PONTIFICIA UNIVERSIDAD CATOLICA DE CHILE
FACULTAD DE FISICA
INSTITUTO DE FISICA

ON THE SPECTRAL STABILITY OF THE NONLINEAR DIRAC EQUATION OF SOLER TYPE

by

Danko Aldunate Bascuñan

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Advisor	:	Dr. Edgardo Stockmeyer (PUC Chile)
Co-advisor	:	Dra. Hanne van den Bosch (U. de Chile)
Committee	:	Dr. Rafael Benguria (PUC Chile)
	:	Dr. Luis Morales (PUC Chile)

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Abstract

We study the spectral stability of the solitary wave solutions to the nonlinear Dirac equation in $(1+1)$ dimension. We focus on a Soler type nonlinear model, where the nonlinearity is given by $(\overline{\psi}\psi)^p$. The method we use consists in perturbe the solutions with a sufficiently small function ρ, finding a time evolution equation for this perturbation where this equation depends on the spectrum of the linearized operator H_μ. We will say that the solitary wave solutions are stable if the spectrum of H_μ does not have eigenvalues with imaginary part other than zero. We were only able to provide bounds for the real and imaginary part of the discrete spectrum of H_μ. In the end, we summarize what is known about $\sigma(H_\mu)$.

Contents

Chapter 1

Introduction

1.1 Motivation and context of the equation

The Dirac equation was proposed by the British physicist Paul Dirac in 1928, in his work *"The quantum theory of the electron"* [26], where he treats the description of the relativistic motion of a spin-1/2 particle in $\mathbb{R}^3$. A full overview of its applications would take us too far. In particular, our interest is the description of materials that share a fundamental similarity: their low-energy fermionic excitations behave as massless Dirac particles. This class of materials, called Dirac materials, exhibits unusual characteristics like Klein tunneling, chiral symmetries and impurity resonances [64].

The interest in Dirac materials grew since the experimental discovery of graphene in 2004 [44]. This discovery gave the Nobel prize in physics to Andre Geim and Konstantin Novoselov in 2010. This is a two dimensional material and it is so interesting given its electronic properties, see [16, 52, 63].

Nonlinear models are been extensively investigated by many researchers in different areas of physics. Most authors study nonlinear Schrödinger equations and there is many literature on the topic. The physical motivation for studying nonlinear models is that stable pulses (solitary waves) have been observed experimentally. For example, Fermi et al. studied the nonlinear phenomena and found periodic behavior at least when the energy is not too high, and that stable pulses (solitary waves) propagate in nonlinear continuous systems. These facts led it to conclude that there will be some nonlinear lattice which admits rigorous periodic waves, and that certain pulses (lattice solitary waves) will be stable there, see [60, 20, 32].

With respect to the nonlinear Dirac equation, there are many types of nonlinearities. In 1938 the Russian physicist Dimitri Ivanenko considered the non-linearity $\overline{\psi}\psi$ [36], then Weyl in 1950 [66] and Heisenberg in 1953 [35] also studied the same type of nonlinearity. But it was Mario Soler [55] who was the first to investigate the stationary states of the nonlinear Dirac field with the scalar fourth order self coupling, proposing them as a model of elementary extended fermions.

The massive Thirring nonlinear model [59] is also well known. The main difference between this model and the Soler model is that the Thirring model is completely integrable and the Soler model is not. Nonlinear generalizations of the Dirac equation, have emerged naturally as a practical model in many physical systems, a few examples are: extended particles, the gap solitary waves in nonlinear optics, light solitary waves in waveguide arrays and experimental realization of an optical analog for relativistic quantum mechanics, Bose-Einstein condensates in honeycomb optical lattices, phenomenological models of quantum chromodynamics, as well as matter influencing the evolution of the Universe in cosmology, etc. see [40, 41].

The Spanish physicist M. Soler re-introduced the Ivanenko type nonlinearity in 1970, since then the model has been studied in many aspects. The n-dimensional Soler model is given by the equation

$$i\partial_t\psi(x,t) = \left(D_0 + \beta\left(m - f(\overline{\psi}\psi)\right)\right)\psi(x,t), \quad \psi(x,t) \in L^2(\mathbb{R}\times\mathbb{R}^n, \mathbb{C}^N), \tag{1.1}$$

where $f(s)$ is a smooth function, $\overline{\psi}\psi = \psi\beta\psi$ and D_0 is the n-dimensional massless free Dirac operator. We are going to consider the case $f(s) = s^p$ with $p > 0$, giving particular results to $p = 1$. The Soler model with this power like nonlinearity was exactly solved only in the one dimensional case [42, 24]. Also, many other areas have been studied, applications, etc., see [25, 24, 14, 22, 6, 15, 21].

There are also variations of this model, like the generalization introduced by Poddubny and Smirnova in 2018 for the two-dimensional case [47]. Most of our results can be extended to this generalization see the appendix A. Basically, this model is a twisted Soler model manipulated by a real constant α and is equal to the Soler model when $\alpha = 1$. In this way, by controlling the parameter α, other nonlinearities can be obtained, for example the case $\alpha = -1$ was considered by William Borrelli who showed the existence of solutions and others general properties [8, 10, 9].

The existence of localized states (solitary waves) and its stability is so important because this type of nonlinearity could be reproduced in laboratory. In the two-dimensional generalization of Poddubny and Smirnova they show one way of how the cubic nonlinearity is reproducible. They even gave several examples where this two-dimensional model could be implemented, like optofluidic platform with photonics crystal fibers filled by liquids, glass fibers, etc. see [47, 43, 28, 48, 62, 23, 2, 18, 54, 19, 51].

The stability of the solitary waves solutions has a huge importance, determines the possibility of giving a physic application to the model. In our case we are talking about applications in Dirac materials, like the graphene, etc.

1.2 On the mathematical side

The first natural question is about the existence of solutions. For the three-dimensional Soler model, existence of stationary states $\psi(x,t) = \phi(x)\mathrm{e}^{-i\omega t}$ with $\omega \in (0, m)$, was proved by Thierry Cazenave and Luis Vazquez in 1986 using ODE arguments [17]. Then, the existence of excited states was proved in [3]. Even the existence for more singular nonlinearities has been proved [4].

In 1995 Maria Esteban and Éric Séré proved the existence of stationary states of the Nonlinear Dirac Equation using variational methods [30]. These techniques have been improved, see [27, 29]. Once we have solved the existence problem, the next natural question is about the stability of the solutions and this can be studied in many different ways.

In particular we are going to consider the nonlinearity $f(s) = s^p$ with $p > 0$, so the equation (1.1) becomes

$$i\partial_t \psi(x,t) = \left(D_0 + \beta\left(m - (\overline{\psi}\psi)^p\right)\right)\psi(x,t), \quad \psi(x,t) \in L^2(\mathbb{R}\times\mathbb{R}^n, \mathbb{C}^N). \tag{1.2}$$

The soler model with this p-power nonlinearity is exactly solvable in the one dimensional case for any $p > 0$, see [42]. For other dimensions, no explicit solutions are known [24]. We will concentrate on the stability of the solutions for the one-dimensional case.

The mathematical meaning of stability is the condition in which a small disturbance in a system does not produce a big effect on that system. In terms of the solution of a differential equation, a solution $\psi(x)$ is said to be stable if any other solution of the equation $\tilde{\psi}(x)$ that starts out sufficiently close to it when $x = 0$, remains close to it for succeeding values of x. If the difference between the solutions approaches zero as x increases, the solution is called stable. If a solution does not have this property, it is called unstable.

In other words, assume that $\psi(x,t) = \phi(x)\mathrm{e}^{-i\omega t}$ is a solution of (1.2) where $\phi(x) \in L^2(\mathbb{R}^n, \mathbb{C}^N)$ is a localized function. Then, the main goal is to prove that $\phi(x)$ is stable, so consider the perturbed solution $\tilde{\psi}(x,t) = (\phi(x) + \rho(x,t))\,\mathrm{e}^{-i\omega t}$, which starts close to ψ, and then we can study how the perturbations $\rho(x,t)$ behaves in time.

There are many criteria for analyzing stability, in chapter 2.3 we briefly summarize the most typical stability criteria. A few examples of stability criteria applied to the solitary wave solutions of the Soler model are: Vakhitov-Kolokolov criterium [21], asymptotic stability [22], numerically stability analysis [25], dilatations stability [56] and spectral stability in [14, 6, 15]. We are going to focus on the spectral stability for the one-dimensional case, this is a work in collaboration between Edgardo Stockmeyer, Hanne van den Bosch, Julien Ricaud and the present author.

Although the solutions in the one-dimensional case are known, their stability is still an open problem. There is a lack of understanding of the problem and only partial results have been given, see chapter 4. Throughout the thesis we will concentrate on trying to solve the spectral stability problem.

Note the difficulty of the problem, first of all is a mathematical model of the year 1970 and the stability of the solitary wave solutions is until now an open problem. There have been many attempts to resolve the stability problem, see [25, 24, 14, 22, 6, 15, 21]. The strategies followed have even presented controversies, for the cubic nonlinearity in $(1+1)$ dimensions through a semi-analytical study Berkolaiko and Comech in [6] conclude that the soliton associated is linearly stable for all values of its frequency ω. Then, this results was validated by the results of [45].

The results of [6] together with the results of [1], which also claim soliton stability, came into controversy with the results of [7], which reported instability of the soliton for $\omega = 0.5$. This fact pushed the numerical study [53] who questioned the results of [6]. Lakoba in 2018 [38] presented this controversy in detail and resolved it concluding that the soliton is linearly stable for small values of frequencies up to $\omega = 0.01$.

Mathematically, the principal difficulty of this model is the unboundedness and negativity of the Dirac operator. In the nonlinear Schrödinger case, the positivity of the operator facilitates the stability study, for example orbital stability is proved in [67] for unbounded potentials, spectral stability is studied in [34] for cubic nonlinearities, etc.

1.3 Organization of the thesis

To help the understanding of the thesis, here is summarize the content of each chapter. In this way the reader can understand the narrative throughout the document.

Chapter 2

In the chapter 2, the basic notions about the Dirac equation and non-linearities are presented. This chapter is organized as follows.

Section 2.1: Starts by deriving the Dirac equation from the energy-moment relation. Then, the Klein-Gordon equation is found. Next, the algebra of the Pauli matrices is introduced, arriving at the Dirac equation. Finally, the Dirac free operator are defined.

Section 2.2: The basic properties of the Dirac operator are studied. We describe the Hilbert space in which we will define this operator. Also the notation that we will use throughout the thesis is also defined. In the end, we found the essential spectrum of the free Dirac operator using the Fourier transform. This spectrum will be important later, since the operators resulting from linearization turn out to have the same.

Section 2.3: A brief historical review of the non-linear Dirac equation is given. The n-dimensional Soler model is introduced summarizing its general properties. Finally, we give a brief summary of the stability criteria to contextualize the reader in the main problem of this thesis.

Chapter 3

In the chapter 3, the stability of the solutions of the Soler model in one dimension is studied. The chapter is organized as follows.

Section 3.1: First, the main equations of one-dimensional Soler model are introduced. Starting with the Dirac equation in a dimension disturbed by a general Soler-type non-linearity. Then, it is shown that the spinor components satisfy a Hamiltonian system, where the Hamiltonian turns out to be 0.

Section 3.2: It is briefly discussed how the problem of the existence of solutions for this model has been dealt with in the literature. Then, it is shown in detail how the solutions for the 1D Soler model are found when the non-linearity is $f(s) = s$. Also, the solutions for the general p-power nonlinearity $f(s) = s^p$ are shown.

Section 3.3: The stability of solutions is studied. Here the solutions are perturbed by means of the Bogoliubov-deGennes linearization stability analysis. Here the operator resulting from the linearization H_μ is found and the notion of spectral stability is defined, in terms of the spectra of this operator. This operator depends on two other operators L_0 and L_μ, which are analyzed extensively in the next section.

Section 3.4: The spectra of the operators L_0 and L_μ are studied extensively. The basic properties of their spectra are shown, properties that can be found in the literature and new results. Some of the new result are for example: in the case $p = 1$ we will show it by an oscillation argument that $\sigma_d(L_0) = \{-2\omega, 0\}$; the eigenvalues of L_0 and L_μ are simple, and so on. These results can be found summarized in the next chapter.

Section 3.5: The spectrum of the H_μ operator is analyzed. This is the most important operator, since the spectral stability of the Soler model solutions depends on their spectrum. The section begins by finding the main symmetries of the H_μ spectrum. Next, the spectral projections of L_0 are studied. These spectral projections are later used to give bounds on the spectrum of the H_μ^2 operator.

Section 3.6: Here you can find the main results of this thesis. Bounds on the imaginary and real part of $\sigma_p(H_\mu)$ are proved. These bounds allow us to exclude a region of the complex plane where the eigenvalues cannot be. Therefore, the result is partial, we could not exclude all complex eigenvalues that lead to instability. Therefore, the problem remains open.

Chapter 4

The chapter 4 summarizes the characterizations for the spectra of the operators L_0, L_μ and the most important H_μ. The main known results are presented first and then the main results of our work. It also indicates where this information can be found in the literature. Not all the new results present in this chapter are in this thesis, they are part of a joint work with E. Stockmeyer, H. van den Bosch and J. Ricaud. For those results that are not proved here, a brief explanation of how they are obtained, is given.

Our results can be synthesized as follows: for suitable ω and p we have $\sigma_d(L_0) = \{-2\omega, 0\}$. The operator L_2 has exactly three negative eigenvalues, i.e. $\sigma_d(L_2) = \{-2\omega, \lambda, 0\}$ where $\lambda \in (-2\omega, 0)$. For proper ω and p, L_μ with $\mu \in (0, 2]$ has exactly one eigenvalue between

$(-2\omega, 0)$ and none at $(-m-\omega, -2\omega)$. For $z \in \sigma(H_\mu) \setminus \{\mathbb{R} \cup i\mathbb{R}\}$ we give bounds on the imaginary and real part of z. Also, for suitable ω and p there are no eigenvalues on the imaginary axis.

Chapter 2

Nonlinear Dirac equation

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Section 3.6: Here you can find the main results of this thesis. Bounds on the imaginary and real part of $\sigma_p(H_\mu)$ are proved. These bounds allow us to exclude a region of the complex plane where the eigenvalues cannot be. Therefore, the result is partial, we could not exclude all complex eigenvalues that lead to instability. Therefore, the problem remains open.

2.1 Derivation of the Dirac equation

The free Dirac equation describes the dynamics of a massive spin-$\frac{1}{2}$ relativistic fermion in (3+1) space-time dimensions. The usual way to derive this equation is to use the fact that classical quantities correspond to differential or multiplication operators in quantum mechanics. The goal is to obtain an equation for quantum particles compatible with the relativistic energy-momentum relation.

So far, we know these two relations:

$$a) \quad E = \sqrt{c^2p^2 + m^2c^4} \qquad\qquad b) \quad E \to i\hbar\frac{\partial}{\partial t} \tag{2.1}$$

The first one is the relativistic energy-momentum relation and the second one is the quantum interpretation of the energy as operator. This relativistic energy needs to be an eigenvalue of the quantum energy operator, and if we write this using the operator form of the momentum, then we have the square-root Klein-Gordon equation:

$$i\hbar\frac{\partial}{\partial t}\varphi(t,\mathbf{x}) = \sqrt{-c^2\hbar^2\Delta + m^2c^4}\varphi(t,\mathbf{x}), \quad t \in \mathbb{R}, \quad \mathbf{x} \in \mathbb{R}^3 \tag{2.2}$$

If we apply the energy operator over each side, then obtain the Klein-Gordon equation. The problem here is that this equation is second order in time and a quantum mechanical evolution equation should be first order. Moreover this equation does not include the spin structure because the scalar wave function [58]. So, Dirac's idea was to change the structure of the energy introducing new factors such that

$$E = c\sum_{i=1}^{3} \alpha_i p_i + \beta mc^2 \equiv c\boldsymbol{\alpha}\cdot\mathbf{p} + \beta mc^2 \tag{2.3}$$

with $\boldsymbol{\alpha}\cdot\mathbf{p} = \sum_{i=1}^{3} \alpha_j \frac{\partial}{\partial x_j}$, $\hbar = c = 1$ and comparing this with the energy-momentum relation we find that the following anti-commutative relations between these new quantities must hold the following relations

$$\{\alpha_i, \alpha_k\} = 2\delta_{ik}\mathbf{I}_4, \quad \{\alpha_i, \beta_k\} = 0, \quad \beta^2 = \mathbf{I}_4, \quad i, j = 1, 2, 3 \tag{2.4}$$

with $\mathbf{1}_4$ being the 4×4 identity matrix and α y β are anti-commutating quantities and can be represented by $n\times n$ matrices. This representation is not unique, the most common choice is using the Pauli matrices given by (see [58], [26]),

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.5}$$

In the case of massless particles, the term with β vanish and the matrices can be used in a direct way similarly happen in the case of one and two dimension where de 2×2 matrices are sufficient. In dimension three, we need write the matrices in the following way, which is the standard representation introduced by Dirac,

$$\beta = \begin{pmatrix} \mathbb{1} & \mathbf{0} \\ \mathbf{0} & -\mathbb{1} \end{pmatrix} \quad \alpha_i = \begin{pmatrix} \mathbf{0} & \sigma_i \\ \sigma_i & \mathbf{0} \end{pmatrix} \quad i = 1, 2, 3 \tag{2.6}$$

these matrices have a lot of properties and play a very important role in field theory. So, using (2.3) we have the free Dirac equation:

$$i\hbar\frac{\partial}{\partial t}\varphi(t,\mathbf{x}) = D_m\varphi(t,\mathbf{x}), \quad \varphi(t,\mathbf{x}) \in \mathbb{C}^4\,. \tag{2.7}$$

Using the triplets $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, D_m is given explicitly by

$$D_m = -i\hbar c \boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta mc^2 = \begin{pmatrix} mc^2 \mathbf{1} & -i\hbar c \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \\ -i\hbar c \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} & -mc^2 \mathbf{1} \end{pmatrix} . \tag{2.8}$$

D_m is called the free Dirac operator and for the case $D_{m=0} := D_0$ is called the massless free Dirac operator.

