. We discuss now each of them.

1.2.1 Equality and difference of quenched and annealed rate functions for RWRE

Recall the questions we posed in Questions 1.1.1 and 1.1.2. One of the limitations of current methods that deals with studying the equality (and difference) between I_q and I_a is that they only tell us information in a neighborhood close to the velocity. In particular, we first need to make sure that such velocity even exists. Thus, the articles discussed so far assume some form of ballisticity, such as condition (T). In our work, we tackle both problems at once, introducing the concept of *disorder*. In simple terms, this concept measures how far an RWRE is from being deterministic. As a simple example, consider a random perturbation of a simple random walk, namely,

$$\omega_\varepsilon(x,e) := \frac{1}{2d} + \varepsilon\xi(x,e),\ \varepsilon \geq 0, \tag{1.2.1}$$

for some i.i.d random vectors $(\xi(x))_{x\in\mathbb{Z}^d}$ that satisfies suitable conditions so that $\omega_\varepsilon = (\omega_\varepsilon(x,e))_{x\in\mathbb{Z}^d,e\in\mathbb{V}}$ is an environment. In this case, one may say that the disorder is ε. Indeed, if $\xi(x,e)\in[-1,1]$, one can deduce that, for each $e\in\mathbb{V}$, $|\omega_\varepsilon(x,e)-(2d)^{-1}|<\varepsilon$. Alternatively, instead of considering the difference, one may take quotients instead,

$$\frac{\omega_\varepsilon(x,e)}{\frac{1}{2d}} = 1 + 2d\varepsilon\xi(x,e), \tag{1.2.2}$$

which, if one again assumes that $\xi(x,e) \in [-1,1]$ and $\varepsilon' := 2d\varepsilon$, we can get the bound

$$\frac{\omega_\varepsilon(x,e)}{\frac{1}{2d}} \in [1-\varepsilon', 1+\varepsilon'].$$

We will take the second point of view to define the disorder of any environment that fulfills both **(UE)** and **(IID)**.

Definition 1.2.1. (disorder, $\mathrm{dis}(\mathbb{P})$) Given any environmental law $\mathbb{P}$ that satisfies **(UE)** and **(IID)**, we define its disorder as

$$\mathrm{dis}(\mathbb{P}) := \inf\left\{\varepsilon > 0 : \frac{\omega(x,e)}{\mathbb{E}[\omega(x,e)]} \in [1-\varepsilon, 1+\varepsilon] \ \mathbb{P}\text{-a.s. for all } x \in \mathbb{Z}^d \text{ and } e \in \mathbb{V}\right\}. \quad (1.2.3)$$

Remark 1.2.1. We mention some works that have used a similar notion of disorder.

(i) In [Szn4], Sznitman considers random perturbations of the simple symmetric random walk and shows conditions for which the walk is ballistic. Moreover, he gives examples of walks that satisfies condition (T) but not Kalikow's condition. More examples are given in the articles [RS] and [FR].

(ii) In [Sab], Sabot considers environments of the type $\omega_\gamma(x,e) = p_0(e)+\gamma\xi(x,e)$, and shows that, under certain conditions, for γ small enough the walk is ballistic. Moreover, he gives an expansion formula for the asymptotic speed.

Our first main result is the equality between I_q and I_a for small disorder on the interior of $\mathbb{D}$.

Theorem 1.2.1. *Let $d \geq 4$, $\kappa > 0$, and a compact set $\mathcal{K} \subseteq \mathrm{int}(\mathbb{D}) \setminus \{0\}$. Then there exists $\varepsilon = \varepsilon(d,\kappa,\mathcal{K}) > 0$ such that, for any RWRE with environmental law $\mathbb{P}$ satisfying* ***(UE)*** *with ellipticity constant κ and* ***(IID)****, if $\mathrm{dis}(\mathbb{P}) < \varepsilon$, then $I_q(x) = I_a(x)$ for each $x \in \mathcal{K}$.*

An analogous result is valid on the boundary $\partial\mathbb{D} = \{x \in \mathbb{R}^d : |x|_1 = 1\}$. We need to introduce some additional notation first. Given $s \in \{\pm 1\}^d$, let

$$\partial\mathbb{D}(s) := \{x \in \partial\mathbb{D} : x_j s_j \geq 0 \text{ for all } 1 \leq j \leq d\},$$
$$\partial\mathbb{D}_{d-2} := \{x \in \partial\mathbb{D} : x_j = 0 \text{ for some } 1 \leq j \leq d\}.$$

Now we can state our result on the boundary.

Theorem 1.2.2. *Let $d \geq 4$, $\kappa > 0$, and a compact set $\mathcal{K} \subseteq \partial\mathbb{D} \setminus \partial\mathbb{D}_{d-2}$. Then there exists $\varepsilon = \varepsilon(d, \kappa, \mathcal{K}) > 0$ such that, for any RWRE with environmental law $\mathbb{P}$ satisfying **(UE)** with ellipticity constant κ and **(IID)**, if $\mathrm{dis}(\mathbb{P}) < \varepsilon$, then $I_q(x) = I_a(x)$ for each $x \in \mathcal{K}$.*

Additionally, we obtain an equality result on the boundary under a weaker condition we call *imbalance*.

Definition 1.2.2. (imbalance, $imb_s(\mathbb{P})$) Given any $s \in \{\pm 1\}^d$, the imbalance of the environmental law $\mathbb{P}$ on the face $\partial\mathbb{D}(s)$ is

$$imb_s(\mathbb{P}) := \inf \left\{ \varepsilon > 0 : \frac{\sum_{i=1}^d \omega(x, s_i e_i)}{\sum_{i=1}^d \mathbb{E}[\omega(x, s_i e_i)]} \in [1-\varepsilon, 1+\varepsilon] \ \mathbb{P}\text{-a.s. for all } x \in \mathbb{Z}^d \right\}. \quad (1.2.4)$$

The next theorem is about equality of I_q and I_a as soon as the imbalance is sufficiently small.

Theorem 1.2.3. *Let $d \geq 4, \kappa > 0$ and $s \in \{\pm 1\}^d$. Then there exists $\varepsilon^* = \varepsilon^*(d, \kappa) > 0$ and a compact set $\mathcal{K}^* = \mathcal{K}^*(d, \kappa)$ such that, for any RWRE with environmental law $\mathbb{P}$ satisfying **(UE)** with ellipticity constant κ and **(IID)**, if $imb_s(\mathbb{P}) < \varepsilon^*$, then $I_q(x) = I_a(x)$ for all $x \in \mathcal{K}^*$.*

One of the interesting consequences of the results on the boundary, is that we can obtain as a corollary an explicit formula for the quenched rate function on the boundary. Indeed, it is not difficult to show that (see the proof of Lemma 3.3.5) for any $x \in \partial\mathbb{D}$ one has the formula (with the convention that $0 \log 0 = 0$)

$$I_a(x) = \sum_{i=1}^{d} |x_i| \log \frac{|x_i|}{\mathbb{E}[\omega(0, \frac{x_i}{|x_i|} e_i)]}. \tag{1.2.5}$$

Therefore, whenever equality holds on the boundary (i.e. either in Theorem 1.2.2 or Theorem 1.2.3), we have a simple expression for I_q. As we saw before (e.g., Eq. (1.1.10)), obtaining such formulas, in general, is not an easy endeavor.

The general conclusion at this point is that low disorder implies equality of the rate functions. One would like to show the opposite: if the disorder is large enough, $I_a < I_q$. In statistical mechanics terms, is there some critical value ε_c such that equality holds if the disorder is less than ε_c, and inequality otherwise? We deal with this problem for a parametrized family of environments, in the spirit of ω_ε from (1.2.1), that makes suitable to study the map $\varepsilon \to I_a(x,\varepsilon) - I_q(x,\varepsilon)$ for fixed $x \in \partial\mathbb{D}$ (here, $I_q(\cdot,\varepsilon)$ is the quenched rate function with environment ω_ε, and the same for I_a). In order to define the parametrization, we need to introduce some additional notation.

Given $\alpha \in \mathcal{M}_1(\mathbb{V})$ with positive coordinates, let

$$\mathcal{E}_\alpha := \left\{ (r(e))_{e\in\mathbb{V}} \in [-1,1]^{\mathbb{V}} : \sum_{e\in\mathbb{V}} \alpha(e) r(e) = 0 \text{ and } \sup_{e\in\mathbb{V}} |r(e)| = 1 \right\}. \tag{1.2.6}$$

Let $\mathbb{Q}$ be any product measure on the space $\Gamma := \mathcal{E}_\alpha^{\mathbb{Z}^d}$. Similarly as we did with Ω, we write for a typical element $\eta \in \Gamma_\alpha$, $\eta = (\eta(x,e))_{x\in\mathbb{Z}^d, e\in\mathbb{V}}$. Finally, define the family of environments $(\omega_\varepsilon)_{\varepsilon\in[0,1)}$ as

$$\omega_\varepsilon(x,e) := \alpha(e)(1 + \varepsilon\eta(x,e)). \tag{1.2.7}$$

We will make the following assumption on the measure $\mathbb{Q}$:

Assumption 1.2.1.

- $\mathbb{E}[\xi(x,e)] = 0$ for any $x \in \mathbb{Z}^d, e \in \mathbb{V}$, and

- the family $(\eta(x,\cdot))_{x\in\mathbb{Z}^d}$ is independent under $\mathbb{Q}$.

Under these assumptions, and by definition of $\mathcal{E}_\alpha$, if $\mathbb{P}_\varepsilon$ is the law of the environment ω_ε, we deduce the following: for $\varepsilon > 0$, $\mathbb{P}_\varepsilon$ satisfies **(UE)** with ellipticity constant $\kappa = (1-\varepsilon)\min_{e\in\mathbb{V}}\alpha(e)$, **(IID)**, and $\mathrm{dis}(\mathbb{P}) = \varepsilon$.

Now we can state our monotonicity result. Let $I_q(\cdot,\varepsilon)$ (resp. $I_a(\cdot,\varepsilon)$) be the quenched (resp. average) rate function for an RWRE with environmental law $\mathbb{P}_\varepsilon$ as above.

Theorem 1.2.4. *Let $d \geq 2$, $\alpha \in \mathcal{M}_1(\mathbb{V})$ with positive coordinates, and a measure $\mathbb{Q}$ on Γ_α satisfying Assumption 1.2.1. Then the following holds:*

(i) For each $x \in \partial\mathbb{D}$, the map $[0,1) \ni \varepsilon \to I_a(x,\varepsilon) - I_q(x,\varepsilon)$ is non-increasing and continuous. In particular, there exists some $\varepsilon_c(x) \geq 0$ such that if $0 \leq \varepsilon \leq \varepsilon_c(x)$ we have $I_a(x) = I_q(x)$, while for $\varepsilon_c(x) < \varepsilon < 1$ we have $I_a(x) < I_q(x)$.

(ii) If $d \geq 4$, there exists a compact set $\mathcal{K} \subseteq \partial\mathbb{D} \setminus \partial\mathbb{D}_{d-2}$ such that for any $x \in \mathcal{K}$ one has $0 < \varepsilon_c(x) < 1$. That is, there is a genuine phase transition.

The results on the interior will be shown in Chapter 2, while the boundary case will be treated in Chapter 3, together with further discussions about the results.

1.2.2 Localization at the boundary for RWRE

We now will summarize our results about localization. First, let us comment upon the concept of localization, as used in the RWRE literature. Its use has been made only in the one-dimensional case. The first important contribution in this matter is the work of Sinai [Sin], where he studies the recurrent case (i.e. case 3. in Theorem 1.1.1). He shows that there exists a random process $(m_n)_{n\in\mathbb{N}}$ such that the walk is concentrated in a neighborhood of size $o(\log^2 n)$ around m_n at time n, as $n \to \infty$. Golosov in [Gol] improves Sinai's result, showing that the neighborhood around m_n is of bounded size as $n \to \infty$.

On the other hand, the notion of localization is very familiar in other models such as the *parabolic Anderson model* and *directed polymers in random environment*. Indeed, we will adapt the definition from the polymer model to our setup. Nevertheless, the concept is similar to the RWRE counterpart; we say the walk is localized if there exists a *favorite path* for which the walk moves with high (relative) probability. We now define the concept of localization rigorously.

We will restrict ourselves to trajectories at the boundary, that is, for walks $(X_n)_{n\in\mathbb{N}}$ such that $|X_n|_1 = n$ for all $n \in \mathbb{N}$. More precisely, we will fix a face of the boundary $\partial\mathbb{D}(s)$ for some $s \in \{\pm 1\}^d$. For simplicity, let us consider the face $\partial\mathbb{D}(\overline{s})$, where $\overline{s} := \{1, 1, \cdots, 1\}$. Also define

$$\partial R_n := \{x \in \mathbb{Z}^d : |x|_1 = n \text{ and } x_j \geq 0 \text{ for all } 1 \leq j \leq d\}. \tag{1.2.8}$$

Finally, let $\mathcal{A}_n := \{X_n - X_0 \in \partial R_n\}$.

Definition 1.2.3. Let $J_n := \max_{x\in\mathbb{Z}^d} P_{0,\omega}(X_{n-1} = x|\mathcal{A}_n)$. We say the RWRE $(X_n)_{n\in\mathbb{N}}$ is localized at the boundary if

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} J_n > 0 \ \mathbb{P}\text{-a.s.} \tag{1.2.9}$$

Similarly, we say the walk is delocalized at the boundary if

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} J_n = 0 \ \mathbb{P}\text{-a.s.} \tag{1.2.10}$$

One of the main results we show in Chapter 4 is that localization occurs almost always when $d = 2$ or 3. The assumption we need is the following:

Assumption 1.2.2. The measure $\mathbb{P}$ satisfies

$$\mathbb{P}\left(\sum_{e\in\mathbb{V}^+}\omega(x,e)=\sum_{e\in\mathbb{V}^+}\mathbb{E}[\omega(0,e)]\right)<1. \tag{1.2.11}$$

Theorem 1.2.5. *Suppose $\mathbb{P}$ satisfies **(UE)**, **(IID)** and Assumption 1.2.11. Then, if $d=2$ or 3, the walk is localized at the boundary. If Assumption 1.2.11 is not satisfied, then the walk is delocalized. Delocalization also holds for any dimension $d\geq 2$ when Assumption 1.2.11 is not fulfilled.*

When $d \geq 4$, we consider the same parametrization we used in (1.2.7) to deduce a phase transition for the map $\varepsilon \to I_a(x,\varepsilon) - I_q(x,\varepsilon)$ to show a phase transition for the localization/delocalization phenomenon. Before stating the result, we require some more notation. Given $\alpha \in \mathcal{M}_1(\mathbb{V}^+)$ and $\kappa > 0$, we define $\varepsilon_{max} := 1 - \frac{\kappa}{\min_{e\in\mathbb{V}}\alpha(e)}$, the maximum disorder parameter so that the walk is uniform elliptic with ellipticity constant at most κ. Finally, we say the walk is ε-localized (resp. ε-delocalized) if (1.2.9) (resp. (1.2.10)) holds with the measure $\mathbb{P}_\varepsilon$.

Theorem 1.2.6. *Let $d \geq 2$, $\alpha \in \mathcal{M}_1(\mathbb{V}^+)$, $\kappa > 0$ and a measure $\mathbb{Q}$ on Γ_α satisfying Assumption 1.2.1. Then there exists some $\overline{\varepsilon} \in [0, \varepsilon_{max}]$ such that the walk is ε-delocalized for $0 \leq \varepsilon \leq \overline{\varepsilon}$, and ε-localized for $\overline{\varepsilon} < \varepsilon \leq \varepsilon_{max}$. Moreover,*

(i) If Assumption 1.2.11 does not hold, then $\overline{\varepsilon} = \varepsilon_{max}$. Otherwise,

(ii) if $d = 2$ or 3, then $\overline{\varepsilon} = 0$;

(iii) if $d \geq 4$, $\overline{\varepsilon} > 0$. Also, there are examples of walks that satisfy $\overline{\varepsilon} < \varepsilon_{max}$.

Chapters 2 and 3 are joint work with Chiranjib Mukherjee, Alejandro F. Ramírez and Santiago Saglietti.

Chapter 2

Equality of quenched and averaged large deviations for RWRE: the impact of the disorder on the interior

2.1 Introduction and background.

The model of a *random walk in a random environment* (RWRE) provides a natural setting for studying "statistical mechanics in random media" and has enjoyed a profound upsurge of interest within physicists and mathematicians in the recent years. RWRE-s were first considered by Solomon [Sol] and extended later by Sinai [Sin] which provided a very efficient methodology for studying the one-dimensional case which is by now fairly well-understood, and exhibits behaviors that are very different from that of the simple random walk. On the other hand, multi-dimensional RWRE turns out to be much more difficult than the one-dimensional model, and even some of the very fundamental questions have remained quite challenging till date.

RWRE can be described as a two-layer process. First, consider a sequence $\omega = (\omega(x))_{x\in\mathbb{Z}^d}$

of probability vectors on $\mathbb{V} := \{x \in \mathbb{Z}^d : |x|_1 = 1\} = \{\pm e_1, \ldots, \pm e_d\}$ indexed by the sites of the lattice, i.e. $\omega(x) = (\omega(x,e))_{e\in\mathbb{V}}$ is a probability vector on $\mathbb{V}$ for each $x \in \mathbb{Z}^d$. Any such sequence ω will be called an *environment* and the space Ω of all such sequences will be called the *environment space*. Then, the first layer of our process consists of, for a fixed $\omega \in \Omega$, a random walk on the lattice whose jump probabilities are given by the environment ω, i.e. for each $x \in \mathbb{Z}^d$ the law $P_{x,\omega}$ of this random walk $(X_n)_{n\geq 0}$ starting at x is prescribed by

$$P_{x,\omega}(X_0 = x) = 1 \qquad \text{and} \qquad P_{x,\omega}(X_{n+1} = y + e \,|\, X_n = y) = \omega(y,e) \quad \forall\, y \in \mathbb{Z}^d\,,\, e \in \mathbb{V}.$$

We call $P_{x,\omega}$ the *quenched law* of the RWRE. The second layer of our process is then obtained when the environment ω is chosen at random according to some Borel probability measure $\mathbb{P}$ on Ω (when endowed with the usual product topology). We call any such $\mathbb{P}$ an *environmental law*. Averaging $P_{x,\omega}$ over ω then produces a probability measure on $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}$ given by the formula

$$P_x(A \times B) = \int_A P_{x,\omega}(B)\mathrm{d}\mathbb{P} \qquad \forall\, A \in \mathcal{B}(\Omega)\,,\, B \in \mathcal{B}((\mathbb{Z}^d)^{\mathbb{N}}).$$

We call the measure P_x the *averaged* or *annealed* law of the RWRE (starting at x) and the sequence $X = (X_n)_{n\in\mathbb{N}}$ under P_x a RWRE with environmental law $\mathbb{P}$.

Given any RWRE, it is natural to ask whether classical limit theorems can hold for its quenched and annealed measures. The law of large numbers (LLN) for the quenched distribution, if valid, takes the form $\mathbb{P}\big(\omega\colon\ \lim_{n\to\infty} \frac{X_n}{n} = v \quad \text{a.e. w.r.t. } P_{x,\omega}\big) = 1$ for some $v \in \mathbb{R}^d$. The latter display is equivalent to the validity of $P_x(\omega\colon\ \lim_{n\to\infty} X_n/n = v) = 1$ which translates to the LLN for the annealed measure. We refer to the literature[Law, PV, Koz2, KV, Ber1, BZ, RAS1] where both LLN and central limit theorems (CLT) have been investigated quite successfully whenever the law $\mathbb{P}$ of the ambient environment enjoys some special properties like the existence of an invariant density for the *environment viewed from the particle* or that of *strong transience* conditions.

While RWRE exhibit the same behavior in the quenched and the annealed setting on the level of LLN, the resulting scenarios for the two cases could be very different for regimes concerning CLTs or large deviation principles (LDP). The latter statement concerns investigating the (formally written) asymptotic behavior

$$\lim_{n\to\infty}\frac{1}{n}\log P_{0,\omega}\left(\frac{X_n}{n}\approx x\right)\simeq -I_q(x),\quad \text{and}\qquad \lim_{n\to\infty}\frac{1}{n}\log P_0\left(\frac{X_n}{n}\approx x\right)\simeq -I_a(x), \tag{2.1.1}$$

where the former statement holds for $\mathbb{P}$-a.e. ω, while I_q and I_a are the *quenched* and *annealed large deviation rate functions*, respectively. From Jensen's inequality and Fatou's lemma it follows that $I_a(\cdot)\leq I_q(\cdot)$. However, a deeper connection between the two rate functions is closely intertwined with the profound interplay between the random walk and the underlying impurities of the environment. Indeed, if at a large time n, the RWRE were to find itself at an atypical location, one could wonder if such unlikely scenario resulted from a strange behavior of the particle in that environment or if the particle actually encountered an atypical environment. Such questions are intimately linked with the relationship between the quenched and the annealed rate functions. The incentive to study and relate these two rate functions therefore becomes quite natural. In this vein, the main result of the current article (stated below formally in Theorem 2.2.1) is that, for $d\geq 4$ and *any* compact subset $\mathcal{K}$ of the *open* ℓ^1-unit ball (not containing the origin), the two rate functions I_q and I_a of *any* RWRE in a uniformly elliptic and i.i.d. environment agree on $\mathcal{K}$, if the *disorder* of the environment remains sufficiently small. Apart from the result itself, this work introduces a novel point of view from which to study the problem of equality of the rate functions, namely that of the disorder of the environment. Indeed, our result suggests that, unless one is focused on very particular regions of the domain (such as the corners in its boundary or neighborhoods around the velocity whenever the RWRE is ballistic), disorder should play an essential role in whether equality between the two rate functions holds, in the sense that equality should hold below and fail above a certain threshold disorder. This intuition has been confirmed when looking at the rate functions at the boundary of their domain for a certain wide family of environments in a separate work [BMRS2], see Remark 2.2.5. We turn to a precise statement of the main result of the article.

2.2 Main result

In the sequel we shall work with environmental laws $\mathbb{P}$ satisfying the following assumption:
Assumption A: the environment is *i.i.d.* (i.e. the random vectors $(\omega(x))_{x\in\mathbb{Z}^d}$ are independent and identically distributed under $\mathbb{P}$) and *uniformly elliptic* under $\mathbb{P}$, i.e., there is a constant $\kappa > 0$ such that, for all $x \in \mathbb{Z}^d$ and $e \in \mathbb{V}$,

$$\mathbb{P}(\omega(x,e) \geq \kappa) = 1. \tag{2.2.1}$$

Given any environmental law $\mathbb{P}$ satisfying Assumption A, we now define its *disorder* as

$$\mathrm{dis}(\mathbb{P}) := \inf\left\{\varepsilon > 0 : \xi(x,e) \in [1-\varepsilon, 1+\varepsilon],\ \mathbb{P}\text{-a.s. for all } e \in \mathbb{V} \text{ and } x \in \mathbb{Z}^d\right\}, \tag{2.2.2}$$

$$\text{with}\quad \xi(x,e) := \frac{\omega(x,e)}{\alpha(e)} \quad \text{and} \quad \alpha(e) := \mathbb{E}[\omega(x,e)] \qquad \forall\, e \in \mathbb{V}, \tag{2.2.3}$$

where $\mathbb{E}$ denotes expectation w.r.t. $\mathbb{P}$ and the definition of $\alpha(e)$ does not depend on $x \in \mathbb{Z}^d$ by Assumption A. Moreover, both $\xi(x,e)$ and $\mathrm{dis}(\mathbb{P})$ are well-defined since $\mathbb{P}$ satisfies Assumption A and $\mathrm{dis}(\mathbb{P})$ can be seen as the $L^\infty(\mathbb{P})$-norm of the random vector $(\xi(x,e)-1)_{e\in\mathbb{V}}$ for any $x \in \mathbb{Z}^d$. In [Var], Varadhan proved that, under Assumption A, both the quenched distribution $P_{0,\omega}\left(\frac{X_n}{n}\right)^{-1}$ and its averaged version $P_0\left(\frac{X_n}{n}\right)^{-1}$ satisfy a large deviations principle, that is, that there exist two lower-semicontinuous functions $I_a, I_q : \mathbb{R}^d \to [0,\infty]$ such that for any $G \subseteq \mathbb{R}^d$ with interior $\mathrm{int}(G)$ and closure $\overline{G}$,

$$-\inf_{x\in\mathrm{int}(G)} I_q(x) \le \liminf_{n\to\infty} \frac{1}{n}\log P_{0,\omega}\left(\frac{X_n}{n}\in G\right) \le \limsup_{n\to\infty} \frac{1}{n}\log P_{0,\omega}\left(\frac{X_n}{n}\in G\right) \le -\inf_{x\in\overline{G}} I_q(x)$$

for $\mathbb{P}$-almost every $\omega \in \Omega$ (and that the analogous statement obtained by replacing $P_{0,\omega}$ and I_q by P_0 and I_a also holds), see Remark 2.2.3 for a brief overview of the literature on large deviations for RWRE. If $|x|_1$ denotes the ℓ^1 norm of $x \in \mathbb{R}^d$, and we write $\mathbb{D} := \{x \in \mathbb{R}^d \colon |x|_1 \le 1\}$ for the closed ℓ^1-unit ball and $\mathrm{int}(\mathbb{D}) := \{x \in \mathbb{R}^d \colon |x|_1 < 1\}$ for its interior, it can be shown that the rate functions I_q and I_a are both convex and are finite if and only if $x \in \mathbb{D}$. Being also lower semicontinuous, this implies that both I_q and I_a are continuous functions on $\mathbb{D}$, see [Roc, Theorem 10.2]. Furthermore, for any RWRE satisfying Assumption A, regardless of the disorder and in any $d \ge 2$, we always have $I_q(0) = I_a(0)$ and $\{I_q = 0\} = \{I_a = 0\}$ (see [Var, Theorem 8.1] and also Theorem 7.1 there for a formula for $I_a(0)$) and it is also well-known that one always has the inequality $I_a \le I_q$. Here is our main result.

Theorem 2.2.1. *For any $d \ge 4$, $\kappa > 0$ and compact set $\mathcal{K} \subseteq \mathrm{int}(\mathbb{D}) \setminus \{0\}$, there exists $\varepsilon = \varepsilon(d,\kappa,\mathcal{K}) > 0$ such that, for any RWRE satisfying Assumption* A *with ellipticity constant κ, if $\mathrm{dis}(\mathbb{P}) < \varepsilon$ then we have the equality*

$$I_q(x) = I_a(x) \qquad \textit{for all } x \in \mathcal{K}.$$

Let us make some comments about the result.

Remark 2.2.1 (The region of equality). As mentioned above, since $I_q(0) = I_a(0)$, for any $d \ge 4$ and $\kappa > 0$ and for any $x \in \mathrm{int}(\mathbb{D})$, the above result implies that there is $\varepsilon > 0$ such that $I_a(x) = I_q(x)$ for $\mathrm{dis}(\mathbb{P}) < \varepsilon$, so we can think of the result above as saying that the region of equality $\{x \in \mathrm{int}(\mathbb{D}) : I_q(x) = I_a(x)\}$ covers the entirety of $\mathrm{int}(\mathbb{D})$ in the limit as $\text{dis}(\mathbb{P}) \to 0$, uniformly over all environmental laws $\mathbb{P}$ with a uniform ellipticity constant bounded from below by some $\kappa > 0$. However, we point out that, unless $\mathbb{P}$ is degenerate, I_a and I_q can never be equal everywhere in $\text{int}(\mathbb{D})$ for a *fixed environmental law* $\mathbb{P}$, see [Yil4, Proposition 4]. Finally, for $d \in \{2, 3\}$, such an identity between the two rate functions is not expected to be true for general RWRE, as shown in [YZ]: for $d = 2, 3$ there is a class of non-nestling random walks in uniformly elliptic and i.i.d. environments such that I_a and I_q are never identical on any open neighborhood of the velocity.

Remark 2.2.2 (An auxiliary random walk). One of the novelties of our approach is the introduction of an auxiliary random walk (in a deterministic environment) satisfying the following key properties: (i) (a particular version of) its logarithmic moment generating function is intimately related with those of the RWRE (see Section 2.2.1 for further details) and (ii) this walk is ballistic and possesses a strong regeneration structure. By means of this auxiliary walk, we are able to study the LDP properties of the original RWRE using techniques available for ballistic walks, even if our original RWRE is not ballistic itself.

Remark 2.2.3 (Literature remarks). Large deviations for RWRE for $d = 1$ were handled by Greven and den Hollander [GdH] in the quenched setting and by Comets, Gantert and Zeitouni [CGZ] (see also [GZ]) in both quenched and annealed settings (including a variational formula relating the the two rate functions. For $d \geq 1$, using sub-additive arguments, Zerner [Zer] (see also Sznitman [Szn1]) proved a quenched LDP for "nestling environments"[1], while Varadhan [Var] dropped the latter assumption on the environment and proved both the quenched and annealed LDP. Kosygina, Rezakhanlou and Varadhan [KRV] developed a novel method for obtaining quenched LDP for elliptic diffusions with a random drift based on a convex variational approach, which was adapted by Rosenbluth [Ros] for elliptic RWRE in $d \geq 1$ and developed further by Yilmaz [Yil2] and by Rassoul-Agha Sepäläinen [RAS2].

[1] A RWRE is called nestling if the origin lies in the interior of the convex hull of the support of the local drift $\sum_{e \in \mathbb{V}} e\omega(0, e)$ around the origin.

The latter approach was extended to non-elliptic models like random walks on percolation clusters including long-range correlations in [BMO] (see also Kubota [Kub] and Mourrat [Mou] for sub-additive approaches).

Remark 2.2.4 (Previous results under condition (T)). To put our work into context, let us now comment on a strong ballisticity criterion known as *condition* (T), introduced by Sznitman [Szn2], which is the main assumption for all previously known results on the equality of the rate functions (at least for standard RWRE in dimensions $d \geq 4$). Given a direction $\ell \in \mathbb{S}^{d-1}$, the RWRE is said to satisfy condition (T) if for some $\gamma \in (0,1]$ (or, equivalently, if for any such γ) there exists a neighborhood V of ℓ such that, for all $\ell' \in V$,

$$\lim_{L\to\infty} L^{-\gamma} \log P_0\big[\langle X_{T_{U_{\ell',L}}}, \ell'\rangle < 0\big] < 0, \tag{2.2.4}$$

where $T_{U_{\ell',L}} := \inf\{n \geq 0\colon X_n \notin U_{\ell',L}\}$ is the exit time from $U_{\ell',L} := \{x \in \mathbb{Z}^d : -L < \langle x, \ell'\rangle < L\}$, see also [GR]. Under Assumption A, condition (T) implies that: i) a law of large numbers $\lim_{n\to\infty} \frac{X_n}{n} = v$ holds P_0-a.s. with a non-zero velocity v and ii) there exist regeneration times (with finite moments) such that the RWRE segments embedded between these times are an i.i.d. sequence under P_0. This regeneration structure has proved very fruitful tool in the study of LDP for RWRE (see e.g. [PZ, Yil3, Ber2]). However, there are prominent RWRE models which do not satisfy this condition; examples of which include random walks in a balanced random environment, the random conductance model, random walks on various percolation clusters as well as random walks on random graphs and trees for which the limiting velocity, or the *expected local drift* happens to be zero, denying any ballistic march of the random walk along any direction.

Under Assumption A and condition (T), it was shown in [Yil4] that when $d \geq 4$, $I_a = I_q$ on some (possibly small) neighborhood of the non-zero velocity (which, as mentioned above, always exists under (2.2.4)). Note that this result does not require the disorder of the environment to be small, but in return only yields equality in a (possibly small) neighborhood of a very specific point in the domain. In contrast, Theorem 2.2.1 does not require the walk to be ballistic nor are we restricted to neighborhoods around given points, as long as the disorder of the environment is maintained low. In particular, our result applies to random walks in *balanced* random environments, i.e. such that $P(\omega(x,e) = \omega(x,-e)\,\forall\, x,e) = 1$, a model where Sznitman's condition (T) fails to hold. Finally, we recall that, as shown in [Var], we always have the equality $I_a(0) = I_q(0)$ under Assumption A, regardless of the disorder. However, except for some results in specific scenarios (see [Yil4, Theorem 5-(iv)]), both our approach and that in [Yil4] seem unfit to study the equality of the rate functions in *neighborhoods* of the origin.

Remark 2.2.5 (Relation between I_q and I_a on the boundary of $\mathbb{D}$). In a separate work [BMRS2], we show the analogue of Theorem 2.2.1 for compact sets on the boundary $\partial\mathbb{D} := \{x \in \mathbb{R}^d : |x|_1 = 1\}$ (not intersecting any of the $(d-2)$-dimensional facets of $\partial\mathbb{D}$). As a consequence, we obtain that both I_q and I_a admit simple explicit formulas on the boundary $\partial\mathbb{D}$ for sufficiently small disorder. We refer to [RASY2, RASY3] for an alternative variational representation of I_q.

2.2.1 Outline of the proof

For conceptual clarity and convenience of the reader, we now present a brief outline of the proof of Theorem 2.2.1, highlighting the main novelties of our approach as well as the similarities and differences with that of earlier works in the same vein.

The proof of Theorem 2.2.1 consists of three parts, which we summarize below. We first notice that, in order to obtain Theorem 2.2.1, it will suffice to show that, for any $y \in \mathrm{int}(\mathbb{D}) \setminus \{0\}$ and $\kappa > 0$, if $d \geq 4$ then there exist $\varepsilon = \varepsilon(y,d,\kappa), r = r(y,d,\kappa) > 0$ such that $I_q = I_a$ on $B_r(y)$, the ℓ^1-ball of radius r centered at y, for any RWRE with $\mathrm{dis}(\mathbb{P}) < \varepsilon$ which satisfies Assumption A with ellipticity constant κ (see Theorem 2.3.1 below). Thus, in the following we explain the steps towards showing this variant of Theorem 2.2.1 for a henceforth fixed $y \in \mathrm{int}(\mathbb{D}) \setminus \{0\}$ and $\kappa > 0$.

Step 1: The first building block of the proof, which is one of the main novelties of our approach, is the construction of an auxiliary random walk in a *deterministic* environment verifying that:

Q1. It is ballistic with velocity y and, furthermore, possesses strong regeneration properties;

Q2. If we denote its law when starting from 0 by Q_0 and define its "quenched" limiting logarithmic moment generating functions as

$$\overline{\Lambda}_q(\theta) := \lim_{n\to\infty} \frac{1}{n} \log E_0^Q\Big(\mathrm{e}^{\langle\theta, X_n\rangle} \prod_{j=1}^{n} \xi(X_{j-1}, \Delta_j(X))\Big) \quad \theta \in \mathbb{R}^d,$$

and its "averaged" counterpart $\overline{\Lambda}_a(\theta)$ being defined analogously with $\prod_{j=1}^{n} \xi(X_{j-1}, \Delta_j(X))$ replaced by its averages $\mathbb{E}\prod_{j=1}^{n} \xi(X_{j-1}, \Delta_j(X))$ (here E_0^Q denotes expectation w.r.t. Q_0, ω is the random environment from our original RWRE and ξ is given by (2.2.3)), then essentially (see Section 2.4.2 for details)

$$I_q - I_a = \tilde{I}_q - \tilde{I}_a, \tag{2.2.5}$$

where $\tilde{I}_q$ and $\tilde{I}_a$ are the Fenchel-Legendre transforms of $\overline{\Lambda}_q$ and $\overline{\Lambda}_a$ respectively, i.e.

$$\tilde{I}_q(x) = \sup_{\theta\in\mathbb{R}^d}[\langle\theta, x\rangle - \overline{\Lambda}_q(\theta)] \qquad \text{and} \qquad \tilde{I}_a(x) = \sup_{\theta\in\mathbb{R}^d}[\langle\theta, x\rangle - \overline{\Lambda}_a(\theta)].$$

Thus, we see from (2.2.5) that, in order to establish that $I_q = I_a$, it will suffice to show that $\tilde{I}_q = \tilde{I}_a$. Noting that $\tilde{I}_q$ and $\tilde{I}_a$ are essentially "quenched" and "averaged" versions of a random perturbation determined by ξ of the rate function for this auxiliary walk,[2] and in light of

[2]Indeed, $\tilde{I}_q$ and $\tilde{I}_a$ are simply the Gartner-Ellis representations of the rate function for this auxiliary

(Q1) above, one could try to adapt the method from [Yil4] originally devised for RWRE with strong regeneration properties to our auxiliary walk in order to show that $\tilde{I}_q = \tilde{I}_a$. However, note that these two settings are not the same[3] and so there is no reason to believe a priori that an approach as in [Yil4] could work here. Thus, one of the main challenges of our work is to show that it actually does: we must control these random perturbations well enough so that one is able to carry out arguments in the spirit of [Yil4] and, furthermore, we must do so *uniformly* over all environmental laws with a uniform ellipticity constant bounded from below by κ. We outline all the necessary steps next.

Step 2: As stated above, we must prove that there exist $\varepsilon, r > 0$, depending only on y, d and κ, such that $\tilde{I}_a(x) = \tilde{I}_q(x)$ for all $x \in B_r(y)$ and any RWRE with environmental law $\mathbb{P}$ with $\mathrm{dis}(\mathbb{P}) < \varepsilon$. As a first step towards this, we show that $\overline{\Lambda}_q(\theta) = \overline{\Lambda}_a(\theta)$ for all θ with $|\theta|_1 < r_1$ and if $\mathrm{dis}(\mathbb{P}) < \varepsilon_1$, for $r_1, \varepsilon_1 > 0$ depending only on y, d and κ. The main step for showing this, which is customary in this line of problems, is establishing the $L^2(\mathbb{P})$-boundedness of a particular sequence $(\Phi_n)_{n\in\mathbb{N}}$, which is often closely related (if not equal) to a martingale. In our case, the sequence of interest is

$$\Phi_n(\theta,\omega) = \overline{E}_0^Q\left(\mathrm{e}^{\langle \theta, X_{L_n}\rangle - \overline{\Lambda}_a(\theta)L_n} \prod_{j=1}^{L_n} \xi(X_{j-1}, X_j - X_{j-1})\,;\, L_n \text{ is a regeneration time}\right), \tag{2.2.6}$$

where L_n denotes the hitting time of the hyperplane $\{x : \langle x, \ell\rangle = n\}$ and $\overline{E}_0^Q$ denotes expectation w.r.t. Q_0 conditional on the event that $\inf\{n : \langle X_n - X_0, \ell\rangle < 0\} = \infty$ for $\ell \in \mathbb{V}$ some particular direction satisfying that $\langle y, \ell\rangle > 0$ (and in terms of which the regeneration structure of the auxiliary random walk is defined, see Section 2.3 for details). To see that $(\Phi_n)_{n\in\mathbb{N}}$ is bounded in $L^2(\mathbb{P})$, one writes Φ_n^2 as an expectation of a certain function of two

walk, where the usual limiting logarithmic moment generating function is replaced by the perturbations $\overline{\Lambda}_q$ and $\overline{\Lambda}_a$, respectively.

[3]Indeed, we have a deterministic environment as opposed to a random one as in [Yil4], and we work with random perturbations of logarithmic MGFs instead of actual MGFs.

independent random walks with law Q_0 and argues that, in order to bound its second moment whenever the disorder is sufficiently small, it is enough to suitably estimate the number of times the trajectories of these two independent walks intersect and then use this to control the perturbation term $\prod_{j=1}^{L_n} \xi(X_{j-1}, X_j - X_{j-1})$ from (2.2.6). The desired control reduces to a suitable bound for the probability of non-intersection of two random walks in the same deterministic environment. For this purpose, and in contrast to [Yil4] where an estimate by Berger-Zeitouni [BZ] has been used (to control the probability of non-intersection of two walks in the same *random environment*), we invoke the bounds developed by Bolthausen and Sznitman [BS1] which are better suited to our setting.

Step 3: The last task in the proof is to show that the equality of $\overline{\Lambda}_q$ and $\overline{\Lambda}_a$ in a neighborhood of the origin translates into the equality of $\tilde{I}_q$ and $\tilde{I}_q$ in a neighborhood of y. To do this, we note that, by standard arguments, we have that for any $x \in \mathbb{R}^d$,

$$\tilde{I}_q(x) = \langle \theta_{x,q}, x \rangle - \overline{\Lambda}_q(\theta_{x,q}) \qquad \text{and} \qquad \tilde{I}_a(x) = \langle \theta_{x,a}, x \rangle - \overline{\Lambda}_a(\theta_{x,a})$$

for any $\theta_{x,q}, \theta_{x,a} \in \mathbb{R}^d$ such that $\nabla \overline{\Lambda}_q(\theta_{x,q}) = x = \nabla \overline{\Lambda}_a(\theta_{x,a})$. In particular, if we can take $\theta_{x,q} = \theta_{x,a}$ then this readily implies that $\tilde{I}_q(x) = \tilde{I}_a(x)$. Thus, since $\nabla \overline{\Lambda}_q(\theta) = \nabla \overline{\Lambda}_a(\theta)$ for $|\theta|_1 < r_1$ if $\text{dis}(\mathbb{P}) < \varepsilon_1$ by Step 2, if we show that there exist $0 < \varepsilon(y, d, \kappa) < \varepsilon_1$ and $r(y, d, \kappa) > 0$ such that

$$B_r(y) \subseteq \{\nabla \overline{\Lambda}_a(\theta) : |\theta|_1 < r_1\} \tag{2.2.7}$$

whenever $\text{dis}(\mathbb{P}) < \varepsilon$ then for each $x \in B_r(y)$ we would have $\nabla \overline{\Lambda}_q(\theta_x) = x = \nabla \overline{\Lambda}_a(\theta_x)$ for some θ_x and hence that $\tilde{I}_q(x) = \tilde{I}_a(x)$ immediately follows. The key point here is that we must show that r in (2.2.7) can be taken to be *independent* of the law $\mathbb{P}$, as long as its disorder is sufficiently low and its uniformly ellipticity constant is bounded from below by κ. We achieve this by using a *uniform* inverse function theorem for families of differentiable functions (Theorem 2.4.5 below), which requires us to obtain uniform estimates (over $\mathbb{P}$) on the modulus of continuity at $\theta = 0$ of the Hessian H_a of $\overline{\Lambda}_a$ as well as a uniform upper bound on the norm of its inverse H_a^{-1}. To obtain such estimates, we rely on a representation of H_a analogous to the one given in [Yil3].

2.2.2 Organization of the article:

The rest of the paper is organized as follows. The construction of the auxiliary random walk as well as the study of its properties is carried out in Section 2.3. Also in Section 2.3 the reader will find proof of the equality $\overline{\Lambda}_q = \overline{\Lambda}_a$ in a neighborhood of the origin, assuming the $L^2(\mathbb{P})$-boundedness of the sequence $(\Phi_n)_{n\in\mathbb{N}}$ in (2.2.6), the proof of which is deferred to Section 2.5. Finally, Step 3 in the above discussion is carried out in Section 2.4.

2.3 An auxiliary random walk and equality of its limiting log-MGFs

As stated in Section 2.2.1, Theorem 2.2.1 is a direct consequence of the following more specific result.

Theorem 2.3.1. *For any $y \in \mathrm{int}(\mathbb{D}) \setminus \{0\}$, $d \geq 4$ and $\kappa > 0$, there exist $\varepsilon = \varepsilon(y, d, \kappa), r = r(y, d, \kappa) > 0$ such that, for any RWRE satisfying Assumption* A *with ellipticity constant κ, if* $\mathrm{dis}(\mathbb{P}) < \varepsilon$ *then then we have the equality*

$$I_q(x) = I_a(x) \qquad \textit{for all } x \in B_r(y) := \{z \in \mathbb{R}^d : |z - y|_1 < r\}.$$

Therefore, here and in the coming sections we shall focus only on proving Theorem 2.3.1. The goal in this particular section is to begin the proof by showing equality between the averaged and quenched limiting logarithmic moment generating functions (log-MGFs, for short) for small enough disorder. A key building block to this end will be the construction of an auxiliary random walk and a detailed investigation of its properties.

