Some Applications of Thermodynamic Formalism to Numerical Systems

By
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Introduction

In this thesis, we will be interested in the dynamics and questions related to the algorithms of different numerical systems. We understand a numerical system as a way of writing a real number x. We can find, associated with a numerical system, a set of *digits* necessary to construct that writing. For example, our decimal system daily used has $\{0, 1, \ldots, 9\}$ as the set of digits. In the algorithm of the expansion we can find associated to it, a map T (usually defined on the interval $[0, 1]$). The dynamics of T will allows us to recover the writing of some x as well as the set of digits.

On the other hand, there are examples of properties in number theory which holds for a full Lebesgue measure set, hence the set of numbers that do not satisfy such property have zero Lebesgue measure. We will be interested in describing the size of particular sets which are number-theoretically defined, where size means the Hasudorff dimension. The ergodic properties of T will be useful to our purposes by means of thermodynamic formalism.

The structure of this thesis is the following. In Chapter 1, we will review all the numeric expansions used along the whole text: continued fractions and generalizations, base-b and Cantor expansions, and finally, Lüroth and Q- Lüroth expansions. We take advantage of any expansion introduced to study dynamical and ergodic properties of the associated transformation. As an example, for any $k > 0$, the dynamics of the interval map defined by

$$T_k(x) = \frac{k(1-x)}{x} - \left[\frac{k(1-x)}{x}\right]$$

allows to construct expansions of the form

$$[a_1, a_2, \ldots,]_k := \cfrac{k}{a_1 + k + \cfrac{k}{a_2 + k + \cfrac{k}{\ddots}}}.$$

called k-continued fractions. Note that, when $k = 1$ we recover the Gauss map. The T_k maps were previously considered in [HM], where the authors studied a more general class of Möbius transformations. Also, a linear version of the map T_k is contemplated in Chapter 1. The k-Lüroth maps $L_k : [0,1] \to [0,1]$ defined as

$$L_k(x) := \begin{cases} x\frac{(n+k)(n+k+1)}{k} - (n+k), & \text{if } x \in \left[\frac{k}{n+k+1}, \frac{k}{n+k}\right), n \in \mathbb{N}_0 \\ 0, & \text{if } x = 0 \end{cases}$$

guarantees the expansion of suitable $x \in [0,1]$ in the form

$$x = \sum_{n=1}^{\infty} \frac{k^n}{(a_1 + k)(a_1 - 1 + k) \cdots (a_{n-1} + k)(a_{n-1} - 1 + k)(a_n + k)}.$$

In general, note that a numerical expansion is essentially defined by the common arithmetic structure (which is the same for all x). Moreover, the sequence of digits and their positions added to the structure of the expansion, characterizes (up to a countable set) the expansion of any real number. From a dynamical point of view, that last means that both maps T_k and L_k are modeled symbolically by a fullshift on countable symbols. The fullshift with alphabet $A \subseteq \mathbb{N}_0$ is defined by the pair (Σ_A, σ) where

$$\Sigma_A := \{(x_n)_{n \in \mathbb{N}_0} : x_n \in A\}$$

and $\sigma : \Sigma_A \to \Sigma_A$ is the shift-map $\sigma(x_0, x_1, x_3, \ldots) := (x_1, x_2, \ldots)$. The elements of A

are called the symbols. We have that this space endowed with the topology generated by *cylinders* is a non-compact space. That will be one of the main difficulties on the thesis, since ergodic theory on non-compact spaces is more subtle than the compact case.

In Chapter 2 we collect main properties of thermodynamic formalism, a branch of ergodic theory which has been vastly studied during the last fifty years. It allows, among other things, to choose a remarkable kind of invariant measures. We will follow mainly [Sar1, MU2, Wal]. One of the fundamental objects in this context is the *pressure map* defined as follows. If $\varphi : \Sigma_{\mathbb{N}_0} \to \mathbb{R}$ satisfies some regularity assumptions, then the *pressure of* φ is

$$P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n x = x} \exp\left(\sum_{i=0}^{n-1} \varphi(\sigma^i x)\right)$$

when the limit exists. Pressure function and its properties can be applied to dimension theory. For instance, and as we will see in Chapter 3, it allows to describe the Lyapunov spectra for the T_k maps. In other words, we show that the function

$$\alpha \mapsto \dim_H \left\{ x \in [0,1] : \lim_{n\to\infty} \frac{1}{n} \log |(T_k^{(n)})'(x)| = \alpha \right\}$$

has a domain of the form $[\alpha_{\min}, \infty)$, and moreover, it is real analytic there (see Theorem 3.2.4). It is important to note that the results obtained in Chapter 3 are highly supported by those from [Iom], where the author studied the Lyapunov spectra for a larger class of interval maps.

In the case of Lüroth maps L_k, the Lyapunov spectra is also analytic in some interval of the form $[\alpha^k_{min}, \infty)$. Note that we have a family of real analytic curves indexed on $k > 0$. In Chapter 4, we will interested in the behavior of this family of curves when the dynamics is perturbated. In fact, we show the following theorem.

Theorem (Theorem 4.3.2)**.** *Let $M > 0$ and fix $\alpha > 0$ such that the Lyapunov spectra for L_k*

is defined on α, for all $k \in (0, M]$. Then, the function

$$\begin{aligned} (0, M] &\to \mathbb{R} \\ k &\mapsto \dim_H \left\{ x \in [0,1] : \lim_{n\to\infty} \frac{1}{n} \log |(T_k^{(n)})'(x)| = \alpha \right\} \end{aligned}$$

is real analytic.

From the dynamical systems point of view, this theorem describes how does the multifractal spectrum of Lyapunov exponents varies along a one parameter family of dynamical systems. On the other hand, we show in fact that Lyapunov exponents are measuring the speed of approximation of the partial sums involved in the Lüroth expansion. Therefore, this theorem characterizes how does the size of the set of points with same speed of approximations by their n-approximants varies in the different numerical systems provided by the k-Lüroth.

Further in Chapter 5, we will study normality in a new numeric expansion inspired in Cantor series [Can]. Let $Q = \{q_n\}_{n\geq 1}$ be a sequence of positive real numbers $q_n > 0$. Consider the family of Lüroth maps $L_Q := \{L_{q_n}\}_{n\geq 1}$. We define the non-autonomous dynamical system generated by the sequence. In other words, the orbits of some $x \in [0,1]$ are given by $L_Q^n(x) := L_{q_n} \circ L_{q_{n-1}} \circ \cdots \circ L_{q_1}(x)$ for $n \geq 1$. Again, for suitable x, we have the expansion (see Section 1.7 for further details)

$$x = \sum_{n=1}^{\infty} \frac{q_1 q_2 \cdots q_n}{(a_1 - 1 + q_1)(a_1 + q_1) \cdots (a_n - 1 + q_n)}$$

for some unique sequence of positive integers $\{a_n\}_{n\geq 1}$. We say that $x \in [0,1]$ is *normal* with respect to the Q-Lüroth expansion, if for every $a \in \mathbb{N}$,

$$\lim_{n\to\infty} \frac{\#\{1 \leq i \leq n : a_i(x) = a\}}{\sum_{i=1}^{n} |I_a^{q_n}|} = 1$$

where $I_n^k := \left[\frac{k}{n+k+1}, \frac{k}{n+k}\right)$. We prove the following theorem which is an analogous to the *Borel's theorem on normal numbers* (see Section 1.1).

Theorem (Theorem 5.3.1)**.** *Let $Q = \{q_n\}_{n\geq 1}$ be a sequence of positive real numbers. Then, Lebesgue almost every real number in $[0,1]$ is normal with respect to the Q-Lüroth expansion if and only if for all $a \geq 1$, the series $\sum_n |I_a^{q_n}|$ is divergent.*

The main difference with the autonomous case is that there is no similar result to Birkhoff ergodic theorem for non-autonomous dynamical systems. We follow the probabilistic techniques used in [Man2, Rén, Rév] where the authors reached comparable results in the context of Cantor series. In addition, using tools from thermodynamic formalism in the setting of non-autonomous dynamics [RGU], we prove the following theorem.

Theorem (Theorem 5.1.2)**.** *The set of non-normal numbers in the Q-Lüroth expansion has Hausdorff dimension equal to one.*

Finally in Chapter 6, we will deal with part of fractal analysis of the derivative of conjugacies between any two maps T_{k_1}, T_{k_2} for $k_1, k_2 > 0$. Let us explain about this. Recall that for every $k > 0$, the map T_k is topologically conjugated to the full-shift on countable symbols. Denote by π_k this conjugacy. Note that π_k acts sending any coding $(x_n)_{n\geq 1} \in \Sigma_{\mathbb{N}_0}$ to the k-continued fraction expansion $[x_1, x_2, \ldots]_k$. Then, given two positive numbers k_1, k_2, we can construct a function $\pi_{k_1,k_2} : [0,1] \to [0,1]$ defined by $\pi_{k_1} \circ \pi_{k_2}^{-1}$. Note that π_{k_1,k_2} sends any continued fraction of the form $[x_1, x_2, \ldots]_{k_2}$ to $[x_1, x_2, \ldots]_{k_1}$. We will be interested in the derivative of π_{k_1,k_2}. In particular, we will prove that it is a singular function, which means that, π_{k_1,k_2} is non-constant and $\pi'_{k_1,k_2}(x) = 0$ holds Lebesgue a.e. in $[0,1]$. So, from a dimension theory point of view, the following problem can be posed: finding the Hausdorff dimension of the sets

$$\mathcal{D}_\infty := \{x \in [0,1] : \pi'_{k_1,k_2}(x) = \infty\}$$

and

$$\mathcal{D}_{\sim} := \{x \in [0,1] : \pi'_{k_1,k_2}(x) \text{ does not exists}\}.$$

In the literature, we can find similar questions in different contexts. For example, in [KS1, Mun] the authors studied the Hausdorff dimension of those sets for the Minkowski's question mark function, which is the conjugation between the Farey map and the Tent map. See also [JMS], where were considered for conjugacies of maps that converge pointwise to some map on the interval. In all of these articles mentioned, thermodynamic formalism tools has been used. In our case, we obtain the following theorem.

Theorem. *Let k_1, k_2 be two positive numbers. Then the sets $\mathcal{D}_\infty, \mathcal{D}_\sim$ defined as above have the following Hausdorff dimensions:*

$$1/2 < \dim_H(\mathcal{D}_\infty) = \dim_H(\mathcal{D}_\sim) = \delta_0 < 1$$

where

$$\delta_0 := \sup\{\delta \in (1/2, 1] : \textit{ for all } q \in \mathbb{R}, P(q\psi - \delta \log |T'_{k_2}|) > 0\}.$$

Here $P(\cdot)$ denotes the pressure function associated to T_{k_2}.

The following articles are part of this thesis and correspond to Chapters 4,5,6, respectively.

- **Erik Contreras**. k-rational approximations in k-Lüroth expansions. *Far East Journal of Mathematical Sciences (FJMS)*, 128(1):67–87, dec 2020.
- **Erik Contreras**. Normal numbers in Q-Lüroth expansions. *Submitted*, 2020.
- **Erik Contreras** and **Thomas Jordan**. The Dimension of non-differentiability points of conjugacies between Gauss-like maps. *Work in progress*, 2021.

Chapter 1

Classical Numeric Expansions

A numeration system encompasses a way of representing a number. The base 10 *numeric expansion is the most widely used numeration system and it has several advantages that, aside form the fact that we have* 10 *fingers, has established it as the standard representation. It is simple to perform arithmetic operations in base* 10 *as we learn from an early age. There are simple analogues of this system in which the base* 10 *is replaced by other positive integer* b*. Another well known numeration system, based on the Euclidean algorithm, is that of continued fractions. This system has the advantage that several arithmetic properties, such rationality or irrationality, are readily seen in the expansion. It also provides, in a simple fashion, the best rational approximations of an irrational number. However, performing simple arithmetic operations is rather difficult. While these are the best known numerical systems there exists a wide range of other systems with particular features. In this chapter we will not only survey base* b *and continued fraction expansion, but we will also study Lüroth and Cantor expansions. Moreover, central to the study developed in this thesis will be generalizations of these systems such as the* k*-continued fractions and* *the Q-Lüroth expansions. The later one, introduced by the author in [Con1].*

A common feature to all of these numeration systems is that it is possible to associate them a (maybe non-autonomous) dynamical system. Iterations of the system provide a way to obtain the representation. More interestingly, the whole theory of dynamical systems can be used to describe in detail the properties of each numeration system. A classical example along these lines is the simple proof of the Borel Theorem of Normal Numbers provided by Riesz as a direct application of the Ergodic Theorem.

In this chapter we present different numeration systems and the dynamical system associated to the them. This will pave the way to a deeper study of the arithmetic properties of the numbers and their representations.

1.1 Base-b expansions

Let $b \geq 2$ be an integer. Every real number x can be written in base-b as a the series

$$x = \sum_{n=1}^{\infty} \frac{\epsilon_n(x)}{b^n}$$

where, for all $n \geq 1, \epsilon_n(x) \in \{0, 1, \ldots, b-1\}$. We call $\epsilon_n(x)$ the *digits* of the expansion of x in the base-b expansion. A very well known fact is that the base-b expansion is closely related to the dynamics of the map $T_b : [0,1] \to [0,1]$, defined by $T_b(x) := bx - [bx] = \{bx\}$, where $[\cdot]$ denotes the *integer part* of a number. Observe that digits in the expansion of a number x can be obtained by the formula $\epsilon_n(x) = [bT_b^{n-1}x]$.

There is a wide range of interesting questions that the relation with dynamical systems suggest. We will begin with that of the frequency of the digits. Given a set A, we denote its cardinality by $\#A$.

Definition 1.1.1. Let $b \geq 2$ be an integer and $x \in \mathbb{R}$. Given $d \in \{0, 1, \ldots, b-1\}$, we call the frequency of appearance of the digit d in the base-b expansion of x, to

$$f_b(x,d) := \lim_{n\to\infty} \frac{1}{n} \# \left\{1 \leq i \leq n, \epsilon_i(x) = d\right\},$$

whenever the limit exists.

From the formula $\epsilon_n(x) = [bT_b^{n-1}x]$, we have that the frequency of appearance of d can be written as the following average

$$f_b(x,d) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_{I_d}(T_b^i x),$$

where $\mathbb{1}_A$ denotes the characteristic function of some set A and $I_d = [d/b, (d+1)/b)$. Thus, the frequency can be seen as an average of the characteristic function $\mathbb{1}_{I_d}$ along the orbit of

x under the map T_b. Such expressions are called *Birkhoff averages* and actually are one of the main objects in Ergodic Theory. In 1931, G. Birkhoff proved one of the most important results of this theory that now bear his name. The Birkhoff ergodic theorem proves that, under an ergodicity assumption, the time averages coincide with the space average of the system. It will be useful to understand the behavior of the function $f_b(x,d)$. Before to state the theorem, we recall some definitions. Let $(X, \mathcal{B}, \mu)$ be a probability space.

Definition 1.1.2. A map $T : X \to X$ is called *measure preserving* if T is measurable and $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathcal{B}$. When this occurs, μ is called a T-invariant measure.

As an example, for any positive integer b, the map T_b preserves the Lebesgue measure. From now on, $|A|$ will denote the Lebesgue measure of a Borel subset of $\mathbb{R}$. In fact, if $(c,d] \subset [0,1]$, then

$$T^{-1}(c,d] = \bigcup_{i=0}^{b-1} \left(\frac{c-i}{b}, \frac{d-i}{b} \right]$$

and then we have that $|T^{-1}(c,d]| = |(c,d]|$.

Definition 1.1.3. Let $T : X \to X$ be a measure-preserving transformation with respect to μ. We call T ergodic if for any $B \in \mathcal{B}$ such that $T^{-1}B = B$ then we have that $\mu(A) = 0$ or $\mu(B) = 1$.

The map T_b is ergodic with respect to the Lebesgue measure (see [Wal, p. 30]).

Theorem 1.1.1 (Birkhoff, 1931)**.** *If $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is an ergodic map and f is μ-integrable, then for μ-almost every $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) = \int_X f d\mu.$$

We have now the following corollary regarding the frequency of appearance of a digit d in base-b expansions.

Corollary 1.1.2. *For any digit $d \in \{0, 1, \dots, b-1\}$, we have that*

$$f_b(x, d) = \lim_{n\to\infty} \frac{1}{n} \#\{1 \le i \le n, \epsilon_i(x) = d\} = \frac{1}{b}$$

for almost every x with respect to the Lebesgue measure.