2.2 Basic spectral properties of the free Dirac Operator

We are going to introduce the Dirac operator following the presentation in [65, Section 10.6]. Consider the Hilbert space given by

$$H^1(\mathbb{R}^n)^N = L^2(\mathbb{R}^n)^N := \bigoplus^N L^2(\mathbb{R}^n),$$

where $N := N(n) = 2^{[(n+1)/2]}$ and $n \geq 1$ is the dimension. The Hilbert space is also equipped with the scalar product

$$\langle \psi, \varphi \rangle = \sum_{n=1}^{N} \int \psi_n^*(x) \varphi_n(x) \, \mathrm{d}x = \int (\psi, \varphi)_{\mathbb{C}^N} \, \mathrm{d}x, \quad x \in \mathbb{R}^n ,$$

where $(\cdot, \cdot)_{\mathbb{C}^N}$ denotes the scalar product in $\mathbb{C}^N$ and if $\psi \in L_2(\mathbb{R}^n)^N$ means that $\psi_i \in L^2(\mathbb{R}^n)$ for $i = 1, ..., N$. Then for any continuously differentiable function $\psi : \mathbb{R}^n \to \mathbb{C}^N$ we can define the free Dirac operator as follows

$$D_m \psi(x) = \left(\hbar \sum_{i=1}^{N} \alpha_i D_i + m\beta \right) \psi(x), \qquad x \in \mathbb{R}^n \tag{2.9}$$

where α_i and β are the $N \times N$ Pauli matrices (2.6). D_i is a first order differential operator given by

$$(D_i\psi)(x) = -i\left(\frac{\partial}{\partial x_i}\psi_1, \frac{\partial}{\partial x_i}\psi_2, ..., \frac{\partial}{\partial x_i}\psi_N\right)^\top .$$

The free Dirac operator is essentially self-adjoint in $C_0^\infty(\mathbb{R}^n)$ and self-adjoint on the Sobolev space $H^1(\mathbb{R}^n)^N \subset L^2(\mathbb{R}^n)^N$ [58, Theorem 1.1]. Moreover, its spectrum is purely absolutely continuous and is given by

$$\sigma_{ess}(D_m) = (-\infty, -mc^2] \cup [mc^2, \infty). \tag{2.10}$$

This can be easily proved taking the Fourier transform of D_m. In three dimensions the transform is given by

$$\mathcal{F}\psi_k(k) \equiv \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \mathrm{e}^{-ip\cdot x}\psi_k(x)\,\mathrm{d}^3x$$

where $k = 1, 2, 3, 4$. Is defined for integrable functions, unitary and maps the coordinate space to the momentum space, i.e. $\mathcal{F}L^2(\mathbb{R}^3,\, \mathrm{d}^3x)^4 = L^2(\mathbb{R}^3,\, \mathrm{d}^3p)^4$. Any matrix differential operator with constant coefficients in $L^2(\mathbb{R}^3,\, \mathrm{d}^3x)^4$ is transformed via $\mathcal{F}$ to a multiplication operator in $L^2(\mathbb{R}^3,\, \mathrm{d}^3p)^4$, see [58]. So, for the Dirac operator (2.8) we have

$$(\mathcal{F}D_m\mathcal{F})\,(p) = \begin{pmatrix} mc^2\mathbf{1} & \hbar c\boldsymbol{\sigma}\cdot p \\ \hbar c\boldsymbol{\sigma}\cdot p & -mc^2\mathbf{1} \end{pmatrix}$$

which is a matrix with eigenvalues depending on p with modulus $\lambda(p)^2 = c^2p^2 + m^2c^4$, this result is expected because we construct everything from (2.1). Then, the spectrum of D_m is given by the range of $\lambda_\pm(p) = \pm\sqrt{c^2p^2 + m^2c^4}$ with $p \in \mathbb{R}^3$, therefore the spectrum is $(-\infty, -mc^2] \cup [mc^2, \infty)$. From this expression, it is clear that the spectrum is purely a.c. and therefore essential. Is easy to see that the essential spectrum is equal at least for $n = 1, 2, 3$.

2.3 The Soler model

In 1938 the Russian physicist Dimitri Ivanenko considered a self interacting model, including the non-linearity $\overline{\psi}\psi$ [36] in the Hamiltonian. Then Weyl in 1950 [66] and Heisenberg in 1953 [35] also studied the same type of nonlinearity. But it was Mario Soler [55] who was the first to investigate the stationary states of the nonlinear Dirac field with the scalar fourth order self coupling, proposing them as a model of elementary extended fermions. This term transforms as a scalar under Lorentz transformations. Nonlinear generalizations of the Dirac equation, have emerged naturally as a practical model in many physical systems, a few examples are: extended particles, the gap solitary waves in nonlinear optics, light solitary waves in waveguide arrays and experimental realization of an optical analog for relativistic quantum mechanics, Bose-Einstein condensates in honeycomb optical lattices, phenomenological models of quantum chromodynamics, as well as matter influencing the evolution of the Universe in cosmology, etc. see [40, 41].

Soler studied this model from a classical point of view, and considered extended nucleons interacting with their own electromagnetic field [24]. The model describes a self-interaction of the fermions with their own lattice, in the case of graphene this phenomenon is well known, see [16]. The nonlinearity in the Soler model is a smooth function of Ivanenko's nonlinearity $f(\overline{\psi}\psi)$. In natural units the Soler equation is given by

$$i\partial_t \psi(x,t) = \left(D_0 + \beta\left(m - f(\overline{\psi}\psi)\right)\right)\psi(x,t), \quad \psi(x,t) \in L^2(\mathbb{R}\times\mathbb{R}^n, \mathbb{C}^N), \tag{2.11}$$

where D_0 is the massless free Dirac operator (2.9), n is the dimension and N comes from the choice of the matrices (2.4).

The Soler model is a relativistically invariant Hamiltonian system. The Hamiltonian is represented by

$$\mathcal{H}_{Soler}(\psi) = \psi^* D_m \psi - F(\psi^*\beta\psi), \quad F(x) = \int_0^x f(t)\,\mathrm{d}t.$$

Analogously we can characterize the Soler model with the following Lagrangian density

$$\mathcal{L}_{Soler} = \overline{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi + F(\overline{\psi}\psi)\,,$$

where we use the notation $\overline{\psi}\psi = \psi^*\beta\psi$. Given the symmetry of this system we have conservation of the value of the charge functional

$$Q(\psi(t)) = \int_{\mathbb{R}^n} \psi(x,t)^*\psi(x,t)\,\mathrm{d}x\,,$$

which is conserved in time assuming smooth enough solutions, see [42, 24]. Also, the energy functional is conserved in time if the solutions are sufficiently smooth, and is given by

$$E(\psi(t)) = \int \mathcal{H}_{Soler}(\psi(x,t))\,\mathrm{d}x.$$

Soler computed that this quantities, $E(\psi(t))$ and $Q(\psi(t))$, both have minima at $\omega_c = 0.936$ [55]. Strauss and Vazquez in 1986 [56] studied the stability of this model in two dimensions under dilatations. They found that the solutions are dilatation unstable for $0.936 < \omega < 1$ and conjectured that are dilatation stable for $0 < \omega < 0.936$.

We are interested in solitary-wave solutions of the form

$$\psi(x,t) = \phi(x)\mathrm{e}^{-i\omega t} \tag{2.12}$$

for some localized $\phi(x) \in L^2(\mathbb{R}^n, \mathbb{C}^N)$. This kind of solutions are known as solitons or solitary waves if they are stable, see [37, 24]. Solitary wave solutions or particle-like solutions are localized solutions with finite energy and charge [5]. This kind of solutions only are known explicitly in the one-dimensional case [24].

The main goal is to prove that $\phi(x)$ is stable, so consider the perturbed solution $\tilde{\psi}(x,t) = (\phi(x) + \rho(x,t))\,\mathrm{e}^{-i\omega t}$, which starts close to ψ, and then we can study how the perturbations

$\rho(x,t)$ behaves in time. For the uniqueness based on the Picard-Lindelöf theorem, we also need an initial data $\psi(t=0)$ so the perturbation starts close to $\psi(t=0)$.

The general stability case can be reeded mathematically as: $\forall\epsilon > 0$, there exists a $\delta > 0$ such that every solution $\tilde{\psi}$ having initial conditions within distance δ respect to $\psi(t=0)$, i.e. that starts close to $\psi(t=0)$ in norm $\left\|\psi(t=0)-\tilde{\psi}\right\| < \delta$, remains within distance ϵ for all $t \geq 0$, i.e. $\left\|\psi(t)-\tilde{\psi}\right\| < \epsilon$. This requirement in generally too strong, that is why other stability criteria are studied when problems become more complicated, for example:

Orbital stability: Exist $\epsilon > 0$, and $\delta_0 > 0$ such that $\forall\delta < \delta_0$, if $\left\|\psi(t=0)-\tilde{\psi}\right\| < \delta$ then $\forall t$ we have that $\left\|\psi - \mathrm{e}^{-i\theta(t)}\tilde{\psi}\right\| < \epsilon\delta$. So, this tells us that if the perturbed solutions have initial data within distance δ with respect to $\psi(t=0)$ then $\tilde{\psi}$ multiplied by a phase of module one, remains close to ψ in time. For example, in [46], Pelinovsky y Shimabukuro proved the orbital stability for the solitary waves of the cubic massive Thirring model.

Asymptotic stability: Exist $\delta_0 > 0$ such that $\forall\delta < \delta_0$ we have that $\left\|\psi(t=0)-\psi(\tilde{t},\omega)\right\| \leq \delta$, then exist $\theta(t)$ and $\tilde{\omega}$ such that $\left\|\psi(t)-\mathrm{e}^{i\theta(t)}\psi(t,\tilde{\omega})\right\| \to 0$ as $t \to 0$ and $\dot{\theta}(t) \to \tilde{\omega}$ as $t \to \infty$. This means that the solution converges to a solitary wave with frequency $\tilde{\omega}$. This stability criteria are very strong and is false to completely integrable systems as Lax pairs, Korteweg-de Vries and the massive Thirring model.

Spectral stability: Let $\tilde{\psi} = (\phi(x)+\rho(x,t))\,\mathrm{e}^{-i\omega t}$ be the perturbed solution of (1.1), where $\rho(x,t) = (\rho_1,\rho_2)^\top \in \mathbb{C}^2$ is small enough to drop the quadratic terms. Recalling that $\psi = \mathrm{e}^{-i\omega t}\phi(x)$ solve (1.1) we obtain an evolution type equation for $\rho(x,t)$, so we can analyze how it behaves in time. This procedure is known as the Bogoliubov-deGennes linearization analysis. The time-evolution equation involves linearized operators as follow, $i\partial_t\rho = A\rho$ where A is the linearized operator, therefore the stability depends on its spectrum. The idea is, if we perturb the solutions and the perturbation goes to zero in a finite time or it keeps oscillating, then this means that $\tilde{\psi}$ tends to ψ or keeps oscillating close to ψ, respectively. Then we can say that the solution is linearly stable, conversely if the perturbation ρ grows in time we have that $\tilde{\psi}$ moves away from ψ and the solution is called linearly unstable.

The existence of this solitary wave solutions was already proved in [17, 30] and can be extended to the Poddubny-Smirnova generalization see the appendix A. In the next chapters we are goin to focus on the stability of this solitary waves using the Bogoliubov-deGennes linearization criterium.

Chapter 3

One dimensional Soler model

In this chapter the stability of the solutions of the Soler model in one dimension is studied. The Soler model in one spatial dimension is also known as the Gross-Neveu model [33]. The chapter is organized as follows.

Section 3.1: First, the main equations of one-dimensional Soler model are introduced. Starting with the Dirac equation in a dimension disturbed by a general Soler-type non-linearity. Then, it is shown that the spinor components satisfy a Hamiltonian system, where the Hamiltonian turns out to be 0.

Section 3.2: It is briefly discussed how the problem of the existence of solutions for this model has been dealt with in the literature. Then, it is shown in detail how the solutions for the 1D Soler model are found when the non-linearity is $f(s) = s$. Also, the solutions for the general p-power nonlinearity $f(s) = s^p$ are shown.

Section 3.3: The stability of solutions is studied. Here the solutions are perturbed by means of the Bogoliubov-deGennes linearization stability analysis. Here the operator resulting from the linearization H_μ is found and the notion of spectral stability is defined, in terms of the spectra of this operator. This operator depends on two other operators L_0 and L_μ, which are analyzed extensively in the next section.

Section 3.4: The spectra of the operators L_0 and L_μ are studied extensively. The basic properties of their spectra are shown, properties that can be found in the literature and new results. Some of the new result are for example: in the case $p = 1$ we will show it by an oscillation argument that $\sigma_d(L_0) = \{-2\omega, 0\}$; the eigenvalues of L_0 and L_μ are simple, and so on. These results can be found summarized in the next chapter.

Section 3.5: The spectrum of the H_μ operator is analyzed. This is the most important operator, since the spectral stability of the Soler model solutions depends on their spectrum. The section begins by finding the main symmetries of the H_μ spectrum. Next, the spectral projections of L_0 are studied. These spectral projections are later used to give bounds on the spectrum of the H_μ^2 operator.

Section 3.6: Here you can find the main results of this thesis. Bounds on the imaginary and real part of $\sigma_p(H_\mu)$ are proved. These bounds allow us to exclude a region of the complex plane where the eigenvalues cannot be. Therefore, the result is partial, we could not exclude all complex eigenvalues that lead to instability. Therefore, the problem remains open.

3.1 Setting the problem

In this case one can represent the equation (2.11) with $N = 2$ choosing $\alpha_1 = \sigma_2$ and $\beta = \sigma_3$ the standard Pauli matrices (2.5). So, the one-dimensional time-dependent nonlinear Dirac equation is given by

$$i\partial_t\psi(x,t) = \left(i\sigma_2\partial_x + \sigma_3\left(m - f(\overline{\psi}\psi)\right)\right)\psi(x,t), \quad \psi(x,t) \in L^2(\mathbb{R}\times\mathbb{R},\mathbb{C}^2) \tag{3.1}$$

In this case, if $\psi = (\psi_1, \psi_2)^\top$, the argument of the function f becomes $\overline{\psi}\psi = \psi^\dagger\sigma_3\psi = |\psi_1|^2 - |\psi_2|^2$. This equation gives us a coupled partial differential equation of the form

$$\begin{aligned} i\partial_t\psi_1 &= \partial_x\psi_2 + (m - f(|\psi_1|^2 - |\psi_2|^2))\psi_1, \\ i\partial_t\psi_2 &= -\partial_x\psi_1 - (m - f(|\psi_1|^2 - |\psi_2|^2))\psi_2 \end{aligned} \tag{3.2}$$

We are interested in solitary-wave solutions of the form

$$\psi(x,t) = \phi(x)\mathrm{e}^{-i\omega t} \tag{3.3}$$

for some localized $\phi(x) = (v,u)^\top \in H^1(\mathbb{R},\mathbb{R}^2)$. Inserting this in (3.1):

$$\omega\phi(x) = \left(i\sigma_2\partial_x + \sigma_3\left(m - f(\phi\sigma_3\phi)\right)\right)\phi(x). \tag{3.4}$$

Equivalently we have the following coupled system

$$\begin{cases} \partial_x v(x) & = -\Big(\omega + m - f(|v|^2 - |u|^2)\Big)\, u(x) \\ \partial_x u(x) & = \Big(\omega - m + f(|v|^2 - |u|^2)\Big)\, v(x) \end{cases} \tag{3.5}$$

following the strategy of Cazenave and Vazquez [17] we can write (3.5) as a Hamiltonian system as follows

$$\begin{aligned} \partial_x v(x) &= -\partial_u H(u,v), \\ \partial_x u(x) &= \partial_v H(u,v). \end{aligned} \tag{3.6}$$

Here, x is playing the role of the time and the Hamiltonian is given by

$$H(u,v) = \frac{1}{2}\left(\omega(v^2+u^2) - G(v^2-u^2)\right), \tag{3.7}$$

where

$$G(x) = \int_0^x g(t)\,\mathrm{d}t, \quad \text{with} \quad g(t) = m - f(t). \tag{3.8}$$

The solitary waves that we are looking belongs to the Sobolev space $H^1(\mathbb{R},\mathbb{R})$, so are such that vanish at infinity, that is

$$\lim_{x\to\pm\infty} v(x) = \lim_{x\to\pm\infty} u(x) = 0. \tag{3.9}$$

So, this implies that $\lim_{x\to\pm\infty} v^2+u^2 = 0$ and therefore $H(u,v) \equiv 0$. A completely analysis of this problem can be found in [6]. So, from (3.7) we can conclude that

$$G(v^2 - u^2) = \omega(v^2 + u^2). \tag{3.10}$$

In the general case $f(s) = s^p$ with $p > 0$, the antiderivative take the form $G(s) = ms - \frac{s^{p+1}}{p+1}$ and this last relation becomes

$$m(v^2 - u^2) - \frac{(v^2 - u^2)^{p+1}}{p+1} = \omega(v^2 + u^2). \tag{3.11}$$

The analysis of the Hamiltonian is useful to prove the existences of solutions in the one and two-dimensional Soler model. In fact, we can extract a lot of information of the solutions analyzing the level sets of the Hamiltonian, later we will study this Hamiltonian more thoroughly.