Before we begin we introduce some further notation to be used throughout the sequel. Given $\kappa > 0$, we define

$$\mathcal{M}_1^{(\kappa)}(\mathbb{V}) := \{p \in \mathcal{M}_1(\mathbb{V}) : p(e) \geq \kappa \text{ for all } e \in \mathbb{V}\}$$

with $\mathcal{M}_1(\mathbb{V})$ the space of all probability vectors on $\mathbb{V}$, together with the class of environmental laws

$$\mathcal{P}_\kappa := \{\mathbb{P} \in \mathcal{M}_1(\Omega) : \mathbb{P} \text{ satisfies Assumption A with ellipticity constant } \kappa\},$$

where $\mathcal{M}_1(\Omega)$ is the space of all environmental laws. Finally, for $\varepsilon > 0$, we define

$$\mathcal{P}_\kappa(\varepsilon) := \{\mathbb{P} \in \mathcal{P}_\kappa : \text{dis}(\mathbb{P}) < \varepsilon\}.$$

We are now ready to present this auxiliary random walk and study its properties.

2.3.1 Introducing the Q-random walk and its limiting log-MGFs

Let us fix $y \in \text{int}(\mathbb{D}) \setminus \{0\}$ and $\mathbb{P} \in \mathcal{P}_\kappa$. Notice that, if we define the function $f : [0, \infty) \to [0, \infty)$ as

$$f(C) := \sum_{i=1}^{d} \sqrt{|\langle y, e_i \rangle|^2 + 4C\alpha(e_i)\alpha(-e_i)}. \tag{2.3.1}$$

Since f is strictly increasing and continuous, with $f(0) = |y|_1 < 1$ and $\lim_{C \to \infty} f(C) = \infty$, there exists a unique $C_{y,\alpha} \in (0, \infty)$ such that $f(C_{y,\alpha}) = 1$. With this, we may define for each

$e \in \mathbb{V}$ the probability weight

$$u(e) := \frac{\langle y, e\rangle}{2} + \frac{1}{2}\sqrt{|\langle y, e\rangle|^2 + 4C_{y,\alpha}\alpha(e)\alpha(-e)}. \tag{2.3.2}$$

Observe that $u(e) \geq 0$ and $\sum_{e\in\mathbb{V}} u(e) = 1$, so that $u := (u(e))_{e\in\mathbb{V}}$ truly is a probability vector. Central to the proof of Theorem 2.3.1 will be the following auxiliary random walk (in a deterministic environment) on $\mathbb{Z}^d$, whose law we denote by Q, which is given by the transition probabilities

$$Q(X_{n+1} = x + e \,|\, X_n = x) = u(e)$$

for each $e \in \mathbb{V}$ and $x \in \mathbb{Z}^d$, with $u(e)$ as in (2.3.2). We call this auxiliary walk the *Q-random walk*. We will write Q_x to denote the law of this walk starting from a fixed $x \in \mathbb{Z}^d$ and E_x^Q to denote expectations with respect to Q_x. Notice that Q_x depends exclusively on x, y and $\mathbb{P}$, but it depends on $\mathbb{P}$ only through the average weights α. In general, we will omit the dependence on y and α from the notation, but occasionally we will write $Q(y, \alpha)$ instead of Q if we wish to make it explicit. Furthermore, the weights u have been particularly chosen so that this Q-random walk satisfies the properties in Lemma 2.3.2 below.

Lemma 2.3.2. *With this choice of probability weights $u = (u(e))_{e\in\mathbb{V}}$, the following properties hold:*

P1. Given $\kappa > 0$ there exists $c_\kappa > 0$ such that $u(e) \geq c_\kappa$ for all $e \in \mathbb{V}$ and $\mathbb{P} \in \mathcal{P}_\kappa$.

P2. $E_x^Q(X_{n+1} - X_n) = y$ for all $n \in \mathbb{N}$ and $x \in Z^d$.

P3. For any $x \in \mathbb{Z}^d$ and all environments ω, we have

$$E_0^Q\Big(\mathrm{e}^{\langle\theta, X_n\rangle}\prod_{j=1}^{n}\xi(X_{j-1}, \Delta_j(X))\Big) = (C_{y,\alpha})^{\frac{n}{2}} E_{0,\omega}\big(\mathrm{e}^{\langle\theta+\theta_{y,\alpha}, X_n\rangle}\big) \tag{2.3.3}$$

and

$$E_0^Q\Big(\mathrm{e}^{\langle\theta,X_n\rangle}\mathbb{E}\prod_{j=1}^n\xi(X_{j-1},\Delta_j(X))\Big)=(C_{y,\alpha})^{\frac{n}{2}}E_0\big(\mathrm{e}^{\langle\theta+\theta_{y,\alpha},X_n\rangle}\big), \tag{2.3.4}$$

where $C_{y,\alpha}$ *is as in* (2.3.2)*, the vector* $\theta_{y,\alpha}\in\mathbb{R}^d$ *is given by the formulas*

$$\langle\theta_{y,\alpha},e_i\rangle:=\log\left(\frac{u(e_i)}{\alpha(e_i)\sqrt{C_{y,\alpha}}}\right)\qquad i=1,\dots,d \tag{2.3.5}$$

and we use the notation $\Delta_j(X):=X_j-X_{j-1}$ *for* $j=1,\dots,n$.

Proof. Since the mapping $\alpha\mapsto C_{y,\alpha}$ is continuous on $\mathcal{M}_1^{(\kappa)}(\mathbb{V})$ (by the implicit function theorem, for example), we see that $\alpha\mapsto u(e)$ is also continuous for each $e\in\mathbb{V}$. In particular, since $\mathcal{M}_1^{(\kappa)}(\mathbb{V})$ is compact, we see that $\inf_{\mathbb{P}\in\mathcal{P}_\kappa}u(e)=\inf_{\alpha\in\mathcal{M}_1^{(\kappa)}(\mathbb{V})}u(e)>0$ for each $e\in\mathbb{V}$, which readily implies (P1). On the other hand, (P2) is immediate from the definition of the weights u in (2.3.2). Therefore, we focus on proving (P3). Notice that it will be enough to show (2.3.3), as (2.3.4) follows immediately upon taking expectations on (2.3.3) with respect to $\mathbb{P}$. To show (2.3.3), we introduce yet another auxiliary random walk, whose law we will denote by Q^u, given by the transition probabilities

$$Q^u(X_{n+1}=x+e\,|\,X_n=x)=\frac{c_{y,\alpha}u(e)}{\alpha(e)} \tag{2.3.6}$$

for each $e\in\mathbb{V}$ and $x\in\mathbb{Z}^d$, where the weights $u(e)$ are the same as before and $c_{y,\alpha}>0$ is a normalizing constant so that the transition probabilities for Q^u in (2.3.6) add up to 1. As before, we write Q^u_x to denote the law of this random walk starting from a fixed $x\in\mathbb{Z}^d$ and use E^u_x to denote the expectation with respect to Q^u_x.

Having introduced this second auxiliary random walk, the first step will be to show that

$$E_0^u\Big(\mathrm{e}^{\langle\theta,X_n\rangle}\prod_{j=1}^n\omega(X_{j-1},\Delta_j(X))\,;\,X_n=x\Big)=(c_{y,\alpha}\sqrt{C_{y,\alpha}})^nE_{0,\omega}(\mathrm{e}^{\langle\theta+\theta_{y,\alpha},X_n\rangle}\,;\,X_n=x), \tag{2.3.7}$$

for every $\theta \in \mathbb{R}^d$ and $x \in \mathbb{Z}^d$, where $c_{y,\alpha}$ is as in (2.3.6), $C_{y,\alpha}$ as in (2.3.2) and $\theta_{y,\alpha}$ is given by (2.3.5). To this end, let us define a *path of length* n to be any sequence $\bar{x} = (x_0, \ldots, x_n)$ of $n+1$ sites in $\mathbb{Z}^d$ satisfying that x_j and x_{j-1} are nearest neighbors for all $j = 1, \ldots, n$. Then observe that, for (2.3.7) to hold, it is enough to show that

$$Q_0^u((X_0, \ldots, X_n) = \bar{x}) = (c_{y,\alpha}\sqrt{C_{y,\alpha}})^n \mathrm{e}^{\langle \theta_{y,\alpha}, x\rangle} \tag{2.3.8}$$

for all paths $\bar{x}$ of length n with $x_0 = 0$ and $x_n = x$. To check (2.3.8), let us fix such a path $\bar{x}$ and denote by $\bar{x}_i^+$ the number of steps made by this path in direction e_i and by $\bar{x}_i^-$ the number of those in direction $-e_i$. Then, since $\bar{x}_i^+ = \bar{x}_i^- + \langle x, e_i\rangle$, by the Markov property we have that

$$\begin{aligned} Q_0^{u,v}((X_0, \ldots, X_n) = \bar{x}) &= c_{y,\alpha}^n \prod_{i=1}^d \left(\frac{u(e_i)}{\alpha(e_i)}\right)^{\bar{x}_i^+} \prod_{i=1}^d \left(\frac{u(-e_i)}{\alpha(-e_i)}\right)^{\bar{x}_i^-} \\ &= c_{y,\alpha}^n \prod_{i=1}^d \left(\frac{u(e_i)u(-e_i)}{\alpha(e_i)\alpha(-e_i)}\right)^{\bar{x}_i^-} \prod_{i=1}^d \left(\frac{u(e_i)}{\alpha(e_i)}\right)^{\langle x, e_i\rangle}. \end{aligned}$$

Notice that, by construction of the weights u, one has that $\frac{u(e_i)u(-e_i)}{\alpha(e_i)\alpha(-e_i)} = C_{y,\alpha}$ holds. Moreover, from the restriction $\sum_{i=1}^d (\bar{x}_i^+ + \bar{x}_i^-) = n$ and the relation $\bar{x}_i^+ = \bar{x}_i^- + \langle x, e_i\rangle$ for every $i = 1, \ldots, d$, it follows that $\sum_{i=1}^d \bar{x}_i^- = \frac{1}{2}(n - \sum_{i=1}^d \langle x, e_i\rangle)$. Hence, we obtain

$$Q_0^u((X_0, \ldots, X_n) = \bar{x}) = (c_{y,\alpha}\sqrt{C_{y,\alpha}})^n \prod_{i=1}^d \left(\frac{u(e_i)}{\alpha(e_i)\sqrt{C_{y,\alpha}}}\right)^{\langle x, e_i\rangle} = (c_{y,\alpha}\sqrt{C_{y,\alpha}})^n \mathrm{e}^{\langle \theta_{y,\alpha}, x\rangle}. \tag{2.3.9}$$

Summing (2.3.7) over all $x \in \mathbb{Z}^d$ yields

$$E_0^u\Big(\mathrm{e}^{\langle \theta, X_n\rangle} \prod_{j=1}^n \omega(X_{j-1}, \Delta_j(X))\Big) = (c_{y,\alpha}\sqrt{C_{y,\alpha}})^n E_{0,\omega}(\mathrm{e}^{\langle \theta + \theta_{y,\alpha}, X_n\rangle}). \tag{2.3.10}$$

Finally, we conclude (2.3.3) from (2.3.10) upon noticing that, by definition of Q and Q^u,

$$E_0^u\Big(\mathrm{e}^{\langle\theta,X_n\rangle}\prod_{j=1}^n \omega(X_{j-1},\Delta_j(X))\Big) = c_{y,\alpha}^n E_0^Q\Big(\mathrm{e}^{\langle\theta,X_n\rangle}\prod_{j=1}^n \xi(X_{j-1},\Delta_j(X))\Big).$$

This completes the proof. $\square$

As a consequence of Lemma 2.3.2, we immediately get the following corollary.

Corollary 2.3.3. *For $\theta \in \mathbb{R}^d$, the quantities*

$$\overline{\Lambda}_q(\theta) := \lim_{n\to\infty}\frac{1}{n}\log E_0^Q\Big(\mathrm{e}^{\langle\theta,X_n\rangle}\prod_{j=1}^n \xi(X_{j-1},\Delta_j(X))\Big), \tag{2.3.11}$$

and

$$\overline{\Lambda}_a(\theta) := \lim_{n\to\infty}\frac{1}{n}\log E_0^Q\Big(\mathrm{e}^{\langle\theta,X_n\rangle}\mathbb{E}\prod_{j=1}^n \xi(X_{j-1},\Delta_j(X))\Big) \tag{2.3.12}$$

are well-defined, i.e. the limits in (2.3.11) *and* (2.3.12) *both exist, are finite and the right-hand side of* (2.3.11) *is $\mathbb{P}$-almost surely constant.*

Proof. It follows from (2.3.3) and (2.3.4) that, for any $\theta \in \mathbb{R}^d$,

$$\overline{\Lambda}_q(\theta) = \log(\sqrt{C_{y,\alpha}}) + \Lambda_q(\theta+\theta_{y,\alpha}) \qquad \text{and} \qquad \overline{\Lambda}_a(\theta) = \log(\sqrt{C_{y,\alpha}}) + \Lambda_a(\theta+\theta_{y,\alpha}), \tag{2.3.13}$$

where $\Lambda_q(\theta) := \lim_{n\to\infty}\frac{1}{n}\log E_{0,\omega}(\mathrm{e}^{\langle\theta,X_n\rangle})$ and $\Lambda_a(\theta) := \lim_{n\to\infty}\frac{1}{n}\log E_0(\mathrm{e}^{\langle\theta,X_n\rangle})$ respectively denote the quenched and annealed limiting logarithmic moment generating functions associated with the RWRE. Since both Λ_q and Λ_a are well-defined in the sense described in the statement of Corollary 2.3.3 (see [RAS3, Theorem 2.6] for the quenched case and, in the annealed case, this follows from [Var, Theorem 3.2] and Varadhan's Lemma [DZ, Theorem 4.3.1]), we see that $\overline{\Lambda}_q(\theta)$ and $\overline{\Lambda}_a(\theta)$ are so as well. $\square$

The following remark contains some crucial estimates that we will use extensively in the sequel.

Remark 2.3.1. Given any $\theta, \theta' \in \mathbb{R}^d$ and environmental law $\mathbb{P}$, for any $n \geq 1$ we have

$$|\overline{\Lambda}_a(\theta) - \overline{\Lambda}_a(\theta')| \leq |\theta - \theta'|_1 \qquad \text{and} \qquad \mathrm{e}^{-\overline{\Lambda}_a(0)n} \prod_{j=1}^{n} \xi(X_{j-1}, \Delta_j(X)) \leq \mathrm{e}^{h(\mathrm{dis}(\mathbb{P}))n} \quad \mathbb{P}\text{-a.s.},$$

where $h(x) := \log\left(\frac{1+x}{1-x}\right)$ for $x \in [0,1)$. The proof of these inequalities is elementary, so we omit it. Nevertheless, from now onwards we will assume that $\mathrm{dis}(\mathbb{P}) < 1$ so that the expression $h(\mathrm{dis}(\mathbb{P})) < 1$, which will appear numerous times in the sequel, is always well-defined. This does not represent any real loss of generality since we shall always be interested in environmental laws with small enough disorder.

The main objective in Section 2.3 is to show that $\overline{\Lambda}_a(\theta) = \overline{\Lambda}_q(\theta)$ for θ close enough to 0, whenever the disorder of the environment is sufficiently low. We will later see in Section 2.4 that, in turn, this will imply that $I_q(x) = I_a(x)$ for x sufficiently close to y. To carry out all this, we shall exploit a renewal structure available for the Q-random walk. We introduce this renewal structure next.

2.3.2 A renewal structure for the Q-random walk

Let us first fix a direction $\ell \in \mathbb{V}$ such that $E_0^Q(\langle X_1, \ell\rangle) > 0$. Notice that such a direction always exists since $E_0^Q(\langle X_1, \ell\rangle) = \langle y, \ell\rangle$ by Lemma 2.3.2 and $y \neq 0$ by assumption. We then set for $u \in \mathbb{R}$,

$$H_u := \inf\{n \geq 1 : \langle X_n, \ell\rangle > u\}, \;\; S_0 := 0, \quad \beta_0 := \inf\{n \geq 1 : \langle X_n, \ell\rangle < \langle X_0, \ell\rangle\}, \;\; R_0 := \langle X_0, \ell\rangle$$

and define the sequences of stopping times $(S_k)_{k\in\mathbb{N}}, (\beta_k)_{k\in\mathbb{N}}$ and $(R_k)_{k\in\mathbb{N}}$ inductively as

$$S_{k+1} := H_{R_k}, \quad \beta_{k+1} := \inf\{n > S_{k+1} : \langle X_n, \ell\rangle < \langle X_{S_{k+1}}, \ell\rangle\},$$

$$R_{k+1} := \begin{cases} \sup\{\langle X_n, \ell\rangle : 0 \leq n \leq \beta_{k+1}\} & \text{if } \beta_{k+1} < \infty \\ \langle X_{S_{k+1}}, \ell\rangle & \text{if } \beta_{k+1} = \infty, \end{cases}$$

with the convention that $\inf \emptyset = \infty$. Observe that, by choice of ℓ and the law of large numbers, we have $\lim_{n\to\infty}\langle X_n, \ell\rangle = \infty$ Q-almost surely. In particular, this implies that

$$R_k < \infty \ Q\text{-a.s.} \Longrightarrow S_{k+1} < \infty \ Q\text{-a.s.} \Longrightarrow R_{k+1} < \infty \ Q\text{-a.s.},$$

so that by induction all S_k and R_k are finite Q-almost surely. However, the β_k will not all be. Thus, we define the sequence $(\tau_k)_{k\in\mathbb{N}}$ of *renewal times* as

$$\tau_k := S_{W_k},$$

where $(W_k)_{k\in\mathbb{N}}$ is defined inductively by first taking $W_0 := 0$ and then setting

$$W_{k+1} := \inf\{n > W_k : \beta_n = \infty\}.$$

That the renewal times τ_k are well-defined is a consequence of the fact that all W_k are Q-a.s. finite, which in turn follows from the Markov property and Lemma 2.3.4 below.

Lemma 2.3.4. *There exists $\bar{c} = \bar{c}(y) > 0$ such that $Q(\beta_0 = \infty) > \bar{c}$ for any $\mathbb{P} \in \mathcal{P}_\kappa$, where $Q = Q(y,\alpha)$ is the law of the $Q(y,\alpha)$-random walk with jump weights given by* (2.3.2).

Proof. Let $(Z_n)_{n\in\mathbb{N}}$ be the random walk on $\mathbb{Z}$ which starts from $\langle X_0, \ell\rangle$ and, at each step $n \in \mathbb{N}$, jumps one unit to the left with probability q and one to the right with probability

$p := 1 - q$, where

$$q := -\frac{\langle y, \ell\rangle}{2} + \frac{1}{2}\sqrt{|\langle y, \ell\rangle|^2 + 1}.$$

Observe that, since $f(1/4\alpha(\ell)\alpha(-\ell)) > 1$ where f is as in (2.3.1), we have that $C_{y,\alpha} < \frac{1}{4\alpha(\ell)\alpha(-\ell)}$ and thus $u(\ell) \leq q$. It follows that we may couple $(Z_n)_{n\in\mathbb{N}}$ with our Q-random walk in such a way that, for all $n \in \mathbb{N}$,

$$Z_n \leq \langle X_n, \ell\rangle \Longrightarrow Z_{n+1} \leq \langle X_{n+1}, \ell\rangle.$$

In particular, if we denote $\beta_0(Z) := \inf\{n \geq 1 : Z_n < Z_0\}$ then $P(\beta_0(Z) = \infty) \leq Q(\beta_0 = \infty)$. But, since $q < \frac{1}{2}$ by Minkowski's inequality, by standard gambler's ruin estimates we have

$$P(\beta_0(Z) = \infty) = 1 - \tfrac{q}{p} =: \bar{c}.$$

This concludes the proof. □

It follows from this construction above that all renewal times τ_k are Q-a.s. finite, that (X_{τ_1}, τ_1) is independent of the sequence $(X_{\tau_{k+1}} - X_{\tau_k}, \tau_{k+1} - \tau_k)_{k\geq 1}$ and that this last sequence is i.i.d. with common law given by that of (X_{τ_1}, τ_1) conditioned on the event $\{\beta_0 = \infty\}$. We now investigate some (uniform in $\mathbb{P}$) integrability properties of these renewal times.

Lemma 2.3.5. *There exists $\rho_1 = \rho_1(y) > 0$ such that $E_0^Q(\mathrm{e}^{\rho\langle X_{\tau_1}, \ell\rangle}) \leq 3\bar{c}^{-1}$ for all $\rho < \rho_1$ and any $\mathbb{P} \in \mathcal{P}_\kappa$, where $\bar{c}$ is the constant from Lemma 2.3.4.*

Proof. By splitting $E_0^Q(\mathrm{e}^{\rho\langle X_{\tau_1}, \ell\rangle})$ according to the value of W_1, we obtain the bound

$$E_0^Q(\mathrm{e}^{\rho\langle X_{\tau_1}, \ell\rangle}) = \sum_{k=1}^{\infty} E_0^Q(\mathrm{e}^{\rho\langle X_{\tau_1}, \ell\rangle}; W_1 = k) \leq \sum_{k=1}^{\infty} E_0^Q(\mathrm{e}^{\rho\langle X_{S_k}, \ell\rangle}; \beta_j < \infty \text{ for } j = 1, \ldots, k). \tag{2.3.14}$$

Observe that $|\langle X_{S_1}, \ell\rangle| \leq 1 + |X_0|$ by definition of R_0 and the fact that the walk is nearest neighbor, so that the first term in the sum on the right-hand side of (2.3.14) is bounded from above by e^{ρ}.

On the other hand, since $R_{k-1} = \sup\{\langle X_n, \ell\rangle : S_{k-1} \le n \le \beta_{k-1}\}$ when $\beta_{k-1} < \infty$ and $k \ge 2$, by writing $\langle X_{S_k}, \ell\rangle = \langle X_{S_{k-1}}, \ell\rangle + \langle X_{S_k} - X_{S_{k-1}}, \ell\rangle$ and using the Markov property at time S_{k-1}, we see that for $k \ge 2$ the k-th term in the right-hand side of (2.3.14) is bounded from above by

$$E_0^Q(e^{\rho\langle X_{S_{k-1}},\ell\rangle}\,;\,\beta_j < \infty \text{ for } j = 1, \ldots, k-1)E_0^Q(e^{\rho(1+\mathcal{R})}\,;\,\beta_0 < \infty),$$

where $\mathcal{R} := \sup\{\langle X_n, \ell\rangle : 0 \le n \le \beta_0\}$. Repeating this argument all the way down to $\langle X_{S_1}, \ell\rangle$ and then using the bound for the case $k = 1$ yields the bound

$$E_0^Q(e^{\rho\langle X_{\tau_1},\ell\rangle}) \le e^{\rho}\sum_{k=1}^{\infty}\left(E_0^Q\left(e^{\rho(1+\mathcal{R})}\,;\,\beta_0 < \infty\right)\right)^{k-1}. \tag{2.3.15}$$

Therefore, in order to complete the proof we only need to show that, for ρ small enough depending only on y, we have

$$E_0^Q\left(e^{\rho(1+\mathcal{R})}\,;\,\beta_0 < \infty\right) < 1 - \frac{\bar{c}}{2}. \tag{2.3.16}$$

But, by the union bound and Lemma 2.3.4, for any $N \ge 1$ the expectation on the left-hand side of (2.3.16) is bounded from above by

$$\begin{aligned} e^{\rho N}Q_0(\beta_0 < \infty) &+ \sum_{n=N}^{\infty} e^{\rho(2+n)}Q_0(n \le \mathcal{R} < n+1\,,\,\beta_0 < \infty) \\ &\le e^{\rho N}(1-\bar{c}) + \sum_{n=N}^{\infty} e^{\rho(2+n)}Q_0(\mathcal{R} \ge n\,,\,\beta_0 < \infty). \end{aligned}$$

Now, observe that for $n \geq 1$

$$Q_0(\mathcal{R} \geq n\,,\, \beta_0 < \infty) \leq Q_0(n \leq \beta_0 < \infty)$$
$$= \sum_{k=n}^{\infty} Q_0(\beta_0 = i) \leq \sum_{k=n}^{\infty} Q_0(\langle X_k, \ell\rangle < 0) \leq \frac{\mathrm{e}^{-\frac{1}{8}|\langle y,\ell\rangle|^2 n}}{1 - \mathrm{e}^{-\frac{1}{8}|\langle y,\ell\rangle|^2}},$$

where to obtain the last inequality we have used the bound $Q_0(\langle X_k, \ell\rangle < 0) \leq \mathrm{e}^{-\frac{1}{8}|\langle y,\ell\rangle|^2 k}$, which follows from the (one-sided) Azuma-Hoeffding inequality for the martingale $(M_n)_{n\in\mathbb{N}}$ given by $M_n := \langle X_n, \ell\rangle - n\langle y, \ell\rangle$ (whose increments are bounded by 2). Thus, we see that, for any $N \geq 1$,

$$E_0^Q\left(\mathrm{e}^{\rho(1+\mathcal{R})}\,;\, \beta_0 < \infty\right) \leq \mathrm{e}^{\rho(N\wedge 2)}\left(1 - \bar{c} + \frac{\mathrm{e}^{-\frac{1}{8}|\langle y,\ell\rangle|^2 N}}{\left(1 - \mathrm{e}^{-\frac{1}{8}|\langle y,\ell\rangle|^2}\right)^2}\right)$$

from where (2.3.16) now follows by taking first N sufficiently large and then ρ accordingly small. □

As a consequence of Lemma 2.3.5, we obtain (uniform in $\mathbb{P}$) exponential moments for τ_1.

Proposition 2.3.6. *There exists $\gamma_0 = \gamma_0(y) > 0$ such that $E_0^Q(\mathrm{e}^{\gamma\tau_1}) \leq 2$ for all $\gamma \leq \gamma_0$ and any $\mathbb{P} \in \mathcal{P}_\kappa$.*

Proof. For $n \geq 1$, by the union bound we have

$$Q_0(\tau_1 > n) \leq Q_0\left(\langle X_{\tau_1}, \ell\rangle > \frac{\langle y, \ell\rangle}{2}\right) + Q_0\left(\tau_1 > n\,,\, \langle X_{\tau_1}, \ell\rangle \leq \frac{\langle y, \ell\rangle n}{2}\right).$$

Using the exponential Tchebychev inequality and Lemma 2.3.5, we have

$$Q_0\left(\langle X_{\tau_1}, \ell\rangle > \frac{\langle y, \ell\rangle}{2}\right) \leq \mathrm{e}^{-\rho\langle y,\ell\rangle n} E_0^Q(\mathrm{e}^{2\rho\langle X_{\tau_1},\ell\rangle}) \leq C\mathrm{e}^{-\rho\langle y,\ell\rangle n}$$

for some $C, \rho > 0$ depending only on y. On the other hand, by definition of τ_1 we have

$$Q_0\left(\tau_1 > n\,,\, \langle X_{\tau_1}, \ell\rangle \leq \frac{\langle y, \ell\rangle n}{2}\right) \leq Q_0\left(\langle X_n, \ell\rangle \leq \frac{\langle y, \ell\rangle n}{2}\right) \leq \mathrm{e}^{-\frac{1}{32}|\langle y, \ell\rangle|^2 n},$$

where to obtain the last inequality we have used the (one-sided) Azuma-Hoeffding inequality for the martingale $(M_n)_{n\in\mathbb{N}}$ as in the proof of Lemma 2.3.5. Hence, we see that there exist $C, \gamma > 0$ depending only on y such that $Q_0(\tau_1 > n) \leq C\mathrm{e}^{-\gamma n}$ for all $n \geq 1$. From this the result now follows by an argument similar to the one used to derive (2.3.16). □

Finally, the above regeneration structure, together with Remark 2.3.1, allows us to deduce analyticity of $\overline{\Lambda}_a$.

Proposition 2.3.7. *There exists* $\gamma_1 > 0$ *(determined by Proposition 2.3.8 below), if* $\mathrm{dis}(\mathbb{P}) <$ γ_1 *then the mapping* $\theta \mapsto \overline{\Lambda}_a(\theta)$ *is analytic on the set* $\{\theta : |\theta|_1 < \gamma_1\}$.

Proof. We follow an idea similar to [Yil3, Lemma 6]. Consider the function $\Psi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ defined as

$$\Psi(\theta, r) := \overline{E}_0^Q\left(\mathrm{e}^{\langle\theta, X_{\tau_1}\rangle - r\tau_1}\mathbb{E}\prod_{j=1}^{\tau_1}\xi(X_{j-1}, \Delta_j(X))\right)$$

where $\overline{E}_0^Q$ above stands for expectation with respect to $\overline{Q}_0$, the law Q_0 conditioned on the event $\{\beta_0 = \infty\}$. By Remark 2.3.1 we have that, whenever $r = \overline{\Lambda}_a(\theta) + \delta$ for some $\delta \in \mathbb{R}$,

$$\left|\langle\theta, X_{\tau_1}\rangle - r\tau_1 + \log\mathbb{E}\prod_{j=1}^{\tau_1}\xi(X_{j-1}, \Delta_j(X))\right| \leq (2|\theta|_1 + h(\mathrm{dis}(\mathbb{P})) + |\delta|)\tau_1$$

so that, by choice of γ_1 (see the proof of Lemma 2.5.2 for details), we have

$$\overline{E}_0^Q\left(\tau_1 \exp\left\{|\langle\theta, X_{\tau_1}\rangle - (\overline{\Lambda}_a(\theta) + \delta)\tau_1 + \log\mathbb{E}\prod_{j=1}^{\tau_1}\xi(X_{j-1}, \Delta_j(X))|\right\}\right) < \infty \qquad (2.3.17)$$

whenever $|\theta|_1 \vee \mathrm{dis}(\mathbb{P}) < \gamma_1$ and $|\delta| < \delta_c$ for some $\delta_c = \delta_c(y) > 0$ small enough. It then follows from (2.3.17), dominated convergence and Remark 2.3.1 once again that, when $\mathrm{dis}(\mathbb{P}) < \gamma_1$, Ψ is analytic on the open set $\mathcal{C}_y := \{(\theta, r) : |\theta|_1 < \gamma_1\,,\ |r - \overline{\Lambda}_a(\theta)| < \delta_c\}$ with series expansion given by

$$\Psi(\theta, r) = \sum_{n=0}^{\infty} \frac{\overline{E}_0^Q((\langle \theta, X_{\tau_1}\rangle - r\tau_1)^n)}{n!}$$

and $\partial_r \Psi$ given by

$$\partial_r \Psi(\theta, r) = -\overline{E}_0^Q\Big(\tau_1 \mathrm{e}^{\langle \theta, X_{\tau_1}\rangle - r\tau_1} \mathbb{E} \prod_{j=1}^{\tau_1} \xi(X_{j-1}, \Delta_j(X))\Big). \tag{2.3.18}$$

But observe that $\Psi(\theta, \overline{\Lambda}_a(\theta)) = 1$ whenever $|\theta|_1 \vee \mathrm{dis}(\mathbb{P}) < \gamma_1$ by Proposition 2.3.8, which in turn implies that $-\partial_r \Psi(\theta, \overline{\Lambda}_a(\theta)) \geq \Psi(\theta, \overline{\Lambda}_a(\theta)) = 1 > 0$ by (2.3.18). Therefore, the analyticity of $\overline{\Lambda}_a(\theta)$ for $|\theta|_1 < \gamma_1$ whenever $\mathrm{dis}(\mathbb{P}) < \gamma_1$ now follows from the analytic implicit function theorem, see [KP, Theorem 6.1.2]. □

2.3.3 Equality of $\overline{\Lambda}_q$ and $\overline{\Lambda}_a$: the main argument

We now describe the main steps in the proof of the equality of $\overline{\Lambda}_a(\theta)$ and $\overline{\Lambda}_q(\theta)$ for θ close enough to 0, whenever the disorder of the environment is sufficiently low. The more technical details are deferred to a separate section. We begin by introducing the key object in our analysis.

Definition 2.3.1. Given $n \geq 1$, $\theta \in \mathbb{R}^d$ and an environment ω, we define

$$\Phi_n(\theta, \omega) := \overline{E}_0^Q\Big(\mathrm{e}^{\langle \theta, X_{L_n}\rangle - \overline{\Lambda}_a(\theta) L_n} \prod_{j=1}^{L_n} \xi(X_{j-1}, \Delta_j(X)), L_n = \tau_k \text{ for some } k \geq 1\Big), \tag{2.3.19}$$

where, as before, $\overline{E}_0^Q$ above stands for expectation with respect to $\overline{Q}_0$, the law Q_0 conditioned on the event $\{\beta_0 = \infty\}$, and $L_n := \inf\{n \geq 1 : \langle X_n - X_0, \ell\rangle = n\}$. Throughout the sequel we shall write $\Phi_n(\theta)$ instead of $\Phi_n(\theta, \omega)$ whenever we think of ω as being random (and therefore of $\Phi_n(\theta)$ as being a random variable).

The following two propositions contain the crucial information about the random variable Φ_n.

Proposition 2.3.8. *There exists $\gamma_1 = \gamma_1(y) > 0$ such that, for any $\mathbb{P} \in \mathcal{P}_\kappa$, whenever $|\theta|_1 \vee \mathrm{dis}(\mathbb{P}) < \gamma_1$ we have*

$$\overline{E}_0^Q\Big(\mathrm{e}^{\langle\theta, X_{\tau_1}\rangle - \overline{\Lambda}_a(\theta)\tau_1}\mathbb{E}\prod_{j=1}^{\tau_1}\xi(X_{j-1}, \Delta_j(X))\Big) = 1 \tag{2.3.20}$$

and

$$\lim_{n\to\infty}\mathbb{E}\Phi_n(\theta) > 0.$$

Proposition 2.3.9. *There exists $\gamma_2 = \gamma_2(y, d, \kappa) > 0$ such that, for any $\mathbb{P} \in \mathcal{P}_\kappa$, whenever $|\theta|_1 \vee \mathrm{dis}(\mathbb{P}) < \gamma_2$ we have* $\sup_{n\geq 1}\mathbb{E}(\Phi_n(\theta))^2 < \infty$.

The proofs of these propositions are deferred to Section 2.5. Let us first conclude

Proof of $\overline{\Lambda}_q = \overline{\Lambda}_a$ (assuming Proposition 2.3.8 and Proposition 2.3.9): Note that by Propositions 2.3.8-2.3.9, whenever $|\theta|_1 \vee \mathrm{dis}(\mathbb{P}) < \gamma_1 \wedge \gamma_2$ we have

$$\mathbb{P}\Big(\lim_{n\to\infty}\Phi_n(\theta) = 0\Big) < 1. \tag{2.3.21}$$

Indeed, if $\Phi_n(\theta) \to 0$ $\mathbb{P}$-a.s. then $\lim_{n\to\infty}\mathbb{E}\Phi_n(\theta) = 0$ since $(\Phi_n(\theta))_{n\geq 1}$ is uniformly integrable by Proposition 2.3.9. However, this is in contradiction with Proposition 2.3.8 and thus (2.3.21) must hold. Furthermore, we also have the following.

Lemma 2.3.10. *For any $\theta \in \mathbb{R}^d$ and $\delta > 0$, we have*

$$\mathbb{P}\left(\lim_{n\to\infty}E_0^Q\Big(\mathrm{e}^{\langle\theta, X_{L_n}\rangle - (\overline{\Lambda}_q(\theta)+\delta)L_n}\prod_{j=1}^{L_n}\xi(X_{j-1}, \Delta_j(X))\Big) = 0\right) = 1. \tag{2.3.22}$$

Proof. Let us write $\lambda_{\theta,\delta} := \overline{\Lambda}_q(\theta) + \delta$ in the sequel for simplicity. Then, by splitting the expectation on the left-hand side of (2.3.22) according to the different possible values for L_n, we can bound it from above by

$$\sum_{k=n}^{\infty} E_0^Q\Big(\mathrm{e}^{\langle\theta,X_k\rangle-\lambda_{\theta,\delta}k}\prod_{j=1}^{k}\xi(X_{j-1},\Delta_j(X))\Big) = \sum_{k=n}^{\infty}\mathrm{e}^{-\lambda_{\theta,\delta}k}E_0^Q\Big(\mathrm{e}^{\langle\theta,X_k\rangle}\prod_{j=1}^{k}\xi(X_{j-1},\Delta_j(X))\Big). \tag{2.3.23}$$

Now, since for $\mathbb{P}$-almost every ω we have

$$E_0^Q\Big(\mathrm{e}^{\langle\theta,X_k\rangle}\prod_{j=1}^{k}\xi(X_{j-1},\Delta_j(X))\Big) = \mathrm{e}^{(\overline{\Lambda}_q(\theta)+o_\omega(1))k}$$

for some $o_\omega(1)\to 0$ as $k\to\infty$, from (2.3.23) we obtain that for all n sufficiently large and $\mathbb{P}$-a.e. ω,

$$E_0^Q\Big(\mathrm{e}^{\langle\theta,X_{L_n}\rangle-(\overline{\Lambda}_q(\theta)+\delta)L_n}\prod_{j=1}^{L_n}\xi(X_{i-1},\Delta_j(X))\Big) \leq \sum_{k=n}^{\infty}\mathrm{e}^{-\frac{\delta}{2}k} \leq \frac{\mathrm{e}^{-\frac{\delta}{2}n}}{1-\mathrm{e}^{-\delta/2}}.$$

Taking $n\to\infty$ on this inequality now allows us to conclude. $\square$

Combined with (2.3.21), Lemma 2.3.10 yields the equality $\overline{\Lambda}_a(\theta) = \overline{\Lambda}_q(\theta)$ whenever $|\theta|_1 \vee \mathrm{dis}(\mathbb{P}) < \gamma_1\wedge\gamma_2$. We state and prove this in a separate proposition for future reference.

Proposition 2.3.11. *Define $\overline{\gamma} = \gamma_1\wedge\gamma_2$, for γ_1 and γ_2 as in Propositions 2.3.8 and 2.3.9, respectively. Then, for any $\mathbb{P}\in\mathcal{P}_\kappa$, whenever $|\theta|_1\vee\mathrm{dis}(\mathbb{P}) < \overline{\gamma}$ we have $\overline{\Lambda}_q(\theta) = \overline{\Lambda}_a(\theta)$.*

Proof. Observe that (2.3.21) implies that, for $|\theta|_1\vee\mathrm{dis}(\mathbb{P}) < \overline{\gamma}$,

$$\mathbb{P}\left(\limsup_{n\to\infty} E_0^Q\Big(\mathrm{e}^{\langle\theta,X_{L_n}\rangle-(\overline{\Lambda}_a(\theta))L_n}\prod_{j=1}^{L_n}\xi(X_{j-1},\Delta_j(X))\Big) > 0\right) > 0.$$

In conjunction with (2.3.22), this yields the existence of an environment ω and $n\geq 1$ such that

$$E_0^Q\Big(\mathrm{e}^{\langle\theta,X_{L_n}\rangle-(\overline{\Lambda}_q(\theta)+\delta)L_n}\prod_{j=1}^{L_n}\xi(X_{i-1},\Delta_j(X))\Big)<E_0^Q\Big(\mathrm{e}^{\langle\theta,X_{L_n}\rangle-(\overline{\Lambda}_a(\theta))L_n}\prod_{j=1}^{L_n}\xi(X_{j-1},\Delta_j(X))\Big),$$

from where it follows that $\overline{\Lambda}_q(\theta)+\delta>\overline{\Lambda}_a(\theta)$. Letting $\delta\to 0$ yields the inequality $\overline{\Lambda}_q(\theta)\geq\overline{\Lambda}_a(\theta)$. But, since $\overline{\Lambda}_q(\theta)\leq\overline{\Lambda}_a(\theta)$ for all $\theta\in\mathbb{R}^d$ by Jensen's inequality, we deduce that $\overline{\Lambda}_q(\theta)=\overline{\Lambda}_a(\theta)$ whenever $|\theta|_1\vee\mathrm{dis}(\mathbb{P})<\overline{\gamma}$, which concludes the proof. □

Thus, to complete the argument it only remains to prove Propositions 2.3.8 and 2.3.9. We will do this later in Section 2.5.

2.4 Proof of Theorem 2.2.1 and Theorem 2.3.1: Deducing $I_q=I_a$ from $\overline{\Lambda}_q=\overline{\Lambda}_a$

We now show how to conclude Theorem 2.3.1 (and therefore, Theorem 2.2.1) from the results in the previous section by proving that the equality of $\overline{\Lambda}_q$ and $\overline{\Lambda}_a$ in a neighborhood of the origin implies, for sufficiently small disorder, the equality of the rate functions I_q and I_a in a neighborhood of y. The task will be carried out in three steps, spanning Section 2.4.1-Section 2.4.3.

2.4.1 Uniform closeness of y and $\nabla\overline{\Lambda}_a(0)$.

As already remarked earlier, we would like to argue that, given $y\neq 0$, for all environmental laws with a small enough disorder, y is close to the gradient $\nabla\overline{\Lambda}_a(0)$. Recall that by Proposition 2.3.8 we have that, for any $\mathbb{P}\in\mathcal{P}_\kappa$, if $|\theta|_1\vee\mathrm{dis}(\mathbb{P})<\gamma_1$ then

$$\overline{E}_0^Q\Big(\mathrm{e}^{\langle\theta,X_{\tau_1}\rangle-\overline{\Lambda}_a(\theta)\tau_1}\mathbb{E}\prod_{j=1}^{\tau_1}\xi(X_{j-1},\Delta_j(X))\Big)=1.$$

In particular, taking gradient on both sides (which we can do by dominated convergence, using Proposition 2.3.7 and the control in (2.5.12)), we obtain that whenever $|\theta|_1 \vee \mathrm{dis}(\mathbb{P}) < \gamma_1$,

$$\overline{E}_0^Q\Big((X_{\tau_1} - \nabla\overline{\Lambda}_a(\theta)\tau_1)\mathrm{e}^{\langle\theta, X_{\tau_1}\rangle - \overline{\Lambda}_a(\theta)\tau_1}\mathbb{E}\prod_{j=1}^{\tau_1}\xi(X_{j-1}, \Delta_j(X))\Big) = 0, \tag{2.4.1}$$

which yields the representation

$$\nabla\overline{\Lambda}_a(\theta) = \frac{\overline{E}_0^Q\Big(X_{\tau_1}\mathrm{e}^{\langle\theta, X_{\tau_1}\rangle - \overline{\Lambda}_a(\theta)\tau_1}\mathbb{E}\prod_{j=1}^{\tau_1}\xi(X_{j-1}, \Delta_j(X))\Big)}{\overline{E}_0^Q\Big(\tau_1\mathrm{e}^{\langle\theta, X_{\tau_1}\rangle - \overline{\Lambda}_a(\theta)\tau_1}\mathbb{E}\prod_{j=1}^{\tau_1}\xi(X_{j-1}, \Delta_j(X))\Big)}. \tag{2.4.2}$$

In particular, notice that whenever $\mathrm{dis}(\mathbb{P}) = 0$, i.e. $\mathbb{P}$-a.s. $\omega(x, e) = \alpha(e)$ for all $e \in \mathbb{V}$ and $x \in \mathbb{Z}^d$, we have $\overline{\Lambda}_a(0) = 0$ so that

$$\nabla\overline{\Lambda}_a(0) = \frac{\overline{E}_0^Q(X_{\tau_1})}{\overline{E}_0^Q(\tau_1)}. \tag{2.4.3}$$

On the other hand, by the renewal structure, the law of large numbers for the Q-random walk and (P2) in Lemma 2.3.2 we have that, for any environmental law $\mathbb{P}$ (with not necessarily zero disorder),

$$\frac{\overline{E}_0^Q(X_{\tau_1})}{\overline{E}_0^Q(\tau_1)} = y. \tag{2.4.4}$$

In particular, in the zero disorder case we conclude that $\nabla\overline{\Lambda}_a(0) = y$. In the general case, whenever $\mathrm{dis}(\mathbb{P})$ is sufficiently small $\nabla\overline{\Lambda}_a(0)$ will be close to y. More precisely, we have the following.

