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A method for solving the multi-objective transit frequency optimization problem

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SUMMARY

Frequency setting takes place at the strategic and tactical planning stages of public transportation systems. The problem consists in determining the time interval between subsequent vehicles for a given set of lines, taking into account interests of users and operators. The result of this stage is considered as input at the operational level. In general, the problem faced by planners is how to distribute a given fleet of buses among a set of given lines. The corresponding decisions determine the frequency of each line, which impacts directly on the waiting time of the users and operator costs. In this work, we consider frequency setting as the problem of minimizing simultaneously users' total travel time and fleet size, which represents the interest of operators. There is a trade-off between these two measures; therefore, we face a multi-objective problem. We extend an existing single-objective formulation to account explicitly for this trade-off, and propose a Tabu Search solving method to handle efficiently this multi-objective variant of the problem. The proposed methodology is then applied to a real medium-sized problem instance, using data of Puerto Montt, Chile. We consider two data sets corresponding to morning-peak and off-peak periods. The results obtained show that the proposed methodology is able to improve the current solution in terms of total travel time and fleet size. In addition, the proposed method is able to efficiently suggest (in computational terms) different trade-off solutions regarding the conflicting objectives of users and operators. Copyright © 2017 John Wiley & Sons, Ltd.

KEY WORDS: transit frequency optimization; multi-objective optimization; Tabu Search; case study

1. INTRODUCTION

Transit service design entails making decisions concerning several aspects of public transportation systems, namely, line design, timetable construction, fleet/crew assignment, and fare determination, among others. Ideally, these decisions should be taken optimally, in the sense of the interests of the whole society. In this work, we focus on the task of frequency setting, known as the problem of determining the number of buses per time unit on each line of a public transportation system. This problem arises in both strategic and tactical planning; in the former case, as part of the transit network design, while for the latter, as a way to adjust the services to variations in the demand or the route network [1, 2]. In these cases, the main decision maker is a centralized planning and regulating entity. The output of the frequency setting is taken as input to create the operational plans for each line, including, for example, the timetable construction. Here, decisions are usually made by individual operators, which are in charge of running a subset of lines from the whole system.

When considered as an optimization problem, frequency setting should take into account the interests of the main actors involved in the system: users and operators. At the strategic and tactical levels, the planner is represented by a transit agency that is in charge of taking care of the interests of both actors. The existing optimization models usually seek to maximize the level of service offered

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to the users and minimize the level of required resources delivered by the operators. Problem data is given by the itinerary of each line and origin–destination (OD) demand within a specific time horizon. An important component of the model is the *assignment sub-model*, which represents the behavior of the users with respect to a set of lines and frequencies. This sub-model is needed to measure the performance of the system with respect to the users, that is, the level of service.

The literature concerning transit frequency optimization can be classified into (i) analytical models that admit closed-form solutions and (ii) mathematical programming formulations either explicit or not, with associated solution algorithms. In the first group, there are formulations that characterize the system in terms of few variables and allow getting a full description of the optimal solution. Although these models make considerable simplifications of the real system, they allow obtaining practical guidelines that are theoretically well founded [3, 4], for example, the well-known *rule of the square root* [5, 6]. The other important stream of work is based on a detailed characterization of the transit system, in terms of the route network and the demand that should be transported over it. These studies formulate the optimization problem in terms of a graph model, where decision variables are the capacities of the arcs (represented by the frequencies) and the flows that represent the OD demand that is governed by the assignment sub-model. In the most general case, the objective function seeks to minimize the total travel time, while a maximum fleet size constraint is the main component of the formulation that bounds the increase of the frequency, which is desirable from the users' viewpoint since it reduces the waiting time. Early work can be found in [7], which states the minimization of the walking and on-board travel time plus the waiting time subject to a constraint on the maximum fleet size; the resulting nonlinear formulation is solved approximately using a descent strategy and the methodology is tested with a small city comprising six lines. In [8], a nonlinear bilevel formulation that minimizes overall travel time subject to a fleet size constraint is proposed and solved approximately using a gradient descent method that exploits the problem structure; the methodology is tested by using several test cases ranging from 38 to 115 transit lines. More recently, in [9], a reformulation of the problem stated in [8] is proposed and solved using mixed-integer linear techniques for small-sized cases (seven lines) and using a Tabu Search metaheuristic for instances comprising up to 133 lines. In these two studies, special effort has been done in solving the problem to optimality and quantifying the accuracy of approximate methods in the sense of distance from their results to optimal solutions; moreover, these studies are based on the well-known *optimal strategies* assignment sub-model [10], which is widely studied and used at both academic and practice fields. Other variants explored in the literature include joint optimization of frequencies and bus size [11–13], joint optimization of frequencies and fare [14], joint optimization and data collection methods [15], consideration of stochastic demand and travel time [16], consideration of equity concerns [17] and the case in which elastic demand is considered [18–21].

The existing models usually have real variables that represent line frequencies and passenger flows. In most cases, the mathematical formulations are nonlinear, because of the relationships between the frequency with the waiting time and the passenger flows [7]. Moreover, the representation of the passenger behavior makes it difficult to state an explicit formulation, which precludes the identification of the model structure and therefore the determination of an effective solution method [22]. In addition, the realistic modeling of bus capacity and congestion introduces even more difficulties due to bilevel and/or equilibrium formulations [8, 23]. Therefore, all solution methods rely on heuristic algorithms either driven by mathematical formulations [8] or purely heuristics [24]. These methods have been tested with different cases, including real ones comprising up to 140 lines approximately [9, 19].

The transit frequency optimization problem has several aspects that have been studied with different emphasis by different authors, as can be noted from the previous paragraphs. In this work, we focus on the multi-objective aspect of the problem and its efficient solution. Particularly, we consider user and operator objectives under general assumptions of fixed and deterministic demand, deterministic travel times, single fare, homogeneous fleet, single operation pattern, and uncongested conditions. Under these hypothesis, conflicting interests of users and operators can be clearly identified: higher frequencies entail lower travel time (particularly its waiting component) and vice-versa. Note this trade-off cannot be as easily identified in cases of demand elasticity, congestion, or different fare patters, where special care has to be taken in the formulation of the objective function and constraints. We resume the work undertaken in [8, 9] in a multi-objective optimization framework. This means that

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in order to solve the problem we look for a set of non-dominated solutions (also known as Pareto front or set of *efficient solutions* [25]) in terms of the conflicting objectives of users and operators. Roughly speaking, we can say that two solutions (settings of frequencies) are non-dominated if none of them can improve the other in both objectives. To the best of our knowledge, this approach has not been applied before to transit frequency optimization, even though the existing models consider interests of users and operators in their formulations. Few models join both objectives into a single function by using weighting values [23]. Moreover, most models optimize users' objective under a budgetary constraint representing the fleet size (e.g. [7, 8]). The usual way to obtain different non-dominated solutions using this kind of formulations is by repeating the execution of the single-objective model for different weighting or constraint values. These are in fact two classical methods for solving multi-objective problems [25]: the *Weighted-Sum* and the *ε-Constraint* methods. Previous experiences with a related problem regarding transit network design [26, 27] have shown that the multi-objective problem can be tackled as such, by using specific solution algorithms designed specially to deliver an entire set of non-dominated solutions in a single run. The main advantages of this approach are the significant reduction in total execution time and the ability of finding solutions that cannot be found by using classical multi-objective optimization methods (non-supported efficient solutions, [25]). That stream of work is resumed in the current proposal.

