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Zero separation results for solutions of second order linear differential equations[☆]

Martin Chuaqui[a], Janne Gröhn[b], Janne Heittokangas[b,*], Jouni Rättyä[b]

[a] *Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Casilla 306, Correo 22, Santiago, Chile*
[b] *Department of Physics and Mathematics, University of Eastern Finland, P.O. Box 111, FI-80101 Joensuu, Finland*

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Abstract

The oscillation of solutions of $f'' + Af = 0$ is discussed by focusing on four separate situations. In the complex case A is assumed to be either analytic in the unit disc $\mathbb{D}$ or entire, while in the real case A is continuous either on $(-1, 1)$ or on $(0, \infty)$. In all situations A is expected to grow beyond bounds that ensure finite oscillation for all (non-trivial) solutions, and the separation between distinct zeros of solutions is considered.

In the complex case, it is shown that the growth of the maximum modulus of A determines the minimal separation of zeros of all solutions, and vice versa. This gives rise to new concepts called *zero separation exponents*, which measure the separation of zeros of either all solutions or of individual analytic functions. In $\mathbb{D}$ these quantities are defined in terms of the hyperbolic distance, while in the complex plane the Euclidean distance is used. As a by-product of these findings, the 1955-result of B. Schwarz, which asserts that $\sup_{z\in\mathbb{D}} |A(z)|(1 - |z|^2)^2 < \infty$ if and only if the zero-sequences of all solutions are separated in the hyperbolic sense, is rediscovered. The striking plane analogue established reveals that the Euclidean distance between all distinct zeros of every solution is uniformly bounded away from zero if and only if A is a constant. As an outgrowth of the results, new information on the zero distribution of solutions in the classical polynomial coefficient case is also obtained. The main results are proved by using a method of

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* Corresponding author.
E-mail addresses: mchuaqui@mat.puc.cl (M. Chuaqui), janne.grohn@uef.fi (J. Gröhn), janne.heittokangas@uef.fi (J. Heittokangas), jouni.rattya@uef.fi (J. Rättyä).

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localization, which naturally induces characterizations of certain subclasses of locally univalent functions in terms of the growth of their pre-Schwarzian and Schwarzian derivatives.

In the real case, it is shown that the separation of zeros of non-trivial solutions is restricted according to the growth of A, but not conversely.

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1. Introduction

The purpose of this paper is to offer a unified and consistent discussion on the oscillation of solutions of the linear differential equation

$$f'' + Af = 0 \tag{1.1}$$

in different situations. The methods employed here give a new approach to this classical topic. In the real case, $A = A(x)$ is assumed to be continuous either on a finite open interval or on a half-bounded interval. In the complex case, $A = A(z)$ is analytic either in the open unit disc $\mathbb{D}$ or in the whole complex plane $\mathbb{C}$. Under these assumptions all zeros of all non-trivial solutions of (1.1) are simple. The treatment that follows produces estimates for the separation of zeros of non-trivial solutions in terms of the growth of the coefficient, whenever the coefficient grows sufficiently fast permitting infinite oscillation for non-trivial solutions. In the instance of complex differential equations the converse problem is also addressed. It turns out that lower bounds for the separation of zeros induce growth restrictions to the coefficient. Therefore, in particular, a one-to-one correspondence between the separation of zeros of solutions and the growth of the maximum modulus of the coefficient is obtained. The proofs of the main results rest upon a method of localization providing with an effective tool that takes full advantage of classical results due to Kraus, Nehari and Sturm, which are the foundation of this study. The results and classifications obtained are discussed by means of several non-trivial examples that also illustrate the variety of different phenomena, with regard to the distribution of zeros of solutions, that may occur.

Our motivation originates from the Euler differential equation

$$f'' + \frac{c}{x^2} f = 0, \quad x \in (0, \infty). \tag{1.2}$$

By [12, p. 20] it is known that, if $c \le 1/4$, then (1.2) is *disconjugate*, which means that every non-trivial solution vanishes at most once. If $c > 1/4$, then all solutions have infinitely many zeros by Sturm's theorem on interlacing zeros, see [5, Chapter 2]. In fact, the fine line between disconjugacy and infinite oscillation can be refined by means of logarithmic terms, see [12, p. 20]. The differential equation

$$f'' + \frac{c}{(1-x^2)^2} f = 0, \quad x \in (-1, 1), \tag{1.3}$$

is an analogue of (1.2) on the interval $(-1, 1)$, see [8, p. 161] and [35, p. 162]. Now (1.3) is disconjugate for $c \le 1$, and for $c > 1$ every solution vanishes infinitely many times. Moreover, the hyperbolic distance between two consecutive zeros of any non-trivial solution of (1.3) is exactly $\pi/\sqrt{c-1}$, which is easily verified by means of [8, Eq. (2)].

In order to make more refined statements, one is led to consider coefficients of the form

$$A(x) = \frac{1+\epsilon(|x|)}{(1-x^2)^2}, \quad x \in (-1,1),$$

and to determine the order to which the continuous function $\epsilon(r) > 0$ must decay to zero as $r \to 1^-$ to give distinction between finite and infinite oscillation. By [8, Theorem 1] this distinction is given by $\epsilon(r) = \log^{-2}(1-x)$. Again the constant coefficient is optimal in the sense that (1.1) becomes *oscillatory*, i.e., it possesses a non-trivial solution having infinitely many zeros, if $\epsilon(r) = c\log^{-2}(1-x)$ for any $c > 1$. Our intention is to analyse those cases when the coefficient $A(x)$ grows essentially faster than $(1-x^2)^{-2}$; say $A(x) = (1-x^2)^{-2-p}$ for $p > 0$. Since (1.1) is then oscillatory, it is sensible to determine lower bounds for the hyperbolic distance between consecutive zeros. These lower bounds tend to zero near the endpoints of the interval $(-1,1)$, at rates depending on the growth of $A(x)(1-x^2)^2$ near $x = \pm 1$, and hence we consider coefficients satisfying the growth restriction

$$A(x) \le \frac{1}{\psi(|x|)^2(1-x^2)^2}, \quad x \in (-1,1),$$

where $\psi = \psi(r)$ is a positive function approaching to zero as $r \to 1^-$. The utilized techniques are based on localization, and they also render optimal results for half-bounded intervals, which are modelled by $(0,\infty)$.

Considerations in $\mathbb{D}$ run parallel to the ones on $(-1,1)$. For example, if

$$|A(z)| \le \frac{1}{(1-|z|^2)^2}, \quad z \in \mathbb{D}, \tag{1.4}$$

then every non-trivial solution of (1.1) vanishes at most once in $\mathbb{D}$. In another form, this corresponds to the well-known theorem of Z. Nehari [33, Theorem 1], which provides with a sufficient condition for injectivity of a locally univalent meromorphic function in $\mathbb{D}$ in terms of the growth of its Schwarzian derivative. If (1.4) holds only in an annulus $\{z \in \mathbb{D} : a < |z| < 1\}$ for some $a \in (0,1)$, then by a result of B. Schwarz [35, Theorem 1] all non-trivial solutions vanish at most finitely many times in $\mathbb{D}$. A quantitative version of Schwarz's theorem [9, Theorem 1] shows that the number of zeros of non-trivial solutions of (1.1) in $\mathbb{D}$ is then at most $O(1/(1-a))$. As in the real case, infinite oscillation is possible provided that the numerator in (1.4) is replaced by any constant strictly greater than 1.

The distinction between finite and infinite oscillation in the complex case is not as well understood as in the real case. In particular, it is no longer true that

$$|A(z)| \le \frac{1+\log^{-2}(1-|z|)}{(1-|z|^2)^2}, \quad z \in \mathbb{D},$$

implies finite oscillation for non-trivial solutions of (1.1). In fact, if $\epsilon : [0,1) \to (0,\infty)$ is a continuous function satisfying $\epsilon(r)/(1-r) \to \infty$, as $r \to 1^-$, then there exists an analytic coefficient A, depending on ϵ, such that

$$|A(z)| \le \frac{1+\epsilon(|z|)}{(1-|z|^2)^2}, \quad z \in \mathbb{D},$$

and (1.1) is oscillatory, see [9, Theorem 5]. An important discovery [35, Theorems 3 and 4], also due to B. Schwarz, characterizes the separation of distinct zeros in terms of the growth of the coefficient in a certain special case. To be more precise, this neat result states that the distance between distinct zeros of non-trivial solutions of (1.1) is uniformly bounded away from zero in the hyperbolic sense if and only if

$$\sup_{z\in\mathbb{D}} |A(z)|(1-|z|^2)^2 < \infty. \tag{1.5}$$

For the best possible constant lower bound for the separation of zeros, under the restriction (1.5), we refer to [26]. Corresponding to the preceding case of the real interval, we investigate coefficients that satisfy the growth restriction

$$|A(z)| \le \frac{1}{\psi(|z|)^2(1-|z|^2)^2}, \quad z \in \mathbb{D},$$

where $\psi = \psi(r)$ decays to zero as $r \to 1^-$. The method of localization employed on the interval $(-1, 1)$ for the real differential equation (1.1) not only provides with results for the separation of zeros of non-trivial solutions of (1.1) in $\mathbb{D}$ with respect to the hyperbolic metric, but also addresses via Kraus' theorem [31] the converse direction resulting growth restrictions for the coefficient A. This reveals the following new discovery with regards to oscillatory equations: The growth of the maximum modulus of A determines the minimal separation of zeros of all solutions, and vice versa. Our main result in the unit disc case is therefore a true generalization of Schwarz's characterization.

The counterpart of Nehari's result in the case of the complex plane is the trivial condition $A \equiv 0$. In fact, this condition corresponds to the sole disconjugate differential equation (1.1) with an entire coefficient, see Lemma 24 below. Our analysis on the equation (1.1) with an entire coefficient $A \not\equiv 0$ results in a one-to-one correspondence between the Euclidean distance between zeros of non-trivial solutions and the growth of the maximum modulus of A. As an immediate consequence of the results obtained the following striking analogue of Schwarz's characterization [35, Theorems 3 and 4] for the complex plane case is established: The Euclidean distance between distinct zeros of non-trivial solutions of (1.1) is uniformly bounded away from zero if and only if the entire coefficient A is a constant. It is worth mentioning that our analysis on the plane case is based on the same methods that we employ in $\mathbb{D}$, and therefore we strongly rely on the classical theorems of Nehari and Kraus related to univalent functions in $\mathbb{D}$.

As a byproduct of our analysis, we end up characterizing certain classes of locally univalent functions in terms of the growth of their pre-Schwarzian and Schwarzian derivatives. These classes generalize so-called locally uniformly univalent functions, and they are of independent interest.

Finally, we mention that our results regarding the complex case give a new point of view on the connection between the coefficient and solutions of (1.1). It is well-known that the three concepts — the growth of the coefficient, the growth of solutions and the quantity of zeros of solutions — are closely related [23]. In this regard, our contribution is to introduce the notion of *zero separation exponents*, which measure the separation of zeros either of all non-trivial solutions of (1.1) or of individual analytic functions. In $\mathbb{D}$ these concepts $\Lambda_{\mathrm{DE}}(A)$ and $\Lambda(f)$ are defined in terms of the hyperbolic distance, while in $\mathbb{C}$ the corresponding quantities $\Upsilon_{\mathrm{DE}}(A)$ and $\Upsilon(f)$ involve the Euclidean distance. Our findings clearly show that in view of differential equations, the separation of zeros of non-trivial solutions gives a fourth quantity, which is firmly linked to the previously found three other quantities.

2. Real intervals

In this section, we study the oscillation of solutions of

$$f'' + A(x)f = 0 \tag{2.1}$$

on an interval I of the real line, assuming that $A(x)$ is continuous on I. The standard choices for I are finite intervals and half-bounded intervals. Without loss of generality, we limit our analysis to $(-1, 1)$ and $(0, \infty)$.

2.1. *Interval* $(-1, 1)$

Our point of departure is Theorem 1 below, which concerns the hyperbolic distance between consecutive zeros of solutions of (2.1) on the open interval $(-1, 1)$. This result is stated in terms of an auxiliary function ψ satisfying the technical condition (2.2), which is studied in detail in Section 2.3.

Recall that, for any complex numbers z_1 and z_2 in the unit disc $\mathbb{D}$, the *pseudo-hyperbolic distance* $\varrho_p(z_1, z_2)$ and the *hyperbolic distance* $\varrho_h(z_1, z_2)$ between z_1 and z_2 are given by

$$\varrho_p(z_1, z_2) = \left|\varphi_{z_1}(z_2)\right| \quad \text{and} \quad \varrho_h(z_1, z_2) = \frac{1}{2}\log\frac{1 + \varrho_p(z_1, z_2)}{1 - \varrho_p(z_1, z_2)},$$

where $\varphi_a(z) = (a - z)/(1 - \overline{a}z)$. Correspondingly,

$$\Delta_p(a, r) = \left\{z \in \mathbb{D} : \varrho_p(z, a) < r\right\} \quad \text{and} \quad \Delta_h(a, r) = \left\{z \in \mathbb{D} : \varrho_h(z, a) < r\right\}$$

are the *pseudo-hyperbolic* and the *hyperbolic open discs* of radius $r > 0$ centred at $a \in \mathbb{D}$, respectively. We employ the same notation with obvious modifications also in the real case.

Theorem 1. *Let A be a continuous function in $(-1, 1)$, and let $\psi : [0, 1) \to (0, 1)$ be a non-increasing function such that*

$$K = \sup_{0\le x<1} \frac{\psi(x)}{\psi\left(\frac{x+\psi(x)}{1+x\psi(x)}\right)} < \infty. \tag{2.2}$$

If

$$A(x)\big(\psi(|x|)(1 - x^2)\big)^2 \le M < \infty, \quad x \in (-1, 1), \tag{2.3}$$

then the hyperbolic distance between any distinct zeros x_1 and x_2 of any non-trivial solution of (2.1) satisfies

$$\varrho_h(x_1, x_2) \ge \log \frac{1 + \frac{\psi(|t_h(x_1,x_2)|)}{\max\left\{K\sqrt{M},1\right\}}}{1 - \frac{\psi(|t_h(x_1,x_2)|)}{\max\left\{K\sqrt{M},1\right\}}}, \tag{2.4}$$

where $t_h(x_1, x_2)$ is the hyperbolic mid-point of x_1 and x_2.

Proof. Let $\{f_1, f_2\}$ be a solution base of (2.1), and set $h = f_1/f_2$. Then the Schwarzian derivative

$$S_h = \left(\frac{h''}{h'}\right)' - \frac{1}{2}\left(\frac{h''}{h'}\right)^2$$

satisfies the identity $S_h = 2A$. For $a \in (-1, 1)$, set $g_a(x) = (h \circ \varphi_a)\big(\psi(|a|)rx\big)$, where $r = \min\big\{(K\sqrt{M})^{-1}, 1\big\}$. Then the assumption (2.3) yields

$$\begin{aligned} S_{g_a}(x)(1-x^2)^2 &= S_h\big(\varphi_a(\psi(|a|)rx)\big)\left(\varphi_a'(\psi(|a|)rx)\right)^2\left(\psi(|a|)r\right)^2(1-x^2)^2 \\ &\le 2M\left(\frac{\varphi_a'\big(\psi(|a|)rx\big)\,\psi(|a|)\,(1-x^2)}{\big(1-\left(\varphi_a(\psi(|a|)rx)\right)^2\big)\psi\big(|\varphi_a(\psi(|a|)rx)|\big)}\right)^2 r^2 \\ &= 2M\left(\frac{1-x^2}{1-\left(\psi(|a|)rx\right)^2}\right)^2\left(\frac{\psi(|a|)}{\psi\big(|\varphi_a(\psi(|a|)rx)|\big)}\right)^2 r^2 \\ &\le 2M\left(\sup_{a\in(-1,1)}\frac{\psi(|a|)}{\psi\left(\frac{|a|+\psi(|a|)}{1+|a|\psi(|a|)}\right)}\right)^2 r^2 \le 2 \end{aligned}$$

for all $x \in (-1, 1)$.

Since $v'' + (1-x^2)^{-2}v = 0$ has a non-vanishing solution $\sqrt{1-x^2}$ on $(-1, 1)$, every non-trivial solution of $u'' + \frac{1}{2}S_{g_a}(x)u = 0$ has at most one zero in $(-1, 1)$ by Sturm's comparison theorem [5, Chapter 2]. In particular, this is true for

$$u(x) = \frac{(f \circ \varphi_a)\big(\psi(|a|)rx\big)}{\sqrt{\varphi_a'\big(\psi(|a|)rx\big)\,\psi(|a|)r}}, \quad x \in (-1, 1),$$

where f is any non-trivial solution of the Eq. (2.1). We conclude that every non-trivial solution f of (2.1) has at most one zero in

$$\Delta_p\big(a, \psi(|a|)r\big) = \Delta_h\left(a, \frac{1}{2}\log\frac{1+\psi(|a|)r}{1-\psi(|a|)r}\right), \quad a \in (-1, 1). \tag{2.5}$$

Assume that f is a non-trivial solution of (2.1) having two distinct zeros $x_1 < x_2$ in $(-1, 1)$, and take $a = t_h(x_1, x_2)$. The above argumentation shows that

$$\varrho_h(x_1, x_2) \ge \log\frac{1+\psi(|a|)r}{1-\psi(|a|)r} = \log\frac{1+\psi(|t_h(x_1, x_2)|)r}{1-\psi(|t_h(x_1, x_2)|)r},$$

for otherwise (2.5) for $a = t_h(x_1, x_2)$ would contain two distinct zeros of f. The claim (2.4) follows by substituting the value of r.

We note that a zero-separation result parallel to Theorem 1, without the condition (2.2), can be obtained by applying Sturm's comparison theorem in the line segments between distinct zeros. For example, if A is a continuous function in $(-1, 1)$ satisfying (2.3), where $\psi : [0, 1) \to \left(0, \sqrt{M}/\pi\right)$ is non-increasing, then an application of [5, Theorem 8, p. 47] yields

$$\varrho_h(x_1, x_2) \ge \frac{1}{2}\log\frac{1+\frac{2\pi\,\psi(\max\{|x_1|,|x_2|\})}{\sqrt{M}\left(\sqrt{1+4\pi^2\,\psi(\max\{|x_1|,|x_2|\})^2/M}+1\right)}}{1-\frac{2\pi\,\psi(\max\{|x_1|,|x_2|\})}{\sqrt{M}\left(\sqrt{1+4\pi^2\,\psi(\max\{|x_1|,|x_2|\})^2/M}+1\right)}} \tag{2.6}$$

for all distinct zeros x_1 and x_2 of every non-trivial solution f of (2.1).

With regard to Theorem 1, it is known that the separation of zeros of non-trivial solutions of (2.1) does not restrict the growth of the coefficient A. This follows from [20, Corollary 5, p. 346], which implies that (2.1) is disconjugate whenever $\int_{-1}^{1} \max\{A(x), 0\}\, dx \le 2$. Therefore, if A is chosen appropriately, then $\max_{|x|\le r} A(x)$ exceeds any pre-given function in growth, while (2.1) is disconjugate.