Proposition 2.4.1. *Given $\delta > 0$ there exists $\varepsilon_1 = \varepsilon_1(y, \delta) > 0$ such that, for any $\mathbb{P} \in \mathcal{P}_\kappa$, if $\mathrm{dis}(\mathbb{P}) < \varepsilon_1$ then $|\nabla\overline{\Lambda}_a(0) - y|_1 < \delta$.*

Proof. It follows from (2.4.2) that

$$\nabla\overline{\Lambda}_a(0) = \frac{\overline{E}_0^Q\Big(X_{\tau_1}\mathrm{e}^{-\overline{\Lambda}_a(0)\tau_1}\mathbb{E}\prod_{j=1}^{\tau_1}\xi(X_{j-1},\Delta_j(X))\Big)}{\overline{E}_0^Q\Big(\tau_1\mathrm{e}^{-\overline{\Lambda}_a(0)\tau_1}\mathbb{E}\prod_{j=1}^{\tau_1}\xi(X_{j-1},\Delta_j(X))\Big)}.$$

Thus, in light of (2.4.4) and since $\overline{E}_0^Q(\tau_1) \geq 1$, in order to prove the result it will suffice to show that given $\delta' > 0$ there exists $\varepsilon_1' = \varepsilon_1'(y,\delta') > 0$ such that, for any $\mathbb{P} \in \mathcal{P}_\kappa$, if $\mathrm{dis}(\mathbb{P}) < \varepsilon_1'$ then

$$\left|\overline{E}_0^Q\Big(X_{\tau_1}\mathrm{e}^{-\overline{\Lambda}_a(0)\tau_1}\mathbb{E}\prod_{j=1}^{\tau_1}\xi(X_{j-1},\Delta_j(X))\Big) - \overline{E}_0^Q(X_{\tau_1})\right|_1 \leq \delta' \tag{2.4.5}$$

and

$$\left|\overline{E}_0^Q\Big(\tau_1\mathrm{e}^{-\overline{\Lambda}_a(0)\tau_1}\mathbb{E}\prod_{j=1}^{\tau_1}\xi(X_{j-1},\Delta_j(X))\Big) - \overline{E}_0^Q(\tau_1)\right| \leq \delta'. \tag{2.4.6}$$

But by Remark 2.3.1 and the the mean value theorem we have that

$$\left|\overline{E}_0^Q\Big(X_{\tau_1}\mathrm{e}^{-\overline{\Lambda}_a(0)\tau_1}\mathbb{E}\prod_{j=1}^{\tau_1}\xi(X_{j-1},\Delta_j(X))\Big) - \overline{E}_0^Q(X_{\tau_1})\right|_1 \leq h(\mathrm{dis}(\mathbb{P}))\overline{E}_0^Q\Big(|X_{\tau_1}|_1\tau_1\mathrm{e}^{h(\mathrm{dis}(\mathbb{P}))\tau_1}\Big),$$

so that (2.4.5) now follows from the bound $|X_{\tau_1}|_1 \leq \tau_1$, Lemma 2.3.4 and Proposition 2.3.6 upon taking $\mathrm{dis}(\mathbb{P})$ small enough (depending only on y and δ'). Since (2.4.6) also follows in a similar way, this concludes the proof. □

Next, we consider the set

$$\mathcal{A}_{y,\mathbb{P}} := \{\nabla\overline{\Lambda}_a(\theta) : |\theta|_1 < \overline{\gamma}\},$$

with $\overline{\gamma}$ as in Proposition 2.3.11. Observe that this set depends on both y and $\mathbb{P}$ (and we stress this dependence in the notation). The next proposition shows that this set is open when $\mathrm{dis}(\mathbb{P}) < \gamma_1$.

Proposition 2.4.2. *For any $\mathbb{P} \in \mathcal{P}_\kappa$, whenever $|\theta|_1 \vee \mathrm{dis}(\mathbb{P}) < \gamma_1$, with $\gamma_1 > 0$ given by Proposition 2.3.8, the Hessian $H_a(\theta)$ of $\overline{\Lambda}_a$ at the point θ is given by the formula*

$$H_a(\theta) = \frac{\overline{E}_0^Q\left((X_{\tau_1} - \nabla\overline{\Lambda}_a(\theta)\tau_1)^T(X_{\tau_1} - \nabla\overline{\Lambda}_a(\theta)\tau_1)\mathrm{e}^{\langle\theta, X_{\tau_1}\rangle - \overline{\Lambda}_a(\theta)\tau_1}\mathbb{E}\prod_{j=1}^{\tau_1}\xi(X_{j-1}, \Delta_j(X))\right)}{\overline{E}_0^Q\left(\tau_1\mathrm{e}^{\langle\theta, X_{\tau_1}\rangle - \overline{\Lambda}_a(\theta)\tau_1}\mathbb{E}\prod_{j=1}^{\tau_1}\xi(X_{j-1}, \Delta_j(X))\right)} \tag{2.4.7}$$

and is positive definite. In particular, whenever $\mathrm{dis}(\mathbb{P}) < \gamma_1$ the set $\mathcal{A}_{y,\mathbb{P}}$ is open.

Proof. Taking derivatives on (2.4.1) (which again we can do by using Proposition 2.3.7 and (2.5.12)) and proceeding as for (2.4.2) immediately yields (2.4.7). On the other hand, for any column vector $v \in \mathbb{R}^{n\times 1}$ we have

$$\langle v, H_a(\theta)\cdot v\rangle = \frac{\overline{E}_0^Q\left(|\langle X_{\tau_1} - \nabla\overline{\Lambda}_a(\theta)\tau_1, v\rangle|^2\mathrm{e}^{\langle\theta, X_{\tau_1}\rangle - \overline{\Lambda}_a(\theta)\tau_1}\mathbb{E}\prod_{j=1}^{\tau_1}\xi(X_{j-1}, \Delta_j(X))\right)}{\overline{E}_0^Q\left(\tau_1\mathrm{e}^{\langle\theta, X_{\tau_1}\rangle - \overline{\Lambda}_a(\theta)\tau_1}\mathbb{E}\prod_{j=1}^{\tau_1}\xi(X_{j-1}, \Delta_j(X))\right)},$$

so that $\langle v, H_a(\theta)\cdot v\rangle \geq 0$ and the equality holds if and only if $\langle X_{\tau_1} - \nabla\overline{\Lambda}_a(\theta)\tau_1, v\rangle = 0$ $\overline{Q}_0$-a.s. or, equivalently, if $\langle \frac{X_{\tau_1}}{\tau_1}, v\rangle$ is $\overline{Q}_0$-almost surely constant. However, since $\inf_{e\in\mathbb{V}}\alpha(e) > 0$, it is not hard to check that if $v \neq 0$ then $\langle \frac{X_{\tau_1}}{\tau_1}, v\rangle$ cannot be constant. Hence, we see that in this case v must be zero and therefore $H_a(\theta)$ is positive definite. Finally, that $\mathcal{A}_{y,\mathbb{P}}$ is open follows from this and the inverse function theorem. □

The next proposition states that, whenever the disorder is small enough, the set $\mathcal{A}_{y,\mathbb{P}}$ contains a ball centered at $\nabla\overline{\Lambda}_a(0)$ whose radius is independent of $\mathbb{P}$.

Proposition 2.4.3. *There exist $\varepsilon_2 = \varepsilon_2(y, d, \kappa), r_2 = r_2(y, d, \kappa) > 0$ such that, for any $\mathbb{P} \in \mathcal{P}_\kappa$, if $\mathrm{dis}(\mathbb{P}) < \varepsilon_2$ then $B_{r_2}(\nabla\overline{\Lambda}_a(0)) \subseteq \mathcal{A}_{y,\mathbb{P}}$.*

The proof of Proposition 2.4.3 will be carried out in Subsection 2.4.3 . As a consequence of Propositions 2.4.1 and 2.4.3, we immediately obtain the following corollary.

Corollary 2.4.4. *There exist $\varepsilon = \varepsilon(y, d, \kappa), r = r(y, d, \kappa) > 0$ such that, for any $\mathbb{P} \in \mathcal{P}_\kappa$, if $\mathrm{dis}(\mathbb{P}) < \varepsilon$ then $B_r(y) \subseteq \mathcal{A}_{y,\mathbb{P}}$.*

2.4.2 Proof of Theorem 2.2.1 and Theorem 2.3.1 (Assuming Proposition 2.4.3):

Now, for $x \in B_r(y)$ with r as in Corollary 2.4.4, define the quantities

$$\tilde{I}_q(x) := \sup_{\theta \in \mathbb{R}^d} [\langle \theta, x\rangle - \overline{\Lambda}_q(\theta)] \qquad \text{and} \qquad \tilde{I}_a(x) := \sup_{\theta \in \mathbb{R}^d} [\langle \theta, x\rangle - \overline{\Lambda}_a(\theta)].$$

It is standard to show that (see [DZ, Lemma 2.3.9] for details)

$$\tilde{I}_q(x) = \langle \theta_{x,q}, y\rangle - \overline{\Lambda}_q(\theta_{x,q}) \qquad \text{and} \qquad \tilde{I}_a(x) = \langle \theta_{x,a}, y\rangle - \overline{\Lambda}_a(\theta_{x,a}) \tag{2.4.8}$$

for any $\theta_{x,q}$ and $\theta_{x,a}$ respectively satisfying

$$\nabla\overline{\Lambda}_q(\theta_{x,q}) = x \qquad \text{and} \qquad \nabla\overline{\Lambda}_a(\theta_{x,a}) = x.$$

Notice that such $\theta_{x,a}$ exists and satisfies $|\theta_{x,a}| < \overline{\gamma}$ since $x \in \mathcal{A}_{y,\mathbb{P}}$ by choice of x. Furthermore, such $\theta_{x,q}$ also exists and in fact can be taken equal to $\theta_{x,a}$, since both $\overline{\Lambda}_q(\theta)$ and $\overline{\Lambda}_a(\theta)$ coincide for $|\theta|_1 < \overline{\gamma}$ by Proposition 2.3.11. Hence, from (2.4.8) and the fact that $\theta_{x,q} = \theta_{x,a}$, we obtain that $\tilde{I}_q(x) = \tilde{I}_a(x)$ for all $x \in B_r(y)$. We may then conclude Theorem 2.3.1 once we show this implies that $I_q(x) = I_a(x)$. But, from (2.3.13) and the definition of $\tilde{I}_q$ and $\tilde{I}_a$, for $x \in B_r(y)$ we have that

$$\tilde{I}_q(x) + \log(\sqrt{C_{y,\alpha}}) + \langle \theta_{y,\alpha}, x\rangle = \sup_{\theta \in \mathbb{R}^d} [\langle \theta + \theta_{y,\alpha}, x\rangle - \Lambda_q(\theta + \theta_{y,\alpha})] = \sup_{\theta \in \mathbb{R}^d} [\langle \theta, x\rangle - \Lambda_q(\theta)] = I_q(x) \tag{2.4.9}$$

and

$$\tilde{I}_a(x)+\log(\sqrt{C_{y,\alpha}})+\langle\theta_{y,\alpha},x\rangle = \sup_{\theta\in\mathbb{R}^d}[\langle\theta+\theta_{y,\alpha},x\rangle-\Lambda_a(\theta+\theta_{y,\alpha})] = \sup_{\theta\in\mathbb{R}^d}[\langle\theta,x\rangle-\Lambda_a(\theta)] = I_a(x), \tag{2.4.10}$$

where the rightmost equalities in (2.4.9) and (2.4.10) follow from standard arguments (see [DZ, Section 2.3] for details) using that Λ_q and Λ_a are well-defined in the sense of Corollary 2.3.3 and that $B_r(y)$ is contained in the set of exposed points of the Fenchel-Legendre transforms of both Λ_q and Λ_a by (2.4.8) and (2.3.13). Therefore, as $\tilde{I}_q$ and $\tilde{I}_a$ agree on $B_r(y)$, we see that the same holds for I_q, I_a and thus we obtain Theorem 2.3.1.

Then, in order to complete the proof, it only remains to prove Proposition 2.4.3. We do this next.

2.4.3 Proof of Proposition 2.4.3

The key ingredient in the proof of Proposition 2.4.3 is the following uniform version of the inverse function theorem.

Theorem 2.4.5 (Uniform inverse function theorem)**.** *Let $\mathcal{F}$ be a family of C^1-functions $f : G \to \mathbb{R}^d$ defined on some neighborhood $G \subseteq \mathbb{R}^d$ of 0 such that the differential matrix $Df(0) \in \mathbb{R}^{d\times d}$ is invertible for every $f \in \mathcal{F}$. Then, if there exist constants $c, \delta > 0$ such that $\{\theta : |\theta|_1 < \delta\} \subseteq G$ and*

I1. $\sup_{f\in\mathcal{F}} ||Df(0)^{-1}|| < c$,

I2. $\sup_{f\in\mathcal{F},\,|\theta|_1<\delta} ||Df(\theta) - Df(0)|| < \frac{1}{2c}$,

where $||\cdot||$ denotes the operator 1-norm, there exists ρ (depending only on c and δ) such that for all $f \in \mathcal{F}$,

$$B_\rho(f(0)) \subseteq \{f(\theta) : |\theta|_1 < \delta\}.$$

The proof of Theorem 2.4.5 is obtained by simply mimicking (part of) the proof of the standard inverse function theorem (see e.g. [Rud, Theorem 9.24]), replacing the usual estimates with uniform bounds. Therefore, we omit the proof and leave the details to the reader.

In light of Theorem 2.4.5, to obtain Proposition 2.4.3 it will suffice to show that there exists $\varepsilon_2 > 0$ depending only on y, d and κ such that the family of C^1-functions

$$\mathcal{F}_y := \{\nabla\overline{\Lambda}_a : \mathbb{P} \in \mathcal{P}_\kappa \text{ with } \mathrm{dis}(\mathbb{P}) < \varepsilon_2\}$$

satisfies the hypotheses of Theorem 2.4.5. By Proposition 2.4.2, we only need to check conditions (I1) and (I2). For this, we will need three auxiliary lemmas. The first one asserts that $\nabla\overline{\Lambda}_a(\theta)$ is close to $\nabla\overline{\Lambda}_a(0)$ (uniformly over $\mathbb{P}$) whenever θ is close to 0 and the disorder is sufficiently small.

Lemma 2.4.6. *Given $c > 0$, there exist $\varepsilon_3 = \varepsilon_3(y, c), \delta = \delta(y, c) > 0$ such that, for any $\mathbb{P} \in \mathcal{P}_\kappa$, if $\mathrm{dis}(\mathbb{P}) < \varepsilon_3$ then*

$$\sup_{|\theta|_1 < \delta} |\nabla\overline{\Lambda}_a(\theta) - \nabla\overline{\Lambda}_a(0)|_1 < c.$$

Proof. In view of (2.4.6) and the fact that $\overline{E}_0^Q(\tau_1) \geq 1$, it will be enough to check that, given $c' > 0$, there exist $\varepsilon' = \varepsilon'(y, c'), \delta' = \delta'(y, c') > 0$ such that, for any $\mathbb{P} \in \mathcal{P}_\kappa$, if $\mathrm{dis}(\mathbb{P}) < \varepsilon'$ then

$$\sup_{|\theta|_1 < \delta'} \left| \overline{E}_0^Q \left(X_{\tau_1} \mathbb{E} \prod_{j=1}^{\tau_1} \xi(X_{j-1}, \Delta_j(X)) \left(\mathrm{e}^{\langle \theta, X_{\tau_1} \rangle - \overline{\Lambda}_a(\theta)\tau_1} - \mathrm{e}^{-\overline{\Lambda}_a(0)\tau_1} \right) \right) \right|_1 < c'$$

and

$$\sup_{|\theta|_1 < \delta'} \left| \overline{E}_0^Q \left(\tau_1 \mathbb{E} \prod_{j=1}^{\tau_1} \xi(X_{j-1}, \Delta_j(X)) \left(\mathrm{e}^{\langle \theta, X_{\tau_1} \rangle - \overline{\Lambda}_a(\theta)\tau_1} - \mathrm{e}^{-\overline{\Lambda}_a(0)\tau_1} \right) \right) \right| < c'.$$

But this can be done exactly as in the proof of (2.4.5)-(2.4.6), using now the inequality

$$|\langle\theta, X_{\tau_1}\rangle - \overline{\Lambda}_a(\theta)\tau_1| + |\overline{\Lambda}_a(0)\tau_1| \leq 2\big(|\theta|_1 + h(\mathrm{dis}(\mathbb{P}))\big)\tau_1,$$

where h is as in Remark 2.3.1, which follows in the same way as the inequalities in this last remark. We omit the details. $\square$

The second lemma is the analogue of Proposition 2.4.1 but for the Hessian H_a, which states that whenever $\mathrm{dis}(\mathbb{P})$ is sufficiently small $H_a(0)$ will be close to the corresponding Hessian for the case of zero disorder. We denote by $||\cdot||$ to the operator 1-norm on $\mathbb{R}^{d\times d}$ matrices.

Lemma 2.4.7. *Given $c > 0$, there exist $\varepsilon_4 = \varepsilon_4(y, c) > 0$ such that, for any $\mathbb{P} \in \mathcal{P}_\kappa$, if $\mathrm{dis}(\mathbb{P}) < \varepsilon_4$ then*

$$\big\|H_a(0) - H_a^*(0)\big\| < c,$$

where

$$H_a^*(0) := \frac{\overline{E}_0^Q((X_{\tau_1} - y\tau_1)^T(X_{\tau_1} - y\tau_1))}{\overline{E}_0^Q(\tau_1)}. \tag{2.4.11}$$

Proof. For simplicity, let us set $\Gamma(v) := (X_{\tau_1} - v\tau_1)^T(X_{\tau_1} - v\tau_1)$ for $v \in \mathbb{R}^d$. Then, in view of (2.4.6), the fact that $\overline{E}_0^Q(\tau_1) \geq 1$ and since

$$\|\overline{E}_0^Q(\Gamma(y))\| \leq \overline{E}_0^Q(|X_{\tau_1} - y\tau_1|^2) \leq (1 + |y|_1)^2\overline{E}_0^Q(\tau_1^2),$$

by Proposition 2.3.6 (which can be used to bound the second moment of τ_1 uniformly in $\mathbb{P}$) we see that it will suffice to show that the numerators of both matrices are close, i.e. that given any $c' > 0$, there exists $\varepsilon' = \varepsilon'(y, c') > 0$ such that if $\mathrm{dis}(\mathbb{P}) < \varepsilon'$ then

$$\left\|\overline{E}_0^Q\Big(\Gamma(\nabla\overline{\Lambda}_a(0))\mathrm{e}^{-\overline{\Lambda}_a(0)\tau_1}\mathbb{E}\prod_{j=1}^{\tau_1}\xi(X_{j-1}, \Delta_j(X))\Big) - \overline{E}_0^Q(\Gamma(y))\right\| < c'. \tag{2.4.12}$$

Now, writing $\Xi_a(0) := \mathrm{e}^{-\overline{\Lambda}_a(0)\tau_1}\mathbb{E}\prod_{j=1}^{\tau_1}\xi(X_{j-1},\Delta_j(X))$ for simplicity, observe that we can bound the left-hand side of (2.4.12) from above by

$$\overline{E}_0^Q\Big(\|\Gamma(\nabla\overline{\Lambda}_a(0))-\Gamma(y)\|\,|\Xi_a(0)|\Big)+\overline{E}_0^Q\Big(\|\Gamma(y)\|\,|\Xi_a(0)-1|\Big).$$

Since by Remark 2.3.1 we have

$$|\Xi_a(0)-1|\leq h(\mathrm{dis}(\mathbb{P}))\tau_1\mathrm{e}^{h(\mathrm{dis}(\mathbb{P}))\tau_1}, \tag{2.4.13}$$

and, furthermore, it is straightforward to verify that

$$\|\Gamma(\nabla\overline{\Lambda}_a(0))-\Gamma(y)\|\leq 5(|\nabla\overline{\Lambda}_a(0)-y|\vee 1)\tau_1^2 \tag{2.4.14}$$

and

$$\|\Gamma(y)\|\leq|X_{\tau_1}-y\tau_1|^2\leq(1+|y|_1)^2\tau_1^2, \tag{2.4.15}$$

(2.4.12) follows at once from (2.4.13)-(2.4.14)-(2.4.15) by using Propositions 2.3.6 and 2.4.1.

□

The last auxiliary lemma states that $\|(H_a^*(0))^{-1}\|$ is uniformly bounded over $\mathcal{P}_\kappa$.

Lemma 2.4.8. *The mapping* $\alpha\mapsto H_a^*(0)$ *is continuous on* $\mathcal{M}_1^*(\mathbb{V}):=\{\alpha\in\mathcal{M}_1(\mathbb{V}):\inf_{e\in\mathbb{V}}\alpha(e)>0\}$*. In particular, for any* $\kappa>0$ *we have* $\sup_{\mathbb{P}\in\mathcal{P}_k}\|(H_a^*(0))^{-1}\|<\infty$*.*

Proof. By definition of $H_a^*(0)$, it suffices to check that the mappings

$$\alpha\mapsto\overline{E}_0^Q(\tau_1)\quad\text{and}\quad\alpha\mapsto\overline{E}_0^Q((X_{\tau_1}-y\tau_1)^T(X_{\tau_1}-y\tau_1))$$

are continuous on $\mathcal{M}_1^*(\mathbb{V})$. The proof for both mappings is similar, so we only show the continuity of $\alpha\mapsto\overline{E}_0^Q(\tau_1)$. To this end, since $\overline{E}_0^Q(\tau_1\mathbb{1}_{\{\tau_1>N\}})\to 0$ as $N\to\infty$ uniformly over

$\mathcal{M}_1^*(\mathbb{V})$ by Proposition 2.3.6, it will be enough to show that $\alpha \mapsto \overline{E}_0^Q(\tau_1 \mathbb{1}_{\{\tau_1 = N\}})$ is continuous for every $N \geq 1$. But, using the Markov property together with the fact that $Q_x(\beta_0 = \infty)$ does not depend on x, it is not difficult to see that $\overline{E}_0^Q(\tau_1 \mathbb{1}_{\{\tau_1 = N\}})$ is a polynomial of degree N in the weights $u = (u(e))_{e \in \mathbb{V}}$ from (2.3.2). Indeed, we have

$$\overline{E}_0^Q(\tau_1 \mathbb{1}_{\{\tau_1 = N\}}) = \sum_{\bar{x}_n} \prod_{j=1}^{n} \alpha(\Delta_j(\bar{x}_n)),$$

where the sum is over all paths $\bar{x}_n$ of length n which start at 0 and be extended to an infinite path $\bar{x}_\infty$ such that $\tau_1(\bar{x}_\infty) = n$, where $\tau_1(\bar{x}_\infty)$ denotes the analogue of τ_1 but for $\bar{x}_\infty$. Therefore, since the weights $u(e)$ all depend continuously on α, the continuity of $\alpha \mapsto \overline{E}_0^Q(\tau_1 \mathbb{1}_{\{\tau_1 = N\}})$ follows.

Finally, to check the last statement, we first notice that $\alpha \mapsto \|(H_a^*(0))^{-1}\|$ is also continuous on $\mathcal{M}_1^*(\mathbb{V})$ by Proposition 2.4.2, since the mappings $A \mapsto A^{-1}$ and $A \mapsto \|A\|$ are also continuous in their respective domains. Hence, since $\mathcal{M}_1^{(\kappa)}(\mathbb{V})$ is compact for any $\kappa > 0$ and

$$\sup_{\mathbb{P} \in \mathcal{P}_k} \|(H_a^*(0))^{-1}\| = \sup_{\alpha \in \mathcal{M}_1^{(\kappa)}(\mathbb{V})} \|(H_a^*(0))^{-1}\|,$$

the last statement now follows. $\square$

We are now ready to show (I1) and (I2). To check (I1), using Lemmas 2.4.7-2.4.8 we may choose $\varepsilon_2 > 0$ depending only on y, d and κ such that if $\mathrm{dis}(\mathbb{P}) < \varepsilon_2$ then

$$\|H_a(0) - H_a^*(0)\| \leq \frac{1}{2 \sup_{\mathbb{P} \in \mathcal{P}_k} \|(H_a^*(0))^{-1}\|}.$$

Then, using the identity $A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$ for any invertible matrices $A, B \in$

$\mathbb{R}^{d\times d}$, we have that, for any $\mathbb{P} \in \mathcal{P}_k$, if $\mathrm{dis}(\mathbb{P}) < \varepsilon_2$ then

$$\left\|(H_a(0))^{-1} - (H_a^*(0))^{-1}\right\| \leq \left\|(H_a(0))^{-1}\right\| \|H_a(0) - H_a^*(0)\| \left\|(H_a^*(0))^{-1}\right\| < \frac{1}{2}\left\|(H_a(0))^{-1}\right\|,$$

so that by the triangle inequality

$$\left\|H_a(0)^{-1}\right\| \leq \frac{1}{2}\left\|H_a(0)^{-1}\right\| + \left\|(H_a^*(0))^{-1}\right\|$$

and thus

$$\left\|H_a(0)^{-1}\right\| \leq 2\left\|(H_a^*(0))^{-1}\right\| \leq 2\sup_{\mathbb{P}\in\mathcal{P}_k}\|(H_a^*(0))^{-1}\|.$$

This shows (I1) for $c := 2\sup_{\mathbb{P}\in\mathcal{P}_k}\|(H_a^*(0))^{-1}\|$. It remains to check (I2).

By arguing as in the proof of Lemma 2.4.7, to check (I2) it will suffice to show that, given $c' > 0$, one can find $\varepsilon_2' = \varepsilon_2'(y, c'), \delta = \delta(y, c') > 0$ such that if $\mathrm{dis}(\mathbb{P}) < \varepsilon_2'$ then

$$\sup_{|\theta|_1<\delta}\left\|\overline{E}_0^Q\big(\Gamma(\nabla\overline{\Lambda}_a(\theta))\Xi_a(\theta)\big) - \overline{E}_0^Q\big(\Gamma(\nabla\overline{\Lambda}_a(0))\Xi_a(0)\big)\right\| < c'$$

where, for $v, \theta \in \mathbb{R}^d$, we set

$$\Gamma(v) := (X_{\tau_1} - v\tau_1)^T(X_{\tau_1} - v\tau_1) \qquad \text{and} \qquad \Xi_a(\theta) := \mathrm{e}^{\langle\theta, X_{\tau_1}\rangle - \overline{\Lambda}_a(\theta)\tau_1}\mathbb{E}\prod_{j=1}^{\tau_1}\xi(X_{j-1}, \Delta_j(X)).$$

But this can be done as in the proof of Lemma 2.4.7, by using Lemma 2.4.6 and (2.4.13)-(2.4.14)-(2.4.15) together with the inequalities

$$\|\Gamma(\nabla\overline{\Lambda}_a(\theta)) - \Gamma(\nabla\overline{\Lambda}_a(0))\| \leq 5(|\nabla\overline{\Lambda}_a(\theta) - \nabla\overline{\Lambda}_a(0)|_1 \vee 1)\tau_1^2$$

and

$$|\Xi_a(\theta) - \Xi_a(0)| \leq 2(|\theta|_1 + h(\mathrm{dis}(\mathbb{P})))\tau_1\mathrm{e}^{2(|\theta|_1 + h(\mathrm{dis}(\mathbb{P})))\tau_1}$$

for h as in Remark 2.3.1, which are both straightforward to check. This shows (I2) and therefore completes the proof of Proposition 2.4.3.

2.5 Non-triviality of $\lim_{n\to\infty}\Phi_n(\theta)$ - proof of Propositions 2.3.8 and 2.3.9

2.5.1 Proof of Proposition 2.3.8

The first step in the proof will be to show that there exists $\gamma_1 = \gamma_1(y) > 0$ such that, for any $\mathbb{P} \in \mathcal{P}_\kappa$, whenever $|\theta|_1 \vee \mathrm{dis}(\mathbb{P}) < \gamma_1$ we have that (2.3.20) holds. This will be a consequence of the following two lemmas.

Lemma 2.5.1. *For all $\theta \in \mathbb{R}^d$,*

$$\overline{E}_0^Q\Big(\mathrm{e}^{\langle\theta,X_{\tau_1}\rangle-\overline{\Lambda}_a(\theta)\tau_1}\mathbb{E}\prod_{j=1}^{\tau_1}\xi(X_{j-1},\Delta_j(X))\Big) \leq 1.$$

Lemma 2.5.2. *There exists $\gamma_1 = \gamma_1(y) > 0$ such that, for any $\mathbb{P} \in \mathcal{P}_\kappa$, whenever $|\theta|_1 \vee \mathrm{dis}(\mathbb{P}) < \gamma_1$,*

$$\overline{E}_0^Q\Big(\mathrm{e}^{\langle\theta,X_{\tau_1}\rangle-\overline{\Lambda}_a(\theta)\tau_1}\mathbb{E}\prod_{j=1}^{\tau_1}\xi(X_{j-1},\Delta_j(X))\Big) \geq 1. \tag{2.5.1}$$

Postponing the proofs of these lemmas for a moment, let us finish the proof of Proposition 2.3.8. For $\theta \in \mathbb{R}^d, \mathbb{P} \in \mathcal{P}_\kappa$ such that $|\theta|_1 \vee \mathrm{dis}(\mathbb{P}) < \gamma_1$ we may define the probability measure $\mu^{(\theta)}$ on $\mathbb{Z}^d$ as

$$\mu^{(\theta)}(x) := \overline{E}_0^Q\Big(\mathrm{e}^{\langle\theta,X_{\tau_1}\rangle-\overline{\Lambda}_a(\theta)\tau_1}\mathbb{E}\prod_{j=1}^{\tau_1}\xi(X_{j-1},\Delta_j(X))\,;\,X_{\tau_1}=x\Big) \tag{2.5.2}$$

and consider the random walk $Y^{(\theta)} = (Y_n^{(\theta)})_{n\in\mathbb{N}}$ with jump distribution $\mu^{(\theta)}$. Then, if $\widehat{E}_0^{(\theta)}$

denotes expectation with respect to $\widehat{P}^{(\theta)}$, the law of $Y^{(\theta)}$ starting from 0, we have that

$$\lim_{n\to\infty} \mathbb{E}\Phi_n(\theta) = \frac{1}{\widehat{E}_0^{(\theta)}(\langle Y_1, \ell\rangle)}. \tag{2.5.3}$$

Indeed, using (2.3.20) and the renewal structure of the Q-random walk, for each $n \geq 1$ we have

$$\begin{aligned}\mathbb{E}\Phi_n(\theta) &= \sum_{k=1}^{\infty} \overline{E}_0^Q \Big(\mathrm{e}^{\langle \theta, X_{\tau_k}\rangle - \overline{\Lambda}_a(\theta)\tau_k} \mathbb{E} \prod_{j=1}^{\tau_k} \xi(X_{j-1}, \Delta_j(X))\,;\, L_n = \tau_k \Big) \\ &= \sum_{k=1}^{\infty} \widehat{P}_0^{(\theta)}(\langle Y_k, \ell\rangle = n) = \widehat{P}_0^{(\theta)}(\langle Y_k, \ell\rangle = n \text{ for some } k \geq 1) \end{aligned} \tag{2.5.4}$$

so that (2.5.3) is now a consequence of the renewal theorem for the sequence $(\langle Y_k - Y_{k-1}, \ell\rangle)_{k\geq 1}$. Finally, Proposition 2.3.8 then follows from (2.5.3) and the next lemma.

Lemma 2.5.3. *There exists $\gamma_1 = \gamma_1(y) > 0$ such that, for any $\mathbb{P} \in \mathcal{P}_\kappa$, whenever $|\theta|_1 \vee \mathrm{dis}(\mathbb{P}) < \gamma_1$,*

$$\widehat{E}_0^{(\theta)}(\langle Y_1, \ell\rangle) < \infty.$$

Thus, in order to complete the proof of Proposition 2.3.8 we only need to prove Lemmas 2.5.1, 2.5.2 and 2.5.3 above. The rest of this subsection is devoted to this.

Proof of Lemma 2.5.1. Given $\delta > 0$, let us write $\eta_{\theta,\delta} := \overline{\Lambda}_a(\theta) + \delta$ for simplicity and for $n \geq 1$ define

$$\Upsilon_{n,\delta}(\theta) := E_0^Q \Big(\mathrm{e}^{\langle \theta, X_{\tau_n}\rangle - \eta_{\theta,\delta}\tau_n} \mathbb{E} \prod_{j=1}^{\tau_n} \xi(X_{j-1}, \Delta_j(X)) \Big).$$

Then, by splitting the expectation in the definition of $\Upsilon_{n,\delta}(\theta)$ according to the different possible values for τ_n, we have as in (2.3.23) that

$$\Upsilon_{n,\delta}(\theta) \leq \sum_{k=n}^{\infty} \mathrm{e}^{-\eta_{\theta,\delta}k} E_0^Q \Big(\mathrm{e}^{\langle \theta, X_k\rangle} \mathbb{E} \prod_{j=1}^{k} \xi(X_{j-1}, \Delta_j(X)) \Big). \tag{2.5.5}$$

Since, for some $o(1) \to 0$ as $k \to \infty$ we have

$$E_0^Q \Big(\mathrm{e}^{\langle \theta, X_k \rangle} \mathbb{E} \prod_{j=1}^{k} \xi(X_{j-1}, \Delta_j(X)) \Big) = \mathrm{e}^{(\overline{\Lambda}_a(\theta) + o(1))k}, \tag{2.5.6}$$

from (2.5.5) we obtain that for all n sufficiently large (depending on δ)

$$\Upsilon_{n,\delta}(\theta) \le \sum_{k=n}^{\infty} \mathrm{e}^{-\frac{\delta}{2}k} = \frac{\mathrm{e}^{-\frac{\delta}{2}n}}{1 - \mathrm{e}^{-\frac{\delta}{2}}}. \tag{2.5.7}$$

On the other hand, by the renewal structure, we have Q_0-almost surely,

$$\mathbb{E} \prod_{j=1}^{\tau_n} \xi(X_{j-1}, \Delta_j(X)) = \prod_{i=0}^{n-1} \left(\mathbb{E} \prod_{j=\tau_i+1}^{\tau_{i+1}} \xi(X_{j-1}, \Delta_j(X)) \right). \tag{2.5.8}$$

From this, using the renewal structure once again together with the translation invariance of $\mathbb{P}$, we see that for all $n \ge 1$

$$\Upsilon_{n,\delta}(\theta) = \Upsilon_{1,\delta}(\theta) \left(\overline{E}_0^Q \Big(\mathrm{e}^{\langle \theta, X_{\tau_1} \rangle - \eta_{\theta,\delta}\tau_1} \mathbb{E} \prod_{j=1}^{\tau_1} \xi(X_{j-1}, \Delta_j(X)) \Big) \right)^{n-1}. \tag{2.5.9}$$

Since $\Upsilon_{1,\delta}(\theta) > 0$, in light of (2.5.7) we conclude that

$$\overline{E}_0^Q \Big(\mathrm{e}^{\langle \theta, X_{\tau_1} \rangle - \overline{\Lambda}_a(\theta)\tau_1} \mathbb{E} \prod_{j=1}^{\tau_1} \xi(X_{j-1}, \Delta_j(X)) \Big) \le \mathrm{e}^{-\frac{\delta}{2}}.$$

Letting $\delta \searrow 0$, by monotone convergence we get the desired result. □

Proof of Lemma 2.5.2. Given $\theta \in \mathbb{R}^d$, $n \ge 1$ and $r \in \mathbb{R}$, let us write

$$\Xi_{n,r}(\theta) := \mathrm{e}^{\langle \theta, X_n \rangle - rn} \mathbb{E} \prod_{j=1}^{n} \xi(X_{j-1}, \Delta_j(X)). \tag{2.5.10}$$

Then, by splitting $E_0^Q(\Xi_{n,r}(\theta))$ according to the different events $\{n \in (\tau_m, \tau_{m+1}] \,, \, n = \tau_m + i\}$

for $m = 0, \dots, n-1$ and $i = 1, \dots, n$ and using the Markov property at τ_m, we see that

$$
\begin{aligned}
E_0^Q(\Xi_{n,r}(\theta)) &\leq \sum_{m=0}^{n-1}\sum_{i=1}^{n} E_0^Q(\Xi_{\tau_m,r}(\theta)\,;\,\tau_m = n-i)\overline{E}_0^Q(\Xi_{i,r}(\theta)\,;\,\tau_1 > i)\\
&\leq \sum_{m=0}^{n-1} E_0^Q(\Xi_{\tau_m,r}(\theta))\overline{E}_0^Q\Big(\sup_{i\leq\tau_1}\Xi_{i,r}(\theta)\Big)\\
&\leq \overline{E}_0^Q\Big(\sup_{i\leq\tau_1}\Xi_{i,r}(\theta)\Big)\left(1 + E_0^Q\Big(\sup_{i\leq\tau_1}\Xi_{i,r}(\theta)\Big)\sum_{m=1}^{\infty}\Big(\overline{E}_0^Q(\Xi_{\tau_1,r})\Big)^{m-1}\right)
\end{aligned}
\tag{2.5.11}
$$

where, in order to obtain the last inequality, we have used that for $m \geq 1$,

$$
E_0^Q(\Xi_{\tau_m,r}(\theta)) = E_0^Q(\Xi_{\tau_1,r})\Big(\overline{E}_0^Q(\Xi_{\tau_1,r})\Big)^{m-1}
$$

which follows from the renewal structure as in (2.5.9).

Now, if we take then $r = \overline{\Lambda}_a(\theta) - \delta$ for some $\delta > 0$ then by Remark 2.3.1 we have, for any $i \geq 1$,

$$
\Xi_{i,r}(\theta) \leq \exp\big(\big(2|\theta|_1 + h(\mathrm{dis}(\mathbb{P})) + \delta\big)i\big)\,.
$$

If we choose γ_1 and δ small enough (but depending only on y) so that $2|\theta|_1 + h(\mathrm{dis}(\mathbb{P})) + \delta < \frac{\gamma_0}{2}$ whenever $|\theta|_1 \vee \mathrm{dis}(\mathbb{P}) < \gamma_1$, where γ_0 is as in Proposition 2.3.6, then we obtain that

$$
E_0^Q\Big(\sup_{i\leq\tau_1}\Xi_{i,r}(\theta)\Big) \leq E_0^Q\Big(\mathrm{e}^{\frac{\gamma_0}{2}\tau_1}\Big) < \infty,
\tag{2.5.12}
$$

and combining (2.5.12) with Lemma 2.3.4 shows that $\overline{E}_0^Q(\sup_{i\leq\tau_1}\Xi_{i,r}(\theta)) < \infty$ as well. Thus, since the bound in (2.5.11) is uniform in n, if $\overline{E}_0^Q\big(\Xi_{\tau_1,r}(\theta)\big) < 1$ then we would have $\sup_{n\geq 1} E_0^Q(\Xi_{n,r}(\theta)) < \infty$, and this in turn would imply that

$$
\lim_{n\to\infty}\frac{1}{n}\log E_0^Q(\Xi_{n,r}(\theta)) = 0.
$$

However, observe that by choice of r, definition of $_{n,r}$ and (2.5.6), we have that

$$\lim_{n\to\infty}\frac{1}{n}\log E_0^Q(\Xi_{n,r}(\theta))=\delta$$

so that in reality whenever $|\theta|_1 \vee \mathrm{dis}(\mathbb{P}) < \gamma_1$ we must have

$$1\leq \overline{E}_0^Q\big(\Xi_{1,r}(\theta)\big)=\overline{E}_0^Q\Big(\mathrm{e}^{\langle\theta,X_{\tau_1}\rangle-(\overline{\Lambda}_a(\theta)-\delta)\tau_1}\mathbb{E}\prod_{j=1}^{\tau_1}\xi(X_{j-1},\Delta_j(X))\Big).$$

Letting $\delta\searrow 0$, by dominated convergence we get the desired result (note that we can indeed use dominated convergence since $\overline{E}_0^Q\big(\Xi_{\tau_1,r}(\theta)\big)<\infty$ for $r=\overline{\Lambda}_a(\theta)-\delta$ and $\delta>0$ sufficiently small, by (2.5.12) and choice of γ_0). This concludes the proof. □

Proof of Lemma 2.5.3. Since $\langle Y_1,\ell\rangle\leq\tau_1$ by definition of τ_1, using also that $\tau_1\leq\frac{1}{\delta}\mathrm{e}^{\delta\tau_1}$ for any $\delta>0$, we see that

$$\widehat{E}_0^{(\theta)}(\langle Y_1,\ell\rangle)\leq\frac{1}{\delta}\overline{E}_0^Q\Big(\mathrm{e}^{\langle\theta,X_{\tau_1}\rangle-(\overline{\Lambda}_a(\theta)-\delta)\tau_1}\mathbb{E}\prod_{j=1}^{\tau_1}\xi(X_{j-1},\Delta_j(X))\Big)$$

and so the lemma now follows as in the proof of (2.5.12). □

2.5.2 Proof of Proposition 2.3.9

We will show that there exists a constant $\gamma_2>0$, depending only on y,d and κ such that, for any $\mathbb{P}\in\mathcal{P}_\kappa$, if $\mathrm{dis}(\mathbb{P})<\gamma_2$ then

$$\sup_{n\geq 1\,,\,|\theta|_1<\gamma_2}\mathbb{E}(\Phi_n(\theta))^2<\infty.$$

This is equivalent to showing that

$$\sup_{n\geq 1\,,\,|\theta|_1<\gamma_2} \overline{E}^{Q}_{0,0}\Big(\mathrm{e}^{\langle\theta, X_{L_n}+\widetilde{X}_{\widetilde{L}_n}\rangle-\overline{\Lambda}_a(\theta)(L_n+\widetilde{L}_n)}\mathbb{E}\prod_{j=1}^{L_n}\xi(X_{j-1},\Delta_j(X))\prod_{j=1}^{\widetilde{L}_n}\xi(\widetilde{X}_{j-1},\Delta_j(\widetilde{X}))\,;\,n\in\mathcal{L}\Big)<\infty, \tag{2.5.13}$$

where $X=(X_n)_{n\in\mathbb{N}}$ and $\widetilde{X}=(\widetilde{X}_n)_{n\in\mathbb{N}}$ are independent copies of the conditioned random walk with law $\overline{Q}_0$, $\widetilde{L}_n$ and $\widetilde{\tau}_n$ are the analogues of L_n τ_n but for $\widetilde{X}$, and

$$\mathcal{L}:=\{n\geq 0: \langle X_i,\ell\rangle\geq n \text{ for all } i\geq L_n\,,\,\langle\widetilde{X}_j,\ell\rangle\geq n\text{ for all } j\geq\widetilde{L}_n\} \tag{2.5.14}$$

are the so-called *common renewal levels*. In the sequel, we shall write $\overline{Q}_{x,\widetilde{x}}:=\overline{Q}_x\times\overline{Q}_{\widetilde{x}}$ and $\overline{E}^{Q}_{x,\widetilde{x}}$ to denote expectation with respect to $\overline{Q}_{x,\widetilde{x}}$.