Proof. This is a direct consequence of Birkhoff's ergodic theorem. Since

$$f_b(x, d) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_{I_d}(T_b^i x);$$

the integrability of χ_{I_d} and the ergodicity of T_b with respect to the Lebesgue measure, we have that, $f_b(x, d) \to \int_{[0,1]} \chi_{I_d}(x)dx = \frac{1}{b}$, for Lebesgue-almost every point $x \in [0, 1]$. $\square$

This simple consequence of the Birkhoff's theorem is coherent with classical work by Borel [Bor]. Indeed, in 1909 Borel defined the notion of *normality*: a number $x \in \mathbb{R}$ is called *normal* in base b if $f_b(x, d) = 1/b$ for every digit $d \in \{0, 1, \dots, b\}$. Hence, Corollary 1.1.2 can be rephrased as the form of *Borel's theorem on normal numbers:*

Theorem 1.1.3. *Lebesgue-almost every number $x \in [0, 1]$ is normal.*

Typically in x any digit appears with frequency $1/b$ in the expansion. So, a natural question can be posed: are there points $x \in [0, 1]$ for which $f_b(x, d) \ne 1/b$, when $d \in \{0, 1, \dots, b-1\}$? The answer is positive, and it is not difficult to construct such kind of numbers. For example, in base 4, if x is defined by

$$\epsilon_{4k+1}(x) = 0, \quad \epsilon_{4k+2}(x) = \epsilon_{4k+1}(x) = 3; \quad \epsilon_{4k+4}(x) = 2$$

for all $k \in \mathbb{N}_0$, then we have that $f_4(x, 0) = 1/4, f_4(x, 1) = 0, f_4(x, 2) = 1/2$ and $f_4(x, 3) = 1/4$. Moreover, in [Bes, Egg] the authors gave an explicit formula for the *"size"* of the set of points having a prescribed vector of frequencies. Since any of these set is of null Lebesgue measure, we mean "size" by the *Hausdorff dimension* .

1.2 Cantor expansions

In 1869, Cantor [Can] generalized the notion of b-expansion in the following direction. Let $B = \{b_n\}_{n\geq 1}$ be a sequence of integers each of which is greater than 2. Cantor showed that every real number $x \in [0, 1)$ can be written as infinite series of the form

$$x = \sum_{n=1}^{\infty} \frac{c_n}{b_1 b_2 \cdots b_n},$$

with $c_n \in \{0, 1, \ldots, b_n - 1\}$. Observe that if for every $n \in \mathbb{N}$ we have $b_n = b$ then we recover the base b-expansion. As in the case of base $b-$expansion, the Cantor series is related to a dynamical system. However, in this case it is a non-autonomous system. Indeed, consider the maps defined in $[0, 1]$ by $T_{b_n}(x) = \{b_n x\}$. The iteration is defined by

$$T_B^n(x) = T_{b_n} \circ T_{b_{n-1}} \circ \cdots \circ T_{b_1}(x).$$

The dynamics is, therefore, obtained applying different maps T_{b_i} at prescribed times. Note that, as in the case of the base $b-$expansion, we have $c_n = [b_n T_B^{n-1}]$. Unfortunately, there is no analog of Birkhoff's ergodic theorem for non-autonomous systems. Therefore, questions related to frequencies of digits as Corollary 1.3.5 or Theorem 1.1.3 cannot be solved by means of Birkhoff averages. For instance the question about normality in this setting has to be addressed with different methods. It was actually shown by Renyi [Rén] the following result.

Theorem 1.2.1. *Lebesgue almost every number is normal for $B = \{b_n\}_{n\geq 1}$ if and only if $\sum_{n=1}^{\infty} 1/b_n = \infty$*

More recently, constructions and properties of normal numbers for Cantor series have been studied by Mance [Man2].

1.3 Continued Fractions

This section is devoted to the study of continued fractions and their properties. We start recalling some definitions. We follow [EW, Kin].

Definition 1.3.1. Given a sequence of positive integers $\{a_n\}_{n\in\mathbb{N}}$, we define a *infinite continued fraction* (or simply, continued fraction) as the formal expression

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

which will be denoted by $[a_1, a_2, a_3, \ldots]$. A *finite continued fraction* is given by the rational number

$$[a_1, a_2, a_3, \ldots, a_n] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}} \tag{1.3.1}$$

.

A priori, the concept of infinite continued fraction is merely formal since an infinite iterative process is implicitly involved. Moreover, the last definition suggests thinking infinite continued fraction as a limit of finite continued fractions. We will see that indeed this is true.

Lemma 1.3.1. *Let* $\{a_n\}_{n\in\mathbb{N}}$ *be a sequence of positive integers. For* $n \geq 1$ *let* p_n, q_n *the coprime numerator and denominator of the irreductible fraction*

$$\frac{p_n}{q_n} = [a_1, a_2, \ldots, a_n].$$

Then, for all $n \geq 1$

$$\begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} = \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} \tag{1.3.2}$$

with $p_0 := a_0$ *and* $q_0 := 1$.

Proof. [EW, p. 71]. □

Let $[a_1, a_2, \ldots]$ be a a continued fraction. For all $n \geq 1, a_n$ is called a *digit* of the continued fraction and p_n/q_n is called a *convergent* of the continued fraction. The next proposition summarizes some properties about digits and convergents.

Proposition 1.3.2. *If* p_n/q_n *are the convergents associated to* $[a_1, a_2, \ldots]$*, then for all* $n \geq 1$*, we have the following properties*

1. $p_{n+1} = a_{n+1}p_n + p_{n-1}$
2. $q_{n+1} = a_{n+1}q_n + q_{n-1}$
3. $p_nq_{n-1} - p_{n-1}q_n = (-1)^{n+1}$
4. $p_n \geq 2^{(n-2)/2}$
5. $q_n \geq 2^{(n-2)/2}$.

Proof. [EW, p. 71]. □

It is possible to show from the Proposition 1.3.2 that a continued fraction defines a real number. More precisely, we have that

$$[a_1, a_2, \ldots] = \lim_{n\to\infty} [a_1, a_2, \ldots, a_n] = \lim_{n\to\infty} \frac{p_n}{q_n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{q_{n-1}q_n}$$

where the last serie is absolutely convergent [EW, p. 72].

Definition 1.3.2. The Gauss map is the function $G : [0,1] \to [0,1]$ defined by $G(0) := 0$ and

$$G(x) := \frac{1}{x} - \left[\frac{1}{x}\right]$$

for $x \neq 0$.

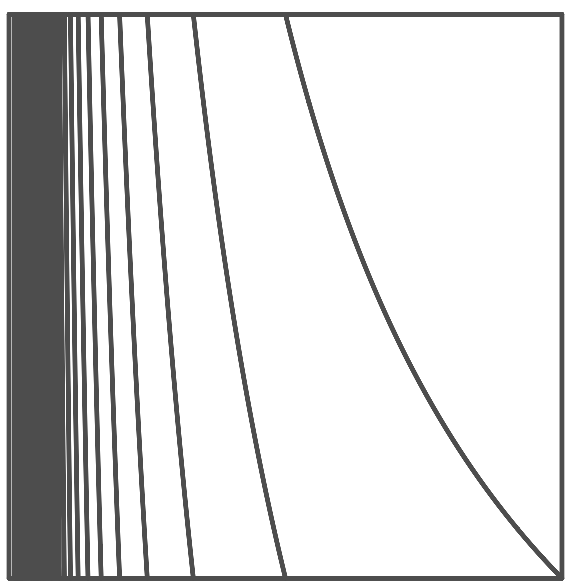

Figure 1.1: Graphic of Gauss map

The interaction between the Gauss map and continued fractions is explained in the next proposition. More details in [HW, p. 135, *continued fraction algorithm*].

Proposition 1.3.3. *If $x \in [0,1] \setminus \mathbb{Q}$ then the digits of the expansion in continued fractions of x are given by*

$$a_n = \left[\frac{1}{G^{n-1}(x)}\right], \ n \geq 1.$$

Proof. [EW, p. 78]. □

We now study ergodic properties of G. In contrast with the base-b maps T_b, the Lebesgue measure is not invariant for the Gauss map. In fact, we have that

$$G^{-1}\left(0, \frac{1}{2}\right) = \bigcup_{n=1}^{\infty} \left(\frac{2}{2n+1} - \frac{1}{n}\right)$$

and

$$\left|G^{-1}\left(0, \frac{1}{2}\right)\right| = 2 - 2\log 2 \neq \frac{1}{2}.$$

However, Gauss proved that there exists a G-invariant measure absolutely continuous with respect to the Lebesgue measure, that we now call the *Gauss measure*, defined for any Borel set $A \subset [0,1]$ by

$$\mu_G(A) = \frac{1}{\log 2} \int_A \frac{dx}{x+1}.$$

Theorem 1.3.4. *The Gauss map preserves the measure μ_G and it is μ_G-ergodic.*

Proof. [EW, p. 77-79]. □

The ergodic theorem implies the following results. For a proof see [EW, p. 82].

Corollary 1.3.5. *For Lebesgue almost every $x = [a_1, a_2, \ldots] \in [0,1]$ we have that*

1. *Any digit $d \in \mathbb{N}$ appears with frequency*

$$\log \frac{(d+1)^2}{d(d+2)}$$

2. The arithmetic averages of the digits diverges

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}a_i=\infty$$

3. The exponential growth of the denominators is given by

$$\lim_{n\to\infty}\frac{1}{n}\log q_n(x)=\frac{\pi^2}{12\log 2} \tag{1.3.3}$$

4. The exponential speed of approximation by the convergents is

$$\lim_{n\to\infty}\frac{1}{n}\log\left|x-\frac{p_n}{q_n}\right|=-\frac{\pi^2}{6\log 2} \tag{1.3.4}$$

The right side of 1.3.3 is known as the *Kintchine-Lévy constant.* The name of Kintchine comes since he proved in 1935 that the limit in 1.3.3 is constant almost everywhere and Lévy gave the explicit expression for that limit. On the other hand, the identity 1.3.4 says that convergents approaches to x with speed of approximation $e^{-n\frac{\pi^2}{6\log 2}}$. Nevertheless, it is possible to extract more dynamical information from 1.3.4 (see Chapter 3 for further details).

1.4 Generalized Continued Fractions

In this section we will study a family of maps defined in $[0,1]$ which generalize the Gauss map G. From the dynamics of each map a new continued fraction expansion of $x\in[0,1]$ arises. We follow [HM].

Definition 1.4.1. A 2×2 matrix

$$C=\begin{bmatrix} a & b\\ c & d\end{bmatrix}$$

with real entries and determinant $ad-bc=\pm 1$ acts on the Riemann sphere $\mathbb{C}\cup\{\infty\}$ as a

Moebius transformation by

$$C(z) := \frac{az + bc}{cz + d}, \quad C(\infty) = \frac{a}{c} \quad \text{and} \quad C(-d/c) = \infty.$$

Remark 1.4.1. Given an interval $I \subset \mathbb{R}$ and a matrix C with the conditions established in Definition 1.4.1 we denote by CI the set $CI := \{Cx : x \in I\}$.

For $k > 0$, we will consider the matrices

$$A_k = \begin{bmatrix} \frac{k}{\sqrt{k}} & \frac{-k}{\sqrt{k}} \\ \frac{-1}{\sqrt{k}} & 0 \end{bmatrix}$$

and the family of Moebius transformations parametrized by k, given by

$$A_k(x) = \frac{k(1-x)}{x}.$$

Then we can consider the corresponding family of transformations $T_k : [0,1] \to [0,1]$, defined by $T_k(0) = 0$ and

$$T_k(x) = A_k(x) - [A_k(x)],$$

for $x \neq 0$. Each map T_k is called *Gauss-like transformations.* If we denote the fractional part of a real number w by $\langle w \rangle = w - [w]$, then we shall be writing $T_k(x) = \langle A_k(x) \rangle$. We stress that this is in fact a generalization, in the sense of that we can recover the Gauss map when $k = 1$. Figure 1.2 shows a comparison between the graphs of T_k for three values of k with respect to Gauss map.

We now define the *k-continued fraction.* For $k > 0$, the set of *k-digits* is defined as

$$D_k = \{l \in \mathbb{Z} : [A_k(x)] = l \text{ for some } x \in (0,1)\}.$$

Remark 1.4.2. It is not difficult to prove that $D_k = \mathbb{N}_0$ for all $k > 0$.

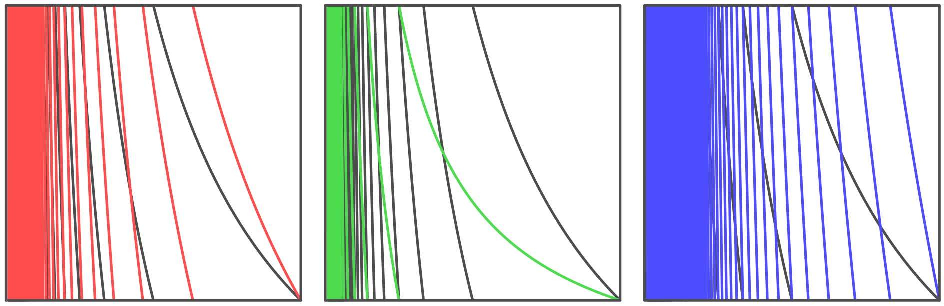

Figure 1.2: Graph of T_k when $k = \sqrt{3}, \frac{1}{3}, 5$ respectively. The black graph is that of Gauss map.

Definition 1.4.2. Given any finite sequence of integers in D_k we define the *cylinder of level* n as the subset of $[0,1]$ given by

$$\Delta^{(n)}_{a_1 a_2 \ldots a_n} = A_k^{-1} B^{a_1} A_k^{-1} B^{a_2} \cdots A_k^{-1} B^{a_n}(0,1), \text{ with } B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

In [HM], the authors proved the following proposition.

Proposition 1.4.1. *For each $n \geq 1$, we have the following properties related to the cylinders of level n,*

1. *The cylinders $\Delta^{(n)}_{a_1 a_2 \ldots a_n}$ are the maximal open subintervals of $(0,1)$, on which the n-th iterate of T_k is a homeomorphism.*

2. *On each $\Delta^{(n)}_{a_1 a_2 \ldots a_n}$, the map T_k acts as a shift, that is, $T_k \Delta^{(n)}_{a_1 a_2 \ldots a_n} = \Delta^{(n-1)}_{a_2 a_3 \ldots a_n}$.*

3. *T_k^n restricted to the cylinder of level n is equal to the Möbius transformation C_n given by*

$$C_n = B^{-a_n} A_k \cdots B^{-a_1} A_k$$

which maps $\Delta^{(n)}_{a_1 a_2 \ldots a_n}$ onto $(0,1)$.

Proof. [HM, Propositions 1, 3]. □

Definition 1.4.3. If $a_1, \ldots, a_n$ is a finite sequence of k-digits, then we define the finite k-continued fraction expansion by

$$[a_1, a_2, \ldots, a_n] = A_k^{-1} B^{a_1} A_k^{-1} \cdots A_k^{-1} B^{a_n} A_k^{-1}(\infty) = C_n^{-1} A_k^{-1}(\infty).$$

Consequently, if we write

$$C_n^{-1} A_k^{-1} := \begin{bmatrix} p_k & r_n \\ q_n & s_n \end{bmatrix} \tag{1.4.1}$$

then $[a_1, a_2, \ldots, a_n] = C_n^{-1} A_k^{-1}(\infty) = \frac{p_n}{q_n}$. We now define an infinite continued fraction.

Definition 1.4.4. Given a infinite sequence of k-digits $\{a_n\}_{n \in \mathbb{N}}$ we define the *infinite k-continued fraction* by the limit

$$[a_1, a_2, \ldots]_k = \lim_{n \to \infty} [a_1, a_2, \ldots, a_n] = \lim_{n \to \infty} \frac{p_n}{q_n}.$$

Remark 1.4.3. This last limit is well defined since the fractions $\frac{p_n}{q_n}$ are always a endpoint of the closed interval $\overline{\Delta}^{(n)}_{a_1 a_2 \ldots a_n}$ which is a sequence of nested closed sets [HM, page 2856].

As in the classical setting, we can partition de interval $(0, 1)$ in cylinders at level n modulo a countable set. Given $n \geq 1$, let $\mathbb{Q}_k^{(n)}$ be the set

$$\mathbb{Q}_k^{(n)} := \{x \in [0, 1] : T_k^m(x) = 0 \text{ for some } m \leq n\}.$$

Proposition 1.4.2. *For each $n \geq 1$*

$$[0, 1] = \left(\bigcup_{a_1, \ldots, a_n \in D_k} \Delta^{(n)}_{a_1, a_2, \ldots, a_n} \right) \cup \mathbb{Q}_k^{(n)}.$$

Proof. [HM, Page 2855, Proposition 2]. □

Definition 1.4.5. We will call the set of *k-rational numbers* to the union

$$\mathbb{Q}_k := \bigcup_{n=1}^{\infty} \mathbb{Q}_k^n.$$

The complement of $\mathbb{Q}_k$ in $[0,1]$ is called the set of *k-irrational numbers.*

Proposition 1.4.3. *Each k-irrational x has a unique, infinite k-expansion in continued fractions. We have k-expansion $x = [a_1, a_2, \ldots, a_n, \ldots]$ if and only if*

$$x \in \Delta^{(n)}_{a_1,a_2,\ldots,a_n} \textit{ for all } n \geq 1,$$

if and only if

$$T_k^{n-1}(x) \in \Delta^{(1)}_{a_n} \textit{ for all } n \geq 1.$$

Proof. [HM, Page 2856, Proposition 3]. □

Proposition 6.2.3 allows to write a k-irrational x as a limit of k-rationals

$$x = \lim_{n\to\infty} \frac{p_n}{q_n}.$$

Definition 1.4.6. For each $n \geq 1$, the rationals $\frac{p_n}{q_n}$ are called the *convergents* of the k-continued fraction.