Franz Mertens et al, in 2012 [42] considered the Lagrangian nonlinearity $\frac{g^2}{p+1}(\overline{\psi}\psi)^{p+1}$ and were able to find the exact solutions. In the next chapter we are going to show how the solutions are get for $p = 1$, following the strategy of [6]. Is interesting note that there are more nonlinearities in the literature, Walter Thirring in 1958 [59] considered the Lagrangian nonlinearity $\lambda\psi^*\psi\psi^*\psi$ and found the 1D solutions. Pelinovsky and Shimabukuro in [46] proved the *orbital stability* for the Thirring model. Many other nonlinearities can be found and most of them are exactly solvable in the 1-d case.

3.2 Existence and solutions of the 1D Soler model

One of the first proof of existence was given by Cazenave and Vazquez [17] in 1986 using ODE arguments. We will follow the lines of the proof to extend it to the generalization of Poddubny-Smirnova for some range of the parameters in the appendix A. In 1995 Maria Esteban and Éric Séré proved the existence of stationary states using variational methods [30]. Then, improve the knowledge about the variational methods [27, 29], where the existence problem for this type of nonlinearity is well summarize.

The one-dimensional Soler model is exactly solvable for any nonlinearity of the form $f(s) = s^p$, see [40, 41, 42]. But, in [14, Theorem 2.11] and [61] they show the absence of solitary waves if $|\omega| > m$, even more, in the first one proves the exponencial decay of the solutions.

Now we are going to show how Berkolaiko and Comech in [6] found the solutions in the case $v^2 - u^2 > 0$ and $0 < \omega < m \in \mathbb{R}$. The coupled system (3.5) take the form

$$\begin{cases} \partial_x v(x) & = -\left(\omega + m - v^2 + u^2\right) u(x) \\ \partial_x u(x) & = \left(\omega - m + v^2 - u^2\right) v(x). \end{cases} \tag{3.12}$$

To solve this we need to define some new variables and study his derivatives

$$h(x) = v(x)^2 - u(x)^2, \qquad p(x) = v(x)u(x). \tag{3.13}$$

Using (3.5) the derivatives are

$$h(x)' = 2\left(vv' - uu'\right) = -4\omega vu = -4\omega p(x), \tag{3.14}$$

$$p(x)' = v'u + vu' = -\left(m - f(v^2 - u^2)\right)(v^2 + u^2) + \omega(v^2 - u^2). \tag{3.15}$$

Notice that since (3.10) the derivative of $p(x)$, recalling that $g(s) = m - f(t)$, can be written in terms of $h(x)$ and these two new variables solve

$$\begin{cases} h(x)' = -4\omega p(x), \\ p(x)' = -\frac{1}{\omega} g(h)G(h) + \omega h. \end{cases} \tag{3.16}$$

Now the second derivative of $h(x)$:

$$h(x)'' = 4g(h)G(h) - 4\omega^2 h = \partial_h \left(2G(h)^2 - 2\omega^2 h^2\right), \tag{3.17}$$

integrating this expression, we obtain the equation that will be useful to find the exact solutions of this model and its given by

$$\left(h(x)'\right)^2 = 4G(h)^2 - 4\omega^2 h^2, \tag{3.18}$$

this implies that is needed $G(x) > \omega x$, due the left side of the equation is positive. Moreover, realize that this equation is separable and, with the appropriate change of variable, exactly solvable.

Until now the nonlinearity f introduced in (2.11) is a smooth function of the spinor and the solutions found in [6] are specifically for the case of the Gross-Neveu model, which is the same that the one-dimensional Soler model. These authors used $m = 1$ but it is easy to do the same calculation for any $m > \omega > 0$.

The antiderivative of the nonlinearity (3.8) for this model is given by:

$$G(x) = mx - \frac{x^2}{2}, \tag{3.19}$$

replacing this in (3.18), choosing the negative square root and separating the derivatives we obtain

$$\mathrm{d}x = -\frac{\mathrm{d}h}{2h\sqrt{(m-h/2)^2-\omega^2}}. \tag{3.20}$$

Now, using the substitution used in [6],

$$m - \frac{h}{2} = \frac{\omega}{\cos 2\theta} \implies \quad h = 2\left(m - \frac{\omega}{\cos 2\theta}\right), \tag{3.21}$$

so $\mathrm{d}h = -\frac{4\omega \sin 2\theta}{\cos^2 2\theta}\,\mathrm{d}\theta$. Then

$$\mathrm{d}x = \frac{\frac{4\omega\sin 2\theta}{\cos^2 2\theta}\,\mathrm{d}\theta}{4\omega\left(m - \frac{\omega}{\cos 2\theta}\right)\frac{\sin 2\theta}{\cos 2\theta}} = \frac{\mathrm{d}\theta}{m\cos 2\theta - \omega} \tag{3.22}$$

this equation is integrable and it is easy to verify that the solution is

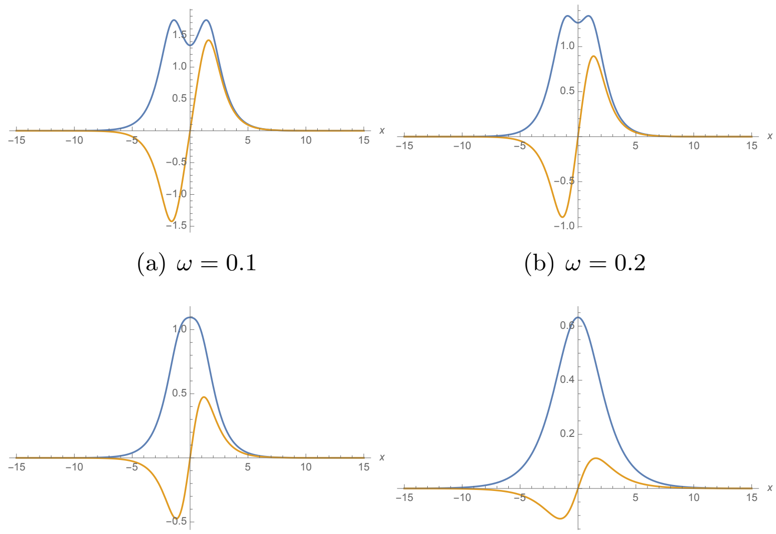
$$x = \frac{1}{2\kappa}\ln\left|\frac{\sqrt{\nu}+\tan\theta}{\sqrt{\nu}-\tan\theta}\right| \tag{3.23}$$

where $\kappa = \sqrt{m^2-\omega^2}$ and $\nu = \frac{m-\omega}{m+\omega}$. The following steps are use trigonometric identities to compute and combine the variables (3.13) and v^2+u^2, all in terms of $\theta(x)$. Finally, they get the following expressions for the spinor components:

$$v(x) = \frac{\sqrt{2(1-\omega)}}{\left(1-\nu\tanh^2\kappa x\right)\cosh\kappa x}, \quad u(x) = \frac{\sqrt{2\nu(1-\omega)}\tanh\kappa x}{\left(1-\nu\tanh^2\kappa x\right)\cosh\kappa x}. \tag{3.24}$$

Note that $u(x) = \sqrt{\nu}\tanh(kx)v(x)$. These functions decays exponentially to zero as $|x| \to \infty$, see [6, Remark 3.5].

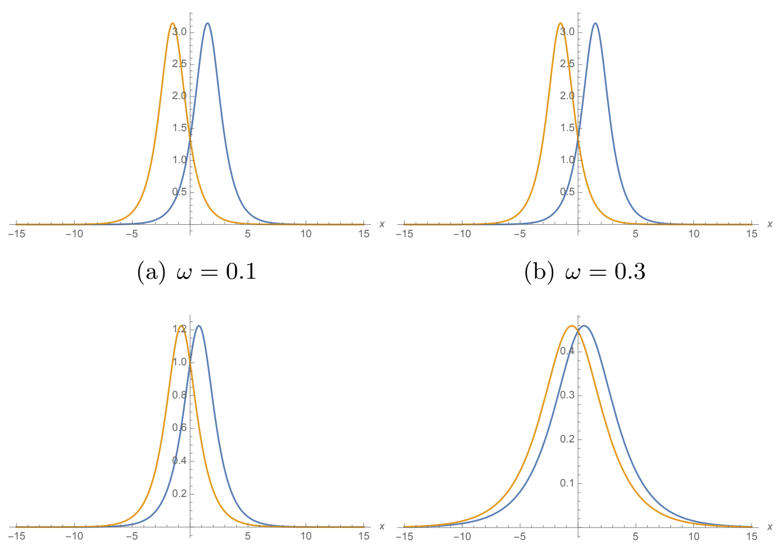
The graph of this functions and of $v(x)+u(x)$ and $v(x)-u(x)$ will be useful for reasons that we will explain latter. So, the graphics for $m=1$ and some values of ω are given by:



(a) $\omega = 0.1$ (b) $\omega = 0.2$

(c) $\omega = 0.4$ (d) $\omega = 0.8$

Figure 3.1: Graph of functions $v(x)$ in blue and $u(x)$ in yellow for $\omega = \{0.1, 0.2, 0.4, 0.8\}$ respectively.



(a) $\omega = 0.1$ (b) $\omega = 0.3$

(c) $\omega = 0.5$ (d) $\omega = 0.9$

Figure 3.2: Graph of functions $v(x) + u(x)$ in blue and $v(x) - u(x)$ in yellow for $\omega = \{0.1, 0.3, 0.5, 0.9\}$ respectively.

In the case where the nonlinearity is given by $f(s) = s^p$, with $p > 0$ and $\omega \in (0, m)$, is also exactly solvable, see [24]. The solitary wave solutions is given by $\phi_0(x) := \phi_0(p, \omega, m, x) :=$

$\begin{pmatrix} v(p,\omega,m,x) \\ u(p,\omega,m,x) \end{pmatrix}$ where the components are

$$v(p,\omega,m,x) := \frac{1}{\sqrt{1-\nu\tanh^2(p\kappa x)}}\left[(p+1)(m-\omega)\frac{1-\tanh^2(p\kappa x)}{1-\nu\tanh^2(p\kappa x)}\right]^{\frac{1}{2p}}, \tag{3.25}$$

$$u(p,\omega,m,x) := \sqrt{\nu}\tanh(p\kappa x)\, v(p,\omega,m,x)\,, \tag{3.26}$$

we have introduced the parameters

$$\kappa = \sqrt{m^2-\omega^2} \quad \text{and} \quad \nu = \frac{m-\omega}{m+\omega} \in (0,1).$$

We will often omit the arguments p, ω and m written them as $v(x)$ and $u(x)$, when this causes no confusion. The solution of (3.4) is unique up to multiplication by a constant phase and translations in space. The solution $\phi_0 = (v,u)^\top$ is (up to complex multiples) the unique solution such that the first entry is even, the second entry odd.

Moreover, given the explicit spatial depends in $p\kappa x$ of ϕ_0 and because it will often be convenient for shortness and clarity, we will use the convention that a tilde means a spatial rescaling by a factor $p\kappa$. Namely, if f is a function on $\mathbb{R}$, then $\tilde{f}$ is defined through $f(x) = \tilde{f}(p\kappa x)$ for any $x \in \mathbb{R}$. For example,

$$\tilde{v} = \frac{1}{\sqrt{1-\nu\tanh^2}}\left[(p+1)(m-\omega)\frac{1-\tanh^2}{1-\nu\tanh^2}\right]^{\frac{1}{2p}}. \tag{3.27}$$

Remark 3.1. Note that the function $v(x)+u(x)$ and $v(x)-u(x)$ does not change sign for $\omega \in (0,1)$ and $p > 0$. Because we have the relation $u(x) = \sqrt{\nu}\tanh(p\kappa x)v(x)$, where $\nu = \frac{1-\frac{\omega}{m}}{1+\frac{\omega}{m}} < 1$ and the function $\tanh(z) \in (-1,1) \quad \forall z \in \mathbb{R}$. See figure 3.2.

3.3 Perturbation and Linear Stability

The linear stability is considered by the Bogoliubov-deGennes linearization analysis, see [24, 13]. General nonlinearities have been studied using the same linearization [24, 13, 14, 6]. In [15] gives a partial result on spectral stability for a range of parameters. Consider another solution of the one-dimensional soler model (3.1) which is close to ψ (3.3) and is given by

$$\tilde{\psi} = (\phi(x) + \rho(x,t))\,\mathrm{e}^{-i\omega t}, \quad \rho(x,t) = (\rho_1, \rho_2)^\top \in \mathbb{C}^2 \tag{3.28}$$

and found a differential equation for the perturbation $\rho(x,t)$. In the process you need to consider that this perturbation is small enough to drop quadratic terms and this is called the linearization. With this equation we can study the time evolution of the perturbation and we hope, if the system is stable, that the norm is not growing in time. In fact, if the system is stable this perturbation is only a oscillation and $\tilde{\psi}$ tend to ψ in a finite time.

Inserting this perturbation in (3.1)and dropping terms quadratic in ρ we obtain

$$i\partial_t\,(\phi(x) + \rho(x,t))\,\mathrm{e}^{-i\omega t} = (i\sigma_2\partial_x + \sigma_3\,(m - f(s)))\,(\phi(x) + \rho(x,t))\,\mathrm{e}^{-i\omega t} \tag{3.29}$$

where the argument of the nonlinearity is $s = v^2 - u^2 + 2\,(v\,\mathrm{Re}(\rho_1) - u\,\mathrm{Re}(\rho_2))$. For the case $f(s) = s$ use that ϕ solve the equation (3.4):

$$i\partial_t\rho = \left(i\sigma_2\partial_x + \sigma_3\left(m - v^2 + u^2\right) - \omega\right)\rho - 2\,(v\,\mathrm{Re}(\rho_1) - u\,\mathrm{Re}(\rho_2))\,\sigma_3\rho \tag{3.30}$$

$$i\partial_t\rho := L_0(\omega)\rho - 2Q(\omega)\,\mathrm{Re}(\rho) \tag{3.31}$$

where we have defined the following two operators

$$L_0(\omega) \equiv L_0 = i\sigma_2\partial_x + \sigma_3\left(m - v^2 + u^2\right) - \omega \tag{3.32}$$

$$Q(\omega) \equiv Q = \begin{pmatrix} v^2 & -uv \\ -uv & u^2 \end{pmatrix}. \tag{3.33}$$

Remark 3.2. Observe that Q acts as a projection on $(v, -u)^\top$: For all $\varphi = (\varphi_1, \varphi_2)^\top$,

$$Q\varphi = (v\varphi_1 - u\varphi_2)\begin{pmatrix} v \\ -u \end{pmatrix}. \tag{3.34}$$

In particular if $\varphi = (u, v)^\top$ then $Q\varphi = 0$. We may also write $Q = V^*V$ with

$$V = \begin{pmatrix} v & -u \\ 0 & 0 \end{pmatrix}, \quad V^* = \begin{pmatrix} v & 0 \\ -u & 0 \end{pmatrix}. \tag{3.35}$$

These operators play a crucial role in the stability problem, the evolution equation of the perturbation is determined by the spectrum of this operator as we will see next. Substituting $\rho = \mathrm{Re}(\rho) + i\,\mathrm{Im}(\rho)$ in (3.30)

$$i\partial_t \begin{pmatrix} \mathrm{Re}\,\rho \\ \mathrm{Im}\,\rho \end{pmatrix} = \begin{pmatrix} 0 & iL_0 \\ -i\left(L_0 - 2Q\right) & 0 \end{pmatrix}\begin{pmatrix} \mathrm{Re}\,\rho \\ \mathrm{Im}\,\rho \end{pmatrix} =: \tilde{H}(\omega)\begin{pmatrix} \mathrm{Re}\,\rho \\ \mathrm{Im}\,\rho \end{pmatrix}. \tag{3.36}$$

This is a time evolution equation for the perturbation and it gives us an idea of how have to be the eigenvalues of the operator $\tilde{H}(\omega)$ for the system to be stable, this is known as the linearized equation. For example, if the operator has no complex eigenvalues, in some sense, the solution is only an oscillatory wave and the original system after being perturbed it stays stable. Following the idea described in [14] we are going to define the stability as follows,

Definition 3.1. We will say that a particular solitary wave is spectrally stable if the spectrum of the equation linearized at this wave does not contain points with imaginary part.