The following example illustrates Theorem 1. It is useful to notice that, if $x_1, x_2 \in (-1, 1)$ such that $x_1 < x_2$, then the hyperbolic mid-point $t_h(x_1, x_2)$ is given by

$$t_h(x_1, x_2) = \frac{\exp\bigl(\varrho_h(x_1, x_2)\bigr)(1 + x_1) - (1 - x_1)}{\exp\bigl(\varrho_h(x_1, x_2)\bigr)(1 + x_1) + (1 - x_1)}. \tag{2.7}$$

Example 2. Consider the differential equation

$$f'' + \frac{1}{(1 - x)^4} f = 0 \tag{2.8}$$

on the interval $(-1, 1)$. In this case infinite oscillation of non-trivial solutions occurs only near the point $x = 1$. A solution base $\{f_1, f_2\}$ of (2.8) is given by means of the functions

$$f_1(x) = (1 - x) \cos \frac{1}{1 - x} \quad \text{and} \quad f_2(x) = (1 - x) \sin \frac{1}{1 - x},$$

and hence the zeros x_k of $f = \alpha f_1 + \beta f_2$ are the solutions of $\tan\bigl(1/(1-x_k)\bigr) = -\beta/\alpha$. Evidently, $x_k = 1 - (\mu + k\pi)^{-1} \in (-1, 1)$, $k \in \mathbb{Z}$, for some $\mu \in [-\pi/2, \pi/2)$ depending on the value $-\beta/\alpha$. Direct computation shows that

$$\varrho_h(x_k, x_{k+1}) = \frac{1}{2} \log\left(\frac{2\mu + 2\pi(1 + k) - 1}{2\mu + 2\pi k - 1}\right) \sim \frac{1}{2k}, \quad k \to \infty,$$

where $A \sim B$ means that $A/B \to 1$ in the limit process in question. If we define $\psi(x) = (1 - x)/2$, then

$$(1 - x^2)^2 \psi(x)^2 A(x) = \frac{(1 + x)^2}{4} \le 1 = M, \quad x \in (-1, 1),$$

and further,

$$K = \sup_{0\le x<1} \frac{\psi(x)}{\psi\left(\frac{x+\psi(x)}{1+x\psi(x)}\right)} = \sup_{0\le x<1} (2 - x) = 2.$$

Theorem 1 implies that

$$\varrho_h(x_k, x_{k+1}) \ge \log \frac{1 + \frac{1}{2}\psi\bigl(t_h(x_k, x_{k+1})\bigr)}{1 - \frac{1}{2}\psi\bigl(t_h(x_k, x_{k+1})\bigr)} = \log\left(\frac{T + 3}{T + 1}\right) \sim \frac{1}{2k\pi}, \quad k \to \infty,$$

where $T = 2\sqrt{(2\mu + 2\pi k - 1)(2\mu + 2\pi(1 + k) - 1)}$.

The following example generalizes the analysis in Example 2.

Example 3. Suppose that $p : (-1, 1) \to (0, \infty)$ is a continuously differentiable function with a strictly negative derivative, and $\lim_{x\to 1^-} p(x) = 0$. Now, the functions

$$f_1 = \frac{p}{\sqrt{-p'}} \cos \frac{1}{p} \quad \text{and} \quad f_2 = \frac{p}{\sqrt{-p'}} \sin \frac{1}{p}$$

are linearly independent solutions of (2.1) with $A = (p')^2/p^4 + (1/2)S_p$.

If we choose $p(x) = (1-x)^\alpha, \alpha > 0$, then

$$A(x) = \frac{\alpha^2}{(1-x)^{2+2\alpha}} + \frac{1}{4}\frac{1-\alpha^2}{(1-x)^2}.$$

If $\psi(x) = 2^{-1}(1-x)^\alpha$, then both the actual distance between consecutive zeros x_k and x_{k+1}, and the estimate from Theorem 1, are asymptotically equal to a constant multiple of $1/k$. We omit the details of these computations.

The last example concerning the interval $(-1, 1)$ deals with damped harmonic oscillators. Note that the weight functions $\psi(x) = \exp\left(-(1-x)^{-1}\right)$ and $\psi(x) = (1-x)^\alpha$ for $\alpha > 0$ satisfy the assumption $(1-x^2)^2\psi(x)^2\xi(x) \to 0$, as $x \to 1^-$, in Example 4.

Example 4. Let ψ be a twice continuously differentiable positive function on $(-1, 1)$, and define

$$\xi(x) = -\frac{1}{2}\frac{\psi''(x)}{\psi(x)} + \frac{x}{1-x^2}\frac{\psi'(x)}{\psi(x)} + \frac{1}{4}\left(\frac{\psi'(x)}{\psi(x)}\right)^2 + \frac{1}{(1-x^2)^2}.$$

If $(1-x^2)^2\psi(x)^2\xi(x) \to 0$, as $x \to 1^-$, then there exists a continuous function A on $(-1, 1)$ with

$$A(x) \sim \frac{1}{(1-x^2)^2\psi(x)^2}, \quad x \to 1^-, \tag{2.9}$$

such that (2.1) possesses linearly independent solutions

$$\begin{aligned} f_1(x) &= \sqrt{(1-x^2)\psi(x)}\sin\left(\int_0^x \frac{dt}{(1-t^2)\psi(t)}\right), \\ f_2(x) &= \sqrt{(1-x^2)\psi(x)}\cos\left(\int_0^x \frac{dt}{(1-t^2)\psi(t)}\right). \end{aligned} \tag{2.10}$$

To prove the existence of A with the property (2.9), we argue as follows. Let $F(x)$ be a positive and twice continuously differentiable function on $(-1, 1)$. Then the functions

$$\begin{aligned} y_1(x) &= F(x)^{-1/4}\sin\left(\int_0^x F(t)^{1/2}\,dt\right), \\ y_2(x) &= F(x)^{-1/4}\cos\left(\int_0^x F(t)^{1/2}\,dt\right), \end{aligned}$$

are linearly independent solutions of

$$y'' + B(x)y = 0, \quad B(x) = F(x) + \frac{1}{4}\frac{F''(x)}{F(x)} - \frac{5}{16}\left(\frac{F'(x)}{F(x)}\right)^2,$$

see [28, pp. 478–479]. Choose $F(x) = (1-x^2)^{-2}\psi(x)^{-2}$. Then the functions y_1 and y_2 are respectively equal to the functions f_1 and f_2 in (2.10). A simple computation yields $B(x) = F(x)+\xi(x)$. Since $\xi(x)/F(x) \to 0$, as $x \to 1^-$, by assumption, we have $B(x) \sim F(x)$, as $x \to 1^-$. Note that, if $(1-x^2)\psi(x) \to 0$, as $x \to 1^-$, then all solutions of (2.1) decay to zero as $x \to 1^-$, no matter how fast the coefficient grows as $x \to 1^-$.

2.2. Positive real axis

By a suitable Möbius transformation, Theorem 1 can be translated to the half-bounded interval $(0, \infty)$. Note that, if we are interested in the oscillation of solutions near the infinity, the property

$\Psi(x) = \Psi\left(\frac{1}{x}\right)$ in Theorem 5 is not needed. As in the case of Theorem 1, the separation of zeros of non-trivial solutions of (2.1) does not restrict the growth of the coefficient A on $(0, \infty)$.

Theorem 5. *Let A be a continuous function on the interval $(0, \infty)$, and let $\Psi : (0, \infty) \to (0, 1)$ be non-increasing on $[1, \infty)$ such that $\Psi(x) = \Psi\left(\frac{1}{x}\right)$ for all $x \in (0, \infty)$, and*

$$K = \sup_{1 \le x < \infty} \frac{\Psi(x)}{\Psi\left(x \frac{1+\Psi(x)}{1-\Psi(x)}\right)} < \infty. \tag{2.11}$$

If

$$A(x)\big(\Psi(x)x\big)^2 \le M < \infty, \quad x \in (0, \infty), \tag{2.12}$$

then the Euclidean distance between any distinct zeros x_1 and x_2 of any non-trivial solution of (2.1) satisfies

$$|x_1 - x_2| \ge 2 \min\left\{\left(2K\sqrt{M}\right)^{-1}, 1\right\} t_a(x_1, x_2)\, \Psi\big(t_g(x_1, x_2)\big),$$

where $t_a(x_1, x_2)$ and $t_g(x_1, x_2)$ are the arithmetic and the geometric mean value of x_1 and x_2, respectively.

Proof. Assume that f is a non-trivial solution of (2.1) having two zeros $0 < x_1 < x_2 < \infty$. Let $T(y) = (1 + y)/(1 - y)$. Consequently, $g(y) = f(T(y))(T'(y))^{-\frac{1}{2}}$ is a solution of

$$g'' + B(y)g = 0, \quad B(y) = A\big(T(y)\big)\big(T'(y)\big)^2 = A\big(T(y)\big)\frac{4}{(1-y)^4}, \tag{2.13}$$

having two zeros $-1 < y_1 < y_2 < 1$, where $y_1 = T^{-1}(x_1)$ and $y_2 = T^{-1}(x_2)$. We proceed to show that (2.13) satisfies the hypothesis of Theorem 1, if the non-increasing function $\psi : [0, 1) \to (0, 1)$ is defined by $\psi = \Psi \circ T$. By (2.11),

$$\sup_{0 \le y < 1} \frac{\psi(y)}{\psi\left(\frac{y+\psi(y)}{1+y\psi(y)}\right)} = \sup_{0 \le y < 1} \frac{\Psi\big(T(y)\big)}{\Psi\left(T(y)\frac{1+\Psi(T(y))}{1-\Psi(T(y))}\right)} = K < \infty.$$

Moreover, (2.12) implies

$$B(y)\big(\psi(|y|)(1 - y^2)\big)^2 = 4\, A\big(T(y)\big)\left(\Psi\big(T(|y|)\big)T(y)\right)^2 \le 4M, \quad y \in (-1, 1).$$

Note that, if y is negative, then $\Psi\big(T(|y|)\big) = \Psi\big(1/T(y)\big) = \Psi\big(T(y)\big)$, while if y is positive, then $\Psi\big(T(|y|)\big) = \Psi\big(T(y)\big)$ trivially.

We conclude from Theorem 1 that the zeros y_1 and y_2 satisfy (2.4), and hence

$$\left|\frac{y_2 - y_1}{1 - y_2 y_1}\right| \ge \frac{2\psi\big(|t_h(y_1, y_2)|\big)r}{1 + \psi\big(|t_h(y_1, y_2)|\big)^2 r^2} > \psi\big(|t_h(y_1, y_2)|\big)r, \quad r = \frac{1}{\max\left\{2K\sqrt{M}, 1\right\}},$$

where $t_h(y_1, y_2)$ is the hyperbolic mid-point of y_1 and y_2. By means of (2.7) we get

$$\psi\big(|t_h(y_1, y_2)|\big) = \Psi\big(T(|t_h(y_1, y_2)|)\big) = \Psi\big(T(t_h(y_1, y_2))\big) = \Psi\left(\sqrt{x_1 x_2}\right).$$

Therefore,

$$\left|\frac{x_2 - x_1}{x_1 + x_2}\right| \geq \frac{\Psi\left(\sqrt{x_1 x_2}\right)}{\max\left\{2K\sqrt{M}, 1\right\}},$$

which proves the claim.

We note that a zero-separation result parallel to Theorem 5, without the condition (2.11), can be obtained with the aid of (2.6) and the Möbius transformation $T(y) = (1+y)/(1-y)$. The details are omitted.

The following example illustrates Theorem 5.

Example 6. We discuss the oscillation of solutions of (2.1) on the infinite end of $[1, \infty)$ by considering $A(x) \equiv 1$ with Ψ defined on $[1, \infty)$ by $\Psi(x) = 1/(2x)$. Then

$$A(x)\big(\Psi(x)x\big)^2 = \frac{1}{4} = M,$$

and a simple computation shows that $K = 3$ in (2.11). Consecutive zeros $x_k < x_{k+1}$ of non-trivial solutions of (2.1) are separated in the Euclidean sense by π, since these solutions are linear combinations of sin and cos, and hence $x_{k+1} = x_k + \pi$. In this case, Theorem 5 yields

$$|x_k - x_{k+1}| \geq \frac{2}{3}\,\frac{2x_k + \pi}{2}\,\frac{1}{2\sqrt{x_k(x_k+\pi)}} \to \frac{1}{3}, \qquad k \to \infty.$$

We close this section with two examples concerning the asymptotic growth of the coefficient in the infinite end of positive real axis. Note that weight functions $\Psi(x) = e^{-x}$ and $\Psi(x) = x^{-\alpha}$ for $\alpha > 0$ satisfy the assumption $x^2\,\Psi(x)^2\xi(x) \to 0$, as $x \to \infty$, in Example 7.

Example 7. Let Ψ be a twice continuously differentiable positive function on $[1, \infty)$, and define

$$\xi(x) = -\frac{1}{2}\,\frac{\Psi''(x)}{\Psi(x)} - \frac{1}{2x}\,\frac{\Psi'(x)}{\Psi(x)} + \frac{1}{4}\left(\frac{\Psi'(x)}{\Psi(x)}\right)^2 + \frac{1}{4x^2}.$$

If $x^2\,\Psi(x)^2\xi(x) \to 0$, as $x \to \infty$, then there exists a continuous function A on $[1, \infty)$ with $A(x) \sim x^{-2}\,\Psi(x)^{-2}$, as $x \to \infty$, such that (2.1) possesses linearly independent solutions

$$f_1(x) = \sqrt{x\,\Psi(x)}\,\sin\left(\int_1^x \frac{dt}{t\,\Psi(t)}\right), \qquad f_2(x) = \sqrt{x\,\Psi(x)}\,\cos\left(\int_1^x \frac{dt}{t\,\Psi(t)}\right).$$

The proof is similar to that in Example 4. Note that, if $x\,\Psi(x) \to 0$, as $x \to \infty$, then all solutions of (2.1) decay to zero as $x \to \infty$, no matter how fast the coefficient grows as $x \to \infty$.

Example 8. Suppose that $A(x) = x^{-2}\,\Psi(x)^{-2}$, where $\Psi(x) = x^{-\alpha}$ and $\alpha > 1$. On the one hand, Theorem 5 shows that any two zeros $x_{n-1} < x_n < x_{n-1} + 1$ of any non-trivial solution of (2.1) satisfy

$$|x_{n-1} - x_n| \gtrsim t_a(x_{n-1}, x_n)\,\Psi\big(t_g(x_{n-1}, x_n)\big) \gtrsim x_n\,\Psi(x_n) = x_n^{1-\alpha}, \qquad x_n \to \infty.$$

The notation $h(r) \gtrsim g(r)$ means that there exists a constant $C > 0$ such that $h(r) \geq C\,g(r)$ for all sufficiently large r. Note that the lower bound for the separation of zeros induces an upper bound for the number of zeros. In particular, we conclude that each non-trivial solution f of

(2.1) has $n(r, f, 0) \lesssim r^\alpha$ zeros on the interval $[1, r)$ for $1 < r < \infty$. On the other hand, since $A'(x)A(x)^{-3/2} \to 0$, as $x \to \infty$, [16, Lemma 4] shows that

$$n(r, f, 0) \sim \frac{1}{\pi}\int_1^r \sqrt{A(x)}\,dx \sim \frac{1}{\pi\alpha} r^\alpha, \quad r \to \infty.$$

Consequently, the estimate for the counting function of zeros resulting from Theorem 5 is of the correct order of magnitude.

2.3. Discussion on the weight functions

In this section we consider the weight function ψ appearing in Theorem 1. The proof of Theorem 1 shows that the hypothesis on ψ can be relaxed. In particular, if $\psi : (-1, 1) \to (0, 1)$ satisfies

$$K = \sup_{a\in(-1,1)} \sup_{x\in(-1,1)} \frac{\psi(a)}{\psi(\varphi_a(\psi(a)x))} < \infty, \tag{2.14}$$

and $A(x)\big(\psi(x)(1-x^2)\big)^2 \le M$ on $(-1, 1)$, then we deduce (2.4) with $t_h(x_1, x_2)$ in place of $|t_h(x_1, x_2)|$. For example, in the case of Example 2 this implies the finite oscillation of non-trivial solutions near the point $x = -1$. By using (2.14) instead of (2.2) we get a result more general than Theorem 1, but not too much is gained by this generalization. This is due to the fact that (2.2) for $\psi : [0, 1) \to (0, 1)$ non-increasing is not very restrictive, because it permits ψ to either decrease arbitrarily fast or arbitrarily slowly. Namely, if $\psi : [0, 1) \to (0, 1)$ is differentiable and convex such that $\psi(x) \to 0^+$, as $x \to 1^-$, then

$$\begin{aligned}\psi\left(\frac{x+\psi(x)}{1+x\psi(x)}\right) &\ge \psi(x) + \psi'(x)\left(\frac{x+\psi(x)}{1+x\psi(x)} - x\right)\\ &\ge \psi(x)\left(1 + \psi'(x)\frac{1-x^2}{1+x\psi(x)}\right),\end{aligned}$$

where

$$\lim_{x\to 1^-}\left(1 + \psi'(x)\frac{1-x^2}{1+x\psi(x)}\right) = 1.$$

It follows that $K < \infty$ in (2.2). On the other hand, if $\psi : [0, 1) \to (0, 1)$ is concave and $\psi(x) \to 0^+$, as $x \to 1^-$, then the image of $[x, 1)$ under ψ lies above the line segment joining $(x, \psi(x))$ and $(1, 0)$. Hence

$$\frac{\psi(x)}{1-x}\left(1 - \frac{x+\psi(x)}{1+x\psi(x)}\right) \le \psi\left(\frac{x+\psi(x)}{1+x\psi(x)}\right),$$

which implies

$$\frac{\psi(x)}{\psi\left(\frac{x+\psi(x)}{1+x\psi(x)}\right)} \le \frac{1+x\psi(x)}{1-\psi(x)} \to 1^+, \quad x \to 1^-,$$

and we again deduce (2.2). It is also worth noticing that

$$\left|\frac{\psi\left(\frac{x+\psi(x)}{1+x\psi(x)}\right)}{\psi(x)} - 1\right| = \left|\frac{\psi(x) - \psi\left(\frac{x+\psi(x)}{1+x\psi(x)}\right)}{x - \frac{x+\psi(x)}{1+x\psi(x)}}\right| \frac{1-x^2}{1+x\psi(x)},$$

and hence (2.2) holds, if the Lipschitz condition

$$\sup_{0<s<t<1} \left| \frac{\psi(s) - \psi(t)}{s - t} \right| < \infty$$

is satisfied.

