Phase-difference fluctuations of the quantum-beat laser

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We calculate the phase-difference and phase-sum fluctuations of the quantum-beat laser, using the nonlinear model of Bergou *et al.* [Phys. Rev. A **38**, 754 (1988)], and also making use of the phase operators defined by Pegg and Barnett [Phys. Rev. A **39**, 1665 (1989)]. We found a vanishing phase-difference diffusion constant, in agreement with Bergou *et al.* However, the present formalism is more general, and includes shot noise as well as saturation effects for the relative and sum phase fluctuations.

Recently, Pegg and Barnett[1,2] defined a well-behaved Hermitian phase operator $\hat{\phi}_\theta$, constructed in a finite-dimensional Hilbert space. After the calculations of various moments, the dimension is allowed to tend to infinity. Some of the applications of this phase operator, so far are (a) to show that a highly excited coherent state has a minimum uncertainty product of the photon number and phase fluctuations;[2] (b) in a low photon number field, the phase fluctuations are in agreement with the experimental results;[3,4] (c) to study the phase properties of squeezed states;[5,6] (d) to obtain in a single curve, the phase fluctuations versus time of a laser with atomic memory effects.[7]

Experimentally, one only measures phase differences, typically, a signal whose phase is referred to as a local oscillator. We could also have a phase difference in a beat signal in correlated physical systems such as the quantum-beat laser.

In the present paper, we study the relative phase fluctuations in a quantum-beat laser, using the Hermitian phase operator (HPO).

In a recent paper, the nonlinear theory of the quantum-beat laser was developed.[8] The main result of that paper is that the relative phase-diffusion coefficient vanished to all orders. This was also in agreement with a previous linear theory.[9]

Our results are in agreement with Ref. 8. In addition, we also determine the constant value of the phase fluctuations of the relative phase.

Sharp phase states[10] can be defined as

$$|\theta_m\rangle = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} e^{i\theta_m n}|n\rangle \ , \tag{1}$$

with

$$\theta_m = \theta_0 + \frac{2\pi}{s+1} m, \quad m = 0, 1, \ldots, s \tag{2}$$

and θ_0 is arbitrary. These $|\theta_m\rangle$ states form an orthonormal basis in the $(s+1)$-dimensional Hilbert space and the electron of θ_0 defines a particular basis set.

From Eq. (1), a phase operator can be defined as

$$\hat{\phi}_\theta = \sum_{m=0}^{s} \theta_m |\theta_m\rangle\langle\theta_m| \ . \tag{3}$$

In order to deal with the quantum-beat laser (QBL), definition (1) will have to be extended for two modes, that is,

$$|\theta_{m_1}, \theta_{m_2}\rangle = \frac{1}{s+1} \sum_{n_1, n_2 = 0}^{s} e^{i(\theta_{m_1} n_1 + \theta_{m_2} n_2)} |n_1, n_2\rangle \ , \tag{4}$$

where one assumes that the dimension associated with each mode is the same.

The fluctuations of the phase-difference operator,[11] can now be written as

$$\langle[\Delta(\hat{\phi}_{\theta_1} - \hat{\phi}_{\theta_2})]^2\rangle = \sum_{m_1, m_2 = 0}^{s} (\theta_{m_1} - \theta_{m_2})^2 \langle\theta_{m_1}, \theta_{m_2}|\rho|\theta_{m_1}, \theta_{m_2}\rangle - \left[\sum_{m_1, m_2 = 0}^{s} (\theta_{m_1} - \theta_{m_2}) \langle\theta_{m_1}, \theta_{m_2}|\rho|\theta_{m_1}, \theta_{m_2}\rangle\right]^2 . \tag{5}$$

We notice in Eq. (5) that $\langle\theta_{m_1}, \theta_{m_2}|\rho|\theta_{m_1}, \theta_{m_2}\rangle$ is the probability of having a relative phase $\theta_{m_1} - \theta_{m_2}$. This probability is normalized. We will now calculate $\langle\theta_{m_1}, \theta_{m_2}|\rho|\theta_{m_1}, \theta_{m_2}\rangle$ when ρ is the density matrix of the QBL.

In Ref. 8, the quantum-beat laser was described by a model consisting of three-level atoms pumped into their upper state, at a rate R. The transitions from the two upper states to the lower one ($a \to c, b \to c$) are assumed dipole allowed. Also, the two upper states (a, b) are coupled by an external intense classical field. There is a double cavity tuned at frequencies ν_1 and ν_2 corresponding to the $a \to c$ and $b \to c$ transitions, respectively. These two modes are quantum-mechanically treated.