Convergents have similar properties that in the case of classical continued fractions. The following proposition summarizes some of them which will be useful for our purposes.

Proposition 1.4.4. *Let $x = [a_1, a_2, \ldots]$ be a k-irrational. Then, the following properties related to convergents are satisfied:*

1. $p_n = \frac{1}{\sqrt{k}}((a_n + k)p_{n-1} + \sqrt{k}p_{n-2}), \qquad n \geq 2$

2. $q_n = \frac{1}{\sqrt{k}}((a_n + k)q_{n-1} + \sqrt{k}q_{n-2}), \qquad n \geq 2$

3. $s_n = kq_n + \sqrt{k}q_{n-1}$

4. $r_n = kp_n + \sqrt{k}p_{n-1}$

5. $|p_n q_{n-1} - q_n p_{n-1}| = \frac{1}{\sqrt{k}}$

Proof. This follow from Lemma 2; Propositions 4 and 5 of [HM] . $\square$

Now, we relate the k-continued fractions with the transformations T_k.

Proposition 1.4.5. *If $x = [a_1, a_2, \ldots]_k$ is a k-irrational continued fraction, then we have*

$$a_n = [A_k(T_k^{n-1}(x))]$$

and

$$x = \cfrac{k}{a_1 + k + \cfrac{k}{a_2 + k + \cfrac{k}{\ldots + \cfrac{k}{a_n + k + T_k^n(x)}}}} \tag{1.4.2}$$

for all $n \geq 1$.

Proof. By Proposition 6.2.3 we have $x = [a_1, a_2, \ldots]_k$ if and only if $x \in \Delta^{(n)}_{a_1 a_2 \ldots a_n}$ for all $n \geq 1$, and this implies $T_k^{n-1}(x) \in \Delta^{(1)}_{a_n}$. In particular, $A_k(T^{n-1}(x)) \in B^{a_n}(0,1) = (a_n, a_n + 1)$ which means $a_n = [A_k(T^{n-1}(x))], n \geq 1$. On the other hand, we have $T_k(x) = \{A_k(x)\} = A_k(x) + a_1$. Solving for x in this last equation we obtain

$$x = \frac{k}{a_1 + k + T_k(x)}.$$

Now let suppose (1.4.2) for some $n \geq 1$. Since $a_{n+1} = [A_k(T_k^n(x))]$ then $T_k^n(x) = \frac{k}{a_{n+1}+k+T^{n+1}(x)}$. Replacing in (1.4.2) we obtain the equality for $n+1$. $\square$

We conclude this section with a relation between k-continued fractions and their *tails*. Given a k-irrational number $x = [a_1, a_2 \ldots, a_n, \ldots]_k \in [0,1]$, we define its n^{th} *tail* by $x_n := [a_{n+1}, a_{n+2}, \ldots]_k$. The main property of the tail corresponds to the following proposition.

Proposition 1.4.6. *Let $x = [a_1, a_2, a_3, \ldots]_k \in I$ be a k-irrational. If $x_n = [a_{n+1}, a_{n+2}, \ldots]_k$ denotes the n^{th}-tail of x, then*

$$x = \frac{p_n x_{n+1} + p_n a_{n+1} + r_n}{q_n x_{n+1} + q_n a_{n+1} + s_n}$$

for all $n \geq 1$.

Proof. Using the definition of $C_n^{-1}A_k^{-1}$ we note that if $j \geq 1$ is an integer, then

$$\begin{aligned}
\begin{bmatrix} p_{n+j} \\ q_{n+j} \end{bmatrix} &= C_{n+j}^{-1}A_k^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= A_k^{-1}B^{a_1}A_k^{-1}B^{a_2}\cdots A_k^{-1}B^{a_n}A_k^{-1}B^{a_{n+1}}\cdots A_k^{-1}B^{a_{n+j}}A_k^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= C_n^{-1}A_k^{-1}B^{a_{n+1}}\cdots A_k^{-1}B^{a_{n+j}}A_k^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= C_n^{-1}A_k^{-1}B^{a_{n+1}}\begin{bmatrix} p_{j-1}(x_{n+1}) & r_{j-1}(x_{n+1}) \\ q_{j-1}(x_{n+1}) & s_{j-1}(x_{n+1}) \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}
\end{aligned}$$

where $p_{j-1}(x_{n+1}), r_{j-1}(x_{n+1}), q_{j-1}(x_{n+1}), s_{j-1}(x_{n+1})$ are the entries of the matrix $C_j^{-1}A_k^{-1}$ for the continued fraction x_{n+1}. Therefore

$$
\begin{aligned}
\frac{p_{n+j}}{q_{n+j}} &= \frac{p_n p_{j-1}(x_{n+1}) + p_n a_{n+1} q_{j-1}(x_{n+1}) + r_n q_{j-1}(x_{n+1})}{q_n p_{j-1}(x_{n+1}) + q_n a_{n+1} q_{j-1}(x_{n+1}) + s_n q_{j-1}(x_{n+1})} \\
&= \frac{p_n \frac{p_{j-1}(x_{n+1})}{q_{j-1}(x_{n+1})} + p_n a_{n+1} + r_n}{q_n \frac{p_{j-1}(x_{n+1})}{q_{j-1}(x_{n+1})} + q_n a_{n+1} + s_n}
\end{aligned}
$$

and we conclude the proposition doing $j \to \infty$. $\square$

1.5 Ergodic properties of T_k and their consequences

As the Gauss map, the T_k transformations have dynamical and ergodic related to continued fractions. In [HM] it was shown that there exists a measure absolutely continuous to the Lebesgue measure for which each transformation T_k is ergodic.

Theorem 1.5.1. *Let $k > 0$. The transformation $T_k : [0,1] \to [0,1]$ preserve the measure μ^k defined on Borel subset of $[0,1]$ as*

$$\mu^k(A) = \int_A \frac{c_k}{x+k} dx$$

where $c_k = \left(\log \frac{k+1}{k}\right)^{-1}$. Moreover, μ^k is T_k-ergodic.

We have the following consequences from Birkohff's ergodic theorem.

Proposition 1.5.2. *For all $k > 0$ and for Lebesgue almost every k-irrational $x = [a_1, a_2, \dots] \in [0,1]$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} a_i = \infty,$$

$$\lim_{n\to\infty} \sqrt[n]{(a_1+1)(a_2+1)\cdots(a_n+1)} = \prod_{i=2}^{\infty} \left(\frac{(i+k)^2}{(i+k)^2 - 1} \right)^{\log i / \log(\frac{k+1}{k})}.$$

1.5.1 Diophantine Approximation

The following proposition corresponds to a version for k-continued fractions of the classical Dirichlet theorem.

Proposition 1.5.3. *For each k-irrational number $x \in (0,1)$ the convergents $\frac{p_n}{q_n}$ satisfy*

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{kq_n^2}.$$

Proof. [HM, Corollary 1]. □

Also, the exponential growth of the denominators.

Theorem 1.5.4. *For $k > 0$ and Lebesgue almost every all $x \in I$ we have*

$$\lim_{n\to\infty} \frac{\log q_n}{n} = \log \sqrt{k} - \left(\log \frac{k+1}{k} \right)^{-1} \mathcal{L}_2 \left(-\frac{1}{k} \right),$$

where $\mathcal{L}_2(z) = \int_z^0 \frac{\log(1-t)}{t} dt$ is the Euler dilogarithm.

Proof. [HM, Theorem 4]. □

We know that one end point of the cylinder at level n is $\frac{p_n}{q_n}$. From [HM, Proposition 5], it is possible to show that the other endpoint is

$$\frac{\sqrt{k}p_n + p_{n-1}}{\sqrt{k}q_n + q_{n-1}}.$$

Therefore, the length of $\Delta^{(n)}_{a_1,a_2,\ldots,a_n}$ is given by

$$|\Delta^{(n)}_{a_1,a_2,\ldots,a_n}| = \left| \frac{p_n}{q_n} - \frac{\sqrt{k}p_n + p_{n-1}}{\sqrt{k}q_n + q_{n-1}} \right| = \frac{1}{q_n(\sqrt{k}q_n + q_{n-1})}.$$

As a corollary we obtain the exponential growth of the length of the cylinders.

Corollary 1.5.5. *For $k > 0$ and Lebesgue almost all $x \in [0,1]$*

$$\lim_{n\to\infty} \frac{1}{n} \log |\Delta^{(n)}_{a_1,a_2,\ldots,a_n}| = -\log k + 2\left(\log \frac{k+1}{k}\right)^{-1} \mathcal{L}_2\left(-\frac{1}{k}\right)$$

Proof. [HM, Corollary 2]. □

Finally, we finish this section with the exponential speed of approximation by convergents.

Theorem 1.5.6. *For $k > 0$ and Lebesgue almost all $x \in [0,1]$*

$$\lim_{n\to\infty} \frac{1}{n} \log \left| x - \frac{p_n^k}{q_n^k} \right| = -\log k + 2\left(\log \frac{k+1}{k}\right)^{-1} \mathcal{L}_2\left(-\frac{1}{k}\right)$$

Proof. [HM, Theorem 5]. □

Remark 1.5.1. Note that, for $k > 0$ and Lebesgue almost $x \in [0,1]$

$$2 \lim_{n\to\infty} \frac{\log q_n}{n} = -\lim_{n\to\infty} \frac{1}{n} \log \left| x - \frac{p_n^k}{q_n^k} \right| = \lim_{n\to\infty} \frac{1}{n} \log |\Delta^{(n)}_{a_1,a_2,\ldots,a_n}|$$

1.6 Lüroth expansions and generalizations

In 1883, J. Lüroth [Lür] proved that every real number $x \in (0,1]$ can be written in the form

$$\begin{aligned} x &= \frac{1}{a_1} + \frac{1}{a_1(a_1-1)a_2} + \ldots + \frac{1}{a_1(a_1-1)\cdots a_{n-1}(a_{n-1}-1)a_n} + \cdots \\ &= \sum_{n=1}^{\infty} \frac{1}{a_1(a_1-1)\cdots a_{n-1}(a_{n-1}-1)a_n} \end{aligned}$$

where $a_n \geq 2$, for all $n \geq 1$. This expansion is called the *Lüroth series* of x and the numbers $\{a_n\}_{n\in\mathbb{N}}$ are called the *digits*. Arithmetic properties of the numbers can be read from its corresponding series, and moreover, it is closely related to the dynamical properties of the transformation $L : [0,1) \to [0,1)$ defined by

$$L(x) := \begin{cases} n(n+1)x - n & \text{if } x \in \left[\frac{1}{n+1}, \frac{1}{n}\right), n \in \mathbb{N} \\ 0 & \text{if } x = 0. \end{cases}$$

If x has a Lüroth expansion with digits $\{a_n\}_{n\in\mathbb{N}}$ then the following relation holds:

$$a_n = a_1(L^{n-1}(x)) \quad \text{for} \quad n \geq 1, \quad \text{where} \quad a_1(u) := n+1 \quad \text{if} \quad u \in \left[\frac{1}{n+1}, \frac{1}{n}\right).$$

Partial sums in the Lüroth series of an irrational number x can be thought of, in analogy to the continued fractions, as rationals approximations for the irrational number x. For $n \geq 1$, denote those partial sums by p_n/q_n, that is,

$$\frac{p_n}{q_n} = \frac{1}{a_1} + \frac{1}{a_1(a_1-1)a_2} + \cdots + \frac{1}{a_1(a_1-1)a_2 \cdots a_{n-1}(a_{n-1}-1)a_n}.$$

The number p_n/q_n is called *n-th approximant* of x. We stress that the Lüroth map can be thought of as a linear version of the Gauss map and that the Lüroth series is analogous to the continued fraction expansion.

Note that L can be thought as a linear version of the Gauss map. Furthermore, linear versions of T_k maps can be defined. For each $k > 0$, the *k-Lüroth map* $L_k : [0,1) \to [0,1)$ is defined by

$$L_k(x) := \begin{cases} x\frac{(n+k)(n+k+1)}{k} - (n+k), & \text{if } x \in \left[\frac{k}{n+k+1}, \frac{k}{n+k}\right), n \in \mathbb{N}_0 \\ 0, & \text{if } x = 0 \end{cases}$$

Each k-Lüroth map induces a series expansion of every $x \in [0,1]$ whose n-th iterated $L_k^n(x)$ different from zero, and a set playing the role of rational numbers in the Lüroth expansion. Proposition 4.2.1 gives that if $L_k^n(x) \neq 0$ for all $n \geq 1$, then we have the expansion

$$x = \sum_{n=1}^{\infty} \frac{k^n}{(a_1 + k)(a_1 - 1 + k) \cdots (a_{n-1} + k)(a_{n-1} - 1 + k)(a_n + k)}$$

where

$$a_n = a_1(L_k^{n-1}(x)), n \geq 1,$$

and

$$a_1(u) := n + 1 \text{ if } u \in \left[\frac{k}{n+1+k}, \frac{k}{n+k}\right), n \geq 0.$$

Further details about k-Lüroth expansions, see Chapter 4.

1.7 Q-Lüroth expansions

This section introduces a new numeric expansion which combines the $k-$Lüroth maps and the Cantor series expansion. Let $Q = \{q_n\}_{n\geq 1}$ be a sequence of positive real numbers $q_n > 0$. Consider the family of Lüroth maps $\{L_{q_n}\}_{n\geq 1}$, that we will denote by L_Q. This family of transformations induces a non-autonomous dynamical system $([0,1), L_Q)$ in a similar way as in Section 1.2. Indeed, the time evolution of the system is defined by composing the maps L_{q_n} in the prescribed order given by the sequence $Q = \{q_n\}_{n\geq 1}$. In other words, for all $n \geq 1$, we define:

$$L_Q^n := L_{q_n} \circ L_{q_{n-1}} \circ \cdots \circ L_{q_1}.$$

The *orbit* of $x \in [0,1]$ is the sequence $\{L_{q_n}(x)\}_{n\geq 1}$. The Q-Lüroth expansion is given by the following theorem.

Theorem 1.7.1. *Each $x \in [0,1)$ such that $L_Q^n(x) \neq 0$ for all $n \geq 0$, can be expanded uniquely* *in a infinite series of the form*

$$x = \sum_{n=1}^{\infty} \frac{q_1 q_2 \cdots q_n}{(a_1 - 1 + q_1)(a_1 + q_1) \cdots (a_n - 1 + q_n)}$$

where $a_n = \left[\frac{q_n}{L_Q^{n-1}(x)} - q_n\right] + 1$.

See Chapter 5 for further details about Q-Lüroth expansions.

Chapter 2

Thermodynamic Formalism

In this chapter we recall notions and results from Thermodynamic Formalism which is a set of tools and methods brought into hyperbolic dynamics with great success in the early seventies from statistical physics. It allows for the selection of relevant measures from the, sometimes very large, set of invariant measures. It has been used as tool in the dimension theory of dynamical systems at least since the work of Bowen in the 70s [Bow], where the author developed the theory on compact spaces and in particular for fullshifts on finitely many symbols. Section 2.2 is devoted to this theory in the compact case, we will follow [Wal]. On the other hand, thermodynamic formalism for dynamical systems defined in non-compact spaces has been studied and developed over the last 20 years. The particular case of the fullshift on countable many symbols has been throughly studied, see [BS, MU2, Sar2]. In Section 2.3 we recall the main definitions and results. Finally, we apply the theory of Section 2 for the case of EMR maps, which are transformations of the unit interval $[0,1]$ *modeled by the fullshift.*

2.1 Entropy

2.1.1 Metric Entropy

Let $(X, \mathcal{A}, \mu)$ be a probability space. A partition of X is a collection of measurable disjoint sets whose union is equal to X.

Definition 2.1.1. The entropy of a countable (or finite) partition α of X is given by

$$H_\mu(\alpha) := -\sum_{A\in\alpha} \mu(A)\log\mu(A)$$

with the convention $0\log 0 := 0$.