In this work, we contribute in the context of transit frequency optimization on the following specific directions: (i) We present an extension and improvement from our previous single-objective model and Tabu Search solution method [9], conceived to solve efficiently the multi-objective variant of the problem, and (ii) an application to a real case concerning a medium-sized city. A preliminary version of this work was presented in [28].

Concerning the first contribution, we propose an extension of our previous formulation and solution method to handle a multi-objective variant of the transit frequency optimization problem, considering the conflicting objectives of users and operators. There are two motivations for generating a multi-objective transit frequency model. The first one is based on the fact that the frequency setting problem arises at the strategic and tactical planning levels of public transportation systems. In these contexts, especially in the strategic one, the planner may be interested in exploring a range of different solutions. For example, if the transit agency is planning to deploy more resources (buses and drivers) to the system, the hypothetical new (optimized) frequencies and their effect over the level of service is likely to be analyzed for different levels of resources. Moreover, because the transit frequency optimization problem is computationally hard to solve, heuristics have been identified as efficient methods to obtain near-optimal solutions [9]. Multi-objective metaheuristics [29] take advantage of the exploration of the search space in order to generate efficiently an entire set of trade-off solutions in a single execution of the algorithm. This is particularly useful in the context of multi-objective transit frequency optimization because it reduces the time taken by the method to deliver solutions to the planner.

Concerning the case study, we use data from the city of Puerto Montt, Chile. This is a medium-sized city of 230 000 inhabitants approximately, with a transit system comprising 20 lines. We use data provided by SECTRA (the transportation planning agency of Chile), which allow building very realistic scenarios because the corresponding information is systematically updated and validated. We use morning-peak and off-peak data, which are not usual in studies concerning transit frequency optimization. It is worth mentioning that we do not attempt to model a multi-period problem, instead we illustrate the application and capabilities of the proposed methodology for different demand scenarios.

The article is structured as follows. Section 2 provides a description of the methodology, including both model and solution algorithm. In Section 3, we describe the main characteristics of the case study, present the experiments performed, and discuss their results. Finally, Section 4 draws some conclusions and lines for future research.

2. METHODOLOGY

The setting of transit frequencies is modeled as a combinatorial optimization problem, where we must assign a frequency value (taken from a given discrete set) to every line given as input data. In the single-objective variant of the problem, any setting of frequencies must respect an upper limit on the

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available fleet size, while the objective is the minimization of the overall travel time of the users. Each solution is evaluated according to an assignment sub-model, in order to compute its objective value. In [9], this problem is formulated as a mixed-integer linear programming one, which can be solved to optimality for small-sized instances. For larger instances, a Tabu Search metaheuristic was proposed and implemented, and it is the base methodology applied in this work. As an extension of the existing methodology, we propose a variation of the heuristic that solves efficiently a multi-objective version of the problem, considering the conflicting objectives of users and operators.

In the following, we state the main hypothesis considered with respect to the transit system. This includes the representation of demand and supply, as well as the relevant aspects of the assignment sub-model. Then, we formally state the transit frequency optimization problem and the main concepts of the metaheuristic approach used to solve it. Sections 2.1 to 2.3 present concepts and notation already introduced in our previous paper [9]. Finally, we present a more detailed description of the extension proposed in this work.

2.1. Transit system and passenger behavior

We model the underlying structure of the transit system as a directed graph $G = (N, A)$. The set of nodes N represents either bus stops, endpoints of street segments, or zone centroids (fictitious points that concentrate the demand). The arcs in A are classified in three types:

- Travel arcs that represent the bus movements along street segments and the on-board component of the passengers' trips.
- Walking arcs that represent passenger paths between stop nodes or to/from centroids.
- Boarding and alighting of passengers to/from the bus, which represent connections between the places where demand is generated and the transit network.

Both travel and walking arcs are labeled by a fixed (deterministic) cost that represents an average time value taken by both buses and users to traverse the corresponding arc. Boarding and alighting arcs are labeled by the frequency of the corresponding bus line. Figure 1 illustrates this structure. A transit line consists of forward and backward routes or a single route if it is circular. Each route is a sequence of contiguous arcs in G. Moreover, the demand is represented as an OD matrix, where each non-null entry (called OD pair) has origin and destination centroids (nodes of G) and the number of trips per time unit within a specific time horizon. Trip rates represent mean values in steady state. Moreover, this demand is fixed over the entire horizon under consideration.

The assignment sub-model takes as input the graph representing the transit lines with a setting of frequencies and the OD demand. Its output is a distribution of demand flows representing trajectories between origins and destinations, obtained by applying the hypothesis about the passenger behavior with respect to the given lines and frequencies. We adopt the optimal strategies assignment sub-model [10]. A strategy is defined as a set of rules that when applied, enables the user to reach his destination. In terms of the graph G and for a given OD pair, a strategy can be seen as a subset of arcs in A that represents all the lines that the user identifies *a priori*, for traveling from its origin to its destination. The model assumes that a given user selects the strategy that minimizes his total travel time. To do this, he will select *a priori* (i.e., before leaving the place where the trip originates) a set of attractive lines among all the possible lines that connect its origin and destination bus stops (even including transfers). While waiting at the bus stop, the user will take the first bus passing by that stop, belonging to the set of attractive lines determined *a priori*. An application of the aforementioned model for a single OD pair over a graph G computes

- The distribution of the corresponding demand, as an assignment of flows v_a for each arc $a \in A$.

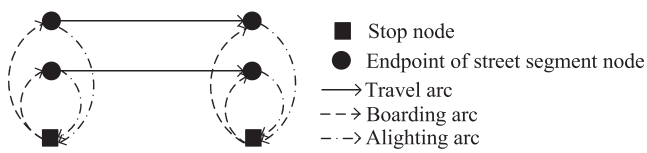

Figure 1. Representation of the transit system.

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- The waiting time WT_n at each stop node n, calculated as $WT_n = 1/\Sigma_{l \in L(n)} f_l$ where $L(n)$ is the set of lines passing by n, corresponding to the attractive set identified *a priori* by the user and f_l is the frequency (buses per time unit) of line l.

As a consequence, the result of the assignment sub-model enables computing the total travel time of the system, which is a performance measure of such a system from the point of view of the users. Note that both flows and waiting times depend on the frequencies, which are decision variables of our problem. Thus, the assignment sub-model must be applied to every new setting of frequencies. It is worth to mention that the adopted assigned sub-model does not consider the effect of congestion over the users' travel time. Congestion may occur because of traffic conditions, boarding/alighting times, and lack of bus capacity. In our model, we assume the system is isolated from the traffic, dwell time due to boarding and alighting is negligible, and the bus capacity is sufficient to load every passenger who wants to board each bus.