Example 9. We construct a non-increasing function ψ for which (2.2) fails. We only have to fix the value of ψ at a point sequence tending to 1, while the behaviour of ψ elsewhere is not of the essence. In particular, ψ can be made continuous or differentiable on $[0, 1)$ if needed. Let $x_k = 1 - 2^{-k}$, and let $\varepsilon_k \in (0, 1)$ be a strictly decreasing sequence such that $\varepsilon_k \to 0^+$, as $k \to \infty$. Define $\psi(x_1) = 1/4$, and define the values y_k, $\psi(y_k)$ and $\psi(x_{k+1})$ inductively by

$$y_k = \frac{x_k + \psi(x_k)}{1 + x_k\psi(x_k)}, \qquad \psi(y_k) = \varepsilon_k\psi(x_k) \quad \text{and} \quad \psi(x_{k+1}) = \psi(y_k), \quad k \in \mathbb{N}.$$

Since $\psi(x_k) < 1/3 < (2 + x_k)^{-1}$, we conclude $x_k < y_k < x_{k+1}$ for all $k \in \mathbb{N}$. By the construction, we have

$$\frac{\psi(x_k)}{\psi\left(\frac{x_k+\psi(x_k)}{1+x_k\psi(x_k)}\right)} = \frac{\psi(x_k)}{\psi(y_k)} = \frac{1}{\varepsilon_k} \to \infty, \quad k \to \infty,$$

and hence (2.2) fails. Note that $\psi(y_k)$ lies below the line segment joining $(x_k, \psi(x_k))$ and $(x_{k+1}, \psi(x_{k+1}))$, which implies that ψ is not concave, while $\psi(x_k)$ lies above the line segment joining $(y_{k-1}, \psi(y_{k-1}))$ and $(y_k, \psi(y_k))$, which shows that ψ is not convex.

We finish this section with a result, which may be of independent interest. It shows that for all non-increasing and continuous functions $\psi : [0, 1) \to (0, 1)$, the expression $\psi(x)\psi\left(\frac{x+\psi(x)}{1+x\psi(x)}\right)^{-1}$ in (2.2) is bounded outside of a relatively small exceptional set. The proof is influenced by the standard proof of Borel's lemma needed in Nevanlinna theory.

Theorem 10. *Let $\psi : [0, 1) \to (0, 1)$ be a non-increasing and continuous function, and let $k > 1$. Then there exists a constant $C > 0$, depending on k, such that*

$$\psi(x) < k\,\psi\left(\frac{x + \psi(x)}{1 + x\psi(x)}\right) \tag{2.15}$$

outside a set $E \subset [0, 1)$ of x-values satisfying $\int_E dx/(1 - x) \le C < \infty$.

Proof. Let $E \subset [0, 1)$ be the set of x-values for which (2.15) is false. If $E = \emptyset$ or if $\sup E < 1$, then there is nothing to prove. Hence, we may suppose that $\sup E = 1$.

Let $x_1 = \inf E$. We proceed to define a sequence $\{x_n\}$ of points in $[0, 1)$ inductively. Suppose that x_n is defined for some $n \in \mathbb{N}$, and write

$$x_n' = \frac{x_n + \psi(x_n)}{1 + x_n\psi(x_n)}.$$

It is clear that $x_n < x_n' < 1$. Let then $x_{n+1} = \inf\big(E \setminus [0, x_n')\big)$. Since each set $E \setminus [0, x_n')$ is non-empty by the assumption $\sup E = 1$, this process produces an infinite sequence $\{x_n\}$. Note that by the continuity of ψ the inequality (2.15) is false at each point $x = x_n$, $n \in \mathbb{N}$. Thus E contains the sequence $\{x_n\}$. From the definition of x_{n+1} it follows that there are no points of E in (x_n', x_{n+1}). Hence $E \subset \bigcup_{n=1}^{\infty}[x_n, x_n']$.

The sequence $\{x_n\}$ is increasing by the construction. We prove that $x_n \to 1^-$. Suppose on the contrary that $x_n \to r^-$, as $n \to \infty$, for some $r \in (0, 1)$. Since $x_n < x_n' \leq x_{n+1}$, we have $x_n' \to r^-$. Now

$$x_n' - x_n = \frac{(1-x_n^2)\psi(x_n)}{1+x_n\psi(x_n)} \to \frac{(1-r^2)\psi(r)}{1+r\psi(r)} > 0, \quad n \to \infty,$$

by the continuity of ψ. This contradicts the fact that $x_n' - x_n \to 0^+$, as $n \to \infty$.

It remains to estimate the logarithmic measure of the set $\bigcup_{n=1}^{\infty}[x_n, x_n']$. Since ψ is non-increasing and $x_n \in E$, we have

$$\psi(x_{n+1}) \leq \psi(x_n') = \psi\left(\frac{x_n + \psi(x_n)}{1+x_n\psi(x_n)}\right) \leq \frac{1}{k}\psi(x_n).$$

Inductively,

$$\psi(x_{n+1}) \leq \frac{1}{k}\psi(x_n) \leq \cdots \leq \frac{1}{k^n}\psi(x_1) \leq \frac{1}{k^n},$$

and hence

$$\begin{aligned}\int_{x_{n+1}}^{x_{n+1}'} \frac{dx}{1-x} &= \log\frac{1-x_{n+1}}{1-x_{n+1}'} = \log\frac{1+x_{n+1}\psi(x_{n+1})}{1-\psi(x_{n+1})} \leq \log\frac{1+\psi(x_{n+1})}{1-\psi(x_{n+1})} \\ &\leq \log\frac{k^n+1}{k^n-1} \leq \frac{k^n+1}{k^n-1} - 1 = \frac{2}{k^n-1}.\end{aligned}$$

Finally,

$$\int_E \frac{dx}{1-x} \leq \sum_{n=0}^{\infty}\int_{x_{n+1}}^{x_{n+1}'}\frac{dx}{1-x} \leq \sum_{n=0}^{\infty}\frac{2}{k^n-1} < \infty,$$

since $k > 1$, and we are done.

The function ψ in Example 9 shows that Theorem 10 is no longer true without an exceptional set.

3. Unit disc

In this section, we discuss the oscillation of solutions of

$$f'' + A(z)f = 0 \tag{3.1}$$

in the unit disc, assuming that the analytic coefficient A grows essentially faster than $(1-|z|^2)^{-2}$ near the boundary of $\mathbb{D}$. In particular, the following considerations fall on the interplay between the growth of the coefficient and the separation of zeros of solutions of (3.1). This section gives a detailed account of both radial and non-radial estimates, and ties new results to the existing oscillation theory.

For the convenience of the reader, we begin with some elementary observations in hyperbolic geometry. If z and $z^\star$ are two points in any pseudo-hyperbolic disc $\Delta_p(a, r)$, where $a \in \mathbb{D}$ and $r \in (0, 1)$, then the hyperbolic mid-point $t_h(z, z^\star) \in \Delta_p(a, r)$; the same is obviously true for all hyperbolic discs. The following assertions, which explore the geometric position of $t_h(z, z^\star)$ in terms of $z, z^\star \in \mathbb{D}$, are needed later. Suppose that $\{z_n\}$ and $\{z_n^\star\}$ are sequences of points in $\mathbb{D}$, and $\zeta \in \partial\mathbb{D}$:

(i) If $z_n, z_n^\star \to \zeta$, then $t_h(z_n, z_n^\star) \to \zeta$, as $n \to \infty$;

(ii) If $t_h(z_n, z_n^\star) \to \zeta$ and $|z_n - z_n^\star| \to 0^+$, then $z_n, z_n^\star \to \zeta$, as $n \to \infty$;
(iii) If $|t_h(z_n, z_n^\star)| \to 1^-$ and $|z_n - z_n^\star| \to 0^+$, then $|z_n|, |z_n^\star| \to 1^-$, as $n \to \infty$.

3.1. Radial weights

Our first result concerning the unit disc case resembles Theorem 1. It shows that the separation of zeros of non-trivial solutions of (3.1) is essentially dictated by the boundary behaviour of the coefficient, and, in contrast to the real case, also vice versa. Note that $R^\star$ in (3.3) is a discontinuous function of R; if $R = 0$, then the assertion concerns $\mathbb{D}$, while if $R > 0$, then the result relates to certain annuli in $\mathbb{D}$. See Section 2.3 for a detailed study of the condition (3.2).

Theorem 11. *Let A be analytic in $\mathbb{D}$, $R \in [0, 1)$, and let $\psi : [R, 1) \to (0, 1)$ be a non-increasing function such that*

$$K = \sup_{R^\star \le r < 1} \frac{\psi(r)}{\psi\left(\frac{r+\psi(r)}{1+r\psi(r)}\right)} < \infty, \tag{3.2}$$

where

$$R^\star = \begin{cases} \dfrac{\psi(R) + R}{1 + \psi(R)R}, & \text{if } 0 < R < 1, \\ 0, & \text{if } R = 0. \end{cases} \tag{3.3}$$

(i) *If the coefficient A satisfies*

$$|A(z)|\big(\psi(|z|)(1 - |z|^2)\big)^2 \le M < \infty, \quad R \le |z| < 1, \tag{3.4}$$

then the hyperbolic distance between any distinct zeros z_1 and z_2 of any non-trivial solution of (3.1), for which $|t_h(z_1, z_2)| \ge R^\star$, satisfies

$$\varrho_h(z_1, z_2) \ge \log \frac{1 + \frac{\psi(|t_h(z_1,z_2)|)}{\max\left\{K\sqrt{M},1\right\}}}{1 - \frac{\psi(|t_h(z_1,z_2)|)}{\max\left\{K\sqrt{M},1\right\}}}. \tag{3.5}$$

(ii) *Conversely, if (3.5) is satisfied for all distinct zeros z_1 and z_2 of every non-trivial solution of (3.1), for which $|t_h(z_1, z_2)| \ge R$, then the coefficient A satisfies*

$$|A(z)|\big(\psi(|z|)(1 - |z|^2)\big)^2 < 3K^2 \max\{K^2 M, 1\}, \quad R^\star \le |z| < 1. \tag{3.6}$$

Proof. (i) Let $\{f_1, f_2\}$ be a solution base of (3.1), and set $h = f_1/f_2$ so that $S_h = 2A$. For $a \in \mathbb{D}$, set $g_a(z) = (h \circ \varphi_a)\big(\psi(|a|)rz\big)$, where $r = 1/\max\left\{K\sqrt{M}, 1\right\}$. If $|a| \ge R^\star$, where $R^\star$ is given by (3.3), then

$$\varphi_a\big(\psi(|a|)rz\big) \in \Delta_p\big(a, \psi(|a|)r\big) \subset \Delta_p\big(a, \psi(R)\big) \subset \big\{z \in \mathbb{D} : R \le |z|\big\}, \quad z \in \mathbb{D},$$

and for these values of a, the assumption (3.4) yields

$$\begin{aligned} |S_{g_a}(z)|(1 - |z|^2)^2 &= \big|S_h\big(\varphi_a(\psi(|a|)rz)\big)\big| \big|\varphi_a'\big(\psi(|a|)rz\big)\big|^2 \big(\psi(|a|)r\big)^2 (1 - |z|^2)^2 \\ &\le 2M \left(\frac{\big|\varphi_a'\big(\psi(|a|)rz\big)\big|\psi(|a|)(1 - |z|^2)}{\big(1 - |\varphi_a\big(\psi(|a|)rz\big)|^2\big)\psi\big(|\varphi_a(\psi(|a|)rz)|\big)}\right)^2 r^2 \end{aligned}$$

$$= 2M\left(\frac{1-|z|^2}{1-|\psi(|a|)rz|^2}\right)^2\left(\frac{\psi(|a|)}{\psi\big(|\varphi_a(\psi(|a|)rz)|\big)}\right)^2 r^2$$

$$\leq 2M\left(\sup_{|a|\geq R^\star}\frac{\psi(|a|)}{\psi\left(\frac{|a|+\psi(|a|)}{1+|a|\psi(|a|)}\right)}\right)^2 r^2 \leq 2$$

for all $z\in\mathbb{D}$. Therefore g_a is univalent in $\mathbb{D}$ for any $|a|\geq R^\star$, and we conclude that $h=f_1/f_2$ is univalent in each hyperbolic disc

$$\Delta_h\left(a,\frac{1}{2}\log\frac{1+\psi(|a|)r}{1-\psi(|a|)r}\right),\qquad |a|\geq R^\star.$$

Assume now that f is a non-trivial solution of (3.1) having two distinct zeros $z_1,z_2\in\mathbb{D}$, for which $|t_h(z_1,z_2)|\geq R^\star$, and take $a=t_h(z_1,z_2)$. Since $\varrho_h(z_1,a)=\varrho_h(z_2,a)=(1/2)\varrho_h(z_1,z_2)$, it follows that

$$\varrho_h(z_1,z_2)=2\varrho_h(z_1,a)\geq\log\frac{1+\psi(|a|)r}{1-\psi(|a|)r}.$$

The claim (3.5) follows by substituting the values of a and r.

(ii) Assume that all distinct zeros $z_1,z_2\in\mathbb{D}$ of every non-trivial solution of (3.1), for which $|t_h(z_1,z_2)|\geq R$, satisfy (3.5). First, we show that each non-trivial solution of (3.1) vanishes at most once in

$$\Delta_h\left(a,\frac{1}{2}\log\frac{1+R_a}{1-R_a}\right)=\Delta_p(a,R_a),\qquad R_a=\frac{\psi(|a|)}{K\max\left\{K\sqrt{M},1\right\}},$$

for $|a|\geq R^\star$, where $R^\star$ is given by (3.3). Assume on the contrary, that there exists a non-trivial solution of (3.1) having two distinct zeros $z_1,z_2\in\Delta_p(a,R_a)$ for some $|a|\geq R^\star$. It follows that $t_h(z_1,z_2)\in\Delta_p(a,R_a)$, and consequently,

$$|t_h(z_1,z_2)|<\frac{|a|+R_a}{1+|a|R_a}\leq\frac{|a|+\psi(|a|)}{1+|a|\psi(|a|)}.$$

Hence

$$\frac{\psi(|a|)}{\psi\big(|t_h(z_1,z_2)|\big)}\leq\frac{\psi(|a|)}{\psi\left(\frac{|a|+\psi(|a|)}{1+|a|\psi(|a|)}\right)}\leq K \tag{3.7}$$

by (3.2). We deduce from the antithesis and (3.7) that

$$\varrho_h(z_1,z_2)<\log\frac{1+R_a}{1-R_a}=\log\frac{1+\frac{\psi(|a|)}{K\max\left\{K\sqrt{M},1\right\}}}{1-\frac{\psi(|a|)}{K\max\left\{K\sqrt{M},1\right\}}}\leq\log\frac{1+\frac{\psi(|t_h(z_1,z_2)|)}{\max\left\{K\sqrt{M},1\right\}}}{1-\frac{\psi(|t_h(z_1,z_2)|)}{\max\left\{K\sqrt{M},1\right\}}}.$$

This estimate contradicts (3.5), since $t_h(z_1,z_2)\in\Delta_p\big(a,\psi(R)\big)\subset\{z\in\mathbb{D}:|z|\geq R\}$. Hence each non-trivial solution of (3.1) vanishes at most once in $\Delta_p(a,R_a)$ for every $|a|\geq R^\star$.

Second, since $z\mapsto\varphi_a(R_a z)$ maps $\mathbb{D}$ onto $\Delta_p(a,R_a)$, the discussion above shows that $g_a(z)=h\big(\varphi_a(R_a z)\big)$ is univalent in $\mathbb{D}$ for all $|a|\geq R^\star$. For these values of a, we have

$$|S_{g_a}(z)|(1-|z|^2)^2=\left|S_h\big(\varphi_a(R_a z)\big)\right||\varphi_a'(R_a z)|^2R_a^2(1-|z|^2)^2\leq 6,\qquad z\in\mathbb{D},$$

by Kraus' theorem [31]. For another reference, see [33, p. 545]. Take $z = 0$ and use $S_h = 2A$ to obtain

$$2|A(a)|(1-|a|^2)^2 \frac{\psi(|a|)^2}{K^2 \max\{K^2 M, 1\}} \le 6, \quad |a| \ge R^\star.$$

The assertion (3.6) follows.

We point out that a zero-separation result parallel to Theorem 11(i), without the condition (3.2), can be obtained by applying Sturm's comparison theorem rather than applying Nehari's univalence criteria. For example, if A is an analytic function in $\mathbb{D}$ satisfying (3.4) for $R = 0$, where $\Psi : [0,1) \to \left(0, \sqrt{M}/\pi\right)$ is non-increasing and continuous, then a straightforward application of [28, Corollary on p. 579] yields $\varrho_h(z_1, z_2) \ge 2^{-1} \log\bigl((1+\rho)/(1-\rho)\bigr)$, where

$$\rho = \frac{\pi}{\sqrt{M}}\,\psi\left(\frac{\max\{|z_1|,|z_2|\} + \frac{\pi}{\sqrt{M}}\,\psi\bigl(\max\{|z_1|,|z_2|\}\bigr)}{1 + \max\{|z_1|,|z_2|\}\frac{\pi}{\sqrt{M}}\,\psi\bigl(\max\{|z_1|,|z_2|\}\bigr)}\right) \cdot \left(1 - \left(\frac{\pi}{\sqrt{M}}\,\psi\bigl(\max\{|z_1|,|z_2|\}\bigr)\right)^2\right),$$

for all distinct zeros z_1 and z_2 of every non-trivial solution f of (3.1). We may also apply Sturm's comparison theorem directly on the hyperbolic geodesics between distinct zeros, and then obtain a slightly different lower bound for the separation of zeros corresponding the estimate (2.6). In this approach the weight function ψ is not required to be continuous. For a similar reasoning, see [7, p. 19].

Conversely, by a modification of the proof of Theorem 11(ii), if $M \in (0,\infty)$ and $\psi : [0,1) \to \left(0, \sqrt{M}\right)$ is non-increasing, and if all distinct zeros z_1 and z_2 of every non-trivial solution satisfy

$$\varrho_h(z_1,z_2) \ge \log \frac{1 + \psi\bigl(\max\{|z_1|,|z_2|\}\bigr)/\sqrt{M}}{1 - \psi\bigl(\max\{|z_1|,|z_2|\}\bigr)/\sqrt{M}},$$

then

$$|A(z)| \left(\psi\left(\frac{|z| + \psi(|z|)/\sqrt{M}}{1 + |z|\psi(|z|)/\sqrt{M}}\right)(1-|z|^2)\right)^2 \le 3M, \quad z \in \mathbb{D}.$$

The advantage of these results, when compared to Theorem 11, is the fact that the technical condition (3.2) is not needed. However, when (3.2) is satisfied, then one may use either these results or Theorem 11, and find the most useful estimate for each purpose by studying the different constants appearing in the statements and the behaviour of the weight function ψ in the points in question.