Hence, we need to focus in study the spectrum of the operator $\tilde{H}(\omega)$. For this purpose we need analyze the following eigenvalue problem, let be $\tilde{\psi} = (\tilde{\varphi}_1, \tilde{\varphi}_2)^\top \in H^1(\mathbb{R}, \mathbb{C}^4)$ an non-zero eigenfunction such that

$$\tilde{H}(\omega)\tilde{\psi} = z\tilde{\psi}\,. \tag{3.37}$$

Notice that equation (3.37) is satisfied if and only if z is an eigenvalue of

$$H(\omega) := \begin{pmatrix} 0 & L_0 \\ L_0 - 2Q & 0 \end{pmatrix}, \tag{3.38}$$

with eigenfunction $\psi = (\varphi_1, \varphi_2)^\top = (\tilde{\varphi}_1, i\tilde{\varphi}_2)^\top$. It might be useful to define, for $\mu \in \mathbb{R}$, the operators $L_\mu = L_0 - \mu Q$ and

$$H(\mu, \omega) = \begin{pmatrix} 0 & L_0 \\ L_\mu & 0 \end{pmatrix}. \tag{3.39}$$

Hence, our analysis is directed towards the existence of $z \in \mathbb{C} \setminus \mathbb{R}$, for the case $\mu = 2$, such that

$$L_0\varphi_2 = z\varphi_1 \tag{3.40}$$

$$L_\mu\varphi_1 = z\varphi_2\,. \tag{3.41}$$

Finally, for the general case where $f(s) = s^p$ the operators take the following form (see [24])

$$L_0 = \begin{pmatrix} m - \omega - (v^2 - u^2)^p & \partial_x \\ -\partial_x & -m - \omega + (v^2 - u^2)^p \end{pmatrix}, \tag{3.42}$$

$$L_\mu = L_0 - \mu p(v^2 - u^2)^{p-1} \begin{pmatrix} v^2 & -uv \\ -uv & u^2 \end{pmatrix} \equiv L_0 - \mu Q. \tag{3.43}$$

To avoid confusion, in the next chapters we will always specify which operator we are referring to. Or analogously, what power p do we refer to. If we do not say anything, is because the result works for every p or the statement is clear. Furthermore, note that there is only one difference between the operators in the case of $p = 1$ and any p, that is we need to add the coefficient $p(v^2 - u^2)^{p-1}$ in front of Q.

Remark 3.3. The operator $H_{\mu=2}(\omega)$ with domain $H^1(\mathbb{R}, \mathbb{C}^4)$ is closed in $L^2(\mathbb{R}, \mathbb{C})^4$.

Is useful study some properties of the eigenfunction components φ_1 and φ_2. We are going to use several times the eigenvalues equations (3.40) and (3.41). The first easy property is that either φ_1 or φ_2 can not be zero and his norms can not be equal to one. In order to avoid confusions in the following properties z always going to be the eigenvalue of $H_\mu(\omega)$ associated to the normalized eigenfunction $(\varphi_1, \varphi_2)^\top$ and we are going to call $a = \mathrm{Re}(z)$ and $b = \mathrm{Im}(z)$.

Remark 3.4. Let $(\varphi_1, \varphi_2)^\top$ be a normalized eigenfunction of $H_\mu(\omega)$ associated with eigenvalue $z \in \mathbb{C} \setminus \{0\}$. Then, $\varphi_1 \neq 0 \neq \varphi_2$ and $\|\varphi_1\|^2 \neq 1 \neq \|\varphi_2\|^2$.

Proof. From the eigenvalue equations (3.40) and (3.41) is straightforward. By contradiction, if $\|\varphi_1\|^2 = 1$ then, since $(\varphi_1, \varphi_2)^\top$ is normalized, $\varphi_2 = 0$ a.e., but $z\varphi_1 = L_0\varphi_2 = 0$. The same follows for $\|\varphi_2\|^2 = 1$. $\square$

3.4 Properties of L_0 and L_μ

Let us first note that the essential spectrum of L_0,

$$\sigma_{\mathrm{ess}}(L_0) = (-\infty, -m - \omega] \cup [m - \omega, \infty) . \tag{3.44}$$

Since Q decays sufficiently fast at infinity these are relatively compact perturbation to $D_m - \omega$ and therefore the essential spectrum remains equal, see [49, Theorem XIII.14]. The same is true for L_μ [6, Lemma 5.1].

An important fact about the spectrum of L_0 is that it is symmetric with respect to $-\omega$, this holds for any nonlinearity $g(s)$, see [6, Lemma 5.2]. To verify this, suppose that $\psi = (R, S)^\top$ satisfies the eigenvalue equation $(L_0 - \lambda)\psi = 0$. Then $\tilde{\psi} = (S, R)^\top$ satisfies $(L_0 + (2\omega + \omega))\tilde{\psi} = 0$.

In the case $p = 1$ in the power of the nonlinearity, Berkolaiko and Comech found explicit expression for resonances at the thresholds, that is, uniformly bounded eigenfunctions in the endpoints of the essential spectrum [6, Lemma 5.5]. They forgot the factor $-\frac{m-\omega}{2\omega}$ in front of $S(x)$ and the expressions they give are for $m = 1$.

Remark 3.5. For $p = 1$, the values $\lambda = m - \omega$ and $\lambda = -m - \omega$ are resonances of L_0. The generalized eigenfunctions corresponding to $\lambda = m - \omega$ are

$$\psi(x) = \begin{pmatrix} R(x) \\ S(x) \end{pmatrix}, \quad \text{where} \quad R(x) = \frac{vu}{v^2 - u^2}, \quad S(x) = -\frac{m-\omega}{2\omega}\frac{v^2 - \frac{m+\omega}{m-\omega}u^2}{v^2 - u^2}. \tag{3.45}$$

By the symmetry of the spectrum, the eigenfunction corresponding to $\lambda = -m - \omega$ is $\psi = (S, R)^\top$.

Proof. Write the system given by $L_0\psi(x) = (m - \omega)\psi$, so

$$\partial_x S = (v^2 - u^2)R\,, \tag{3.46}$$

$$\partial_x R = -(2m - (v^2 - u^2))S. \tag{3.47}$$

To verify the first one, using (3.11) and taking $p = 1$, we can rewrite S as

$$S(x) = -\frac{1}{4\omega}(v^2 - u^2)$$

and using (3.5) to take the derivative, is straightforward that

$$S' = uv \equiv (v^2 - u^2)R\,.$$

Now using the same equations, we have that $(v^2-u^2)' = -4\omega vu$ and

$$(uv)' = -\frac{1}{\omega}(m(v^2-u^2) - \frac{(v^2-u^2)^2}{2})(m-(v^2-u^2)) + \omega(v^2-u^2), \tag{3.48}$$

and the last useful relation is

$$4\omega^2 v^2 u^2 = (m(v^2-u^2) - \frac{(v^2-u^2)^2}{2})^2 - \omega^2(v^2-u^2)^2. \tag{3.49}$$

So, the derivative of $R(x)$ is

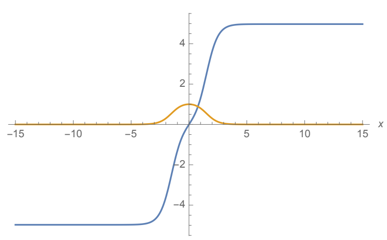
$$\begin{aligned}
R' &= \frac{(uv)'(v^2-u^2) + 4\omega u^2 v^2}{(v^2-u^2)^2} \\
&= -\frac{1}{\omega}(m - \frac{(v^2-u^2)}{2})(m-(v^2-u^2)) + \omega + \frac{1}{\omega}(m - \frac{(v^2-u^2)}{2})^2 - \omega \\
&= -\frac{1}{\omega}(m - \frac{(v^2-u^2)}{2})(m-(v^2-u^2) - m + \frac{(v^2-u^2)}{2}) \\
&= -(2m-(v^2-u^2))\frac{-1}{4\omega}(v^2-u^2).
\end{aligned}$$

□

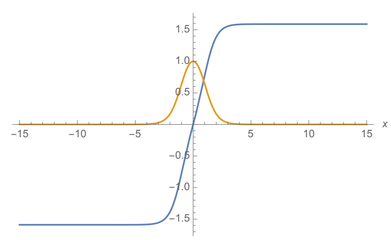
Then, since the symmetry of the spectrum, we have the following system:

$$
\begin{aligned}
(L_0 - (m-\omega)) \begin{pmatrix} R \\ S \end{pmatrix} &= 0, \\
(L_0 - (-m-\omega)) \begin{pmatrix} S \\ R \end{pmatrix} &= 0, \\
\implies (L_0 + \omega) \begin{pmatrix} R-S \\ S-R \end{pmatrix} &= m \begin{pmatrix} R-S \\ S-R \end{pmatrix}.
\end{aligned}
$$

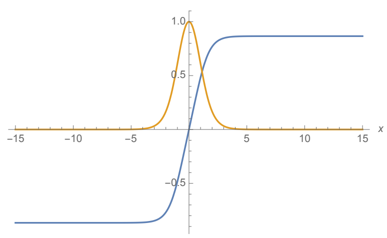
Hence m is an eigenvalue of the operator $A := L_0 + \omega$ with eigenfunction $(R-S, S-R)^\top$. Note that the spectrum of A is symmetric with respect to zero. The graphic of this functions for some values of ω are given in the next figures.



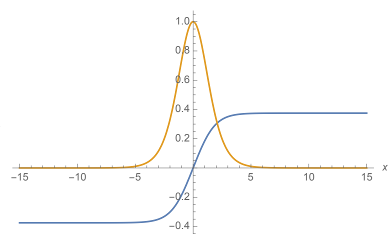
(a) $\omega = 0.1$



(b) $\omega = 0.3$

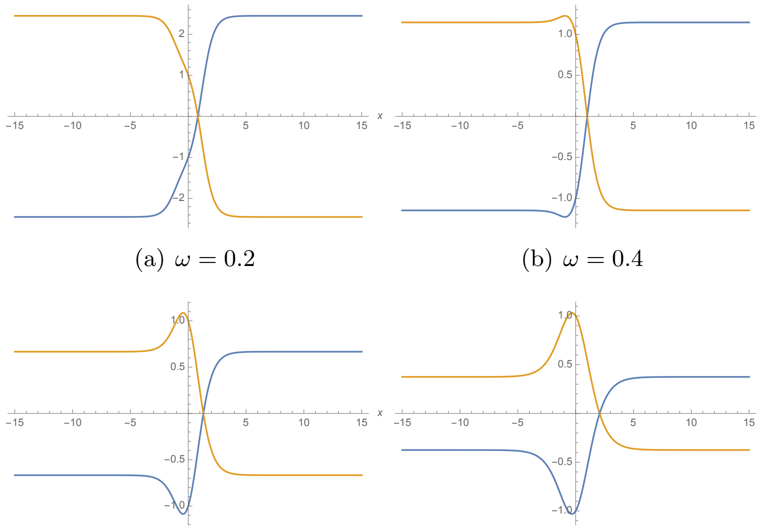


(c) $\omega = 0.5$



(d) $\omega = 0.8$

Figure 3.3: Graph of functions $R(x)$ in blue and $S(x)$ in yellow for $\omega = \{0.1, 0.3, 0.5, 0.8\}$ respectively.



(a) $\omega = 0.2$ (b) $\omega = 0.4$

(c) $\omega = 0.6$ (d) $\omega = 0.8$

Figure 3.4: Graph of functions $R(x) - S(x)$ in blue and $S(x) - R(x)$ in yellow for $\omega = \{0.2, 0.4, 0.6, 0.8\}$ respectively.

Corollary. *For all eigenvalue $z \in \mathbb{C} \setminus i\mathbb{R} \cup \mathbb{R}$ of $H_\mu(\omega)$, associated to the eigenfunction $(\varphi_1, \varphi_2)^\top$, we have that φ_1 cant be eigenfunction of L_μ and φ_2 cant be eigenfunction of L_0.*

Proof. If $L_\mu \varphi_1 = z_1 \varphi_1$ we have that $z\varphi_2 = z_1\varphi_1$ and therefore they can not be orthogonal. The same argument holds to $L_0\varphi_2 = z_2\varphi_2 = z\varphi_1$. □

For the nexts statements we are going to use the following relation that follows from (3.40)–(3.41):

$$\langle \mu Q\varphi_1, \varphi_2\rangle = \langle L_0\varphi_1 - z\varphi_2, \varphi_2\rangle = z\left\|\varphi_1\right\|^2 - \overline{z}\left\|\varphi_2\right\|^2 = a(\left\|\varphi_1\right\|^2 - \left\|\varphi_2\right\|^2) + bi\,. \quad (3.50)$$

Corollary. *For $z \in C \setminus \mathbb{R} \cup i\mathbb{R}$ we have that φ_1 can not be eigenfunction of L_0 and φ_2 can not be eigenfunction of L_μ. Even more, neither φ_1 nor φ_2 can be eigenfunction of μQ.*

Proof. By contradiction, let z_1 be the eigenvalue of L_0 associated to φ_1, then $L_0\varphi_1 = z_1\varphi_1 = z\varphi_2 + \mu Q\varphi_1$, so we have that

$$\langle \mu Q\varphi_1, \varphi_2 \rangle = \langle z_1\varphi_1 - z\varphi_2, \varphi_2 \rangle = \overline{z} \left\| \varphi_2 \right\|^2$$

therefore, from (3.50) we can conclude that $bi = -bi \left\| \varphi_2 \right\|^2 \implies b = 0$, which is a contradiction. Analogously, $L_\mu \varphi_2 = z_2\varphi_2 = z\varphi_1 - \mu Q\varphi_2 \implies \mu Q\varphi_2 = z\varphi_1 - z_2\varphi_2$ and using again (3.50) we can conclude again that b must be zero.

On the other hand, if φ_1 or φ_2 are eigenfunctions of μQ, then $\langle \mu Q\varphi_1, \varphi_2 \rangle = 0$ and from (3.50) we get the contradiction that $b = 0$. □

Remark 3.6. Let $z \in \mathbb{C} \setminus \mathbb{R} \cup i\mathbb{R}$ be an eigenvalue of H_μ, then we have that

1. $\langle \varphi_1, [L_\mu, \mu Q]\varphi_1 \rangle = -4ab \left\| \varphi_1 \right\|^2 i$
2. $\langle \varphi_2, [L_\mu, \mu Q]\varphi_2 \rangle = 4ab \left\| \varphi_2 \right\|^2 i$

Proof. Using $\mu Q\varphi_2 = z\varphi_1 - L_\mu \varphi_2$ is straightforward:

1.

$$\begin{aligned}
\langle \varphi_1, [L_\mu, \mu Q]\varphi_1 \rangle &= \langle \varphi_1, L_\mu \mu Q\varphi_1 \rangle - \langle \varphi_1, \mu Q L_\mu \varphi_1 \rangle \\
&= \langle z\varphi_2, \mu Q\varphi_1 \rangle - \langle \varphi_1, z\mu Q\varphi_2 \rangle \\
&= \overline{z} \langle \varphi_2, \mu Q\varphi_1 \rangle - z \langle \varphi_1, \mu Q\varphi_2 \rangle \\
&= \overline{z} \langle z\varphi_1 - L_\mu \varphi_2, \varphi_1 \rangle - z \langle \varphi_1, z\varphi_1 - L_\mu \varphi_2 \rangle \\
&= \overline{z}^2 \left\| \varphi_1 \right\|^2 - |z|^2 \left\| \varphi_2 \right\|^2 - z^2 \left\| \varphi_1 \right\|^2 + |z|^2 \left\| \varphi_2 \right\|^2 \\
&= \left\| \varphi_1 \right\|^2 (\overline{z}^2 - z^2).
\end{aligned}$$

2.

$$\begin{aligned}
\langle \varphi_2, [L_\mu, \mu Q]\varphi_2 \rangle &= \langle \varphi_2, [L_0, \mu Q]\varphi_2 \rangle \\
&= \langle z\varphi_1, \mu Q \varphi_2 \rangle - \langle \varphi_2, z\mu Q \varphi_1 \rangle \\
&= \overline{z} \langle \varphi_1, \mu Q \varphi_2 \rangle - z \langle \varphi_2, \mu Q \varphi_1 \rangle \\
&= \overline{z} \langle \varphi_1, z\varphi_1 - L_\mu \varphi_2 \rangle - z \langle z\varphi_1 - L_\mu \varphi_2, \varphi_1 \rangle \\
&= |z|^2 \|\varphi_1\|^2 - \overline{z}^2 \|\varphi_2\|^2 - |z|^2 \|\varphi_1\|^2 + z^2 \|\varphi_2\|^2 \\
&= \|\varphi_2\|^2 (z^2 - \overline{z}^2).
\end{aligned}$$

□

Remark 3.7. Note that the functions $R(x) - S(x)$ and $R(x) + S(x)$ they have a single zero for $\omega \in (0, 1)$.

Proposition 3.1. *The operator L_0 has only two eigenvalues -2ω and 0, with*

$$L_0 \begin{pmatrix} v \\ u \end{pmatrix} = 0\,, \tag{3.51}$$

$$L_0 \begin{pmatrix} u \\ v \end{pmatrix} = -2\omega \begin{pmatrix} u \\ v \end{pmatrix} . \tag{3.52}$$

Proof. Clearly by (3.12) and (3.32),

$$L_0 (v, u)^\top = \big(i\sigma_2 \partial_x + \sigma_3 (m - v^2 + u^2) - \omega\big) (v, u)^\top = 0.$$

We write $L_0 := A - \omega$. Then, using (2.4) is clear that σ_1 anti-commutes with A, so we see that

$$0 = \sigma_1 L_0 \begin{pmatrix} v \\ u \end{pmatrix} = -[A + \omega]\sigma_1 \begin{pmatrix} v \\ u \end{pmatrix} = -(L_0 + 2\omega) \begin{pmatrix} u \\ v \end{pmatrix} .$$

see [6, Lemma 5.4]. Notice that from (3.52), $(u, v)^\top$ is an eigenfunction of A with eigenvalue

$-\omega$.