2.2. Multi-objective optimization problem

We start from the single-objective mixed-integer linear programming formulation proposed in [9], and we consider its multi-objective variant resulting from casting the fleet size constraint into a conflicting objective with respect to the original objective function. This new model is aimed to provide more flexibility to the decision maker and (as it is explained in Section 2.4) allows for a more efficient implementation of the solution method. Regarding the multi-objective characteristic of the model, note that in transit frequency optimization, an increase in the line frequencies always has a positive impact over the users (waiting time decreases) and a negative impact over the operators (the number of required resources increases). This is true under hypothesis of inelastic demand (therefore, constant revenue) and no congestion (sufficient bus capacity and constant travel time over the network), which is part of the framework of this study. Also, note that we assume sufficient capacity in the infrastructure (streets or corridors) to potentially accommodate several lines with high frequencies.

According to [25], the solution of (1)–(8) is not a single optimal one. Instead, it is a set of non-dominated solutions in the space of objectives (1) and (2). This set is usually referred as *Pareto front*. There are different alternatives for arriving to a single solution of a multi-objective problem, namely, *a priori*, *a posteriori*, and *interactive* methods [29]. The *a priori* method fixes the parameters that determine a concrete trade-off between the conflictive objectives, and then it solves a single-objective problem. The *a posteriori* method finds the complete Pareto front, and then selects a single solution from it. The *interactive* method incorporates into the optimization method the information that guides the selection of the desired trade-off level. In every case, that information is an additional input, usually provided by the decision maker. In this work, we adopt the *a posteriori* method.

Table I summarizes the notation used in formulation (1)–(8), which assumes a given set of lines L and a predetermined set of frequencies $\Theta = \{\theta_1,\ldots,\theta_m\}$, which is a discrete domain of values that

Table I. Symbols used in the mathematical formulation.

Symbol	Meaning
A (A^B)	Set of arcs (boarding arcs) with generic element a
N (N^P)	Set of nodes (stop nodes) with generic element n
K	Set of OD pairs with generic element k
L	Set of lines with generic element l
Θ	Set of frequencies indexed by f in the range $1..m$
$out(n)$, $in(n)$	Set of outgoing and incoming arcs of node n, respectively
δ_k	Amount of demand corresponding to OD pair k
b_{nk}	Constant value equal to 1 (−1) if node n is the origin (destination) of OD pair k and 0 otherwise
c_a	Cost of arc a
v_{ak}	Flow of OD pair k over arc a
w_{nk}	Waiting time multiplied by flow of OD pair k in node n
y_{lf}	Equal to 1 if frequency f is assigned to line l
$l(a)$, $f(a)$	Line and frequency corresponding to arc a, respectively

OD, origin–destination.

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can be assigned to each line. This set can contain reasonable values identified by the decision maker, for example, those values that are suitable for coordinating transfers. Also, the first and last values of the set impose implicitly the lower and upper bounds respectively on the frequencies.

Objective function (1) states the minimization of the total travel time of users, accumulated for all OD pairs. Its value is computed by the assignment sub-model once the frequencies are determined. Note that frequencies by themselves impact directly over the user waiting time. But because passengers consider all the travel time components to decide the lines to board, a change in the line frequencies also impacts over the whole travel time of each passenger.

Objective function (2) is a proxy for operators' cost. Several transit planning models adopt this expression [30], because the real operator cost depends on each particular case. For example, there can be subsidies and revenues based on the tickets effectively sold. Moreover, because our model is not conceived for operational planning, changes of buses among different lines are not considered; that characteristic is usually taken into account in vehicle scheduling models [1]. Our proxy for operators' cost is the number of buses required to travel simultaneously in the network. Note that this measure can be fractional, and it is directly proportional to the vehicle operation cost by distance unit and the driver cost by time unit. We assume the same cost for all the buses in the system; therefore, we are considering a homogenous fleet.

Constraint (3) states that exactly one value of θ is assigned to each line, while constraint (4) is a conservation flow expression that ensures the demand is routed from origin to destination. Constraint (5) is a flow-splitting expression coming from the optimal strategies assignment sub-model; it states the frequency-share rule [31] that distributes the passenger flow at bus stops, among the common lines that lead to destination. Constraint (6) is a technical one, which states that demand can flow only through arcs enabled according to the frequency assigned to the corresponding line. Finally, constraint (7) states the non-negative nature of flow values, and constraint (8) states the binary nature of variables that indicate the setting of frequencies. For more details concerning the reasoning and justification of this formulation, we refer to [9].

$$
\begin{array}{llll}
\min_{y,v,w} & \sum_{k \in K} \left(\sum_{a \in A} c_a v_{ak} + \sum_{n \in N^P} w_{nk}\right) & & (1)\\
\min_{y,v,w} & \sum_{l \in L} \sum_{f \in 1..m} \theta_f y_{lf} \sum_{a \in l} c_a) & & (2)\\
\text{s.t.} & \sum_{f \in 1..m} y_{lf} = 1 & \forall\, l \in L, & (3)\\
 & \sum_{a \in out(n)} v_{ak} - \sum_{a \in in(n)} v_{ak} = b_{nk} & \forall\, n \in N,\, k \in K, & (4)\\
 & v_{ak} \le \theta_{f(a)} w_{nk} & \forall\, n \in N^P,\, a \in out(n),\, k \in K, & (5)\\
 & v_{ak} \le \delta_k\, y_{l(a)f(a)} & \forall\, a \in A^B,\, k \in K, & (6)\\
 & v_{ak} \ge 0 & \forall\, a \in A,\, k \in K, & (7)\\
 & y_{lf} \in \{0,1\} & \forall\, l \in L, f \in 1..m. & (8)
\end{array}
$$

2.3. *Metaheuristic solution method*

The combinatorial aspect of the problem denoted by (1)–(8) is approached through a metaheuristic solution method based on Tabu Search [32]. The original implementation [9] performs a local search on the discrete domain determined by $\Theta^{|L|}$ looking for the best possible values of y in the context of the single-objective problem. The search advances according to a compound move (which defines a neighborhood structure) that changes the frequencies of two lines in L: one decrease and one increase. A line can change its frequency only to a value that is contiguous in Θ to its current frequency. Thus, the compound move can be seen as a redistribution of the available buses among the lines of the system. Note this is an intuitive procedure that has a direct link with the real problem. Also, systematic methods like the Hook and Jeeves algorithm have been applied to transit frequency optimization [11], performing similar operations to the ones adopted by our method. However, this intuitive move in the space of frequencies has the challenge of avoiding getting trapped in local optima. Therefore, we apply tabu concepts to the basic local search performed through compound increase and decrease moves, in order to escape from local optima and to increase the chances of reaching the global optimum. Tabu Search is a metaheuristic that has been successful in solving several optimization problems [33], particularly the transit frequency optimization problem [11]. Moreover, the local search based on the proposed moves can be easily extended to include the specific features of Tabu Search.