Theorem 11(i) is proved by means of [33, Theorem I], although we could equally use [33, Theorem II], and estimate the growth of $|S_{g_a}(z)|$ without the weight $(1-|z|^2)^2$. This is due to the fact that ψ does not attain the value 1. The same is also true for Theorems 15(i) and 25(i) below. Moreover, Schwarz's results [35, Theorems 3 and 4] (see also [26]) follow from Theorem 11(i) by choosing $R = 0$ and $\psi \equiv C \in (0,1)$ sufficiently large.

We turn to consider the situation in which the quantity $\sup_{z\in\mathbb{D}} |A(z)|(1-|z|^2)^2$ is no longer finite.

Example 12. Let $q \in (1, \infty)$, and consider the locally univalent analytic function $p(z) = \big(\log(e/(1-z))\big)^q$ in $\mathbb{D}$. The functions

$$f_1(z) = \frac{1}{\sqrt{p'(z)}} \sin\big(p(z)\big), \qquad f_2(z) = \frac{1}{\sqrt{p'(z)}} \cos\big(p(z)\big)$$

are linearly independent solutions of (3.1) with

$$\begin{aligned} A(z) &= \big(p'(z)\big)^2 + \frac{1}{2} S_p(z) \\ &= \frac{1}{(1-z)^2} \left(q^2 \left(\log \frac{e}{1-z}\right)^{2(q-1)} + \frac{1}{4} + \frac{1}{4} \frac{1-q^2}{\left(\log \frac{e}{1-z}\right)^2} \right). \end{aligned}$$

Zeros of f_1 are real, and they are given by $z_k = 1 - \exp\big(1 - (k\pi)^{1/q}\big)$, $k \in \mathbb{Z}$. Evidently,

$$\varrho_h(z_k, z_{k+1}) = \frac{1}{2} \log \frac{1 + \varrho_p(z_k, z_{k+1})}{1 - \varrho_p(z_k, z_{k+1})} \sim \frac{\pi}{2q} (\pi k)^{1/q-1}, \quad k \to \infty.$$

An application of Theorem 11 with $\psi(r) = (1/2)\big(\log(e/(1-r))\big)^{1-q}$, satisfying (3.2) for $K = \big(\log(2e)\big)^{q-1}$ yields

$$\varrho_h(z_k, z_{k+1}) \gtrsim \psi\big(|t_h(z_k, z_{k+1})|\big) \sim \frac{1}{2} (\pi k)^{1/q-1}, \quad k \to \infty.$$

We conclude that the estimate resulting from Theorem 11 is of the correct order of magnitude.

The following two examples concern the case when $|A(z)|$ grows at most like a negative power of $1 - |z|$, as $|z| \to 1^-$. The set of all such analytic functions is known as the *Korenblum space* $A^{-\infty}$, whose theory is rich. For example, $A^{-\infty}$ contains all classical Hardy and Bergman spaces of the disc. It is also well-known that in the sense of differential equations, functions in $A^{-\infty}$ play a similar role in $\mathbb{D}$ as polynomials do in $\mathbb{C}$; see Section 4, and for example [11,22].

Example 13. For $\beta > 0$, the functions

$$f_j(z) = (1-z)^{\frac{\beta+1}{2}} \exp\left(\frac{(-1)^{j+1} i}{(1-z)^\beta} \right), \quad j = 1, 2, \tag{3.8}$$

are linearly independent solutions of (3.1), where

$$A(z) = \frac{\beta^2}{(1-z)^{2\beta+2}} + \frac{1}{4} \frac{1-\beta^2}{(1-z)^2}. \tag{3.9}$$

Functions f_1 and f_2 are non-vanishing, and the zeros of $f = \alpha f_1 + \beta f_2$, where $\alpha\beta \neq 0$, are given by

$$z_k = 1 - \left(\frac{2}{c + 2\pi k} \right)^{1/\beta}, \quad k \in \mathbb{Z}.$$

Here c is a complex constant, which agrees with the principal value of $-i \log(-\beta/\alpha)$.

Take $\alpha = -1$ and $\beta = e^i$, which imply that $c = 1$, and further, all zeros z_k of $f = \alpha f_1 + \beta f_2$ are real. Moreover, $\varrho_h(z_k, z_{k+1}) \sim (2\beta k)^{-1}$, as $k \to \infty$, while Theorem 11 with $\psi(r) =$

$(1/2)(1-r)^\beta$, $K = 2^\beta$ and $M = \beta^2 + 1/16$ gives us the estimate

$$\varrho_h(z_k, z_{k+1}) \geq \frac{\psi\big(|t_h(z_k, z_{k+1})|\big)}{\max\left\{K\sqrt{M}, 1\right\}} \sim \frac{1}{\max\left\{2^\beta\sqrt{\beta^2+1/16}, 1\right\} 2\pi k}, \quad k \to \infty.$$

Differential equations in Examples 12 and 13 are solvable, and therefore we are able to verify that the bounds for the separation of zeros of solutions given by Theorem 11 are of the correct order of magnitude. In the following example we investigate a differential equation, which is too complicated to be solved explicitly. However, a notable amount of information about the solutions can be retrieved whenever the behaviour of the coefficient is known. The coefficient A in Example 14 is given as an infinite product, which has regularly spaced zeros in $\mathbb{D}$. It turns out that A grows regularly in a large subset of $\mathbb{D}$, and possesses similar growth properties to what is commonly seen with lacunary series. The weights ω satisfying (3.10) are known as regular weights, see for example [37]. We write $g(r) \asymp h(r)$, if there are positive constants C_1 and C_2, which are independent of r, such that $C_1\, g(r) \leq h(r) \leq C_2\, g(r)$ for all sufficiently large r.

Example 14. Let $0 < p < \infty$ and let $\omega : [0, 1) \to (0, \infty)$ be a continuous function such that $\int_0^1 \omega(r)\,dr < 1$,

$$\frac{1}{\omega(r)} \int_r^1 \omega(s)\,ds \asymp (1-r), \tag{3.10}$$

and

$$\frac{\left(\int_r^1 \omega(s)\,ds\right)^{\frac{p}{2}}}{1-r} \to 0^+, \quad r \to 1^-.$$

By [37, Lemma 1.1(i)] there are constants $0 < \alpha \leq \beta$, depending on ω, such that

$$\begin{aligned}\left(\frac{1-r}{1-t}\right)^\alpha \int_t^1 \omega(s)\,ds &\leq \int_r^1 \omega(s)\,ds \\ &\leq \left(\frac{1-r}{1-t}\right)^\beta \int_t^1 \omega(s)\,ds, \quad 0 \leq r \leq t < 1.\end{aligned} \tag{3.11}$$

Following [15], let A be the infinite product defined by

$$A(z) = \prod_{k=1}^\infty F_k(z) = \prod_{k=1}^\infty \frac{1 + a_k z^{n_k}}{1 + a_k^{-1} z^{n_k}}, \quad z \in \mathbb{D},\ a_k = \left(\frac{\int_{1-n_k^{-1}}^1 \omega(s)\,ds}{\int_{1-n_{k+1}^{-1}}^1 \omega(s)\,ds}\right)^p,$$

where $n_{k+1}/n_k = q$ for all $k \in \mathbb{N}$, and q is any fixed natural number strictly greater than β/α. Define $L = q^{\alpha p}$ and $U = q^{\beta p}$. By choosing $r = 1 - n_k^{-1}$ and $t = 1 - n_{k+1}^{-1}$ in (3.11) we conclude $1 < L < a_k < U < \infty$ for all $k \in \mathbb{N}$. Moreover, note that $L^q > U$, which is needed later. We begin with a discussion on the properties of A, and then proceed to consider solutions of (3.1).

Function A is analytic in $\mathbb{D}$, since $|F_k(z)| = |a_k|\, \varrho_p\big(a_k^{-1}, -z^{n_k}\big) < U < \infty$ for all $z \in \mathbb{D}$, and

$$\sum_{k=1}^\infty |F_k(z) - 1| \leq \sum_{k=1}^\infty \frac{|a_k - a_k^{-1}|\,|z|^{n_k}}{1 - a_k^{-1}|z|^{n_k}} \leq \frac{a_k - a_k^{-1}}{1 - a_k^{-1}} \sum_{k=1}^\infty |z|^{n_k} \leq (U+1) \sum_{k=1}^\infty |z|^{n_k}$$

converges uniformly on compact subsets of $\mathbb{D}$; see [38, Theorem 15.4]. Regarding the growth of A, we conclude from [15] that

$$M(r, A) \lesssim \left(\int_r^1 \omega(s)\, ds \right)^{-p}, \quad 0 \le r < 1, \tag{3.12}$$

where $M(r, A) = \max_{|z|=r} |A(z)|$ is the maximum modulus of A. In addition, by a modification of [15], there exists a subset of $[0, 1)$, namely

$$F = \bigcup_{k=1}^{\infty} \left[a_k^{-n_k^{-1}(1-\delta)} a_{k+1}^{-n_{k+1}^{-1}\delta}, a_k^{-n_k^{-1}\delta} a_{k+1}^{-n_{k+1}^{-1}(1-\delta)} \right], \quad \delta \in (0, 1/2),$$

such that

$$M(r, A) \asymp \left(\int_r^1 \omega(s)\, ds \right)^{-p}, \quad r \in F. \tag{3.13}$$

The lower density $\underline{d}(F)$ of F is positive, provided that δ is sufficiently small, since

$$\begin{aligned}
\underline{d}(F) &= \liminf_{r\to 1^-} \frac{m\big(F \cap [r, 1)\big)}{1-r} \ge \lim_{k\to\infty} \frac{a_{k+1}^{-n_{k+1}^{-1}\delta} a_{k+2}^{-n_{k+2}^{-1}(1-\delta)} - a_{k+1}^{-n_{k+1}^{-1}(1-\delta)} a_{k+2}^{-n_{k+2}^{-1}\delta}}{1 - a_k^{-n_k^{-1}\delta} a_{k+1}^{-n_{k+1}^{-1}(1-\delta)}} \\
&\ge \lim_{k\to\infty} \frac{\left(U^{-\delta/q-(1-\delta)/q^2}\right)^{1/n_k} - \left(L^{-(1-\delta)/q-\delta/q^2}\right)^{1/n_k}}{1 - \left(U^{-\delta-(1-\delta)/q}\right)^{1/n_k}} \\
&= \frac{\left(\frac{1-\delta}{q} + \frac{\delta}{q^2}\right)\log L - \left(\frac{\delta}{q} + \frac{1-\delta}{q^2}\right)\log U}{\left(\delta + \frac{1-\delta}{q}\right)\log U} \longrightarrow \frac{\log L^q - \log U}{q \log U} > 0, \quad \delta \to 0^+,
\end{aligned}$$

by Bernoulli–l'Hospital's rule.

The following discussion deals with a-points of A, where $a \in \mathbb{C}$. The zeros of A can be found explicitly, and in particular, there are exactly n_k distinct zeros on each circle $\{z \in \mathbb{D} : |z| = a_k^{-1/n_k}\}$. This implies that the non-integrated counting function of zeros in $\{z \in \mathbb{D} : |z| \le r\}$ satisfies $n(r, A, 0) \asymp (1-r)^{-1}$, as $r \to 1^-$. Similarly as in the proof of [13, Theorem 3], a laborious calculation shows that

$$T(r, A) \sim \log M(r, A) \sim N(r, A, a) \sim -p \log \left(\int_r^1 \omega(s)\, ds \right), \quad r \to 1^-,$$

for all $a \in \mathbb{C}$. Here $T(r, A)$ is the Nevanlinna characteristic of A, and $N(r, A, a)$ is the integrated counting function for a-points of A.

We turn to consider the properties of solutions of the differential equation (3.1). Let $\psi : [0, 1) \to (0, 1)$ be a non-increasing function such that

$$\psi(r) \asymp \frac{\left(\int_r^1 \omega(s)\, ds\right)^{\frac{p}{2}}}{1-r}, \quad r \to 1^-. \tag{3.14}$$

By choosing $t_r = \frac{r+\psi(r)}{1+r\psi(r)}$ in (3.11), we obtain

$$\sup_{0\le r<1} \frac{\psi(r)}{\psi(t_r)} \le \sup_{0\le r<1} \left(\frac{1-t_r}{1-r}\right)^{1-\beta p/2} = \sup_{0\le r<1} \left(\frac{1-\psi(r)}{1+r\psi(r)}\right)^{1-\beta p/2} < \infty,$$

from which (3.2) follows. Now Theorem 11 ensures two things. First, the pseudo-hyperbolic distance between any two distinct zeros z_1 and z_2 of any non-trivial solution f of (3.1) satisfies $\varrho_p(z_1, z_2) \gtrsim \psi(|t_h(z_1, z_2)|)$. Second, there must exist a sequence f_n of non-trivial solutions of (3.1), such that each f_n possesses two distinct zeros $z_n, z_n^\star \in \mathbb{D}$, with $|t_h(z_n, z_n^\star)| > 1 - 1/n$ and

$$\psi\big(|t_h(z_n, z_n^\star)|\big) \lesssim \varrho_p(z_n, z_n^\star) \lesssim \tau\big(|t_h(z_n, z_n^\star)|\big), \quad n \in \mathbb{N}, \tag{3.15}$$

where $\tau\colon [0, 1) \to (0, 1)$ is any non-increasing function, which satisfies (3.2) and the requirement $1 < \tau(r)/\psi(r) \to \infty$, as $r \to 1^-$. To prove the second inequality in (3.15), suppose that there exists $n \in \mathbb{N}$ such that (3.5) with τ in place of ψ is valid for all distinct zeros z and $z^\star$ of every non-trivial solution of (3.1), for which $|t_h(z, z^\star)| > 1 - 1/n$. Then, (3.14) and (3.13) imply that

$$\left(\frac{\tau(r)}{\psi(r)}\right)^2 \asymp \frac{\big(\tau(r)(1-r)\big)^2}{\left(\int_r^1 \omega(s)\,ds\right)^p} \asymp M(r, A)\left(\tau(r)(1-r^2)\right)^2, \quad r \in F,$$

where $M(r, A)\left(\tau(r)(1-r^2)\right)^2$ is uniformly bounded for all $r \in (R^\star, 1)$ by Theorem 11(ii), and where $R^\star$ is given by (3.3) with $R = 1 - 1/n$. This is clearly a contradiction, which proves that for each $n \in \mathbb{N}$ there corresponds a non-trivial solution f_n of (3.1) having two distinct zeros $z_n, z_n^\star \in \mathbb{D}$ such that $|t_h(z_n, z_n^\star)| > 1 - 1/n$ and

$$\varrho_h(z_n, z_n^\star) < \log \frac{1 + C\,\tau\big(|t_h(z_n, z_n^\star)|\big)}{1 - C\,\tau\big(|t_h(z_n, z_n^\star)|\big)}, \tag{3.16}$$

where $C > 0$ is a constant independent of n. The second inequality in (3.15) follows. Note that, if $\tau(r)$ decays to zero as $r \to 1^-$, then by (3.16) we have $|z_n|, |z_n^\star| \to 1^-$, as $n \to \infty$.

3.2. *Non-radial weights*

We take the opportunity to state a slightly more general version of Theorem 11. This modification utilizes non-radial weights, and it preserves the local information on zeros of solutions more accurately than its radial counterpart. It is useful to keep in mind the geometric properties of hyperbolic mid-points, which were mentioned in the beginning of Section 3, since these elementary observations allow us to pinpoint the accumulation points of the sequences of zero-pairs with minimal separation. Note that a condition similar to (3.17) arises naturally in a study of certain classes of meromorphic functions closely related to the normal functions [2].

Theorem 15. *Let A be analytic in $\mathbb{D}$, and let $\psi : \mathbb{D} \to (0, 1/2)$ be such that*

$$K = \sup_{a,z\in\mathbb{D}} \frac{\psi(a)}{\psi\big(\varphi_a(\psi(a)z)\big)} < \infty. \tag{3.17}$$

Let $\zeta \in \partial\mathbb{D}$, $R \in (0, 2]$, and

$$R^\star = \begin{cases} R/5, & \text{if } 0 < R < 2, \\ 2, & \text{if } R = 2. \end{cases} \tag{3.18}$$

(i) *If the coefficient A satisfies*

$$|A(z)|\big(\psi(z)(1-|z|^2)\big)^2 \leq M < \infty, \quad z \in \mathbb{D} \cap D(\zeta, R), \tag{3.19}$$

then the hyperbolic distance between any distinct zeros z_1 and z_2 of any non-trivial solution of (3.1), for which $|\zeta - t_h(z_1, z_2)| < R^\star$, satisfies

$$\varrho_h(z_1, z_2) \geq \log \frac{1 + \frac{\psi(t_h(z_1,z_2))}{\max\left\{K\sqrt{M},1\right\}}}{1 - \frac{\psi(t_h(z_1,z_2))}{\max\left\{K\sqrt{M},1\right\}}}. \tag{3.20}$$

(ii) *Conversely, if (3.20) is satisfied for all distinct zeros z_1 and z_2 of every non-trivial solution of (3.1), for which $|\zeta - t_h(z_1, z_2)| < R$, then the coefficient A satisfies*

$$|A(z)|\big(\psi(z)(1-|z|^2)\big)^2 < 3K^2 \max\{K^2 M, 1\}, \quad z \in \mathbb{D} \cap D(\zeta, R^\star). \tag{3.21}$$

Proof. The proof of Theorem 15 is similar to that of Theorem 11, and hence we content ourselves to merely indicate the necessary changes.

(i) Let $g_a(z) = (h \circ \varphi_a)\big(\psi(a) r z\big)$, where h is a quotient of any two linearly independent solutions f_1 and f_2 of (3.1), and $r = 1/\max\left\{K\sqrt{M}, 1\right\}$. If $a \in \mathbb{D}$ and $|\zeta - a| < R^\star$, then

$$\varphi_a\big(\psi(a) r z\big) \in \Delta_p\big(a, \psi(a) r\big) \subset \Delta_p(a, 1/2) \subset \mathbb{D} \cap D(\zeta, R), \quad z \in \mathbb{D}. \tag{3.22}$$

If $a \in \mathbb{D}$ and $|\zeta - a| < R^\star$, then (3.19) and (3.22) imply that

$$|S_{g_a}(z)|(1-|z|^2)^2 \leq 2M \left(\frac{1-|z|^2}{1-|\psi(a) r z|^2}\right)^2 \left(\frac{\psi(a)}{\psi\big(\varphi_a(\psi(a) r z)\big)}\right)^2 r^2 \leq 2, \quad z \in \mathbb{D}.$$

We conclude that g_a is univalent in $\mathbb{D}$ for any $a \in \mathbb{D}$ satisfying $|\zeta - a| < R^\star$. Consequently, h is univalent in each hyperbolic disc

$$\Delta_h\left(a, \frac{1}{2}\log\frac{1+\psi(a) r}{1-\psi(a) r}\right), \quad a \in \mathbb{D},\ |\zeta - a| < R^\star,$$

and the claim (3.20) follows by choosing $a = t_h(z_1, z_2)$, where z_1 and z_2 are any two distinct zeros of any non-trivial solution of (3.1), for which $|\zeta - t_h(z_1, z_2)| < R^\star$.