In order to check (2.5.13), let us introduce, for $x\in\mathbb{Z}^d$, $e\in\mathbb{V}$ and $n\geq 1$, the quantities

$$N_{x,e}(n):=\#\{j\in\{1,\dots,n\}: X_{j-1}=x\,,\,\Delta_j(X)=e\}=\sum_{j=1}^{n}\mathbb{1}_x(X_{j-1})\mathbb{1}_e(\Delta_j(X))$$

and

$$N_x(n):=\#\{j\in\{1,\dots,n\}:X_{j-1}=x\}=\sum_{e\in\mathbb{V}}N_{x,e}(n),$$

as well as the corresponding analogues $\widetilde{N}_{x,e}(n)$ and $\widetilde{N}_x(n)$ for $\widetilde{X}$. Then, using that by definition of $\mathrm{dis}(\mathbb{P})$ we have that, for all $x\in\mathbb{Z}^d$, $e\in\mathbb{V}$ and h as in Remark 2.3.1, the inequality

$$\omega(x,e)\leq\widetilde{\omega}(x,e)\mathrm{e}^{h(\mathrm{dis}(\mathbb{P}))}$$

holds almost surely for any pair of independent environments ω and $\widetilde{\omega}$ with law $\mathbb{P}$, we have

$$\begin{aligned}\mathbb{E}\prod_{j=1}^{L_n}\omega(X_{j-1},\Delta_j(X))\prod_{j=1}^{\widetilde{L}_n}\omega(\widetilde{X}_{j-1},\Delta_j(\widetilde{X})) &= \prod_{x\in\mathbb{Z}^d}\mathbb{E}\prod_{e\in\mathbb{V}}\omega(x,e)^{N_{x,e}(L_n)+\widetilde{N}_{x,e}(\widetilde{L}_n)} \\ &\leq \prod_{x\in\mathbb{Z}^d}\mathbb{E}\left[\prod_{e\in\mathbb{V}}\omega(x,e)^{N_{x,e}(L_n)}\right]\mathbb{E}\left[\prod_{e\in\mathbb{V}}\omega(x,e)^{\widetilde{N}_{x,e}(\widetilde{L}_n)}\right]\mathrm{e}^{h(\mathrm{dis}(\mathbb{P}))[N_x(L_n)\wedge\widetilde{N}_x(\widetilde{L}_n)]} \\ &= \mathbb{E}\left[\prod_{j=1}^{L_n}\omega(X_{j-1},\Delta_j(X))\right]\mathbb{E}\left[\prod_{j=1}^{\widetilde{L}_n}\omega(\widetilde{X}_{j-1},\Delta_j(\widetilde{X}))\right]\mathrm{e}^{h(\mathrm{dis}(\mathbb{P}))I_n},\end{aligned}$$

where

$$I_n := \sum_{x\in\mathbb{Z}^d}[N_x(L_n)\wedge\widetilde{N}_x(\widetilde{L}_n)].$$

Hence, we conclude that the supremum in (2.5.13) is bounded from above by

$$A := \sup_{n\geq 1\,,\,|\theta|_1<\gamma_2\,,\,z\in\mathbb{V}_d} A_{z,n}(\theta), \tag{2.5.15}$$

where, for $z\in\mathbb{V}_d := \{z\in\mathbb{Z}^d : \langle z,\ell\rangle = 0\}$ and $n\geq 1$, we define

$$A_{z,n}(\theta) := \overline{E}^{Q}_{0,z}\big(F_n(\theta)\,;\,n\in\mathcal{L}\big) \tag{2.5.16}$$

with

$$F_n(\theta) := \Phi_n(\theta)\widetilde{\Phi}_n(\theta)\mathrm{e}^{h(\mathrm{dis}(\mathbb{P}))I_n},$$

where

$$\Phi_n(\theta) := \mathrm{e}^{\langle\theta,X_{L_n}-X_0\rangle-\overline{\Lambda}_a(\theta)L_n}\mathbb{E}\prod_{j=1}^{L_n}\xi(X_{j-1}-X_0,\Delta_j(X)) \tag{2.5.17}$$

and $\widetilde{\Phi}_n(\theta)$ is defined analogously but interchanging (X,L_n) with $(\widetilde{X},\widetilde{L}_n)$.

In order to prove Proposition 2.3.9, we will show that A is finite provided that $\theta\vee\mathrm{dis}(\mathbb{P})$

is taken sufficiently small (depending only on y, d and κ). To this end, let us set

$$\zeta := \inf\{m \geq 0 : \exists\, i, j \geq 1 \text{ such that } X_i = \widetilde{X}_j \text{ and } \langle X_i, \ell\rangle = m\}, \tag{2.5.18}$$

i.e. the first level in which both walks intersect at a time other than zero. Observe that whenever $1 \leq n \leq \zeta$ we have $X_i \neq \widetilde{X}_j$ for all $i < L_n$ and $j < \widetilde{L}_n$, so that $I_n = 1 \leq 1$, with the only possible non-vanishing term being $x = 0$. In particular, by virtue of independence and the definition of $\mathcal{L}$, we obtain that, for $\gamma_1 = \gamma_1(y) > 0$ as in the proof of Proposition 2.3.8 and any $\mathbb{P} \in \mathcal{P}_\kappa$, whenever $|\theta|_1 \vee \mathrm{dis}(\mathbb{P}) < \gamma_1 \wedge \frac{1}{2}$ we have

$$\begin{aligned} \overline{E}^{Q}_{0,z}\big(F_n(\theta)\,;\, n \in \mathcal{L}\,,\, n \leq \zeta\big) &\leq \overline{E}^{Q}_{0,z}\Big(\Phi_n(\theta)\widetilde{\Phi}_n(\theta)\mathrm{e}^{h(1/2)}, n \in \mathcal{L}\Big) \\ &= \mathrm{e}^{h(1/2)}\Big[\mathbb{E}\Phi_n(\theta)\Big]^2 \leq \mathrm{e}^{h(1/2)} \end{aligned}$$

where for the last inequality we have used that $\mathbb{E}\Phi_n(\theta) \leq 1$ since it coincides with a probability by (2.5.4). In light of this bound we see that, in order to show that A is finite, it only remains to obtain a suitable control on the expectation

$$\overline{E}^{Q}_{0,z}\Big(F_n(\theta)\,;\, n \in \mathcal{L}\,,\, n > \zeta\Big). \tag{2.5.19}$$

To this end, define

$$\sigma := \inf\{k \in \mathcal{L} : k > \zeta\}, \tag{2.5.20}$$

i.e. the first common renewal level after the walks first intersect (at a time other than zero). Then, by (2.5.8), the Markov property and translation invariance, (2.5.19) can be rewritten as

$$\sum_{k=1}^{n} \overline{E}^{Q}_{0,z}\Big(F_n(\theta)\,;\, n\in\mathcal{L}\,,\, \sigma=k\Big)$$

$$= \sum_{k=1}^{n}\sum_{z'\in\mathbb{V}_d} \overline{E}^{Q}_{0,z}\Big(F_k(\theta)\,;\,\sigma=k\,,\, \widetilde{X}_{\widetilde{L}_k}-X_{L_k}=z'\Big)\overline{E}^{Q}_{0,z'}\Big(F_{n-k}(\theta)\,;\, n-k\in\mathcal{L}\Big)$$

$$\leq \sum_{k=1}^{n} \overline{E}^{Q}_{0,z}\Big(F_k(\theta)\,;\,\sigma=k\Big)\sup_{z'\in\mathbb{V}_d} A_{z',n-k}(\theta),$$

where we use the convention $A_{z',0}(\theta):=1$ and, to obtain the first equality, we have used that $N_x(L_k)=N_x(L_n)$ whenever $\langle x,\ell\rangle<k$ and $N_x(L_k)=0$ whenever $\langle x,\ell\rangle\geq k$ (and the analogous statements for $\widetilde{N}_x$). Now, if we set

$$B_{z,n}(\theta) := \overline{E}^{Q}_{0,z}\big(F_n(\theta)\,;\,\sigma=n\big), \tag{2.5.21}$$

then by the arguments above, for any $\mathbb{P}\in\mathcal{P}_\kappa$, $n\geq 1$ and $z\in\mathbb{V}_d$, whenever $|\theta|_1\vee \mathrm{dis}(\mathbb{P})<\gamma_1\wedge\frac{1}{2}$ we have

$$A_{z,n}(\theta)\leq \mathrm{e}^{h(1/2)}+\sum_{k=1}^{n} B_{z,k}(\theta)\sup_{z'\in\mathbb{V}_d} A_{z',n-k}(\theta). \tag{2.5.22}$$

The next lemma will be crucial to conclude the proof.

Lemma 2.5.4. *There exists $\gamma_3=\gamma_3(y,d,\kappa)>0$ such that, for any $\mathbb{P}\in\mathcal{P}_\kappa$, whenever $|\theta|_1\vee\mathrm{dis}(\mathbb{P})<\gamma_3$,*

$$B:=\sup_{z\in\mathbb{V}_d}\sum_{n=1}^{\infty} B_{z,n}(\theta)<1.$$

Completing proof of Proposition 2.3.9 (Assuming Lemma 2.5.4): By (2.5.22), if we fix $N\geq 1$ then for any $n\leq N$ we have

$$A_{z,n}(\theta)\leq \mathrm{e}^{h(1/2)}+\Big(\sup_{m\leq N, z\in\mathbb{V}_d} A_{z,m}(\theta)\Big)\sum_{k=1}^{N} B_{z,k}(\theta),$$

so that, upon taking suprema, we find

$$\Big(1-\sum_{k=1}^{N} B_{z,k}(\theta)\Big)\sup_{n\leq N, z\in\mathbb{V}_d} A_{z,n}(\theta) \leq \mathrm{e}^{h(1/2)}.$$

Hence, whenever $|\theta|_1 \vee \mathrm{dis}(\mathbb{P}) < \gamma_3 \wedge \gamma_1 \wedge \frac{1}{2} =: \gamma_2$, letting $N \to \infty$ we conclude by Lemma 2.5.4 that $A \leq \frac{\mathrm{e}^{h(1/2)}}{(1-B)} < \infty$ and thus Proposition 2.3.9 follows. □

Hence, it only remains to prove Lemma 2.5.4.

2.5.3 Proof of Lemma 2.5.4.

We will need the aid of three additional lemmas. Before stating these, we introduce $B^*_{z,n}(\theta)$, the zero-disorder version of $B_{z,n}(\theta)$, given by the formula

$$B^*_{z,n}(\theta) := \overline{E}^{Q}_{0,z}\left(\mathrm{e}^{\langle \theta, X_{L_n} + (\widetilde{X}_{\widetilde{L}_n} - z)\rangle - \overline{\Lambda}^*_a(\theta)(L_n + \widetilde{L}_n)}\,;\, \sigma = n\right),$$

where $\overline{\Lambda}^*_a(\theta) := \lim_{n\to\infty} \frac{1}{n}\log E^{Q}_0(\mathrm{e}^{\langle\theta, X_n\rangle})$ (note that this limit exists by Corollary 2.3.3 applied to the particular case of zero-disorder environmental laws). The three additional lemmas we need are then the following:

Lemma 2.5.5. *Given $\kappa > 0$, there exists $\delta = \delta(y,d,\kappa) > 0$ such that, for any $\mathbb{P} \in \mathcal{P}_\kappa$,*

$$\sup_{z\in\mathbb{V}_d}\sum_{n=1}^{\infty} B^*_{z,n}(0) = \sup_{z\in\mathbb{V}_d} \overline{Q}_{0,z}(\sigma < \infty) < 1-\delta.$$

Lemma 2.5.6. *Given $\kappa > 0$, there exist $\gamma_4 = \gamma_4(y,d,\kappa), K_0 = K_0(y,d,\kappa) > 0$ such that*

$$\sum_{n=1}^{\infty}\left[\sup_{\mathbb{P}\in\mathcal{P}_\kappa(\gamma_4)\,,\,|\theta|_1<\gamma_4\,,\,z\in\mathbb{V}_d} B_{z,n}(\theta)\right] \leq K_0,$$

where $\mathcal{P}_\kappa(\gamma_4) := \{\mathbb{P}\in\mathcal{P}_\kappa : \mathrm{dis}(\mathbb{P}) < \gamma_4\}$.

Lemma 2.5.7. *For every $n \geq 1$ and $\eta > 0$ there exists $\gamma_5 = \gamma_5(y, n, \eta) > 0$ such that, for any $\mathbb{P} \in \mathcal{P}_\kappa$, whenever* $\mathrm{dis}(\mathbb{P}) < \gamma_5$ *one has*

$$\sup_{|\theta|_1 < \gamma_5 \,,\, z \in \mathbb{V}_d} \left[B_{z,n}(\theta) - B^*_{z,n}(0) \right] < \eta.$$

Proofs of Lemma 2.5.5 - Lemma 2.5.7 span Section 2.5.4 - Section 2.5.6. Assuming these, let us first complete

Proof of Lemma 2.5.4 (assuming Lemma 2.5.5-Lemma 2.5.7):

Take $\delta = \delta(y, d, \kappa) > 0$ as in Lemma 2.5.5. Since $B^*_{z,n}(\theta) \geq 0$, by Lemma 2.5.6 there exists $\gamma_4 = \gamma_4(y, d, \kappa) > 0$ and $N = N(y, d, \kappa, \delta) \geq 1$ such that, for any $\mathbb{P} \in \mathcal{P}_\kappa$, if $\mathrm{dis}(\mathbb{P}) < \gamma_4$ then

$$\sum_{n > N} \left(\sup_{|\theta|_1 < \gamma_4 \,,\, z \in \mathbb{V}_d} [B_{z,n}(\theta) - B^*_{z,n}(0)] \right) \leq \sum_{n > N} \left(\sup_{|\theta|_1 < \gamma_4 \,,\, z \in \mathbb{V}_d} B_{z,n}(\theta) \right) < \frac{\delta}{4}. \tag{2.5.23}$$

Furthermore, by Lemma 2.5.7 there exists $\gamma_5 = \gamma_5(y, d, \kappa, N, \delta) > 0$ such that, for any $\mathbb{P} \in \mathcal{P}_\kappa$, whenever $\mathrm{dis}(\mathbb{P}) < \gamma_5$ we have

$$\sum_{n=1}^{N} \sup_{|\theta|_1 < \gamma_5 \,,\, z \in \mathbb{V}_d} [B_{z,n}(\theta) - B^*_{z,n}(0)] < \frac{\delta}{4}. \tag{2.5.24}$$

Combined with (2.5.23) and (2.5.24), Lemma 2.5.5 then yields the bound

$$B \leq \sup_{z \in \mathbb{V}^d} \sum_{n=1}^{\infty} B^*_{z,n}(0) + \sum_{n=1}^{\infty} \sup_{|\theta|_1 < \gamma_2 \,,\, z \in \mathbb{V}_d} \left[B_{z,n}(\theta) - B^*_{z,n}(0) \right] < 1 - \frac{\delta}{2}$$

for any $\mathbb{P} \in \mathcal{P}_\kappa$ such that $\mathrm{dis}(\mathbb{P}) < \gamma_3 := \gamma_4 \wedge \gamma_5$. □

2.5.4 Proof of of Lemma 2.5.7.

For $z \in \mathbb{V}_d$ and $n \geq 1$, by Cauchy-Schwarz inequality we have

$$\begin{aligned} B_{z,n}(\theta) - B^*_{z,n}(0) &= \overline{E}^Q_{0,z}\left(F_n(\theta) - 1\,;\, \sigma = n\right) \\ &\leq \left[\overline{E}^Q_{0,z}\left((F_n(\theta)-1)^2\right)\right]^{\frac{1}{2}} \left[\overline{Q}_{0,z}(\sigma = n)\right]^{\frac{1}{2}} \\ &\leq \left[\overline{E}^Q_{0,z}\big(\,(F_n(\theta))^2\,\big) + 1\right]^{\frac{1}{2}} \left[\overline{Q}_{0,z}(\sigma = n)\right]^{\frac{1}{2}}. \end{aligned}$$

Now, on the one hand, by Remark 2.3.1, the bounds $I_n \leq L_n \leq \tau_n$ and the renewal structure, whenever $|\theta|_1 \vee h(\mathrm{dis}(\mathbb{P})) < \frac{\gamma_0}{16}$, where γ_0 is the constant from Proposition 2.3.6, we have that

$$\overline{E}^Q_{0,z}\big(\big(F_n(\theta)\big)^2\big) \leq \overline{E}^Q_{0,z}\left(\mathrm{e}^{4(|\theta|_1 + h(\mathrm{dis}(\mathbb{P})))\tau_n}\right) = \left[\overline{E}^Q_{0}\left(\mathrm{e}^{4(|\theta|_1 + h(\mathrm{dis}(\mathbb{P})))\tau_1}\right)\right]^n \leq \left[\frac{2}{\overline{c}}\right]^n \tag{2.5.25}$$

where $\overline{c} > 0$ is the constant from Lemma 2.3.4. On the other hand, by the nature of renewal times, on the event that $\sigma = n$ there exist some $k \in \{1, \cdots, L_n\}$ and $k' \in \{1, \ldots, \widetilde{L}_n\}$ such that $X_k = \widetilde{X}_{k'}$. In particular, it follows that

$$\begin{aligned} \overline{Q}_{0,z}(\sigma = n) &\leq \overline{Q}_0\left(\sup_{1\leq k\leq L_n} |X_k|_1 \geq \frac{|z|_1}{2}\right) + \overline{Q}_z\left(\sup_{1\leq k'\leq \widetilde{L}_n} |X_{k'} - z|_1 \geq \frac{|z|_1}{2}\right) \\ &= 2\overline{Q}_0\left(\sup_{1\leq k\leq L_n} |X_k|_1 \geq \frac{|z|_1}{2}\right) \\ &\leq 2\overline{Q}_0\left(\tau_n \geq \frac{|z|_1}{2}\right) \leq 4\frac{\overline{E}^Q_0(\tau_n)}{|z|_1} = 4\frac{[\overline{E}^Q_0(\tau_1)]^n}{|z|_1}. \end{aligned} \tag{2.5.26}$$

From (2.5.25) and (2.5.26), using Lemma 2.3.4 and Proposition 2.3.6 it is straightforward to check that there exists $R_0 = R_0(y, n, \eta) > 0$ such that if $\mathrm{dis}(\mathbb{P}) < h^{-1}(\frac{\gamma_0}{16})$ then

$$\sup_{|\theta|_1 < \frac{\gamma_0}{16},\, |z|_1 > R_0} [B_{z,n}(\theta) - B^*_{z,n}(0)] < \eta. \tag{2.5.27}$$

Finally, by an argument similar to the one used for (2.5.25), Remark 2.3.1 and the mean value theorem together yield that

$$|B_{z,n}(\theta) - B^*_{z,n}(0)| \leq 2(|\theta|_1 + h(\mathrm{dis}(\mathbb{P})))\overline{E}^Q_{0,z}\big(\tau_n \mathrm{e}^{2(|\theta|_1 + h(\mathrm{dis}(\mathbb{P})))\tau_n}\big)$$

for any fixed $z \in \mathbb{Z}^d$. In particular, by Lemma 2.3.4 and Proposition 2.3.6 it follows that for any $R > 0$ there exists $\gamma_R = \gamma_R(y, n, R, \eta) > 0$ such that if $|\theta|_1 \vee \mathrm{dis}(\mathbb{P}) < \gamma_R$ then

$$\sup_{|\theta|_1 < \gamma_R\,,\, |z|_1 \leq R} [B_{z,n}(\theta) - B^*_{z,n}(0)] < \eta.$$

Together with (2.5.27), this yields the result with $\gamma_5 := h^{-1}(\frac{\gamma_0}{16}) \wedge \gamma_{R_0}$.

2.5.5 Proof of Lemma 2.5.6.

Next, we prove Lemma 2.5.6.

Proof of Lemma 2.5.6. If we set $\Psi := \sup\{n \in \mathcal{L} : n \leq \tau\}$ then, similarly to (2.5.19), we can decompose

$$\begin{aligned} B_{z,n}(\theta) &= \sum_{j=0}^{n-1} \overline{E}^Q_{0,z}\Big(F_n(\theta)\,;\, \sigma = n\,,\, \Psi = j\Big) \\ &= \sum_{j=0}^{n-1} \sum_{z' \in \mathbb{V}_d} \overline{E}^Q_{0,z}\Big(F_j(\theta)\,;\; \widetilde{X}_{\widetilde{L}_j} - X_{L_j} = z'\,,\, \Psi = j\Big)\overline{E}^Q_{0,z'}\Big(F_{n-j}(\theta)\,;\, n-j = \inf\{k \in \mathcal{L} : k > 0\} > \tau\Big) \\ &\leq \sum_{j=0}^{n-1} \left[\sup_{z' \in \mathbb{V}_d} \overline{E}^Q_{0,z}\Big(F_j(\theta)\,;\, ,\widetilde{X}_{\widetilde{L}_j} - X_{L_j} = z'\,, \Psi = j\Big)\right] \sum_{z' \in \mathbb{V}_d} D_{n-j,z'}(\theta), \end{aligned} \tag{2.5.28}$$

where, for $n \geq 1$ and $z' \in \mathbb{V}_d$, we write

$$D_{n,z'}(\theta) := \overline{E}^Q_{0,z'}\Big(F_n(\theta)\,;\, n = \inf\{k \in \mathcal{L} : k > 0\} > \tau\Big). \tag{2.5.29}$$

Note that $\Psi = j$ implies that $I_j \leq 1$ so that, recalling the random walk $Y^{(\theta)}$ with law $\widehat{P}_0^{(\theta)}$ defined in the proof of Proposition 2.3.8, if we write $\widehat{P}_{0,0}^{(\theta)} := \widehat{P}_0^{(\theta)} \times \widehat{P}_0^{(\theta)}$ then for any $j \geq 1$ we have

$$
\begin{aligned}
\overline{E}_{0,z}^{Q}\Big(F_j(\theta)\,;\, \widetilde{X}_{\widetilde{L}_j} - X_{L_j} = z'\,,\, \Psi = j\Big) &\leq \mathrm{e}^{h(\mathrm{dis}(\mathbb{P}))}\overline{E}_{0,z}^{Q}\Big(\Phi_j(\theta)\widetilde{\Phi}_j(\theta)\,;\, \widetilde{X}_{\widetilde{L}_j} - X_{L_j} = z'\,,\, j \in \mathcal{L}\Big) \\
&= \mathrm{e}^{h(\mathrm{dis}(\mathbb{P}))}\widehat{P}_{0,0}^{(\theta)}\Big(\exists k, m : \langle Y_k, \ell\rangle = j\,,\, \widetilde{Y}_m - Y_k = z' - z\Big) \\
&\leq \mathrm{e}^{h(\mathrm{dis}(\mathbb{P}))} \sum_{\langle x,\ell\rangle = j} \widehat{P}_0^{(\theta)}\Big(\exists k : \langle Y_k, \ell\rangle = x\Big)\widehat{P}_0^{(\theta)}\Big(\exists m : \langle \widetilde{Y}_m, \ell\rangle = x + z' - z\Big) \\
&\leq \mathrm{e}^{h(\mathrm{dis}(\mathbb{P}))}\left[\sup_{\langle x,\ell\rangle = j} \widehat{P}_0^{(\theta)}\Big(\exists k : \langle Y_k, \ell\rangle = x\Big)\right] \sum_{\langle x,\ell\rangle = j} \widehat{P}_0^{(\theta)}\Big(\exists m : \langle \widetilde{Y}_m, \ell\rangle = x + z' - z\Big)
\end{aligned}
\tag{2.5.30}
$$

$$
\begin{aligned}
&= \mathrm{e}^{h(\mathrm{dis}(\mathbb{P}))}\left[\sup_{\langle x,\ell\rangle = j} \sum_{k\in\mathbb{N}} \widehat{P}_0^{(\theta)}\Big(\langle Y_k, \ell\rangle = x\Big)\right] \widehat{P}_0^{(\theta)}(\exists\, m : \langle \widetilde{Y}_m, \ell\rangle = j) \\
&\leq \mathrm{e}^{h(\mathrm{dis}(\mathbb{P}))} \sup_{\langle x,\ell\rangle = j} \sum_{k\in\mathbb{N}} \mu_k^{(\theta)}(x),
\end{aligned}
\tag{2.5.31}
$$

where $\mu^{(\theta)}$ is as in (2.5.2) and, given any probability measure μ, μ_k denotes its k-fold convolution. Observe that for $j = 0$ we obtain directly from (2.5.30) the upper bound $\mathrm{e}^{h(\mathrm{dis}(\mathbb{P}))}$.

Now, in the proof of [BS1, Theorem 5.1] it is shown that, whenever $d \geq 4$, given any $c_1, c_2, c_3 > 0$ there exists $K_1 = K_1(d, c_1, c_2, c_3) > 1$ such that for any $j \geq 1$

$$
\sup_{\langle x,\ell\rangle = j} \sum_{k\in\mathbb{N}} \mu_k(x) \leq \frac{K_1}{(1+j)^{(d-1)/2}} \tag{2.5.32}
$$

holds uniformly over all probability measures μ on $\mathbb{Z}^d$ satisfying

C1. $\sum_{x\in\mathbb{Z}^d} \mu(x)\mathrm{e}^{c_1|x|_1} \leq 2$,

C2. $\Sigma_\mu \geq c_2 I_d$, where I_d denotes the $d \times d$ identity matrix,

C3. $|\sum_{x\in\mathbb{Z}^d}\langle x, \ell\rangle\mu(x)| > c_3$.

More precisely, it is shown that for any measure μ satisfying these conditions and $k \in \mathbb{N}$ one has the estimate

$$\mu_k(x) \leq C(\varphi_k^{(1)}(x) + \varphi_k^{(2)}(x))$$

for some constant $C = C(d, c_1, c_2, c_3) > 0$, where

$$\sum_{k \in \mathbb{N}} \varphi_k^{(1)}(x) \leq \frac{K_1'}{(1+|x|_1)^{-(d-1)/2}} \qquad \text{and} \qquad \sum_{k \in \mathbb{N}} \varphi_k^{(2)}(x) \leq K_1'' \mathrm{e}^{-\delta |x|_1}$$

for some constants $\delta, K_1', K_1'' > 0$ depending only on d, c_1, c_2 and c_3.

Thus, to bound (2.5.31) we will show that there exists $\nu = \nu(y, d, \kappa) > 0$ such that, for any $\mathbb{P} \in \mathcal{P}_\kappa$, whenever $|\theta|_1 \vee \mathrm{dis}(\mathbb{P}) < \nu$ the measure $\mu^{(\theta)}$ satisfies (C1)-(C2)-(C3) above for some $c_1, c_2, c_3 > 0$ depending only on y, d and κ. Indeed, by the same type of argument leading to (2.5.25), we have

$$\Big| \sum_{x \in \mathbb{Z}^d} \mu^{(\theta)}(x) \mathrm{e}^{c_1 |x|_1} - 1 \Big| \leq (2|\theta|_1 + h(\mathrm{dis}(\mathbb{P})) + c_1) \overline{E}_0^Q (\tau_1 \mathrm{e}^{(2|\theta|_1 + h(\mathrm{dis}(\mathbb{P})) + c_1)\tau_1})$$

so that, by Lemma 2.3.4 and Proposition 2.3.6, there exists $\nu_1 = \nu_1(y) > 0$ such that if $c_1 > 0$ is taken small enough (depending only on y) then (C1) holds when $|\theta|_1 \vee \mathrm{dis}(\mathbb{P}) < \nu_1$. On the other hand, since $\langle X_{\tau_1} - X_0, \ell \rangle \geq +1$ by definition of τ_1, it follows that

$$\left| \sum_{x \in \mathbb{Z}^d} x \mu^{(\theta)}(x) \right| \geq \overline{E}_0^Q (\Phi_1(\theta) \langle X_{\tau_1}, \ell \rangle) \geq \overline{E}_0^Q (\Phi_1(\theta)) = 1$$

and so (C3) is satisfied with $c_3 := 1$. Finally, to check (C2) we first notice that by (2.4.1) and (2.4.7),

$$\Sigma_{\mu^{(\theta)}} = H_a(\theta) \overline{E}_0^Q \big(\tau_1 \Phi_1(\theta) \big).$$

Since $\Sigma_{\mu^{(\theta)}}$ is a positive definite matrix whenever $|\theta|_1 \vee \mathrm{dis}(\mathbb{P}) < \gamma_1$ by Proposition 2.4.2, to obtain (C2) it will suffice to show that there exists $\nu_2 = \nu_2(y, d, \kappa) > 0$ such that if

$|\theta|_1 \vee \mathrm{dis}(\mathbb{P}) < \nu_2$ then

$$\inf_{\mathbb{P} \in \mathcal{P}_\kappa(\nu_2)\,,\,|\theta|_1 < \nu_2} \sigma_{\min}(\Sigma_{\mu^{(\theta)}}) \geq c_2 \tag{2.5.33}$$

for some constant $c_2 > 0$ depending only on y, d and κ, where $\sigma_{\min}(A)$ above denotes the smallest singular value of a matrix A. Since $\overline{E}_0^Q(\tau_1 G(1,\theta)) \geq 1$ and $1/\sigma_{\min}(A) = \|A^{-1}\|_2 \leq \sqrt{d}\|A^{-1}\|$ for any invertible $A \in \mathbb{R}^{d \times d}$, where $\|\cdot\|_2$ and $\|\cdot\|$ denote the operator 2-norm and 1-norm respectively, we see that (2.5.33) will hold if we show that for some $\nu_2 = \nu_2(y,d,\kappa) > 0$ we have

$$\sup_{\mathbb{P} \in \mathcal{P}_\kappa(\nu_2)\,,\,|\theta|_1 < \nu_2} \|(H_a(\theta))^{-1}\| < \infty$$

and take $c_2 := (\sqrt{d} \sup_{\mathbb{P} \in \mathcal{P}_\kappa(\nu_2)\,,\,|\theta|_1 < \nu_2} \|(H_a(\theta))^{-1}\|)^{-1}$. Using once again the identity $A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$ for invertible matrices $A, B \in \mathbb{R}^{d \times d}$, we have

$$\|(H_a(\theta))^{-1} - (H_a(0))^{-1}\| \leq \|(H_a(\theta))^{-1}\| \|H_a(\theta) - H_a(0)\| \|(H_a(0))^{-1}\|. \tag{2.5.34}$$

But then, by the proof of Proposition 2.4.3 there exist $\nu_2 = \nu_2(y,d,\kappa), c = c(y,d,\kappa) > 0$ such that

$$\sup_{\mathbb{P} \in \mathcal{P}_\kappa(\nu_2)} \|(H_a(0))^{-1}\| \leq c \qquad \text{and} \qquad \sup_{\mathbb{P} \in \mathcal{P}_\kappa(\nu_2)\,,\,|\theta|_1 < \nu_2} \|H_a(\theta) - H_a(0)\| < \frac{1}{2c},$$

which by (2.5.34) and the triangle inequality implies that

$$\sup_{\mathbb{P} \in \mathcal{P}_\kappa(\nu_2)\,,\,|\theta|_1 < \nu_2} \|(H_a(\theta))^{-1}\| \leq 2c < \infty$$

and so (C2) follows. Thus, we see that for $\nu := \nu_1 \wedge \nu_2 \wedge \frac{1}{2}$ we have by (2.5.28), (2.5.31) and

(2.5.32)

$$\left[\sup_{\mathbb{P}\in\mathcal{P}_\kappa(\nu)\,,\,|\theta|_1<\nu\,,\,z\in\mathbb{V}_d} B_{z,n}(\theta)\right] \le \mathrm{e}^{h(1/2)} K_1 \sum_{j=0}^{n-1} \frac{1}{(1+j)^{(d-1)/2}} \sum_{z'\in\mathbb{V}_d} \sup_{\mathbb{P}\in\mathcal{P}_\kappa(\nu)\,,\,|\theta|_1<\nu} D_{n-j,z'}(\theta)$$

so that

$$\sum_{n=1}^{\infty}\left[\sup_{\mathbb{P}\in\mathcal{P}_\kappa(\nu)\,,\,|\theta|_1<\nu\,,\,z\in\mathbb{V}_d} B_{z,n}(\theta)\right] \le \mathrm{e}^{h(1/2)} K_1 \sum_{j=0}^{\infty} \frac{1}{(1+j)^{(d-1)/2}} \sum_{n=1}^{\infty}\sum_{z\in\mathbb{V}_d} \sup_{\mathbb{P}\in\mathcal{P}_\kappa(\nu)\,,\,|\theta|_1<\nu} D_{n,z}(\theta).$$

The proof of Lemma 2.5.6 will then be complete once we prove the result stated below. □

Lemma 2.5.8. *There exist* $\gamma_6 = \gamma_6(y), K' = K'(y,d) > 0$ *such that*

$$\sum_{n=1}^{\infty}\sum_{z\in\mathbb{V}_d} \sup_{\mathbb{P}\in\mathcal{P}_\kappa(\gamma_6)\,,\,|\theta|_1<\gamma_6} D_{n,z}(\theta) \le K'.$$

Proof. By Cauchy-Schwarz inequality,

$$D_{n,z}(\theta) \le \left(\overline{E}^{Q}_{0,z}\big((F_n(\theta))^2\big)\right)^{1/2}\left(\overline{P}^{Q}_{0,z}(n=\inf\{k\in\mathcal{L}:k>0\})\right)^{1/4}\left(\overline{P}^{Q}_{0,z}(n>\tau)\right)^{1/4}. \quad (2.5.35)$$

As in (2.5.25), the first factor on the right-hand side of (2.5.35) can be bounded from above by

$$\left[\overline{E}^{Q}_{0}\left(\mathrm{e}^{4(|\theta|_1+h(\mathrm{dis}(\mathbb{P})))\tau_1}\right)\right]^{n/2} \le \left[\overline{E}^{Q}_{0}\left(\mathrm{e}^{\frac{\gamma_0}{2}\tau_1}\right)\right]^{\frac{4(|\theta|_1+h(\mathrm{dis}(\mathbb{P})))}{\gamma_0}n} \le \mathrm{e}^{\log(2/\overline{c})\frac{4(|\theta|_1+h(\mathrm{dis}(\mathbb{P})))}{\gamma_0}n} \quad (2.5.36)$$

whenever $|\theta|_1 \vee h(\mathrm{dis}(\mathbb{P})) < \frac{\gamma_0}{16}$, with γ_0 as in Proposition 2.3.6, by Jensen's inequality.

On the other hand, to deal with the third factor we notice that if $z \neq 0$ then whenever $n > \tau$ then $X_i = \widetilde{X}_j$ for some $1 \le i \le \tau_n$ and $1 \le j \le \widetilde{\tau}_n$ so that, in particular, we must have $\tau_n \vee \widetilde{\tau}_n \ge \frac{|z|_1}{2}$. Then, using the inequality $(a_1+\cdots+a_n)^m \le n^{m-1}(a_1^m+\cdots+a_n^m)$, valid for positive $(a_i)_{1\leq i\leq n}$ and $m\geq 1$, by the union bound we obtain

$$\overline{P}^{Q}_{0,z}(n>\tau)\leq 2\overline{P}^{Q}_{0}\Big(\tau_n\geq\frac{|z|_1}{2}\Big)\leq 2\Big(\frac{2}{|z|_1}\Big)^{4d+1}\overline{E}^{Q}_{0}(\tau_n^{4d+1})\leq 2\Big(\frac{2}{|z|_1}\Big)^{4d+1}n^{4d}\overline{E}^{Q}_{0}(\tau_1^{4d+1}).$$

From this, by the trivial bound $\overline{P}^{Q}_{0,0}(n>\tau)\leq 1$ and Proposition 2.3.6 we conclude that there exists $K_1'=K_1'(d,y)>0$ such that, for any $n\geq 1$ and $z\in\mathbb{V}_d$,

$$\Big(\overline{P}^{Q}_{0,z}(n>\tau)\Big)^{1/4}\leq K_1'n^d(1\vee|z|_1)^{-\left(d+\frac{1}{4}\right)}. \tag{2.5.37}$$

Finally, to control the middle factor in the right-hand side of (2.5.35), we will show that there exist $c=c(y),K_2'=K_2'(y)>0$ such that, for any $\mathbb{P}\in\mathcal{P}_\kappa$,

$$\sup_{z\in\mathbb{V}_d}\overline{E}^{Q}_{0,z}(\mathrm{e}^{4c\lambda^*})<(K_2')^4 \tag{2.5.38}$$

where $\lambda^*:=\inf\{k\in\mathcal{L}:k>0\}$, so that

$$\Big(\overline{P}^{Q}_{0,z}(n=\inf\{k\in\mathcal{L}:k>0\})\Big)^{1/4}\leq\Big(\overline{P}^{Q}_{0,z}(\lambda^*\geq n)\Big)^{1/4}\leq K_2'\mathrm{e}^{-cn}. \tag{2.5.39}$$

To this end, for $m\geq 0$ define

$$\beta(m):=\inf\{n\geq L_m:\langle X_n,\ell\rangle<m\}\qquad\text{and}\qquad R(m):=\sup\{\langle X_n,\ell\rangle:L_m\leq n<\beta(m)\},$$

together with the corresponding quantities $\widetilde{\beta}(m)$,$\widetilde{R}(m)$ for $\widetilde{X}$ and consider the sequence $(\lambda_j)_{j\geq 1}$ defined inductively by first taking $\lambda_1:=1$ and then setting

$$\lambda_{j+1}=\begin{cases}R(\lambda_j)\wedge\widetilde{R}(\lambda_j)+1 & \text{if } \lambda_j<\infty\\ \infty & \text{if } \lambda_j=\infty.\end{cases}$$

It is not hard to check that $\lambda^* = \sup\{\lambda_j : \lambda_j < \infty\}$. We will use this representation of λ^* to estimate its exponential moments and show (2.5.38). In order to do this, let us first observe that if we define $\lambda := R(0) \wedge \widetilde{R}(0) + 1$ then, for any $z \in \mathbb{V}_d$ and $\hat{c} \in (0, \gamma_0)$ (with γ_0 as in Proposition 2.3.6), we have by Hölder's inequality that

$$E^Q_{0,z}(\mathrm{e}^{\hat{c}\lambda}\,;\,\lambda < \infty) \leq \left[E^Q_{0,z}(\mathrm{e}^{\gamma_0\lambda}\,;\,\lambda < \infty)\right]^{\frac{\hat{c}}{\gamma_0}} \left[Q_{0,z}(\lambda < \infty)\right]^{1-\frac{\hat{c}}{\gamma_0}}.$$

Since Q_0-a.s. we have $R(0) + 1 \leq \tau_1$ on the event that $\beta_0 < \infty$ (observe that $\beta_0 = \beta(0)$ Q_0-a.s.), then by Proposition 2.3.6

$$E^Q_{0,z}(\mathrm{e}^{\gamma_0\lambda}\,;\,\lambda < \infty) \leq E^Q_{0,z}(\mathrm{e}^{\gamma_0(R(0)+1)}\,;\,\beta_0 < \infty)+E^Q_{0,z}(\mathrm{e}^{\gamma_0(\widetilde{R}(0)+1)}\,;\,\widetilde{\beta}_0 < \infty) \leq 2E^Q_{0,z}(\mathrm{e}^{\gamma_0\tau_1}) \leq 4.$$

On the other hand, by Lemma 2.3.4 we have

$$Q_{0,z}(\lambda < \infty) \leq Q_{0,z}(\beta_0 < \infty \text{ or } \widetilde{\beta}_0 < \infty) = 1 - (Q_0(\beta_0 = \infty))^2 < 1 - \overline{c}^2.$$

It follows that for some $\hat{c} = \hat{c}(y) \in (0, \gamma_0)$ sufficiently small we have

$$\sup_{z \in \mathbb{V}_d} E^Q_{0,z}(\mathrm{e}^{\hat{c}\lambda}\,;\,\lambda < \infty) \leq 1 - \frac{\overline{c}^2}{2}.$$

With this, using the Markov property and translation invariance, for $z \in \mathbb{V}_d$ we may compute

$$\begin{aligned}
E^Q_{0,z}(\mathrm{e}^{\hat{c}\lambda^*}) &= \sum_{j=1}^{\infty} E^Q_{0,z}(\mathrm{e}^{\hat{c}\lambda_j}\,;\,\lambda^* = \lambda_j) \leq \sum_{j=1}^{\infty} E^Q_{0,z}(\mathrm{e}^{\hat{c}\lambda_j}\,;\,\lambda_j < \infty)\\
&= \sum_{j=1}^{\infty} \sum_{z' \in \mathbb{V}_d} E^Q_{0,z}(\mathrm{e}^{\hat{c}\lambda_{j-1}}\,;\,\lambda_{j-1} < \infty, \widetilde{X}_{\widetilde{L}_{\lambda_{j-1}}} - X_{L_{\lambda_j}} = z')E^Q_{0,z'}(\mathrm{e}^{\hat{c}\lambda}\,;\,\lambda < \infty)\\
&\leq \sum_{j=1}^{\infty} E^Q_{0,z}(\mathrm{e}^{\hat{c}\lambda_{j-1}}\,;\,\lambda_{j-1} < \infty)(1 - \tfrac{\overline{c}^2}{2}),
\end{aligned}$$

so that by induction we conclude that

$$\sup_{z\in\mathbb{V}^d} E^Q_{0,z}(\mathrm{e}^{\hat{c}\lambda^*}) \leq \mathrm{e}^{\hat{c}}\sum_{j=1}^{\infty}(1-\tfrac{\overline{c}^2}{2})^{j-1} = \frac{2\mathrm{e}^{\hat{c}}}{\overline{c}^2},$$

and so (2.5.38) follows.

Gathering (2.5.36), (2.5.39) and (2.5.37), from (2.5.35) we see that if $\gamma_6 > 0$ is chosen sufficiently small so that $|\theta|_1 \vee h(\mathrm{dis}(\mathbb{P})) < \frac{\gamma_0}{16}$ and $\log(2/\overline{c})\frac{4(|\theta|_1+h(\mathrm{dis}(\mathbb{P})))}{\gamma_0} < \frac{c}{2}$ with c as in (2.5.39) (which can be done depending only on y), then

$$\sum_{n=1}^{\infty}\sum_{z\in\mathbb{V}_d}\sup_{\mathbb{P}\in\mathcal{P}_\kappa(\gamma_6)\,,\,|\theta|_1<\gamma_6} D_{n,z}(\theta) \leq K_1'K_2'\left[\sum_{n=1}^{\infty} n^d \mathrm{e}^{-\frac{c}{2}n}\right]\left[\sum_{z\in\mathbb{V}_d}(1\vee|z|_1)^{-\left(d+\frac{1}{4}\right)}\right] =: K' < \infty$$

and this completes the proof. $\square$

2.5.6 Proof of Lemma 2.5.5.

We finish by giving the proof of Lemma 2.5.5.

Proof of Lemma 2.5.5. We first notice that there exist constants $\eta_1, \eta_2, \eta_3 > 0$, all depending only on y, d and κ such that, for any $\mathbb{P} \in \mathcal{P}_\kappa$,

D1. $Q_0(\beta_0 = \infty) > \eta_1$,

D2. $E^Q_0(\tau_1^9) < \eta_2$,

D3. $\sup_{z\in\mathbb{Z}^d} \overline{Q}_0(X_{\tau_n} = z) \leq \eta_3 n^{-d/2}$ for any $n \geq 1$.

Indeed, (D1)-(D2) follow immediately from Lemma 2.3.4 and Proposition 2.3.6, respectively. To check (D3), note that for any $\mathbb{P} \in \mathcal{P}_\kappa$ the law μ^* of X_{τ_1} under $\overline{Q}_0$ satisfies conditions (C1)-(C2)-(C3) in the proof of Lemma 2.5.6 for some constants $c_1, c_2, c_3 > 0$ which depend only on y, d and κ. Indeed, this follows from the proof of Lemma 2.5.6 upon noticing that

μ^* coincides with $\mu^{(0)}$ for the zero-disorder law $\mathbb{P}_\alpha$ with marginals $\alpha \in \mathcal{M}_1^{(\kappa)}(\mathbb{V})$. By [BS1, Eq. 5.5], this gives (D3) for some η_3 depending only on y, d and κ.