Now, we have to show that there are no other eigenvalues. It suffices to show that A has no more eigenvalues. It is convenient to use the *change of basis*

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} (\sigma_1 + \sigma_3)$$

to obtain the unitary equivalence, using the properties of the Pauli matrices we have that

$$UAU = -i\partial_x \sigma_2 + M\sigma_1 = \begin{pmatrix} 0 & -\partial_x + M \\ \partial_x + M & 0 \end{pmatrix}.$$

Where $M := m + u^2 - v^2$, thus we find that A^2 is unitarily equivalent to

$$\begin{aligned}(UAU)^2 &= \begin{pmatrix} (-\partial_x + M)(\partial_x + M) & 0 \\ 0 & (\partial_x + M)(-\partial_x + M) \end{pmatrix} \\ &= \begin{pmatrix} -\partial_x^2 + M^2 - M' & 0 \\ 0 & -\partial_x^2 + M^2 + M' \end{pmatrix},\end{aligned}$$

which is a diagonal matrix with two Schrödinger operators on the diagonal, which are isospectral away from zero. By applying the coordinate transformation to $(v, u)^\top$, we find that $v + u$ is an eigenfunction for $-\partial_x^2 + M^2 - M'$ associated to the eigenvalue ω^2.

From the remark 3.1 and the figure 3.2, we see that $v + u$ does not change sign, hence it must be the groundstate. Analogously holds that $v - u$ is the groundstate of $-\partial_x^2 + M^2 + M'$ with eigenenergy ω^2.

This shows the absence of eigenvalues for A in $(-\omega, \omega)$. To proceed further, we use the resonances found explicitly in [6, Lemma 5.5]. After the coordinate transformation, these resonances give a bounded solution to the equation

$$(-\partial_x^2 + M^2 - M' - m^2)f = 0.$$

By plotting the explicit expression, we see that f has a single zero 3.7. By an oscillation argument [31, Theorem 3.5], this shows that there are no eigenvalues of L^2 in the interval (ω^2, m^2). □

Now we can obtain eigenfunctions of the operator L_μ, known by Berkolaiko and Comech [6, Lemma 5.4].

Proposition 3.2. *We have*

$$L_\mu \begin{pmatrix} u \\ v \end{pmatrix} = -2\omega \begin{pmatrix} u \\ v \end{pmatrix}, \quad \mu \in \mathbb{R}, \tag{3.53}$$

$$L_\mu \begin{pmatrix} v' \\ u' \end{pmatrix} = 0, \quad \mu = 2. \tag{3.54}$$

Proof. Equation (3.53) is clear in view of Proposition 3.1 and the fact that $Q(u,v)^\top = 0$ by (3.34). We show (3.54), on the one hand, using again (3.34) and computing

$$Q \begin{pmatrix} v' \\ u' \end{pmatrix} = (vv' - uu') \begin{pmatrix} v \\ -u \end{pmatrix} = -\frac{M'}{2} \begin{pmatrix} v \\ -u \end{pmatrix}.$$

On the other hand, since $L_0(v,u)^\top = 0$, we have

$$L_0 \begin{pmatrix} v' \\ u' \end{pmatrix} = L_0 \partial_x \begin{pmatrix} v \\ u \end{pmatrix} = \partial_x L_0 \begin{pmatrix} v \\ u \end{pmatrix} - M'\sigma_3 \begin{pmatrix} v \\ u \end{pmatrix} = -M' \begin{pmatrix} v \\ -u \end{pmatrix}.$$

□

From now on, we denote

$$\varphi_0 := \begin{pmatrix} v \\ u \end{pmatrix} = (v,u)^\top \tag{3.55}$$

and

$$\varphi_{-2\omega} := \begin{pmatrix} u \\ v \end{pmatrix} = (u, v)^\top, \tag{3.56}$$

with φ_0' and $\varphi_{-2\omega}'$ their respective derivatives:

$$\varphi_0' = \begin{pmatrix} v' \\ u' \end{pmatrix}, \tag{3.57}$$

$$\varphi_{-2\omega}' = \begin{pmatrix} u' \\ v' \end{pmatrix}. \tag{3.58}$$

Lemma 3.3. *For any $\mu \in \mathbb{R}$, $p > 0$ and $0 < \omega < m$, the eigenvalues of $L_\mu(p, \omega, m)$ and $L_0(p, \omega, m)$ are simple.*

Proof. Let $\lambda \in \mathbb{R}$ be an eigenvalue of L_μ and assume $\phi_1 = (f_1, g_1)^\top$ and $\phi_2 = (f_2, g_2)^\top$ to be eigenfunctions associated to λ. The equation $L_\mu \phi_j = \lambda \phi_j$ can be rewritten as

$$\partial_x \phi_j = -i\lambda\sigma_2\phi_j - M(x)\sigma_1\phi_j - \mu\sigma_2 Q(x)\phi_j,$$

with $M(x)$ defined as $M(x) := m - \big(v^2(x) - u^2(x)\big)^p$. Furthermore, we can decompose $\mu Q(x) = q_0(x)\mathbb{1}_{\mathbb{C}^2} + q_1\sigma_1 + q_2\sigma_3$, for some functions q_0, q_1 and q_3 whose explicit expressions are not necessary to complete the proof. Using the identity

$$\sigma_m\sigma_k = i\epsilon_{mkl}\sigma_l$$

we finally rewrite the eigenvalue equation as

$$\partial_x \phi_j = (-i\lambda + \mu q_0)\sigma_2\phi_j - (M(x) + q_2(x))\sigma_1\phi_j - \mu q_1(x)\sigma_3\phi_j.$$

Now, define the determinant

$$W(x) := \det\Big(\phi_1(x)|\phi_2(x)\Big)$$

and compute

$$\begin{aligned}
W' &= \det\left(\phi_1'|\phi_2\right) + \det\left(\phi_1|\phi_2'\right) \\
&= (-i\lambda + \mu q_0)\left(\det\left(\sigma_2\phi_1|\phi_2\right) + \det\left(\phi_1|\sigma_2\phi_2\right)\right) \\
&\quad + (M(x) + q_2(x))\left(\det\left(\sigma_1\phi_1|\phi_2\right) + \det\left(\phi_1|\sigma_1\phi_2\right)\right) \\
&\quad - \mu q_1(x)\left(\det\left(\sigma_3\phi_1|\phi_2\right) + \det\left(\phi_1|\sigma_3\phi_2\right)\right)
\end{aligned}$$

For $k = 1, 2, 3$, we compute

$$\det\left(\sigma_k\phi_1|\phi_2\right) = \det\left(\sigma_k\phi_1|\sigma_k^2\phi_2\right) = \det(\sigma_k)\det\left(\phi_1|\sigma_k\phi_2\right) = -\det\left(\phi_1|\sigma_k\phi_2\right),$$

so we conclude $W' \equiv 0$ because $\phi_i \in L^2(\mathbb{R})$, even if $\mu = 0$ that is the analogously case to L_0. This implies that $W \equiv 0$. Now we have to exclude the case where ϕ_1 is proportional to ϕ_2 for every $x \in \mathbb{R}$, i.e. there exists a function $\alpha : \mathbb{R} \to \mathbb{C}$ s.t. $\phi_1(x) = \alpha(x)\phi_2(x)$ for any $x \in \mathbb{R}$. In particular, $\phi_1(0) = \alpha(0)\phi_2(0)$.

Since, by Picard-Lindelöf theorem (see [57, Theorem 2.2]), the trivial solution $x \mapsto (0,0)^\top$ is the unique solution to the Cauchy problem

$$\begin{cases} L_\mu\Psi = \lambda\Psi \\ \Psi(0) = (0,0)^\top, \end{cases}$$

we have $\phi_i(0) \neq 0$ as, otherwise, we would have $\phi_i \equiv (0,0)^\top$, which would contradict that ϕ_i is an eigenfunction. Therefore $\alpha(0) \neq 0$ and we have that ϕ_1 and $x \mapsto \alpha(0)\phi_2(x)$ are both the unique solution to the Cauchy problem

$$\begin{cases} L_\mu\Psi = \lambda\Psi \\ \Psi(0) = \phi_1(0) = \alpha(0)\phi_2(0), \end{cases}$$

This proves that $\phi_1(x) = \alpha(0)\phi_2(x)$ for any $x \in \mathbb{R}$, ending the proof of simplicity of eigenvalues. For more details about Picard-Lindelöf theorem see the Appendix B. $\square$

Remark 3.8. Notice that, since

$$\left\langle \begin{pmatrix} v \\ u \end{pmatrix}, \begin{pmatrix} v' \\ u' \end{pmatrix} \right\rangle = 0$$

we have that the kernels of L_0 and L_μ (for $\mu = 2$) are orthogonal.

Lemma 3.4. *The operator Q which acts as point-wise multiplication by the two by two matrix, is positive definite and satisfies*

$$|||Q||| = \begin{cases} 2p(p+1)m\dfrac{\nu}{1+\nu} = p(p+1)(m-\omega), & \text{if } \nu < \dfrac{1}{3} \Leftrightarrow \omega > \dfrac{m}{2}, \\ p\dfrac{p+1}{2}\dfrac{m}{2}\dfrac{1+\nu}{1-\nu} = p\dfrac{p+1}{2}\dfrac{m^2}{2\omega}, & \text{if } \nu \geq \dfrac{1}{3} \Leftrightarrow \omega \leq \dfrac{m}{2}. \end{cases} \tag{3.59}$$

Proof. For all $x \in \mathbb{R}$, the matrix $Q(x)$ is positive semidefinite with eigenvalues 0 and $p(v^2-u^2)^{p-1}(v^2+u^2)$. We have

$$|||Q||| = p\left\|\left(v^2-u^2\right)^{p-1}\left(v^2+u^2\right)\right\|_\infty = p\left\|\left(\tilde{v}^2-\tilde{u}^2\right)^{p-1}\left(\tilde{v}^2+\tilde{u}^2\right)\right\|_\infty.$$

Where the tilde notation is the spatial rescaling by a factor $p\kappa$ form (3.27). Note that the argument of the norm is a positive function and the rescaling does not affect since the supremum of a real valued function remains equal under translations on the x axis. On the other hand, we have

$$\left(\tilde{v}^2-\tilde{u}^2\right)^{p-1}\left(\tilde{v}^2+\tilde{u}^2\right) = \left(\tilde{v}^2-\tilde{u}^2\right)^p\frac{\tilde{v}^2+\tilde{u}^2}{\tilde{v}^2-\tilde{u}^2} = 2m(p+1)\frac{\nu}{1+\nu}F,$$

where F is the even function

$$F(t) := \frac{1-\tanh^2(t)}{1-\nu\tanh^2(t)}\frac{1+\nu\tanh^2(t)}{1-\nu\tanh^2(t)}.$$

Its derivative satisfies

$$\left(1-\nu\tanh^2\right)^3F' = 2\left(3\nu-1-\nu(3-\nu)\tanh^2\right)\left(1-\tanh^2\right)\tanh.$$

Therefore, on $\mathbb{R}_+$,

$$F'(t) > 0 \Leftrightarrow 3\nu - 1 > \nu(3-\nu)\tanh^2(t)$$
$$\Leftrightarrow \begin{cases} t < t_\nu := \operatorname{arctanh}\sqrt{\frac{3\nu-1}{\nu(3-\nu)}}, & \text{if} \quad \nu \geq \frac{1}{3} \Leftrightarrow \omega \leq \frac{m}{2}, \\ \emptyset, & \text{if} \quad \nu < \frac{1}{3} \Leftrightarrow \omega > \frac{m}{2}. \end{cases}$$

Consequently, $\|F\|_\infty = F(0) = 1$ and $\left\|(v^2-u^2)^{p-1}(v^2+u^2)\right\|_\infty = 2m(p+1)\frac{\nu}{1+\nu}$ if $\nu < \frac{1}{3}$, otherwise $\|F\|_\infty = F(t_\nu) = \frac{1}{8}\frac{1+\nu}{\nu}\frac{1+\nu}{1-\nu}$ and $\left\|(v^2-u^2)^{p-1}(v^2+u^2)\right\|_\infty = \frac{m}{2}\frac{p+1}{2}\frac{1+\nu}{1-\nu}$. We therefore have proved (3.59).

□

3.5 Properties of $H_\mu(\omega)$

It is easy to see that

$$\sigma_{\mathrm{ess}}(H_\mu(\omega)) = \sigma_{\mathrm{ess}}(H_0(\omega)) = (-\infty, -m-\omega] \cup [m-\omega, \infty)\,.$$

Since $u(x)$ and $v(x)$ decays sufficiently fast at infinity these are relatively compact perturbation to a free Dirac-like operator.

3.5.1 Symmetries

Proposition 3.5. *The spectrum of $H(\mu,\omega)$ is symmetric with respect to both the real and the imaginary axis.*

Proof. This an immediate consequence of the eigenvalue equation

$$H(\mu,\omega)\psi = z\psi\,, \tag{3.60}$$

combined with the fact that H is equal to its complex conjugate and that $\boldsymbol{\beta} H = -H\boldsymbol{\beta}$, where

$$\boldsymbol{\beta} = \begin{pmatrix} 1_{2\times 2} & 0 \\ 0 & -1_{2\times 2} \end{pmatrix} . \tag{3.61}$$

Let ψ be an eigenfunction of $H(\mu, \omega)$ with eigenvalue λ, then $\boldsymbol{\beta} H\psi = -H\boldsymbol{\beta}\psi = -\lambda\boldsymbol{\beta}\psi$. $\square$

There is a further symmetry that relates H with its adjoint. Define

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & 1_{2\times 2} \\ 1_{2\times 2} & 0 \end{pmatrix} , \tag{3.62}$$

and note that $\boldsymbol{\alpha} H = H^* \boldsymbol{\alpha}$.

Proposition 3.6. *Assume that* $0 \neq \psi = (\varphi_1, \varphi_2)^\top$ *satisfies* $H(\mu, \omega)\psi = z\psi$ *for some* $z \in \mathbb{C} \setminus \{0\}$. *Then,*

$$\mathrm{Im}(z)\,\mathrm{Re}\langle \varphi_2, \varphi_1 \rangle = 0\,, \quad \textit{and} \quad \mathrm{Re}(z)\,\mathrm{Im}\langle \varphi_2, \varphi_1 \rangle = 0\,. \tag{3.63}$$

Proof. From doing scalar product of (3.40) with φ_2 we get that

$$z\langle \varphi_2, \varphi_1 \rangle \in \mathbb{R}\,. \tag{3.64}$$

This implies that

$$\mathrm{Im}(z)\mathrm{Re}\langle \varphi_2, \varphi_1 \rangle + \mathrm{Im}\langle \varphi_2, \varphi_1 \rangle \mathrm{Re}(z) = 0\,. \tag{3.65}$$

On the other hand, using (3.60) we have that

$$z\langle \psi, \boldsymbol{\alpha}\psi \rangle = \langle \psi, \boldsymbol{\alpha} H\psi \rangle = \langle H\psi, \boldsymbol{\alpha}\psi \rangle = \overline{z}\langle \psi, \boldsymbol{\alpha}\psi \rangle\,. \tag{3.66}$$

Then, $(z - \overline{z})\langle \psi, \boldsymbol{\alpha}\psi \rangle = 0$, which together with (3.65) implies (3.63). $\square$

This implies that if an eigenvalue z is away from the real and imaginary axis, then its associated φ_1 and φ_2 must be orthogonal.

Corollary. *Let $z \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$ be an eigenvalue of $H_\mu(\omega)$ with associated eigenfunction $(\varphi_1, \varphi_2)^\top$. Then,*

$$\langle \varphi_2, \varphi_1 \rangle = 0.$$

Consequently,

$$\langle \varphi_2, L_0 \varphi_2 \rangle = 0 = \langle \varphi_1, L_\mu \varphi_1 \rangle \, .$$

Proof. The orthogonality $\langle \varphi_2, \varphi_1 \rangle = 0$ of the φ's follows immediately from (3.65) since $\operatorname{Re} z \neq 0$ and $\operatorname{Im} z \neq 0$. The second part is then readily obtained from (3.40)–(3.41). □

3.5.2 Basics of the eigenvalue equation

Let us denote by P_{-1}, P_0 the orthogonal projections onto the eigenspaces of L_0 associated to the eigenvalues $-2\omega, 0$, respectively. Moreover, let $P_+ := E_{[m-\omega,\infty)}(L_0)$ and $P_- := E_{(-\infty,-m-\omega]}(L_0)$ are the projections onto the spectral subspaces associated to the positive and negative energies in the essential spectrum of L_0. We have the decomposition

$$1 = P_- + P_{-1} + P_0 + P_+ \, . \tag{3.67}$$

We further write $P_0(\mu)$ to be the eigenprojection onto the kernel of L_μ.