Given two partitions α, β we define their *join* by $\alpha \vee \beta := \{A \cap B : A \in \alpha, B \in \beta\}$. Also, if $T : X \to X$ is a μ-invariant map and $n \geq 0$ is an integer, then we define $T^{-n}\alpha$ as the partition $T^{-n}\alpha := \{T^{-n}A : A \in \alpha\}$. It follows that $H_\mu(T^{-n}\alpha) = H(\alpha)$.

Note that if α, β are two partitions, then $H_\mu(\alpha \vee \beta) \leq H_\mu(\alpha) + H_\mu(\beta)$. Therefore, if $H_\mu(\alpha) < \infty$, then $H_\mu(\bigvee_{i=0}^{n-1} T^{-i}\alpha) \leq nH_\mu(\alpha) < \infty$, for all $n \geq 0$.

Proposition 2.1.1. *Let α be a countable partition of X such that $H_\mu(\alpha) < \infty$. Then, the limit*

$$\lim_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right)$$

exists and it is equals to $\inf_n \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right)$.

Proof. [Wal, Corollary 4.9.1; p. 96, Remark 1] □

Denote the limit stated above by

$$h_\mu(T, \alpha) := \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right).$$

Definition 2.1.2. Let $(X, \mathcal{A}, \mu)$ be a probability space and T a map preserving μ. We define the *metric entropy* of T as the supremum

$$h_\mu(T) = \sup\{h_\mu(T, \alpha) : \alpha \text{ is a countable measurable partition with } H_\mu(\alpha) < \infty\}.$$

2.1.2 Topological Entropy

Let (X, d) be a metric space and $T : X \to X$ a continuous transformation.

Definition 2.1.3. Let n be a natural number $\varepsilon > 0$ and $K \subset X$ be a compact subset. A subset $E \subset K$ is called (n, ε)-*separated* subset of K with respect to T if $x, y \in E, x \neq y$ implies $d(T^j x, T^j y) > \varepsilon$ for some $j \in \{0, \ldots, n-1\}$.

We call $s_n(\varepsilon, K, T)$ the largest cardinality of any (n, ε)-separated subset of K with respect to T. Also, denote by

$$s(\varepsilon, K, T) := \limsup_{n\to\infty} \frac{1}{n} \log s_n(\varepsilon, K, T)$$

and

$$h(K, T) := \lim_{\varepsilon\to 0} s(\varepsilon, K, T).$$

Definition 2.1.4. We define the *topological entropy* of T as the supremum

$$h(T) = \sup\{h(K, T) : K \subset X \text{ is a compact subset of } X\}$$

The relationship between the topological entropy and the metric entropy is given by the following theorem known as the *variational principle.* Denote by $\mathcal{M}_T(X)$ the space of $T-$invariant probability measures.

Theorem 2.1.2. *Let $T : X \to X$ be a continuous map of a compact metric space X. Then $h(T) = \sup\{h_\mu(T) : \mu \in \mathcal{M}_T(X)\}$.*

Proof. [Wal, Theorem 8.6]. ☐

The variational principle provides a natural way to distinguish particular measures in $\mathcal{M}_T(X)$. If there is a measure that attains the supremum in Theorem 2.1.2, we call it a maximal entropy measure. See [Wal, Section 8.3] for further details.

2.2 Thermodynamic Formalism: the compact case

2.2.1 Topological Pressure

Let (X, d) be a metric space and $T : X \to X$ a continuous transformation and consider $\varphi : X \to \mathbb{R}$ a continuous function.

Definition 2.2.1. For each $n \geq 1$ and $\varepsilon > 0$, we define

$$P_n(T, \varphi, \varepsilon) = \sup \left\{ \sum_{x \in E} e^{S_n \varphi(x)} : E \text{ is an } (n, \varepsilon)\text{-separated set for } X \right\}$$

and

$$P(T, \varphi, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log P_n(T, \varphi, \varepsilon).$$

Theorem 2.2.1. *If $\varphi : X \to \mathbb{R}$ is continuous, then the limit $\lim_{\varepsilon \to 0} P(T, \varphi, \varepsilon)$ exists.*

Proof. [Wal, Theorem 9.1] ☐

We define the *topological pressure* or only *pressure* of the potential φ as

$$P(T, \varphi) = \lim_{\varepsilon \to 0} P(T, \varphi, \varepsilon).$$

Note that when $\varphi \equiv 0$, then we recover the topological entropy of T. In other words, $P(T, 0) = h(T)$. So, we can think the topological pressure as a *weighted* entropy, where any x in a separated set contributes with "weight" $S_n\varphi(x)$.

Denote by $C^0(X)$ the space of continued functions on X to $\mathbb{R}$ and consider $C^0(X)$ with the supremum norm $\|\varphi\| = \sup\{|\varphi(x)| : x \in X\}$. The next proposition gives properties of $P(f, \cdot)$ seen as a function from $C^0(X)$ to $\mathbb{R} \cup \{\infty\}$.

Proposition 2.2.2. *The pressure function $P(f, \cdot) : C^0(X) \to \mathbb{R} \cup \{\infty\}$ is*

1. *a Lipschitz function, that is, $|P(f, \varphi) - P(f, \psi)| \leq \|\varphi - \psi\|$, for all potentials φ, ψ;*

2. *a convex function, that is $P(f, (1-t)\varphi + t\psi) \leq (1-t)P(f, \varphi) + tP(f, \psi)$, for all potentials φ, ψ and $t \in [0, 1]$.*

Proof. [Wal, Theorem 9.7] □

Similarly as Theorem 2.1.2, the pressure satisfies also a variational principle.

Theorem 2.2.3 (Variational Principle). *Let $T : X \to X$ be a continuous function. Then, for all continuous potential $\varphi : X \to \mathbb{R}$, we have*

$$P(\varphi, T) = \sup\left\{h_\mu(T) + \int \varphi d\mu : \mu \in \mathcal{M}_T\right\} \tag{2.2.1}$$

Proof. [Wal, Theorem 9.10] □

Observe that if $\varphi \equiv 0$, then we recover the variational principle for the entropy of T. Also, Theorem 2.2.3 motivates the study of measures on which the supremum (2.2.1) is attained. Such measures are called *equilibrium states.* The existence of equilibrium measures is a non-trivial question. In fact, stronger assumptions on regularity of φ are required. Moreover, the transfer operator theory is one of the tools used to prove the existence of such measures [PP].

2.3 Fullshift on countable symbols

This section is devoted to the study of thermodynamical formalism for the fullshift on countable symbols. We follow [Sar1, Sar5] in which the author developed the theory for a larger class of dynamical systems called *Topological Markov Shifts*. The main difference with the result obtained in the previous sections is that the space is no longer assumed to be compact.

Let S be a countable set and $A = (a_{ij})_{i,j\in S}$ be a matrix with entries equal to 0 or 1 but not having rows or columns identically zero.

Definition 2.3.1. The *topological Markov shift* generated by the matrix $A = (a_{ij})_{i,j\in S}$ is the pair (X, σ_A), where X is the set defined by

$$X := \{x \in S^{\mathbb{N}_0} : a_{x_i x_{i+1}} = 1, \text{ for all } i \geq 0\}$$

equipped with the topology generated by subsets of the form

$$C_{a_0,\ldots,a_{n-1}} := \{x \in X : x_i = a_i, 0 \leq i \leq n-1\},$$

where $n \in \mathbb{N}, a_0, \ldots a_{n-1} \in S$. Note that the set may be empty. Also, $\sigma_A : X \to X$ is the map given by $\sigma_A(x_0, x_1, \ldots) = (x_1, x_2, \ldots)$. We call S the *alphabet*, $C_{a_0,\ldots,a_{n-1}}$ a *cylinder* (of length n) and σ_A the *shift map*.

Remark 2.3.1. On a topological Markov shift we define the metric $d(x, y) := 2^{-\min\{n\geq 0: x_n \neq y_n\}}$ when $x \neq y$, and $d(x, y) := 0$ when $x = y$. The topology generated by this metric is equivalent to that generated by the cylinders.

Definition 2.3.2. The *fullshift* with alphabet $S = \mathbb{N}_0$ is the pair (Σ, σ) defined by the topological Markov shift $\Sigma := X$ generated by the matrix $A = (a_{ij})_{i,j\in S}$ with $a_{i,j} = 1$ for all $(i, j) \in S \times S$.

The fullshift endowed with the topology generated by the cylinders (as in Definition 2.3.1) is not compact. Hence, the thermodynamic formalism theory of Section 2.2 does not apply to this setting.

2.3.1 Regularity of functions

Let $\varphi : \Sigma \to \mathbb{R}$ be a function defined on the fullshift.

Definition 2.3.3. Let $n \in \mathbb{N}$. We define the *n-th variation* of φ by the supremum

$$V_n(\varphi) := \sup\{|\varphi(x) - \varphi(y)| : x, y \in \Sigma, x_i = y_i, 0 \leq i \leq n-1\}.$$

Definition 2.3.4. We say that φ is *weakly Hölder* if there exists $\theta \in (0,1)$ and a constant $C > 0$ such that, for all $n \geq 2, V_n(\varphi) \leq C\theta^n$. If in addition $V_1(\varphi) < \infty$, then we say that φ is *locally Hölder.*

Definition 2.3.5. We say that φ has *summable variations* if

$$\sum_{n=2}^{\infty} V_n(\varphi) < \infty.$$

Proposition 2.3.1. *Let $\varphi : \Sigma \to \mathbb{R}$. Then*

1. *If ϕ is weakly Hölder continuous, then ϕ is of summable variations.*

2. *If ϕ is of summable variations, then ϕ is uniformly continuous.*

3. *If $q, t \in \mathbb{R}$ and φ, ψ are locally Hölder, then $q\varphi + t\psi$ is locally Hölder.*

Proof. The first implication follows directly from definitions 2.3.5 and 2.3.4. If ϕ is of summable variations, there exists $n \in \mathbb{N}$ such that $V_n(\phi) < \varepsilon$. Therefore, if $x, y \in \Sigma$ with

$d(x,y) < 2^{-n}$, then $|\phi(x) - \phi(y)| < \varepsilon$. On the other hand, if $q, t \in \mathbb{R}$ then

$$V_n(q\varphi + t\psi) \leq qV_n(\varphi) + tV_n(\psi) \leq qC_1\theta_1^n + tC_2\theta_2^n \leq (qC_1 + tC_2)\theta^n$$

where $C_1, C_2 > 0; \theta_1, \theta_2 \in (0,1)$ and $\theta = \max\{\theta_1, \theta_2\}$. □

Definition 2.3.6. We say that two functions $\varphi, \psi : \Sigma \to \mathbb{R}$ of summable variations are *cohomologous* if there exists $h : \Sigma \to \mathbb{R}$ with summable variations such that $\varphi = \psi + h - h \circ T$. When a function φ is cohomologous to the function identically zero, then we say that φ is a coboundary.

Theorem 2.3.2 (Livsic)**.** *Suppose that $\varphi, \psi : \Sigma \to \mathbb{R}$ have summable variations. Then φ, ψ are cohomologous if and only if for all $x \in X$ and $n \in \mathbb{N}$ such that $\sigma^n x = x$, then $S_n\phi(x) = S_n\psi(x)$.*

Proof. See [Sar4] □

2.3.2 Definition and properties

Thermodynamic formalism on countable Markov shifts has been developed initially by Mauldin, Urbanski [MU2] and Sarig [Sar2]. We will follow [Sar2] to describe the theory. For any $i_0 \in \mathbb{N}_0$, denote by $\mathbb{1}_{C_{i_0}}$ the characteristic function of the cylinder C_{i_0}.

In what follows (Σ, σ) denotes the full shift on the alphabet $\mathbb{N}$.

Proposition 2.3.3. *Suppose that $\varphi : \Sigma \to \mathbb{R}$ is a function of summable variations. Then, the limit*

$$P_G(\varphi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n x = x} \exp\left(\sum_{i=0}^{n-1} \varphi(\sigma^i x)\right) \mathbb{1}_{C_{i_0}}(x)$$

exists and is independent of $i_0 \in \mathbb{N}$. Moreover, $P_G(\varphi) > -\infty$.

Definition 2.3.7. Let $\varphi : \Sigma \to \mathbb{R}$ a function of summable variations. We call $P_G(\varphi)$ the *Gurevich pressure of φ.*

Proposition 2.3.4. *Suppose that $\psi, \varphi : \Sigma \to \mathbb{R}$ have summable variations. Then, the following holds*

1. *If $c \in \mathbb{R}$, then $P_G(\varphi + c) = P_G(\varphi) + c$*

2. *The Gurevich pressure is convex: for every $t \in [0,1], P_G(t\varphi + (1-t)\psi) \le tP_G(\varphi) + (1-t)P_G(\psi)$*

3. *The Gurevich pressure is invariant under cohomologous functions: if φ, ψ are cohomologous, then $P_G(\varphi) = P_G(\psi)$.*

Proof. Let $a \in \mathbb{N}, c \in \mathbb{R}$ and $t \in [0,1]$. Then

$$
\begin{aligned}
P_G(\varphi + c) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n x = x} \exp\left(\sum_{i=0}^{n-1} (\varphi + c)(\sigma^i x)\right) \mathbb{1}_{C_{i_0}}(x) \\
&= c + \lim_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n x = x} \exp\left(\sum_{i=0}^{n-1} \varphi(\sigma^i x)\right) \mathbb{1}_{C_{i_0}}(x) \\
&= c + P_G(\varphi).
\end{aligned}
$$

The convexity follows from convexity of exponential function and Hölder's inequality ,

$$
\begin{aligned}
\sum_{\substack{\sigma^n x = x \\ x_0 = a}} \exp\left(\sum_{i=0}^{n-1} (t\varphi + (1-t)\psi)(\sigma^i x)\right) &\le \sum_{\substack{\sigma^n x = x \\ x_0 = a}} \exp\left(t \sum_{i=0}^{n-1} \varphi(\sigma^i x)\right) \exp\left((1-t) \sum_{i=0}^{n-1} \psi(\sigma^i x)\right) \\
&\le \left(\sum_{\substack{\sigma^n x = x \\ x_0 = a}} \exp \sum_{i=0}^{n-1} \varphi(\sigma^i x)\right)^t \left(\sum_{\substack{\sigma^n x = x \\ x_0 = a}} \exp \sum_{i=0}^{n-1} \psi(\sigma^i x)\right)^{1-t}
\end{aligned}
$$

which implies that $P_G(t\varphi + (1-t)\psi) \le tP_G(\varphi) + (1-t)P_G(\psi)$. Finally, the invariance on cohomologous functions follows from Livsic's Theorem 2.3.2. □

The following proposition was proved in [Sar3, p. 1755]

Proposition 2.3.5. *Suppose that $\varphi : \Sigma \to \mathbb{R}$ has summable variations and $V_1(\varphi) < \infty$. Then*

$$P_G(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n x = x} \exp\left(\sum_{i=0}^{n-1} \varphi(\sigma^i x)\right). \tag{2.3.1}$$

Remark 2.3.2. In [MU1, p. 117] and [MU2, p. 7] the authors developed the theory using a different definition of the pressure. Indeed, they defined the pressure basically as the right side of 2.3.1. Hence, Proposition 2.3.5 shows that both definitions are equivalent in the fullshift.

Henceforth, we will denote P instead of P_G that will be named simply as *pressure of* φ. The following theorem relates this last definition with pressure defined in Section 2.2 for compact spaces [Sar1, Corollary 1].

Theorem 2.3.6. *Let $\varphi : \Sigma \to \mathbb{R}$ be a weakly Hölder continuous potential. If $\mathcal{K} = \{K \subset \Sigma : K$ compact and σ-invariant, $K \neq \emptyset\}$ then*

$$P(\varphi) = \sup\{P(\varphi|K) : K \in \mathcal{K}\}$$

where $P(\varphi|K)$ is the pressure defined as in Section 2.2 for $\varphi|_K : K \to \mathbb{R}$.

Theorem 2.3.7 (Variational Principle)**.** *Assume that $\varphi : \Sigma \to \mathbb{R}$ has summable variations. Then*

$$P(\varphi) = \sup\left\{h(\mu) + \int \varphi d\mu : \mu \in \mathcal{M}_\sigma \text{ and } -\int \varphi d\mu < \infty\right\}.$$

Proof. [Sar5, Theorem 5.3] and [IJT, Lemma 2.9]. □

2.3.3 Equilibrium measures

Definition 2.3.8. A measure $\mu \in \mathcal{M}_\sigma$ is called an *equilibrium state* for φ if μ attains the supremum in the Variational Principle, that is, μ is a σ-invariant measure such that $-\int \varphi < \infty$ and

$$P(\varphi) = h(\mu) + \int \varphi d\mu.$$

Theorem 2.3.8. *Assume that $\varphi : \Sigma \to \mathbb{R}$ has summable variations, $V_1(\varphi) < \infty$ and $P(\varphi) < \infty$. Then there exists at most one equilibrium measure for the potential φ.*

Proof. See [BS, Theorem 1.1]. □

Definition 2.3.9. Given a potential $\varphi : \Sigma \to \mathbb{R}$, we say that a measure μ on Σ is a *Gibbs measure* if there exists numbers $C > 0$ and $P \in \mathbb{R}$ such that for every cylinder $C_{i_0 i_1 \ldots i_{n-1}}$ we have

$$\frac{1}{C} \leq \frac{\mu(C_{i_0 i_1 \ldots i_{n-1}})}{\exp(-nP + \sum_{i=0}^{n-1} \varphi(\sigma^i x))} \leq C$$

for all $x \in C_{i_0 i_1 \ldots i_{n-1}}$.