The master equation for the quantum-beat laser is given by Eq. (34) of Ref. 8, that is,

$$\dot{\rho}^{(A)}_{N,N'}=\left[\frac{\sqrt{NN'}\mathcal{A}}{1+\mathcal{N}_{N-1,N'-1}\dfrac{\mathcal{B}}{\mathcal{A}}}\rho^{(A)}_{N-1,N'-1}-\frac{\mathcal{N}'_{N,N'}\mathcal{A}}{1+\mathcal{N}_{N,N'}\dfrac{\mathcal{B}}{\mathcal{A}}}\rho^{(A)}_{N,N'}\right]$$
$$-\frac{\gamma_c}{2}[(N+N')\rho^{(A)}_{N,N'}-2\sqrt{(N+1)(N'+1)}\rho^{(A)}_{N+1,N'+1}] , \tag{6}$$

and

$$\dot{\rho}^{(B)}_{N,N'}=-\frac{\gamma_c}{2}[(N+N')\rho^{(B)}_{N,N'}-2\sqrt{(N+1)(N'+1)}\rho^{(B)}_{N+1,N'+1}] , \tag{7}$$

where the A and B modes are the dressed modes of the cavity, defined as

$$A=\frac{a_1+a_2}{\sqrt{2}} , \tag{8}$$

$$B=\frac{a_1-a_2}{\sqrt{2}} , \tag{9}$$

with a_1 and a_2 being the annihilation operators of the two original modes. The various parameters appearing in Eq. (6) are just the usual parameters from the Scully-Lamb[12,13] laser theory, that is,

$$\gamma_c=\frac{\nu_1}{Q_1}=\frac{\nu_1}{Q_2} , \tag{10a}$$

where Q_1 and Q_2 are the quality factors of the two cavities,

$$\mathcal{A}=4R\left[\frac{g}{\gamma}\right]^2 , \tag{10b}$$

$$\mathcal{B}=8\left[\frac{g}{\gamma}\right]^2\mathcal{A} , \tag{10c}$$

$$\mathcal{N}_{n,n'}=\tfrac{1}{2}(n+1+n'+1)+\tfrac{1}{16}(n-n')^2\frac{\mathcal{B}}{\mathcal{A}} , \tag{10d}$$

$$\mathcal{N}'_{n,n'}=\tfrac{1}{2}(n+1+n'+1)+\tfrac{1}{8}(n-n')^2\frac{\mathcal{B}}{\mathcal{A}} , \tag{10e}$$

and R is the pump rate; g is the coupling constant; γ is the atomic loss rate.

As we can see from Eqs. (6) and (7), there are two dressed independent modes A and B. The A mode satisfies the Scully-Lamb master equation and corresponds to an ordinary laser mode, while the B mode master equation contains only a loss term and therefore, in steady state $\rho^{(B)}_{N,N}=\delta_{N,0}$.

We assume, now, as an initial condition, that the "laser" in A mode is in a coherent state.[14] Then, the density operator at time t is given by

$$\rho(t)=\sum_{N_A,N'_A=0}^{s}\rho_{N_A,N'_A}(t)|N_A,0\rangle\langle N'_A,0| , \tag{11}$$

with

$$\rho_{N_A,N'_A}(t)=\frac{r^{(N_A+N'_A)}}{\sqrt{N_A!N'_A!}}\exp\left[-r^2+i\zeta_0(N_A-N'_A)-\frac{D_At}{2}(N_A-N'_A)^2\right] , \tag{12}$$

where r and ζ_0 are, respectively, the amplitude and phase of the initial coherent state, and D_A is the decay constant of the off-diagonal matrix elements. Now, using Eqs. (8) and (9), we obtain the density matrix in the true modes of the electromagnetic field. The result is

$$\rho(t)=\sum_{\substack{n_1,l_1\\n_2,l_2}}\rho_{\substack{n_1,l_1\\n_2,l_2}}(t)|n_1,n_2\rangle\langle l_1,l_2| , \tag{13}$$

where

$$\rho_{\substack{n_1,l_1\\n_2,l_2}}(t)=e^{-r^2}\left[\frac{r}{\sqrt{2}}\right]^{n_1+n_2+l_1+l_2}\frac{\exp[i\zeta_0(n_1+n_2-l_1-l_2)-(D_At/2)(n_1+n_2-l_1-l_2)^2]}{\sqrt{N_1!n_2!l_1!l_2!}} . \tag{14}$$