Proposition 3.7. *Assume that $0 \neq z \in \sigma_{\mathrm{d}}(H(\mu, \omega))$ and let $\psi = (\varphi_1, \varphi_2)^\top$ be a corresponding eigenfunction. Then, $P_0 \varphi_1 = 0 = P_0(\mu) \varphi_2$. If, in addition $z^2 \neq 4\omega^2$, then $P_{-1}\varphi_1 = P_{-1}\varphi_2 = 0$.*

Proof. That $P_0 \varphi_1 = 0$ follows by multiplying (3.40) by P_0. We proceed analogously for $P_0(\mu)\varphi_2$. For the statement on P_{-1} write $\phi_- = (u, v)^\top$ and recall that $Q\phi_- = 0$. Since the eigenvalues of L_0 are simple $P_{-1} = \langle \phi_-, \cdot \rangle \phi_-$. Hence, $QP_{-1} = 0$ and, by taking the adjoint, $P_{-1}Q = 0$. Therefore, after multiplying (3.40) and (3.41) by P_{-1}, we get that $(-2\omega)P_{-1}\varphi_2 = zP_{-1}\varphi_1$ and $(-2\omega)P_{-1}\varphi_1 = zP_{-1}\varphi_2$. This concludes the proof since $z^2 \notin \{0, 4\omega^2\}$. □

Proposition 3.8. *Let $z \in \mathbb{C} \setminus \{0\}$. Then, the following statements are equivalent* *(i)* $z \in \sigma_{\mathrm{d}}(H(\mu,\omega))$,

(ii) $z^2 \in \sigma_{\mathrm{d}}(H(\mu,\omega)^2)$,

(iii) $z^2 \in \sigma_{\mathrm{d}}(L_0 L_\mu)$,

(iv) $z^2 \in \sigma_{\mathrm{d}}(L_\mu L_0)$.

Proof. It follows by inspection that the latter three statements are equivalent and that (i) implies (ii). Moreover, let $z^2 \neq 0$ be such that $L_0 L_\mu \phi = z^2 \phi$, for some $\phi \neq 0$. Then, one can verify that

$$\begin{pmatrix} 0 & L_0 \\ L_\mu & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \frac{1}{z} L_\mu \phi \end{pmatrix} = z \begin{pmatrix} \phi \\ \frac{1}{z} L_\mu \phi \end{pmatrix} .$$

□

Lemma 3.9. *Let $P^\perp = P_- + P_+$ and $z \in \mathbb{C} \setminus \{0, \pm 2\omega\}$ be an eigenvalue of $H(\mu,\omega)$ satisfying* (3.40) *and* (3.41)*. Then, we have that $\varphi_1 = P^\perp \varphi_1$,*

$$P^\perp L_\mu P^\perp \varphi_1 = z^2 P^\perp \frac{1}{L_0} \varphi_1 . \tag{3.68}$$

Moreover, z^2 satisfies the eigenvalue eqution,

$$|L_0|^{1/2} \left(P_+ L_\mu P_+ - P_- L_\mu P_- - \mu(Q_{+-} - Q_{-+})\right) |L_0|^{1/2} \psi_1 = z^2 \psi_1, \tag{3.69}$$

where $\psi_1 = |L_0|^{-1/2} \varphi_1$, $Q_{+-} = P_+ Q P_-$, and $Q_{-+} = P_- Q P_+$.

Proof. By Proposition 3.7 we see that $\varphi_1 = P^\perp \varphi_1$. Multiplying (3.41) by $P^\perp$ we readly get (3.68):

$$P^\perp L_\mu P^\perp \varphi_1 = z P^\perp \varphi_2 = z^2 L_0^{-1} \varphi_1 ,$$

where we use (3.40) in the last equality. Next, we multiply (3.68) by P_+ and P_-, to get

$$P_+L_\mu P_+\varphi_1 - \mu Q_{+-}\varphi_1 = z^2 \left|L_0\right|^{-1} P_+\varphi_1 \,, \tag{3.70}$$

$$P_-L_\mu P_-\varphi_1 - \mu Q_{-+}\varphi_1 = -z^2 \left|L_0\right|^{-1} P_-\varphi_1 \,. \tag{3.71}$$

Substracting (3.71) from (3.70) we get that

$$\big(P_+L_\mu P_+ - P_-L_\mu P_- - \mu(Q_{+-} - Q_{-+})\big)\varphi_1 = z^2 \left|L_0\right|^{-1} \varphi_1 \,.$$

From this follows (3.69) after multiplying by $\left|L_0\right|^{1/2}$. □

From the previous result we get

$$z^2\big\langle\varphi_1, \left|L_0\right|^{-1}\varphi_1\big\rangle = \big\langle\varphi_1, \big(P_+L_\mu P_+ - P_-L_\mu P_- - \mu(Q_{+-} - Q_{-+})\big)\varphi_1\big\rangle \,. \tag{3.72}$$

and, in particular, since $Q_{+-} - Q_{-+} \in \mathbb{C}$ we have that

$$\mathrm{Re}(z^2) = \frac{\big\langle\varphi_1, \big(P_+L_\mu P_+ - P_-L_\mu P_-\big)\varphi_1\big\rangle}{\Big\langle\varphi_1, \left|L_0\right|^{-1}\varphi_1\Big\rangle} \,. \tag{3.73}$$

3.6 Bounds to the eigenvalues of H_μ

Lemma 3.10. *Let $0 < \omega < m$ and $\mu \geq 0$. If $z \in \mathbb{C}$ is an eigenvalue of $H_\mu(\omega, p)$, then*

$$|\mathrm{Im}\, z| \leq \frac{\mu}{2} \left\|\!\left|Q\right|\!\right\| \,. \tag{3.74}$$

Proof. Denoting by $(\varphi_1, \varphi_2)^\top$ the normalized eigenfunction of H_μ associated to the eigenvalue z, taking the scalar products of (3.40) and (3.41) respectively with φ_1 and φ_2 gives

$$\langle\varphi_1, L_0\varphi_2\rangle + \langle\varphi_2, L_0\varphi_1\rangle - \langle\varphi_2, \mu Q\varphi_1\rangle = z\left(\|\varphi_1\|^2 + \|\varphi_2\|^2\right) = z.$$

But L_0 being self-adjoint, we have $\langle \varphi_1, L_0\varphi_2\rangle + \langle \varphi_2, L_0\varphi_1\rangle \in \mathbb{R}$ and

$$\operatorname{Im} z = \mu \operatorname{Im} \langle Q\varphi_1, \varphi_2\rangle \ .$$

Therefore, by Cauchy–Schwarz inequality

$$|\operatorname{Im} z| \le \mu \, |||Q||| \, \|\varphi_1\| \, \|\varphi_2\| \le \frac{\mu}{2} \, |||Q||| \ .$$

□

We can use Corollary 3.5.1 onto (3.73) in order to obtain bounds on $\operatorname{Re}(z^2)$. To do that we are going to use the following bound

Corollary. *Let $z \in \mathbb{C} \setminus \mathbb{R} \cup i\mathbb{R}$ be an eigenvalue of $H_\mu(\omega)$, then we have that*

$$\langle \varphi_1, |L_0|^{-1} \varphi_1\rangle = 2 \left\| |L_0|^{-1/2} P_+\varphi_1 \right\|^2 = 2 \left\| |L_0|^{-1/2} P_-\varphi_1 \right\|^2 \le 2(m+\omega)^{-1} \|P_-\varphi_1\|^2 \ . \tag{3.75}$$

Proof. First of all observe that $0 = \langle \varphi_2, L_0\varphi_2\rangle$ so, using that $\varphi_2 = zL_0^{-1}\varphi_1$ implies $0 = \langle \varphi_1, L_0^{-1}\varphi_1\rangle$, hence

$$0 = \langle \varphi_1, L_0^{-1}P_+\varphi_1\rangle + \langle \varphi_1, L_0^{-1}P_-\varphi_1\rangle = \langle \varphi_1, P_+ |L_0|^{-1} P_+\varphi_1\rangle - \langle \varphi_1, P_- |L_0|^{-1} P_-\varphi_1\rangle$$

Thus, we have that

$$\langle \varphi_1, |L_0|^{-1} \varphi_1\rangle = 2 \left\| |L_0|^{-1/2} P_+\varphi_1 \right\|^2 = 2 \left\| |L_0|^{-1/2} P_-\varphi_1 \right\|^2 \ . \tag{3.76}$$

Finally, recalling that P_- projects over the negative essential spectrum of L_0 and using the spectral theorem, we have the bound $\langle \varphi_1, |L_0|^{-1} \varphi_1\rangle \le 2(m+\omega)^{-1} \|P_-\varphi_1\|^2$. □

Moreover, we also know from Corollary 3.5.1 that

$$0 = \langle \varphi_1, L_\mu \varphi_1 \rangle = \langle \varphi_1, (P_+ L_\mu P_+ + P_- L_\mu P_- - \mu(Q_{+-} + Q_{-+}))\, \varphi_1 \rangle . \tag{3.77}$$

For the followings bounds, define $\eta \in \mathbb{R}^+$ as

$$\eta = \frac{\|P_+ \varphi_1\|^2}{\|P_- \varphi_1\|^2}. \tag{3.78}$$

Remark 3.9. Since the spectrum of L_0 is symmetric with respect $-\omega$, $\|P_- \varphi_1\|^2$ cannot be zero, because $\|P_+ \varphi_1\|^2$ it would also have to be zero and then it would not make sense to define η.

Thus, by combining this with (3.73), we obtain the following result

Lemma 3.11. *Assume that $z \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$ is an eigenvalue of $H_\mu(\omega)$, with eigenfunction $(\varphi_1, \varphi_2)^\top$. We have the bounds*

$$\operatorname{Re}(z^2) \geq (m+\omega)^2 \left(1 - \frac{\mu\eta \,|||Q|||}{4(m+\omega)}\right), \qquad \text{if } \mu\eta \,|||Q||| \leq 4(m+\omega), \tag{3.79}$$

$$\operatorname{Re}(z^2) \geq (m+\omega)^2 \eta \left(\frac{\eta + \eta^{-1}}{2} - \frac{\mu \,|||Q|||}{2(m+\omega)}\right), \qquad \text{if } \mu \,|||Q||| \leq (\eta + \eta^{-1})(m+\omega). \tag{3.80}$$

Proof.

$$\begin{aligned}
\langle \varphi_1, |L_0|^{-1} \varphi_1 \rangle \operatorname{Re}(z^2) &= -2 \langle \varphi_1, P_- L_\mu P_- \varphi_1 \rangle + \mu \langle \varphi_1, (Q_{+-} + Q_{-+}) \varphi_1 \rangle \\
&= 2 \langle \varphi_1, P_- |L_0| P_- \varphi_1 \rangle + \mu \langle \varphi_1, (2Q_{--} + Q_{+-} + Q_{-+}) \varphi_1 \rangle \\
&= 2 \langle \varphi_1, P_- |L_0| P_- \varphi_1 \rangle + 2\mu \langle (P_- + P_+/2) \varphi_1, Q (P_- + P_+/2) \varphi_1 \rangle \\
&\qquad - \frac{\mu}{2} \langle P_+ \varphi_1, Q P_+ \varphi_1 \rangle \\
&\geq 2(m+\omega) \|P_- \varphi_1\|^2 - \frac{\mu \,|||Q|||}{2} \|P_+ \varphi_1\|^2 \\
&= 2(m+\omega) \left(1 - \frac{\eta\mu \,|||Q|||}{4(m+\omega)}\right) \|P_- \varphi_1\|^2 .
\end{aligned}$$

If the quantity in parenthesis is non-negative, we combine with the bound

$$\langle \varphi_1, |L_0|^{-1} \varphi_1 \rangle = 2 \langle P_- \varphi_1, |L_0|^{-1} P_- \varphi_1 \rangle \leq 2(m+\omega)^{-1} \|P_- \varphi_1\|^2 ,$$

to conclude (3.79). Now we look at the second case, which seems trickier. Note that this is the bound for η large, so the positive projection of φ_1 dominates. In this case, however, φ_1 can not live very close to the $(m-\omega)$ treshold, because we still need (3.75), which implies

$$\langle P_+\varphi_1, |L_0|^{-1} P_+\varphi_1\rangle = \langle P_-\varphi_1, |L_0|^{-1} P_-\varphi_1\rangle \leq (m+\omega)^{-1} \|P_-\varphi_1\|^2 . \tag{3.81}$$

To obtain a statement about the expectation value of L_0 rather than its inverse, we use Jensen's inequality. According to Lieb-Loss Analysis book [39], for a convex function f, it holds that

$$E[f(t)] \geq f(E[t]).$$

By the spectral theorem, for a self-adjoint operator A and any nonzero ψ, there exists a probability measure μ_ψ such that

$$\frac{\langle\psi, f(A)\psi\rangle}{\|\psi\|^2} = E_{\mu_\psi}[f(\lambda)]$$

for nice enough f and ψ, we can use the inequality for Rayleigh quotients. For more details see Appendix C. We apply this with $A = P_+L_0^{-1}P_+$, $f(t) = t^{-1}$ and $\psi = P_+\varphi_1$ to conclude

$$\frac{\langle P_+\varphi_1, |L_0|\, P_+\varphi_1\rangle}{\|P_+\varphi_1\|^2} \geq \frac{\|P_+\varphi_1\|^2}{\langle P_+\varphi_1, |L_0|^{-1} P_+\varphi_1\rangle} \geq (m+\omega)\frac{\|P_+\varphi_1\|^2}{\|P_-\varphi_1\|^2} = \eta(m+\omega), \tag{3.82}$$

where the last line follows from (3.81). Now the remainder is easy. We start again from (3.73) and bound

$$\begin{aligned}
\langle\varphi_1, |L_0|^{-1}\varphi_1\rangle\mathrm{Re}(z^2) &\geq \langle P_+\varphi_1, |L_0|\, P_+\varphi_1\rangle - \mu\langle P_+\varphi_1, QP_+\varphi_1\rangle + \langle P_-\varphi_1, |L_0|\, P_-\varphi_1\rangle \\
&\geq (\eta(m+\omega) - \mu\, |||Q|||)\, \|P_+\varphi_1\|^2 + (m+\omega)\, \|P_-\varphi_1\|^2 \\
&= \left(\left(\eta+\eta^{-1}\right)(m+\omega) - \mu\, |||Q|||\right) \|P_+\varphi_1\|^2 .
\end{aligned}$$

If the quantity in parentheses is positive, we use again (3.81) for the left hand side:

$$\langle\varphi_1, |L_0|^{-1}\varphi_1\rangle \leq 2(m+\omega)^{-1} \|P_-\varphi_1\|^2 = 2\eta^{-1}(m+\omega)^{-1} \|P_+\varphi_1\|^2 .$$

□

We can improve this result seeing when the conditions of Lemma 3.11 are not satisfied. In other words, for which η do we have

$$\eta > \frac{4(m+\omega)}{\mu \, |||Q|||} \tag{3.83}$$

$$\eta^2 - \frac{\mu \, |||Q|||}{(m+\omega)}\eta + 1 < 0 \tag{3.84}$$

the condition (3.84) is never fulfilled if $\frac{\mu|||Q|||}{(m+\omega)} \leq 2$ because the coefficient of η^2 is positive and the the discriminant would be positive, thus the quadratic will never be negative and the second bound from Lemma 3.11 would be satisfied. For others η, let $x := \frac{\mu|||Q|||}{(m+\omega)} > 0$ and we have that

$$\eta > \frac{4}{x} \quad \text{ and } \quad \frac{x - \sqrt{x^2-4}}{2} < \eta < \frac{x + \sqrt{x^2-4}}{2}.$$

where $x = \frac{\mu|||Q|||}{(m+\omega)} > 0$. But for those η that satisfies (3.84) we want to see when also satisfies (3.83) to exclude them. Since $x > 0$ the existence of such η is equivalent to

$$\frac{x + \sqrt{x^2-4}}{2} > \frac{4}{x} \qquad \Leftrightarrow \qquad x > \frac{4}{\sqrt{3}} \approx 2.31.$$

Conclusion, Lemma 3.11 gives us a non-trivial lower bound on $\mathrm{Re}(z^2)$ if, and only if,

$$x = \mu\frac{|||Q|||}{m+\omega} \leq \frac{4}{\sqrt{3}}, \tag{3.85}$$

and in this case, the lower bound is given by the second bound of Lemma 3.11 when $0 < x \leq 2$ and the lemma still gives us a nontrivial bound when $2 < x \leq \frac{4}{\sqrt{3}}$. More precisely, for this sort of x, one or both conditions are satisfied.

Since x depends on the norm of Q we have to separate in two cases:

Case 1: $\omega > \frac{m}{2}$. Then $x = \mu p(p+1)\frac{m-\omega}{m+\omega} < \frac{\mu p(p+1)}{3} < \frac{4}{\sqrt{3}}$ for any $\mu \le 2$ and $p < 1.42$, since $|||Q||| = p(p+1)(m-\omega)$ by Lemma 3.59. From here to the next case, we are considering this set of parameters unless we say otherwise.