Under these conditions, since $\sigma < \infty$ implies that the two walks need to intersect at a time other than zero, by essentially repeating the proofs of [BZ, Propositions 3.1 and 3.4] (but using instead the estimates in (D1)-(D2)-(D3) which are uniform over $\mathbb{P} \in \mathcal{P}_\kappa$), it can be shown that there exists $N = N(y, d, \kappa) \geq 1$ such that, for any $\mathbb{P} \in \mathcal{P}_\kappa$,

$$\sup_{|z|_1 \geq 2N} \overline{Q}_{0,z}(\sigma < \infty) \leq \frac{1}{2}.$$

To deal with $z \in \mathbb{Z}^d$ such that $|z|_1 < 2N$, take any such z together with $e^* \in \mathbb{V} \setminus \{\ell, -\ell\}$ and assume without loss of generality that $\langle z, e^* \rangle \geq 0$. Then consider the events

$$E_1 := \{X_N = -Ne^*\,,\, \widetilde{X}_N = \widetilde{X}_0 + Ne^*\} \qquad E_2 := \{X_i \neq \widetilde{X}_j \text{ for all } i, j\ > N\}.$$

Since $|z+2Ne^*|_1 \geq 2N$ by choice of e^* and on E_1 we have both $\langle X_i - X_0, \ell \rangle = \langle \widetilde{X}_j - \widetilde{X}_0, \ell \rangle = 0$ and $X_i \neq \widetilde{X}_j$ for all $1 \leq i, j \leq N$, using (P1) from Lemma 2.3.2 and translation invariance, we obtain

$$\overline{Q}_{0,z}(\sigma = \infty) \geq \overline{Q}_{0,z}(E_1 \cap E_2) \geq c_\kappa^{2N} \inf_{|y|_1 \geq 2N} \overline{Q}_{0,y}(\sigma = \infty) > \frac{1}{2} c_\kappa^{2N} > 0$$

for any $\mathbb{P} \in \mathcal{P}_\kappa$, so that now the result follows upon taking $\delta := \frac{1}{2} c_\kappa^{2N}$. $\square$

Chapter 3

Equality and difference of quenched and averaged large deviations for RWRE: the impact of the disorder at the boundary

3.1 Introduction and background

The model of a random walk in a random environment (RWRE) can be described as follows. Let $|x|_1$ denote the ℓ^1-norm of any $x \in \mathbb{R}^d$ and define $\mathbb{V} := \{x \in \mathbb{Z}^d : |x|_1 = 1\} = \{\pm e_1, \ldots, \pm e_d\}$, the set of all unit vectors in $\mathbb{Z}^d$, along with $\mathcal{M}_1(\mathbb{V}) := \big\{\vec{p} = (p(e))_{e\in\mathbb{V}} \in [0,1]^{\mathbb{V}} : \sum_{e\in\mathbb{V}} p(e) = 1\big\}$, the space of all probability vectors therein and the product space $\Omega := (\mathcal{M}_1(\mathbb{V}))^{\mathbb{Z}^d}$ with the usual product topology. Any element $\omega \in \Omega$ will be called an *environment*, i.e. each $\omega = (\omega(x))_{x\in\mathbb{Z}^d}$ is a sequence of probability vectors $\omega(x) = (\omega(x,e))_{e\in\mathbb{V}}$ on $\mathbb{V}$ indexed by the sites in the lattice. Given any $x \in \mathbb{Z}^d$ and $\omega \in \Omega$, the *random walk in the environment ω starting at x is defined as the Markov chain $(X_n)_{n\in\mathbb{N}}$ on $\mathbb{Z}^d$ whose law*

$P_{x,\omega}$ is given by

$$P_{x,\omega}(X_0 = x) = 1 \quad \text{and} \quad P_{x,\omega}(X_{n+1} = y + e \,|\, X_n = y) = \omega(y, e) \quad \forall\, y \in \mathbb{Z}^d\,,\ e \in \mathbb{V}.$$

We call $P_{x,\omega}$ the quenched law of the RWRE. Then, if the environment ω is now chosen at random according to some Borel probability measure $\mathbb{P}$ on Ω, we now obtain the measure P_x on $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}$ defined as

$$P_x(A \times B) := \int_A P_{x,\omega}(B) \mathrm{d}\mathbb{P}(\omega) \qquad \forall\, A \in \mathcal{B}(\Omega)\,,\ B \in \mathcal{B}((\mathbb{Z}^d)^{\mathbb{N}}).$$

We call P_x the annealed law of the RWRE and, in general, we will call the sequence $X = (X_n)_{n\in\mathbb{N}}$ under P_x a RWRE with environmental law $\mathbb{P}$. In the sequel, we shall work with environmental laws satisfying the following assumption:

Assumption A: Under $\mathbb{P}$, the environment is *i.i.d.* (the random vectors $(\omega(x))_{x\in\mathbb{Z}^d}$ are independent and identically distributed) and *uniformly elliptic*, i.e., there is a constant $\kappa > 0$ such that

$$\mathbb{P}(\omega(x, e) \geq \kappa \text{ for all } x \in \mathbb{Z}^d \text{ and } e \in \mathbb{V}) = 1. \tag{3.1.1}$$

In [Var], Varadhan proved that, for any $d \geq 1$ and under Assumption A, both the quenched law $P_{0,\omega}\big(\frac{X_n}{n} \in \cdot\big)$ and its annealed version $P_0\big(\frac{X_n}{n} \in \cdot\big)$ satisfy a *large deviations principle* (LDP), i.e. there exist lower-semicontinuous functions $I_a, I_q : \mathbb{R}^d \to [0, \infty]$ such that for any $G \subseteq \mathbb{R}^d$ with interior $\mathrm{int}(G)$ and closure $\overline{G}$,

$$-\inf_{x\in\mathrm{int}(G)} I_q(x) \leq \liminf_{n\to\infty} \frac{1}{n} \log P_{0,\omega}\left(\frac{X_n}{n} \in G\right) \leq \limsup_{n\to\infty} \frac{1}{n} \log P_{0,\omega}\left(\frac{X_n}{n} \in G\right) \leq -\inf_{x\in\overline{G}} I_q(x) \tag{3.1.2}$$

$$-\inf_{x\in\mathrm{int}(G)} I_a(x) \leq \liminf_{n\to\infty} \frac{1}{n} \log P_0\left(\frac{X_n}{n} \in G\right) \leq \limsup_{n\to\infty} \frac{1}{n} \log P_0\left(\frac{X_n}{n} \in G\right) \leq -\inf_{x\in\overline{G}} I_a(x) \tag{3.1.3}$$

with the first assertion being true for $\mathbb{P}$-almost every $\omega \in \Omega$. It can be shown that the rate functions I_q and I_a are both convex and are finite if and only if $x \in \mathbb{D} := \{x \in \mathbb{R}^d : |x|_1 \leq 1\}$. Being also lower semicontinuous, the former implies that I_q and I_a are continuous on $\mathbb{D}$, see [Roc, Theorem 10.2]. Moreover, by Jensen's inequality and Fatou's lemma, we always have the dominance $I_a(\cdot) \leq I_q(\cdot)$. In [Var] it was also shown that, for $d \geq 2$, $I_a(0) = I_q(0)$ and both rate functions have the same zero-sets, leaving open the question of whether both rate functions are in fact equal in other parts of their domain. In this regard, Yilmaz showed later in [Yil4] that, for RWRE with $d \geq 4$ satisfying Assumption A, both rate functions agree on *some* neighborhood of the non-zero velocity, whenever the random walk satisfies Sznitman's *condition* (T) for ballisticity, see [Szn2] for a precise definition.[1] Recently in [BMRS1], we have shown that for $d \geq 4$ the two rate functions agree on any compact set in the interior of $\mathbb{D}$ which does not contain zero, provided that the disorder of the environment is low enough and regardless of whether the RWRE is ballistic. In the current work, we show that, despite the behavior of the RWRE on the boundary $\partial\mathbb{D}$ of $\mathbb{D}$ being quite different than in its interior, the above low-disorder phenomenon extends also to $\partial\mathbb{D}$. Indeed, we show that $I_q = I_a$ holds on any compact set contained in $\partial\mathbb{D}$ (avoiding its $(d-2)$-dimensional facets), provided that the disorder of the environment is sufficiently low. As a consequence, we obtain a simple explicit formula for the quenched rate function on $\partial\mathbb{D}$ at low disorder. Finally, for a general parametrized family of environments, we show that the strength of disorder determines a phase transition in the equality of both rate functions, in the sense that for each $x \in \partial\mathbb{D}$ there exists ε_x such that the two rate functions agree at x when the disorder is smaller than ε_x and disagree when its larger. We turn to the precise statements of these results.

[1] In contrast, this has been shown to be false in [YZ] for dimensions $d \in \{2, 3\}$: there exists a class of *non-nestling* random walks in i.i.d. and uniformly elliptic environments verifying that there is no neighborhood of the velocity on which the two rate functions are identical.

3.2 Main result: Quenched and Annealed rate functions on the boundary

Given any such environmental law $\mathbb{P}$, we define its *disorder* as

$$\operatorname{dis}(\mathbb{P}) := \inf\big\{\varepsilon > 0 : \xi(x,e) \in [1-\varepsilon, 1+\varepsilon],\ \mathbb{P}\text{-a.s. for all } e \in \mathbb{V} \text{ and } x \in \mathbb{Z}^d\big\}, \tag{3.2.1}$$

$$\text{with}\ \ \xi(x,e) := \frac{\omega(x,e)}{\alpha(e)} \quad \text{and} \quad \alpha(e) := \mathbb{E}[\omega(x,e)] \qquad \forall\, e \in \mathbb{V}, \tag{3.2.2}$$

where $\mathbb{E}$ denotes expectation w.r.t. $\mathbb{P}$ and the definition of $\alpha(e)$ does not depend on $x \in \mathbb{Z}^d$ by Assumption A. Moreover, both $\xi(x,e)$ and $\operatorname{dis}(\mathbb{P})$ are well-defined since $\mathbb{P}$ satisfies Assumption A, whereas $\operatorname{dis}(\mathbb{P})$ is the $L^\infty(\mathbb{P})$-norm of the random vector $(\xi(x,e)-1)_{e\in\mathbb{V}}$ for any $x \in \mathbb{Z}^d$.

We set $\partial\mathbb{D} = \{x \in \mathbb{Z}^d \colon |x|_1 = 1\}$ for the boundary of the unit ball and write

$$\begin{aligned} &\partial\mathbb{D}(s) := \{x \in \mathbb{R}^d : |x|_1 = 1 \text{ and } x_j s_j \ge 0 \text{ for all } 1 \le j \le d\} \quad \text{and also,} \\ &\partial\mathbb{D}_{d-2} := \{x \in \partial\mathbb{D} : x_j = 0 \text{ for some } 1 \le j \le d\}. \end{aligned} \tag{3.2.3}$$

Notice that the subsets $\partial\mathbb{D}(s)$ for $s \in \{\pm 1\}^d$ correspond to the different *faces* of the boundary $\partial\mathbb{D}$.

3.2.1 Equality of I_a and I_q for small disorder.

Here is our first main result.

Theorem 3.2.1. *For any $d \ge 4$, $\kappa > 0$ and compact set $\mathcal{K} \subseteq \partial\mathbb{D} \setminus \partial\mathbb{D}_{d-2}$ there exists $\varepsilon = \varepsilon(d,\kappa,\mathcal{K}) > 0$ such that, for any RWRE satisfying Assumption* A *with ellipticity constant κ, if*

$$\operatorname{dis}(\mathbb{P}) < \varepsilon \tag{3.2.4}$$

then we have the equality $I_q(x) = I_a(x)$ for all $x \in \mathcal{K}$.

Remark 3.2.1. One can think of Theorem 3.2.1 above as saying that part of the region of equality in the boundary $\{x \in \partial\mathbb{D} : I_q(x) = I_a(x)\}$ covers the whole of $\partial\mathbb{D} \backslash \partial\mathbb{D}_{d-2}$ in the limit as $\mathrm{dis}(\mathbb{P}) \to 0$ uniformly over all environmental laws $\mathbb{P}$ with a uniform ellipticity constant bounded from below by some $\kappa > 0$. However, we remark that, for a *fixed* environmental law $\mathbb{P}$, I_a and I_q can never be equal everywhere in $\partial\mathbb{D}$ unless $\mathbb{P}$ is degenerate (i.e. ω is non-random under $\mathbb{P}$), see [Yil4, Proposition 4]. □

Our next result states that there exists *at least one* open neighborhood on which there is equality, whenever the environment satisfies the weaker condition of small enough *imbalance*. More precisely, given $s \in \{\pm 1\}^d$ we define the *imbalance of* $\mathbb{P}$ on the face $\partial\mathbb{D}(s)$ as

$$\mathrm{imb}_s(\mathbb{P}) := \inf\left\{\varepsilon > 0 : \zeta_s(x) \in [1-\varepsilon, 1+\varepsilon],\ \mathbb{P}\text{-a.s. for all } x \in \mathbb{Z}^d\right\},$$
$$\text{with } \zeta_s(x) := \frac{\sum_{i=1}^d \omega(x, s_i e_i)}{\sum_{i=1}^d \alpha(s_i e_i)},$$

or, equivalently, $\mathrm{imb}_s(\mathbb{P})$ is the $L^\infty(\mathbb{P})$-norm of the random variable $\zeta_s(x) - 1$, for any given $x \in \mathbb{Z}^d$. Here is the statement of our next main result.

Theorem 3.2.2. *For any $d \geq 4$, $\kappa > 0$ and $s \in \{\pm 1\}^d$, there exists $\varepsilon^\star = \varepsilon^\star(d, \kappa) > 0$ such that, for any RWRE satisfying Assumption* A *with ellipticity constant κ, if*

$$\mathrm{imb}_s(\mathbb{P}) < \varepsilon^\star, \tag{3.2.5}$$

then the following statements hold:

- *I_a and I_q have the same minimum over $\partial\mathbb{D}(s)$,*

$$\min_{x \in \partial\mathbb{D}(s)} I_q(x) = \min_{x \in \partial\mathbb{D}(s)} I_a(x) = -\log \sum_{i=1}^d \alpha(s_i e_i). \tag{3.2.6}$$

- *I_a and I_q have the same unique minimizer,*

$$\arg\min_{x\in\partial\mathbb{D}(s)} I_q(x) = \arg\min_{x\in\partial\mathbb{D}(s)} I_a(x) = \frac{\sum_{i=1}^d \alpha(s_i e_i) s_i e_i}{\sum_{i=1}^d \alpha(s_i e_i)} =: \overline{x}_s.$$

- *There exists a neighborhood $\mathcal{O} \subsetneq \partial\mathbb{D}(s)$ of $\overline{x}_s$ such that I_a and I_q agree on $\mathcal{O}$,*

$$I_q(x) = I_a(x) \qquad \text{for all } x \in \mathcal{O}. \tag{3.2.7}$$

Moreover, the set $\mathcal{O}$ can be taken to be uniform over all environmental laws $\mathbb{P}$ satisfying Assumption A *with ellipticity constant κ in the following sense: there exists $r = r(d,\kappa) > 0$ such that, for any $\mathbb{P}$ satisfying Assumption* A *with ellipticity constant κ, if $\mathrm{imb}_s(\mathbb{P}) < \varepsilon^\star$ (with $\varepsilon^\star$ as above) then*

$$I_q(x) = I_a(x) \qquad \text{for all } x \in B_r(\overline{x}_s) \cap \partial\mathbb{D}(s).$$

(The point being that r is independent of $\overline{x}_s$ and uniform over $\mathbb{P}$.)

Remark 3.2.2. Note that in the current general setup, we do not require the RWRE to possess any limiting velocity, nor do we impose any ballisticity condition on the RWRE. However, one can show that, whenever (3.2.5) holds, the unique minimizer $\overline{x}_s$ in Theorem 3.2.2 is the velocity of $(X_n)_{n\in\mathbb{N}}$ under the annealed conditional measure $P_0(\frac{X_n}{n} \in \cdot \mid \frac{X_n}{n} \in \partial\mathbb{D}(s))$ and the quenched and annealed rate functions of the walk under this conditioning can be seen to equal $I_q - I_q(\overline{x}_s)$ and $I_a - I_a(\overline{x}_s)$, respectively. Also, under this conditioning, the set-up bears some resemblance to a random walk in a *space time i.i.d. environment* ([Yil1]) which corresponds to the case when $\mathrm{imb}_s(\mathbb{P}) = 0$. The latter choice is included as a particular case of Theorem 3.2.2 (certainly Theorem 3.2.2 also covers the case when $\mathrm{imb}_s(\mathbb{P})$ is sufficiently small, not necessarily zero). Also, from this viewpoint, our Theorem 3.2.1 then indicates that previously known equality results for dynamic random environments (available for neighborhoods of the velocity) can be extended to neighborhoods of *arbitrary points* in the domain, provided that the disorder of the environment is sufficiently low (a fact which can be proved rigorously by an adaptation of our method). □

Remark 3.2.3. Notice that if $\mathbb{P}$ is the law of a *balanced random environment*, i.e. $\mathbb{P}$ is such that $\mathbb{P}(\omega(x,e) = \omega(x,-e)$ for all $x,e) = 1$, then $\mathrm{imb}_s(\mathbb{P}) = 0$ for any $s \in \{\pm 1\}^d$. In particular, such environments, as well as small perturbations of them, readily satisfy the hypotheses of Theorem 3.2.2. Observe also that balanced random environments never satisfy condition (T) and, as such, had not been considered before in the study of equality of the rate functions for standard RWRE. □

3.2.2 Formulas for I_q and I_a on the boundary.

Using the observation that the rate functions on the boundary $\partial\mathbb{D}$ can be studied as that of a random process in a space-time i.i.d. environment, Theorem 3.2.1 and Theorem 3.2.2 now provide a simple formula for the quenched rate function I_q. Define the moment generating function $\lambda : \mathbb{R}^d \to \mathbb{R}$ as

$$\lambda(\theta) := \sum_{e \in \mathbb{V}} \alpha(e) \mathrm{e}^{\langle \theta, e \rangle}. \tag{3.2.8}$$

Here is our next main result.

Theorem 3.2.3. *Fix $d \geq 4$ and $\kappa > 0$. Then:*

(i) Given any compact set $\mathcal{K} \subseteq \partial\mathbb{D} \setminus \partial\mathbb{D}_{d-2}$ there exists $\varepsilon = \varepsilon(d,\kappa,\mathcal{K}) > 0$ such that, for any RWRE satisfying Assumption A *with ellipticity constant κ, whenever* (3.2.4) *holds we have*

$$I_a(x) = I_q(x) = \sup_{\theta \in \mathbb{R}^d} \left(\langle \theta, x \rangle - \log \lambda(\theta) \right) = \sum_{i=1}^{d} |x_i| \log \frac{|x_i|}{\alpha(s_i e_i)} \qquad \textit{for all } x \in \mathcal{K}. \tag{3.2.9}$$

(ii) Given any $s \in \{\pm 1\}^d$ there exists $\varepsilon^\star = \varepsilon^\star(d,\kappa) > 0$ such that, for any RWRE satisfying Assumption A *with ellipticity constant κ, whenever* (3.2.5) *holds there exists a non-empty open subset $\mathcal{O} \subsetneq \partial\mathbb{D}(s) \setminus \partial\mathbb{D}_{d-2}$ such that the representation in* (3.2.9) *holds for all $x \in \mathcal{O}$. This open subset is the same from Theorem 3.2.2 and hence can be taken to be uniform over all $\mathbb{P}$ satisfying Assumption* A *with ellipticity constant κ.*

Remark 3.2.4. As a matter of fact, the formula

$$I_a(x) = \sup_{\theta\in\mathbb{R}^d}(\langle\theta,x\rangle - \log\lambda(\theta)) = \sum_{i=1}^d |x_i| \log\frac{|x_i|}{\alpha(\frac{x_i}{|x_i|}e_i)}$$

(with the convention that $0\log 0 = 0$, used whenever $|x_i| = 0$) in (3.2.9) above holds for *all* $x \in \partial\mathbb{D}$, not just for x belonging to $\mathcal{K}$ or $\mathcal{O}$ (it is the equality with I_q which only holds in $\mathcal{K}$ or $\mathcal{O}$, respectively). This will be evident from the proof of Theorem 3.2.3. □

Remark 3.2.5. The annealed rate function I_a was shown in [Var] to admit a variational formula involving entropy, which was analyzed further in [PZ, Yil3, Ber2] under the additional assumption of condition (T). On the other hand, the quenched LDP in [Var] was derived using sub-additivity methods which did not lead to any formula for I_q (see also [Zer] for the quenched LDP in the case of nestling environments in $d \geq 1$ and [GdH, CGZ] for the $d = 1$ case). Later, based on the method in [KRV], the following variational formula for I_q was shown in [Ros] for elliptic RWRE:

$$\begin{aligned} I_q(x) &= F^\star(x) \stackrel{\text{def}}{=} \sup_{\theta\in\mathbb{R}^d}[\langle\theta,x\rangle - F(\theta)] \quad \text{where,} \\ F(\theta) &= \inf_G \operatorname{ess\,sup}_{\mathbb{P}} \log\Bigg(\sum_{|e|=1}\omega(0,e)\mathrm{e}^{G(\omega,e)+\langle\theta,e\rangle}\Bigg), \end{aligned} \tag{3.2.10}$$

where the infimum above is taken over a class of mean-zero gradients satisfying a certain moment condition. We also refer to [Yil2, RAS2] for extensions of the above result to level-2 and level-3 LDP for elliptic RWRE, and to [BMO] for a similar representation for non-elliptic

RWRE including random walks on percolation clusters. Finally, we refer to [RASY2, RASY3] for another variational representation of the quenched rate function. Notice that the Cramér-type representation in (3.2.9) simplifies its earlier antecedents significantly. □

3.2.3 Monotonicity in the disorder and phase transition in the equality of rate functions.

We now turn to the statement that provides a phase transition in the behavior of the difference $I_a(x,\cdot) - I_q(x,\cdot)$ as a function of the underlying disorder. We first need some further notation. Given a probability vector [2] $\alpha \in \mathcal{M}_1(\mathbb{V})$ with strictly positive entries, let

$$\mathcal{E}_\alpha := \left\{ (r(e))_{e\in\mathbb{V}} \in [-1,1]^{\mathbb{V}} : \sum_{e\in\mathbb{V}} \alpha(e) r(e) = 0 \text{ and } \sup_{e\in\mathbb{V}} |r(e)| = 1 \right\}.$$

We denote probability measures on the space $\Gamma_\alpha := \mathcal{E}_\alpha^{\mathbb{Z}^d}$ by $\mathbb{Q}$. We also write $\eta = (\eta(x))_{x\in\mathbb{Z}^d} \in \Gamma_\alpha$, with $\eta(x) = (\eta(x,e))_{e\in\mathbb{V}}$ being a typical element of the space $\mathcal{E}_\alpha$. Since α will remain fixed in the remainder of this subsection, we will omit the dependence on α of $\mathbb{Q}$ and η from the notation.

Now, given a probability vector $\alpha \in \mathcal{M}_1(\mathbb{V})$ with strictly positive entries and a probability measure $\mathbb{Q}$ on Γ_α, let us consider the parametrized family of random environments $\{\omega_\varepsilon\}_{\varepsilon\in[0,1)}$ given by

$$\omega_\varepsilon(x,e) := \alpha(e)(1 + \varepsilon\eta(x,e)).$$

We will make the following assumptions on $\mathbb{Q}$:

Assumption B. The probability measure $\mathbb{Q}$ satisfies the following three properties:

- The support of $\mathbb{Q}$ is not a singleton.[3]

[2]This is an abuse of notation with the α defined in (3.2.2). However, from the context will be clear to which α we are referring.

[3]As matter of fact, this condition is already implied by the third one since $\sup_{e\in\mathbb{V}} |\xi(x,e)| = 1$ by definition

- The family $(\eta(x))_{x\in\mathbb{Z}^d}$ is i.i.d. under $\mathbb{Q}$.

- $\mathbb{E}\eta(x,e)=0$ for all $e\in\mathbb{V}$ and $x\in\mathbb{Z}^d$.

The assumption that the support of $\mathbb{Q}$ is not a singleton is made to ensure that there exists some true randomness in the environments ω_ε for $\varepsilon>0$. On the other hand, the other two assumptions guarantee that for each $\varepsilon\in[0,1)$ the law $\mathbb{P}_\varepsilon$ of the environment ω_ε satisfies Assumption A with ellipticity constant $\kappa:=(1-\varepsilon)(\min_{e\in\mathbb{V}}\alpha(e))>0$ and $\mathrm{dis}(\mathbb{P}_\varepsilon)=\varepsilon$, with $\mathbb{E}(\omega_\varepsilon(x,e))=\alpha(e)$ for all $e\in\mathbb{V}$ and $x\in\mathbb{Z}^d$. In this context, we will denote by $I_a(\cdot,\varepsilon)$ and $I_q(\cdot,\varepsilon)$ to be the annealed and quenched rate functions, respectively. Recall that $I_a(x,\varepsilon)\leq I_q(x,\varepsilon)$ for all $x\in\mathbb{Z}^d$ and $\varepsilon\geq 0$ by Jensen's inequality. Our next main result establishes the monotonicity property for the difference of these two rate functions $I_a(x,\cdot)-I_q(x,\cdot)$.

Theorem 3.2.4. *Fix $d\geq 4$. Then, for any probability vector $\alpha\in\mathcal{M}_1(\mathbb{V})$ with strictly positive entries and probability measure $\mathbb{Q}$ on Γ_α satisfying Assumption* B*, the following assertions hold:*

- *For each $x\in\partial\mathbb{D}$, the map*

$$[0,1)\ni\varepsilon\mapsto I_a(x,\varepsilon)-I_q(x,\varepsilon)$$

is non-increasing and continuous. In particular, there is $\varepsilon_c(x)\geq 0$ such that for $\varepsilon\in[0,1)$,

$$\begin{cases} I_a(x,\varepsilon)=I_q(x,\varepsilon) & \textit{if} \quad \varepsilon\leq\varepsilon_c(x) \\ I_a(x,\varepsilon)<I_q(x,\varepsilon) & \textit{if} \quad \varepsilon>\varepsilon_c(x). \end{cases} \tag{3.2.11}$$

of $\mathcal{E}_\alpha$. Nevertheless, we still include it for clarity purposes.

- *Furthermore, there exists an open subset $\mathcal{O} \subsetneq \partial\mathbb{D} \setminus \partial\mathbb{D}_{d-2}$ such that for all $x \in \mathcal{O}$,*

$$0 < \varepsilon_c(x) < 1. \tag{3.2.12}$$

Remark 3.2.6. It follows from Theorem 3.2.2 that for any $x \in \partial\mathbb{D} \setminus \partial\mathbb{D}_{d-2}$ one always has $\varepsilon_c(x) > 0$. What is (in principle) only true for $x \in \mathcal{O}$ is the additional requirement in (3.2.12) that $\varepsilon_c(x) < 1$, which together with $\varepsilon_c(x) > 0$ implies the existence of a true phase transition in the disorder ε. □

Remark 3.2.7. We emphasize that the family of random environments considered presently is quite general and contains several widely studied models for RWRE (see [CR, Sab]). Furthermore, consideration of such a parametrization is in fact quite natural. Indeed, there are two basic questions that one can ask regarding this point. Namely,

Q1. Given $x \in (\partial\mathbb{D} \setminus \partial\mathbb{D}_{d-2})$, is it true that there exists ε_x such that the equality $I_a(x) = I_q(x)$ holds for *any* model with disorder less than ε_x and fails to hold for all larger disorders?

Q2. Given $x \in (\partial\mathbb{D} \setminus \partial\mathbb{D}_{d-2})$, is the mapping $\varepsilon \mapsto I_a(x,\varepsilon) - I_q(x,\varepsilon)$ monotonic?

Clearly the affirmation of (2) implies the same for (1). However, (2) does not make sense in general. Indeed, I_a and I_q need not be functions of the underlying disorder, only perhaps when dealing with *parametrized families of environments* as in Theorem 3.2.4. On the other hand, (1) does make sense in general, but it seems out of reach with our current method and we are not sure even if it is true. The difference with our Theorem 3.2.4 is that for us the "source of randomness" is fixed beforehand, so that when we make its influence smaller and smaller by taking the limit $\varepsilon_x \to 0$ then it is natural to expect equality to hold. However, we do not know whether there exists some *universal* ε_x which works simultaneously for all possible sources of randomness (as the affirmation of (1) would imply).

3.2.4 Outline of the proofs

For the sake of conceptual transparency and also to provide guidance to the reader, we find it convenient to present a brief description of the method of proof developed in the present article. This will then also underline the technical novelty of our contribution.

To treat the boundary behavior of I_q and I_a, we shall develop a somewhat different approach to the one used in [BMRS1] to deal with the behavior in the interior of $\mathbb{D}$. The method in the interior used there relied on the construction of an auxiliary random walk in a deterministic environment possessing a regeneration structure and showing that its large deviation properties are intimately related to those of the true RWRE. Since the RWRE behaves differently on the boundary,[4] here we develop an alternative approach which is conceptually more transparent and is based on a novel application of the *martingale method* developed originally by Bolthausen [Bol] in the context of directed polymers [Com]. The key idea is to construct the "renormalized partition function" or the *polymer martingale* in the context of general RWRE scenario even in the absence of "directed" structure. To this end, first we observe that it is enough to show equality of the rate functions holds on each face separately, i.e. for compact sets $\mathcal{K} \subseteq \partial\mathbb{D} \setminus \partial\mathbb{D}_{d-2}$ contained in $\partial\mathbb{D}(s)$ for some $s = (s_1, \ldots, s_d) \in \{\pm 1\}^d$, where

$$\partial\mathbb{D}(s) := \{x \in \partial\mathbb{D} \colon s_j x_j \geq 0 \ \forall j = 1, \ldots, d\}.$$

At this point, we make the following crucial observation: for each $s \in \{\pm\}^d$, on the event

$$\mathbb{B}_n(s) := \left\{\tfrac{1}{n}(X_n - X_0) \in \partial\mathbb{D}(s)\right\},$$

[4] While it might be possible to again define an auxiliary walk and study its regeneration times on the boundary, many technical problems now appear due to the non-positive definiteness of the Hessian of (the averaged) logarithmic moment generating function as the support of the first step for the auxiliary walk on the boundary is contained in a $(d-1)$-dimensional hyperplane, in addition to the reduced dimension $d-1$ leading to additional difficulties in using the approach of [BMRS1] which requires that the dimension be at least four.

one has that for all $j = 1, \dots, n$

$$X_j - X_{j-1} \in \mathbb{V}(s) := \{s_i e_i : i = 1, \dots, d\} \tag{3.2.13}$$

and, as a consequence, that for any $j, j' \in \{0, \dots, n\}$

$$X_j = X_{j'} \Longleftrightarrow j = j'. \tag{3.2.14}$$

In particular, if for an affine transformation π mapping the hyperplane $\{x \colon \sum_{j=1}^{d} s_j x_j = 1\}$ which contains $\partial\mathbb{D}(s)$ onto $\{x \colon x_d = 0\}$ we define the *projected RWRE* $S_n := \sum_{j=0}^{n-1} \pi(X_{j+1} - X_j)$ then, on the event $\mathbb{B}_n(s)$, the walk S_n satisfies the following two important properties:

- By (3.2.13), the path $(S_1, \dots, S_n)$ falls entirely on the hyperplane $\{x \colon x_d = 0\} = \mathbb{R}^{d-1} \times \{0\}$, and therefore we may view it as a $(d-1)$-dimensional walk. Moreover, since the jumps $(\pi(e))_{e \in \mathbb{V}(s)}$ of S_n span all of $\{x : x_d = 0\}$, it has effective dimension $d - 1$.

- For each $j = 1, \dots, n$, the weights used by S_j to decide where to jump next are given by the random probability vector $\omega(X_{j-1}, X_j - X_{j-1})$. By the i.i.d. structure of the environment, (3.2.14) yields that these vectors $(\omega(X_{j-1}, X_j - X_{j-1}))_{j=1,\dots,n}$ are independent. Furthermore, by uniform ellipticity, all these weights are uniformly bounded away from 0.

These crucial facts now allow us to construct a non-negative martingale *on the event* $\mathbb{B}_n(s)$ which in our context translates to

$$\mathscr{Z}_{n,\theta}(\omega, x) := \psi^{-n}(\theta) E_{x,\omega}\big[\mathrm{e}^{\langle \theta, S_n \rangle}\, \mathbb{1}_{\mathbb{B}_n(s)}\big], \qquad \text{with} \quad \psi(\theta) := \sum_{e \in \mathbb{V}(s)} \alpha(e) \mathrm{e}^{\langle \theta, \pi(e) \rangle}.$$

The above structure seems to be a natural way to construct the "renormalized partition function" in the context of general RWRE. However since the above extra ubiquitous conditions (e.g. restriction to paths on $\mathbb{B}_n(s)$) manifest throughout the entire analysis, the actual leveraging of the martingale method in our context of Theorem 3.2.1 (cf. Section 3.3 for its proof) and Theorem 3.2.2 (cf. Section 3.4 for its proof) is quite different from earlier approaches. Theorem 3.2.3 then follows from the proof of the two earlier results, while the proof of Theorem 3.2.4 builds on a method relying on the FKG inequality, see Section 3.5 for the proofs of these two results.

3.3 Equality on the boundary $\partial\mathbb{D}$ - Proof of Theorem 3.2.1

We first remark that the boundary $\partial\mathbb{D}$ of the unit ball $\mathbb{D}$ can be decomposed into (non-overlapping) *faces* $\partial\mathbb{D}(s)$, $s = (s_1, \ldots, s_d) \in \{-1, 1\}^d$, defined as

$$\partial\mathbb{D}(s) := \{x \in \partial\mathbb{D} : s_j x_j \geq 0 \text{ for all } j = 1, \ldots, d\}.$$

We will prove the equality of rate functions

$$I_q(x) = I_a(x) \tag{3.3.1}$$

under the assumptions of Theorem 3.2.1 on each face $\partial\mathbb{D}(s)$ separately. Since the proof is exactly the same for all faces, from now on we will fix a face $s := (s_1, \ldots, s_d)$ and prove (3.3.1) for $x \in \partial\mathbb{D}(s)$. For simplicity, in the sequel we will also sometimes remove the dependence on s from the notation.

Our proof of (3.3.1) is divided into four steps, each occupying a separate subsection. Before we begin, let us introduce some further notation to be used throughout the sequel.

Given $\kappa > 0$, we define

$$\mathcal{M}_1^{(\kappa)}(\mathbb{V}) := \{p \in \mathcal{M}_1(\mathbb{V}) : p(e) \geq \kappa \text{ for all } e \in \mathbb{V}\},$$

together with the class of environmental laws

$$\mathcal{P}_\kappa := \{\mathbb{P} \in \mathcal{M}_1(\Omega) : \mathbb{P} \text{ satisfies Assumption A with ellipticity constant } \kappa\},$$

where $\mathcal{M}_1(\Omega)$ is the space of all environmental laws. We are now ready to begin the proof.

3.3.1 Projecting on a $(d-1)$-dimensional hyperplane.

For each $n \in \mathbb{N}$ let us define

$$\partial R_n := \{x \in \mathbb{Z}^d : |x|_1 = n\,,\ s_j x_j \geq 0 \text{ for all } j = 1, \ldots, d\} = n \cdot \partial\mathbb{D}(s). \tag{3.3.2}$$

and for each $x \in \mathbb{Z}^d$ set

$$\partial R_n(x) := x + \partial R_n.$$

Also, define the set $\mathbb{V}(s)$ of s-allowed jumps as

$$\mathbb{V}(s) = \{s_j e_j : j = 1, \ldots, d\} \subseteq \mathbb{V}.$$

Given $n \geq 1$, recall that a sequence $z := (z_0, \ldots, z_n)$ of sites in $\mathbb{Z}^d$ is a *path of length* n if $z_j - z_{j-1} \in \mathbb{V}$ for all $j = 1, \ldots, n$. For $x \in \mathbb{Z}^d$, let $\mathcal{R}_n(x)$ denote the set of all paths of length n such that $z_0 = x$ and $z_n \in \partial R_n(x)$. Notice that a path $z = (z_0, \ldots, z_n)$ of length n belongs to $\partial R_n(z_0)$ if and only if all of its jumps belong to $\mathbb{V}(s)$, i.e. if we define the j-th jump of the path z by

$$\Delta_j(z) := z_j - z_{j-1}, \tag{3.3.3}$$

then

$$z = (z_0, \ldots, z_n) \in \mathcal{R}_n(z_0) \Longleftrightarrow \Delta_j(z) \in \mathbb{V}(s) \text{ for all } j = 1, \ldots, n, \tag{3.3.4}$$

from where we easily deduce that

$$z = (z_0, \ldots, z_n) \in \mathcal{R}_n(z_0) \Longleftrightarrow (z_0, \ldots, z_{n-1}) \in \mathcal{R}_{n-1}(z_0) \text{ and } \Delta_n(z) \in \mathbb{V}(s). \tag{3.3.5}$$

Now, notice that $\{x : s_1x_1 + \cdots + s_dx_d = 1\}$ is the unique hyperplane which contains $\mathbb{V}(s)$, which is (affinely) generated by the vectors $(s_ie_i)_{i=1,\ldots,d}$, and let $\pi : \mathbb{R}^d \to \mathbb{R}^d$ be the affine transformation mapping $\{x : s_1x_1 + \cdots + s_dx_d = 1\} \longrightarrow \{x : x_d = 0\}$ given by

$$\pi(x) = \begin{cases} e_i & \text{if } x = s_ie_i \text{ for } i = 1, \ldots, d-1 \\ -(e_1 + \cdots + e_{d-1}) & \text{if } x = s_de_d \\ \frac{d-1}{d}e_d & \text{if } x = s. \end{cases} \tag{3.3.6}$$

We then define then the *projected walk* $(S_n)_{n\in\mathbb{N}}$ by the formula

$$S_n := \sum_{j=1}^{n} \pi(X_j - X_{j-1}), \qquad k \in \mathbb{N}, \tag{3.3.7}$$

where $X = (X_n)_{n\in\mathbb{N}}$ is our original RWRE, and for each $n \geq 1$ consider the event

$$\mathbb{B}_n := \{\Delta_j(X) \in \mathbb{V}(s) \text{ for all } j = 1, \ldots, n\} = \{(X_0, \ldots, X_n) \in \partial R_n(X_0)\}. \tag{3.3.8}$$

Notice that, on the event $\mathbb{B}_n$, the projected walk S_n belongs to the hyperplane $\{x \in \mathbb{R}^d : x_d = 0\}$, which we can (and will henceforth) identify with $\mathbb{R}^{d-1}$.Thus, if for $\theta \in \mathbb{R}^{d-1}$ we define

$$\psi(\theta) := \sum_{e\in\mathbb{V}(s)} \alpha(e)\mathrm{e}^{\langle\theta,\pi(e)\rangle} = \sum_{i=1}^{d-1} \alpha(s_ie_i)\mathrm{e}^{\theta_i} + \alpha(s_de_d)\mathrm{e}^{-(\theta_1+\cdots+\theta_{d-1})}, \tag{3.3.9}$$

with the identification $\{x \in \mathbb{R}^d : x_d = 0\} = \mathbb{R}^{d-1}$ in mind we may define for $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$,

$$\begin{aligned}\mathscr{Z}_{n,\theta}(\omega, x) &:= \frac{E_{x,\omega}(\mathrm{e}^{\langle \theta, S_n\rangle} \mathbb{1}_{\mathbb{B}_n})}{\psi^n(\theta)} \\ &= \frac{\sum_{z\in\mathcal{R}_n(x)} \mathrm{e}^{\langle \theta, \sum_{j=1}^n \pi(\Delta_j(z))\rangle} \prod_{i=1}^n \omega(z_{j-1}, \Delta_j(z))}{\psi^n(\theta)}.\end{aligned} \tag{3.3.10}$$

for $\Delta_j(z)$ as in (3.3.3). Now a simple computation using (3.3.5) and the definition of ψ shows that

$$\mathscr{Z}_\theta(\cdot) = (\mathscr{Z}_{n,\theta}(\cdot, x))_{n\in\mathbb{N}}$$

is a $\mathbb{P}$-martingale for any θ and x. Being also nonnegative, we know it has an $\mathbb{P}$-almost sure limit:

$$\mathscr{Z}_{\infty,\theta}(\cdot, x) \overset{\text{a.s.}}{=} \lim_{n\to\infty} \mathscr{Z}_{n,\theta}(\cdot, x). \tag{3.3.11}$$

3.3.2 Martingale convergence in L^2.

Our goal is now to show that the converge in (3.3.11) holds also in $L^2(\mathbb{P})$. The following assertion, providing the desired $L^2(\mathbb{P})$-convergence, will furthermore imply that the limit $\mathscr{Z}_{\infty,\theta}$ is also strictly positive.

Recall the definition of disorder $\mathrm{dis}(\mathbb{P})$ from (3.2.1).

Lemma 3.3.1. *Given $d \geq 4$, $\kappa > 0$ and a compact set $\Theta \subseteq \mathbb{R}^{d-1}$, there exists $\varepsilon' = \varepsilon'(d, \kappa, \Theta) > 0$ such that, for any RWRE in dimension d with $\mathbb{P} \in \mathcal{P}_\kappa$, if $\mathrm{dis}(\mathbb{P}) < \varepsilon'$ then for any $x \in \mathbb{Z}^d$*

$$\sup_{n\in\mathbb{N}\,,\,\theta\in\Theta} \|\mathscr{Z}_{n,\theta}(x)\|_{L^2(\mathbb{P})} < \infty.$$

For the proof of Lemma 3.3.1 we shall need the following result, which is (a particular version of) the well-known Khas'minskii's lemma. We include the short proof to keep the material self-contained.

Lemma 3.3.2. *Let $Z = (Z_i)_{i\in\mathbb{N}}$ be a random walk on $\mathbb{Z}^d$ starting at the origin, whose law is denoted by* P_0 *with expectation* E_0*. If we define*

$$\eta := \mathrm{E}_0\left(\sum_{i=0}^{\infty} \mathbb{1}_{\{Z_i=0\}}\right) = \sum_{i=0}^{\infty} \mathrm{P}_0(Z_i = 0)$$

then for any $C > 0$ such that $C\eta < 1$ we have

$$\mathrm{E}_0\left(\exp\left\{C\sum_{i=0}^{\infty} \mathbb{1}_{\{Z_i=0\}}\right\}\right) \le \frac{1}{1-C\eta}. \tag{3.3.12}$$

Proof. By expanding the exponential on the left-hand side in (3.3.12) we can write

$$\begin{aligned}
\mathrm{E}_0\left(\exp\left\{C\sum_{i=0}^{\infty} \mathbb{1}_{\{Z_i=0\}}\right\}\right) &= \sum_{n=0}^{\infty} \frac{C^n}{n!}\mathrm{E}_0\left[\left(\sum_{i=0}^{\infty} \mathbb{1}_{\{Z_i=0\}}\right)^n\right] \\
&\le \sum_{n=0}^{\infty} C^n \sum_{0\le i_1\le\cdots\le i_n} \mathrm{P}_0(Z_{i_1} = 0, \dots, Z_{i_n} = 0) \\
&= \sum_{n=0}^{\infty} C^n \sum_{0\le i_1\le\cdots\le i_{n-1}} \mathrm{P}_0(Z_{i_1} = 0, \dots, Z_{i_{n-1}} = 0) \sum_{i_n=i_{n-1}}^{\infty} \mathrm{P}_0(Z_{i_n-i_{n-1}} = 0) \\
&= \sum_{n=0}^{\infty} C^n\eta \sum_{0\le i_1\le\cdots\le i_{n-1}} \mathrm{P}_0(Z_{i_1} = 0, \dots, Z_{i_{n-1}} = 0) \\
&= \sum_{n=0}^{\infty} (C\eta)^n = \frac{1}{1-C\eta} \qquad \text{if} \quad C\eta < 1,
\end{aligned}$$

where in the upper bound above we have used symmetry, while the next identities follow by successive use of the Markov property. □

We are now ready to prove Lemma 3.3.1.