The diagonal matrix elements of ρ in two-mode phase states can be calculated in a straightforward manner. The result is

$$\langle\theta_{m_1},\theta_{m_2}|\rho|\theta_{m_1},\theta_{m_2}\rangle=\frac{e^{-r^2}}{(s+1)^2}\sum_{\substack{n_1,n_2\\l_1,l_2}}\left[\frac{r}{\sqrt{2}}\right]^{n_1+n_2+l_1+l_2}\frac{\exp[i(\zeta_0-\theta_{m_1})(n_1-l_1)+i(\zeta_0-\theta_{m_2})(n_2-l_2)-(D_At/2)(n_1+n_2-l_1-l_2)^2]}{\sqrt{n_1!n_2!l_1!l_2!}} . \tag{15}$$

Now, we will assume that we have a highly excited initial coherent state, namely, $r^2\gg 1$. Then, one can approximate

$$f(k)=\frac{b^{2k}}{k!}e^{-b^2}\simeq\frac{1}{\sqrt{2\pi b^2}}e^{-(b^2-k)^2/2b^2}\ . \tag{16}$$

Taking $b=r/\sqrt{2}$ and substituting the square root of Eq. (16) into Eq. (15), and transforming the sums into integrals, we readily get

$$\langle\theta_{m_1},\theta_{m_2}|\rho|\theta_{m_1},\theta_{m_2}\rangle=\frac{\pi r\sqrt{8}}{a(s+1)^2}\exp\left[\frac{-[\zeta_0-(\theta_1+\theta_2)/2]^2}{a^2}\right]\exp\left[-2r^2\left[\frac{\theta_{m_1}-\theta_{m_2}}{2}\right]^2\right], \tag{17}$$

with

$$a^2=2\left[\frac{1}{4r^2}+D_A t\right]\ . \tag{18}$$

Taking the continuous limit in Eq. (5), we can define a phase probability density distribution as

$$P(\theta_1,\theta_2)=\frac{(s+1)^2}{4\pi^2}\langle\theta_1,\theta_2|\rho|\theta_1,\theta_2\rangle\ . \tag{19}$$

This probability is normalized in the range $-\infty\le\theta_{m_1},\theta_{m_2}\le\infty$. Originally $\langle\theta_{m_1},\theta_{m_2}|\rho|\theta_{m_1},\theta_{m_2}\rangle$ was normalized in the $[\theta_{0_1},\theta_{0_1}+2\pi]\times[\theta_{0_2},\theta_{0_2}+2\pi]$ interval. From Eq. (17), we notice immediately that $P(\theta_1,\theta_2)$ is not invariant under the $\zeta_0\to\zeta_0+2\pi k$ translation, with k integer. In order to recover both properties, we define the probability density:

$$P^{(\zeta_0)}(\theta_1,\theta_2)=\frac{1}{2}P_0^{(\zeta_0)}\left[\frac{\theta_1+\theta_2}{2},a^2\right]\times P_0^{(0)}\left[\frac{\theta_1-\theta_2}{2},\frac{1}{2r^2}\right], \tag{20}$$

with

$$P_0^{(\gamma)}(\delta,g^2)=\frac{1}{\sqrt{\pi g^2}}\sum_{k=-\infty}^{\infty}\exp\left[-\frac{(\gamma+2\pi k-\delta)^2}{g^2}\right]\ . \tag{21}$$

Using the properties of Gaussians, it is simple to prove that

$$\int_{\theta_{0_1}}^{\theta_{0_1}+2\pi}d\theta_1\int_{\theta_{0_2}}^{\theta_{0_2}+2\pi}d\theta_2 P^{(\zeta_0)}(\theta_1,\theta_2)=1 \tag{22}$$

and

$$P^{(\zeta_0+2\pi k)}(\theta_1,\theta_2)=P^{(\zeta_0)}(\theta_1,\theta_2)\ ,\quad k=0,1,2,\ldots\ . \tag{23}$$