This definition gives a description of the measure of a cylinder in the sense of that we can compare it with $\exp(-nP + \sum_{i=0}^{n-1} \varphi(\sigma^i x))$ which involves the Birkhoff sum of the potential φ.

Remark 2.3.3. When we consider Gibbs measures, open sets have positive measure.

Theorem 2.3.9. *Suppose that $\varphi : \Sigma \to \mathbb{R}$ has summable variations and $V_1(\varphi) < \infty$. Assume that $P(\varphi) < \infty$. Then, there exists a unique Gibbs measure for φ. Denote by μ_φ such measure. Moreover, if $\int \varphi d\mu_\varphi > -\infty$ then μ_φ is the unique equilibrium measure.*

Proof. The existence was proved in a more strong result in [Sar3, Theorem 1]: there, the author proved the result for topological Markov shifts satisfying the *big images and preimages (BIP) property* (see [Sar3] for a definition), which is the case of the fullshift. The uniqueness of the Gibbs measure was proved in [MU2, Theorem 2.2.4] and the last part of the theorem, correspond to [MU2, Theorem 2.2.9]. □

2.3.4 Regularity of the pressure and consequences

The following result was proved in [Sar3, Corollary 4] and [MU2, Theorem 2.6.12].

Theorem 2.3.10. *Suppose that $\varphi, \psi : \Sigma \to \mathbb{R}$ have summable variations and finite first variation. Let $I \subset \mathbb{R}$ an open interval such that $P(\varphi + t\psi) < \infty$ for all $t \in I$. Then, the function $t \mapsto P(\varphi + t\psi)$ is real analytic in I.*

Theorem 2.3.11. *Let $\varphi, \psi : \Sigma \to \mathbb{R}$ be two functions with summable variations and having finite first variations. Moreover, suppose that $P(\varphi + t_0\psi) < \infty$ for some $t_0 \in \mathbb{R}$. Let μ_{t_0} be the Gibbs measure for $\varphi + t_0\psi$ and suppose that $\int -(\varphi + t\psi) d\mu_{t_0} < \infty$ for all t in an open neighborhood of t_0. Then*

$$\frac{d}{dt} P(\varphi + t\psi)\Big|_{t=t_0} = \int \psi d\mu_{t_0}$$

and

$$\frac{d^2}{dt^2} P(\varphi + t\psi)\Big|_{t=t_0} = \sigma^2_{t_0}(\varphi, \psi)$$

where

$$\sigma^2_t(\varphi, \psi) = \lim_{n\to\infty} \frac{1}{n} \int S_n \left(\varphi - \int \varphi d\mu_t \right) S_n \left(\psi - \int \psi d\mu_t \right) d\mu_t.$$

Proof. [MU2, Proposition 2.6.14] □

Example. Suppose that $\varphi : \Sigma \to \mathbb{R}$ is a negative locally Hölder potential such that $P(\varphi) < \infty$. Then, there exists a critical value $t^* \in (0, 1]$ such that

$$P(t\varphi) \text{ is } \begin{cases} \text{infinite}, & \text{if } t < t^* \\ \text{finite}, & \text{if } t > t^*. \end{cases}$$

Moreover, when $t > t^*$ the pressure function $t \mapsto P(t\varphi)$ is real analytic and convex. Observe that if $t > t^*$, there exists a Gibbs measure for $t\varphi$. Let μ_t be such measure. If φ is μ_t-integrable, then μ_t is an equilibrium measure for $t\varphi$ and $t \mapsto P(t\varphi)$ is a strictly decreasing function with first derivative given by

$$\frac{d}{dt}P(t\varphi) = \int \varphi d\mu_t.$$

Assuming also that φ is not a coboundary, then $t \mapsto P(t\varphi)$ is strictly convex.

2.4 Thermodynamic formalism for EMR maps

Denote by $I = [0,1]$. This section is devoted to studying thermodynamic formalism for dynamical systems $T : I \to I$ modeled by a fullshift on countable symbols. Gauss-like maps or Lüroth maps are examples of such dynamics. Since the fullshift is a non-compact space, the results of Section 2.3 will be used. Moreover, as regularity assumptions on potentials are required to define the pressure, we need to put some conditions on T to get comparable results for potentials on I. In [PW] a special class of maps $T : I \to I$ called *Expanding-Markov-Rényi maps* (EMR) was studied. We start recalling the definition of Markov map.

Definition 2.4.1. We say that $T : I \to I$ is a *Markov map* if there exists a countable (or finite) collection $\{O_n\}_{n\in\mathcal{A}\subset\mathbb{N}_0}$ of open non-empty subintervals of I satisfying the following properties

1. $T\big|_{\overline{O_n}} : \overline{O_n} \to T(\overline{O_n})$ is a homeomorphism,
2. $O_n \cap O_m = \emptyset$ for $n \neq m$,
3. If for some $n \neq m, T(O_n) \cap O_m \neq \emptyset$ them $O_m \subset T(O_n)$. The collection $\{O_n\}_{n\in\mathbb{N}}$ is called a *Markov partition*.

Remark 2.4.1. Along this thesis, the maps considered satisfies in general that $(0,1] \subset \bigcup_{n\in\mathcal{A}} \overline{O_n}$. Further details on Markov maps on the interval, see [KMS2].

Example. The T_k maps defined on Section 1.4 are Markov maps with Markov partition

$$\{I_n^k\}_{n\geq 0} := \left\{ \left(\frac{k}{n+k+1}, \frac{k}{n+k} \right] \right\}_{n\geq 0}.$$

Definition 2.4.2. A map $T : I \to I$ is an *EMR map*, if there exists a countable family $\{I_i\}_i$ of closed intervals (with disjoint interiors $\mathrm{int}(I_n)$), with $I_n \subset I$ for every $i \in \mathbb{N}$, satisfying

1. If $I_n = [a_n, b_n]$, then a_n, b_n are decreasing sequences, $b_1 = 1$, and $b_n \to 0$.

2. The map is C^2 on $\bigcup_{i=1}^{\infty} \mathrm{int}(I_i)$.

3. (Expansiveness) There exists a constant $\alpha > 1$ and $N \in \mathbb{N}$ such that for every $x \in \bigcup_{i=1}^{\infty} \mathrm{int}(I_i)$, we have $|(T^N)'(x)| > \alpha$.

4. (Markov) The sequence $\{\mathrm{int}(I_n)\}_{n\geq 1}$ is a Markov partition for T.

5. (Rényi) There exists a positive number $K > 0$ such that

$$\sup_{n\in\mathbb{N}} \sup_{x,y,z\in I_n} \frac{|T''(x)|}{|T'(y)||T'(z)|} \leq K.$$

We will be interested in the points of I such that all the orbit is well defined. We call the repeller of T to the set

$$\Lambda := \left\{ x \in \bigcup_{i=1}^{\infty} I_i : T^n(x) \text{ is well defined for every } n \in \mathbb{N} \right\}.$$

The Markov assumption in definition of EMR maps allows us to codify the system in a well defined way. More precisely, we can represent the system $T : \Lambda \to \Lambda$ by a fullshift on a countable alphabet (Σ, σ), with a continuous map $\pi : \Sigma \to \Lambda$ such that $\pi \circ \sigma = T \circ \pi$. In fact, if we define the set E of end points of the partition $\{I_n\}_n$, then we have that the map $\pi : \Sigma \to \Lambda \setminus \bigcup_{n\in\mathbb{N}} T^{-n}E$ is an homeomorphism. If $C_{a_1,\dots,a_n}$ denotes a typical cylinder on the fullshift (Σ, σ), we define $I(a_1, \dots, a_n) := \pi(C_{a_1,\dots,a_n})$ a cylinder of level n for T.

The Rényi condition gives the following relation between the derivative along the orbit of points that belong to a same cylinder of level n (see [CFS, Chapter 7, Section 4]).

Proposition 2.4.1 (Bounded distortion property)**.** *There exists a positive constant $C > 0$ such that for all $n \geq 1$ and for every $x \in I(a_1, \dots, a_n)$ the following holds*

$$\frac{1}{C} \leq \left| \frac{(T^n)'(x)}{(T^n)'(y)} \right| \leq C$$

for all $y \in I(a_1, \dots, a_n)$.

Moreover, the expansiveness together with the bounded distortion property, allows to estimate the length of the cylinders $I(a_1, \dots, a_n)$.

Corollary 2.4.2. *Let $n \geq 1$. If N and α are the constants involved in the expansiveness condition in Definition 2.4.2, then the length of the cylinder $I(a_, \dots, a_n)$ is bounded above, up to a positive factor, by $\alpha^{n/N}$.*

Proof. First, note that, for all $n \geq 1, x \in I(a_1, \dots, a_n)$, then

$$\frac{1}{C} \leq |(T^n)'(x)||I(a_1, \dots, a_n)| \leq C.$$

If $n = mN$ for some $m \geq 1$ then $|(T^n)'(x)| > \alpha^m = \alpha^{n/N}$. In particular, $|I(a_1, \dots, a_n)| < C\alpha^{-n/N}$. Assume that $n/N > 1$ is not an integer. Then

$$|I(a_1, \dots, a_n)| \leq |I(a_1, \dots, a_{[n/N]N})| \leq |(f^{[n/N]N})'|^{-1} < C\alpha^{-[n/N]} \leq C\alpha^{1-n/N}.$$

When $0 < n/N < 1$, then $|I(a_1, \dots, a_n)| < 1 = \alpha^{-[n/N]} \leq \alpha^{1-n/N}$. □

We conclude this subsection with the definition of the pressure for a EMR map T. Denote by $\tilde{\Lambda} := \Lambda \setminus \bigcup_{n\in\mathbb{N}} T^{-n}E$.

Definition 2.4.3. Let $\varphi : \tilde{\Lambda} \to \mathbb{R}$ a function such that $\varphi \circ \pi : \Sigma \to \mathbb{R}$ has summable variations and $V_1(\varphi \circ \pi) < \infty$. We define the *pressure of φ with respect to T* by

$$P_T(\varphi) := P_G(\varphi \circ T) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{T^n x = x} \exp\left(\sum_{i=0}^{n-1} \varphi(T^i x)\right).$$

If there is not risk of confusion we write P instead of P_T.

Examples. In the following examples, we assume that T is an EMR map with intervals $\{I_n\}_n$.

1. Let $\varphi : \tilde{\Lambda} \to \mathbb{R}$ to be constant at cylinders of level 1 for T, of the form $\varphi\big|_{I_{a_i}} = \log \lambda_i$, for all i. In particular $\varphi \circ \pi$ is constant (equal to $\log \lambda_i$) at cylinders of level 1 in (Σ, σ). In this case, $\varphi \circ \pi$ is called *locally constant* and note that it satisfies all regularity assumptions for results from Section 2.3. Therefore

$$P(\varphi) = \log \sum_{n=1}^{\infty} \lambda_n.$$

2. (*pressure function*) Let $\varphi : \tilde{\Lambda} \to \mathbb{R}$ given by $\varphi = -\log |T'|$. We will prove that $\varphi \circ \pi$ is a locally Hölder potential. In fact, if $x, y \in \Sigma$ with $x_0 = y_0, \dots x_{n-1} = y_{n-1}$ then, there exist w between $\pi(x)$ and $\pi(y)$ such that

$$\begin{aligned} |\log |T'|(\pi(x)) - \log |T'|(\pi(y))| &= \frac{|T''(w)|}{|T'(w)|}|\pi(x) - \pi(y)| \leq \frac{|T''(w)|}{|T'(w)|}|\pi(C_{x_0\dots x_{n-1}})| \\ &= \frac{|T''(w)|}{|T'(w)|}|I(x_0 \dots x_{n-1})| \leq C\frac{|T''(w)|}{|T'(w)|}|(T^n)'(z)|^{-1} \\ &\leq C\frac{|T''(w)|}{|T'(w)||T'(T^{n-1}(z))|}|(T^{n-1})'(z)|^{-1} \end{aligned}$$

for any $z \in I(x_0 \dots x_{n-1})$. The Rényi hypothesis of T and Corollary 2.4.2 implies the existence of a positive constant $M > 0$ (independent of the cylinder) such that $|\log|T'(\pi(x))| - \log|T'(\pi(y))|| \leq M\alpha^{-(n-1)/N}$. Thus, for all $n \geq 1$

$$V_n(\varphi \circ \pi) \leq M\alpha^{1/N}\alpha^{-n/N}.$$

Note that the same calculations are valid for $\varphi_t = -t\log|T'|$ and $t > 0$. The pressure of this potential is given by

$$P(\varphi_t) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{T^n x = x} \exp \sum_{i=0}^{n-1} \log|T'(T^i x)|^{-t} = \lim_{n\to\infty} \frac{1}{n} \log \sum_{T^n x = x} \prod_{i=0}^{n-1} |T'(T^i x)|^{-t}$$

and we can apply results from Section 2.3 when it is finite. From now on, we will call $t \mapsto P(\varphi_t)$ the *pressure function*.

Chapter 3

Lyapunov Spectrum for T_k maps

In this chapter we will apply the tools from thermodynamic formalism to multifractal analysis theory of Lyapunov exponents of T_k *maps. Given* $\alpha \in \mathbb{R}$ *we will be interested in the Hausdorff dimension of level sets for the Lyapunov exponents, that is, in the set of points* $x \in [0, 1]$ *having Lyapunov exponent equal to* α*. Then, the Lyapunov spectrum consist in studying the dimension of the level sets as a function in* α*. When* $k = 1$ *(i.e. the Gauss map), the Lyapunov spectrum was completely determined by Pollicott-Weiss and Kessebömer-Stratmann [PW, KS2]. In concrete, we will prove that the Lyapunov spectrum is real analytic. We use results from [Iom].*

As we saw in Chapter 1, one of the consequences of Birkhoff ergodic theorem applied to the Gauss map G is that Lebesgue almost every $x = [a_1, a_2, \ldots] \in [0,1]$ the exponential speed of approximation by the convergents is

$$\lim_{n\to\infty} \frac{1}{n} \log \left| x - \frac{p_n}{q_n} \right| = -\frac{\pi^2}{6\log 2}. \tag{3.0.1}$$

This implies that the lengths of the cylinders associated to Gauss map G tends to zero exponentially fast since the LHS of 3.0.1 is equal to $\lim_{n\to\infty} \frac{1}{n} \log |I_n(x)|$. From a dynamical point of view, the bounded distortion property applied to G, allows to know the behavior of the orbits by means of the equality

$$\lim_{n\to\infty} \frac{1}{n} \log |(G^n)'(x)| = -\lim_{n\to\infty} \frac{1}{n} \log |I_n(x)|.$$

Definition 3.0.1. Let $(T, [0,1])$ be a piecewise differentiable dynamical system. The Lyapunov exponent of x with respect to T is defined by the limit

$$\lambda_T(x) = \lim_{n\to\infty} \frac{1}{n} \log |(T^n)'(x)| = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log |T'(T^i(x))|$$

whenever exists.

From 3.0.1 we have that, Lebesgue almost every $x \in [0,1]$ has Lyapunov exponent $\lambda_G(x) = \frac{\pi^2}{6\log 2}$ which is not the only possible value: if x_0 is a fixed point of G, then $\lambda_G(x_0) = \log |G'(x_0)| = -2\log x_0$. In fact, the range of all possible values of Lyapunov exponents for the Gauss map is the interval $[-2\log \frac{1+\sqrt{5}}{2}, \infty)$ [PW]. On the other hand, the Lyapunov exponent of any rational number x does not exists since $G^n(x) = 0$ for some $n \geq 1$. Also, the Liouville's number $x = \sum_{k=1}^{\infty} 10^{-k!}$ is a non-trivial example for which its Lyapunov exponent does not exists [PW, p.164].

In [PW, KS2] the authors gave a complete description of the multifractal analysis for

Lyapunov exponents for the Gauss map. Multifractal analysis is a branch of the dimension theory of dynamical systems. It typically involves decomposing the phase space into level sets where some local quantity takes a fixed value. Questions that are usually addressed are determining the the size of each of the level sets and how does this dimension varies with the parameter. Let us explain how this analysis works in the case of Lyapunov exponents for the Gauss map. Given $\alpha \in [2 \log \frac{1+\sqrt{5}}{2}, \infty)$ define the *level set*

$$J(\alpha) := \{x \in [0,1] : \lambda_G(x) = \alpha\}.$$

Then we get a decomposition

$$[0,1] = \bigcup_{\alpha} J(\alpha) \cup J'$$

where $J' := \{x \in [0,1] : \lambda_G(x) \text{ does not exists}\}$ is called the irregular set. Observe that if $\alpha \neq \frac{\pi^2}{6 \log 2}$, then $J(\alpha)$ has null Lebesgue measure therefore a good way to measure those sets is using the Hausdorff dimension (see Section 5.4 or [Fal, Chapter 2] for further details).