(ii) Assume that all distinct zeros $z_1, z_2 \in \mathbb{D}$ of every non-trivial solution of (3.1), for which $|\zeta - t_h(z_1, z_2)| < R$, satisfy (3.20). First, we prove that each non-trivial solution of (3.1) vanish at most once in

$$\Delta_h\left(a, \frac{1}{2}\log\frac{1+R_a}{1-R_a}\right) = \Delta_p(a, R_a), \quad R_a = \frac{\psi(a)}{K \max\left\{K\sqrt{M}, 1\right\}},$$

for any $a \in \mathbb{D}$ satisfying $|\zeta - a| < R^\star$, where $R^\star$ is given by (3.18). Assume on the contrary, that there exists a non-trivial solution of (3.1) having two distinct zeros $z_1, z_2 \in \Delta_p(a, R_a)$ for such a. Since $t_h(z_1, z_2) \in \Delta_p(a, R_a) \subset \Delta_p\big(a, \psi(a)\big)$, and further, $z \mapsto \varphi_a(\psi(a) z)$ maps $\mathbb{D}$ onto $\Delta_p\big(a, \psi(a)\big)$, we obtain

$$\frac{\psi(a)}{\psi\big(t_h(z_1, z_2)\big)} \leq \sup_{a \in \mathbb{D}} \left(\sup_{z \in \Delta_p(a, \psi(a))} \frac{\psi(a)}{\psi(z)}\right) = \sup_{a, z \in \mathbb{D}} \frac{\psi(a)}{\psi\big(\varphi_a(\psi(a) z)\big)} = K.$$

Now

$$\varrho_h(z_1,z_2) < \log\frac{1+R_a}{1-R_a} = \log\frac{1+\frac{\psi(a)}{K\max\left\{K\sqrt{M},1\right\}}}{1-\frac{\psi(a)}{K\max\left\{K\sqrt{M},1\right\}}} \le \log\frac{1+\frac{\psi(t_h(z_1,z_2))}{\max\left\{K\sqrt{M},1\right\}}}{1-\frac{\psi(t_h(z_1,z_2))}{\max\left\{K\sqrt{M},1\right\}}},$$

which contradicts (3.20), since $t_h(z_1,z_2)\in\Delta_p\big(a,1/2\big)\subset\mathbb{D}\cap D(\zeta,R)$.

Second, since $z\mapsto\varphi_a(R_a z)$ maps $\mathbb{D}$ onto $\Delta_p(a,R_a)$, the discussion above shows that $g_a(z)=h\big(\varphi_a(R_a z)\big)$ is univalent in $\mathbb{D}$ for all $a\in\mathbb{D}\cap D(\zeta,R^\star)$. Now

$$2|A(a)|(1-|a|^2)^2\frac{\psi(a)^2}{K^2\max\{K^2M,1\}}\le 6,\quad a\in\mathbb{D}\cap D(\zeta,R^\star),$$

by Kraus' theorem [31], or alternatively [33, p. 545], which proves the assertion (3.21).

The strength of Theorem 15 is demonstrated in the following example.

Example 16. Let A be an analytic function in $\mathbb{D}$, and assume that

$$p=\inf\left\{\alpha\ge 0:\sup_{z\in\mathbb{D}}|A(z)|\big(|1-z|^\alpha(1-|z|^2)\big)^2<\infty\right\}>0. \tag{3.23}$$

Let $q>p$, and denote $\psi(z)=|1-z|^q/2^{q+1}$. Now $\psi:\mathbb{D}\to(0,1/2)$, and K in (3.17) satisfies

$$K=\sup_{a\in\mathbb{D}}\left(\sup_{z\in\Delta_p(a,\psi(a))}\frac{\psi(a)}{\psi(z)}\right)\le\sup_{a\in\mathbb{D}}\left(\sup_{z\in\Delta_p(a,1/2)}\left|\frac{1-a}{1-z}\right|^q\right)\le 3^q.$$

According to (3.23) there exists a positive constant M such that (3.19) holds.

First, zero-sequences of non-trivial solutions of (3.1), which are contained in $\mathbb{D}\setminus D(1,r)$ for some $r>0$, are separated in the hyperbolic sense by Theorem 15. This follows from the fact that ψ is bounded away from zero in $\mathbb{D}\setminus D(1,r)$. In particular, this means that any zero-sequence converging to any $\zeta\in\partial\mathbb{D}\setminus\{1\}$ is separated in the hyperbolic metric. Second, there exists an infinite sequence of zero-pairs of non-trivial solutions of (3.1) such that the separation between the zeros in each pair is minimal. To make this vague statement more precise, let $\varepsilon\in(0,p)$. For each $n\in\mathbb{N}$ there corresponds a pair of zeros $(z_n,z_n^\star)$ of a non-trivial solution f_n of (3.1) such that $|1-t_h(z_n,z_n^\star)|<1/n$, and

$$\log\frac{1+C_\varepsilon|1-t_h(z_n,z_n^\star)|^{p+\varepsilon}}{1-C_\varepsilon|1-t_h(z_n,z_n^\star)|^{p+\varepsilon}}\le\varrho_h(z_n,z_n^\star)<\log\frac{1+C_\varepsilon|1-t_h(z_n,z_n^\star)|^{p-\varepsilon}}{1-C_\varepsilon|1-t_h(z_n,z_n^\star)|^{p-\varepsilon}}, \tag{3.24}$$

where C_ε is a constant depending only on ε. The first inequality in (3.24) follows from the first assertion of Theorem 15 with $R=2$, and it is valid for all zero-pairs of all non-trivial solutions of (3.1). Assume on the contrary, that there exists $n\in\mathbb{N}$ such that the second inequality in (3.24) in false. That is, all zero-pairs $(z,z^\star)$ of every non-trivial solution of (3.1), for which $|1-t_h(z,z^\star)|<1/n$, satisfy

$$\varrho_h(z,z^\star)\ge\log\frac{1+C_\varepsilon|1-t_h(z,z^\star)|^{p-\varepsilon}}{1-C_\varepsilon|1-t_h(z,z^\star)|^{p-\varepsilon}}.$$

The second assertion of Theorem 15 now implies that $p<p-\varepsilon$, which is obviously impossible. Remark that $t_h(z_n,z_n^\star)\to 1$, as $n\to\infty$, and hence $z_n,z_n^\star\to 1$, as $n\to\infty$, by (3.24).

Note that, if $q \geq 0$, and the coefficient A satisfies $|A(z)|\,|1-z|^{2q}\,(1-|z|^2)^2 \to 0^+$, as $|z| \to 1^-$, then $z = 1$ is the only possible accumulation point of the zeros of non-trivial solutions of (3.1) by [21, Theorem 8]. However, condition (3.23) allows zeros to accumulate to any point on $\partial\mathbb{D}$.

The last result in this section is a local version of Theorem 15. The proof is an easy modification of that of Theorem 15, and hence is omitted.

Theorem 17. *Let A be analytic in $\mathbb{D}$, $\zeta \in \partial\mathbb{D}$, and let $\Omega(\zeta) \subset \mathbb{D}$ be a simply connected domain such that $\overline{\Omega(\zeta)} \cap \overline{\mathbb{D}} = \{\zeta\}$. Let $s \in (0,1)$, and let $\psi : \mathbb{D} \to (0,s)$ be such that*

$$K = \sup_{a\in\Omega(\zeta)} \sup_{z\in\mathbb{D}} \frac{\psi(a)}{\psi\big(\varphi_a(\psi(a)z)\big)} < \infty.$$

(i) *If the coefficient A satisfies $|A(z)|\big(\psi(z)(1-|z|^2)\big)^2 \leq M < \infty$ for all $z \in \bigcup_{w\in\Omega(\zeta)} \Delta_p(w,s)$, then the hyperbolic distance between any distinct zeros z_1 and z_2 of any non-trivial solution of (3.1), for which $t_h(z_1,z_2) \in \Omega(\zeta)$, satisfies (3.20).*

(ii) *Conversely, if (3.20) is satisfied for all distinct zeros z_1 and z_2 of every non-trivial solution of (3.1), for which $t_h(z_1,z_2) \in \bigcup_{w\in\Omega(\zeta)} \Delta_p(w,s)$, then the coefficient A satisfies $|A(z)|\big(\psi(z)(1-|z|^2)\big)^2 < 3K^2 \max\{K^2M, 1\}$ for all $z \in \Omega(\zeta)$.*

Theorem 17(ii) implies that, if the analytic coefficient A has a singularity of the type $(\zeta - z)^{-\alpha}$ for some $\alpha > 2$ at a point $z = \zeta \in \partial\mathbb{D}$, then each simply connected domain $\Omega(\zeta) \subset \mathbb{D}$ satisfying $\overline{\Omega(\zeta)} \cap \overline{\mathbb{D}} = \{\zeta\}$ contains infinitely many zero-pairs of non-trivial solutions of (3.1) with minimal separation; compare to Corollary 32 below.

3.3. Connection to the existing oscillation theory

We state the following observations on Theorem 11: If A is analytic in $\mathbb{D}$, and $\psi : [0,1) \to (0,1)$ is a non-increasing function satisfying (3.2), then

$$\sup_{z\in\mathbb{D}} |A(z)|\big(\psi(|z|)(1-|z|^2)\big)^2 < \infty \iff \inf_{(z_j,z_k)\in\Gamma_0(A)} \frac{\varrho_p(z_j,z_k)}{\psi(|t_h(z_j,z_k)|)} > 0, \tag{3.25}$$

where we define $\Gamma_r(A)$ for $r \in [0,1)$ to be the set of pairs (z_1,z_2) such that $z_1, z_2 \in \mathbb{D}$ are distinct zeros of the same non-trivial solution of (3.1), and $|t_h(z_1,z_2)| \geq r$. In particular, if the coefficient A is analytic in $\mathbb{D}$ such that the expression $\sup_{z\in\mathbb{D}} |A(z)|\big(\psi(|z|)(1-|z|^2)\big)^2$ is finite, and if there exists a solution f of (3.1) whose zero-sequence has a subsequence $\{z_n\}$, which satisfies $\varrho_p(z_n,z_{n+1})/\psi\big(|t_h(z_n,z_{n+1})|\big) \to 0^+$, as $n \to \infty$, then $f \equiv 0$.

If $\psi : [0,1) \to (0,1)$ is a non-increasing function such that $\lim_{r\to 1^-} \psi(r) = 0$, and there exists a constant $t \in (0,1)$ for which

$$\sup_{0\leq r<1} \frac{\psi(r)}{\psi\left(\frac{r+t}{1+rt}\right)} < \infty, \tag{3.26}$$

then by modifying the proof of Theorem 11, one can show that

$$\lim_{|z|\to 1^-} |A(z)|(\psi(|z|)(1-|z|^2))^2 = 0 \iff \lim_{r\to 1^-} \inf_{(z_j,z_k)\in\Gamma_r(A)} \frac{\varrho_p(z_j,z_k)}{\psi(|t_h(z_j,z_k)|)} = \infty.$$

For example, $\psi(r) = (1-r)^\alpha/2$, $\alpha > 0$, satisfies (3.26) for any fixed $t \in (0,1)$.

Depending on the assumptions on the auxiliary function ψ, the separation condition in (3.25) may also be expressed in terms of φ-separation, which was introduced in [2]. Note that $|t_h(z_1, z_2)| \le \max\{|z_1|, |z_2|\}$ for all $z_1, z_2 \in \mathbb{D}$.

We proceed to compare Theorem 11 with existing results in the literature. To this end, several definitions and observations are required. For $p > 0$, the *growth space* H_p^∞ consists of those analytic functions g in $\mathbb{D}$, for which

$$\|g\|_{H_p^\infty} = \sup_{z\in\mathbb{D}} |g(z)|(1-|z|^2)^p < \infty.$$

Recall that the union of all these spaces is the Korenblum space $A^{-\infty}$. Moreover, we say that $g \in \mathbb{H}_p^\infty$ provided that $p = \inf\{q > 0 : g \in H_q^\infty\}$. The *order of growth* $\sigma_M(g)$ of an analytic function g in $\mathbb{D}$, with respect to the maximum modulus, is given by

$$\sigma_M(g) = \limsup_{r\to 1^-} \frac{\log^+ \log^+ M(r, g)}{-\log(1-r)}.$$

If A is an analytic function in $\mathbb{D}$, and (3.1) admits a non-trivial solution f which vanishes at distinct points z_n and $z_n^\star$ satisfying $|z_n - z_n^\star| < \varepsilon_n$ for all $n \in \mathbb{N}$, then the following affirmations are valid by Theorem 11:

(i) If $0 < \varepsilon_n < C_1 \exp\big(-C_2/(1-|z_n|)\big)$ for some $C_1, C_2 > 0$, then the infimum part of (3.25) fails for the weight $\psi(r) = (1-r)^\alpha/2$ for all $\alpha > 0$, and hence $A \notin A^{-\infty}$. As a consequence we get [14, Theorem 5].

(ii) If $0 < \varepsilon_n < C_1 \exp\big(-C_2 \exp\big(C_3/(1-|z_n|)\big)\big)$ for some $C_1, C_2, C_3 > 0$, then the infimum part of (3.25) fails for the weight $\psi(r) = \exp\big(-1/(1-r)^\alpha\big)$ for all $\alpha > 0$, and hence $\sigma_M(A) = \infty$.

It is well-known that the growth of the coefficient of (3.1) is related to the growth of solutions, and to the number of zeros of solutions. Our aim is to connect the separation of zeros of solutions of (3.1) to these widely studied properties in the case that the coefficient belongs to the Korenblum space. To this end, we introduce a new quantity, which measures the separation of zeros of non-trivial solutions of (3.1). Supposing that A is an analytic function in $\mathbb{D}$, we define the *zero separation exponent* for (3.1) to be

$$\Lambda_{\mathrm{DE}}(A) = \inf\left\{q > 0 : \inf_{(z_j,z_k)\in\Gamma_0(A)} \frac{\varrho_p(z_j, z_k)}{(1-|t_h(z_j, z_k)|)^q} > 0\right\}, \tag{3.27}$$

with the convention that $\Lambda_{\mathrm{DE}}(A) = \infty$ if the infimum in (3.27) is zero for all $q > 0$.

The following result, which is a consequence of Theorem 11, underscores the linkage between existing growth results and the separation of zeros. Note that the equivalence of (i) and (iv) in Corollary 18 below is valid for all $\lambda > 0$.

Corollary 18. *Let A be an analytic function in $\mathbb{D}$, and $\lambda \in (1, \infty)$. Then, the following assertions are equivalent:*

(i) $A \in \mathbb{H}_{2\lambda+2}^\infty$;
(ii) *All non-trivial solutions f of (3.1) satisfy $\sigma_M(f) = \lambda$;*
(iii) *There is a solution f of (3.1) such that $\sigma_M(f) = \lambda$;*
(iv) $\Lambda_{\mathrm{DE}}(A) = \lambda$.

Proof. Equivalence of (i), (ii) and (iii) is essentially known: Implication from (i) to (ii), and hence to (iii), follows from [11, Theorem 1.4]. Assume that (iii) holds. Then [11, Corollary 1.3] and [21, Lemma 2] prove that $A \in H_p^\infty$ for all $p > 2(\lambda+1)$, and (i) follows by [11, Theorem 1.4].

We complete the proof by showing that (i) and (iv) are equivalent. Let $\varepsilon > 0$ be such that $\lambda - \varepsilon/2 > 1$. If (iv) holds, then

$$\inf_{(z_j,z_k)\in\Gamma_0(A)} \frac{\varrho_p(z_j,z_k)}{\psi(|t_h(z_j,z_k)|)} > 0 \quad \text{for } \psi(r) = 2^{-1}(1-r)^{\lambda+\varepsilon/2},$$

and

$$\inf_{(z_j,z_k)\in\Gamma_0(A)} \frac{\varrho_p(z_j,z_k)}{\psi(|t_h(z_j,z_k)|)} = 0 \quad \text{for } \psi(r) = 2^{-1}(1-r)^{\lambda-\varepsilon/2}.$$

In the former case we have $\sup_{z\in\mathbb{D}} |A(z)|\big(\psi(|z|)(1-|z|^2)\big)^2 < \infty$ by (3.25), while in the latter case this supremum is infinite. This proves (iv) $\Rightarrow$ (i). The proof of (i) $\Rightarrow$ (iv) is similar, and hence the details are omitted.

Conditions (i) and (ii) in Corollary 18 are not equivalent for $\lambda = 1$, since (3.1) with $A(z) = -4z/(1-z)^4$ admits a bounded solution $\exp\big(-(1+z)/(1-z)\big)$. Recent findings in [10] show that there is a clever way to measure the growth of slowly growing analytic functions in $\mathbb{D}$, and it seems that the assumption $\lambda \in (1,\infty)$ in Corollary 18 can be relaxed to $\lambda \in (0,\infty)$, provided that the order of growth is defined differently.

When discussing the quantity of zeros of solutions, it is natural to measure the growth of the coefficient by means of integrated estimates. The *order of growth* of an analytic function g in $\mathbb{D}$, with respect to the Nevanlinna characteristic $T(r,g)$, is

$$\sigma_T(g) = \limsup_{r\to 1^-} \frac{\log^+ T(r,g)}{-\log(1-r)}.$$

Moreover, the *exponent of convergence* of the zero-sequence $\{z_n\}$ of g is

$$\lambda(g) = \inf\left\{\beta > 0 : \sum_{n=1}^{\infty}(1-|z_n|)^{\beta+1} < \infty\right\}.$$

Note that $\lambda(g)$ measures the quantity of zeros of g, whereas $\Lambda_{\mathrm{DE}}(A)$ quantifies the minimal separation of zeros of all non-trivial solutions of (3.1). Despite of the apparent differences of these quantities, for non-trivial solutions f of (3.1), $\Lambda_{\mathrm{DE}}(A)$ and $\lambda(f)$ are closely related.

Theorem 19. *Let A be an analytic function in $\mathbb{D}$. Then*

$$\sup_f \lambda(f) \le \Lambda_{\mathrm{DE}}(A) \le 1 + \sup_f \lambda(f), \tag{3.28}$$

where the supremums are taken over all non-trivial solutions f of (3.1). In particular, the quantities $\sup_f \lambda(f)$ and $\Lambda_{\mathrm{DE}}(A)$ are finite or infinite at the same time.

Proof. To prove the first inequality in (3.28), assume that $\Lambda_{\mathrm{DE}}(A) = \lambda < \infty$, for otherwise there is nothing to prove. Then $A \in \mathbb{H}^\infty_{2\lambda+2}$ by Corollary 18, and hence

$$\int_{\mathbb{D}} |A(z)|^{\frac{1}{2}}(1-|z|^2)^{\lambda+\varepsilon}\, dm(z) < \infty, \qquad \varepsilon > 0.$$

Therefore all non-trivial solutions f of (3.1) satisfy $\lambda(f) \le \lambda = \Lambda_{\mathrm{DE}}(A)$ by [23, Theorem 1.5].