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In order to tailor the general framework of Tabu Search to our particular problem, we implement a tabu list that records the last iteration of the search where each line has increased or decreased its frequency. These tabu-active statuses expire once a number of iterations is reached or when there is a reason to generate more valid moves (*aspiration criteria*). Moreover, we should note that the evaluation of each new solution entails an invocation to the assignment sub-model, which may slow down the performance of the overall algorithm when it is invoked repeatedly. Taking into account these observations, we implemented the *aspiration plus strategy* [34] that explores a reasonable number of neighbors (settings of frequencies) considering that too few may constrain the search and too many may slow down the process.

2.4. Multi-objective solution method

The Tabu Search explained in Section 2.3 and implemented in [9] for solving the single-objective variant of the transit frequency optimization problem was successful in finding near-optimal solutions for a small-sized test case. Its extension to the multi-objective case seems very natural once we include the long-term memory feature, which relaxes the fleet size constraint in order to search into a wider set of alternatives.

Our basic Tabu Search uses a *short-term memory* to record moves that should be avoided in subsequent iterations. In this work, in a first step we introduce an extension by introducing a *long-term memory* [34]. This type of memory is expected to bring the opportunity to the algorithm of learning from its achievements in a wider horizon. Thus, the long-term memory can enhance the single-objective Tabu Search by exploring solutions not ordinarily found, and by intensifying the search in regions of promising solutions. Let us define sol_1 (sol_m) as the solution in which every line frequency has value θ_1 (θ_m). Note that sol_1 (sol_m) has the lowest (highest) fleet size and the highest (lowest) travel time. Also, note that we can reach sol_1 (sol_m) by applying a finite number of decrease (increase) operations to any solution s and vice-versa.

We consider an *oscillation boundary* determined by the fleet size expression (2). By using exclusively operations increase and decrease, we may potentially explore the whole solution space between sol_1 and sol_m, whose trajectory may cross the oscillation boundary several times. Whenever the boundary is reached, an intensification phase is applied, by staying in the corresponding fleet size level until a given number of iterations has passed without improvement. At each iteration, the short-term memory strategies are applied (Figure 2a). We also maintain a *critical event memory* that records the history of the most recent *critical events* (crossings of the oscillatory boundary) that happened over the search. The critical events corresponding to the first feasible solution found and every new best solution are recorded in this memory. Thus, we have a measure of the most recent and frequent values of the frequencies of the critical events. Every time sol_1 or sol_m has been reached, we bias the search to take the least frequent solutions using this memory, by penalizing the path to the most frequent solutions over a defined number of iterations. This is intended to guide the search to regions little explored or unexplored in the boundary level (feasible solutions in terms of fleet size), where we repeatedly apply the short-term memory strategy (intensification phase) and we update the critical event memory accordingly. With the combined use of the oscillatory strategy and critical event memory, we diversify the search in an explicit manner, while we intensify implicitly using the local search. We call this variation as *long-term memory Tabu Search*.

Using the algorithmic component described previously, we extend the methodology in order to handle a multi-objective variant of the problem. The proposed solution method performs the following steps:

(1) Find a set of initial solutions using the long-term memory Tabu Search.
(2) Construct some statistic solutions from the initial solutions.
(3) Find new solutions by searching the path between the solutions from steps 1 and 2.
(4) With all the solutions found in previous steps, select the non-dominated ones.

To find the initial solutions we extend the long-term memory Tabu Search to manage multiple critical levels (equal to the number of different frequencies: m). As an initial solution for level i, we select the one having all frequencies equal to θ_i (note that the only solution for level 1 is sol_1 and

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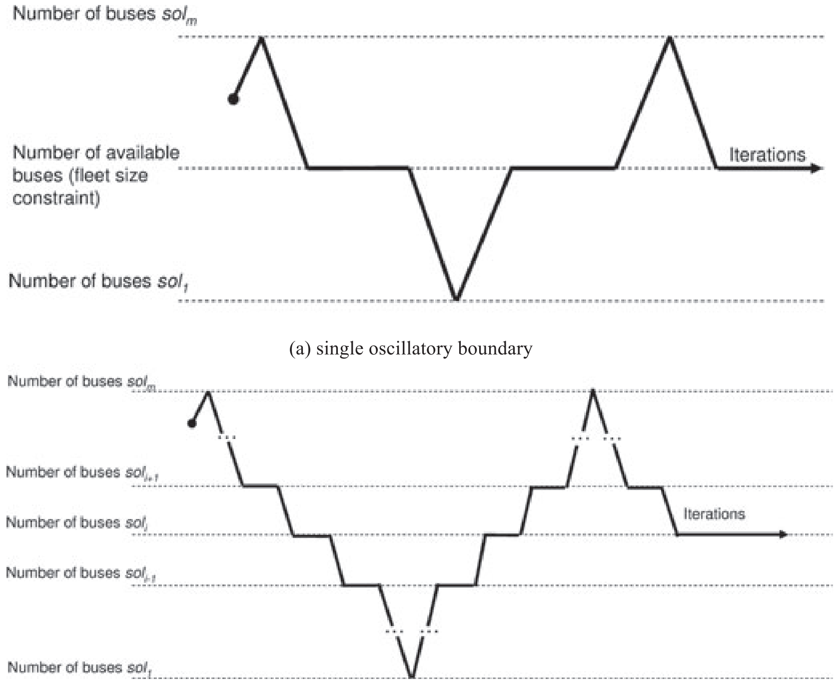

(b) multiple oscillatory boundaries

Figure 2. Long-term memory Tabu Search. (a) single oscillatory boundary; (b) multiple oscillatory boundaries.

for level m is sol_m). After the execution of the long-term memory Tabu Search, we obtain a set of m best solutions, one for each level (including sol_1 and sol_m). To construct the statistic solutions, we use the mean, mode, and median of the best solutions of each level found in the previous step. At step 3, we build paths between the solutions of contiguous levels obtained in the steps 1 and 2. Starting from the solution of level 1, we search towards the solution of level 2, then towards the solution of level 3 and so on, until level m (forward path). Similarly, we search in a downward path (beginning at level m). At each step of both paths, an exhaustive search of all the neighbors is done to select the best one (Figure 2b). Because the fleet size upper limit is set to the fleet size of the current solution, both paths are different (Figure 3). The paths are explored for each type of solution one at a time: best, mean, mode, and median. The potential set of Pareto-optimal solutions is made by the solutions best,

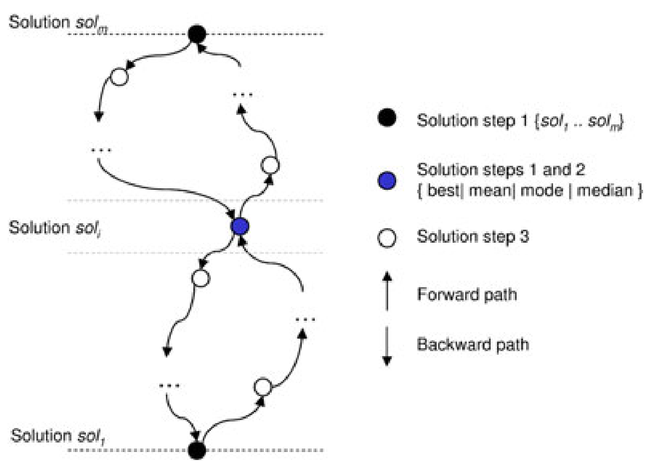

Figure 3. Forward and backward paths in the multi-objective long-term memory Tabu search.