Thus, multifractal analysis study the function $\alpha \mapsto \dim_H J(\alpha)$. The following theorem [PW, KS2] characterize this functions by means of thermodynamic formalism tools.

Theorem 3.0.1 (Pollicott-Weiss, Kesseböhmer-Stratmann)**.** *If $P(\cdot)$ denotes the pressure function for the Gauss maps, then the following holds:*

$$g(\alpha) := \dim_H J(\alpha) = \frac{1}{\alpha} \inf_{t \in \mathbb{R}} (P(-t \log |G'|) + t\alpha).$$

Moreover, the function $g : [2 \log \frac{1+\sqrt{5}}{2}, \infty) \to [0,1]$ is real analytic.

The hidden technical tool in Theorem 3.0.1 which connects dimension theory and thermodynamic formalism is the *Legendre transform.* Let $I \subset \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$

be a convex function. We define the Legendre transformation $\widehat{f}$ of f by

$$\widehat{f}(\alpha) := \sup_{t \in I}\{\alpha t - f(t)\}.$$

Therefore, Theorem 3.0.1 shows that the multifractal analysis for Lyapunov expoenents for the Gauss map is completely understood, essentially, by the Legendre transformation of the pressure function since $g(\alpha) = \widehat{P}(-\alpha)/-\alpha$. Moreover, this formula was extended in [Iom]. We rephrase this result in the case of EMR maps.

Theorem 3.0.2. *Let T be a EMR map, and let $P(\cdot)$ be the pressure function of T. Then the Lyapunov spectrum satisfies*

$$\dim_H\{x \in [0,1] : \lambda_T(x) = \alpha\} = \frac{\widehat{P}(-\alpha)}{-\alpha}$$

for all α in an unbounded interval of the form $[\alpha_{\min}, \infty)$. Moreover on this domain the Lyapunov spectrum is real analytic.

From now on, this chapter is concerned to understand the Lyapunov spectra of T_k maps.

3.1 Pressure function for T_k maps

Along this section we will be interested in studying the pressure function for the maps T_k defined on Section 1.4.

Proposition 3.1.1. *The map T_k is EMR.*

Proof. We will prove the hypothesis given in Definition 2.4.2 for the countable family $\{\overline{I_1^k(n)}\}_{n \in \mathbb{N}_0}$, where

$$I_1^k(n) = \left(\frac{k}{n+k+1}, \frac{k}{n+k}\right].$$

As discussed in Section 2.4, the collection of intervals $\{\mathrm{int}(I_1^k(n))\}_{n\in\mathbb{N}_0}$ is a Markov partition for T_k. Also, note that T_k is a C^2 map. We need to prove the expansiveness of T_k and the Rényi condition. Let us first to prove the expansiveness. Let $n \geq 0$ such that $T_k(x) \in I_1^k(n)$. Note that $T_k''(x) = 2k/x$ which is always positive and therefore T_k' is increasing. Thus

$$(T^2(x))' = T_k'(T_k(x))T_k'(x) > \frac{k}{(k/(n+k+1))^2} \cdot \frac{k}{x^2} > (n+k+1)^2 \geq (k+1)^2.$$

In order to prove property (5) of Definition 2.4.2, we note that $|T_k'(x)| = k/x^2$ and $|T_k''(x)| = k^2/x^3$. Then, on the interval I_n

$$\frac{|T_k''(x)|}{|T_k'(y)||T_k'(z)|} = \frac{y^2z^2}{x^3} \leq \frac{k(n+k+1)^3}{(n+k)^4}$$

then

$$\sup_{n\geq 0} \frac{k(n+k+1)^3}{(n+k)^4} = \frac{(k+1)^3}{k^3} < \infty.$$

$\square$

The last proposition allows us to calculate the pressure for the potential $\varphi = -t\log|T_k'|$. Recall from Example 2.4 that, $P_k(t)$ take the form

$$P_k(t) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{T_k^n x = x} |(T_k^n)'x)|^{-t}. \tag{3.1.1}$$

Proposition 3.1.2. *Let $k > 0$. The pressure function $t \mapsto P(-t\log|T_k'|)$ is finite if $t > 1/2$ and it is equal to ∞ if $t < \frac{1}{2}$. When $t > 1/2$, $P(-t\log|T_k'|)$ is real analytic, strictly decreasing and strictly convex. Moreover $P(-t\log|T_k'|) \to \infty$ when $t \to \frac{1}{2}^+$.*

Proof. For each n, the mean value theorem guarantees the existence of $z \in I_1^k(n)$ such that

$$|T_k'(z)| = \frac{1}{|I_1^k(n)|},$$

then, for all $x \in I_1^k(n)$ we have

$$\frac{1}{C} \leq \frac{|T_k'(x)|}{|I_1^k(n)|^{-1}} \leq C.$$

Therefore

$$C^{-tn} \sum_{(j_1,\dots,j_n)\in\mathbb{N}_0^n} \prod_{i=1}^{n} |I_1^k(j_i)|^t \leq \sum_{T_k^n x = x} \prod_{i=0}^{n-1} |T_k'(T_k^i x)|^{-t} \leq C^{tn} \sum_{(j_1,\dots,j_n)\in\mathbb{N}_0^n} \prod_{i=1}^{n} |I_1^k(j_i)|^t$$

We note that each one of the sums at left and right are equal to $\left(\sum_{j=0}^{\infty} |I_1^k(j)|^t\right)^n$, which implies

$$-tn\log C + n\log\sum_{j=0}^{\infty} |I_1^k(j)|^t \leq \log \sum_{T_k^n x = x} \prod_{i=0}^{n-1} |T_k'(T_k^i x)|^{-t} \leq -tn\log C + n\log\sum_{j=0}^{\infty} |I_1^k(j)|^t$$

and

$$-t\log C + \log\sum_{j=0}^{\infty} |I_1^k(j)|^t \leq P(-t\log|T_k'|) \leq -t\log C + \log\sum_{j=0}^{\infty} |I_1^k(j)|^t \tag{3.1.2}$$

First two assumptions are given by the convergence of series involved in inequality (3.1.2). For the limit, we first note that by Fatou's lemma

$$\liminf_{n\to\infty} \sum_{j=0}^{\infty} \frac{1}{(j+k+1)^{\frac{1}{2}+\frac{1}{n}}(j+k)^{\frac{1}{2}+\frac{1}{n}}} \geq \sum_{j=0}^{\infty} \liminf_{n\to\infty} \frac{1}{(j+k+1)^{\frac{1}{2}+\frac{1}{n}}(j+k)^{\frac{1}{2}+\frac{1}{n}}} = \infty.$$

From Section 2.3.4 $P(-t\log|T_k'|)$ is a real analytic, strictly convex and strictly decreasing function on $(\frac{1}{2},\infty)$ since

$$\frac{dP(-t\log|T_k'|)}{dt} = -\int \log|T_k'|d\mu_t^k < 0.$$

Using that and the inequality (3.1.2) we have that $P(-t\log|T_k'|) \to \infty$, when $t \to 1/2^+$. $\square$

3.2 Lyapunov spectrum

In this section we calcule the Lyapunov exponents of $x \in [0,1]$, for T_k maps. Since the Lyapunov exponent of $x \in [0,1]$ with respect to T_k is defined by the limit

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \log |(T_k^n)'(x)| = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log |T_k'(T_k^i(x))|$$

whenever exists, and by Birkhoff Ergodic Theorem, we have

$$\lambda(x) = \int_0^1 \log |T_k'| d\mu_k$$

for μ_k-a.e. $x \in [0,1]$, where μ_k is the T_k invariant measure defined in section 1.4. From Theorem 1.5.4 and Corollary 1.5.4 we have that, Lebesgue almost every $x \in [0,1]$

$$\lambda(x) = 2 \lim_{n\to\infty} \frac{\log q_n}{n} = 2 \log \sqrt{k} - 2 \left(\log \frac{k+1}{k} \right)^{-1} \mathcal{L}_2 \left(-\frac{1}{k} \right)$$

where

$$\mathcal{L}_2(z) = \int_z^0 \frac{\log(1-t)}{t} dt.$$

Now, we are interested on the range of the function $x \mapsto \lambda(x)$ whenever the Lyapunov exponent exists. Note that, when x is a fixed point of T_k, then

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \left| \frac{-k}{x^2} \right| = \log k - 2 \log x.$$

Let $k > 0$. On each subinterval $I_n, n \geq 0$, there exists a unique fixed point x_n^k given by

$$x_n^k := \frac{-(n+k) + \sqrt{(k+n)^2 + 4k}}{2}.$$

Note that x_n^k is equal to the k-continued fraction having the constant coding $[[\overline{n}]]_k := [n, n, n, \ldots]$. Thus the Lyapunov exponent of each fixed point is

$$\lambda(x_n^k) = \log k - 2\log\left(\frac{-(n+k) + \sqrt{(n+k)^2 + 4k}}{2}\right)$$

for all $n \geq 0$. In particular, we have that $x \mapsto \lambda(x)$ is unbounded.

Lemma 3.2.1. *Let $x \in [0,1]$ such that $\lambda(x)$ exists. Then*

$$\lambda(x) \geq 2\log\left(\frac{\sqrt{k} + \sqrt{k+4}}{2}\right) = \lambda([\overline{0}]_k).$$

Before proving this lemma, we will first a prove a simple property on convergents that will be useful. From Proposition 1.4.4 we have the following recursion

$$q_n = \frac{1}{\sqrt{k}}((a_n + k)q_{n-1} + \sqrt{k}q_{n-2}), n \geq 2;$$

$q_0 = q_1 = 1$. The next simple lemma will be useful for our calculations.

Lemma 3.2.2. *If $x = [a_1, \ldots, a_n, \ldots]_k$ and $y = [b_1, \ldots, b_n, \ldots]_k$ are two k-continued fractions with $a_n \leq b_n$ for all $n \geq 1$, then $q_n(x) \leq q_n(y)$ for all $n \geq 1$. Here $q_n(x)$ represents the denominator of nth convergent associated to x.*

Proof. The proof is by induction. For $n = 1$ we have $q_1(x) = 1 = q_1(y)$. Now, suppose that $q_k(x) \leq q_k(y)$ for all $k \leq n$. Then

$$\begin{aligned} q_{n+1}(x) &= \frac{1}{\sqrt{k}}((a_{n+1} + k)q_n(x) + \sqrt{k}q_{n-1}(x)) \\ &\leq \frac{1}{\sqrt{k}}((b_{n+1} + k)q_n(y) + \sqrt{k}q_{n-1}(y)) \\ &= q_{n+1}(y) \end{aligned}$$

and thus we conclude the proof. $\square$

Proof of Lemma 3.2.1. Let x be a k-irrational number. For any $n \geq 1$, we have that $q_n(x) \geq q_n([\overline{0}]_k)$ where $[\overline{0}]_k$ is the largest fixed point of T_k. By simplicity, denote $q_n := q_n([\overline{0}]_k)$. Note that $q_n = q_{n-1}\sqrt{k} + q_{n-2}$, which is a linear difference equation with initial conditions given by $q_0 = q_1 = 1$. To solve this equation, we observe that $x^2 = \sqrt{k}x + 1$ has two roots given by

$$\phi_k = \frac{\sqrt{k} + \sqrt{k+4}}{2}; \quad \overline{\phi}_k = \frac{\sqrt{k} - \sqrt{k+4}}{2},$$

with $-1 < \overline{\phi}_k < 0 < \phi_k < 1$. Then, there exist $A, B \in \mathbb{R}$ such that, for any $n \geq 1$, we have $q_n = A\phi_k^n + B\overline{\phi}_k^n$. Thus, $q_n \geq C\phi_k^n$ for some constant $C > 0$. Finally,

$$\lambda(x) = 2 \lim_{n\to\infty} \frac{\log q_n(x)}{n} \geq 2 \lim_{n\to\infty} \left(\frac{1}{n}\log C + \log \phi_k\right) = 2\log(\phi_k).$$

$\square$

Denote by $\lambda^k_{min} := \lambda([\overline{0}]_k) = 2\log \phi_k$.

Proposition 3.2.3. *We have that* $\{\lambda(x) \in \mathbb{R} : x \in [0,1]\} = [\lambda^k_{\min}, \infty)$.

Proof. Let $t > 1/2$ and let μ^k_t be the equilibrium state of $P_k(t) := P(-t\log|T_k'|)$. By T_k-ergodicity of μ_k, we have that $\int \log|T_k'|d\mu_t = \lambda(x)$ for some $x \in [0,1]$. Then

$$\left\{\int \log|T_k'|d\mu^k_t : t > 1/2\right\} \subset \{\lambda(x) \in \mathbb{R} : x \in [0,1]\}$$

By Proposition 3.1.2, $t \mapsto P_k(t)$ is analytic on $(1/2, \infty)$ and in particular $t \mapsto P'(t) = -\int \log|T_k'|d\mu_t$ is continuous on $(1/2,\infty)$. Therefore $\{\int \log|T_k'|d\mu^k_t : t > 1/2\}$ is an interval in $\mathbb{R}$ and

$$\inf_{t>1/2} \int \log|T_k'|d\mu^k_t \geq \lambda_{\min}.$$

On the other hand,

$$\begin{aligned}\sup_{t>1/2} -\int \log|T_k'|d\mu_t^k &= \lim_{t\to\infty} P_k'(t)\\ &= \lim_{t\to\infty}\frac{P_k(t)}{t} \geq -\int \log|T_k'|d\delta_{[\overline{0}]_k} = -\lambda_{\min}\end{aligned}$$

where $\delta_{[\overline{0}]_k}$ denotes the Dirac's delta measure supported on the fixed point $[\overline{0}]_k$. Hence $\inf_{t>1/2}\int \log|T_k'|d\mu_t^k \geq \lambda_{\min}$. We conclude that

$$(\lambda_{\min},\infty) = \left\{\int \log|T_k'|d\mu_t^k : t > 1/2\right\}$$

and

$$\{\lambda(x)\in\mathbb{R} : x\in[0,1]\} = [\lambda_{\min},\infty).$$

$\square$

Finally, applying Theorem 3.0.2 we obtain,

Theorem 3.2.4. *The Lyapunov spectrum of T_k is given by the function*

$$\alpha \mapsto \frac{\widehat{P}(-\alpha)}{-\alpha} = \frac{1}{\alpha}\inf_{t\in\mathbb{R}}(P(-t\log|T_k'|)+t\alpha)$$

for all $\alpha\in[\lambda_{\min}^k,\infty)$. Moreover on this domain is a real analytic function.

Chapter 4

k-rational approximations in k-Lüroth expansions

In this chapter we study a one parameter family of numerical systems called k- Lüroth expansions. Every irrational number has an infinite expansion and associated to it there is a sequence of k-rational approximations. We are interested in the size of sets of points having the same exponential speed of approximations by k-rationals for different values of k. We prove that the Hausdorff dimension of these sets varies analytically with respect to the parameter k. Our techniques come from ergodic theory, in particular thermodynamic formalism for countable Markov shifts. The results obtained in this chapter appear in the article [Con2].

4.1 Introduction

In 1883, J. Lüroth [Lür] proved that every real number $x \in (0,1]$ has an expansion in the form

$$
\begin{aligned}
x = \frac{1}{a_1} + \frac{1}{a_1(a_1-1)a_2} + \ldots + \frac{1}{a_1(a_1-1)\cdots a_{n-1}(a_{n-1}-1)a_n} + \cdots \\
= \sum_{n=1}^{\infty} \frac{1}{a_1(a_1-1)\cdots a_{n-1}(a_{n-1}-1)a_n}
\end{aligned}
$$

where $a_n \geq 2$, for all $n \geq 1$. This expansion is called the *Lüroth series* of x and it is denoted by $x = [a_1, a_2, \ldots]_1$. Arithmetic properties of the number can be read form its corresponding series. Indeed, rational numbers of $[0,1]$ are characterized by the fact that its expansion is either finite or periodic. Every irrational number has a unique infinite expansion. Interestingly, the Lüroth series is closely related to the dynamical properties of the transformation $L : [0,1) \to [0,1)$ defined by

$$
L(x) := \begin{cases} n(n+1)x - n & \text{if } x \in \left[\frac{1}{n+1}, \frac{1}{n}\right), n \in \mathbb{N} \\ 0 & \text{if } x = 0. \end{cases}
$$

If $x = [a_1, a_2 \ldots]_1$ then the following relation holds: $a_n = a_1(L^{n-1}(x))$ for $n \geq 1$, where $a_1(u) := n+1$ if $u \in [\frac{1}{n+1}, \frac{1}{n})$. Partial sums in the Lüroth series of an irrational number x can be thought of, in analogy to the continued fractions, as rationals approximations for the irrational number $x = [a_1, \ldots]_1$. For $n \geq 1$, denote by $p_n/q_n := [a_1, \ldots, a_n]_1$ that is,

$$
\frac{p_n}{q_n} = \frac{1}{a_1} + \frac{1}{a_1(a_1-1)a_2} + \cdots + \frac{1}{a_1(a_1-1)a_2\cdots a_{n-1}(a_{n-1}-1)a_n}.
$$

The number p_n/q_n is called *n-th approximant* of x. We stress that the Lüroth map can be thought of as a linear version of the Gauss map and that the Lüroth series is analogous to the continued fraction expansion.