If $1+\sup_f \lambda(f)=\lambda\in[1,\infty)$, then

$$\int_{\mathbb{D}} |A(z)|^{\frac{1}{2}}(1-|z|^2)^{\lambda-1+\varepsilon}\,dm(z)<\infty,\quad \varepsilon>0,$$

by [23, Theorem 1.5]. Hence $A\in H^\infty_p$ for all $p>2+2\lambda$ by the subharmonicity. We deduce that $\Lambda_{\mathrm{DE}}(A)\le\lambda=\sup_f\lambda(f)+1$ by Corollary 18.

Inequalities in (3.28) are reminiscent to the well-known inequalities $\sigma_T(g)\le\sigma_M(g)\le 1+\sigma_T(g)$, which are satisfied for all analytic functions g in $\mathbb{D}$. However, estimates in Theorem 19 concern zeros of functions, which are solutions of the same differential equation (3.1).

3.4. *Zeros of individual functions*

The following argument is a modification of Hille's example in [35, Eq. (2.12)], and it shows that we can find a sequence of zero-pairs of non-trivial solutions, which converges to the boundary singularity of the coefficient, even though all such solutions have at most two zeros in $\mathbb{D}$.

Example 20. Let $A(z)=-8/(1-z^2)^2$. Then A is analytic in $\mathbb{D}$, and differential equation (3.1) admits a non-vanishing solution base $\{f_1,f_2\}$, where

$$f_1(z)=\sqrt{1-z^2}\left(\frac{1-z}{1+z}\right)^{3/2}\quad\text{and}\quad f_2(z)=\sqrt{1-z^2}\left(\frac{1-z}{1+z}\right)^{-3/2}.$$

Since f_1/f_2 assumes every a-point at most twice in $\mathbb{D}$, all non-trivial solutions of (3.1) have at most two zeros in $\mathbb{D}$. For $n\in\mathbb{N}$, we define $f_n=f_1+f_2/n$. Then f_n has exactly two zeros $z_n,z_n^\star\in\mathbb{D}$, which are given by

$$z_n=\frac{1-n^{-1/3}\exp(i\pi/3)}{1+n^{-1/3}\exp(i\pi/3)}\quad\text{and}\quad z_n^\star=\frac{1-n^{-1/3}\exp(i\pi/3)}{1+n^{-1/3}\exp(i\pi/3)}.$$

If we let $n\to\infty$, then z_n and $z_n^\star=\overline{z}_n$ converge to $z=1$ inside the unit disc such that $\varrho_p(z_n,z_n^\star)=\sqrt{3}/2$ for all $n\in\mathbb{N}$. Application of Theorem 11 shows that the pseudo-hyperbolic distance between any distinct zeros of any non-trivial solution of (3.1) is at least $\sqrt{2}/4$.

Since $\Lambda_{\mathrm{DE}}(A)$ measures the separation of zeros of all non-trivial solutions of (3.1), it can be considered as a property of the differential equation (3.1) itself. Alongside with $\Lambda_{\mathrm{DE}}(A)$ we can also consider the separation of zeros of individual functions whether or not they are solutions of a differential equation (3.1). If $\{z_n\}$ is the zero-sequence of an analytic function f in $\mathbb{D}$, then we define the *zero separation exponent* for f to be

$$\Lambda(f)=\inf\left\{q>0:\inf_{j\neq k}\frac{\varrho_p(z_j,z_k)}{(1-|t_h(z_j,z_k)|)^q}>0\right\},\tag{3.29}$$

and set $\Lambda(f)=\infty$, if the infimum in (3.29) is zero for all $q>0$. Further, set $\Lambda(f)=0$ if f has only finitely many zeros in $\mathbb{D}$, or if f has multiple zeros.

Evidently, $\Lambda(f)\le\Lambda_{\mathrm{DE}}(A)$ for all solutions f of (3.1). The next example illustrates that for some non-trivial solutions f of (3.1) we can have $\Lambda(f)=0$, while for other solutions f of (3.1) $\Lambda(f)$ and $\Lambda_{\mathrm{DE}}(A)$ are equal.

Example 21. In the case of Example 13, weights $\psi(r) = 2^{-1}(1-r)^\alpha$ for $\alpha > 0$ together with (3.25) show that $\Lambda_{\mathrm{DE}}(A) = \beta$. However, both base functions f_1 and f_2 in (3.8) are non-vanishing, and hence $\Lambda(f_1) = \Lambda(f_2) = 0$. By Example 13 there is a solution, for example $f = -f_1 + e^i f_2$, for which $\Lambda(f) = \Lambda_{\mathrm{DE}}(A)$. Moreover, the convergence exponent of f satisfies $\lambda(f) = \beta$.

We remark that all non-trivial solutions f of (3.1) are of maximal growth. Namely, [11, Theorem 1.4] and [30, Theorem 2] show that $\sigma_M(f) = \beta$ and $\sigma_T(f) = \max\{0, \beta - 1\}$.

We conclude this discussion by comparing the quantities $\lambda(f)$ and $\Lambda(f)$. On the one hand, it is obvious that $\Lambda(f) \lesssim \lambda(f)$ is not true for all analytic functions in $\mathbb{D}$. For example, one can easily find a Blaschke product B for which $\Lambda(B) = \infty$ and $\lambda(B) = 0$. In fact, $\Lambda(f) \lesssim \lambda(f)$ is not true even for solutions of (3.1), since [14, Theorem 5] shows that we can construct a differential equation (3.1) having a solution f such that $\Lambda(f) = \Lambda_{\mathrm{DE}}(A) = \infty$ while $\lambda(f) = 0$. On the other hand, if $\Lambda(f)$ is finite, then $\lambda(f)$ is finite. The proof of the sharp inequality $\lambda(f) \le 2\Lambda(f)$ is similar to the proof of Theorem 33 below, and hence is omitted.

Note the following observation concerning solutions of differential equation (3.1). If $A \in \mathbb{H}^\infty_{2\lambda+2}$ for $\lambda > 1$, then [23, Corollary 1.6] ensures the existence of a solution f such that $\lambda - 1 \le \lambda(f) \le \lambda$. Now $\Lambda(f) \ge (\lambda - 1)/2$, which means that there is at least one non-trivial solution, whose zero-sequence contains infinitely many zeros, such that the separation between distinct zeros becomes small near the boundary.

3.5. Uniform local univalence

Theorem 11 gives rise to natural subclasses of $\mathcal{U}_{\mathrm{loc}}(\mathbb{D})$, the class of locally univalent analytic functions in $\mathbb{D}$. Namely, let $\psi : [0,1) \to (0,1)$ be a non-increasing function such that (3.2) is satisfied. We write $f \in \mathcal{U}_\psi$ provided that there exists $\delta \in (0,1)$ such that f is univalent in each hyperbolic disc

$$\Delta_h\left(a, \frac{1}{2}\log\frac{1+\psi(|a|)\delta}{1-\psi(|a|)\delta}\right), \quad a \in \mathbb{D}.$$

Functions in $\mathcal{U}_\psi$ are called *ψ-uniformly locally univalent functions* in $\mathbb{D}$. The following theorem characterizes these functions f among $\mathcal{U}_{\mathrm{loc}}(\mathbb{D})$ by means of the growth of their pre-Schwarzian derivatives f''/f' and their Schwarzian derivatives S_f.

Theorem 22. *Let $f \in \mathcal{U}_{\mathrm{loc}}(\mathbb{D})$, and let $\psi : [0,1) \to (0,1)$ be a non-increasing function such that (3.2) is satisfied. Then the following assertions are equivalent:*

(i) $f \in \mathcal{U}_\psi$;
(ii) $\sup_{z\in\mathbb{D}} \left|\frac{f''(z)}{f'(z)}\right| \psi(|z|)(1-|z|^2) < \infty$;
(iii) $\sup_{z\in\mathbb{D}} |S_f(z)|\big(\psi(|z|)(1-|z|^2)\big)^2 < \infty$.

Proof. The implication (iii) $\Rightarrow$ (i) is implicit in the proof of Theorem 11.

If $f \in \mathcal{U}_\psi$, then there exists $\delta \in (0,1)$ such that $g_a(z) = (f \circ \varphi_a)\big(\psi(|a|)\delta z\big)$ is univalent for all $a \in \mathbb{D}$. Take $h_a(z) = \big(g_a(z) - g_a(0)\big)/g_a'(0)$ so that $h_a(0) = 0$ and $h_a'(0) = 1$. Now h_a belongs to the Schlicht class of normalized univalent functions in $\mathbb{D}$, and hence the modulus of the coefficient of z^2 in the Maclaurin expansion of h_a is bounded by two. Consequently,

$$\psi(|a|)\delta\left|\frac{f''(a)}{f'(a)}(1-|a|^2) - 2\overline{a}\right| = \left|\frac{g_a''(0)}{g_a'(0)}\right| = |h_a''(0)| \le 4,$$

and it follows that

$$\left|\frac{f''(a)}{f'(a)}\right| \psi(|a|)(1-|a|^2) \leq 6, \quad a \in \mathbb{D}.$$

Thus (i) implies (ii).

Suppose that the analytic function f''/f' satisfies

$$\sup_{z\in\mathbb{D}} \left|\frac{f''(z)}{f'(z)}\right| \psi(|z|)(1-|z|^2) = M < \infty.$$

Define $\rho = \big(|z| + \psi(|z|)\big)/\big(1 + |z|\psi(|z|)\big) \in (|z|, 1)$. A standard application of Cauchy's integral formula shows that

$$\left|\left(\frac{f''}{f'}\right)'(z)\right| \leq \frac{M\left(\rho, \frac{f''}{f'}\right)}{\rho - |z|}, \quad |z| < \rho. \tag{3.30}$$

Now

$$\left|\left(\frac{f''}{f'}\right)'(z)\right| \left(\psi(|z|)(1-|z|^2)\right)^2 \leq M\left(\rho, \frac{f''}{f'}\right) \psi(\rho)(1-\rho^2) \frac{\left(\psi(|z|)(1-|z|^2)\right)^2}{\psi(\rho)(1-\rho^2)(\rho-|z|)},$$

where

$$\frac{\left(\psi(|z|)(1-|z|^2)\right)^2}{\psi(\rho)(1-\rho^2)(\rho-|z|)} = \frac{\psi(|z|)\big(1+\psi(|z|)|z|\big)^3}{\psi(\rho)(1-\psi(|z|)^2)} \leq \frac{8K}{1-\psi(0)}.$$

It follows that

$$\begin{aligned}
|S_f(z)| \left(\psi(|z|)(1-|z|^2)\right)^2 &\leq \left|\left(\frac{f''}{f'}\right)'(z)\right| \left(\psi(|z|)(1-|z|^2)\right)^2 \\
&\quad + \frac{1}{2}\left(\left|\frac{f''(z)}{f'(z)}\right| \psi(|z|)(1-|z|^2)\right)^2 \\
&\leq \frac{8KM}{1-\psi(0)} + \frac{M^2}{2}
\end{aligned}$$

for all $z \in \mathbb{D}$, and thus (ii) implies (iii).

The following example explores Example 13 in terms of uniform local univalence.

Example 23. Consider the locally univalent analytic function

$$f = \frac{f_1 - f_2}{f_1} = 1 - \exp\left(\frac{-2i}{(1-z)^\beta}\right), \quad \beta > 0,$$

induced by the linearly independent solutions f_1 and f_2 in (3.8) of the differential equation in Example 13. Now

$$\frac{f''(z)}{f'(z)} = \frac{-2i\beta}{(1-z)^{1+\beta}} + \frac{1+\beta}{1-z} \quad \text{and} \quad S_f(z) = 2A(z) = \frac{2\beta^2}{(1-z)^{2+2\beta}} + \frac{1-\beta^2}{2(1-z)^2},$$

where A is the coefficient function in (3.9). Clearly, conditions (ii) and (iii) in Theorem 22 are satisfied for $\psi(r) = 2^{-1}(1-r)^\beta$. We conclude that f is ψ-uniformly locally univalent.

4. Complex plane

In this section, we consider the oscillation of solutions of

$$f'' + A(z)f = 0 \tag{4.1}$$

assuming that the coefficient A is entire. Since some non-trivial solutions of (4.1) with a constant coefficient $A \not\equiv 0$ have infinitely many zeros, no restriction for the growth of A, other than $A \equiv 0$, imply finite oscillation for all non-trivial solutions. In fact, if $A \not\equiv 0$, then Lemma 24 below shows that (4.1) possesses a non-trivial solution f such that $\mu(f) \geq 1$, where

$$\mu(f) = \inf\left\{\beta > 0 : \sum_{n=1}^{\infty} |z_n|^{-\beta} < \infty\right\}$$

denotes the *exponent of convergence* of the zeros $\{z_n\}$ of f. Although the analogy of Nehari's result reduces to the trivial case $A \equiv 0$, differential equation (4.1) can be disconjugate in some unbounded subsets of $\mathbb{C}$. For example, if there exists an unbounded quasi-disk, in where the coefficient A is sufficiently small, then each non-trivial solution f of (4.1) vanishes at most once there [29].

Lemma 24. *Let A be entire. If every non-trivial solution f of (4.1) satisfies $\mu(f) < 1$, then $A \equiv 0$.*

Proof. Let f_1 and f_2 be linearly independent solutions of (4.1), and define $h = f_1/f_2$. By Nevanlinna's second fundamental theorem, we have

$$T(r,h) \leq N(r,h) + N(r,1/h) + N\big(r,1/(h-1)\big) + S(r,h) \tag{4.2}$$

outside an exceptional set E of finite linear measure. Now [6, Theorem 2.5.8] implies that there exists $\varepsilon > 0$ such that $T(r,h) = O(r^{1-\varepsilon})$ for all $r \in [0,\infty) \setminus E$, and hence for all r sufficiently large [32, Lemma 1.1.1]. By applying standard logarithmic derivative estimates [17, Corollary 2] to $2A = S_h$, we conclude that $A \equiv 0$.

4.1. Radial and non-radial weights

The following theorem gives an estimate for the separation of zeros in terms of the growth of coefficient, and vice versa.

Theorem 25. *Let A be entire, $R \in [0,\infty)$, and let $\Psi : [R,\infty) \to (0,\infty)$ be a non-increasing function such that*

$$K = \sup_{R^\star \leq r < \infty} \frac{\Psi(r)}{\Psi\big(r + \Psi(r)\big)} < \infty, \tag{4.3}$$

where

$$R^\star = \begin{cases} R + \Psi(R), & \text{if } 0 < R < \infty, \\ 0, & \text{if } R = 0. \end{cases} \tag{4.4}$$

(i) *If the coefficient A satisfies*

$$|A(z)|\,\Psi(|z|)^2 \leq M < \infty, \quad R \leq |z| < \infty, \tag{4.5}$$

then the Euclidean distance between any distinct zeros z_1 and z_2 of any non-trivial solution of (4.1), for which the Euclidean mid-point $|t_a(z_1, z_2)| \geq R^\star$, satisfies

$$|z_1 - z_2| \geq \frac{2\,\Psi\big(|t_a(z_1, z_2)|\big)}{\max\left\{K\sqrt{M}, 1\right\}}. \tag{4.6}$$

(ii) *Conversely, if (4.6) is satisfied for all distinct zeros z_1 and z_2 of every non-trivial solution of (4.1), for which $|t_a(z_1, z_2)| \geq R$, then the coefficient A satisfies*

$$|A(z)|\ \Psi(|z|)^2 \leq 3K^2 \max\{K^2 M, 1\}, \quad |z| \geq R^\star. \tag{4.7}$$

Proof. (i) Let $\{f_1, f_2\}$ be a solution base of (4.1), and set $h = f_1/f_2$ so that $S_h = 2A$. For $a \in \mathbb{C}$, define $\Phi_a(z) = a + \Psi(|a|)rz$, where $r = \min\left\{\left(K\sqrt{M}\right)^{-1}, 1\right\}$, and consider the function $g_a = h \circ \Phi_a$ in the unit disc $\mathbb{D}$. If $|a| \geq R^\star$, where $R^\star$ is given by (4.4), then $|\Phi_a(z)| \geq R$, and hence the assumption (4.5) yields

$$\begin{aligned}|S_{g_a}(z)|(1-|z|^2)^2 &= \big|S_h\big(\Phi_a(z)\big)\big|\,|\Phi_a'(z)|^2(1-|z|^2)^2 \leq 2M\left(\frac{\Psi(|a|)}{\Psi\big(|\Phi_a(z)|\big)}\right)^2 r^2 \\ &\leq 2M\left(\frac{\Psi(|a|)}{\Psi\big(|a| + \Psi(|a|)\big)}\right)^2 r^2 \leq 2MK^2r^2 \leq 2\end{aligned}$$

for all $z \in \mathbb{D}$. Therefore g_a is univalent in $\mathbb{D}$ for any $|a| \geq R^\star$ by Nehari's univalence criterion [33, Theorem 1]. Hence $h = f_1/f_2$ is univalent in each Euclidean disc $D\big(a,\ \Psi(|a|)r\big)$, $|a| \geq R^\star$, and consequently, condition (4.6) is true for any distinct zeros $z_1, z_2 \in \mathbb{C}$ of any non-trivial solution of (4.1), for which $|t_a(z_1, z_2)| \geq R^\star$.

(ii) Assume that (4.6) holds for all distinct zeros z_1 and z_2 of every non-trivial solution of (4.1), for which $|t_a(z_1, z_2)| \geq R$. As in the proof of Theorem 11, we deduce that each non-trivial solution of (4.1) vanishes at most once in

$$D(a, R_a), \quad R_a = \frac{\Psi(|a|)}{K\max\left\{K\sqrt{M}, 1\right\}}, \quad |a| \geq R^\star,$$

where $R^\star$ is given by (4.4). It follows that $g_a = h \circ \Phi_a$, where h is a quotient of any two linearly independent solutions of (4.1) and $\Phi_a(z) = a + R_a z$, is univalent in $\mathbb{D}$ for all $|a| \geq R^\star$. Now

$$|S_{g_a}(z)|(1-|z|^2)^2 = \big|S_h\big(\Phi_a(z)\big)\big|\,|\Phi_a'(z)|^2(1-|z|^2)^2 \leq 6, \quad z \in \mathbb{D},$$

by Kraus' theorem [31], or [33, p. 545]. Since $S_h = 2A$, by choosing $z = 0$ we get

$$2|A(a)|\,\frac{\Psi(|a|)^2}{K^2\max\{K^2M, 1\}} \leq 6, \quad |a| \geq R^\star,$$

from which (4.7) follows.