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mean, mode, and median for each level (obtained by the short-term memory Tabu Search) and the whole forward and backward paths (obtained by applying only increase and decrease moves). A simple filtering process is performed in step 4, in order to discard dominated solutions.

Notice that the algorithm explained earlier produces an entire set of non-dominated solutions that are either

- Good approximations to optimal solutions in terms of total travel time for different values of fleet size. These solutions are obtained by applying the already existing short-term memory Tabu Search, whose accuracy is evaluated in previous work [9].
- Solutions that connect the previous ones by performing small changes on their frequencies, therefore we can expect they are also good in terms of both objectives.

For the reasons explained previously, we can say that in a single run of the algorithm, we can obtain an approximation to the optimal Pareto front of problem (1)–(8). This is particularly relevant from the computational point of view, because it does not imply solving a single-objective problem several times. Thus, it is a multi-objective metaheuristic [29].

3. EXPERIMENTS AND RESULTS

We tested the methodology proposed in this work, using a real case corresponding to the city of Puerto Montt, Chile. The city has a public transportation system comprising 20 bus lines, each one having forward and backward itineraries, with headways (inverse of frequency) in the range [1.5, 16.1] minutes. All the buses of the transit system have the same capacity, which is consistent with the hypothesis of homogenous fleet of our model. Data were provided by SECTRA, the transportation planning agency of Chile. The underlying graph G comprises 733 nodes and 1662 arcs, including 70 zone centroids and corresponding access (walk) links. Figure 4 shows the bus network of the city. We consider two scenarios corresponding to peak and off-peak conditions, where the first one comprises 3780 OD pairs and 8595 trips per hour while the second one comprises 4335 OD pairs and 3802 trips per hour. As we can expect, the peak scenario entails more trips, concentrated in fewer OD pairs. Although we use two time horizons in the numerical experiments, our proposed methodology is not conceived to deal with a multi-period scenario. Note that issues like transitions between periods belong to the operational plan, while our context is the strategic and tactical planning.

According to the aims of our proposed methodology, we performed a set of validating experiments using the real case study of Puerto Montt. We configured the predetermined set of frequencies as $\Theta = \{1/20, 1/12, 1/6, 1/4, 1/3\}$ (values expressed in 1/minute), which are representative values with respect to the current frequencies of the real system. Notice that there are many different values of frequencies in the current solution, so we restricted the predetermined set to a manageable size, because the execution time of the Tabu Search algorithm grows in direct proportion to this size. Figure 5 plots the objective values (total travel time and fleet size) of the obtained results for the peak and off-peak scenarios, according to the following references:

Figure 4. The bus network of Puerto Montt.

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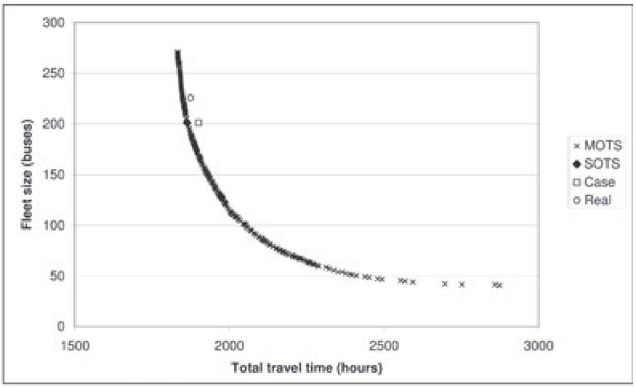
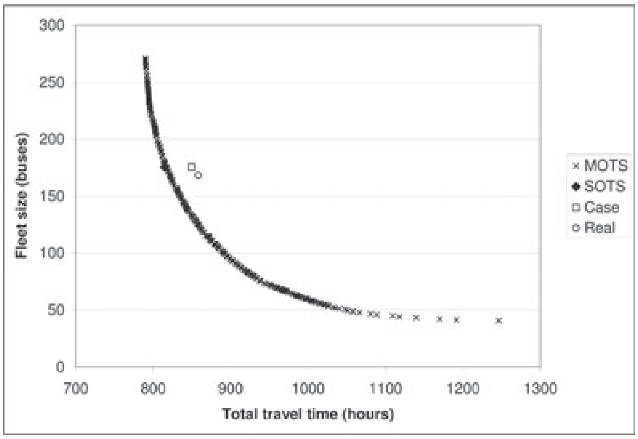


(a) morning peak



(b) off-peak

Figure 5. Objective values of current and optimized solutions. (a) morning peak; (b) off-peak.

- Real: current solution of the city, evaluated using exactly its frequencies.
- Case: current solution, evaluated using the frequencies of Θ that are more similar to the real ones.
- SOTS: solution of the short-term memory Tabu Search for the single-objective problem, proposed and implemented in [9].
- MOTS: solutions of the multi-objective Tabu Search proposed and implemented in this work.

Note that solutions of MOTS are intended to provide a picture of the different trade-off levels between objectives of users and operators, while SOTS solutions seek to improve the current system (in terms of total travel time) for a fixed value of fleet size. In the single-objective approach [9], the independent (decision) variables are the frequencies, while the fleet size can be considered as a factor that has to be varied in order to obtain different (optimized) values of travel time. But in the multi-objective approach, both fleet size and travel time are outputs of the model, both depending on frequencies. Concerning the obtained non-dominated sets, we can observe that the slopes of both curves are similar for high values of fleet size. This means that an increase in the fleet size (which represents an increase in the resources delivered to the transit system) does not have a strong impact in the level of service represented by the total travel time. For medium and low values of fleet size, the impact of an increase has a stronger positive impact on travel time. Note that in this analysis (and in the whole study), we are considering inelastic demand with respect to the frequencies. Moreover, we are not taking into account the effects of the bus capacity neither on the passenger behavior nor in the computation of the system performance.