Remark 3.10. On one hand the function $(1 - \eta x/4)$ from (3.79) decrease with η and applies for any $\eta \in (0, \frac{4}{x})$. So, for $\omega \approx \frac{m}{2}$ we have that $\eta < 3$ and if $\omega \approx m$ this condition applies for $\eta > 0$, therefore applies for any $m/2 < \omega < m$ when $\eta < 3$.
On the other hand the function $\frac{1}{2}(\eta^2 - x\eta + 1)$ from (3.80) decreases until $x/2$ then increases, and applies for any $\eta > 0$ when $p < 1.3$ because $x < 2$.

Now, note that we have two lower bounds depending on a lineal function and a parabola, so we can found the value where the both functions match, i.e. $\frac{1}{2}(\eta^2 - x\eta + 1) = 1 - \eta x/4$, or equivalently when $\eta^2 - \frac{x}{2}\eta - 1 = 0$, thus the match value is given by

$$\eta_\star := \frac{x + \sqrt{x^2+16}}{4} = \frac{\mu}{4}\frac{p(p+1)(m-\omega)}{m+\omega} + \sqrt{\left(\frac{\mu}{4}\frac{p(p+1)(m-\omega)}{m+\omega}\right)^2 + 1} < \sqrt{3},$$

Thus we have a lower bound for any $\omega > m/2$ and obtain the following.

Proposition 3.12. *Let $z \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$ be an eigenvalue of $H(\mu, \omega)$ for $\mu \le 2$, $m/2 < \omega < m$ and $p < 1.3$. Then for $m = 1$,*

$$\mathrm{Re}(z^2) > 0.76\,,$$

in the case $(\mu = 2, p = 1, m = 1)$:

$$\mathrm{Re}\left(z^2\right) \ge 4\omega^2 + (1-\omega)\left(4\omega - \sqrt{2(1+\omega^2)}\right). \tag{3.86}$$

Proof. We have already shown almost everything, note that $\eta_\star < 2$. So we only need evaluate any bound from the Lemma 3.11 because the both of them applies and

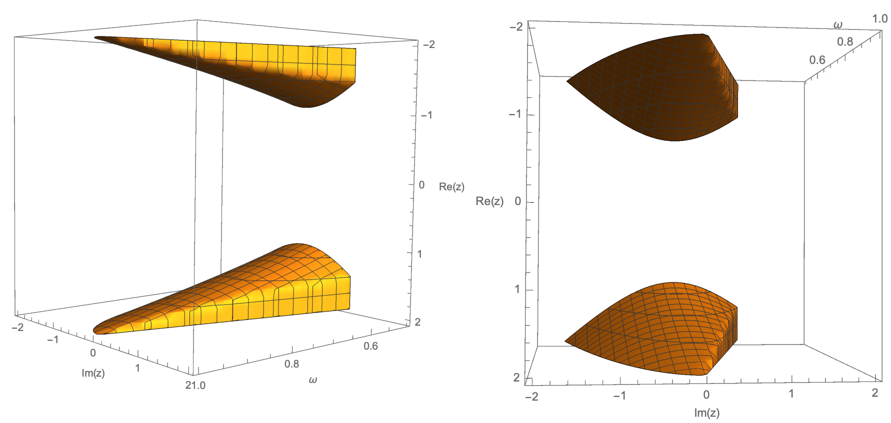
$$\begin{aligned}\mathrm{Re}(z^2) &\geq \frac{(m+\omega)^2}{4}(4-\eta x)|_{\eta=\eta_\star} &(3.87)\\ &= \frac{(m+\omega)^2}{16}\left(16-x^2-x\sqrt{x^2+16}\right) &(3.88)\end{aligned}$$

where the function in parenthesis is a positive decreasing function on x, so computing $x=\sqrt{3}$ we have that $\mathrm{Re}(z^2) > \frac{5.45}{16}(m+\omega)^2 > 0.76$. Replacing $\eta_\star(\mu=2, p=1, m=1, \omega)$ we have that

$$\mathrm{Re}\left(z^2\right) \geq 4\omega^2 + (1-\omega)\left(4\omega - \sqrt{2(1+\omega^2)}\right).$$

□

Combining this previous bound with the bound on $\mathrm{Im}(z)$ of Lemma 3.10, we obtain that for any $\mu \leq 2$, eigenvalues that are not on the axis can only be in the $3D$ region in Figure 3.5. The drawn region is the one for $\mu = 2$ since the region expands when μ increases.



(a) (b)

Figure 3.5: Region where the eigenvalues that are not on the axes can be for $(\mu = 2, p = 1, m = 1, 0.5 < \omega < 1)$.

Case 2: $\omega \leq \frac{m}{2}$. Then for $m = 1$ we have that $x = \frac{\mu p(p+1)}{2}\frac{m^2}{2\omega(m+\omega)} < \frac{p(p+1)}{2}\frac{1}{\omega(1+\omega)}$ for any $\mu \leq 2$ and $p > 0$, since $|||Q||| = \frac{p(p+1)}{2}\frac{m^2}{2\omega}$ by Lemma 3.59.

Remark 3.11. In this case we need found some critical $\tilde{\omega}$ such that $x \leq 4/\sqrt{3}$ or equivalently $0 \leq \omega^2 + \omega - \frac{\sqrt{3}p(p+1)}{8}$, thus $\omega \geq \frac{1}{2}\left(\sqrt{1 + \frac{\sqrt{3}}{2}p(p+1)} - 1\right) := \tilde{\omega}$.

Then, using Remark 3.10 we have that

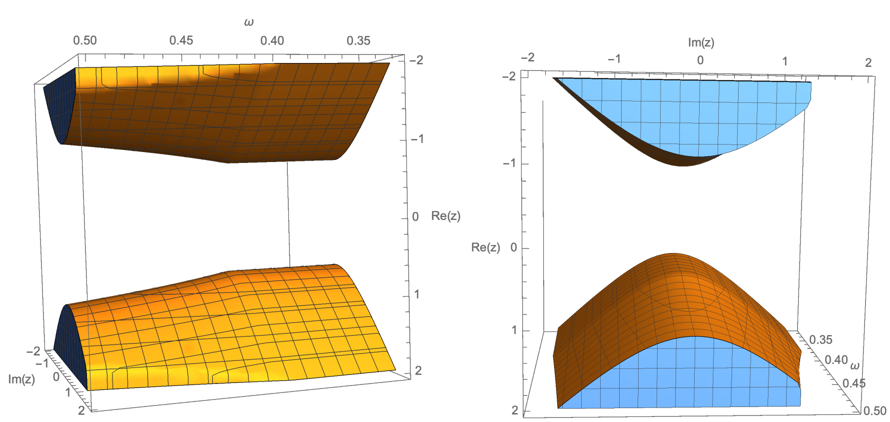
$$\eta_\star := \frac{x + \sqrt{x^2 + 16}}{4} = \frac{\mu}{4}\frac{p(p+1)m^2}{4\omega(m+\omega)} + \sqrt{\left(\frac{\mu}{4}\frac{p(p+1)m^2}{4\omega(m+\omega)}\right)^2 + 1} < \sqrt{3},$$

Proposition 3.13. *Let $z \in \mathbb{C}\backslash(\mathbb{R} \cup i\mathbb{R})$ be an eigenvalue of $H(\mu, \omega)$ for $\mu \leq 2$, $\tilde{\omega} < \omega < m$ and $p < 1.3$. Then for $m = 1$, $\mathrm{Re}(z^2) > 0.597$ and in the case $(\mu = 2, p = 1, m = 1, \tilde{\omega} < \omega \leq m/2)$:*

$$\mathrm{Re}\left(z^2\right) \geq (1+\omega)^2 - \frac{\sqrt{1 + (4\omega(1+\omega))^2} + 1}{16\omega^2}. \tag{3.89}$$

Proof. We already see that for $\tilde{\omega} \leq \omega < m/2$ we have that $x < 4/\sqrt{3}$ so we recover the first part the proof of Proposition 3.12. Now take $p = 1$ to compute $\tilde{\omega} \approx 0.326$, therefore the bound is given by $\mathrm{Re}(z^2) > \frac{5.45}{16}(m+\omega)^2 > 0.597$. The next step is, replace the new $\eta_\star$ in (3.88) and we get (3.89). □

Combining again the bound with the Lemma 3.10, we obtain that for any $\mu \leq 2$, eigenvalues that are not on the axis can only be in the $3D$ region in Figure 3.6.



(a) (b)

Figure 3.6: Region where the eigenvalues that are not on the axes can be for $(\mu = 2, p = 1, m = 1, 0.326 \leq \omega \leq 0.5)$.

Chapter 4

Knowledge compilation

This chapter summarizes the characterizations for the spectra of the operators L_0, L_μ and the most important H_μ. The main known results are presented first and then the main results of our work. It also indicates where this information can be found in the literature. Not all the new results present in this chapter are in this thesis, they are part of a joint work with E. Stockmeyer, H. van den Bosch and J. Ricaud. For those results that are not proved here, a brief explanation of how they are obtained, is given.

Our results can be synthesized as follows: for suitable ω and p we have $\sigma_d(L_0) = \{-2\omega, 0\}$. The operator L_2 has exactly three negative eigenvalues, i.e. $\sigma_d(L_2) = \{-2\omega, \lambda, 0\}$ where $\lambda \in (-2\omega, 0)$. For proper ω and p, L_μ with $\mu \in (0, 2]$ has exactly one eigenvalue between $(-2\omega, 0)$ and none at $(-m - \omega, -2\omega)$. For $z \in \sigma(H_\mu) \setminus \{\mathbb{R} \cup i\mathbb{R}\}$ we give bounds on the imaginary and real part of z. Also, for suitable ω and p there are no eigenvalues on the imaginary axis.

The first 3 theorems synthesize some results on the spectrum of L_0, L_μ and H_μ of the papers [6, 21].

Theorem 4.1 (Spectrum of L_0)**.** *Let $p > 0$, and $\omega \in (0, m)$, and L_0 defined in* (3.42) *with domain $H^1(\mathbb{R}, \mathbb{C}^2)$. Then L_0 is self-adjoint and a bounded, relatively compact perturbation of $D_m - \omega$. We have* *(i)* $\sigma_{\text{ess}}(L_0) = (-\infty, -m-\omega] \cup [m-\omega, +\infty)$, *There are no eigenvalues embedded into the essential spectrum. [6, Lemma 5.1]-[21, Lemma 2.4].*

(ii) *The spectrum is symmetric whit respect to* $-\omega$. *[6, Lemma 5.2].*

(iii) 0 *and* -2ω *are simple eigenvalues of* L_0 *with eigenfunctions* ϕ_0 *and* $\phi_{-2\omega} := \sigma_1\phi_0$ *respectively. [6, Lemma 5.4]-[21, Lemma 2.6].*

(iv) *For* $p = 1$, *the values* $\lambda = m - \omega$ *and* $\lambda = -m - \omega$ *are resonances of* L_0. *[6, Lemma 5.5].*

Theorem 4.2 (Spectrum of L_2)**.** *Let* $p > 0$, $\omega \in (0, m)$, *and* L_2 *defined in* (3.42) *with domain* $H^1(\mathbb{R}, \mathbb{C}^2)$. *Then* L_2 *is a bounded and relatively compact perturbation of* L_0 *and*

(i) $\sigma_{\text{ess}}(L_2) = (-\infty, -m-\omega] \cup [m-\omega, +\infty)$, *There are no eigenvalues embedded into the essential spectrum. [6, Lemma 5.1]-[21, Lemma 2.4].*

(ii) 0 *and* -2ω *are simple eigenvalues of* L_2 *with eigenfunctions* $\partial_x\phi_0$ *and* $\phi_{-2\omega} := \sigma_1\phi_0$ *respectively. [6, Lemma 5.4]-[21, Lemma 2.6].*

Finally, the next Theorem groups the known facts about eigenvalues of the linearization H_2, which is no longer self-adjoint.

Theorem 4.3 (Spectrum of H_2)**.** *Let* $p > 0$. *and* $\omega \in (0, m)$, *and* H_2 *as defined in* (3.39), *seen as an operator with domain* $H^1(\mathbb{R}, \mathbb{C}^4)$.

(i) *Its essential spectrum satisfies* $\sigma_{\text{ess}}(H_2) = (-\infty, -m+\omega] \cup [m-\omega, +\infty)$. *Zero is a double eigenvalue of* H_2 *and* $\pm 2\omega$ *are eigenvalues. [6, Lemma 6.1]-[14, Lemma 2.4]-[21, Corollary 2.8].*

(ii) $\sigma(H_2)$ *is symmetric whit respect to* $\mathbb{R}$ *and* $i\mathbb{R}$ *and has not embedded eigenvalues above* $m + \omega$. *[6, Lemma 6.1]-[14, Lemma 2.1]-[14, Theorem 2.10].*

(iii) *No eigenvalues with* $\mathrm{Im}(z) \neq 0$ *appears from the essential spectrum of* H_2 *above* $m+\omega$. *[14, Theorem 2.18].*

(iv) *The eigenfunctions decays exponentially. [14, Theorem 2.13].*

The essential spectrum is the essential spectrum of the free Dirac operators. Almost everything of these theorems are proved throughout the thesis. We are not able to obtain a complete description of the spectrum of the operators, but we have the following information.

Theorem 4.4 (Spectrum of L_0)**.** *Let $p > 0$, and $\omega \in (0, m)$, and L_0 defined in* (3.42) *with domain $H^1(\mathbb{R}, \mathbb{C}^2)$. Then,*

(i) For $p = 1$, $\sigma(L_0) \cap (-2\omega, 0) = \emptyset$. Proposition 3.1.

(ii) For any $\mu \in \mathbb{R}$, $p > 0$ and $0 < \omega < m$, the eigenvalues are simple. Lemma 3.3.

(iii) For $p = 1$ and any ω, OR $p > 1$ and ω be s.t. $\omega > \frac{p}{p+1}m$ and $\omega \geq \frac{p+1}{2p^2}m$, the only eigenvalues of L_0 are 0 and -2ω.

Theorem 4.5 (Spectrum of L_μ)**.** *Let $p > 0$, $\omega \in (0, m)$, and L_μ defined in* (3.42) *with domain $H^1(\mathbb{R}, \mathbb{C}^2)$. Then,*

(i) For any $\mu \in \mathbb{R}$, $p > 0$ and $0 < \omega < m$, the eigenvalues are simple. Lemma 3.3.

(ii) The operator $L_2(\omega, m)$ has exactly 3 nonpositive eigenvalues $-2\omega = \lambda_0 < \lambda_1 < \lambda_2 = 0$. Moreover, the second eigenvalue verifies $\lambda_1 \geq \max\{-4(m - \omega); -2\omega\}$.

(iii) For $p = 1$ and any $\omega \in (0, m)$, OR for $p > 1$ and ω be s.t. $\omega > \frac{p}{p+1}m$ and $\omega \geq \frac{p+1}{2p^2}m$. L_μ with $\mu \in (0, 2]$ has no eigenvalues in the interval $(-m - \omega, -2\omega)$ and exactly one eigenvalue in $(-2\omega, 0)$.

Where the proof of Theorem 5.4 (iii) and Theorem 5.5 (iii)-(iii) comes from the nonrelativistic limit $\omega \to m$ and after a suitable scaling, we recover a *nonrelativistic* Schrödinger operator related to the linearization of the nonlinear Schrödinger operator with power $2p + 1$. This operator is well known and the conclusions follows from comparing their eigenvalues with the relativistic operator. Since its spectrum is known, we can obtain good bounds on eigenvalues in this limit.

In [12] with the same non-relativistic limit $\omega \to m$ they proved the stability for the quintic nonlinearity, and then summarized their work in [11] where show stability for some critical values of p but excluding the case $p \in (0, 2)$.

Theorem 4.6 (Spectrum of H_μ)**.** *Let $p > 0$. and $\omega \in (0, m)$, and H_2 as defined in* (3.39)*, seen as an operator with domain $H^1(\mathbb{R}, \mathbb{C}^4)$.*

(i) Let $0 < \omega < m$ and $\mu \geq 0$. If $z \in \mathbb{C}$ is an eigenvalue of $H_\mu(\omega, p)$, then

$$|\mathrm{Im}\, z| \leq \frac{\mu}{2} \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert Q \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert , \tag{4.1}$$

see Lemma 3.10.

(ii) Let $z \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$ be an eigenvalue of $H(\mu, \omega)$ for $\mu \leq 2$, $m/2 < \omega < m$ and $p < 1.3$. Then for $m = 1$, $\mathrm{Re}(z^2) > 0.76$ and in the case $(\mu = 2, p = 1, m = 1)$:

$$\mathrm{Re}\left(z^2\right) \geq 4\omega^2 + (1 - \omega)\left(4\omega - \sqrt{2(1 + \omega^2)}\right), \tag{4.2}$$

see Proprosition 3.12.

(iii) Let $z \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$ be an eigenvalue of $H(\mu, \omega)$ for $\mu \leq 2$, $\tilde{\omega} < \omega < m$ and $p < 1.3$. Then for $m = 1$, $\mathrm{Re}(z^2) > 0.597$ and in the case $(\mu = 2, p = 1, m = 1, \tilde{\omega} < \omega \leq m/2)$:

$$\mathrm{Re}\left(z^2\right) \geq (1 + \omega)^2 - \frac{\sqrt{1 + (4\omega(1 + \omega))^2} + 1}{16\omega^2}, \tag{4.3}$$

see Proposition 3.13.