We can obtain a zero-separation result similar to Theorem 25(i), without the condition (4.3), by applying Sturm's comparison theorem rather than Nehari's univalence criteria. For example, if A is entire and it satisfies (4.5) for $R = 0$, where $\Psi : [0, \infty) \to (0, \infty)$ is non-increasing and continuous, then a straightforward application of [28, Corollary on p. 579] yields

$$|z_1 - z_2| \geq \frac{\pi}{\sqrt{M}}\,\Psi\left(\max\{|z_1|, |z_2|\} + \frac{\pi}{\sqrt{M}}\,\Psi\big(\max\{|z_1|, |z_2|\}\big)\right)$$

for all distinct zeros z_1 and z_2 of every non-trivial solution f of (4.1). We may also apply Sturm's comparison theorem directly on the Euclidean geodesics between distinct zeros, and then obtain a slightly different lower bound for the separation of zeros. In this approach the weight function Ψ is not required to be continuous. For a similar reasoning, see [7, p. 19].

Conversely, if $\Psi : [0,\infty) \to (0,\infty)$ is non-increasing, and we assume

$$|z_1 - z_2| \geq \frac{2}{\sqrt{M}}\, \Psi\big(\max\{|z_1|, |z_2|\}\big), \quad 0 < M < \infty,$$

instead of (4.6), then an argument similar to the proof of Theorem 25(ii) gives

$$|A(z)|\, \Psi\left(|z| + \frac{\Psi(|z|)}{\sqrt{M}}\right)^2 \leq 3M, \quad z \in \mathbb{C}.$$

The advantage of these results, when compared to Theorem 25, is the fact that the technical condition (4.3) is not needed. However, when (4.3) is satisfied, then one may use either these results or Theorem 25, and find the most useful estimate for each purpose by studying the different constants appearing in the statements and the behaviour of the weight function Ψ in the points in question.

The following example illustrates Theorem 25.

Example 26. The functions $f_1(z) = \exp\big(-(e^z + z)/2\big)$ and $f_2(z) = \exp\big((e^z - z)/2\big)$ are non-vanishing linearly independent solutions of (4.1) with an entire coefficient $A(z) = -\big(e^{2z}+1\big)/4$, see [4, p. 356]. Moreover, the zeros of $\alpha f_1 + \beta f_2$, where $\alpha\beta \neq 0$, are the points $z \in \mathbb{C}$ for which $\exp\big(e^z\big) = -\alpha/\beta$. Among these points we may pick a subsequence $z_n = \log\big(\log(-\alpha/\beta) + i\,2\pi n\big)$ for $n \in \mathbb{N}$, where log denotes the principal branch of the complex logarithmic. Evidently, $|z_n - z_{n+1}| \sim n^{-1}$, as $n \to \infty$. An application of Theorem 25 with $\Psi(r) = \exp(-r)$, for which $K = e$ in (4.3), yields $|z_n - z_{n+1}| \gtrsim \Psi\big(|t_a(z_n, z_{n+1})|\big) \sim \Psi(\log n) = n^{-1}$, as $n \to \infty$. We conclude that the estimate resulting from Theorem 25 is of the correct order of magnitude.

Corresponding to Section 3.2, the proof of Theorem 25 shows that we can also consider non-radial weights Ψ. Namely, if there are constants $C > 0$ and $R \geq 0$ such that the function $\Psi : \mathbb{C} \to (0, C)$ satisfies

$$K = \sup_{a\in\mathbb{C}} \sup_{z\in\mathbb{D}} \frac{\Psi(a)}{\Psi\big(a + \Psi(a)z\big)} < \infty, \tag{4.8}$$

and $|A(z)|\, \Psi(z)^2 \leq M$ for $R \leq |z| < \infty$, then we deduce (4.6) with $t_a(z_1, z_2)$ in place of $|t_a(z_1, z_2)|$ for any distinct zeros z_1 and z_2 of any non-trivial solution of (4.1), for which $|t_a(z_1, z_2)| \geq R^\star$, where $R^\star$ is given by (4.4). An observation corresponding to above applies for the converse statement.

4.2. Discussion on the weight functions

Weight functions are a subject of more detailed inspection in Section 2.3; here we merely point out a few differences. If $\Psi : [0,\infty) \to (0,\infty)$ is non-increasing, differentiable and convex, then

$$\frac{\Psi(r)}{\Psi\big(r + \Psi(r)\big)} \leq \frac{\Psi(r)}{\Psi(r) + \Psi'(r)\,\Psi(r)} = \frac{1}{1 + \Psi'(r)}$$

for r large enough, and it follows that Ψ satisfies (4.3). However, the only non-increasing and concave mappings from $[0,\infty)$ to $(0,\infty)$ are constants. It is also worth noticing that

$$\frac{\Psi\big(r+\Psi(r)\big)}{\Psi(r)}-1=\frac{\Psi\big(r+\Psi(r)\big)-\Psi(r)}{\big(r+\Psi(r)\big)-r},$$

and hence (4.3) holds, if we assume the Lipschitz condition

$$\sup_{0<s<t<\infty}\left|\frac{\Psi(s)-\Psi(t)}{s-t}\right|<1.$$

We next construct a non-increasing function $\Psi:[0,\infty)\to(0,\infty)$, for which (4.3) fails. The following construction only defines Ψ at a point sequence tending to infinity; Ψ can be made continuous or differentiable on $[0,\infty)$ if needed. Define $r_k=2^k$ for $n\in\mathbb{N}$, and define $\Psi(r_1)=1$. Let $\varepsilon_k\in(0,1)$ be a decreasing sequence such that $\varepsilon_k\to 0^+$, as $k\to\infty$. We define values y_k, $\Psi(y_k)$ and $\Psi(r_{k+1})$ inductively by $y_k=r_k+\Psi(r_k)$, $\Psi(y_k)=\varepsilon_k\Psi(r_k)$ and $\Psi(r_{k+1})=\Psi(y_k)$, respectively. Since Ψ is non-increasing and $\{r_k\}$ is increasing, we deduce that $\Psi(r_k)<r_k$, and further, $r_k<y_k<2r_k=r_{k+1}$ for all $k\in\mathbb{N}$. Moreover,

$$\frac{\Psi(r_k)}{\Psi\big(r_k+\Psi(r_k)\big)}=\frac{1}{\varepsilon_k}\to\infty,\quad k\to\infty.$$

We have the following elementary analogue of Theorem 10: Let $\Psi:[R,\infty)\to(0,\infty)$ be a continuous and non-increasing function, and let $k>1$. Then there exists a constant $C>0$, depending on k, such that $\Psi(x)<k\,\Psi\big(x+\Psi(x)\big)$ outside a set $E\subset[R,\infty)$ of x-values satisfying $\int_E dx\le C<\infty$.

4.3. Observations on Theorem 25

If A is entire, and $\Psi:[0,\infty)\to(0,\infty)$ is a non-increasing function satisfying (4.3), then Theorem 25 implies that

$$\sup_{z\in\mathbb{C}}|A(z)|\,\Psi(|z|)^2<\infty\iff\inf_{(z_j,z_k)\in\Gamma_0(A)}\frac{|z_j-z_k|}{\Psi\big(|t_a(z_j,z_k)|\big)}>0,\tag{4.9}$$

where $\Gamma_r(A)$ for $r\in[0,\infty)$ is the set of pairs (z_1,z_2) such that $z_1,z_2\in\mathbb{C}$ are distinct zeros of the same non-trivial solution of (4.1), and $|t_a(z_1,z_2)|\ge r$. In particular, if A is entire, $\sup_{z\in\mathbb{C}}|A(z)|\,\Psi(|z|)^2$ is finite, and there exists a solution f of (4.1), whose zero-sequence has a subsequence $\{z_n\}$, which satisfies $|z_{n+1}-z_n|/\Psi\big(|t_a(z_n,z_{n+1})|\big)\to 0^+$, as $n\to\infty$, then $f\equiv 0$.

As in the unit disc case, if we assume that $\Psi:[0,\infty)\to(0,\infty)$ is a non-increasing function such that $\lim_{r\to\infty}\Psi(r)=0$, and there exists a constant $t\in(0,\infty)$ for which

$$\sup_{r\in[0,\infty)}\frac{\Psi(r)}{\Psi(r+t)}<\infty,\tag{4.10}$$

then by modifying the proof of Theorem 25, we conclude

$$\lim_{|z|\to\infty}|A(z)|\,\Psi(|z|)^2=0\iff\lim_{r\to\infty}\inf_{(z_j,z_k)\in\Gamma_r(A)}\frac{|z_j-z_k|}{\Psi\big(|t_a(z_j,z_k)|\big)}=\infty.$$

For example, if $\Psi(r)=(1+r)^{-\alpha}$ for $\alpha>0$, then (4.10) holds for any $t\in(0,\infty)$.

Assume that A is entire and (4.1) has a non-trivial solution f which vanishes at distinct points z_n and $z_n^\star$ satisfying $|z_n-z_n^\star|<\varepsilon_n$ for all $n\in\mathbb{N}$. The following claims are immediate consequences of Theorem 25:

(i) If $0 < \varepsilon_n < M_1 \exp\bigl(-M_2|z_n|\bigr)$ for some $M_1, M_2 > 0$, then the infimum part of (4.9) fails for the weight $\Psi(r) = (1+r)^{-\alpha}$ for all $\alpha > 0$, and hence A is not a polynomial.
(ii) If $0 < \varepsilon_n < M_1 \exp\bigl(-M_2 \exp(M_3|z_n|)\bigr)$ for some $M_1, M_2, M_3 > 0$, then the infimum part of (4.9) fails for the weight $\Psi(r) = \exp(-r^{\alpha})$ for all $\alpha > 0$, and hence $\rho(A) = \infty$. Here, as usual,

$$\rho(g) = \limsup_{r\to\infty} \frac{\log\log M(r,g)}{\log r}$$

denotes the order of growth of an entire function g. As a consequence we get [3, Corollary 1]; for further discussion, see [14, p. 347].

4.4. Polynomial coefficients

The special case $\Psi \equiv c > 0$ of Theorem 25 along with Liouville's theorem yield the following corollary, which can be considered as a plane analogue of the classical unit disc result by Schwarz [35, Theorems 3 and 4]. Recall that Schwarz's result gives a characterization, in terms of the growth of the coefficient, to the case when the hyperbolic distance between any distinct zeros of any non-trivial solution is uniformly bounded away from zero.

Corollary 27. *Let A be entire. Then the Euclidean distance between all distinct zeros z_1 and z_2 every non-trivial solution f of (4.1) is uniformly bounded away from zero if and only if A is constant.*

Note that, if the coefficient A is a constant, then we can solve (4.1). It follows that the Euclidean distance between any distinct zeros of any non-trivial solution is uniformly bounded away from zero. An alternative proof of the converse assertion is presented at the end of Section 4.4. The following result goes further than Corollary 27.

Corollary 28. *Let A be entire. The coefficient A is a polynomial of degree n if and only if $|z_1-z_2|\bigl(1+|z_1+z_2|/2\bigr)^{n/2}$ is uniformly bounded away from zero for all distinct zeros $z_1, z_2 \in \mathbb{C}$ of every non-trivial solution of (4.1).*

Proof. If $A(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ is a polynomial, where the leading coefficient $a_n \neq 0$, then $M(r,A) \asymp r^n$ for all sufficiently large r. If we choose $\Psi(r) = (1+r)^{-n/2}$, then

$$\frac{\Psi(r)}{\Psi\bigl(r+\Psi(r)\bigr)} = \left(1 + \frac{1}{(1+r)^{1+n/2}}\right)^{n/2} \le 2^{n/2}, \quad 0 \le r < \infty,$$

and hence (4.3) holds. Therefore, if A is entire, then the assertion follows from Theorem 25.

Example 29 below deals with a-points of a quotient of two linearly independent solutions of (4.1) with a polynomial coefficient A.

Example 29. If P is a polynomial of degree $d > 1$, then for any distinct a-points $z(a)$ and $z^\star(a)$ of the function

$$f(z) = \int_0^z \exp\bigl(-2P(\zeta)\bigr)\,d\zeta, \quad z \in \mathbb{C},$$

the expression $|z(a) - z^\star(a)|\bigl(1 + |t_a(z(a), z^\star(a))|\bigr)^{d-1}$ is uniformly bounded away from zero, and further, for each $\varepsilon > 0$ there corresponds a sequence $\{a_n\}$ of complex numbers, such that each a_n has two preimages $z(a_n)$ and $z^\star(a_n)$ under f, for which $\bigl|t_a\bigl(z(a_n), z^\star(a_n)\bigr)\bigr| \ge n$ for all $n \in \mathbb{N}$, and

$$|z^\star(a_n) - z(a_n)|\Big(1 + \big|t_a\big(z(a_k), z^\star(a_n)\big)\big|\Big)^{d-1-\varepsilon} \to 0^+, \quad n \to \infty.$$

This follows from Theorem 25, since $g_1 = e^P$ and $g_2 = e^P f$ are linearly independent solutions of

$$g'' - \Big(P''(z) + \big(P'(z)\big)^2\Big)g = 0, \qquad \deg\big(P'' + (P')^2\big) = 2(d-1), \tag{4.11}$$

and the a-points of f are exactly the zeros of the solution $g_2 - ag_1$ of (4.11).

If A is entire and $\mu \in [1, \infty)$, then A is a polynomial of $\deg(A) \le 2\mu - 2$ if and only if all non-trivial solutions f of (4.1) satisfy $\rho(f) \le \mu$; see [23, Theorem 1.1] and the original references therein. It is also true that these conditions are equivalent to the requirement that all non-trivial solutions f of (4.1) satisfy $\mu(f) \le \mu$ [23, Theorem 1.3]. Note that $\mu(f)$ measures the quantity of zeros of f, but it does not imply any lower bound for the Euclidean distance between two distinct zeros. Theorem 25 enables us to resolve this matter. Supposing that A is entire, we define the *zero separation exponent* for (4.1) to be

$$\Upsilon_{\mathrm{DE}}(A) = \inf\left\{q > 1 : \inf_{(z_j,z_k)\in\Gamma_0(A)} |z_j - z_k|\big(1 + |t_a(z_j, z_k)|\big)^{q-1} > 0\right\}, \tag{4.12}$$

with the convention that $\Upsilon_{\mathrm{DE}}(A) = \infty$ if the infimum in (4.12) is zero for all $q > 1$.

The following result, which emerges as a corollary of Theorem 25, shows that for solutions f of (4.1), the quantities $\rho(f)$, $\mu(f)$ and $\Upsilon_{\mathrm{DE}}(A)$ are closely related. Note in Corollary 30 that not all values $\mu \in [1, \infty)$ are permitted, since the degree of the polynomial coefficient must be an integer. In particular, if any of the following equivalent conditions is true for some $\mu \in [1, \infty)$, then it follows that all the other conditions are valid, and further, μ belongs to a certain finite set of permitted rational numbers.

Corollary 30. *Let A be entire and $\mu \in [1, \infty)$. Then, the following assertions are equivalent:*

(i) *The coefficient A is a polynomial of* $\deg(A) = 2\mu - 2$*;*
(ii) *All non-trivial solutions f of* (4.1) *satisfy $\rho(f) = \mu$;*
(iii) *There is a solution f of* (4.1) *such that $\rho(f) = \mu$;*
(iv) *All non-trivial solutions f of* (4.1) *satisfy $\mu(f) \le \mu$, and there exists a solution f for which $\mu(f) = \mu$;*
(v) $\Upsilon_{\mathrm{DE}}(A) = \mu$.

Proof. By combining [16, Theorem 5], [23, Corollary 1.4], [32, Proposition 5.1], it is easy to see that conditions (i), (ii) and (iv) are equivalent. To conclude that condition (iii) can be added to this list of equivalent conditions, it is suffices to prove that (iii) $\Rightarrow$ (i), since (ii) $\Rightarrow$ (iii) is trivial. Assume that (iii) holds. Now standard estimates for the logarithmic derivatives [17, Corollary 3] show that A is a polynomial, and (i) follows by [32, Proposition 5.1].

We complete the proof by showing that (i) and (v) are equivalent. Case $\mu = 1$ is evident by Corollary 27. Suppose that $\mu > 1$, and let $\varepsilon > 0$ be such that $\mu - \varepsilon/2 > 1$. If (v) holds, then

$$\inf_{(z_j,z_k)\in\Gamma_0(A)} \frac{|z_j - z_k|}{\Psi(|t_a(z_j, z_k)|)} > 0 \quad \text{for } \Psi(r) = (1 + r)^{1-\mu-\varepsilon/2},$$

and

$$\inf_{(z_j,z_k)\in\Gamma_0(A)} \frac{|z_j - z_k|}{\Psi(|t_a(z_j, z_k)|)} = 0 \quad \text{for } \Psi(r) = (1 + r)^{1-\mu+\varepsilon/2}.$$

In the former case we have $\sup_{z\in\mathbb{C}} |A(z)|\,\Psi(|z|)^2 < \infty$ by (4.9), and hence the entire function A is a polynomial with $\deg(A) \le 2(\mu-1)+\varepsilon$. In the latter case we have $\sup_{z\in\mathbb{C}} |A(z)|\,\Psi(|z|)^2 = \infty$ by (4.9), and hence $\deg(A) \ge 2(\mu-1)-\varepsilon$. This proves (v) $\Rightarrow$ (i). Conversely, if (i) holds, then $\Upsilon_{\mathrm{DE}}(A) \le \mu$ by Corollary 28, while (4.9) yields $\Upsilon_{\mathrm{DE}}(A) \ge \mu - \varepsilon/2$. Thus (i) $\Rightarrow$ (v).

If $A \not\equiv 0$ is entire, then Corollary 30 shows that $\sup_f \mu(f) = \Upsilon_{\mathrm{DE}}(A)$, where the supremum is taken over all non-trivial solutions f of (4.1). Alongside with $\Upsilon_{\mathrm{DE}}(A)$, which can be considered as a property of the differential equation (4.1) itself, we define another property measuring the separation of zeros of individual functions. If $\{z_n\}$ is the zero-sequence of an entire function f, then we define the *zero separation exponent* for f to be

$$\Upsilon(f) = \inf\left\{ q > 1 : \inf_{j\neq k} |z_j - z_k|\big(1 + |t_a(z_j, z_k)|\big)^{q-1} > 0 \right\}, \tag{4.13}$$

and set $\Upsilon(f) = \infty$, if the infimum in (4.13) is zero for all $q > 1$. Further, set $\Upsilon(f) = 1$ if f has only finitely many zeros in $\mathbb{C}$, or if f has multiple zeros.

Evidently $\Upsilon(f) \le \Upsilon_{\mathrm{DE}}(A)$ for all solutions f of (4.1), and the strict inequality is possible, for example, for non-vanishing solutions. As in the corresponding case of the unit disc, $\Upsilon(f) \lesssim \mu(f)$ is not true even for individual solutions of (4.1) with an entire coefficient, see [3, Corollary 1]. The proof of the sharp inequality $\mu(f) \le 2\Upsilon(f)$ is similar to the proof of Theorem 33 below, and hence is omitted.