Concerning the performance of the methodologies, the first observation is that they produce solutions that dominate the current solution of the system. Moreover, we can observe that the SOTS solution lies over the Pareto front of MOTS; this is an expectable result because the multi-objective

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variant of the algorithm is based on the single-objective one (Section 2.4). Table II reports the percentage of improvement in total travel time for a fixed fleet size (column 3), with respect to the current solution (the Case solution, evaluated using the same discrete domain of frequencies used in the optimization methods). Column 4 shows improvements with respect to the other objective, that is, the fleet size; note the comparison in this column is against a different solution of the Pareto front, the one which has the same total travel time as the Case solution. Also, we report the number of solutions found by each algorithm as well as their execution times in seconds (in a Core i5 computer). Regarding improvements in total travel time, we observe small gains, which are consistent with values reported in the literature for this problem [8, 9]; however, the reductions in fleet size for a fixed total travel time are significant. In the peak period, the fleet is reduced from 201 to 169 buses, while in the off-peak period, the fleet is reduced from 176 to 133 buses, almost one quarter of resources. Note that Table II states that the comparison in terms of travel time reduction is valid for both approaches (single and multi-objective); on the other hand, comparison in terms of fleet size is only valid for the multi-objective approach, because the single-objective one only obtains a single solution for a fixed fleet size.

In terms of computational efficiency, a direct comparison of execution times shows a reduction of 70% for the algorithm proposed in this work, with respect to its original variant. Note that this comparison is made between different algorithms whose implementations are based on the same assignment sub-model, the operation that mostly determines the overall execution time; the same approach for comparison has been used in previous studies [27]. Furthermore, given that a single execution of MOTS produces an entire Pareto front that includes the solution delivered by SOTS (as it can be seen in Figure 5), the relative efficiency of the multi-objective metaheuristic is much greater. Note that if we want to obtain different trade-off solutions using the SOTS algorithm, we have to perform different runs, for different values of maximum fleet size. Moreover, given that the SOTS algorithm has proven to be capable of finding solutions very close to the global optimum for the single-objective problem [9], we may expect that solutions found for Puerto Montt are also close to the optimal ones. The exact model proposed in [9] cannot be used to compute optimal values for Puerto Montt, because of the size of the corresponding model. Moreover, in order to explore the variation of the execution time of the MOTS algorithm with respect to problem size, we ran several iterations of the algorithm with the case of Montevideo, main city of Uruguay. This case was previously used in [9]; it comprises 133 bus lines with an underlying network of 4945 nodes, 14 672 arcs, and 7425 OD pairs (Puerto Montt has 733 nodes, 1662 arcs, and around 4000 OD pairs). We observed an increase factor in the order of several hundred times. This means that because the execution of MOTS for Puerto Montt takes about 20 minutes (Table II), a complete execution for Montevideo could take more than 1 day. This is an expectable result, because all algorithms have polynomial (super linear) order of execution time in terms of problem size (underlying graph and OD matrix). Nevertheless, in the context of tactical and strategic planning, and taking into account that the result is a full Pareto front of different trade-off solutions, it can be considered as a manageable execution time.

Table III shows the headways (inverse or frequencies, values taken from set Θ) of the current solution (the Case one, as explained previously) and optimized solutions for peak and off-peak scenarios. The optimized solutions are taken from the Pareto front produced by MOTS in the following way: opt (tt) is the solution that optimizes total travel time for the fleet size equal to the current solution, while opt (fs) is the one that optimizes fleet size for the total travel time equal to the current

Table II. Performance of the methodologies.

Scenario	Solution method	Per cent reduction in total travel time	Per cent reduction in fleet size	No. of solutions	Time (seconds)
Peak	SOTS	1.9	NA	1	4223
	MOTS	1.9	16.1*	241	1235
Off-peak	SOTS	4.1	NA	1	4565
	MOTS	4.1	24.0*	266	1227

NA, not applicable.
*We compare versus the MOTS solution with the same total travel time as the Case solution.

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Table III. Current and optimized headways in minutes.

Line		1		2		3		4		5		6		7		8		9		10	
Peak	current	4	3	3	3	4	4	3	4	3	3	4	4	20	4	12	4	4	4	4	3
	opt(tt)	6	4	4	3	3	3	4	4	4	3	3	3	20	3	6	3	6	6	3	3
	opt(fs)	3	3	3	3	3	3	4	6	4	3	4	4	20	4	6	4	6	6	4	3
Off-peak	current	4	6	6	3	4	4	3	4	4	3	4	4	20	4	20	4	4	4	20	4
	opt(tt)	12	20	12	12	3	3	12	12	6	3	4	4	12	6	12	12	4	4	3	3
	opt(fs)	6	6	6	12	12	12	6	12	4	4	4	4	6	6	6	6	12	12	4	3

Line		11		12		13		14		15		16		17		18		19		20	
Peak	current	4	6	4	20	4	4	4	4	3	4	3	3	6	20	4	4	4	4	4	4
	opt(tt)	3	4	3	12	3	3	4	4	6	3	3	3	3	4	20	6	6	12	12	20
	opt(fs)	3	4	3	12	4	4	4	4	12	4	4	4	3	6	20	6	6	12	12	20
Off-peak	current	4	4	6	20	4	4	4	4	12	4	3	6	12	20	4	4	4	4	4	4
	opt(tt)	3	6	4	12	3	3	3	3	6	12	3	3	4	12	4	4	4	4	4	4
	opt(fs)	4	6	6	12	6	6	4	4	6	6	4	4	4	4	12	6	6	6	6	12

solution. We reported values for both forward and backward itineraries for each one of the 20 lines, in order to be consistent with the current solution, which has this characteristic. We can observe that in general terms, the changes suggested by the optimization method are consistent between both scenarios, but the specific suggested (optimized) values are different. Particularly, the optimized solutions for the off-peak scenario exhibit more changes of headways with respect to the current solution, because there is more room for improvement in such scenario, as can be seen on Figure 5. In all cases, redistribution of resources among bus lines are observed. These results also contribute to validate the proposed methodology as well it shows the relevance of performing the study for different demand scenarios in the context of tactical and strategic planning. The differences between the optimized frequencies of peak and off-peak are explained by the different demand patterns of the respective periods, which exhibit trips between different origins and destinations. Note that because our model assumes sufficient capacity in the buses, peak flows do not influence the values of the resulting frequencies. Finally, it is worth noting that if we fix the fleet size, because the model optimizes total travel time over the whole set of OD pairs, the optimized solution redistributes the available resources. Thus, for some passengers, the service may improve while for some others the service may worsen. As it is currently implemented, our formulation favors travel time reductions on high-demand OD pairs. In transit frequency optimization, an upper limit on travel time increase can be implemented at the level of set of services or at the level of each line. Note that because the assignment sub-model adopted in our work takes into account the effect of combined frequencies in the users' waiting time (the common lines problem), a constraint of maximum allowable waiting time could be included for each OD pair at each bus stop. This is not included in our model, instead we guarantee maximum waiting time by imposing minimum allowable frequencies on individual routes. This constraint is implicitly enforced by the discrete set of possible frequency values.