(iv) For p and ω such that $L_2(p, \omega)$ has exactly one *eigenvalue in $(-2\omega, 0)$. Then, there are no eigenvalues on the imaginary axis.*

The absence of eigenvalues in the imaginary axis comes from studying the algebraic and geometric multiplicity of the Kernel of L_μ and L_0 while studying the flow in $\mu : 0 \to 2$ mixed with an Vakhitov-Kolokolov stability criterion.

Combining these results, we conclude that eigenvalues leading to instability can only appear far away from the imaginary axis. Affirmations *ii* and *iv* are stated separately since they will follow from very different proofs, *ii* from bounds valid for eigenvalues away from the axes in the complex plane, and *iv* from the Vakhitov-Kolokolov criterion *provided* that L_2 has a single eigenvalue in $(-2\omega, 0)$ and *ii* excludes eigenvalues arriving at the imaginary axis from elsewhere in the complex plane.

From Theorem 4.6 and $p = 1$, we can exclude a region of the complex plane for which there are no eigenvalues, see figures 3.5-3.6. The main objective is to show that there are no eigenvalues with an imaginary part other than zero. If there were complex eigenvalues, we know they must be in these regions, then we can focus on excluding them. Therefore, the problem remains open and it is clear that there are still many questions to be resolved. This thesis does not fully reflect the work carried out with E. Stockmeyer, H. van den Bosch and J. Ricaud, but gives an idea of the complexity.

Chapter 5

Conclusions

The nonlinear equation is of great importance in physics, with many applications. The nonlinearity $\overline{\psi}\psi$ was first considered by Russian physicist Dimitri Ivanenko in 1938 [36]. Then, Mario Soler re-introduce this nonlinearity investigating the stationary states [55]. The existence of solitary wave solutions was solved with many techniques but only are known in the one-dimensional case [24, 42].

We concentrate on the one-dimensional case, the solutions of which are known by any power of the nonlinearity. Given the complexity of the model, only partial results are known, which we summarized in the previous chapter. Exist numerically evidence of that stability occur for some values of ω, see [38]. In [6] also claim stability using semi-analytic arguments.

No explicit solitary wave solutions are known for the Soler model in higher dimensions and the linearized operators becomes even more difficult to deal with. For this reason, the studies in these cases are principally using numerical methods. In conclusion, the stability of the solutions of this model is still an open problem.

This is a joint investigation with H. Van den Bosch, E. Stockmeyer and J. Ricaud. There are more results that we do not present here but that we will publish soon, but none of them prove stability yet. We expect, hopefully, be able to extend some of our results to higher dimensions.

Appendix A

Poddubny-Smirnova generalization

The generalization of the Soler model introduced in 2018 by Poddubny and Smirnova [47] was originally in two-dimensions but we are going to adapt the model to the 1D case. The PS model consist in take $f(s) = s$ from (3.5) and control the nonlinear terms with real constants as follows

$$\partial_x v(x) = \left(-\omega - m + b\,|v|^2 + a_2\,|u|^2\right) u(x), \tag{A.1}$$

$$\partial_x u(x) = \left(\omega - m - a_1\,|v|^2 - b\,|u|^2\right) v(x). \tag{A.2}$$

Notice that if $a_1 = a_2 = -b = -1$ we recover the Soler model. For symmetry reasons is convenient consider $a_1 = -1$, $a_2 = -\alpha^2$ and $b = \alpha$ for $\alpha \in \mathbb{R}$. Then we have the following twisted Soler model

$$\begin{cases} \partial_x v(x) &= \left(-\omega - m + \alpha\,|v|^2 - \alpha^2\,|u|^2\right) u(x) \\ \partial_x u(x) &= \left(\omega - m + |v|^2 - \alpha\,|u|^2\right) v(x) \end{cases} \tag{A.3}$$

Now, if $\alpha = 1$ we get (3.5) in the case $f(s) = s$.

Remark A.1. The coupled system (A.3) also satisfies the Hamiltonian system (3.6) where

$H(u, v)$ becomes

$$H(v,u) = \frac{\omega}{2}(v^2+u^2) - \frac{m}{2}(v^2-u^2) + \frac{1}{4}(v^2-\alpha u^2)^2. \tag{A.4}$$

The problem now is, for which α exist solutions $(v,u)^\top \in H^1(\mathbb{R},\mathbb{R}^2)$. The case $\alpha = 1$ is the Soler model, $\alpha = -1$ was proved by William Borrelli [8, 10, 9]. For others α the existence follows from study the properties of the Hamiltonian (A.4) as Cazenave and Vazquez did for general nonlinearities in [17]. The first step is study the behavior of $H(v,u)$ as a function of v and u. If $m > 0$ and $\omega > 0$ then:

Lemma A.1. *The Hamiltonian* (A.4) *is bounded from below by* $-\frac{m^2}{4}$ *and* $H(v,u) \to \infty$ *if* $|v| + |u| \to \infty$ *for* $\alpha \leq \frac{\omega+m}{m}$.

Proof. First of all, let us consider $\alpha \leq 1$ and thus $v^2 - \alpha u^2 \geq v^2 - u^2$. So,

$$H(v,u) \geq \frac{\omega}{2}(v^2+u^2) - \frac{m}{2}(v^2-\alpha u^2) + \frac{1}{4}(v^2-\alpha u^2)^2.$$

The first term is positive and the parabola $\frac{1}{4}x^2 - \frac{m}{2}x$ has its vertex in $(m, -\frac{m^2}{4})$ then $H(v,u) \geq \frac{\omega}{2}(v^2+u^2) - \frac{m^2}{4}$, this implies for the case $\alpha \leq 1$ that the Hamiltonian go to infinity if $|v| + |u| \to \infty$. Is clear that the lower bound is $-\frac{m^2}{4}$.

In the other case $1 < \alpha < \frac{\omega+m}{m}$ we have $v^2 - u^2 \geq v^2 - \alpha u^2$. Then,

$$\begin{aligned}
H(v,u) &= \frac{\omega}{2}(v^2+u^2) - \frac{m}{2}\int_0^{v^2-u^2} \mathrm{d}t + \frac{1}{2}\int_0^{v^2-\alpha u^2} t\,\mathrm{d}t \\
&= \frac{\omega}{2}(v^2+u^2) + \frac{1}{2}\int_0^{v^2-\alpha u^2} t - m\,\mathrm{d}t - \frac{m}{2}\int_{v^2-\alpha u^2}^{v^2-u^2} \mathrm{d}t \\
&= \frac{\omega}{2}(v^2+u^2) - \frac{m}{2}(v^2-\alpha u^2) + \frac{1}{4}(v^2-\alpha u^2)^2 - \frac{m}{2}u^2(\alpha-1) \\
&= \frac{\omega}{2}v^2 + (\omega - m(\alpha-1))\,\frac{u^2}{2} - \frac{m^2}{4}.
\end{aligned}$$

$\square$

Also, using that v and u satisfies the system (3.6) is easy to see that

$$\frac{\mathrm{d}H(v,u)}{\mathrm{d}x} = \frac{\partial H(v,u)}{\partial v}v' + \frac{\partial H(v,u)}{\partial u}u' = 0 \tag{A.5}$$

So, the Hamiltonian as a function of x must be constant almost everywhere and (v,u) give the compact levels sets of $H(v,u)$. Since the Hamiltonian is an increasing function of the spinor components these cannot be arbitrarily large and therefore these cannot be monotonously increasing functions of x.

If we apply the linearization analysis 3.3 to this model we found the same kind of operators as in (3.39) where

$$\tilde{L}_0 = \begin{pmatrix} m-\omega-v^2+\alpha u^2 & \partial_x \\ -\partial_x & -m-\omega+\alpha v^2-\alpha^2u^2 \end{pmatrix}, \tag{A.6}$$

$$\tilde{Q} = \begin{pmatrix} v^2 & -\alpha uv \\ -\alpha uv & \alpha^2 u^2 \end{pmatrix}. \tag{A.7}$$

Lemma A.2. *For any μ, $\alpha \in \mathbb{R}$, $p > 0$ and $0 < \omega < m$, the eigenvalues of L_μ and L_0 are simple.*

Proof. If λ is eigenvalue of $\tilde{L}_0$ associated to the eigenfunction $(w_1, w_2)^\top$. Thus w_1 and w_2 satisfies the coupled system

$$\begin{aligned} \partial_x w_1 &= -(m+\omega+\lambda+\alpha^2u^2-\alpha v^2)w_2 \\ \partial_x w_2 &= (\omega+\lambda-m-\alpha u^2+v^2)w_1 \end{aligned} \tag{A.8}$$

Assume exist h_1 and h_2 such that $L_0(h_1,h_2)^\top = \lambda(h_1,h_2)^\top$ and now compute the determinant

$$W(r) = \det \begin{pmatrix} w_1 & w_2 \\ h_1 & h_2 \end{pmatrix} = w_1 h_2 - w_2 h_1$$

consider the derivative

$$\begin{aligned}
\dot{W} &= \dot{w}_1 h_2 - w_2 \dot{h_1} + w_1 \dot{h_2} - \dot{w}_2 h_1 \\
&= -(m + \omega + \lambda + \alpha^2 u^2 - \alpha v^2) w_2 h_2 + (m + \omega + \lambda + \alpha^2 u^2 - \alpha v^2) h_2 w_2 \\
&\quad + (\omega + \lambda - m - \alpha u^2 + v^2) w_1 h_1 - (\omega + \lambda - m - \alpha u^2 + v^2) w_1 h_1 \\
&= 0.
\end{aligned}$$

The determinant is therefore constant and since it vanishes at infinity (the eigenfunctions are in $L^2(\mathbb{R})$), it is zero everywhere and the two eigenfunctions are linearly dependent. We can exclude the case where the functions are proportionals for each x through a function $\alpha(x)$ in the same way that in Lemma 3.3. Analogously, following the same argument, if λ is an eigenvalue of $L_0 - \mu Q$ associated to the eigenfunction $(w_1, w_2)^\top$. Thus w_1 and w_2 satisfies the next equations

$$\begin{aligned}
\partial_x w_1 &= -(m + \omega + \lambda + \alpha^2 u^2 - \alpha v^2) w_2 - \mu(\alpha u v w_1 + \alpha^2 u^2 w_2) \\
\partial_x w_2 &= (\omega + \lambda - m - \alpha u^2 + v^2) w_1 + \mu(v^2 w_1 - \alpha u v w_2)
\end{aligned} \tag{A.9}$$

Assume exist h_1 and h_2 such that $(L_0 - \mu Q)(h_1, h_2)^\top = \lambda(h_1, h_2)^\top$, defining the same function and its derivative, so

$$
\begin{aligned}
\dot{W} &= \dot{w}_1 h_2 - w_2 \dot{h_1} + w_1 \dot{h_2} - \dot{w}_2 h_1 \\
&= [-(m+\omega+\lambda+\alpha^2 u^2 - \alpha v^2) w_2 - \mu(\alpha u v w_1 + \alpha^2 u^2 w_2)] h_2 \\
&\quad + [(m+\omega+\lambda+\alpha^2 u^2 - \alpha v^2) h_2 + \mu(\alpha u v h_1 + \alpha^2 u^2 h_2)] w_2 \\
&\quad + [(\omega+\lambda-m-\alpha u^2 + v^2) h_1 + \mu(v^2 h_1 - \alpha u v h_2)] w_1 \\
&\quad - [(\omega+\lambda-m-\alpha u^2 + v^2) w_1 + \mu(v^2 w_1 - \alpha u v w_2)] h_1 \\
&= 0.
\end{aligned}
$$

□

Note that the vast majority of section 3.4 onwards can be extended to this model since they do not depend on the explicit form of the L_0 and L_μ operators. This is interesting given the possibility of applying this model to real physical 2D systems, explained by Poddubny and Smirnova. They show one way of how the cubic nonlinearity is reproducible. They even gave several examples where this two-dimensional model could be implemented, like optofluidic platform with photonics crystal fibers filled by liquids, glass fibers, etc. see [47, 43, 28, 48, 62, 23, 2, 18, 54, 19, 51].

Appendix B

Picard-Lindelöf theorem

In this appendix we are going to enunciate the Picard-Lindelöf existence and uniqueness theorem as in [57, Theorem 2.2] for the IVP:

$$\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0, \end{cases} \tag{B.1}$$

Theorem B.1. *Suppose $f \in C(U, \mathbb{R}^n)$ where U is an open subset of $\mathbb{R}^{n+1}$ and $(t_0, x_0) \in U$. If f is locally Lipschitz continuous in the second argument, uniformly with respect to the first, then there exists a unique local solution $\overline{x}(t) \in C^1(I)$ of the IVP* (B.1)*, where I is some interval around t_0.*

More specifically, if $V = [t_0, t_0 + T] \times \overline{B_\delta(x_0)} \subset U$ and M denotes the maximum of $|f|$ on V. Then the solution exist at least for $t \in [t_0, t_0 + T_0]$ and remains in $\overline{B_\delta(x_0)}$, where $T_0 = \min\{T, \frac{\delta}{M}\}$. The analogous result holds for the interval $[t_0 - T_0, t_0]$.

Is well known that a function $f : X \to Y$ such that exists a constant $C > 0$ that satisfies $d_y\left(f(x_1), f(x_2)\right) \leq C d_x\left(x_1, x_2\right)$ for all $x_1, x_2 \in X$, is Lipchitz continuous. Moreover, is locally Lipchitz if each point $x \in X$ is the center of a ball $B_r(x)$ such that the restriction $f|_B$ is Lipchitz.

Applying these for our case since the IVP $L_\mu \Psi = \lambda\Psi$ and

$$L_\mu \equiv L_0 - \mu Q = i\sigma_2\partial_x + \sigma_3\left(m - (v^2 - u^2)^p\right) - \omega - \mu p(v^2 - u^2)^{p-1}\begin{pmatrix} v^2 & -uv \\ -uv & u^2 \end{pmatrix}. \quad \text{(B.2)}$$

Thus, the function f of the Picard-Lindelöf theorem for our case is given by

$$f(x,\omega) := i\sigma_2\sigma_3\left(m - (v^2 - u^2)^p\right) + i\mu p(v^2 - u^2)^{p-1}\sigma_2\begin{pmatrix} v^2 & -uv \\ -uv & u^2 \end{pmatrix} + i\sigma_2(\lambda + \omega)$$

where $u(x,\omega)$ and $v(x,\omega)$ are continuos uniformly bounded $L^2(\mathbb{R} \times (0,m), \mathbb{R})$ functions on x, with exponential decay. Also, the function f is locally Lipchitz on ω, is sufficient take $\omega_\star \in B_\delta(\omega)$ for some fix $\omega \in (0,1)$, bound f by the norm $|||Q|||$ and the supremum of $(v^2 - u^2)^p$ to find the finite Lipchitz constant.

Appendix C

Jensen's inequality

In this appendix we are going to show the details of Jensen's inequality based on Lieb-Loss analysis book [39]. First of all, we are going to introduce the theorem as is presented in the book:

Theorem C.1. *Let $J : \mathbb{R} \to \mathbb{R}$ be a convex function. Let f be a μ-measurable, real-valued function on Ω. Since J is convex, it is continuos and therefore $(J \circ f)(x) := J(f(x))$ is also μ-measurable function on Ω. We assume that $\mu(\Omega) = \int_\Omega \mu(\,\mathrm{d}x)$ is finite.*

Suppose now that $f \in L^1(\Omega)$ and let $\langle f \rangle$ be the average of f, i.e.,

$$\langle f \rangle = \frac{1}{\mu(\Omega)} \int_\Omega f \,\mathrm{d}\mu.$$

Then

(i) $[J \circ f]_-$, the negative part of $[J \circ f]$, is in $L^1(\Omega)$, whence $\int_\Omega (J \circ f)(x)\mu(\,\mathrm{d}x)$ is well defined although it might be $+\infty$.

(ii)

$$\langle J \circ f \rangle \geq J(\langle f \rangle). \tag{C.1}$$

If J is strictly convex at $\langle f \rangle$ there is equality in (C.1) *if and only if f is a constant function.*

We want to applied this theorem to expectation values of the form $\langle\psi, J(A)\psi\rangle$, where J is a convex function and A is a self adjoint operator. So, what we have to do is write this expectation value using the spectral theorem (see [50][Theorem VIII.5]), that is,

$$\langle\psi, J(A)\psi\rangle = \int_{\mathbb{R}} J(x)\, \mathrm{d}\mu_\psi(x)$$

where $\mathrm{d}\mu_\psi(x)$ is the spectral measure and $\psi \in D(A)$ such that $\|\psi\|^2 = 1$. Thus, in the other hand we have that $\mu(\Omega) = \langle\psi, \mathbb{1}\psi\rangle = 1$, so applying the Jensen's inequality and using the spectral theorem again we have that

$$\langle\psi, J(A)\psi\rangle = \int_{\mathbb{R}} J(x)\, \mathrm{d}\mu_\psi(x) \geq J\left(\int_{\mathbb{R}} x\, \mathrm{d}\mu_\psi(x)\right) = J\left(\langle\psi, A\psi\rangle\right).$$

We can rewrite this inequality using that $\psi = \frac{\varphi}{\|\varphi\|}$ and then

$$\frac{\langle\varphi, J(A)\varphi\rangle}{\|\varphi\|^2} \geq J\left(\frac{\langle\varphi, A\varphi\rangle}{\|\varphi\|^2}\right).$$

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