We proceed to consider the geometric distribution of zeros of solutions of (4.1) with a polynomial coefficient A. According to [28, Chapter 7] all but finitely many zeros of any non-trivial solution of (4.1) lie in *critical sectors* constructed over symmetrically spaced radii emanating from the origin, see also [16, Lemma 2]. The number of critical sectors is $\deg(A)+2$.

Let A be a polynomial of $\deg(A) = n$, and let f by any non-trivial solution of (4.1). It follows that f is an entire function of order of growth $\rho(f) = (n+2)/2$, see Corollary 30. Let us consider the growth of $|f(z)|$ in different parts of the complex plane, as $|z| \to \infty$. We expect $|f(z)|$ to be small when z is close to the zeros of f that are located in the critical sectors, with finitely many possible exceptions. However, either $|f(z)| \to \infty$ or $|f(z)| \to 0$, as $z \to \infty$, in between two consecutive critical sectors, see [18, Theorem E]. By Phragmén–Lindelöf theorem there must exist at least one pair of consecutive critical sectors such that $|f(z)| \to \infty$, as $z \to \infty$, in between them, for otherwise f would be a constant. Define $\mathcal{D} = \{z \in \mathbb{C} : |f(z)| \ge 1\}$. Clearly all zeros of f belong to the complement of $\mathcal{D}$. Since $\rho(f) = (n+2)/2$, it follows from [36, Theorem 1] that the two-dimensional upper density of $\mathcal{D}$ satisfies

$$\limsup_{r\to\infty} \frac{m\big(\mathcal{D} \cap D(0,r)\big)}{\pi r^2} \ge \frac{1}{n+2}. \tag{4.14}$$

Since there are $n+2$ symmetrical critical sectors, the lower bound in (4.14) corresponds to the portion of one sectorial domain between two consecutive critical sectors.

For a slightly more precise analysis, let $r\,\theta(r)$ be the length of the longest arc of $|z| = r$ contained in $\mathcal{D}$. Then $\theta(r)$ is the angle in which this arc is seen from the origin. For a fixed $\varepsilon \in (0,2)$, let $F = \{r \in [1,\infty) : \theta(r) \le (2-\varepsilon)\pi/(n+2)\}$. Such angular restriction again corresponds to a sectorial domain between two consecutive critical sectors. Now [1, Theorem 3] shows that

$$\frac{n+2}{2} \ge \limsup_{r\to\infty} \frac{\pi}{\log r} \int_1^r \frac{dt}{t\,\theta(t)} \ge \frac{n+2}{2-\varepsilon} \limsup_{r\to\infty} \frac{\int_{F\cap[1,r]} \frac{dt}{t}}{\log r}.$$

This shows that the upper logarithmic density $\overline{\mathrm{logdens}}(F) \le (2-\varepsilon)/2 < 1$.

The following theorem, whose proof is an easy modification of that of Theorem 25, allows us to supplement the existing knowledge on zero distribution of solutions of (4.1) with polynomial coefficients. If $\theta \in [0, 2\pi)$ and $0 < s < R$, then

$$\Omega_\theta(R, s) = \left\{z \in \mathbb{C} : |z - re^{i\theta}| < s \text{ for some } r > 0\right\} \setminus \overline{D(0, R)}$$

defines an infinite half-strip domain of width $2s$ in the complex plane.

Theorem 31. *Let A be entire, $R \in (0, \infty)$ and $\Psi : [R, \infty) \to (0, \infty)$ be a non-increasing function such that (4.3) holds, where $R^\star = R + \Psi(R)$.*

(i) *If the coefficient A satisfies $|A(z)|\, \Psi(|z|)^2 \le M < \infty$ for all $z \in \Omega_\theta(R, s)$, where $\theta \in [0, 2\pi)$ and $s \in (\Psi(R), R)$, then the Euclidean distance between any distinct zeros z_1 and z_2 of any non-trivial solution of (4.1), for which the Euclidean mid-point $t_a(z_1, z_2) \in \Omega_\theta\big(R + \Psi(R), s - \Psi(R)\big)$, satisfies (4.6).*

(ii) *Conversely, if (4.6) is satisfied for all distinct zeros z_1 and z_2 of every non-trivial solution of (4.1), for which $t_a(z_1, z_2) \in \Omega_\theta(R, s)$, where $s \in (\Psi(R), R)$, then the coefficient A satisfies*

$$|A(z)|\, \Psi(|z|)^2 \le 3K^2 \max\{K^2 M, 1\}$$

for all $z \in \Omega_\theta\big(R + \Psi(R), s - \Psi(R)\big)$.

The following corollary indicates that, although almost all zeros of individual non-trivial solutions of (4.1) lie in a small portion of the complex plane, one finds infinitely many pairs of zeros with minimal separation in each radial direction, provided that all zeros of all non-trivial solutions of (4.1) are taken into account.

Corollary 32. *Let A be a polynomial of $\deg(A) = d$. Then, each $\Omega_\theta(1, s)$, where $\theta \in [0, 2\pi)$ and $s \in (0, 1)$, contains infinitely many pairs of zeros $(z_n, z_n^\star)$ of non-trivial solutions f_n of (4.1), such that $\sup_{n\in\mathbb{N}} |z_n - z_n^\star|\, n^{(d-\varepsilon)/2} < \infty$ for all $\varepsilon > 0$.*

Proof. Let $\theta \in [0, 2\pi)$ and $s \in (0, 1)$ be fixed, and define $\Psi(r) = (1 + r)^{-(d-\varepsilon)/2}$, where $\varepsilon > 0$ is sufficiently small. Take $N \in \mathbb{N}$ large enough such that $\Psi(n) < s/2$ for all natural numbers $n > N$.

Suppose that there exists a natural number $n > N$ such that all distinct zeros z and $z^\star$ of every non-trivial solution of (4.1), for which $t_a(z, z^\star) \in \Omega_\theta(n, s/2)$, satisfy (4.6) with $z_1 = z$ and $z_2 = z^\star$. Theorem 31(ii) now implies that expression $|A(z)|\, \Psi(|z|)^2$ is uniformly bounded for all $z \in \Omega_\theta\big(n + \Psi(n), s/2 - \Psi(n)\big)$. This is clearly a contradiction, since regardless of the argument

$$|A(z)|\, \Psi(|z|)^2 \asymp \frac{|z|^d}{(1 + |z|)^{d-\varepsilon}} \asymp |z|^\varepsilon \to \infty, \quad |z| \to \infty.$$

We conclude that, for each natural number $n > N$ there corresponds a non-trivial solution f_n of (4.1), such that f_n has two distinct zeros $z_n, z_n^\star \in \mathbb{C}$, for which $t_a(z_n, z_n^\star) \in \Omega_\theta(n, s/2)$ and

$$|z_n - z_n^\star| \le \frac{1}{\big(1 + |t_a(z_n, z_n^\star)|\big)^{(d-\varepsilon)/2}} \le \frac{C}{(1 + n)^{(d-\epsilon)/2}},$$

where $C > 0$ is a constant independent of n. Assertion follows, since evidently $z_n, z_n^\star \in \Omega_\theta(1, s)$ for all sufficiently large $n \in \mathbb{N}$.

The following theorem concerns the separation of zeros of individual solutions of (4.1) with a polynomial coefficient.

Theorem 33. *Let A be a polynomial of* $\deg(A) = d$.

(i) *If $0 \leq d \leq 2$, then all non-trivial solutions f of (4.1) satisfy $\Upsilon(f) \leq (d+2)/2$, and there exists a solution f such that $\Upsilon(f) = (d+2)/2$.*

(ii) *If $d > 2$, then all non-trivial solutions f of (4.1) satisfy $\Upsilon(f) \leq (d+2)/2$, and there exists a solution f such that $\Upsilon(f) \geq (d+2)/4 + 1$.*

Proof. The case $d = 0$ follows from Corollary 27. Since all non-trivial solutions f satisfy $\Upsilon(f) \leq \Upsilon_{\mathrm{DE}}(A) = (d+2)/2$ for any d by Corollary 30, it suffices to find a solution f for which $\Upsilon(f)$ has the desired lower bound depending on d.

Let now $d \geq 1$. By Corollary 30 there exists a solution f such that $\mu(f) = (d+2)/2$. In particular, f has infinitely many zeros. By [25, Theorem 1] there are $d+2$ modified half-strips in the complex plane such that these sets contain all but finitely many zeros of f. Moreover, the widths of these modified half-strips tend to zero at a rate depending on $d = \deg(A)$, when approaching the infinity. It follows that at least one of these modified half-strips, say Ω, contains a sequence $\{z_n\}$ of zeros of f such that convergence exponent of $\{z_n\}$ equals to $(d+2)/2 = \mu(f)$. Without loss of generality, we may suppose that Ω belongs to the right half-plane and is symmetric with respect to the positive real axis. Denote

$$R_j := \{z \in \mathbb{C} : j \leq \mathrm{Re}(z) < j+1\}, \quad j \in \mathbb{N},$$

and let $\varepsilon > 0$. By (4.13) there exists a constant $c > 0$ such that, if $K_j = cj^{-(\Upsilon(f)-1+\varepsilon/4)}$, then $D(z_n, K_j) \cap D(z_m, K_j) = \emptyset$ for all distinct zeros $z_n, z_m \in \Omega \cap R_j$ and j large enough.

If $d = 1$, then [25, Theorem 1] yields

$$\operatorname{area}\left(\bigcup_{z \in \Omega \cap R_j} D(z, K_j)\right) \lesssim 2\int_{j-K_j}^{j+1+K_j} \left(x^{-1/2} + K_j\right) dx \lesssim \max\left\{K_j, \frac{1}{j^{1/2}}\right\}.$$

Let N_j denote the number of zeros of f belonging to $\Omega \cap R_j$, so that

$$N_j \lesssim \frac{1}{\pi K_j^2} \operatorname{area}\left(\bigcup_{z \in \Omega \cap R_j} D(z, K_j)\right) \lesssim \max\left\{\frac{1}{K_j}, \frac{1}{j^{1/2} K_j^2}\right\}. \tag{4.15}$$

If the maximum in (4.15) is equal to K_j^{-1}, then

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^{\Upsilon(f)+\varepsilon}} \lesssim \sum_{j=1}^{\infty} \sum_{z_n \in R_j} \frac{1}{|z_n|^{\Upsilon(f)+\varepsilon}} \lesssim \sum_{j=1}^{\infty} \frac{N_j}{j^{\Upsilon(f)+\varepsilon}} < \infty,$$

and hence $3/2 = \mu(f) \leq \Upsilon(f)$, where the first equality follows from Corollary 30. If the maximum in (4.15) is equal to $j^{-1/2}K_j^{-2}$, then

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^{2\Upsilon(f)-3/2+\varepsilon}} \lesssim \sum_{j=1}^{\infty} \sum_{z_n \in R_j} \frac{1}{|z_n|^{2\Upsilon(f)-3/2+\varepsilon}} \lesssim \sum_{j=1}^{\infty} \frac{N_j}{j^{2\Upsilon(f)-3/2+\varepsilon}} < \infty,$$

which implies $\mu(f) \leq 2\Upsilon(f) - 3/2$, and hence $\Upsilon(f) \geq 3/2$. The assertion follows for $d = 1$.

If $d = 2$, then [25, Theorem 1] yields

$$\operatorname{area}\left(\bigcup_{z \in \Omega \cap R_j} D(z, K_j)\right) \lesssim 2\int_{j-K_j}^{j+1+K_j} \left(\frac{\log x}{x} + K_j\right) dx \lesssim \max\left\{K_j, \frac{\log j}{j}\right\},$$

while if $d > 2$, then

$$\operatorname{area}\left(\bigcup_{z\in\Omega\cap R_j} D(z, K_j)\right) \lesssim 2\int_{j-K_j}^{j+1+K_j} \left(x^{-1} + K_j\right) dx \lesssim \max\left\{K_j, \frac{1}{j}\right\}.$$

We may follow the reasoning in the case $d = 1$ to obtain $(d+2)/2 = \mu(f) \le 2\Upsilon(f) - 2$, from which the assertion follows.

The example below concerns the Airy differential equation, whose solutions arise in many practical applications. For a generalization of the Airy integral, see [19].

Example 34. The classical Airy differential equation $f'' - zf = 0$ possesses a special contour integral solution called the *Airy integral* Ai. Let $\{z_n\}$ denote its zero-sequence. It is known that zeros z_n are real and negative for all $n \in \mathbb{N}$ [34, p. 415], and they satisfy [27, Theorem 2] the asymptotic estimate

$$z_n \sim -\left(\frac{3\pi}{8}\left(4n-1\right)\right)^{2/3}, \quad n \to \infty. \tag{4.16}$$

Theorem 25 with $\Psi(r) = (1+r)^{-1/2}$ proves that all distinct zeros z_1 and z_2 of Airy integral satisfy the separation condition $|z_1 - z_2| \ge 2\big(2 + |z_1 + z_2|\big)^{-1/2}$. This estimate is of the correct order of magnitude, since

$$|z_n - z_{n+1}| \sim \left(\frac{2\pi^2}{3n}\right)^{1/3}, \qquad \frac{2}{\sqrt{2+|z_n+z_{n+1}|}} \sim \left(\frac{4\sqrt{2}}{3\pi n}\right)^{1/3}, \quad n \to \infty,$$

by (4.16). Note also that $\Upsilon(\mathrm{Ai}) = 3/2$ by the proof of Theorem 33.

We close this section by giving an alternative proof of the converse assertion of Corollary 27. Suppose that the zeros of every non-trivial solution f of (4.1) are separated in terms of the Euclidean metric; that is, all zeros are simple, and $\Upsilon(f) = 1$ for all non-trivial solutions f of (4.1). Then, for any fixed non-trivial solution f there is a constant $\delta > 0$ such that for any $z_0 \in \mathbb{C}$ the Euclidean disc $D(z_0, \delta)$ contains at most one zero of f. A simple geometric observation reveals that $n(r, f, 0) = O(r^2)$ and $N(r, f, 0) = O(r^2)$, as $r \to \infty$, for all non-trivial solutions f of (4.1).

Let f_1 and f_2 be linearly independent solutions of (4.1), and define $h = f_1/f_2$. By Nevanlinna's second fundamental theorem (4.2) holds outside an exceptional set E of finite linear measure. By the discussion above, $T(r, f) = O(r^2)$ for all $r \in [0,\infty) \setminus E$, and hence for all r sufficiently large [32, Lemma 1.1.1]. By applying standard logarithmic derivative estimates [17, Corollary 2] to $2A = S_h$, we conclude that A is a polynomial of $\deg(A) \le 2$. This is in contradiction with Theorem 33, unless A is a constant function.

4.5. Uniform local univalence

Theorem 25 gives rise to certain natural subclasses of locally univalent functions. Functions f satisfying (i) in Theorem 35 below are called as *Ψ-uniformly locally univalent functions* in $\mathbb{C}$.

Theorem 35. *Let f be a locally univalent entire function, and let $\Psi : [0,\infty) \to (0,\infty)$ be a non-increasing function such that (4.3) is satisfied for $R^\star = 0$. Then the following assertions are equivalent:*

(i) *There exists* $\delta > 0$ *such that* f *is univalent in each disc* $D(a, \Psi(|a|)\delta)$ *for all* $a \in \mathbb{C}$*;*
(ii) $\sup_{z\in\mathbb{C}} \left|\frac{f''(z)}{f'(z)}\right| \Psi(|z|) < \infty$*;*
(iii) $\sup_{z\in\mathbb{C}} |S_f(z)|\, \Psi(|z|)^2 < \infty$.

Proof. The proof of Theorem 25 shows that (iii) implies (i). Moreover, if (i) is satisfied, then the function $g_a = f \circ \Phi_a$, where $\Phi_a(z) = a + \Psi(|a|)\delta z$ is univalent in $\mathbb{D}$ for all $a \in \mathbb{C}$. Take $h_a(z) = (g_a(z) - g_a(0))/g_a'(0)$ so that $h_a(0) = 0$ and $h_a'(0) = 1$. Then

$$\left|\frac{f''(a)}{f'(a)}\right| \Psi(|a|)\delta = \left|\frac{g_a''(0)}{g_a'(0)}\right| = |h_a''(0)| \le 4,$$

and (ii) follows.

Assume that (ii) is satisfied so that

$$\sup_{z\in\mathbb{C}} \left|\frac{f''(z)}{f'(z)}\right| \Psi(|z|) = M < \infty.$$

Let $\rho = |z| + \Psi(|z|)$. An application of the Cauchy formula gives (3.30), and hence

$$\left|\left(\frac{f''}{f'}\right)'(z)\right| \Psi(|z|)^2 \le K\, M\left(\rho, \frac{f''}{f'}\right) \Psi(\rho)\, \frac{\Psi(|z|)}{\rho - |z|} \le KM, \quad z \in \mathbb{C}.$$

It follows that

$$\left|S_f(z)\right| \Psi(|z|)^2 \le \left|\left(\frac{f''}{f'}\right)'(z)\right| \Psi(|z|)^2 + \frac{1}{2}\left(\left|\frac{f''(z)}{f'(z)}\right| \Psi(|z|)\right)^2 \le KM + \frac{1}{2}M^2$$

for all $z \in \mathbb{C}$, and thus (iii) is satisfied.

5. Concluding remarks

The results reported in this paper fall into two distinct categories. In Section 2 we compare the separation of zeros of non-trivial solutions of

$$f'' + Af = 0 \tag{5.1}$$

to the growth of the continuous real-valued coefficient A on a real interval, whereas in Sections 3 and 4 we discuss the corresponding concepts in a complex domain. Even though the approach we take applies in both instances, there are some profound differences between the real and complex cases no matter how similar they may seem. In the complex case, it is well-known that the growth of the coefficient, the growth of solutions and the quantity of zeros of solutions are closely related. By the results obtained, it is justified to say that the separation of zeros of all non-trivial solutions gives the fourth quantity ($\Lambda_{\mathrm{DE}}(A)$ in the disc and $\Upsilon_{\mathrm{DE}}(A)$ in the plane), which is firmly ensconced among the other three. However, these ties are not carried over into the real case, as the following facts show. First, by Examples 4 and 7, an arbitrarily fast growing coefficient may permit all solutions to be bounded. Second, an elementary corollary [5, p. 48] of Sturm's comparison theorem states that, if A is non-positive, then (5.1) is disconjugate. We derive the same conclusion whenever the integral of A is sufficiently small [20, Corollary 5, p. 346]. Therefore the absolute value of the coefficient may grow arbitrarily fast while all non-trivial solutions vanish at most once. Third, Sturm's theorem on interlacing zeros shows that, if one non-trivial solution has infinitely many zeros, then the same is true for all solutions. In particular, if one non-trivial solution has two zeros, then there are no zero-free solutions, to say nothing of zero-free solutions bases. In contrast to this, zero-free solution bases are possible in the complex case [4,24]. For classical results on the oscillation regarding the real case, see, for example, [39] and the references therein. In conclusion, the connections between the growth of the coefficient, the growth of solutions, and the zero separation of solutions of (5.1) in the real case are different from those in the complex case.

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