To close the experimental section, we note that data for transit frequency optimization models are usually subject to variations and errors, particularly OD matrices. Therefore, methodologies for optimizing frequencies should be robust under these conditions. In this context, robustness refers to the ability of the solution method to provide similar results under changes in the input data. Taking into account this observation, we performed a sensitivity analysis by perturbing the values of the OD matrix and observing the results. Particularly, we perturbed each OD value v by choosing a random value in the range $[v(1-p), v(1+p)]$ where p is a parameter in the range [0,1]. We set $p = 0.2$ meaning that 20% of random increase or decrease in trip rates is assumed. Then, we generated two random OD matrices that are given as input to the solution algorithm, obtaining Pareto fronts S1 and S2, respectively. We perform the analysis in terms of both objective and decision space. In the first case, we plot objective values of S1 and S2 as well as the values generated with the original matrix for both morning-peak and off-peak scenarios, labeled as MOTS (Figure 6). We can observe that objective values are very similar in general terms. In the peak scenario, variations exhibit a higher magnitude because they occur over higher OD values. For the analysis in terms of decision space (i.e. frequency

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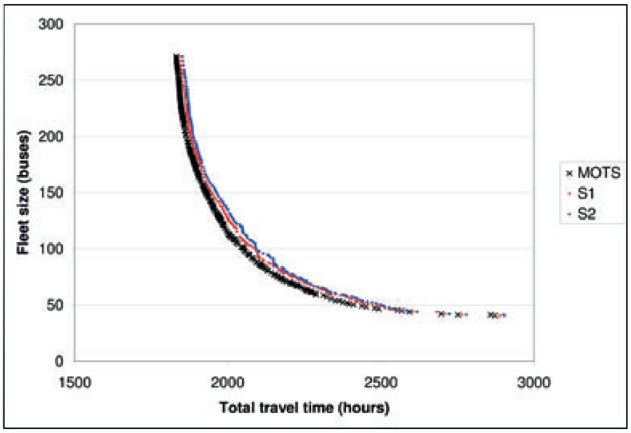
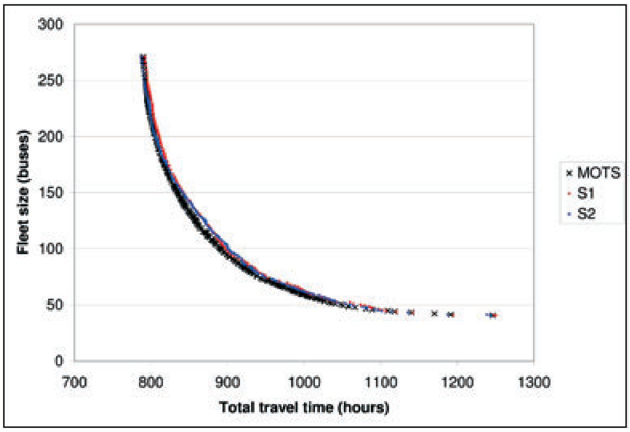


(a) morning peak



(b) off-peak

Figure 6. Sensitivity in objective space under variations in demand data. (a) morning peak; (b) off-peak.

values), we select a single solution from each Pareto front (MOTS, S1, and S2) and we count the number of lines in which their frequencies change to values at different distances in the set Θ. The distance between two frequency values is the difference between their positions in the set Θ. Note that the possible values of distances for a set Θ composed by five different frequencies can be 0, 1, 2, 3, or 4. As an example, for the peak scenario, we consider solutions having a fleet size equal to the current system, that is, 201 buses. Then, line 1 of Table IV states that the solution of S1 having 201 buses differs from the one of MOTS having 201 buses in the following way: 12 lines have the same frequency, 15 differ one step in Θ (contiguous values of frequencies), 13 two steps, and there are no higher differences. The total weighted counts represent the number of changes in frequencies (steps) from one solution to another. As we can see, in the peak scenario, the total number of changes (41 for S1 and 42 for S2) is very similar to the total number of lines (40), meaning that there is in average one change of frequency to a contiguous value in Θ per line when the demand is perturbed. In general terms, this can be considered as an indicator of robustness, because variations of 20% in demand OD

Table IV. Sensitivity in decision space under variations in demand data.

	Steps	0	1	2	3	4	Total (weighted)
Peak	S1	12	15	13	0	0	41
	S2	13	14	11	2	0	42
Off-peak	S1	15	13	8	4	0	41
	S2	14	8	12	6	0	50

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values cause (in average) one change of frequency at each line, to a contiguous value in Θ. This situation changes for the off-peak scenario, where for the fleet size level of 176 buses, the total number of changes for S2 is 50, which can be explained by the (already identified) higher opportunities for improving the system.

4. CONCLUSIONS

In this work, we have proposed a new method for optimizing transit frequencies. We considered a multi-objective variant of an existing model, and we have extended its solution method. Both model and algorithm proposed in this work take advantage of the multi-objective nature of the problem to propose a computationally efficient solving method that produces in a single run of the algorithm, a set of different frequency configurations representing different trade-off levels between the interests of users and operators. Although the proposed algorithm is heuristic, its accuracy is inferred from previous results that show its ability of producing near-optimal solutions.

The proposed method solves a case corresponding to a real system in acceptable execution time. Special effort has been made in solving the multi-objective variant, which entails finding an entire set of non-dominated solutions. The computational efficiency is very relevant in a case such as Puerto Montt, in which there are both a dense network and a dense OD matrix. In this case study, the proposed method improves current solutions in terms of both total travel time and fleet size. The optimized frequencies are not the same for the peak and off-peak scenarios, which show the relevance of performing the optimization for each scenario separately. The approach is mainly conceived for tactical and strategic planning, where usually a less detailed modeling is performed when compared with operational planning. This enables for applying the methodology to large problem instances in an efficient manner. That is the case of the test scenario used in this work, which fulfils the hypothesis of the model. Once the proposed multi-objective methodology is tested, it could be extended and adapted in further research in order to include more detailed modeling that captures more operational aspects of transit systems.

An interesting future study would be the application of transit network optimization methods to the case of Puerto Montt. Reported travel time improvements for medium-sized cities reach 23% [35], which is much higher than the improvements obtained in this work. Nevertheless, we have to mind that transit network optimization methods are more complex and their results usually are harder to implement in practice, given the potential disruptions they may cause in users because of system usability concerns.

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REFERENCES

1. Desaulniers G, Hickman MD. Public transit. In *Transportation*(Eds) Laporte G, Barnhart C. Elsevier: North-Holland, 2008 69–127.
2. Ibarra-Rojas O, Delgado F, Giesen RMuñoz JC. Planning, operation, and control of bus transport systems: A literature review. *Transportation Research Part B* 2015; **77**: 38–75. https://doi.org/10.1016/j.trb.2015.03.002.
3. Hurdle VF. Minimum cost schedules for a public transportation route – I. Theory. *Transportation Science* 1973; **7**(2): 109–137. https://doi.org/10.1287/trsc.7.2.109.
4. Kocur G, Hendrickson C. Design of local bus service with demand equilibration. *Transportation Science* 1982; **16**(2): 149–170. https://doi.org/10.1287/trsc.16.2.149.
5. Jansson JO. A simple bus line model for optimization of service frequency and bus size. *Journal of Transport Economics and Policy* 1980; **14**(1): 53–80.