Proof of Lemma 3.3.1. By the translation invariance of the environment, it will suffice to show the claim for $x = 0$ and, for notational convenience, in the sequel we will abbreviate

$\mathcal{R}_n := \mathcal{R}_n(0)$ and $\mathscr{Z}_{n,\theta} := \mathscr{Z}_{n,\theta}(0)$. Then

$$\begin{aligned}\|\mathscr{Z}_{n,\theta}\|^2_{L^2(\mathbb{P})} &= \frac{\mathbb{E}(E^2_{0,\omega}(\mathrm{e}^{\langle\theta,S_n\rangle}\mathbb{1}_{B_n}))}{\psi^{2n}(\theta)}\\ &= \sum_{z,z'\in\mathcal{R}_n}\mathbb{E}\left(\left(\prod_{j=1}^n\omega(z_{j-1},\Delta_j(z))\frac{\mathrm{e}^{\langle\theta,\pi(\Delta_j(z))\rangle}}{\psi(\theta)}\right)\left(\prod_{k=1}^n\omega(z'_{k-1},\Delta_k(z'))\frac{\mathrm{e}^{\langle\theta,\pi(\Delta_k(z'))\rangle}}{\psi(\theta)}\right)\right).\end{aligned} \tag{3.3.13}$$

Now the following simple observation is crucial for our context. By (3.3.1) we have that

$$z=(0,\ldots,z_n)\in\mathcal{R}_n \Longrightarrow |z_j|=j \text{ for all } j=1,\ldots,n, \tag{3.3.14}$$

so that the z_j must be all distinct and, furthermore, for $z,z'\in\mathcal{R}_n$ one has $z_j=z'_k$ only if $j=k$.

Using that our environment is i.i.d., this allows us to rewrite (3.3.13) as

$$\begin{aligned}\|\mathscr{Z}_{n,\theta}\|^2_{L^2(\mathbb{P})} &= \sum_{z,z'\in\mathcal{R}_n}\mathbb{E}\left(\prod_{j=1}^n\left(\omega(z_{j-1},\Delta_j(z))\omega(z'_{j-1},\Delta_j(z'))\frac{\mathrm{e}^{\langle\theta,\pi(\Delta_j(z))\rangle}}{\psi(\theta)}\frac{\mathrm{e}^{\langle\theta,\pi(\Delta_j(z'))\rangle}}{\psi(\theta)}\right)\right)\\ &= \sum_{z,z'\in\mathcal{R}_n}\prod_{j=1}^n\left(\mathbb{E}\left(\omega(z_{j-1},\Delta_j(z))\omega(z'_{j-1},\Delta_j(z'))\right)\frac{\mathrm{e}^{\langle\theta,\pi(\Delta_j(z))\rangle}}{\psi(\theta)}\frac{\mathrm{e}^{\langle\theta,\pi(\Delta_j(z'))\rangle}}{\psi(\theta)}\right).\end{aligned} \tag{3.3.15}$$

Now, define the probability vector $\vec{\alpha}^{(\theta)}=(\alpha^{(\theta)}(\pi(e)))_{e\in\mathbb{V}(s)}$ on $\mathbb{R}^{d-1}$ by the formula

$$\alpha^{(\theta)}(\pi(e)) := \alpha(e)\frac{\mathrm{e}^{\langle\theta,\pi(e)\rangle}}{\psi(\theta)}, \tag{3.3.16}$$

and $P_0^{(\theta)}$ as the law of the random walk on $\mathbb{R}^{d-1}$ starting from 0 having jump distribution $\vec{\alpha}^{(\theta)}$. Then, since

$$\mathbb{E}\left(\omega(z_{j-1},\Delta_j(z))\omega(z'_{j-1},\Delta_j(z'))\right)=\alpha(\Delta_j(z))\alpha(\Delta_j(z'))$$

holds by independence whenever $z_{j-1}\neq z'_{j-1}$, a straightforward computation yields that one can rewrite (3.3.15) as

$$\|\mathscr{Z}_{n,\theta}\|^2_{L^2(\mathbb{P})} = E_0\left(\exp\left\{\sum_{j=1}^{n} \mathbb{1}_{\{X^{(\theta)}_{j-1}=Y^{(\theta)}_{j-1}\}} V(X^{(\theta)}_j - X^{(\theta)}_{j-1}, Y^{(\theta)}_j - Y^{(\theta)}_{j-1})\right\}\right)$$

where $X^{(\theta)}$ and $Y^{(\theta)}$ are two independent random walks with law $P^{(\theta)}_0$ and expectation $E^{(\theta)}_0$, and for $e, e' \in \mathbb{V}(s)$ we write

$$V(\pi(e), \pi(e')) := \log\left(\frac{\mathbb{E}(\omega(0,e)\omega(0,e'))}{\alpha(e)\alpha(e')}\right).$$

Note that V is well-defined by uniform ellipticity and, moreover, since $\omega(0,e) \leq \alpha(e)(1+\mathrm{dis}(\mathbb{P}))$ for each $e \in V$, we have an upper bound

$$V(\pi(e), \pi(e')) \leq \log(1+\mathrm{dis}(\mathbb{P})) \leq \mathrm{dis}(\mathbb{P}),$$

implying that

$$\|\mathscr{Z}_{n,\theta}\|^2_{L^2(\mathbb{P})} \leq E_0\left(\exp\left\{\mathrm{dis}(\mathbb{P})\sum_{j=0}^{n-1} \mathbb{1}_{\{Z^{(\theta)}_j=0\}}\right\}\right)$$

where, for $j = 0, \ldots, n-1$, we write $Z^{(\theta)}_j = X^{(\theta)}_j - Y^{(\theta)}_j$. In particular, we see that

$$\sup_{n\in\mathbb{N},\,\theta\in\Theta} \|\mathscr{Z}_{n,\theta}\|^2_{L^2(\mathbb{P})} \leq \sup_{\theta\in\Theta} E_0\left(\exp\left\{\mathrm{dis}(\mathbb{P})\sum_{j=0}^{\infty} \mathbb{1}_{\{Z^{(\theta)}_j=0\}}\right\}\right). \tag{3.3.17}$$

By Lemma 3.3.2, the right-hand side of (3.3.17) will be finite if

$$\sup_{\theta\in\Theta}\left(\sum_{j=0}^{\infty} P_0(Z^{(\theta)}_j = 0)\right) < \frac{1}{\mathrm{dis}(\mathbb{P})}.$$

Now, let $\chi_\theta(\xi) = E^{(\theta)}_0[\exp\{\mathbf{i}\langle \xi, Z^{(\theta)}_1\rangle\}]$ denote the characteristic function of $Z^{(\theta)}_1$ (recall that $Z^{(\theta)}_0 = 0$). Since $Z^{(\theta)}_1 = X^{(\theta)}_1 - Y^{(\theta)}_1$ with $X^{(\theta)}_1, Y^{(\theta)}_1$ i.i.d., χ_θ takes only real non-negative values. We claim that there exists a $C_d > 0$ depending only on d such that, for any $\theta \in \mathbb{R}^{d-1}$ and $r > 0$,

$$\sum_{j=0}^{\infty} P_0(Z_j^{(\theta)} = 0) \leq C_d r^{-(d-1)} \int_{B_r(0)} \frac{\mathrm{d}\xi}{1 - \chi_\theta(\xi)}, \tag{3.3.18}$$

where $B_r(0) := \{\xi \in \mathbb{R}^{d-1} : |\xi|_2 \leq r\}$.

We defer the proof of (3.3.18) and continue with the proof of Lemma 3.3.1. Note that the support of $|Z_1^{(\theta)}|_2$ is uniformly bounded in θ. Therefore, by Taylor's expansion we have

$$\chi_\theta(\xi) \leq 1 - \tfrac{1}{2} \sum_{i,k=1} a_{ik}^{\theta} \xi_i \xi_k + C|\xi|_2^3 \tag{3.3.19}$$

for some constant $C > 0$ independent of θ, where $(a_{ik}^{(\theta)})_{i,k}$ is the covariance matrix of $Z_1^{(\theta)}$. Finally, since $(a_{ik}^{(\theta)})_{i,k}$ is positive definite for each θ (since the random walk Z^θ has effective dimension $d-1$) and the maps

$$(\alpha, \theta) \mapsto a_{ik}^{(\theta)}$$

are continuous for all i, k, by proceeding as in the proof of Lemma 2.4.8, it follows from (3.3.19) that for any compact set $\Theta \subseteq \mathbb{R}^{d-1}$ there exist $r_0 = r_0(d, \kappa, \Theta), c_0 = c_0(d, \kappa, \Theta) > 0$ such that

$$c_0 |\xi|_2^2 \leq 1 - \chi_\theta(\xi)$$

for all $\xi \in B_{r_0}(0)$. In particular, from (3.3.18) we see that, since $d \geq 4$, for some constant $\overline{C}_d > 0$ depending only on d we have

$$\sup_{\theta \in \Theta} \sum_{j=0}^{\infty} P_0\big(Z_j^{(\theta)} = 0\big) \leq C_d \frac{r_0^{-(d-1)}}{c_0} \int_{B_{r_0}(0)} \frac{1}{|\xi|_2^2}\, \mathrm{d}\xi = \overline{C}_d \frac{r_0^{-(d-1)}}{c_0} \int_0^{r_0} r^{d-4} =: C_0 < \infty. \tag{3.3.20}$$

Taking $\varepsilon' := \frac{1}{C_0}$ then yields the result. We now owe the reader only the proof of the claim (3.3.18). But this is an immediate consequence of Lemma 3.3.3 below, which is a well-known application of the Fourier inversion formula. $\square$

Lemma 3.3.3. *Let $(Z_n)_{n\geq 0}$ be a random walk in $\mathbb{R}^d$ with law P_0 starting at the origin and assume that χ_μ, the characteristic function of Z_1, takes only real non-negative values. Then for any $r>0$ and $\delta=\sqrt{d}/r$,*

$$\sum_{n\geq 0}\mathrm{P}_0\big[Z_n\in B_\delta(0)\big]\leq \frac{C_d}{r^d}\int_{B_r(0)}\frac{\mathrm{d}\xi}{1-\chi_\mu(\xi)}.$$

Proof. Since we are interested in the event $\{Z_n\leq\delta\}$ we need to consider the function $\prod_{j=1}^d f(x_j/\delta)$ where $f(x_j)=\max(1-|x_j|,0)$. Then we have the Fourier transform of the product

$$\widehat{\prod_{j=1}^d f(x_j)}=\prod_{j=1}^d \widehat{f}(\xi_j)\quad\text{with}\quad \widehat{f}(\xi_j)=\frac{2}{\xi_j^2}(1-\cos\xi_j).$$

If μ denotes the law of Z_1 and $\mu^{\star n}=\mu\star\cdots\star\mu$ its n-fold convolution, then for any $\delta>0$, [5]

$$\int_{\mathbb{R}^d}\widehat{\prod_{j=1}^d f(\frac{x_j}{\delta})}\mu^{\star n}(\mathrm{d}x)=\delta^d\int\prod_{j=1}^d f(\delta\xi_j)(\chi_\mu(\xi))^n\,\mathrm{d}\xi.$$

Therefore, for any $a\in(0,1)$,

$$\int_{\mathbb{R}^d}\widehat{\prod_{j=1}^d f(\frac{x_j}{\delta})}\sum_{n\geq 0}a^n\mu^{\star n}(\mathrm{d}x)=\delta^d\int\frac{\prod_{j=1}^d f(\delta\xi_j)}{1-a\chi_\mu(\xi)}\mathrm{d}\xi, \tag{3.3.21}$$

which implies that, for $\delta=\sqrt{d}/r$ and a suitable constant $C>0$,

$$\begin{aligned}\sum_{n\geq 0}\mathrm{P}_0\big[Z_n\in B_\delta(0)\big]=\sum_{n\geq 0}\mu^{\star n}(B_\delta(0))\leq C\int_{\mathbb{R}^d}\widehat{\prod_{j=1}^d f(\frac{x_j}{\delta})}\sum_{n\geq 0}\mu^{\star n}(\mathrm{d}x)&=C\delta^d\sup_{a\in(0,1)}\int\frac{\prod_{j=1}^d f(\delta\xi_j)}{1-a\chi_\mu(\xi)}\mathrm{d}\xi.\\ &\leq C_d r^{-d}\int_{B_r(0)}\frac{\mathrm{d}\xi}{1-\chi_\mu(\xi)}.\end{aligned}$$

[5]Recall that if μ and ν are two probability measures on $\mathbb{R}^d$ with characteristic functions χ_μ and χ_ν respectively, then $\int\chi_\nu(x)\mu(\mathrm{d}x)=\int\chi_\mu(\xi)\nu(\mathrm{d}\xi)$.

□

3.3.3 Strict positivity of the limit $\mathscr{Z}_{\infty,\theta}$

The next step in the proof is to show the martingale limit $\mathscr{Z}_{\infty,\theta}$ is strictly positive.

Proposition 3.3.4. *Given $d \geq 4$, $\kappa > 0$ and a compact set $\Theta \subseteq \mathbb{R}^{d-1}$ we have that, for any RWRE in dimension d with $\mathbb{P} \in \mathcal{P}_\kappa$, if $\mathrm{dis}(\mathbb{P}) < \varepsilon'$ (with ε' as in Lemma 3.3.1) then for each $\theta \in \Theta$,*

$$\mathbb{P}\big\{\mathscr{Z}_{\infty,\theta}(x) > 0 \text{ for all } x \in \mathbb{Z}^d\big\} = 1.$$

Proof. By (3.3.4) we have

$$z = (0, z_1, \dots, z_n) \in \mathcal{R}_n \Longleftrightarrow \Delta_1(z) \in \mathbb{V}(s) \text{ and } (z_1, \dots, z_n) \in \partial R_{n-1}(z_1)$$

so that, by conditioning on the first step of the walk X_1, a straightforward computation yields that

$$\mathscr{Z}_{n,\theta}(\omega, 0) = \sum_{e \in \mathbb{V}(s)} \omega(0, e) \mathrm{e}^{\langle \theta, \pi(e) \rangle - \log \psi(\theta)} \mathscr{Z}_{n-1,\theta}(\omega, e). \tag{3.3.22}$$

On the other hand, if for $y \in \mathbb{Z}^d$ we define $T_y : \Omega \to \Omega$ to be the translation

$$T_y(\omega)(x) := \omega(x + y), \tag{3.3.23}$$

then it follows that for any $e \in \mathbb{V}$

$$\mathscr{Z}_{n-1,\theta}(\omega, e) = \mathscr{Z}_{n-1,\theta}(T_e(\omega), 0),$$

so that (3.3.22) becomes

$$\mathscr{Z}_{n,\theta}(\omega,0)=\sum_{e\in\mathbb{V}(s)}\omega(0,e)\mathrm{e}^{\langle\theta,\pi(e)\rangle-\log\psi(\theta)}\mathscr{Z}_{n-1,\theta}(T_e(\omega),0). \tag{3.3.24}$$

By translation invariance of $\mathbb{P}$ we know that $\mathscr{Z}_{n,\theta}(T_e(\omega),0)\to\mathscr{Z}_{\infty,\theta}(T_e(\omega),0)$ for $\mathbb{P}$-almost every ω, so that we may take the $\mathbb{P}$-almost sure limit as $n\to\infty$ on (3.3.24) to obtain

$$\mathscr{Z}_{\infty,\theta}(\omega,0)=\sum_{e\in\mathbb{V}(s)}\omega(0,e)\mathrm{e}^{\langle\theta,\pi(e)\rangle-\log\psi(\theta)}\mathscr{Z}_{\infty,\theta}(T_e(\omega),0). \tag{3.3.25}$$

Moreover, it follows from (3.3.25) (and again translation invariance of $\mathbb{P}$) that the event $\{\mathscr{Z}_{\infty,\theta}(0)=0\}$ is almost T_e-invariant for any $e\in\mathbb{V}(s)$ so that, by ergodicity of $\mathbb{P}$, its probability must be either 0 or 1. Since Lemma 3.3.1 dictates that the mean-one martingale $(\mathscr{Z}_{n,\theta}(0))_{n\in\mathbb{N}}$ converges to $\mathscr{Z}_{\infty,\theta}(0)$ in $L^2(\mathbb{P})$, we have $\mathbb{E}(\mathscr{Z}_{\infty,\theta}(0))=1$ and thus it must be $\mathbb{P}(\mathscr{Z}_{\infty,\theta}(0)=0)=0$. By translation invariance of $\mathbb{P}$ we conclude the validity of the last sentence for all $x\in\mathbb{Z}^d$ so that

$$\mathbb{P}\big\{\mathscr{Z}_{\infty,\theta}(x)=0\text{ for some }x\in\mathbb{Z}^d\big\}=0,$$

implying the desired result. □

3.3.4 Concluding the proof of Theorem 3.2.1.

Existence of the LDP limits and properties of moment generating functions.

In order to conclude the proof of Theorem 3.2.1 we shall need Lemma 3.3.5 below, which establishes the existence of certain "point-to-point" free energies (in the terminology of [RAS3]). Throughout the sequel, we will call a sequence $\{x_n\}_{n\in\mathbb{N}}\subseteq\mathbb{Z}^d$ *admissible* if for each $n\in\mathbb{N}$ there exists a path $z=(z_0,z_1,\ldots,z_n)$ of length n with $z_0=0$ and $z_n=x_n$.

Lemma 3.3.5. *Under Assumption* A*, for any* $x \in \partial\mathbb{D}(s)$ *there exists an admissible sequence* $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Z}^d$ *such that* $\frac{x_n}{n} \to x$ *and*

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n} \log P_{0,\omega}(X_n = x_n) = -I_q(x) \qquad \mathbb{P}\text{ - } a.s., \\ &\lim_{n\to\infty} \frac{1}{n} \log P_0(X_n = x_n) = -I_a(x). \end{aligned}$$

The proof of Lemma 3.3.5 is deferred until the end of Section 3.3.4.

Next, recall from (3.3.7) that $S = (S_n)_{n\in\mathbb{N}}$ denotes the projected walk of the RWRE $X = (X_n)_{n\geq 0}$. Now, for each $n \geq 1$, let us set

$$\overline{S}_n := \frac{1}{n} S_n$$

to be the empirical mean and, for each $n \geq 1$ and $\omega \in \Omega$, define the quenched log-moment generating function of $\overline{S}_n$ as

$$A_n^\omega(\theta) := \log E_{0,\omega}\Big(e^{\langle \theta, \overline{S}_n\rangle} \mathbb{1}_{\mathbb{B}_n}\Big), \qquad \theta \in \mathbb{R}^{d-1},$$

where the event $\mathbb{B}_n$ is defined in (3.3.8). Then the *limiting* quenched log-moment generating function is

$$\Lambda^\omega(\theta) = \limsup_{n\to+\infty} \frac{1}{n} A_n^\omega(n\theta). \tag{3.3.26}$$

We recall some qualitative properties of Λ^ω stated in the following result.

Lemma 3.3.6. *For each* $\mathbb{P}$ *satisfying Assumption* A *there exists a full* $\mathbb{P}$*-probability event* $\overline{\Omega} = \overline{\Omega}(\mathbb{P})$ *such that, for any* $\omega \in \overline{\Omega}$*, the following holds:*

(i) The limit in (3.3.26) *exists and is finite for all* $\theta \in \mathbb{R}^{d-1}$*, i.e. for all* $\theta \in \mathbb{R}^{d-1}$

$$\Lambda^\omega(\theta) = \lim_{n\to+\infty} \frac{1}{n} A_n^\omega(n\theta) \in (-\infty, +\infty).$$

(ii) Λ^ω *is convex and continuous on* $\mathbb{R}^{d-1}$*.*

(iii) If $y = \nabla\Lambda^\omega(\eta)$ *for some* $\eta \in \mathbb{R}^{d-1}$*, then*

$$\langle \eta, y\rangle - \Lambda^\omega(\eta) = \sup_{\theta\in\mathbb{R}^{d-1}} [\langle\theta, y\rangle - \Lambda^\omega(\theta)] =: \overline{\Lambda}^\omega(y).$$

Moreover, y *is an exposed point of* $\overline{\Lambda}^\omega$ *and* η *is its exposing hyperplane, i.e. for all* $x \neq y$

$$\langle \eta, y\rangle - \overline{\Lambda}^\omega(y) > \langle \eta, x\rangle - \overline{\Lambda}^\omega(x).$$

(iv) $\overline{\Lambda}^\omega$ *is lower semicontinuous.*

(v) For any closed set $F \subseteq \mathbb{R}^{d-1}$*,*

$$\limsup_{n\to+\infty} \frac{1}{n} \log P_{0,\omega}(\{\overline{S}_n \in F\} \cap \mathbb{B}_n) \leq -\inf_{x\in F} \overline{\Lambda}^\omega(x).$$

(vi) For any open set $G \subseteq \mathbb{R}^{d-1}$*,*

$$\liminf_{n\to+\infty} \frac{1}{n} \log P_{0,\omega}(\{\overline{S}_n \in G\} \cap \mathbb{B}_n) \geq -\inf_{x\in G\cap\mathcal{F}^\omega} \overline{\Lambda}^\omega(x),$$

where $\mathcal{F}^\omega$ *denotes the set of exposed points of* Λ^ω*.*

Proof. All the assertions are found in the standard literature (see [DZ, Section 2.3]) which follows from the existence of a full $\mathbb{P}$-probability event $\overline{\Omega}$ such that, for any $\omega \in \overline{\Omega}$ and all $\theta \in \mathbb{R}^{d-1}$,

$$\Lambda^\omega(\theta) = \lim_{n\to+\infty} \frac{1}{n} A_n^\omega(n\theta) < +\infty. \tag{3.3.27}$$

Alternatively, once we have (3.3.27), one can introduce the conditional probabilities

$$\mu_n := P_{0,\omega}(\overline{S}_n \in \cdot \,|\, \mathbb{B}_n)$$

and deduce the remaining parts of the lemma by applying the standard Gärtner-Ellis theorem for the sequence $(\mu_n)_{n\in\mathbb{N}}$. The existence of the limit (3.3.27) follows from [RAS3, Theorem 2.4-(b)], whereas its finiteness is a consequence of the simple bound $A_n^\omega(n\theta) \leq n|\theta|_1(d-1)$ for all n. $\square$

Remark 3.3.1. As in the quenched set-up, we can define the annealed log-moment generating function

$$A_n(\theta) := \log E_0\Big(\mathrm{e}^{\langle\theta,\overline{S}_n\rangle}\mathbb{1}_{\mathbb{B}_n}\Big) = n\log\psi(\tfrac{\theta}{n}),$$

together with its limiting version

$$\Lambda(\theta) := \lim_{n\to+\infty}\frac{1}{n}A_n(n\theta) = \log\psi(\theta).$$

It is easy to see that an analogue of Lemma 3.3.6 holds for the annealed version Λ, by replacing Λ^ω with Λ and $P_{0,\omega}$ with P_0 everywhere in the statements above. $\square$

Proof of Theorem 3.2.1:

We will now conclude the proof of Theorem 3.2.1 which will be carried out in a few steps. Throughout the following we assume $d \geq 4$ so that Proposition 3.3.4 holds.

Step 1: First, by Proposition 3.3.4, given any $\kappa > 0$ and $R > 0$ there exists $\varepsilon_R = \varepsilon_R(d,\kappa,R) > 0$ such that, for any $\mathbb{P}\in\mathcal{P}_\kappa$, whenever $\mathrm{dis}(\mathbb{P}) < \varepsilon_R$ then, for each

$$\theta_0 \in D_R := \{\theta\in\mathbb{R}^{d-1} : |\theta|_1 \leq R\},$$

we have that $\mathscr{Z}_{\infty,\theta_0}(0)$ is $\mathbb{P}$-a.s. strictly positive. Hence, it follows that for each $\mathbb{P}\in\mathcal{P}_\kappa$ there exists a full $\mathbb{P}$-probability event $\Omega_R = \Omega_R(\mathbb{P},R)$ such that for all $\omega\in\Omega_R$

$$\mathscr{Z}_{\infty,\theta}(\omega,0) > 0 \text{ for all } \theta\in\Theta_R, \tag{3.3.28}$$

where Θ_R is some fixed (but arbitrary) countable dense subset of D_R. Furthermore, without loss of generality we may assume that Ω_R is contained in the event $\overline{\Omega}$ from Lemma 3.3.6. But observe that, if this is the case, for $\omega \in \Omega_R$ and $\theta \in \Theta_R$ we may rewrite

$$\Lambda^\omega(\theta) = \log \psi(\theta) + \lim_{n \to +\infty} \frac{1}{n} \log \mathscr{Z}_{n,\theta}(\omega, 0) = \log \psi(\theta), \tag{3.3.29}$$

where the second equality follows from (3.3.28). Since Λ^ω is continuous on D_R if $\omega \in \Omega_R$ by Lemma 3.3.6, we conclude that for any such ω the equality $\Lambda^\omega(\theta) = \log \psi(\theta)$ in (3.3.29) holds for all θ in D_R. Therefore, we have shown that given any $\kappa, R > 0$ there exists $\varepsilon_R > 0$ such that, for any $\mathbb{P} \in \mathcal{P}_\kappa$, whenever $\mathrm{dis}(\mathbb{P}) < \varepsilon_R$ there exists a full $\mathbb{P}$-probability event Ω_R such that (3.3.29) holds for all $\theta \in D_R$ and $\omega \in \Omega_R$.

Step 2: We now need the following result.

Lemma 3.3.7. *Given $\kappa > 0$ and a compact set $\mathcal{K} \subseteq \partial \mathbb{D}(s) \setminus \partial \mathbb{D}_{d-2}$, there exists $R_{\mathcal{K}} = R_{\mathcal{K}}(d, \kappa, \mathcal{K}) > 0$ such that, for any $\mathbb{P} \in \mathcal{P}_\kappa$, we have*

$$\pi(\mathcal{K}) \subseteq \{\nabla \log \psi(\theta) : \theta \in D_{R_{\mathcal{K}}}\}.$$

We will assume Lemma 3.3.7 for now and continue with the proof of Theorem 3.2.1.

Step 3: By Lemma 3.3.7, it will suffice to show that for any $\kappa, R > 0$ there exists $\varepsilon = \varepsilon(d, \kappa, R) > 0$ such that, for any $\mathbb{P} \in \mathcal{P}_\kappa$, if $\mathrm{dis}(\mathbb{P}) < \varepsilon$ then

$$I_a\big|_{\mathcal{O}_R} \equiv I_q\big|_{\mathcal{O}_R}, \tag{3.3.30}$$

where

$$\mathcal{O}_R := \pi^{-1}(\{\nabla \log \psi(\theta) : \theta \in D_R\}).$$

To this end, let us consider $\varepsilon = \varepsilon_{R+1} > 0$ depending only on d, κ and R such that, for any

$\mathbb{P} \in \mathcal{P}_\kappa$, if $\mathrm{dis}(\mathbb{P}) < \varepsilon$ there exists a full $\mathbb{P}$-probability event $\Omega_{R+1} = \Omega_{R+1}(\mathbb{P}, R)$ satisfying

$$\Lambda^\omega(\theta) = \log \psi(\theta) \tag{3.3.31}$$

for all $\theta \in D_{R+1}$ if $\omega \in \Omega_{R+1}$ (such an ε exists by Step 1). For the remaining steps of the proof, we fix an arbitrary $\mathbb{P} \in \mathcal{P}_\kappa$ satisfying $\mathrm{dis}(\mathbb{P}) < \varepsilon$ and proceed to show (3.3.30) for the RWRE having this environmental law $\mathbb{P}$.

By (3.3.31) and choice of ε, it follows that

$$\pi(\mathcal{O}_R) = \{\nabla \Lambda^\omega(\theta) : \theta \in D_R\}$$

for any $\omega \in \Omega_{R+1}$. By Lemma 3.3.6, it follows that for $\omega \in \Omega_{R+1}$ the sequence $(\overline{S}_n)_{n \in \mathbb{N}}$ under $P_{0,\omega}$ satisfies an LDP inside $\pi(\mathcal{O}_R)$ with rate function

$$\overline{\Lambda}(x) = \langle \theta_x, x \rangle - \log \psi(\theta_x), \tag{3.3.32}$$

where θ_x is defined via the relation $x = \nabla \log \psi(\theta_x)$ (observe that θ_x is well-defined for $x \in \pi(\mathcal{O}_R)$ by definition of $\mathcal{O}_R$). Here the LDP inside $\pi(\mathcal{O}_R)$ is interpreted as:

- For any closed set $F \subseteq \pi(\mathcal{O}_R)$,

$$\limsup_{n \to +\infty} \frac{1}{n} \log P_{0,\omega}(\{\overline{S}_n \in F\} \cap \mathbb{B}_n) \leq - \inf_{x \in F} \overline{\Lambda}(x).$$

- For any open set $G \subseteq \pi(\mathcal{O}_R)$,

$$\liminf_{n \to +\infty} \frac{1}{n} \log P_{0,\omega}(\{\overline{S}_n \in G\} \cap \mathbb{B}_n) \geq - \inf_{x \in G} \overline{\Lambda}(x),$$

where $\overline{\Lambda}(x)$ is given by (3.3.32). But an easy calculation exploiting the fact that π is affine and (3.3.4) shows that, for any set $H \subseteq \partial\mathbb{D}(s)$, we have

$$\{\bar{S}_n \in \pi(H)\} \cap \mathbb{B}_n = \{\tfrac{1}{n}X_n \in H\}, \tag{3.3.33}$$

which implies then that an LDP inside $\mathcal{O}_R$ holds for the distribution of $(\frac{1}{n}X_n)_{n\in\mathbb{N}}$ under $P_{0,\omega}$:

- For any closed set $F \subseteq \mathcal{O}_R$,

$$\limsup_{n\to+\infty} \frac{1}{n} \log P_{0,\omega}(\tfrac{1}{n}X_n \in F) \leq - \inf_{x\in F} \overline{\Lambda}(\pi(x)).$$

- For any open set $G \subseteq \mathcal{O}_R$,

$$\liminf_{n\to+\infty} \frac{1}{n} \log P_{0,\omega}(\tfrac{1}{n}X_n \in G) \geq - \inf_{x\in G} \overline{\Lambda}(\pi(x)),$$

where $\overline{\Lambda}$ is given by (3.3.32).

Step 4: Our next step will be to show that $\overline{\Lambda} \circ \pi \equiv I_q$ on $\mathcal{O}_R$. To this end, suppose first that $\overline{\Lambda}(\pi(x)) < I_q(x)$ for some $x \in \mathcal{O}_R$. By the lower semicontinuity of I_q we may find a neighborhood B of x such that $\inf_{y\in\overline{B}} I_q(y) > \overline{\Lambda}(\pi(x))$, where $\overline{B}$ denotes the closure of B. Observe that the set

$$G_x := \pi(B) \cap \{y \in \mathbb{R}^d : y_d = 0\}$$

is an open set in $\mathbb{R}^{d-1}$. Thus, by Lemma 3.3.6 and (3.3.33), for any $\omega \in \Omega_{R+1}$ we have

$$\begin{aligned} -\overline{\Lambda}(\pi(x)) = -\overline{\Lambda}^{\omega}(\pi(x)) &\leq - \inf_{y\in G_x\cap\mathcal{F}^{\omega}} \overline{\Lambda}^{\omega}(y) \\ &\leq \liminf_{n\to+\infty} \frac{1}{n} \log P_{0,\omega}(\tfrac{1}{n}X_n \in \pi^{-1}(G_x)), \end{aligned}$$

and

$$\liminf_{n\to+\infty}\frac{1}{n}\log P_{0,\omega}(\tfrac{1}{n}X_n\in\pi^{-1}(G_x))\leq\limsup_{n\to+\infty}\frac{1}{n}\log P_{0,\omega}(\tfrac{1}{n}X_n\in\overline{B})\leq-\inf_{y\in\overline{B}}I_q(y)<-\overline{\Lambda}(\pi(x)),$$

which is a contradiction. Thus, we must have $I_q(x)\leq\overline{\Lambda}(\pi(x))$ for all $x\in\mathcal{O}_R$.

On the other hand, if for each $x\in\mathcal{O}_R$ we choose an admissible sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $\frac{x_n}{n}\to x$ as $n\to+\infty$ as in the statement of Lemma 3.3.5. Then, by the aforementioned lemma, (3.3.33) and Lemma 3.3.6, for $\mathbb{P}$-almost every $\omega\in\Omega_{R+1}$ and $\delta>0$ we have

$$-I_q(x)=\lim_{n\to+\infty}\frac{1}{n}\log P_{0,\omega}(\tfrac{1}{n}X_n=\tfrac{1}{n}x_n)\leq\limsup_{n\to+\infty}\frac{1}{n}\log P_{0,\omega}(\tfrac{1}{n}X_n\in\overline{B_\delta(x)})\leq-\inf_{y\in\overline{B_\delta(x)}}\overline{\Lambda}^{\omega}(\pi(y)),$$

with the standard notation $B_\delta(x):=\{y\in\mathbb{R}^d:|y-x|_2<\delta\}$. By the lower semicontinuity of $\overline{\Lambda}^{\omega}$, letting $\delta\to 0$ in the inequality above yields that

$$-I_q(x)\leq-\overline{\Lambda}^{\omega}(\pi(x))=-\overline{\Lambda}(\pi(x)),$$

the last equality being true by (3.3.31) because $x\in\mathcal{O}_R$. Hence, we see that

$$\overline{\Lambda}(\pi(x))\leq I_q(x)\qquad\forall\,x\in\mathcal{O}_R$$

and therefore, since the reverse inequality is also true, we conclude that $I_q\equiv\overline{\Lambda}\circ\pi$ on $\mathcal{O}_R$.

Step 5: Finally, a similar analysis but for the annealed measure now reveals that $I_a\equiv\overline{\Lambda}\circ\pi$ on $\mathcal{O}_R$ as well. Indeed, the key observation to achieve this is that, by the analogue of Lemma 3.3.6 for the annealed measure (recall Remark 3.3.1) the sequence $(\overline{S}_n)_{n\in\mathbb{N}}$ under P_0 satisfies an LDP inside $\pi(\mathcal{O}_R)$ with rate function exactly as in (3.3.32). From here we immediately obtain (3.3.30). Thus, for the proof of Theorem 3.2.1 we only owe the reader the proof of Lemma 3.3.7 as well as Lemma 3.3.5.

Proof of Lemma 3.3.7: Fix any environment law $\mathbb{P}$ satisfying Assumption A with ellipticity constant $\kappa > 0$. Since $\mathscr{Z}_\theta(0)$ is mean-one $\mathbb{P}$-martingale, it follows that

$$E_0(\mathrm{e}^{\langle \theta, S_1\rangle - \log \psi(\theta)} \mathbb{1}_{\mathbb{B}_n}) = 1.$$

From this identity, the methods from [BMRS1, Section 4] now show that the mapping $\theta \mapsto \log \psi(\theta)$ is smooth and has a positive definite Hessian. In particular, it is a smooth strictly convex function on $\mathbb{R}^{d-1}$, so that by [Roc, Theorem 26.5] the sets

$$G_R := \{\nabla \log \psi(\theta) : |\theta|_1 < R\}$$

are open on $\mathbb{R}^{d-1}$ for all $R > 0$.

Therefore, in order to prove the lemma it will be enough to show that for each $x \in \partial\mathbb{D}(s) \setminus \partial\mathbb{D}_{d-2}$ there exist $r_x = r_x(d, \kappa, x), R_x = R_x(d, \kappa, x) > 0$ such that, for any $\mathbb{P} \in \mathcal{P}_\kappa$, we have

$$\pi(B_{r_x}(x) \cap (\partial\mathbb{D}(s) \setminus \partial\mathbb{D}_{d-2})) \in G_{R_x} = \{\nabla \log \psi(\theta) : |\theta|_1 < R_x\}, \tag{3.3.34}$$

where we write $B_{r_x}(x) := \{y \in \mathbb{R}^d : |y - x|_1 < r_x\}$. Indeed, if this is the case then, given any compact set $\mathcal{K} \subseteq \partial\mathbb{D}(s) \setminus \partial\mathbb{D}_{d-2}$, there exists some finite $n_{\mathcal{K}} = n_{\mathcal{K}}(d, \kappa, \mathcal{K}) \geq 1$ and $x_1, \dots, x_{n_{\mathcal{K}}} \in \mathcal{K}$ such that

$$\mathcal{K} \subseteq \bigcup_{j=1}^{n_{\mathcal{K}}} \big(B_{r_{x_j}}(x_j) \cap (\partial\mathbb{D}(s) \setminus \partial\mathbb{D}_{d-2})\big),$$

so that by (3.3.34), if we set $R_{\mathcal{K}} := \max_{j=1,\dots,n_{\mathcal{K}}} r_{x_j} < \infty$ then, for any $\mathbb{P} \in \mathcal{P}_\kappa$, we obtain that

$$\pi(\mathcal{K}) \subseteq G_{R_{\mathcal{K}}}.$$

Hence, we only need to show (3.3.34).

To this end, notice that any $x \in \partial\mathbb{D}(s) \setminus \partial\mathbb{D}_{d-2}$ can be written as

$$x = \sum_{i=1}^{d} \delta_i s_i e_i$$

where $\delta_i > 0$ for all $i = 1, \dots, d$ and $\sum_{i=1} \delta_i = 1$. Since π is affine, it follows that

$$\pi(x) = \sum_{i=1}^{d} \delta_i \pi(s_i e_i) = \sum_{i=1}^{d-1} (\delta_i - \delta_d) e_i.$$

On the other hand, a simple computation shows that for any $\theta \in \mathbb{R}^{d-1}$,

$$\nabla \log \psi(\theta) = \frac{1}{\psi(\theta)} \sum_{i=1}^{d-1} [\alpha(s_i e_i) \mathrm{e}^{\theta_i} - \alpha(s_d e_d) \mathrm{e}^{-(\theta_1 + \cdots + \theta_{d-1})}] e_i.$$

Therefore, in order to check that $\pi(x) \in G_R$ for some $R > 0$, we only need to show that there exists some $\theta(x) = (\theta_1(x), \dots, \theta_{d-1}(x)) \in \mathbb{R}^{d-1}$ such that

$$\delta_i - \delta_d = \frac{1}{\psi(\theta)} \left[\alpha(s_i e_i) \mathrm{e}^{\theta_i(x)} - \alpha(s_d e_d) \mathrm{e}^{-(\theta_1(x) + \cdots + \theta_{d-1}(x))} \right] \tag{3.3.35}$$

for all $i = 1, \dots, d-1$. But it is straightforward to check that, for $\theta(x)$ given by

$$\theta_i(x) := \log \left(\frac{\delta_i C}{\alpha(s_i e_i)} \right) \qquad \text{with} \qquad C := \sqrt[d]{\prod_{i=1}^{d} \frac{\alpha(s_i e_i)}{\delta_i}}$$

for each $i = 1, \dots, d-1$, (3.3.35) is satisfied and so $\pi(x) = \nabla \log \psi(\theta(x))$. Finally, since the mapping

$$(\alpha, x) \mapsto \theta(x)$$

is continuous on $\mathcal{M}_1^{(\kappa)}(\mathbb{V}) \times (\partial\mathbb{D}(s) \setminus \partial\mathbb{D}_{d-2})$, (3.3.34) follows upon taking $r_x := \frac{1}{2} d(x, \partial\mathbb{D}_{d-2}) >$

0 and $R_x := 1 + \sup\{|\theta(y)|_1 : y \in B_{r_x}(x) \cap (\partial\mathbb{D}(s) \setminus \partial\mathbb{D}_{d-2})\} < \infty$. This concludes the proof. $\square$

Remark 3.3.2. Equation (3.3.34) implies that, for any $\mathbb{P} \in \mathcal{P}_\kappa$,

$$\pi(\partial\mathbb{D}(s) \setminus \partial\mathbb{D}_{d-2}) \subseteq \{\nabla \log \psi(\theta) : \theta \in \mathbb{R}^{d-1}\}.$$

From this, we may now repeat the analysis of Step 5 to show that $I_a \equiv \overline{\Lambda} \circ \pi$ holds on $\partial\mathbb{D}(s) \setminus \partial\mathbb{D}_{d-2}$. We will later need this fact for the proof of Theorem 3.2.2.

Proof of Lemma 3.3.5:

We consider the quenched and annealed limits separately.

Case 1: the quenched limit. This is consequence of several results found in [RAS3]. Indeed, in [RAS3, Theorem 2.2] it is proved that $\mathbb{P}$-almost surely for all $x \in \mathbb{D}$, the following limit exists

$$\widehat{I}_q(x) := -\lim_{n\to\infty} \frac{1}{n} \log P_{0,\omega}(X_n = x_n) \in [0, \kappa]$$

for a suitable admissible sequence $\{x_n\}_{n\in\mathbb{N}}$ satisfying $\frac{x_n}{n} \to x$. Moreover, by [RAS3, Theorem 2.4] this limit $\widehat{I}_q(x)$ is deterministic and, by [RAS3, Theorem 3.2-(b)], the map $x \to \widehat{I}_q(x)$ is continuous on $\mathbb{D}$. Finally, [RAS3, Theorem 4.3] shows that $I_q \equiv \widehat{I}_q$ on $\mathrm{int}(\mathbb{D})$. The continuity of both I_q and $\widehat{I}_q$ now allow us to extend the equality to the boundary $\partial\mathbb{D}$, thus proving the quenched case.

Case 2: the annealed limit. First, given $x = (x_1, \ldots, x_d) \in \partial\mathbb{D}(s)$, let us write it as $x = \sum_{i=1}^d s_i |x_i| e_i$. Now, consider any admissible sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Z}^d$ such that:

- $\frac{x_n}{n} \in \partial\mathbb{D}(s)$ for each n, i.e. $x_n = \sum_{i=1}^d s_i n_i e_i$ for some $n_i \geq 0$ with $\sum_{i=1}^d n_i = n$.
- If $x_i = 0$ then $n_i = 0$.

- $\frac{x_n}{n} \to x$ as $n \to \infty$.

It is straightforward to check that such a sequence always exists, see [RAS3] for details.