In this article we study a generalization of this series expansion that will be defined by means of the following family of maps. For each $k > 0$, the *k-Lüroth map* $L_k : [0,1) \to [0,1)$ is defined by

$$L_k(x) := \begin{cases} x\frac{(n+k)(n+k+1)}{k} - (n+k), & \text{if } x \in \left[\frac{k}{n+k+1}, \frac{k}{n+k}\right), n \in \mathbb{N}_0 \\ 0, & \text{if } x = 0 \end{cases}$$

Note that the map $L_1(x)$ corresponds to the Lüroth map. Our interest in this family steams from work of Haas and Molnar [HM], where they studied metrical properties of a family of continued fractions, each of which is defined by an interval map obtained as the fractional part of a Möbius transformation taking the endpoints of the interval to zero and infinity. In particular, considering fractional parts of a family of Möbius tranfsormations $\frac{k(1-x)}{x}, k > 0$. The family of k-Lüroth maps can be thought of as linear versions of the Gauss-like maps studied in [HM]. Families of this type were also studied by Kesseböhmer, Munday and Stratmann in [KMS1].

It turns out that each k-Lüroth map induces a series expansion of every $x \in [0,1]$ whose n-th iterated $L_k^n(x)$ different from zero, and a set playing the role of rational numbers in the Lüroth expansion. Proposition 4.2.1 gives that if $L_k^n(x) \neq 0$ for all $n \geq 1$, then we have the expansion

$$x = \sum_{n=1}^{\infty} \frac{k^n}{(a_1+k)(a_1-1+k)\cdots(a_{n-1}+k)(a_{n-1}-1+k)(a_n+k)}$$

where

$$a_n = a_1(L_k^{n-1}(x)), n \geq 1,$$

and

$$a_1(u) := n+1 \text{ if } u \in \left[\frac{k}{n+1+k}, \frac{k}{n+k}\right), n \geq 0.$$

We stress that a_n depends on k. This series is called the *k-Lüroth series* of x and it will be denoted by $[a_1, a_2 \dots]_k$. The natural numbers a_n are called the *digits* of the expansion. Details are provided in sub-section 4.2.1. As in the case of the Lüroth expansion, for every $x \in [0,1]$ having an infinite k-Lüroth expansion $x = [a_1, \dots]_k$ we can define the n-th *k-approximant* of x by $p_n^k/q_n^k := [a_1, \dots, a_n]_k$. That is,

$$\frac{p_n^k}{q_n^k} = \frac{k}{a_1+k} + \cdots + \frac{k^n}{(a_1+k)\,(a_1-1+k)\cdots(a_{n-1}+k)\,(a_{n-1}-1+k)\,(a_n+k)}.$$

This article is devoted to study the exponential speed of approximations of a number $x \in [0,1]$ by its n-th *k-approximant* $(p_n^k/q_n^k)_n$, as the parameter k varies. More precisely we are interested in the following numbers

$$\lim_{n\to\infty} \frac{1}{n} \log \left| x - \frac{p_n^k}{q_n^k} \right|,$$

whenever the limit exists. We study the range of possible values, the size of the set of elements having a fixed exponential speed of approximation and how do these quantities varies with both, the parameter and the value of k. More precisely, for every $\alpha \geq 0$ we consider the set

$$N_k(\alpha) := \left\{ x \in [0,1) : \lim_{n\to\infty} \frac{1}{n} \log \left| x - \frac{p_n^k}{q_n^k} \right| = \alpha \right\}.$$

Thus, we will be interested in the range of values for which the sets $N_k(\alpha)$ are non-empty and the regularity properties of the maps $\alpha \mapsto \dim_H(N_k(\alpha))$, $k \mapsto \dim_H(N_k(\alpha))$. Here $\dim_H$ denotes the Hausdorff dimension of a set. This is an appropriate way to compute the size of the level sets since, as we will see in section 4.2.4, for every value of α (except for a single value, see Lemma 4.2.7) the Lebesgue measure of the level set is zero.

Our tools are dynamical in nature and are based in the following quantity which measures the exponential rate of divergence of infinitesimally close orbits.

Definition. The *Lyapunov exponent* of the transformation $L_k : [0,1) \to [0,1)$ at the point $x \in [0,1)$ is defined by

$$\lambda_k(x) := \lim_{n\to\infty} \frac{1}{n} \log |(L_k^n)'(x)|$$

whenever the limit exist.

The following relation, which will be proved in Proposition 4.2.8, allows to bring in all the ergodic theory machinery to our problem:

$$\lim_{n\to\infty} \frac{1}{n} \log |(L_k^n)'(x)| = -\lim_{n\to\infty} \frac{1}{n} \log \left| x - \frac{p_n^k}{q_n^k} \right|.$$

Note that it provides arithmetic information by dynamical means. Thus, it will be equivalent and simpler to consider the level sets determined by the Lyapunov exponents. We will therefore study the map

$$\tau_k(\alpha) := \dim_H \left(\{ x \in [0,1) : \lambda_k(x) = \alpha \} \right).$$

For fixed values of k this map was completely described in the work of Barreira and Iommi [BI] and in that of Kesseböhmer, Munday and Stratmann in [KMS1]. Indeed, this corresponds to the multifractal spectrum of Lyapunov exponents and with tools from thermodynamic formalism it can be shown that (in the appropriate domain) the map $\alpha \to \tau_k(\alpha)$ is real analytic. The main novelty of our work, from the dynamical systems point of view, is that we describe how does the multifractal spectrum of Lyapunov exponents varies along a one-parameter family of dynamical systems. That is, we describe how does the Hausdorff dimension of a level set changes with the dynamics. More precisely,

Theorem 4.1.1. *Let $M > 0$ and fix $\alpha \in \mathbb{R}$ such that $\tau_k(\alpha)$ is well defined for all $k \in (0, M]$.*

Then, the function of domain $(0, M]$ defined by

$$k \mapsto \tau_k(\alpha)$$

is real analytic.

From an arithmetic point of view, Theorem 4.1.1 characterizes how does the size of the set of points with same speed of approximations by their n-approximants varies in the different numerical systems provided by the k-Lüroth transformations.

4.2 Ergodic Theory preliminaries and series expansions

4.2.1 Dynamics of k-Lüroth expansions

In this section we discuss arithmetic as well as dynamical properties of the k-Lüroth expansions. Let $k > 0$.

Proposition 4.2.1. *Let $x \in (0, 1)$.*

1. *Let $m \geq 1$ be the smallest positive integer such that $L_k^{m-1}x = 0$. Then*

$$x = \frac{k}{a_1 + k} + \cdots + \frac{k^m}{(a_1 + k)(a_1 - 1 + k) \cdots (a_{m-1} + k)(a_{m-1} - 1 + k)(a_m + k)}.$$

2. *If $L_k^n(x) \neq 0$ for all $n \geq 0$, then*

$$\begin{aligned} x = &\frac{k}{a_1 + k} + \cdots + \frac{k^n}{(a_1 + k)(a_1 - 1 + k) \cdots (a_{n-1} + k)(a_{n-1} - 1 + k)(a_n + k)} + \\ &+ \frac{k^n L_k^n(x)}{(a_1 + k)(a_1 - 1 + k) \cdots (a_{n-1} + k)(a_{n-1} - 1 + k)(a_n + k)(a_n - 1 + k)}. \end{aligned}$$

The proof of Proposition 4.2.1 is obtained inductively similarly as in the Lüroth case (see [DK], pages 38-39).

The subset of *k-rationals*, denoted by $\mathbb{Q}_k$, is defined by

$$\mathbb{Q}_k = \{x \in [0,1) : \text{there is } n \in \mathbb{N}_0 \text{ such that } L_k^n(x) = 0\}.$$

The complement of $\mathbb{Q}_k$ in the unit interval $[0,1)$ is the so-called set of *k-irrationals*. We observe that when $k \in \mathbb{Q}$, then $\mathbb{Q}_k \subset \mathbb{Q}$.

Remark 4.2.1. For every $k > 0$, the set $\mathbb{Q}_k$ is a countable set which contains every number of the form

$$\frac{k}{a_1 + k} + \cdots + \frac{k^n}{(a_1 + k)\,(a_1 - 1 + k) \cdots (a_{n-1} + k)\,(a_{n-1} - 1 + k)\,(a_n + k)},$$

with $a_1, a_2, \ldots, a_n \in \mathbb{N}$ and $n \geq 1$.

As a consequence of Proposition 4.2.1 we obtain the following.

Proposition 4.2.2. *Let $k > 0$. Every k-irrational $x \in [0,1)$ can be expanded in a infinite k-Lüroth expansion, that is*

$$x = \lim_{n \to \infty} [a_1, a_2, \ldots, a_n]_k,$$

where a_n are obtained as in Proposition 4.2.1.

Proof. By Proposition 4.2.1, we have

$$\begin{aligned} \left| x - \frac{p_n^k}{q_n^k} \right| &= \frac{k^n L_k^n(x)}{(a_1 + k)\,(a_1 - 1 + k) \cdots (a_n + k)(a_n - 1 + k)} \\ &\leq \frac{1}{(1 + k)^n} \end{aligned}$$

which goes to zero when n tends to infinity. $\square$

Definition 4.2.1. Let $a_1, a_2, \ldots, a_n$ be positive integers. We define the cylinder of level n corresponding to $(a_1, a_2, \ldots, a_n)$ as the subset of $(0, 1]$ given by

$$\Delta^k(a_1, a_2, \ldots, a_n) = \{x \in [0, 1) : a_1(x) = a_1, a_2(x) = a_2, \ldots, a_n(x) = a_n\}.$$

In other words, $\Delta^k(a_1, a_2, \ldots, a_n)$ is the set of numbers in $[0, 1)$ whose k-Lüroth expansion starts with the digits $a_1, a_2, \ldots, a_n$.

Lemma 4.2.3. *Let $k > 0$. Then $\Delta^k(a_1, a_2, \ldots, a_n)$ is the subinterval of $[0, 1)$ given by*

$$\left[\frac{p_n}{q_n}, \frac{p_n}{q_n} + \frac{k^n}{(a_1+k)\,(a_1-1+k)\cdots(a_n+k)(a_n-1+k)}\right) \tag{4.2.1}$$

where

$$\frac{p_n}{q_n} = \sum_{j=1}^{n} \frac{k^j}{(a_1+k)\,(a_1-1+k)\cdots(a_{j-1}+k)\,(a_{j-1}-1+k)\,(a_j+k)}.$$

Proof. Let I be the interval given in equation (4.2.1). By Proposition 1, we have that $x \in \Delta^k(a_1, \ldots, a_n)$ if and only if

$$x = \frac{p_n}{q_n} + \frac{k^n L_k^n(x)}{(a_1+k)\,(a_1-1+k)\cdots(a_n+k)(a_n-1+k)}$$

which is equivalent that $x \in I$, because $L_k^n : [0, 1) \to [0, 1)$ is onto. In conclusion $\Delta^k(a_1, a_2, \ldots, a_n) = I$. □

As a consequence, each cylinder $\Delta^k(a_1, a_2, \ldots, a_n)$ is a subinterval of $[0, 1]$ with Lebesgue measure equal to

$$|\Delta^k(a_1, a_2, \ldots, a_n)| = \frac{k^n}{(a_1+k)\,(a_1-1+k)\cdots(a_n+k)(a_n-1+k)}.$$

Recall that a probability measure μ in $[0, 1]$ is invariant for the map $T : [0, 1] \to [0, 1]$ if for every Borel set $A \subset [0,1]$ we have $\mu(A) = \mu(T^{-1}A)$. Moreover, we say that an invariant measure μ is ergodic if for every set with the property that $A = T^{-1}A$ we have that $\mu(A) = 0$ or $\mu(A) = 1$, see [Wal, Chapter 1]. It was shown in [KMS1, Lemma 2.4] that:

Proposition 4.2.4. *For every $k > 0$, the map L_k is an ergodic transformation with respect to the Lebesgue measure.*

Therefore, all k-Lüroth maps have Lebesgue measure as a common invariant ergodic measure.

4.2.2 Symbolic Dynamics

The dynamics of the k-Lüroth map can be coded by the *full-shift* on a countable alphabet. This will allow us to reduce the study of the ergodic properties of the map to those of the shift, which are well known. The *full-shift* on a countable alphabet (Σ, σ) is the set

$$\Sigma := \{(x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{N} \text{ for every } n \in \mathbb{N}\},$$

together with the shift map $\sigma : \Sigma \to \Sigma$ defined by $\sigma(x_1, x_2, \dots) = (x_2, x_3, \dots)$. The set $C_{a_1 \dots a_n} := \{(x_n)_n \in \Sigma : x_1 = a_1 \dots x_n = a_n\}$ is called a *symbolic cylinder* of length n. The space Σ endowed with the topology generated by the cylinder sets is a non-compact space. This fact is one of the main difficulties that need to be addressed to develop the theory. The map

$$\pi_k : \Sigma \to [0,1] \setminus \mathbb{Q}_k$$
$$(x_1, x_2, \dots) \mapsto [x_1, x_2, \dots]_k.$$

is a topological conjugacy between the full-shift and the k-Lüroth map.

Remark 4.2.2. We observe that every cylinder is the projection of a symbolic cylinder $C_{a_1,\ldots,a_n}$, that is $\Delta(a_1,\ldots,a_n) = \pi_k(C_{a_1,\ldots,a_n})$.

4.2.3 Thermodynamic formalism

Thermodynamical formalism is a set of tools and methods brought into hyperbolic dynamics with great success in the early seventies from statistical physics. It allows for the selection of relevant measures from the, sometimes very large, set of invariant measures. It has been used as tool in the dimension theory of dynamical systems at least since the work of Bowen in the 70s [Bor]. Thermodynamic formalism for dynamical systems defined in non-compact spaces has been studied and developed over the last 20 years. The particular case of the full-shift on countable many symbols (Σ, σ) has been very well studied, see [BS, MU2, Sar2]. In this section we recall the main definitions and results.

Definition 4.2.2. We say that a potential φ is *locally Hölder* if there exists $\theta \in (0,1)$ such that for all $n \geq 1$, we have

$$\sup \{|\varphi(x) - \varphi(y)| : x, y \in \Sigma, x_i = y_i \text{ for } i = 1, \ldots, n\} \leq C\theta^n$$

for some positive constant C independent of n.

Definition 4.2.3. Let (Σ, σ) be the full-shift on a countable alphabet and $\varphi : \Sigma \to \mathbb{R}$ a locally Hölder function. The *pressure* of φ is defined by

$$P(\varphi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n x = x} \exp\left(\sum_{i=0}^{n-1} \varphi(\sigma^i x)\right).$$

The limit exists, but it can be infinity [BS, MU2, Sar2].

Theorem 4.2.5. *Let $\varphi : \Sigma \to \mathbb{R}$ be a locally Hölder negative function such that $P(\varphi) < \infty$.*

Then, there exists a critical value $t^* \in (0,1]$ such that

$$P(t\varphi) \text{ is } \begin{cases} \textit{infinite}\ , & \textit{if } t < t^* \\ \textit{finite}\ , & \textit{if } t > t^*. \end{cases}$$

Moreover, when $t > t^*$ the pressure function $t \mapsto P(t\varphi)$ is real analytic and strictly convex.

Example. If the function $\varphi : \Sigma \to \mathbb{R}$ is locally constant, that is $\varphi|_{C_{x_i}} = \log \lambda_i$, for every $i \in \mathbb{N}$, then we can explicitly calculate the pressure:

$$\begin{aligned} P(\varphi) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n x = x} \exp\left(\sum_{i=0}^{n-1} \varphi(\sigma^i x)\right) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{(j_0,\ldots,j_{n-1})\in\mathbb{N}^n} \lambda_{j_0}\lambda_{j_1}\cdots\lambda_{j_{n-1}} \\ &= \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{i\in\mathbb{N}_0} \lambda_i\right)^n = \log \sum_{i=1}^{\infty} \lambda_i. \end{aligned}$$

We are interested in the locally constant functions $\varphi_k|_{C_n} = \log \frac{(n+k+1)(n+k)}{k}$, since they correspond to the symbolic version $\log |L_k'|$. We observe that φ_k is a locally Hölder function. In this case, the pressure function is given by

$$P_k(t) := P(-t\varphi_k) = \log \sum_{n=0}^{\infty} \left(\frac{k}{(n+k+1)(n+k)}\right)^t .$$

This explicit expression and Theorem 6.3.1 implies the following result.