6. Newell GF. Dispatching policies for a transportation route. *Transportation Science* 1971; **5**(1): 91–105. https://doi.org/10.1287/trsc.5.1.91.
7. Schéele S. A supply model for public transit services. *Transportation Research Part B* 1981; **14**: 133–146. https://doi.org/10.1016/0191-2615(80)90039-9.
8. Constantin I, Florian M. Optimizing frequencies in a transit network: a nonlinear bi-level programming approach. *International Transactions in Operational Research* 1995; **2**(2): 149–164. https://doi.org/10.1016/0969-6016(94)00023-M.
9. Martínez H, Mauttone AUrquhart ME. Frequency optimization in public transportation systems: formulation and metaheuristic approach. *European Journal of Operational Research* 2014; **236**(1): 27–36. https://doi.org/10.1016/j.ejor.2013.11.007.
10. Spiess H, Florian M. Optimal strategies: a new assignment model for transit networks. *Transportation Research Part B* 1989; **23**(2): 83–102. https://doi.org/10.1016/0191-2615(89)90034-9.
11. Dell'Olio L, Ibeas ARuisánchez F. Optimizing bus-size and headway in transit networks. *Transportation* 2012; **39**: 449–464. https://doi.org/10.1007/s11116-011-9332-2.
12. Ruisánchez F, Dell'Olio LIbeas A. Design of a Tabu Search algorithm for assigning optimal bus sizes and frequencies in urban transport services. *Journal of Advanced Transportation* 2012; **46**: 366–377. https://doi.org/10.1002/atr.1195.
13. Herbon A, Hadas Y. Determining optimal frequency and vehicle capacity for public transit routes: a generalized newsvendor model. *Transportation Research Part B: Methodological* 2015; **71**: 85–99. https://doi.org/10.1016/j.trb.2014.10.007.
14. Chien S, Tsai C. Optimization of fare structure and service frequency for maximum profitability of transit systems. *Transportation Planning and Technology* 2007; **30**(5): 477–500. https://doi.org/10.1080/03081060701599961.
15. Ceder A. Bus frequency determination using passenger count data. *Transportation Research Part A: General* 1984; **18**(5): 439–453. https://doi.org/10.1016/0191-2607(84)90019-0.
16. Hadas Y, Shnaiderman M. Public-transit frequency setting using minimum-cost approach with stochastic demand and travel time. *Transportation Research Part B: Methodological* 2012; **46**(8): 1068–1084. https://doi.org/10.1016/j.trb.2012.02.010.
17. Ferguson E, Duthie J, Unnikrishnan AWaller T. Incorporating equity into the transit frequency-setting problem. *Transportation Research Part A: Policy and Practice* 2012; **46**(1): 190–199. https://doi.org/10.1016/j.tra.2011.06.002.
18. Verbas IÖ, Frei C, Mahmassani HSChan R. Stretching resources: sensitivity of optimal bus frequency allocation to stop-level demand elasticities. *Public Transport* 2015; **7**(1): 1–20. https://doi.org/10.1007/s12469-013-0084-6.
19. Verbas IÖ, Mahmassani HS. Integrated frequency allocation and user assignment in multi-modal transit networks: methodology and application to large-scale urban systems. *Transportation Research Record: Journal of the Transportation Research Board* 2015; **2498**: 37–45. https://doi.org/10.3141/2498-05.
20. Ulusoy Y, Chien S. Optimal bus service patterns and frequencies considering transfer demand elasticity with genetic algorithm. *Transportation Planning and Technology* 2015; **38**(4): 409–424. https://doi.org/10.1080/03081060.2015.1026101.
21. Verbas Ö, Mahmassani H. Exploring trade-offs in frequency allocation in a transit network using bus route patterns: methodology and application to large-scale urban systems. *Transportation Research Part B: Methodological* 2015; **81**(2): 577–595. https://doi.org/10.1016/j.trb.2015.06.018.
22. Han AF, Wilson NM. The allocation of buses in heavily utilized networks with overlapping routes. *Transportation Research Part B* 1982; **13**(3): 221–232. https://doi.org/10.1016/0191-2615(82)90025-X.
23. Gao Z, Sun HShan LL. A continuous equilibrium network design model and algorithm for transit systems. *Transportation Research Part B* 2004; **38**(3): 235–250. https://doi.org/10.1016/S0191-2615(03)00011-0.
24. Yu B, Yang ZYao J. Genetic algorithm for bus frequency optimization. *Journal of Transportation Engineering* 2010; **136**(6): 576–583.
25. Ehrgott M. *Multicriteria Optimization*, Springer: Berlin Heidelberg, 2005.
26. Israeli Y, Ceder A. Transit route design using scheduling and multiobjective programming techniques. In *Proceedings of the Sixth International Workshop on Computer-Aided Scheduling of Public Transport* (Eds) Daduna J, Branco I, Paixão J. Lecture Notes in Economics and Mathematical Systems: Berlin Heidelberg, 1995 56–75.
27. Mauttone A, Urquhart M. A multi-objective metaheuristic approach for the transit network design problem. *Public Transport* 2009; **1**(4): 253–273. https://doi.org/10.1007/s12469-010-0016-7.
28. Giesen A, Martínez H, Mauttone A, Urquhart ME. Multi-objective transit frequency optimization: solution method and its application to a medium-sized city. Presented at the 94th Annual Meeting of the Transportation Research Board, 2015.
29. Ehrgott M, Gandibleux X. Multiobjective combinatorial optimization. In *Multiple Criteria Optimization: State of the Art Annotated Bibliographic Surveys*(Eds) Ehrgott M, Gandibleux X. Springer: US, 2002 369–444.
30. Baaj MH, Mahmassani H. An AI-based approach for transit route system planning and design. *Journal of Advanced Transportation* 1991; **25**(2): 187–210. https://doi.org/10.1002/atr.5670250205.
31. Chriqui C, Robillard P. Common bus lines. *Transportation Science* 1975; **9**(2): 115–121. https://doi.org/10.1287/trsc.9.2.115.
32. Glover F. Tabu Search – Part I. *ORSA Journal on Computing* 1989; **1**(3): 190–206. https://doi.org/10.1287/ijoc.1.3.190.
33. Gendreau M, Potvin J-Y. Tabu Search. In *Handbook of Metaheuristics*(Eds) Gendreau M, Potvin J-Y2010 41–59.
34. Glover F, Laguna M. *Tabu Search*, Springer: US, 1998.
35. Szeto W, Wu Y. A simultaneous bus route design and frequency setting problem for Tin Shui Wai, Hong Kong. *European Journal of Operational Research* 2011; **209**(2): 141–155. https://doi.org/10.1016/j.ejor.2010.08.020.