Observe that, for any such sequence, by (3.3.14) the quantity $\prod_{j=1}^n \alpha(\Delta_j(z))$ is independent of the path z of length n going from 0 to x_n, so that

$$P_0(X_n = x_n) = \#\{z \in \mathcal{R}_n : z_n = x_n\} \prod_{i=1}^d \alpha(s_i e_i)^{n_i} = \frac{n!}{n_1! \cdots n_d!} \prod_{i=1}^d \alpha(s_i e_i)^{n_i}.$$

Taking logarithm and dividing by n, we get

$$\frac{1}{n} \log P_0(X_n = x_n) = \frac{1}{n} \left[\log n! - \sum_{i=1}^d \log n_i! + \sum_{i=1}^d n_i \log \alpha(s_i e_i) \right].$$

Now, since $\frac{x_n}{n} \to x$, we obtain that $\frac{n_i}{n} \to |x_i|$ for all i and thus that as $n \to \infty$,

$$\frac{1}{n} \sum_{i=1}^d n_i \log \alpha(s_i e_i) \to \sum_{i=1}^d |x_i| \log \alpha(s_i e_i).$$

On the other hand, since $\frac{n_i}{n} \to |x_i|$, by Stirling's approximation we have $\log n! = n \log n - n + o(n)$ and $\log n_i! = n_i \log n_i - n_i + o(n)$ (if $x_i = 0$ for some i, the equality still holds since $n_i = 0$), so that

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \left[\log(n!) - \sum_{i=1}^d \log(n_i!) \right] &= \lim_{n\to\infty} \frac{1}{n} \left[n \log n - n - \left(\sum_{i=1}^d n_i \log n_i - n_i \right) \right] \\
&= -\lim_{n\to\infty} \sum_{i=1}^d \frac{n_i}{n} \log \frac{n_i}{n} \\
&= -\sum_{i=1}^d |x_i| \log |x_i|.
\end{aligned}$$

Therefore, we conclude that

$$-\widehat{I}_a(x) := \lim_{n\to\infty} \frac{1}{n} P_0(X_n = x_n) = -\sum_{i=1}^{d} |x_i| \log \frac{|x_i|}{\alpha(s_i e_i)}.$$

To conclude the proof, we must now check that $\widehat{I}_a(x) = I_a(x)$. To this end, define

$$\widetilde{I}_a(x) := \sup_{\theta\in\mathbb{R}^d} (\langle \theta, x\rangle - \log\lambda(\theta)) \tag{3.3.36}$$

for λ as in (3.2.8). It is straightforward to check that $\widetilde{I}_a$ is the annealed rate function corresponding to a random walk $(Y_n)_{n\in\mathbb{N}}$ in a space-time random environment $\bar{\omega} = (\bar{\omega}(n,\cdot))_{n\in\mathbb{N}}$, where the $\bar{\omega}(n,\cdot)$ are i.i.d. having common law $\mathbb{P}$. Furthermore, by standard considerations of Fenchel-Legendre transforms (see Lemma 3.3.6, for instance), it is straightforward to check that for all $x \in \partial\mathbb{D}(s)$ the supremum in (3.3.36) coincides with the expression derived for $\widehat{I}_a(x)$, so that $\widehat{I}_a(x) = \widetilde{I}_a(x)$. Thus, in order to conclude the proof, it will suffice to show that

$$\widetilde{I}_a(x) \le I_a(x) \le \widehat{I}_a(x). \tag{3.3.37}$$

To check the right inequality in (3.3.37) we observe that, by the annealed LDP for the random walk and the fact that $\frac{x_n}{x} \to x$, for any $\delta > 0$ we have

$$-\widehat{I}_a(x) \le \limsup_{n\to\infty} \frac{1}{n} \log P_0(\tfrac{1}{n} X_n \in \overline{B_\delta(x)}) \le -\inf_{y\in\overline{B_\delta(x)}} I_a(y), \tag{3.3.38}$$

where $B_\delta(x) := \{y \in \mathbb{R}^d : |y-x|_2 < \delta\}$. By the lower semicontinuity of I_a, taking $\delta \to 0$ in (3.3.38) then yields the right inequality in (3.3.37).

On the other hand, if Q_0 denotes the law of the random walk $(Y_n)_{n\in\mathbb{N}}$ in a space-time environment introduced previously starting from 0, then for any $\delta > 0$ we have

$$P_0(\tfrac{1}{n}X_n \in B_\delta(x)) \leq \kappa^{-n\delta} Q_0(\tfrac{1}{n}Y_n \in B_\delta(x)). \tag{3.3.39}$$

Indeed, notice that

$$\begin{aligned} P_0(\tfrac{1}{n}X_n \in B_\delta(x)) &= \sum_{z \in \mathcal{R}_n \,:\, \frac{z_n}{n} \in B_\delta(x)} \mathbb{E}\left(\prod_{j=1}^n \omega(z_{j-1}, \Delta_j(z)) \right) \\ &\leq \kappa^{-n\delta} \sum_{z \in \mathcal{R}_n \,:\, \frac{z_n}{n} \in B_\delta(x)} \prod_{j=1}^n \alpha(\Delta_j(z)) = \kappa^{-n\delta} Q_0(\tfrac{1}{n}Y_n \in B_\delta(x)), \end{aligned}$$

where the middle equality follows from the fact that the factors in the product $\prod_{j=1}^n \omega(z_{j-1}, \Delta_j(z))$ are all independent except for at most $n\delta$ of them, but we can majorize these by independent versions at the expense of an additional κ^{-1} factor. It follows from (3.3.39) that

$$\inf_{y \in B_\delta(x)} I_a(y) \geq \inf_{y \in \overline{B_\delta(x)}} \widetilde{I}_a(y) - \delta \log k^{-1}.$$

By the lower semicontinuity of both I_a and $\widetilde{I}_a$, letting $\delta \to 0$ in the last display above reveals that $\widetilde{I}_a(x) \leq I_a(x)$ and thus (3.3.37) is proved. □

Remark 3.3.3. In [RAS3, Theorem 4.3] (see also [CDR$^+$]) it is shown that the sequence $(\frac{1}{n}X_n)_{n\in\mathbb{N}}$ satisfies a quenched LDP on $\partial\mathbb{D}$ with rate function $\widehat{I}_q$. Using this and Case 2 of Lemma 3.3.5, the analysis carried out in Section 3.3.4 (in particular, in Steps 4 and 5) already shows that for dis($\mathbb{P}$) sufficiently small one has $\widehat{I}_q \equiv I_a$ on the boundary $\partial\mathbb{D}$. Some additional effort is required to show that $\widehat{I}_q \equiv I_q$ and thus conclude the result in Theorem 3.2.1, but this is given by the other results from [RAS3] as shown in Case 1 of Lemma 3.3.5.

3.4 Proof of Theorem 3.2.2

Note that the proof of Theorem 3.2.1 in Section 3.3 already reveals that, in order to prove Theorem 3.2.2, it suffices to check that there exist $\varepsilon^\star = \varepsilon^\star(d,\kappa) > 0$ such that, whenever $\mathrm{imb}_s(\mathbb{P})$ is small enough, there exists some $\eta = \eta(d,\kappa) > 0$ such that for each $|\theta|_1 \le \eta$ we have

$$\sup_{n\ge 1} \|\mathscr{Z}_{n,\theta}\|^2_{L^2(\mathbb{P})} < \infty. \tag{3.4.1}$$

The above estimate together with arguments similar to those given for the proof of Theorem 3.2.1 will then imply the desired equality of the rate functions on an open subset of $\partial\mathbb{D}(s) \setminus \partial\mathbb{D}_{d-2}$.

For $x \in \mathbb{Z}^d$, $e \in \mathbb{V}$ and $\theta \in \mathbb{R}^{d-1}$, define

$$W(x,e,\theta) := \omega(x,e)\,\mathrm{e}^{\langle\theta,\pi(e)\rangle}$$

and

$$W_s(x,\theta) := \sum_{e\in\mathbb{V}(s)} W(x,e,\theta) = \sum_{e\in\mathbb{V}(s)} \omega(x,e)\,\mathrm{e}^{\langle\theta,\pi(e)\rangle},$$

where π is the affine mapping from (3.3.6) and we use the identification $\pi(e) \in \mathbb{R}^{d-1}$ for $e \in \mathbb{V}(s)$. Note that, since $\mathbb{E}(W_s(x,\theta)) = \psi(\theta)$ for any $x \in \mathbb{Z}^d$ and, moreover, $\mathbb{P}$-almost surely for all $x \in \mathbb{Z}^d$

$$W_s(x,\theta) \le \mathrm{e}^{(d-1)|\theta|_1}\psi(0)(1+\mathrm{imb}_s(\mathbb{P})) \le \mathrm{e}^{2(d-1)|\theta|_1}\psi(\theta)(1+\mathrm{imb}_s(\mathbb{P})),$$

we have (recall the definition of $\Delta_j(z)$ from (3.3.3) and $\alpha^{(\theta)}$ from (3.3.16)),

$$\begin{aligned}
&\|\mathscr{Z}_{n,\theta}\|^2_{L^2(\mathbb{P})} \\
&\quad = \sum_{z,z'\in\mathcal{R}_n}\prod_{j=1}^{n}\left[\frac{\mathbb{E}\big(W(z_{j-1},\Delta_j(z),\theta)W(z'_{j-1},\Delta_j(z'),\theta)\big)}{\psi^2(\theta)}\right] \\
&\quad = \sum_{z,z'\in\mathcal{R}_{n-1}}\frac{\mathbb{E}(W_s(z_{n-1},\theta)W_s(z'_{n-1},\theta))}{\psi^2(\theta)}\prod_{j=1}^{n-1}\left[\frac{\mathbb{E}\big(W(z_{j-1},\Delta_j(z),\theta)W(z'_{j-1},\Delta_j(z'),\theta)\big)}{\psi^2(\theta)}\right] \\
&\quad \leq \sum_{z,z'\in\mathcal{R}_{n-1}}\mathrm{e}^{\mathscr{V}^{(0)}_{s,\theta}\mathbb{1}_{\{z_{n-1}=z'_{n-1}\}}}\prod_{j=1}^{n-1}\left[\frac{\mathbb{E}\big(W(z_{j-1},\Delta_j(z),\theta)W(z'_{j-1},\Delta_j(z'),\theta)\big)}{\psi^2(\theta)}\right] \\
&\quad = \sum_{z,z'\in\mathcal{R}_n}\alpha^{(\theta)}(\Delta_n(z))\alpha^{(\theta)}(\Delta_n(z'))\mathrm{e}^{\mathscr{V}^{(0)}_{s,\theta}\mathbb{1}_{\{z_{n-1}=z'_{n-1}\}}}\prod_{j=1}^{n-1}\left[\frac{\mathbb{E}\big(W(z_{j-1},\Delta_j(z),\theta)W(z'_{j-1},\Delta_j(z'),\theta)\big)}{\psi^2(\theta)}\right],
\end{aligned} \tag{3.4.2}$$

where

$$\mathscr{V}^{(0)}_{s,\theta} := 2(d-1)|\theta|_1 + \log\left(1+\mathrm{imb}_s(\mathbb{P})\right).$$

We will now continue with an estimate for the sum over z_{n-1} and z'_{n-1}. First, note that whenever $z_{n-2}\neq z'_{n-2}$ we have

$$\begin{aligned}
&\sum_{\Delta_{n-1}(z),\Delta_{n-1}(z')\in\mathbb{V}(s)}\mathrm{e}^{\mathscr{V}^{(0)}_{s,\theta}\mathbb{1}_{\{z_{n-1}=z'_{n-1}\}}}\left[\frac{\mathbb{E}\big(W(z_{n-2},\Delta_{n-1}(z),\theta)W(z'_{n-2},\Delta_{n-1}(z'),\theta)\big)}{\psi^2(\theta)}\right] \\
&\quad = \sum_{\Delta_{n-1}(z),\Delta_{n-1}(z')\in\mathbb{V}(s)}\alpha^{(\theta)}(\Delta_{n-1}(z))\alpha^{(\theta)}(\Delta_{n-1}(z'))\mathrm{e}^{\mathscr{V}^{(0)}_{s,\theta}\mathbb{1}_{\{z_{n-1}=z'_{n-1}\}}}.
\end{aligned} \tag{3.4.3}$$

Next, we claim that if $\mathrm{imb}_s(\mathbb{P}) < (d-2)\kappa$ then $\mathbb{P}$-almost surely for all $x\in\mathbb{Z}^d$ and $e\in\mathbb{V}(s)$,

$$\omega(x,e) \leq (1-\kappa)\psi(0). \tag{3.4.4}$$

Indeed, if (3.4.4) is not satisfied for some $x'\in\mathbb{Z}^d$ and $e'\in\mathbb{V}(s)$ then, on a set of positive $\mathbb{P}$-measure we have that

$$\omega(x',e') > \psi(0) - (d-2)\kappa.$$

Hence, by uniform ellipticity and the trivial bound $\psi(0) \le 1$, we have on a set of positive $\mathbb{P}$-measure,

$$W_s(x',0) = \omega(x',e') + \sum_{e'\neq e\in\mathbb{V}(s)} \omega(x',e) > (1-\kappa)\psi(0) + (d-1)\kappa \ge (1+(d-2)\kappa)\psi(0)$$

which implies that $\mathrm{imb}_s(\mathbb{P}) > (d-2)\kappa$ and thus contradicts our assumptions. Hence, we conclude that, whenever $\mathrm{imb}_s(\mathbb{P}) < (d-2)\kappa$, (3.4.4) holds and thus that $\mathbb{P}$-a.s. for all $x \in \mathbb{Z}^d$ and $e \in \mathbb{V}(s)$,

$$\omega(x,e) \le (1-\kappa)\psi(0) \le \mathrm{e}^{(d-1)|\theta|_1}(1-\kappa)\psi(\theta). \tag{3.4.5}$$

Now, whenever $z_{n-2} = z'_{n-2}$, using (3.4.5) we have

$$\begin{aligned}
&\sum_{\Delta_{n-1}(z),\Delta_{n-1}(z')\in\mathbb{V}(s)} \mathrm{e}^{\mathscr{V}^{(0)}_{s,\theta}\mathbb{1}_{\{z_{n-1}=z'_{n-1}\}}} \left[\frac{\mathbb{E}\big(W(z_{n-2},\Delta_{n-1}(z),\theta)W(z'_{n-2},\Delta_{n-1}(z'),\theta)\big)}{\psi^2(\theta)}\right] \\
&= \sum_{\Delta_{n-1}(z),\Delta_{n-1}(z')\in\mathbb{V}(s)} \left[\frac{\mathbb{E}\big(W(z_{n-2},\Delta_{n-1}(z),\theta)W(z_{n-2},\Delta_{n-1}(z'),\theta)\big)}{\psi^2(\theta)}\right] \\
&\quad + \sum_{\Delta_{n-1}(z),\Delta_{n-1}(z')\in\mathbb{V}(s)} \left[\mathrm{e}^{\mathscr{V}^{(0)}_{s,\theta}\mathbb{1}_{\{z_{n-1}=z'_{n-1}\}}} - 1\right] \left[\frac{\mathbb{E}\big(W(z_{n-2},\Delta_{n-1}(z),\theta)W(z_{n-2},\Delta_{n-1}(z'),\theta)\big)}{\psi^2(\theta)}\right] \\
&= \frac{\mathbb{E}\left(W_s^2(z_{n-2},\theta)\right)}{\psi^2(\theta)} + \sum_{\Delta_{n-1}(z)\in\mathbb{V}(s)} \left(\mathrm{e}^{\mathscr{V}^{(0)}_{s,\theta}} - 1\right) \frac{\mathbb{E}\left(W^2(z_{n-2},\Delta_{n-1}(z),\theta)\right)}{\psi^2(\theta)} \\
&\le \mathrm{e}^{\mathscr{V}^{(0)}_{s,\theta}} + \big(\mathrm{e}^{\mathscr{V}^{(0)}_{s,\theta}} - 1\big)K_{\kappa,\theta} \\
&= \sum_{\Delta_{n-1}(z),\Delta_{n-1}(z')\in\mathbb{V}(s)} \alpha^{(\theta)}(\Delta_{n-1}(z))\alpha^{(\theta)}(\Delta_{n-1}(z'))\mathrm{e}^{\mathscr{V}^{(1)}_{\kappa,s,\theta}} \\
&\le \sum_{\Delta_{n-1}(z),\Delta_{n-1}(z')\in\mathbb{V}(s)} \alpha^{(\theta)}(\Delta_{n-1}(z))\alpha^{(\theta)}(\Delta_{n-1}(z'))\mathrm{e}^{\mathscr{V}^{(1)}_{\kappa,s,\theta}+\mathscr{V}^{(0)}_{s,\theta}\mathbb{1}_{\{z_{n-1}=z'_{n-1}\}}},
\end{aligned} \tag{3.4.6}$$

where

$$K_{\kappa,\theta} := \mathrm{e}^{(d-1)|\theta|_1}(1-\kappa) \qquad \text{and} \qquad \mathrm{e}^{\mathscr{V}^{(1)}_{\kappa,s,\theta}} := \mathrm{e}^{\mathscr{V}^{(0)}_{s,\theta}} + \big(\mathrm{e}^{\mathscr{V}^{(0)}_{s,\theta}} - 1\big)K_{\kappa,\theta}.$$

Combining (3.4.3) with (3.4.6) we see that

$$\sum_{\Delta_{n-1}(z),\Delta_{n-1}(z')\in\mathbb{V}(s)} \mathrm{e}^{\mathscr{V}^{(0)}_{\bar{\varepsilon},\varepsilon}\mathbb{1}_{\{z_{n-1}=z'_{n-1}\}}}\left[\frac{\mathbb{E}\big(W(z_{n-2},\Delta_{n-1}(z),\theta)W(z'_{n-2},\Delta_{n-1}(z'),\theta)\big)}{\psi^2(\theta)}\right]$$
$$\leq \sum_{\Delta_{n-1}(z),\Delta_{n-1}(z')\in\mathbb{V}(s)} \alpha^{(\theta)}(\Delta_{n-1}(z))\alpha^{(\theta)}(\Delta_{n-1}(z'))\exp\left\{\mathscr{V}^{(1)}_{\kappa,s,\theta}\mathbb{1}_{\{z_{n-2}=z'_{n-2}\}}+\mathscr{V}^{(0)}_{s,\theta}\mathbb{1}_{\{z_{n-1}=z'_{n-1}\}}\right\}. \tag{3.4.7}$$

From the above estimate and (3.4.2), we conclude that

$$\|\mathscr{Z}_{n,\theta}\|^2_{L^2(\mathbb{P})} \leq \sum_{z,z'\in\mathcal{R}_n} \mathrm{e}^{\mathscr{V}^{(1)}_{\kappa,s,\theta}\mathbb{1}_{\{z_{n-2}=z'_{n-2}\}}+\mathscr{V}^{(0)}_{s,\theta}\mathbb{1}_{\{z_{n-1}=z'_{n-1}\}}}$$
$$\times\prod_{j=1}^{n-2}\left[\frac{\mathbb{E}\big(W(z_{j-1},\Delta_j(z),\theta)W(z'_{j-1},\Delta_j(z'),\theta)\big)}{\psi^2(\theta)}\right]\prod_{j=n-1}^{n}\alpha^{(\theta)}(\Delta_j(z))\alpha^{(\theta)}(\Delta_j(z')). \tag{3.4.8}$$

By successive application of the above estimate, we get

$$\|\mathscr{Z}_{n,\theta}\|^2_{L^2(\mathbb{P})} \leq E_0^{(\theta)}\left[\exp\left(\sum_{j=0}^{n-1}\mathscr{V}^{(n-1-k)}_{\kappa,s,\theta}\mathbb{1}_{\{X_j^{(\theta)}=Y_j^{(\theta)}\}}\right)\right],$$

where $X^{(\theta)}$ and $Y^{(\theta)}$ are as before two independent random walks starting from 0 with jump distribution given by the probability vector $\vec{\alpha}^{(\theta)}$, we write $\mathscr{V}^{(0)}_{\kappa,s,\theta} := \mathscr{V}^{(0)}_{\bar{\varepsilon},\theta}$ for homogeneity of notation and, for $0\leq k\leq n-1$, we define

$$\mathrm{e}^{\mathscr{V}^{(k+1)}_{\kappa,s,\theta}} := \mathrm{e}^{\mathscr{V}^{(0)}_{\kappa,s,\theta}} + \left(\mathrm{e}^{\mathscr{V}^{(k)}_{\kappa,s,\theta}}-1\right)K_{\kappa,\theta}.$$

Now, since $K_{\kappa,\theta}<1$ for $|\theta|_1$ small enough (depending only on d and κ), for any such θ we have

$$\mathrm{e}^{\mathscr{V}^{(k+1)}_{\kappa,s,\theta}} = \mathrm{e}^{\mathscr{V}^{(0)}_{\kappa,s,\theta}} + \left(\mathrm{e}^{\mathscr{V}^{(0)}_{\kappa,s,\theta}}-1\right)(K_{\kappa,\theta}+K^2_{\kappa,\theta}+\cdots+K^{k+1}_{\kappa,\theta}) \leq \mathrm{e}^{\mathscr{V}^{(0)}_{\kappa,s,\theta}} + \left(\mathrm{e}^{\mathscr{V}^{(0)}_{\kappa,s,\theta}}-1\right)\frac{K_{\kappa,\theta}}{1-K_{\kappa,\theta}}.$$

Hence, we can define $\mathscr{V}^{(\infty)}_{\kappa,s,\theta}$ by the formula

$$\mathrm{e}^{\mathscr{V}^{(\infty)}_{\kappa,s,\theta}} := \mathrm{e}^{\mathscr{V}^{(0)}_{\kappa,s,\theta}} + \left(\mathrm{e}^{\mathscr{V}^{(0)}_{\kappa,s,\theta}} - 1\right)\frac{K_{\kappa,\theta}}{1-K_{\kappa,\theta}},$$

and conclude that

$$\|\mathscr{Z}_{n,\theta}\|^2_{L^2(\mathbb{P})} \leq E^{(\theta)}_0\left[\exp\left(\mathscr{V}^{(\infty)}_{\kappa,s,\theta}\sum_{j=0}^{n-1}\mathbb{1}_{\{Z^{(\theta)}_j=0\}}\right)\right] \tag{3.4.9}$$

where $Z^{(\theta)}_j = X^{(\theta)}_j - Y^{(\theta)}_j$. Moreover, since for $|\theta|_1 \leq \eta_1(d,\kappa)$ we have $K_{\kappa,\theta} \leq 1-\frac{\kappa}{2}$ and $2(d-1)|\theta|_1 \leq 1$, a straightforward calculation yields that

$$\mathscr{V}^{(\infty)}_{\kappa,s,\theta} \leq C_1(|\theta|_1 + \mathrm{imb}_s(\mathbb{P}))(1+\mathrm{imb}_s(\mathbb{P})) \tag{3.4.10}$$

for some constant $C_1 = C_1(d,\kappa) > 0$.

Now, by (3.3.20) there exists $C_0 = C_0(d,\kappa) > 0$ such that

$$\sup_{|\theta|_1\leq 1}\sum_{j=0}^{\infty} P_0\big(Z^{(\theta)}_j = 0\big) \leq C_0. \tag{3.4.11}$$

It then follows from (3.4.10) that there exist $\eta_2 = \eta_2(d,\kappa) \in (0,\eta_1)$ and $\varepsilon' = \varepsilon'(d,\kappa) > 0$ such that, for any $\mathbb{P} \in \mathcal{P}_\kappa$, if $\mathrm{imb}_s(\mathbb{P}) < \varepsilon'$ then $\sup_{|\theta|_1<\eta_2}\mathscr{V}^{(\infty)}_{\kappa,s,\theta} < C_0^{-1}$ which, by Lemma 3.3.2 and (3.4.11), implies

$$\sup_{|\theta|_1<\eta_2\,,\,n\geq 0}\|\mathscr{Z}_{n,\theta}\|^2_{L^2(\mathbb{P})} < \infty.$$

The rest of the proof of (3.2.7) now follows the same line of arguments as that of Theorem 3.2.1. In the end, we obtain that there exist $\eta = \eta(d,\kappa), \varepsilon^* = \varepsilon^\star(d,\kappa) > 0$ such that, for any $\mathbb{P} \in \mathcal{P}_\kappa$, if $\mathrm{imb}_s(\mathbb{P}) < \varepsilon^\star$ then $I_q(x) = I_a(x)$ for all $x \in \mathcal{O}$, where $\mathcal{O} \subseteq \partial\mathbb{D}(s) \setminus \partial\mathbb{D}_{d-2}$ is the open set given by

$$\mathcal{O} := \pi^{-1}(\{\nabla \log\psi(\theta) : |\theta|_1 < \eta\}) \cap (\partial\mathbb{D}(s) \setminus \partial\mathbb{D}_{d-2}). \tag{3.4.12}$$

Since the mapping $(\alpha, \theta) \mapsto \text{Hessian}(\log \psi(\theta))$ is continuous on $\mathcal{M}^{(\kappa)}(\mathbb{V}) \times \mathbb{R}^{d-1}$, by [BMRS1, Theorem 4.5] (see also the proof of [BMRS1, Lemma 4.8]) there exists $r = r(d, \kappa) > 0$ such that, for any $\mathbb{P} \in \mathcal{P}_\kappa$,

$$B_r(\nabla \log \psi(0)) \subseteq \{\nabla \log \psi(\theta) : |\theta|_1 < \eta\}.$$

From this, standard properties of affine transformations show that there exists some $c > 0$ depending only on the transformation π such that

$$B_{cr}(\overline{x}_s) \cap (\partial\mathbb{D}(s) \setminus \partial\mathbb{D}_{d-2}) \subseteq \mathcal{O},$$

for $\overline{x}_s$ defined as

$$\overline{x}_s := \pi^{-1}(\nabla \log \psi(0)) = \frac{1}{\psi(0)} \sum_{i=1}^{d} \alpha(s_i e_i) s_i e_i \in \partial\mathbb{D}(s) \setminus \partial\mathbb{D}_{d-2}. \tag{3.4.13}$$

Finally, to check (3.2.6) we first observe that $I_a \leq I_q$ by Jensen's inequality and Fatou's lemma (or Lemma 3.3.5), so that it will be enough to show that $I_q(x_0) = \min_{x \in \partial\mathbb{D}(s)} I_a(x)$ for some $x_0 \in \partial\mathbb{D}(s)$. Now, by Lemma 3.3.6 and Remarks 3.3.1-3.3.2 we have that $\min_{x \in \partial\mathbb{D}(s)} I_a(x) = I_a(\overline{x}_s)$ for $\overline{x}_s$ as in (3.4.13). Since $\overline{x}_s$ belongs to the set $\mathcal{O}$ in (3.4.12), we see that $I_q(\overline{x}_s) = I_a(\overline{x}_s) = \min_{x \in \partial\mathbb{D}(s)} I_a(x)$ and so (3.2.6) now follows.

3.5 Proofs of Theorem 3.2.3-Theorem 3.2.4

Proof of Theorem 3.2.3. That $I_a(x) = \sup_{\theta \in \mathbb{R}^d}[\langle \theta, x \rangle - \log \lambda(\theta)]$ has been shown already in Case 2 of the proof of Lemma 3.3.5. Theorem 3.2.2-Theorem 3.2.1 then imply the desired identity for I_q. □

Proof of Theorem 3.2.4. Recall that in this context the environments admit the representation

$$\omega_\varepsilon(x,e) := \alpha(e)(1+\varepsilon\eta(x,e)),$$

for $\varepsilon \in [0,1)$ and $\{\eta(x,\cdot)\}_{x\in\mathbb{Z}^d}$ an i.i.d. family of mean-zero random vectors on Γ_α. To emphasize the dependence on the disorder parameter, we henceforth write $I_q(\cdot,\varepsilon)$ and $I_a(\cdot,\varepsilon)$ respectively for the quenched and annealed large deviation rate functions of the random walk in the environment ω_ε. For $x \in \partial\mathbb{D}\setminus\partial\mathbb{D}_{d-2}$ define

$$\varepsilon_c := \sup\{\varepsilon \in [0,1) : I_q(x,\varepsilon) = I_a(x,\varepsilon)\}. \tag{3.5.1}$$

Note that we always have $I_q(\cdot,0) \equiv I_a(\cdot,0)$ since ω_0 is non-random, so that the set in (3.5.1) is always nonempty. Furthermore, by Theorem 3.2.1 we have that $I_q(x,\varepsilon) = I_a(x,\varepsilon)$ for all ε sufficiently small, so that in fact $\varepsilon_c(x) > 0$ for all $x \in \partial\mathbb{D}\setminus\partial\mathbb{D}_{d-2}$. Assuming that the mapping $\varepsilon \mapsto I_a(\cdot,\varepsilon) - I_q(\cdot,\varepsilon)$ is monotone for the moment, let us deduce (3.2.12).

Proof of (3.2.12)**:** Choose any probability measure $\mathbb{Q}$ satisfying Assumption B and $\varepsilon' \in (0,1)$. By [Yil4, Proposition 4], $I_a(x_0,\varepsilon') < I_q(x_0,\varepsilon')$ for some $x_0 \in \partial\mathbb{D}$.[6] As the rate functions are continuous on $\mathbb{D}$, there exists an open set $\mathcal{O} \subsetneq \partial\mathbb{D}\setminus\partial\mathbb{D}_{d-2}$ on which the inequality above holds. Since $\varepsilon_c(x) > 0$ for all $x \in \partial\mathbb{D}\setminus\partial\mathbb{D}_{d-2}$ by Theorem 3.2.1, the monotonicity of the map $\varepsilon \mapsto I_a(x,\varepsilon) - I_q(x,\varepsilon)$ now implies that $0 < \varepsilon_c(x) \le \varepsilon'$ for all $x \in \mathcal{O}$ which, since $\varepsilon' < 1$, shows (3.2.12) and therefore proves the existence of a true phase transition. □

For the proof of Theorem 3.2.4, we now owe the reader the proof of (3.2.11).

Proof of (3.2.11)**:** By the uniform ellipticity of ω_ε, the proof of this part now follows from Lemma 3.3.5, the dominated convergence theorem, and

Lemma 3.5.1. *Fix $x \in \partial\mathbb{D}$ and let $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Z}^d$ be the corresponding admissible sequence*

[6] Even though [Yil4, Proposition 4] states that the strict inequality holds for some interior point $x_0 \in \mathrm{int}(\mathbb{D})$, the proof actually shows that the inequality holds for some $x_0 \in \mathbb{V} \subseteq \partial\mathbb{D}$.

from Lemma 3.3.5. Then, under Assumption B*, for all* $n \in \mathbb{N}$ *the map*

$$\varepsilon \mapsto \frac{1}{n}\Big[\mathbb{E}\log P_{0,\omega_\varepsilon}(X_n = x_n) - \log P_0(X_n = x_n)\Big] =: D_n(\varepsilon)$$

is non-increasing. Moreover, the map $\varepsilon \mapsto I_a(x,\varepsilon) - I_q(x,\varepsilon) = \lim_{n\to\infty} D_n(\varepsilon)$ *is continuous on* $[0,1)$*.*

Proof of Lemma 3.5.1. Fix $n \in \mathbb{N}$ and $x_n \in \mathbb{Z}^d$ with $|x_n|_1 = n$. Then, in the notation of Section 3.3.1, by (3.3.5) and (3.3.14) we can compute explicitly

$$\begin{aligned} P_{0,\omega_\varepsilon}(X_n = x_n) &= \sum_{z=(z_0,\dots,z_{n-1})\in\mathcal{R}_{n-1}^{(x_n)}} \prod_{j=1}^{n} \left(\alpha(\Delta_j(z))(1+\varepsilon\eta(z_{j-1},\Delta_j(z)))\right), \\ P_0(X_n = x_n) &= \sum_{z=(z_0,\dots,z_{n-1})\in\mathcal{R}_{n-1}^{(x_n)}} \prod_{j=1}^{n} \alpha(\Delta_j(z)), \end{aligned}$$

where $\mathcal{R}_{n-1}^{(x_n)}$ is the set of paths z of length $n-1$ which start at 0 and end at some neighbor of x_n, i.e. all paths $z \in \mathcal{R}_{n-1}$ such that $\Delta_n(z) := x_n - z_{n-1} \in \mathbb{V}$.

To show that D_n is non-increasing, it will be enough to show that its derivative $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}$ is non-positive. The second term in D_n does not depend on ε, so by uniform ellipticity we have for $\varepsilon \in (0,1)$

$$\begin{aligned} \frac{\mathrm{d}D_n}{\mathrm{d}\varepsilon} = \frac{1}{n}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big(\mathbb{E}\big[\log P_{0,\omega_\varepsilon}(X_n = x_n)\big]\Big) &= \frac{1}{n}\mathbb{E}\left[\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big(\log P_{0,\omega_\varepsilon}(X_n = x_n)\Big)\right] \\ &= \frac{1}{n}\sum_{z\in\mathcal{R}_{n-1}^{(x_n)}} \mathbb{E}\left[\frac{A_n(z)B_n(z)}{\sum_{z'\in\mathcal{R}_{n-1}^{(x_n)}} A_n(z')}\right], \end{aligned} \tag{3.5.2}$$

where

$$A_n(z) := \prod_{j=1}^{n}\left(\alpha(\Delta_j(z))(1+\varepsilon\eta(z_{j-1},\Delta_j(z)))\right) \quad \text{and} \quad B_n(z) := \sum_{j=1}^{n}\frac{\eta(z_{j-1},\Delta_j(z))}{1+\varepsilon\eta(z_{j-1},\Delta_j(z))}. \tag{3.5.3}$$

Next, for each path $z \in \mathcal{R}_{n-1}^{(x_n)}$ let us define the probability measure P^z given by

$$\mathrm{d}P^z = \frac{A_n(z)}{\prod_{j=1}^n \alpha(\Delta_j(z))}\mathrm{d}\mathbb{Q}.$$

Recalling (3.5.3), this allows us to write the derivative as

$$\frac{\mathrm{d}D_n}{\mathrm{d}\varepsilon} = \frac{1}{n}\sum_{z \in \mathcal{R}_{n-1}^{(x_n)}} \left[\prod_{j=1}^n \alpha(\Delta_j(z))\right] E^z \left[\frac{B_n(z)}{\sum_{z' \in \mathcal{R}_{n-1}^{(x_n)}} A_n(z')}\right].$$

Note that, for each $z \in \mathcal{R}_{n-1}^{(x_n)}$, the random variables $(\eta(z_{j-1}, \Delta_j(z)) : j = 1, \ldots, n)$ are independent under P^z (although not necessarily identically distributed). Furthermore, observe that $A_n(z)$ and $B_n(z)$ are both increasing in η for any path z. Therefore, by uniform ellipticity and the Harris-FKG inequality (see [Har]) we conclude that for any $\varepsilon \in (0,1)$,

$$\begin{aligned}\frac{\mathrm{d}D_n}{\mathrm{d}\varepsilon} &\leq \frac{1}{n}\sum_{z \in \mathcal{R}_{n-1}^{(x_n)}} \left[\prod_{j=1}^n \alpha(\Delta_j(z))\right] E^z(B_n(z)) E^z \left[\frac{1}{\sum_{z' \in \mathcal{R}_{n}^{(x_n)}} A_n(z')}\right] \\ &= \frac{1}{n}\sum_{z \in \mathcal{R}_{n-1}^{(x_n)}} \left(\prod_{j=1}^n \alpha(\Delta_j(z))\right)^{-1} \mathbb{E}(A_n(z)B_n(z))\mathbb{E}\left[\frac{A_n(z)}{\sum_{z' \in \mathcal{R}_{n-1}^{(x_n)}} A_n(z')}\right] = 0,\end{aligned}$$

where the last equality follows from the fact that $\mathbb{E}(A_n(z)B_n(z)) = 0$ since the random variables $(\eta(z_{j-1}, \Delta_j(z)) : j = 1, \ldots, n)$ all have mean zero and are independent under $\mathbb{Q}$ by (3.3.14). Thus, we see that $\frac{\mathrm{d}D_n}{\mathrm{d}\varepsilon} \leq 0$ and therefore D_n is non-increasing on $[0,1)$. Finally, to show that the map $D_\infty(\varepsilon) := I_a(x,\varepsilon) - I_q(x,\varepsilon)$ is continuous we first observe that for any $\varepsilon' \in (0,1)$ there exists some $C_{\varepsilon'} > 0$ such that $\sup_{\varepsilon \leq \varepsilon'} |B_n(z)| \leq C_{\varepsilon'} n$ for all paths $z \in \mathcal{R}_{n-1}^{(x_n)}$. By (3.5.2), this implies that $\sup_{\varepsilon \leq \varepsilon'} \frac{\mathrm{d}D_n}{\mathrm{d}\varepsilon}(\varepsilon) \leq C_{\varepsilon'}$ for any $\varepsilon' \in (0,1)$, and so by the mean value theorem

$$|D_n(\varepsilon_1) - D_n(\varepsilon_2)| \leq C_{\varepsilon'}|\varepsilon_1 - \varepsilon_2| \tag{3.5.4}$$

for any $\varepsilon_1, \varepsilon_2 \in [0, \varepsilon']$. Since ε' can be taken arbitrarily close to 1, the continuity of D_∞ now follows upon taking the limit as $n \to \infty$ on (3.5.4), since $D_\infty(\varepsilon) = \lim_{n\to\infty} D_n(\varepsilon)$ by Lemma 3.3.5. $\square$

Chapter 4

Localization at the boundary for RWRE

4.1 Introduction

Random walks in random environment (RWRE) is a fundamental model in probability used as a prototype for various phenomena. Examples of this include DNA chain replication [Che], crystal growth [Tem], among others. It was introduced in the '70s to study motion in random media. In dimension $d = 1$, the model is well understood. Some of the known results include transience, recurrence, law of large numbers ([Sol],[Ali]), and large deviations ([GdH], [CGZ]). However, when $d \geq 2$, there are several open questions, including how to characterize precisely when the walk is transient or recurrent, or whether directional transience implies ballisticity. We refer the reader to the references [DR] and [Zei] for a complete presentation of the model.

In this paper, we deal with the notion of localization. Informally, we say that the walk is *localized* if its asymptotic trajectory is confined to some region with positive probability. Otherwise, it is *delocalized*. For RWRE, localization has been studied almost entirely in the one-dimensional case (see, for example, the works of Sinai [Sin] and Golosov [Gol]). When the dimension is two or higher, the topic has been practically untouched ([BCR] and [DG] are two somewhat related articles, dealing with local central limit theorems). To motivate this concept, consider first a simple random walk $(S_n)_{n\in\mathbb{N}}$ on $\mathbb{Z}^d$ conditioned to reach the boundary at time n, that is, $|S_n|_1 = n$ for each $n \in \mathbb{N}$. This walk is an example of delocalization, as a consequence of Remark 4.2.1 below. The natural question to ask is if the same situation continues to happen if we perturb the walk in some (random) directions. It turns out that the introduction of a small disorder can change the walk's typical paths so that the perturbed walk has a *favorite trajectory* that it is likely to visit, that is, it is localized. A perturbation of this sort is a perfect illustration of an RWRE. More precisely, if $(\xi(x,\cdot))_{x\in\mathbb{Z}^d}$ is an i.i.d. family of mean-zero random variables, and $(\alpha(e))_{e\in\mathbb{V}}$ are nonnegative numbers such that $\sum_{e\in\mathbb{V}}\alpha(e) = 1$ we can consider environments of the type

$$\omega_\varepsilon(x,e) = \alpha(e)(1+\varepsilon\xi(x,e)). \tag{4.1.1}$$

Under this setting, the question is whether there is localization or delocalization for a given ε. As the case $\varepsilon = 0$ corresponds to delocalization, one foresees that this will also be the case under a low disorder, and for large enough disorder, the opposite may occur. Thus, one might expect the existence of a *phase transition* in terms of the parameter ε. That result is proved in Theorem 4.2.2. However, the phase transition may be "trivial" in two ways:

(i) There is only delocalization at $\varepsilon = 0$. In other words, the walk is always localized unless it is deterministic. We show in Theorem 4.2.1 that this is the case if $d = 2$ or 3. Not only that, but any[1] (non-deterministic) RWRE will be localized.

(ii) There is always delocalization. If $d \geq 4$, we establish that the previous situation cannot hold, namely, only localization. Actually, the opposite may take place. Nonetheless, we show in Subsection 4.5.1 examples when a genuine phase transition occurs.

[1]More precisely, any RWRE that satisfies both Assumptions A and B, see below.

In summary, the main results of this paper are two: localization holds if $d = 2$ or 3, and for $d \geq 4$ there is a phase transition for the localization/delocalization phenomena.

The concept of localization/delocalization is closely related to the equality or difference between the quenched and averaged large deviations for RWRE at the boundary. Without being completely rigorous for now, consider a face F of the set $\mathbb{D} := \{x \in \mathbb{R}^d : |x|_1 = 1\}$. If I_q and I_a are the quenched and annealed rate functions for an RWRE (see (4.3.10) for the definition), then in Theorem 4.3.1 we show that localization in the face F is equivalent to

$$\inf_{x \in F} I_a(x) < \inf_{x \in F} I_q(x) \tag{4.1.2}$$

and delocalization in the same face corresponds to the equality in (4.1.2). This criterion is one of the central technical results since the annealed rate function at the boundary can be computed explicitly (see Remark 2.7 in [BMRS2]). Even though the quenched rate function has not an easy explicit formula (see Theorem 2 in [Ros]), one can obtain estimates for the quenched infimum in (4.1.2) that ensures the strict inequality in the same equation. In Subsection 4.5.1 we exploit this fact to show a part of Theorem 4.2.2.

To finish this introduction, we mention that in the model of *directed polymers in random environment*, the path localization of the walk has been studied vigorously, and several remarkable results have been obtained in the last two decades (see [CSY], [AL], [Bat] to select a few of them). The lectures notes [Com] contains an updated account of some of these articles.

4.2 Definition and statements

4.2.1 The model

Fix $d \in \mathbb{N}$, the dimension where the walk moves. For $x \in \mathbb{R}^d$ and $p \in [1, \infty]$, the ℓ_p norm of x is denoted by $|x|_p$. Define $\mathbb{V} := \{x \in \mathbb{Z}^d : |x|_1 = 1\} = \{\pm e_1, \cdots, \pm e_d\}$ the set of allowed jumps of the walk (as usual, e_i is the vector with zero coordinates excepting the one in the ith position, where it is equal to one). Next, define $\mathcal{M}_1(\mathbb{V})$ as the set of nearest neighbors probability vectors, that is,

$$\mathcal{M}_1(\mathbb{V}) := \{p : \mathbb{V} \to [0,1] : \sum_{e \in \mathbb{V}} p(e) = 1\}.$$

Now we can define the *environments*. An environment is an element ω in the space

$$\Omega := \{\omega : \mathbb{Z}^d \times \mathbb{V} \to [0,1] : \omega(x) \in \mathcal{M}_1(\mathbb{V}) \text{ for all } x \in \mathbb{Z}^d\} = \mathcal{M}_1(\mathbb{V})^{\mathbb{Z}^d}.$$

We usually write $\omega = \{\omega(x,e)\}_{x \in \mathbb{Z}^d, e \in \mathbb{V}}$. Finally, we can define a *random walk in the environment* $\omega \in \Omega$ starting at a point $x \in \mathbb{Z}^d$ as the Markov chain $X = (X_n)_{n \in \mathbb{N}}$ with law $P_{x,\omega}$ that satisfies

$$\begin{aligned} P_{x,\omega}(X_0 = x) &= 1, \\ P_{x,\omega}(X_n + 1 = y + e | X_n = y) &= \begin{cases} \omega(y,e), & \text{if } P_{x,\omega}(X_n = y) > 0 \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{4.2.1}$$

The measure $P_{x,\omega}$ in the literature is known as the *quenched measure*, in contrast to the *annealed (or averaged) measure* we describe next. We use the following notation in the sequel: if $Y = \Omega$ or $Y = (\mathbb{Z}^d)^{\mathbb{N}}$, then $\mathcal{B}(Y)$ is its Borel σ-algebra. In our case, choose a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{B}(\Omega))$. The annealed measure P_x of the RWRE starting at

$x \in \mathbb{Z}^d$ is defined as the measure on $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}$ that satisfies

$$P_x(A \times B) = \int_A P_{x,\omega}(B) d\mathbb{P} \tag{4.2.2}$$

for each $A \in \mathcal{B}(\Omega)$ and $B \in \mathcal{B}((\mathbb{Z}^d)^{\mathbb{N}})$. Expectations with respect to $P_{x,\omega}, P_x$ and $\mathbb{P}$ are denoted by $E_{x,\omega}, E_x$ and $\mathbb{E}$ respectively. The basics assumptions in this work are the following:

Assumption A:

(i) The random vectors $\{\omega(x, \cdot)\}_{x \in \mathbb{Z}^d}$ are i.i.d under $\mathbb{P}$.