Since θ_{0_1} and θ_{0_2} are arbitrary, we choose $\theta_{0_1}=\theta_{0_2}=\zeta_0-\pi$. Now, since we have the probability distribution for the phases, we can calculate the fluctuations of $\hat{\phi}_{\theta_1}-\hat{\phi}_{\theta_2}$ [as in Eq. (5)] and also of $\hat{\phi}_{\theta_1}+\hat{\phi}_{\theta_2}$. The results are

$$\langle[\Delta(\hat{\phi}_{\theta_1}-\hat{\phi}_{\theta_2})]^2\rangle=\frac{1}{r^2}+16\pi^2\sum_{k=1}^{\infty}k^2[\Phi(\sqrt{2\pi^2}(2\pi k+\pi))-\Phi(\sqrt{2r^2}(2\pi k-\pi))]$$
$$-8\sqrt{2\pi/r^2}\sum_{k=1}^{\infty}k\{\exp[-2r^2(2\pi k-\pi)^2]-\exp[-2r^2(2\pi k+\pi)^2]\}\ , \tag{24}$$

$$\langle[\Delta(\hat{\phi}_{\theta_1}+\hat{\phi}_{\theta_2})]^2\rangle=\frac{1}{r^2}+4D_A t+16\pi^2\sum_{k=1}^{\infty}k^2\left[\Phi\left[\frac{2\pi k-\pi}{\sqrt{a^2}}\right]-\Phi\left[\frac{2\pi k-\pi}{\sqrt{a^2}}\right]\right]$$
$$-16\sqrt{\pi a^2}\sum_{k=1}^{\infty}k\left[\exp\left[\frac{-(2\pi k-\pi)^2}{a^2}\right]-\exp\left[\frac{-(2\pi k+\pi)^2}{a^2}\right]\right], \tag{25}$$

where $\Phi(x)$ is the error function, and a^2 is given by Eq. (18). For $r^2\gg1$, we get

$$\langle[\Delta(\hat{\phi}_{\theta_1}-\hat{\phi}_{2\theta_2})]^2\rangle\simeq\frac{1}{r^2}\ . \tag{26}$$

The conclusions are the following.

(a) The phase-difference fluctuations are time independent and extremely small ($r^2\gg1$). This accounts for a vanishing diffusion constant, in full agreement with Ref. 8.

(b) The phase-sum fluctuations are time dependent and large. The corresponding diffusion constant, after conversion from the A to the real modes, gives identical results to the diffusion constant of the sum of the phases as given, again, in Ref. 8.

The present description of the phase fluctuations of a quantum-beat laser using the Pegg and Barnett phase operators is by far more complete than the Fokker-Planck representation, because it not only contains the phase diffusion constants, but it also has the shot noise term and saturation, for short and long measurement times, respectively, both absent in the Fokker-Planck description. Although we have not discussed here the saturation, the last two terms in Eq. (25) have a saturation effect in the phase-sum fluctuations, since for small times, $\langle [\Delta(\hat{\phi}_{\theta_1}+\hat{\phi}_{\theta_2})/2]^2 \rangle$, that is the fluctuations of the average phase increases linearly with time, the curve flattens out as $t \geq 1/D_A$, acquiring a constant value of $\pi^2/3$ when $t \gg 1/D_A$. This value corresponds to the fluctuations of a classical phase, when the probability distribution becomes uniform (random phase) over the whole interval. This last point is discussed in detail in Ref. 7.

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[1]D. T. Pegg and S. M. Barnett, Europhys. Lett. **6**, 483 (1988).

[2]S. M. Barnett and D. T. Pegg, J. Mod. Opt. **36**, 7 (1989).

[3]R. Lynch, Phys. Rev. A **41**, 2841 (1990).

[4]C. C. Gerry and K. E. Urbanski, Phys. Rev. A **42**, 662 (1990).

[5]J. A. Vaccaro and D. T. Pegg, Opt. Commun. **70**, 529 (1989).

[6]N. Grønbeck-Jensen and P. L. Christiansen, J. Opt. Soc. Am. B **6**, 2423 (1989).

[7]M. Orszag and C. Saavedra, Phys. Rev. A **43**, 554 (1991).

[8]J. A. Bergou, M. Orszag, and M. O. Scully, Phys. Rev. A **38**, 754 (1988).

[9]For the original idea of the CEL see M. O. Scully, Phys. Rev. Lett. **55**, 2802 (1985). The linear theory is found in M. O. Scully and M. S. Zubairy, Phys. Rev. A **35**, 752 (1987).

[10]R. Loudon, *The Quantum Theory of Light,* 1st ed. (Oxford University Press, 1973), p. 149.

[11]D. T. Pegg and S. M. Barnett, Phys. Rev. A **39**, 1665 (1989).

[12]M. O. Scully and W. Lamb, Phys. Rev. **159**, 208 (1967).

[13]M. Sargent III, M. O. Scully, and W. E. Lamb, *Laser Physics* (Addison Wesley, Reading MA, 1974).

[14]For example, see Chap. 15 of Ref. 13.