(ii) Uniform ellipticity: there exists a $\kappa > 0$ such that for every $x \in \mathbb{Z}^d$ and $e \in \mathbb{V}$,

$$\mathbb{P}(\omega(x, e) \geq \kappa) = 1. \tag{4.2.3}$$

The two assumptions above are common in the literature. In particular, under assumption (i), we can define

$$\alpha(e) := \mathbb{E}[\omega(0, e)] = \mathbb{E}[\omega(x, e)], \quad x \in \mathbb{Z}^d, e \in \mathbb{V}. \tag{4.2.4}$$

4.2.2 Localization at the boundary

We will look at trajectories $(X_n)_{n \in \mathbb{N}}$ of an RWRE such that $|X_n|_1 = n$ for each n, and study the asymptotic behavior of the *normalized* quenched probability of reaching the boundary at time n, that is, if $x \in \mathbb{Z}^d$ satisfies $|x|_1 = n$,

$$P_{0,\omega}\left(X_n = x \mid |X|_1 = n\right). \tag{4.2.5}$$

Specifically, we are concerned in knowing if for some sequence $(x_n)_{n \in \mathbb{N}} \subseteq \mathbb{Z}^d$ such that $|x|_1 = n$ for all n, the quenched probability (4.2.5) is greater than some constant c, uniformly on n. In this case, the conditioned walk is "localized" around this path (the rigorous definition appears in Definition 4.2.1 below). There is a counterpart in the literature of directed polymers in random environment (see [Com], p. 88). In this model, there is a nice characterization of localization/delocalization depending on the disorder of the environment. For RWRE, the disorder measures how far is the environment $\omega(0,e)$ from its expectation $\alpha(e)$. This allows us to obtain analogous results in our case.

At this point, we proceed to characterize localization rigorously. We decompose $\partial\mathbb{D}$ in faces $\partial\mathbb{D}(s)$, with $s \in \{-1,1\}^d$, defined by

$$\partial\mathbb{D}(s) := \{x \in \partial\mathbb{D} : s_j x_j \geq 0, j = 1, \cdots, d\}. \tag{4.2.6}$$

Let $\overline{s} := (1,1,\cdots 1)$. From now on, we consider only $\partial\mathbb{D}^+ := \partial\mathbb{D}(\overline{s})$. Define the allowed jumps by

$$\mathbb{V}^+ := \{e_1, \cdots, e_d\} \subseteq \mathbb{V}.$$

Next, consider the set

$$\partial R_n := n\partial\mathbb{D}^+ = \{x \in \mathbb{Z}^d : |x|_1 = n, x_j \geq 0 \text{ for all } j \in \{1, \cdots, d\}\}$$

and define R_n as the sets of all paths $(z_0, z_1, \cdots, z_n) \in (\mathbb{Z}^d)^{n+1}$ for which $z_0 = 0$ and $z_n \in \partial R_n$. Note that this happens if and only if $\triangle z_i := z_i - z_{i-1} \in \mathbb{V}^+$ for each $i = 1, \cdots, n$. Subsequently, let $\mathcal{A}_n := \{X_n - X_0 \in \partial R_n\}$. Finally, the sequence of random variables $(J_n)_{n\in\mathbb{N}}$ is defined by $J_1 := 1$, and for $n \geq 2$,

$$J_n := \max_{x\in\mathbb{Z}^d} P_{0,\omega}(X_{n-1} = x|\mathcal{A}_{n-1}). \tag{4.2.7}$$

Definition 4.2.1. An RWRE is localized at the boundary if

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} J_k > 0 \ \ \mathbb{P}-a.s., \tag{4.2.8}$$

and an RWRE is delocalized at the boundary if

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} J_k = 0 \ \ \mathbb{P}-a.s. \tag{4.2.9}$$

Note that *a priori*, the walk can be neither localized nor delocalized. However, in Theorem 4.3.1, we show that this cannot happen for walks that satisfy Assumption A.

A different formulation

Working on the boundary induces a *polymer-like* interpretation that makes more transparent the argument we use below. Given $\omega \in \Omega, x \in \mathbb{Z}^d$, and $e \in \mathbb{V}^+$, define

$$\pi(\omega, x, e) := \frac{\omega(x,e)}{\sum_{e'\in\mathbb{V}^+}\omega(x,e')}, \quad \Psi(\omega,x) := \log\left(\sum_{e\in\mathbb{V}^+}\omega(x,e)\right). \tag{4.2.10}$$

Then, $\omega(x,e) = \pi(\omega,x,e)\mathrm{e}^{\Psi(\omega,x)}$, and π induces an RWRE, with $\mathbb{V}^+$ as the set of allowed jumps. Its quenched measure (starting at x) is $P_{x,\pi}$, and its expectation is $E_{x,\pi}$. Therefore, for fixed $n \in \mathbb{N}$ and $A \in \mathcal{B}((\mathbb{Z}^d)^{\mathbb{N}})$,

$$P_{0,\omega}(A, X_n \in \partial R_n) = E_{0,\pi}\left(\mathrm{e}^{\sum_{i=0}^{n-1}\Psi(\omega,X_i)}, A\right). \tag{4.2.11}$$

The last display leads to define a quenched *polymer measure* $P^{\omega}_{x,n}$ as

$$P^{\omega}_{x,n}(A) := \frac{E_{0,\pi}\left(\mathrm{e}^{\sum_{i=0}^{n-1}\Psi(\omega,X_i)}, A\right)}{E_{0,\pi}\left(\mathrm{e}^{\sum_{i=0}^{n-1}\Psi(\omega,X_i)}\right)}, \quad A \in \mathcal{B}((\mathbb{Z}^d)^{\mathbb{N}}). \tag{4.2.12}$$

This definition resembles the general framework introduced in [RASY1].

Using the polymer measure, it is direct to verify the identity

$$J_n = \max_{x\in\mathbb{Z}^d} P^{\omega}_{0,n-1}(X_{n-1}=x).$$

From now on, we use this scheme (except in Subsection 4.5.1), although, of course, both definitions are equivalent.

4.2.3 Main results

The main results of this paper are that localization holds for (almost) all uniformly elliptic and i.i.d environments in dimensions two and three, and a phase transition in terms of the disorder in dimensions four or higher. Let $c := \sum_{e\in\mathbb{V}^+}\alpha(e)$. The following assumption will play a remarkable role throughout the sequel.

Assumption B: The measure $\mathbb{P}$ satisfies

$$\mathbb{P}(\Psi(\omega,0)=\log(c))<1. \tag{4.2.13}$$

Remark 4.2.1. As a consequence of Theorem 4.3.1, if Assumption B does not hold, then the walk is delocalized at the boundary for any $d\geq 2$.

Theorem 4.2.1. *Let $(X_n)_{n\in\mathbb{N}}$ be an RWRE that satisfies Assumptions A and B. If $d\in\{2,3\}$, then the walk is localized at the boundary.*

A related result in RWRE appears in the article [YZ] of Yilmaz and Zeitouni. They show that for walks that satisfy a certain ballisticity condition[2] , there is a class of measures $\mathbb{P}$ such that the quenched and annealed rate functions differ in a neighborhood of the LLN velocity. In the directed polymer model, Comets and Vargas [CV] prove localization

[2] Called condition (T). This condition is equivalent to the ballisticity conditions (T') and $\mathscr{P}_M$, as showed in [GR]

in dimension $1+1$ (one dimension for time, and one for space), while Lacoin [Lac] proves localization in dimension $1+2$. Berger and Lacoin improved this result in [BL1], where they gave the precise asymptotic behavior for the difference between the quenched and annealed free energies, as $n \to \infty$.

For $d \geq 4$, we consider a certain family of environments, parameterized by $\varepsilon \in [0,1)$. This parameter represents how much the distribution of the jumps in an RWRE differs from a simple random walk.

First, fix a probability vector [3] $\alpha = (\alpha(e))_{e\in\mathbb{V}}$ with strictly positive entries. Define

$$\mathcal{E}_\alpha := \{(r(e))_{e\in\mathbb{V}} \in [-1,1]^{\mathbb{V}} : \sum_{e\in\mathbb{V}} r(e)\alpha(e) = 0, \text{and } \sup_{e\in\mathbb{V}} |r(e)| = 1\} \tag{4.2.14}$$

and consider a probability measure $\mathbb{Q}$ on $\Gamma_\alpha := \mathcal{E}_\alpha^{\mathbb{Z}^d}$ (also fixed from now). Next, pick an i.i.d family of random variables $(\xi(x))_{x\in\mathbb{Z}^d} \in \Gamma_\alpha$ such that $\mathbb{E}[\xi(x,e)] = 0$ for all $e \in \mathbb{V}$. Finally, given $\varepsilon \in [0,1)$, define the environments $(\omega_\varepsilon(x))_{x\in\mathbb{Z}^d}$ as

$$\omega_\varepsilon(x,e) := \alpha(e)(1 + \varepsilon\xi(x,e)). \tag{4.2.15}$$

This framework was originally used in [BMRS2] to study a phase transition of the map

$$\varepsilon \to I_a(x,\cdot) - I_q(x,\cdot),$$

where $I_q(x,\cdot), I_a(x,\cdot)$ are the quenched (respectively annealed) rate functions of an RWRE in the environment ω_ε. The study of RWRE at low disorder has also been considered in [Szn4], [Sab], among others.

For fixed $\varepsilon \in [0,1)$, let $\mathbb{P}_\varepsilon$ be the law of ω_ε. This measure is uniformly elliptic with constant

[3] This is an abuse of notation with the α defined in (4.4.9). However, from the context will be clear to which α we are referring.

$\kappa = (1-\varepsilon)\min_{e\in\mathbb{V}}\alpha(e)$. Conversely, for fixed $\kappa < \frac{1}{\min_{e\in\mathbb{V}}\alpha(e)}$, we define $\varepsilon_{max} := 1 - \frac{\kappa}{\min_{e\in\mathbb{V}}\alpha(e)}$, the maximum parameter so that for all $\varepsilon \le \varepsilon_{max}$, $\mathbb{P}_\varepsilon$ is uniformly elliptic with constant κ.

The last result of the paper is the phase transition for localization/delocalization for parametrized environments. We say that an RWRE is ε-localized (resp. delocalized) if (4.2.8) (resp. (4.2.9)) holds under the measure $\mathbb{P}_\varepsilon$.

Theorem 4.2.2. *For $d \ge 2$, $\alpha = (\alpha(e))_{e\in\mathbb{V}}$, $\mathbb{Q}$ and κ fixed, there exists $\overline{\varepsilon} \in [0, \varepsilon_{max}]$ such that the walk is ε-localized for $0 \le \varepsilon \le \overline{\varepsilon}$, and ε-delocalized for $\overline{\varepsilon} < \varepsilon \le \varepsilon_{max}$. Moreover, if $d \ge 4$, then $\overline{\varepsilon} > 0$. Also, there are examples of walks that satisfy $\overline{\varepsilon} < \varepsilon_{max}$.*

A direct consequence of Theorem 4.2.1 is the following:

Corollary 4.2.3. *Under the same hypotheses and notation of Theorem 4.2.2, if Assumption B does not hold, then $\overline{\varepsilon} = \varepsilon_{max}$. Otherwise, and if also $d = 2$ or 3, then $\overline{\varepsilon} = 0$.*

4.3 An equivalent criterion for localization

In this section, we prove an equivalent criterion of localization/delocalization that will be used throughout the sequel. First, we need to define the following quantities.

Definition 4.3.1. Let $(X_n)_{n\in\mathbb{N}}$ be an RWRE. Define the limits

$$\begin{aligned} p(\omega) &:= \lim_{n\to\infty} \frac{1}{n} \log E_{0,\pi}\left(\mathrm{e}^{\sum_{i=0}^{n-1}\Psi(\omega,X_i)}\right), \\ \lambda &:= \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}E_{0,\pi}\left(\mathrm{e}^{\sum_{i=0}^{n-1}\Psi(\omega,X_i)}\right) = \log(c). \end{aligned} \tag{4.3.1}$$

The last equality holds since the conditioned walk is directed.

In the directed polymer literature, these limits are known as *quenched and annealed* free energy, respectively. We leave the proof of the existence of $p(\omega)$ to the end of the section (see Lemma 4.3.5). Moreover, we will show that it does not depend on the environment, i.e., it is constant $\mathbb{P}$- a.s. Hence, assuming the existence and non-randomness of p for now, by Jensen's inequality, we deduce that $p \leq \lambda$.

Theorem 4.3.1. *Let $(X_n)_{n\in\mathbb{N}}$ be an RWRE that satisfies Assumption A.*

(i) The RWRE is localized at the boundary if and only if $p < \lambda$.

(ii) The RWRE is delocalized at the boundary if and only if $p = \lambda$.

In particular, the walk is either localized or delocalized $\mathbb{P}$-a.s.

4.3.1 Proof of Theorem 4.3.1

In order to prove the result, we need to introduce a couple of definitions. The first is a martingale that is related to p and λ, and the second is a random variable linked to J_n.

Definition 4.3.2. Given an RWRE $(X_n)_{n\in\mathbb{N}}$ that satisfies Assumption A, define the random variable in $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$

$$W_n(\omega) := E_{0,\pi}\left(\mathrm{e}^{\sum_{i=0}^{n-1}\Psi(\omega,X_i)-n\log(c)}\right), \quad n \in \mathbb{N}. \tag{4.3.2}$$

The following lemma is straightforward, so its proof is skipped.

Lemma 4.3.2. *The process $\{W_n\}_{n\in\mathbb{N}}$ is a mean-one $\mathcal{F}_n$-martingale under the filtration $\{\mathcal{F}_n\}_{n\geq 0}$ given by $\mathcal{F}_0 := \{\emptyset, \Omega\}$, and for $n \geq 1, \mathcal{F}_n := \{\omega(x,e) : |x|_1 < n, x \in \mathbb{Z}^d, e \in \mathbb{V}^+\}$.*

The martingale convergence theorem implies that $W_\infty := \lim_{n\to\infty} W_n$ exists and is non-negative $\mathbb{P}$-a.s. Since the event $\{W_\infty = 0\}$ is T_e-invariant $\mathbb{P}$- a.s. for each $e \in \mathbb{V}^+$, the ergodicity of $\mathbb{P}$ implies that $\mathbb{P}(W_\infty = 0) \in \{0, 1\}$. This consequence will be useful in Proposition 4.3.3.

Next, we introduce a second random variable,

$$I_n(\omega) := \sum_{x\in\mathbb{Z}^d} P^{\omega}_{0,n-1}(X_{n-1}=x)^2. \tag{4.3.3}$$

This random variable is $\mathcal{F}_{n-1}$-measurable. Observe that

$$J_n^2 \le I_n \le J_n. \tag{4.3.4}$$

The main ingredient in the proof of Theorem 4.3.1 is the next proposition, which compares W_n and I_n. We use the following notation: for sequences $(a_n),(b_n)$ we say that $a_n = \Theta(b_n)$ if $a_n = O(b_n)$ and $b_n = O(a_n)$.

Proposition 4.3.3. *Let $(X_n)_{n\in\mathbb{N}}$ be an RWRE that satisfies Assumption A. Then the equality*

$$\{W_\infty = 0\} = \left\{\sum_{n=1}^{\infty} I_n = \infty\right\} \tag{4.3.5}$$

holds $\mathbb{P}$-a.s. Furthermore, if $\mathbb{P}(W_\infty = 0) = 1$, there exist constants $c_1(\mathbb{P}), c_2(\mathbb{P}) \in (0,\infty)$ for which $\mathbb{P}$-a.s.,

$$c_1 \sum_{k=1}^{n} I_k \le -\log W_n \le c_2 \sum_{k=1}^{n} I_k \quad \textit{for } n \textit{ large enough,} \tag{4.3.6}$$

that is, $-\log W_n = \Theta(\sum_{k=1}^n I_k)$.

Sketch of the proof of Proposition 4.3.3. The proof of Theorem 2.1 in [CSY] can be adapted to show Proposition 4.3.3. It is based on the Doob's decomposition of the submartingale $-\log W_n$. More precisely, there exist a martingale $\{M_n\}_{n\in\mathbb{N}}$ and an adapted process $\{A_n\}_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$,

$$-\log W_n = M_n + A_n. \tag{4.3.7}$$

Indeed, $A_n = -\sum_{i=1}^n \mathbb{E}\left[\log\left(\frac{W_i}{W_{i-1}}\right)|\mathcal{F}_{i-1}\right]$. Noting that

$$\frac{W_i}{W_{i-1}} = E^{\omega}_{0,i-1}\left[\mathrm{e}^{\Psi(\omega,X_{i-1})-\log(c)}\right] = 1 + E^{\omega}_{0,i-1}\left[\mathrm{e}^{\Psi(\omega,X_{i-1})-\log(c)} - 1\right] =: 1 + U_i,$$

we decompose A_n and M_n as

$$A_n = -\sum_{i=1}^n \mathbb{E}[\log(1+U_i)|\mathcal{F}_{i-1}], \quad M_n = \sum_{i=1}^n \left(-\log(1+U_i) + \mathbb{E}[\log(1+U_i)|\mathcal{F}_{i-1}]\right).$$

Exactly as in the aforementioned result, it is enough to prove that there is a constant $C > 0$ such that for all $n \in \mathbb{N}$,

$$\frac{1}{C} I_n \le \mathbb{E}[-\log(1+U_n)|\mathcal{F}_{n-1}] \le C I_n, \quad \mathbb{E}[\log^2(1+U_n)|\mathcal{F}_{n-1}] \le C I_n. \tag{4.3.8}$$

To check the inequalities above, notice that, by uniform ellipticity, the potential Ψ is bounded $\mathbb{P}$-a.s., so there are constants $0 < C_1 < C_2$ such that $\mathbb{P}$- a.s., for all $n \in \mathbb{N}$, $\frac{W_n}{W_{n-1}} \in (C_1, C_2)$. Therefore,

$$U_n - C_3 U_n^2 \le \log(1+U_n) \le U_n - C_4 U_n^2 \tag{4.3.9}$$

for some constants $C_3, C_4 > 0$. Thus, $\mathbb{E}[-\log(1+U_n)|\mathcal{F}_{n-1}]$ is bounded by above by

$$\begin{aligned}
&\mathbb{E}[-U_n + C_3 U_n^2 | \mathcal{F}_{n-1}] = -C_4 \mathbb{E}[U_n^2 | \mathcal{F}_{n-1}] \\
&= C_3 \sum_{x,x' \in \mathbb{Z}^d} \mathbb{E}\big[E^{\omega}_{0,n-1}\left(\mathrm{e}^{\Psi(\omega,x)-\log(c)} - 1, X_{n-1} = x\right) \times \\
&\qquad\qquad E^{\omega}_{0,n-1}\left(\mathrm{e}^{\Psi(\omega,x')-\log(c)} - 1, X_{n-1} = x'\right) \big| \mathcal{F}_{n-1}\big] \\
&= C_3 \sum_{x,x' \in \mathbb{Z}^d} \mathbb{E}\left[\left(\mathrm{e}^{\Psi(\omega,x)-\log(c)} - 1\right)\left(\mathrm{e}^{\Psi(\omega,x')-\log(c)} - 1\right)\right] \times \\
&\qquad\qquad P^{\omega}_{0,n-1}(X_{n-1} = x) P^{\omega}_{0,n-1}(X_{n-1} = x') \\
&= C_3 \mathbb{E}\left[\left(\mathrm{e}^{\Psi(\omega,0)-\log(c)} - 1\right)^2\right] I_n.
\end{aligned}$$

Similarly we get a lower bound $\mathbb{E}[-\log(1+U_n)|\mathcal{F}_{n-1}] \geq C_4 \mathbb{E}\left[\left(\mathrm{e}^{\Psi(\omega,0)-\log(c)} - 1\right)^2\right] I_n$, and this shows the first inequality in (4.3.9). Finally, noting that for some constant $C_5 > 0$, $\log^2(1+U_n) \leq C_5 U_n^2$, repeating the steps from the last display we get the second inequality on (4.3.9), concluding the proof. $\square$

Proof of Theorem 4.3.1.

Recall that, due to (4.3.4), we have

$$\left(\frac{1}{n}\sum_{k=1}^{n} J_k\right)^2 \leq \frac{1}{n}\sum_{k=1}^{n} J_k^2 \leq \frac{1}{n}\sum_{k=1}^{n} I_k \leq \frac{1}{n}\sum_{k=1}^{n} J_k.$$

Thus, the liminfs of the sequences $(\frac{1}{n}\sum_{k=1}^{n} I_k)_n$ and $(\frac{1}{n}\sum_{k=1}^{n} J_k)_n$ are of the same nature, that is, both are positive $\mathbb{P}$-a.s. or zero $\mathbb{P}$-a.s.

If $p < \lambda$, $W_\infty = 0$ $\mathbb{P}$-a.s. To check this, observe that if $W_\infty > 0$ then $\frac{\log W_n}{n} \to 0$, but at the same time

$$\frac{\log W_n}{n} \to p - \lambda = 0.$$

So, if $p < \lambda$, then $W_\infty = 0$ $\mathbb{P}$-a.s. By (4.3.5), $\sum_n I_n = \infty$ $\mathbb{P}$- a.s. and $-\log W_n = \Theta(\sum_{k=1}^n I_k)$. In particular, $\liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^n I_n > 0$, so the RWRE is localized at the boundary. Reciprocally, if the walk is localized, $\sum_{k=1}^n I_k = \infty$, so by (4.3.5) , $-\log W_n = \Theta(\sum_{k=1}^n I_k)$, and then $-\frac{\log W_n}{n} \to p - \lambda > 0$. This proves (i), and the proof of (ii) is analogous. $\square$

4.3.2 Relation between p and λ with RWRE rate functions

To justify the existence of the first limit in (4.3.1), we relate p (resp. λ) to the quenched (resp. annealed) rate function for random walks in random environment. First, we recall some standard notation. If $G \subseteq \mathbb{R}^d$ is a Borel set (with respect to the usual topology), the interior and closure of G are, respectively, $\mathrm{int}(G)$ and $\overline{G}$. We say that the position of the walk satisfies a *quenched large deviation principle* if there is a lower semicontinuous function $I_q : \mathbb{R}^d \to [0,\infty]$ such that for each Borel set $G \subseteq \mathbb{R}^d$

$$-\inf_{x\in \mathrm{int}(G)} I_q(x) \le \liminf_{n\to\infty} \frac{1}{n}\log P_{0,\omega}(X_n/n \in G) \le \limsup_{n\to\infty} \frac{1}{n}\log P_{0,\omega}(X_n/n \in G) \le -\inf_{x\in\overline{G}} I_q(x). \tag{4.3.10}$$

Analogously, we say that the position of the walk satisfies an *annealed large deviation principle* if there is a lower semicontinuous function $I_a : \mathbb{R}^d \to [0,\infty]$ such that for every Borel set $G \subseteq \mathbb{R}^d$, (4.3.10) holds with P_0 instead of $P_{0,\omega}$. It is well known that the domain of both functions (that is, when $I_q, I_a < \infty$) is the set $\mathbb{D} := \{x \in \mathbb{R}^d : |x|_1 \le 1\}$. Also, by Jensen's inequality and Fatou's lemma, $I_a \le I_q$. Moreover, Varadhan proved in [Var] that both functions exist under i.i.d and uniform elliptic environments, and I_q is deterministic (i.e., it does not depend on ω).

Next, we characterize the rate functions at $\partial\mathbb{D}^+$ (see (4.2.6)).

Lemma 4.3.4. *Under Assumption A, for any $x \in \partial\mathbb{D}^+$ there is a sequence $(x_n)_{n\in\mathbb{N}}$ such that for all $n, x_n \in \mathbb{Z}^d, |x_n|_1 = n, \frac{x_n}{n} \to x$, and*

$$I_q(x) = -\lim_{n\to\infty} \frac{1}{n} \log P_{0,\omega}(X_n = x_n), \quad I_a(x) = -\lim_{n\to\infty} \frac{1}{n} \log P_0(X_n = x_n). \tag{4.3.11}$$

Moreover, the limits are independent of the chosen sequence. This result is Lemma 3.5 in [BMRS2].

Finally, the existence of p is consequence of the lemma below.

Lemma 4.3.5. *For an RWRE that satisfies Assumption A, the following identities hold:*

$$p = -\inf_{x\in\partial\mathbb{D}^+} I_q(x), \quad \lambda = -\inf_{x\in\partial\mathbb{D}^+} I_a(x). \tag{4.3.12}$$

In particular, p is not random (since I_q is deterministic).

The proof of this lemma is standard (see Lemma 16.12 in [RAS4]). As a corollary, we obtain the characterization of localization/delocalization in terms of the difference between the infima of the quenched and annealed rate functions:

Corollary 4.3.6. *For an RWRE that satisfies Assumption A, we have localization at the boundary if and only if*

$$\inf_{x\in\partial\mathbb{D}^+} I_a(x) < \inf_{x\in\partial\mathbb{D}^+} I_q(x).$$

4.4 Proof of Theorem 4.2.1

4.4.1 Preliminaries for the proof of Theorem 4.2.1

The method of fractional moment and change of measure used in the proof was originally introduced by Derrida *et al.* in [DGLT] for the *pinning model*. For directed polymers, Lacoin and Moreno used it for the first time in [LM] (on the hierarchical lattice), and Lacoin in [Lac] (on $\mathbb{Z}^d$). Yilmaz and Zeitouni adapted the technique in [YZ] for random walks in random environment. As the proofs are similar, we only mention the main points of them and refer to the papers above for further details. More precisely, let $\phi(\theta) := \sum_{e\in\mathbb{V}} \alpha(e)\mathrm{e}^{\langle\theta,z\rangle}$. In [YZ], the analog of showing that $p < \lambda$ in the space-time RWRE setting, is to demonstrate that for a sufficiently large set of points $\theta \in \mathbb{R}^d$,

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\log E_{0,\omega}\left[\mathrm{e}^{\langle\theta,X_n\rangle - n\log(\phi(\theta))}\right] < 0, \tag{4.4.1}$$

Comparing with

$$p-\lambda = \lim_{n\to\infty}\frac{1}{n}\mathbb{E}\log[W_n] = \lim_{n\to\infty}\frac{1}{n}\mathbb{E}\log E_{0,\pi}\left(\mathrm{e}^{\sum_{i=0}^{n-1}\Psi(\omega,X_i)-n\log(c)}\right),$$

the main difference between the two models is that the potential $\Psi(\omega,x)$ is replaced by a tilt that depends on the steps of the walk, namely, $\Psi_{st}(\theta,e) := \mathrm{e}^{\langle\theta,e\rangle}$. This introduces a correlation that in our case is not present (see the paragraph below (4.4.9)). Thus, it is natural to apply the methods in [YZ] to deduce the desired result. We sketch the main ideas and differences in the next pages.

First, note that Theorem 4.3.1 implies immediately delocalization when (4.2.13) does not hold. Indeed, in this case, $\mathbb{P}$-a.s $\Psi(\omega,x) = \log(c)$ for all $x\in\mathbb{Z}^d$, so by (4.3.1), $p=\log(c)=\lambda$. Hence, until the end of the proof we assume that (4.2.13) holds.

Let $\{\hat{X}_n\}_{n\in\mathbb{N}}$ be a simple random walk with jumps in $\mathbb{V}^+$ and law $\hat{P}$ that satisfies

$$\hat{P}(\hat{X}_{n+1} = x+e|\hat{X}_n = x) = \frac{\alpha(e)}{\sum_{e'\in\mathbb{V}^+} q(e')}, \quad x\in\partial R_n, e\in\mathbb{V}^+,$$

and define $\mu := \hat{E}(\hat{X}_1)$. Consider $N = nm$ with n fixed (but large enough) and $m\to\infty$. Recall that

$$W_N(\omega) = E_{0,\pi}\left(\mathrm{e}^{\sum_{i=0}^{N-1}\Psi(\omega,X_i)-N\log(c)}\right).$$

We define, for $y \in \mathbb{Z}^d$,

$$J_y := \left((y-\frac{1}{2})\sqrt{n},(y+\frac{1}{2})\sqrt{n}\right) \subseteq \mathbb{R}^d. \tag{4.4.2}$$

Given $Y = (y_1, \cdots, y_m) \in (\mathbb{Z}^d)^m$, let

$$W_N(\omega, Y) := E_{0,\pi}\left(\mathrm{e}^{\sum_{i=0}^{N-1}\Psi(\omega,X_i)-N\log(c)}, X_{jn} - jn\mu \in J_{y_j}, \forall j \le m\right)$$

and decompose

$$W_N(\omega) = \sum_Y W_N(\omega, Y). \tag{4.4.3}$$

The decomposition in (4.4.3) is well-founded, since $\mathbb{Z}^d \subseteq \bigcup_{y\in\Lambda} J_y$. By the inequality $(\sum_i a_i)^{1/2} \le \sum_i a_i^{1/2}$, valid for countable indices, we obtain

$$\mathbb{E}[W_N(\omega)^{1/2}] \le \sum_Y \mathbb{E}[W_N(\omega, Y)^{1/2}].$$

This inequality gives us

$$p - \lambda = \lim_{N\to\infty}\frac{1}{N}\mathbb{E}\log[W_N] \le \liminf_{N\to\infty}\frac{2}{N}\log\mathbb{E}[W_N^{1/2}] \le \liminf_{N\to\infty}\frac{2}{N}\log\left(\sum_Y \mathbb{E}[W_N(\omega,Y)^{1/2}]\right). \tag{4.4.4}$$

Now, we estimate each expectation $\mathbb{E}[W_N(\omega, Y)]^{1/2}$, applying the change of measure. The plan is the following (recall that $N = mn$ with fixed n): fix $j \in \{1, \cdots, m\}$, $Y \in (\mathbb{Z}^d)^m$, and a square integer n. Also, C_1 is a constant to determine, and $y_0 := 0$. Then define

$$B_j := \{(z,i) \in \mathbb{Z}^d \times \mathbb{N} : (j-1)n \le i < jn, |z - i\mu - y_{j-1}\sqrt{n}|_1 \le C_1\sqrt{n}\}. \tag{4.4.5}$$

4.4.2 Proof in the case $d = 2$

The idea is to define a function that depends on the different blocks B_j. Let

$$\tilde{\omega}(y) := \mathrm{e}^{\Psi(\omega,y)} - c, \text{ and } \quad D(B_j) := \sum_{y:(y,|y|_1)\in B_j} \tilde{\omega}(y). \tag{4.4.6}$$

In particular, $\mathbb{E}[D(B_j)] = 0$, and they form an independent family of random variables. It is important to observe that (4.2.13) guarantees that $\tilde{\omega}$ and $D(B_j)$ are non-degenerate random variables. We also define $\delta_n := C_1^{-1/2} n^{-3/4}$. Note that $\delta_n^2 |D(B_1)| = O(1)$. Finally, for $K > 0$ large enough (to determine), define

$$f_K(u) := -K\mathbb{1}_{\{u \ge \mathrm{e}^{K^2}\}}, \qquad g(\omega, Y) := \mathrm{e}^{\sum_{j=1}^m f_K(\delta_n D(B_j))}.$$

By Cauchy-Schwarz inequality,

$$\begin{aligned} &\mathbb{E}[W_N(\omega,Y)^{1/2}] = \mathbb{E}[W_N(\omega,Y)^{1/2} g(\omega,Y)^{1/2} g(\omega,Y)^{-1/2}] \\ &\le \mathbb{E}[W_N(\omega,Y) g(\omega,Y)]^{1/2} \mathbb{E}[g(\omega,Y)^{-1}]^{1/2}. \end{aligned} \tag{4.4.7}$$

One can show that for K large enough, $\mathbb{E}[g(\omega,Y)^{-1}]^{1/2} \le 2^m$. To bound $\mathbb{E}[W_N(\omega,Y)g(\omega,Y)]$, we can follow the estimates in Pages 251-252 from [YZ] to deduce that

$$\mathbb{E}[W_N(\omega,Y)^{1/2}] \le \left(2 \sum_{y\in\mathbb{Z}^2} \max_{x\in J_0} \mathbb{E}E_{x,\pi}\left(\mathrm{e}^{\sum_{i=0}^{n-1}\Psi(\omega,X_i) + f_K(\delta_n D(B_1)) - n\log(c)}; X_n - n\mu \in J_y\right)\right)^m.$$

The bound (4.4.4) tell us that $p - \lambda < 0$ once we are able to prove the following:

Lemma 4.4.1. *For $n, K,$ and C_1 large enough,*

$$\sum_{y\in\mathbb{Z}^2} \max_{x\in J_0} \mathbb{E}E_{x,\pi}\left(\mathrm{e}^{\sum_{i=0}^{n-1}\Psi(\omega,X_i)+f_K(\delta_n D(B_1))-n\log(c)}; X_n - n\mu \in J_y\right) < 1/2.$$

The proof of the lemma above is followed almost exactly from Subsection 2.5 in [YZ]. The main difference rests in display (2.22) in the aforementioned paper. In our case, we need to check that for some $\beta > 0$,

$$\mathbb{E}E_{0,\pi}\left[\mathrm{e}^{\sum_{i=0}^{n-1}\Psi(\omega,X_i)-n\log(c)}\left(\sum_{i=0}^{n-1}\tilde{\omega}(X_i) - \beta\right)^2\right]$$

is $O(n)$.

We can decompose it as

$$\begin{aligned}&\sum_{j=1}^{n-1}\mathbb{E}E_{0,\pi}\left[\mathrm{e}^{\sum_{i=0}^{n-1}\Psi(\omega,X_i)-n\log(c)}(\tilde{\omega}(X_j)-\beta)^2\right]+\\&2\sum_{1\le\ell<j\le n-1}\mathbb{E}E_{0,\pi}\left[\mathrm{e}^{\sum_{i=0}^{n-1}\Psi(\omega,X_i)-n\log(c)}(\tilde{\omega}(X_\ell)-\beta)(\tilde{\omega}(X_j)-\beta)\right].\end{aligned} \tag{4.4.8}$$

The first term is $n\mathbb{E}E_{0,\pi}\left[\mathrm{e}^{\Psi(\omega,X_1)-\log(c)}\left(\tilde{\omega}(X_1)-\beta\right)^2\right]$. As $c_n := \frac{\delta_n^2}{(\mu n\delta_n - A_n - \mathrm{e}^{K^2})^2} = O(n^{-2})$, this expression vanishes as $n \to \infty$. On the other hand, if we choose

$$\beta := \mathbb{E}E_{0,\pi}\left[\mathrm{e}^{\Psi(\omega,X_1)-\log(c)}\tilde{\omega}(X_1)\right] = \frac{\mathbb{E}\left[\mathrm{e}^{2\Psi(\omega,0)}\right]-c^2}{c} > 0 \text{ (by (4.2.13))}, \tag{4.4.9}$$

then by independence, the second term in (4.4.8) is zero. By comparison, the analog of α (called μ in [YZ]) is greater than zero due to a positive correlation that in our case is not needed.

Combining the previous results, such election of constants help us to deduce that Lemma 4.4.1 is true, and therefore $p - \lambda < 0$. $\square$

4.4.3 Proof in case $d = 3$

In this case, the proof in principle is essentially the same, but some technical details need to be adapted to this situation. In particular, we need to redefine δ_n and $D(B_j)$. First, for a constant $C_2 > 0$ to determine, let

$$V(y,z) := \frac{1}{|i-j|}\mathbb{1}_{\{|y-z-(i-j)\mu|_1 < C_2\sqrt{|i-j|}\}} \text{ if } i \neq j, \text{ and } 0 \text{ otherwise.}$$

Also, recall that $\tilde{\omega}$ is defined as in (4.4.6). Then we redefine

$$\delta_n := n^{-1}(\log n)^{-1/2}, \quad D(B_j) := \sum_{\substack{y,z \\ (y,i),(z,j)\in Bj}} V(y,z)\tilde{\omega}(y)\tilde{\omega}(z).$$

The proof of Theorem 1.6 in [YZ] can be followed almost word by word, and our case is a little bit simplified since the correlation issue is not present as in the $d = 2$ case. Details are omitted. $\square$

4.5 Phase transition

Recall the parametrization of the environments $(\omega_\varepsilon)_{\varepsilon\in[0,1)}$ (i.e., (4.2.15)). Let be $p(\varepsilon)$ the limit in (4.3.1) with environment ω_ε. On the other hand, λ is constant over ε, and it is equal to $\log(\sum_{e\in\mathbb{V}^+}\alpha(e))$. The first part of Theorem 4.2.2 is consequence of the lemma below:

Lemma 4.5.1. *For each $n \in \mathbb{N}$, the map*

$$\varepsilon \in [0, \varepsilon_{max}] \to \frac{1}{n}\left[\mathbb{E}\log P_{0,\omega_\varepsilon}(X_n \in \partial R_n) - \log P_0(X_n \in \partial R_n)\right] \text{ is non-increasing.}$$

This is an adaptation of Lemma 5.1 in [BMRS2]. If we let n to infinity, then we deduce that $p(\varepsilon) - \lambda$ is non-increasing. To finish the proof, define

$$\overline{\varepsilon} := \inf\{\varepsilon \in (0, \varepsilon_{max}] : p(\varepsilon) - \lambda < 0\},$$

with the convention that $\inf \emptyset = \varepsilon_{max}$.

The rest of this section is devoted to prove the second part of Theorem 4.2.2. The main ingredient to show that $\overline{\varepsilon} > 0$ is the next lemma, a particular case of Lemma 3.1 with $\theta = 0$ in [BMRS2].

Lemma 4.5.2. *If $\varepsilon > 0$ is small enough, then* $\sup_n |W_n^2|_2 < \infty$.

Recall the following: if $W_\infty(\varepsilon) := W_\infty(\omega_\varepsilon)$, then

$$W_\infty(\omega_\varepsilon) > 0 \text{ implies } p(\varepsilon) = \lambda,$$

the later being equivalent to localization. Indeed, If $W_\infty > 0$, then $\log(W_\infty) = \lim_{n\to\infty} \log(W_n) < \infty$, so

$$p(\varepsilon) = \lim_{n\to\infty} \frac{1}{n} \log P_{0,\omega_\varepsilon}(X_n \in \partial R_n) = \lim_{n\to\infty} \frac{W_n(\omega_\varepsilon)}{n} + \lambda = \lambda.$$

Now pick $\varepsilon > 0$ small enough such that $\sup_n |W_n^2|_2 < \infty$ as in Lemma 4.5.2, and call it ε^*. By the martingale convergence theorem, $W_n(\varepsilon^*) \to W_\infty(\varepsilon^*)$ a.s. and in L^2. As $|W_n|_2 = 1$ for all n, then we necessarily have $W_\infty(\varepsilon^*) > 0$, and therefore $p(\varepsilon^*) = \lambda$. But the map $\varepsilon \to p(\varepsilon) - \lambda$ is non-increasing, so $p = \lambda$ on $[0, \varepsilon^*]$, and thus $\overline{\varepsilon} \geq \varepsilon^* > 0$.

It only remains to show an example in dimension greater or equal than 4 where $0 < \overline{\varepsilon} < \varepsilon_{max}$.

4.5.1 An example on which $\overline{\varepsilon} < \varepsilon_{max}$

For simplicity, we consider $d = 4$, and i.i.d random variables $(\xi(x))_{x\in\mathbb{Z}^d} \in \Gamma_\alpha$ such that $\xi(x,e) = \xi(x,e')$ for all $e, e' \in \mathbb{V}^+$, and $\xi(x,-e) = -\xi(x,e)$. If $y = (y_1, \cdots, y_d) \in \partial\mathbb{D}^+$ is a point to determine, for $i = 1, \cdots, d$, define $\alpha(e_i) = \alpha(-e_i) := \frac{y_i}{2\sum_{i=1}^d y_i}$. Recall that

$$\omega_\varepsilon(x, e_i) = \alpha(e_i)(1 + \varepsilon\xi(x)) \qquad e_i \in \mathbb{V}^+.$$

Moreover, assume that the distribution of $\xi(0)$ under $\mathbb{Q}$ is the Rademacher distribution, namely, $\mathbb{Q}(\xi(0) = 1) = \mathbb{Q}(\xi(0) = -1) = \frac{1}{2}$. By Corollary 4.3.6, localization occurs as soon as

$$\inf_{x\in\partial\mathbb{D}^+} I_a(x) < \inf_{x\in\partial\mathbb{D}^+} I_q(x). \tag{4.5.1}$$

However, in this case, the infimum on the left is exactly $I_a(y)$, and it is achieved only at this point (see Theorem 2.3 in [BMRS2]). On the other hand, by the continuity of I_q, the infimum on the right is also achieved at some point $\overline{x} \in \partial\mathbb{D}^+$. If $\overline{x} \neq y$, then $I_a(y) < I_a(\overline{x}) \leq I_q(\overline{x})$, so we are done. Thus, assume that $\overline{x} = y$. Denote by $(y_n)_{n\in\mathbb{N}}$ any sequence as in Lemma 4.3.4 for the point y. Then we decompose $-I_q(y)$ as

$$\begin{aligned} -I_q(y) &= -I_a(y) + \lim_{n\to\infty} \frac{1}{n} \log \left(\frac{\sum_{0=x_0,x_1,\cdots,x_n=y_n} \prod_{i=1}^n q(\triangle x_i)(1+\varepsilon\xi(x_{i-1}))}{\sum_{0=x_0,x_1,\cdots,x_n=y_n} \prod_{i=1}^n q(\triangle x_i)} \right) \\ &\leq -I_a(y) + \limsup_{n\to\infty} \max_{0=x_0,x_1,\cdots,y_n} \frac{1}{n} \sum_{i=1}^n \log(1+\varepsilon\xi(x_{i-1})). \end{aligned} \tag{4.5.2}$$

Also, the sum and maximum above are over all directed paths $0 = x_0, x_1, \cdots, x_n$ such that $x_n = y_n$. Let $C(y_n)$ be the number of such paths. It's easy to check that there exists some constant $C > 0$ such that, for all $n \in \mathbb{N}, C(y_n) \leq C e^{nf(y) - \frac{d-1}{2}\log n}$, where $f(y) = -\sum_{i=1}^d y_i \log(y_i)$. To estimate the maximum above, we can use Hoeffding inequality (see

Theorem 2.8 in [BLM]) to obtain, for $a > 0$,

$$\mathbb{P}\left(\sum_{i=1}^{n} \log(1+\varepsilon\xi(x_{i-1})) - n\mathbb{E}[\log(1+\varepsilon\xi(0))] > na\right) \leq \exp\left(\frac{-2na^2}{\log\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^2}\right). \tag{4.5.3}$$

Therefore,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{0=x_0,x_1,\cdots,y_n} \sum_{i=1}^{n} \log(1+\varepsilon\xi(x_{i-1})) - n\mathbb{E}[\log(1+\varepsilon\xi(0))] > na\right) \\
&\leq \sum_{n=1}^{\infty} C(y_n) \exp\left(\frac{-2na^2}{\log\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^2}\right) < \infty
\end{aligned}$$

as soon as $a > \log\left(\frac{1+\varepsilon}{1-\varepsilon}\right)\sqrt{f(y)/2}$. By Borel-Cantelli's lemma, (4.5.2) is bounded by

$$\begin{aligned}
&- I_a(y) + \log\left(\frac{1+\varepsilon}{1-\varepsilon}\right)\sqrt{f(y)/2} + \mathbb{E}\left[\log(1+\varepsilon\xi(0))\right] \\
&= -I_a(y) + \log\left(\frac{1+\varepsilon}{1-\varepsilon}\right)\sqrt{f(y)/2} + \frac{1}{2}\left(\log(1+\varepsilon) + \log(1-\varepsilon)\right).
\end{aligned}$$

If $f(y) \leq \frac{9}{50}$, then $\sqrt{f(y)/2} \leq \frac{3}{10}$, and the last display is strictly smaller than $-I_a(y)$ at least for $\varepsilon > \frac{9}{10}$. The required value for $f(y)$ can be achieved, for example, selecting the vector $y = \left(\frac{97}{100}, \frac{1}{100}, \frac{1}{100}, \frac{1}{100}\right)$, so in this case, we can choose $\varepsilon_{max} \approx \frac{9}{10}$ to obtain a true phase transition, with $\kappa \approx \frac{1}{1000}$. $\square$

Remark 4.5.1. The asymmetry in terms of α is needed. Indeed, if $\alpha(e) = \frac{1}{2d}$ for all $e \in \mathbb{V}$, then it is not difficult to show (see pp. 36-37 in [Com]) that under our setting, $\sup_{n\in\mathbb{N}} \mathbb{E}[W_n^2] < \infty$, and therefore, $p(\varepsilon) = \lambda$ for all $\varepsilon \in [0, \varepsilon_{max}]$.

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