Proposition 4.2.6. *For every $k > 0$, the pressure function $t \mapsto P_k(t)$ is finite if $t > \frac{1}{2}$ and infinite if $t \leq 1/2$. When $P_k(t)$ is finite, then it is real analytic and strictly convex.*

We note that the critical value after which the pressure becomes finite is independent of the value of k.

4.2.4 Multifractal analysis for Lyapunov exponents

Multifractal analysis is a branch of the dimension theory of dynamical systems. It typically involves decomposing the phase space into level sets where some local quantity takes a fixed value. Questions that are usually addressed are determining the the size of each of the level sets and how does this dimension varies with the parameter. Thermodynamic formalism has been employed as tool to answer this questions. The theory is well understood for uniformly hyperbolic systems defined over compact phase spaces. In this section we will describe the multifractal spectrum of Lyapunov exponentes for the k-Lüroth maps. We stress in that in this context the phase space is no longer compact. Note that the the Lyapunov exponent of L_k at the point x satisfies

$$\lambda_k(x) := \lim_{n\to\infty} \frac{1}{n} \log |(L_k^n)'(x)| = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log |L_k'(L_k^{i-1}(x))|,$$

whenever the limit exists. In particular, it is a Birkhoff sum. The phase space $[0,1)$ can be decomposed into level sets. Indeed, for $\alpha \in \mathbb{R}$ we define

$$J_k(\alpha) := \{x \in [0,1) : \lambda_k(x) = \alpha\}.$$

Thus,

$$[0,1) = \bigcup_{\alpha} J_k(\alpha) \cup J_k'$$

where $J_k' = \{x \in [0,1) : \lambda_k(x) \text{ does not exists}\}$. We are interested in the multifractal spectrum of Lyapunov exponent which is defined by the function

$$\tau_k(\alpha) := \dim_H J_k(\alpha).$$

As observed in Proposition 4.2.4, the Lebesgue measure in $[0,1)$, that we denote by Leb, is invariant and ergodic for every Lüroth map L_k. It directly follows from the Birkhoff ergodic theorem that:

Lemma 4.2.7. *Let $k > 0$, then for Lebesgue almost every $x \in [0,1)$ the Lyapunov exponent with respect to L_k is given by*

$$\lambda_k(x) = \int_0^1 \log |L_k'| dLeb = \sum_{n=0}^{\infty} \frac{k}{(n+k)(n+k+1)} \log \frac{(n+k)(n+k+1)}{k}.$$

That is, for each fixed map L_k the Lyapunov exponent is constant Lebesgue almost everywhere. Note that $\lambda^k_{min} := \min\{\lambda_k(x) : x \in [0,1)\} = \log(k+1)$ which is attained at the largest fixed point of L_k. As we will see below, the range of values that the Lyapunov exponent can attain is the interval $[\lambda^k_{min}, \infty)$.

The relationship between Lyapunov exponents and the speed of convergence of p_n^k/q_n^k to x is given by the following.

Proposition 4.2.8. *If $x \in [0,1)$ is such that the Lyapunov exponent with respect to L_k exists then*

$$\lambda_k(x) = -\lim_{n\to\infty} \frac{1}{n} \log \left| x - \frac{p_n^k}{q_n^k} \right|.$$

Proof. Let $x \in \Delta^k(a_1, \ldots, a_n)$. Since the map L_k is piecewise linear, we observe that $|(L_k^n)'(x)|$ is constant equal to $1/|\Delta^k(a_1, \ldots, a_n)|$. Therefore

$$\begin{aligned}
\lambda_k(x) &= -\lim_{n\to\infty} \frac{1}{n} \log |\Delta^k(a_1, \ldots, a_n)| \\
&= -\lim_{n\to\infty} \frac{1}{n} \log \frac{k^n}{(a_1+k)\,(a_1-1+k)\cdots(a_n+k)(a_n-1+k)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}\log\left|x-\frac{p_n^k}{q_n^k}\right| &= \lim_{n\to\infty}\frac{1}{n}\log\frac{k^n L_k^n(x)}{(a_1+k)(a_1-1+k)\cdots(a_n+k)(a_n-1+k)} \\
&\leq \lim_{n\to\infty}\frac{1}{n}\log\frac{k^n}{(a_1+k)\,(a_1-1+k)\cdots(a_n+k)(a_n-1+k)} \\
&= -\lambda_k(x).
\end{aligned}$$

To obtain the other inequality, we observe that

$$\left|x-\frac{p_n^k}{q_n^k}\right| = \sum_{j=n+1}^{\infty}\frac{k^j}{(a_1+k)\,(a_1-1+k)\cdots(a_{j-1}+k)\,(a_{j-1}-1+k)\,(a_j+k)}$$

then

$$\begin{aligned}
\left|x-\frac{p_n^k}{q_n^k}\right| &\geq \frac{k^{n+1}}{(a_1+k)\,(a_1-1+k)\cdots(a_n+k)\,(a_n-1+k)\,(a_{n+1}+k)} \\
&\geq \frac{k^{n+2}}{(a_1+k)\,(a_1-1+k)\cdots(a_n+k)\,(a_n-1+k)\,(a_{n+1}+k)(a_{n+1}-1+k)}
\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\frac{1}{n}\log\left|x-\frac{p_n}{q_n}\right| \geq -\lambda_k(x)$$

and we conlcude the proof. □

Therefore, understanding the properties of the function $\tau_k(\alpha)$ corresponds to understand the level sets determined by the exponential speed of approximation of number by the $\mathbb{Q}_k$ approximants. It turns out that a description of the multifractal spectrum of the Lyapunov exponents of the Lüroth maps was done in [BI]. This was later extended to handle generalized Lüroth maps in [KMS1]. In both cases the main result is that the map $\tau_k(\alpha)$ can be described in terms of the Legendre transform of the pressure function. More precisely,

Definition 4.2.4. Let $I \subset \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a convex function. We define the Legendre transformation $\widehat{f}$ of f by

$$\widehat{f}(\alpha) := \sup_{t \in I}\{\alpha t - f(t)\}.$$

Remark 4.2.3. When f is convex and differentiable everywhere in I then

$$\widehat{f}(\alpha) = \alpha(f')^{-1}(\alpha) + f((f')^{-1}(\alpha)).$$

The following was proved in [BI, KMS1] and completely describes the multifractal spectrum.

Theorem 4.2.9. *Fix $k > 0$. Then the following holds:*

$$\tau_k(\alpha) = \frac{\widehat{P}(-\alpha)}{-\alpha} = \frac{1}{\alpha}\inf_{t \in \mathbb{R}}(P(-t\log|L_k'| + t\alpha)).$$

Moreover,

1. *The set $J_k(\alpha)$ is non empty if and only if $\alpha \in [\log(k+1), \infty)$.*
2. *The map $\tau_k : [\log(k+1), \infty) \to [0, 1]$ is real analytic.*
3. *We have that $\lim_{\alpha \to \infty} \tau_k(\alpha) = 1/2$.*
4. *If $\alpha \in [\log(k+1), \infty)$ then the set $J_k(\alpha)$ is dense in $[0, 1]$.*
5. *The irregular set has full Hausdorff dimension, that is $\dim_H J_k' = 1$.*

That is, despite the fact that the decomposition is extremely complicated, each level set is dense, the function that encodes it is as regular as it can be, real analytic.

4.3 Pressure and Lyapunov spectra in terms of the parameter k

In the previous section we described the multfractal spectrum for a fixed dynamical system, L_k. We now address the question as how does this function varies as a function of the dynamics. In this section we prove that, for every t and α fixed, the functions $k \mapsto P_k(t)$ and $k \mapsto \tau_k(\alpha)$ are real analytic functions.

Theorem 4.3.1. *Fix $t > 1/2$, then the function defined in $(0, \infty)$ by $k \mapsto P_k(t)$ is real analytic.*

Proof. Note that

$$P_k(t) = t \log k + \log \sum_{n=0}^{\infty} \frac{1}{(n+k+1)^t (n+k)^t}.$$

Since sums, compositions of real analytic functions is real analytic, it is sufficient to prove the real analyticity of the function

$$\sum_{n=0}^{\infty} \frac{1}{(n+k+1)^t (n+k)^t}.$$

In order to do so, we consider the extension of this series to the complex domain $D = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$. Let $F : D \to \mathbb{C}$ be defined by

$$F(z) := \sum_{n=0}^{\infty} \frac{1}{(n+z+1)^t (n+z)^t}.$$

Observe that F is well defined for every $z \in D$ and furthermore, it is an infinite sum of holomorphic functions in D. As a result of Weierstrass M-test (see [GKR, Corollary 7.3]) we have that $F(z)$ is a holomorphic function. In fact, let $f_n(z) = \frac{1}{(n+z+1)^t (n+z)^t}$ and $r > 0$. For each n we will prove that there exists $M_n > 0$ (possibly depending on r) such that for all $z \in B_r := \{z \in D : |z| < r\}$ we have

$$|f_n(z)| \leq M_n.$$

Moreover, $\sum M_n < \infty$. Let $n > r$ then for every $z \in B_r$ we have

$$\frac{1}{|(n+z+1)(n+z)|^t} \leq \frac{1}{(n+1-|z|)^t(n-|z|)^t} \leq \frac{1}{(n+1-r)^t(n-r)^t}.$$

Let

$$M_n := \frac{1}{(n+1-r)^t(n-r)^t}.$$

Observe now that, since $t > 1/2$ we have

$$\sum \frac{1}{(n+1-r)^t(n-r)^t} < \infty.$$

Thus, we deduce the uniform convergence of $F(z)$ on B_r, for every $r > 0$. Hence, the uniform convergence of $F(z)$ on every compact subset of D implies that $F(z)$ is holomorphic on D. In particular $F\big|_{\mathbb{R}}$ is real analytic. Finally, the pressure function $P_k(t)$ is real analytic in k. $\square$

We now address the question as how the family of Lyapunov spectra $\{\tau_k : [\lambda^k_{min}, \infty) \to \mathbb{R}\}_k$ changes for different values of k. In our next result we prove that it varies real analytically when we fix α in a common domain.

Theorem 4.3.2. *Let $M > 0$ and fix $\alpha \in [\lambda^M_{min}, \infty)$. Then, the function*

$$\begin{aligned} (0, M] &\to \mathbb{R} \\ k &\mapsto \tau_k(\alpha) \end{aligned}$$

is real analytic.

Proof. Recall that the Lyapunov spectrum is given by the following formula:

$$\tau_k(\alpha) = \frac{1}{\alpha} \inf_{t>1/2} (P(-t \log |T_k'|) + t\alpha) = \frac{1}{\alpha} \inf_{t>1/2} (P_k(t) + t\alpha) = \frac{\widehat{P_k}(-\alpha)}{-\alpha}$$

where $\widehat{P}_k$ is the Legendre transform of P_k. By convexity of $P_k(t)$, we have

$$\tau_k(\alpha) = \frac{1}{\alpha}(\alpha (P_k')^{-1}(-\alpha) + (P_k \circ (P_k')^{-1})(-\alpha))$$

where all derivatives are with respect to t. We already have proved that $k \mapsto P_k(t)$ is a real analytic function in k, when we fix t bigger than $1/2$ (see Theorem 4.3.1). Therefore, it is sufficient to show the analyticity of $P_k'(t)$.

Let $t > 1/2$ and recall that

$$P_k(t) = t \log k + \log \sum_{n=0}^{\infty} \frac{1}{(n+k+1)^t (n+k)^t}.$$

Hence

$$\begin{aligned} \frac{d}{dt} P_k(t) &= \log k + \frac{d}{dt} \log \sum_{n=0}^{\infty} \frac{1}{(n+k+1)^t (n+k)^t} \\ &= \log k + \left(\sum_{n=0}^{\infty} \frac{1}{(n+k+1)^t (n+k)^t} \right)^{-1} \frac{d}{dt} \sum_{n=0}^{\infty} \frac{1}{(n+k+1)^t (n+k)^t} \end{aligned} \tag{4.3.1}$$

Claim. We have that

$$\frac{d}{dt} \sum_{n=0}^{\infty} \frac{1}{(n+k+1)^t (n+k)^t} = \sum_{n=0}^{\infty} \frac{d}{dt} \frac{1}{(n+k+1)^t (n+k)^t}$$

Proof of the Claim. According to [Rud, Theorem 7.17], the equality holds when

$$\sum_{n=0}^{\infty} \frac{d}{dt} \frac{1}{(n+k+1)^t (n+k)^t} \tag{4.3.2}$$

converges uniformly on the set $[1/2+\varepsilon, \infty)$, for $\varepsilon > 0$. Note that

$$\frac{d}{dt} \frac{1}{(n+k+1)^t (n+k)^t} = \frac{-\log(n+k+1)(n+k)}{(n+k+1)^t (n+k)^t}$$

then

$$\begin{aligned} \left| \frac{-\log(n+k+1)(n+k)}{(n+k+1)^t (n+k)^t} \right| &\leq \frac{\log(n+k+1)(n+k)}{(n+k+1)^{1/2+\varepsilon} (n+k)^{1/2+\varepsilon}} \\ &\leq \frac{(n+k+1)^{\varepsilon/2} (n+k)^{\varepsilon/2}}{(n+k+1)^{1/2+\varepsilon} (n+k)^{1/2+\varepsilon}} \\ &= \frac{1}{(n+k+1)^{1/2+\varepsilon/2} (n+k)^{1/2+\varepsilon/2}} \leq \frac{1}{n^{1+\varepsilon}} \end{aligned}$$

which implies that

$$\sum \frac{1}{(n+k+1)^{1/2+\varepsilon/2} (n+k)^{1/2+\varepsilon/2}} < \infty$$

and by Weierstrass criterion, we have the uniform convergence of (4.3.2) on the set $[1/2+\varepsilon, \infty)$, for all $\varepsilon > 0$. In consequence

$$\frac{d}{dt} P_k(t) = \log k + \left(\sum_{n=0}^{\infty} \frac{1}{(n+k+1)^t (n+k)^t} \right)^{-1} \sum_{n=0}^{\infty} \frac{-\log(n+k+1)(n+k)}{(n+k+1)^t (n+k)^t}.$$

□

We now prove that $k \mapsto P_k'(t)$ is real analytic in the variable k. By algebra of analytic functions, it is sufficient to prove the analyticity of the series

$$\sum_{n=0}^{\infty} \frac{-\log(n+k+1)(n+k)}{(n+k+1)^t(n+k)^t}.$$

Let $r > 0$. We will work in the subset of complex numbers $B_r = \{|z| < r\}$. Since

$$f_n(k) = \frac{-\log(n+k+1)(n+k)}{(n+k+1)^t(n+k)^t}$$

is analytic, we have to show that the series is uniformly convergent on compact subset of B_r. Note that

$$|f_n(k)| \leq \frac{|\log(n+k+1)(n+k)|}{|(n+k+1)^t(n+k)^t|} \leq \frac{\log(n+r+1)(n+r)+2\pi}{(n+1-r)^t(n-r)^t}$$

for n sufficiently large. The series of this last sequence is convergent, then we conclude the uniform convergence on compacts of B_r, by Weierstrass theorem. In consequence, $k \mapsto P_k'(t)$ is analytic. By the inverse function theorem for analytic functions (see [KP, Theorem 1.8.1]), we obtain the analyticity of $(P_k')^{-1}(t)$. Finally, we obtain that

$$\tau_k(\alpha) = \frac{1}{\alpha}(\alpha(P_k')^{-1}(-\alpha) + (P_k \circ (P_k')^{-1})(-\alpha))$$

is analytic in k. □

Chapter 5

Normal numbers in Q-Lüroth expansions

In analogy to the Cantor series expansions, we introduce the so called $Q-$Lüroth expansion of a real number, where Q is a sequence of positive numbers. We describe some of its properties, define a notion of normal number, and go on to prove an analog of Borel's normal number theorem. That is, we prove that Lebesgue almost every real number is normal for $Q-$Lüroth expansions, if and only if we have a divergence of a series whose summands depends on the sequence $Q = (q_n)$. On the other hand, although normal numbers form a large set with respect to the Lebesgue measure, we prove that its complement in [0,1] has full Hausdorff dimension. Namely, we prove that the Hausdorff dimension of non-normal numbers is equal to one. The results obtained in this chapter appear in the preprint [Con1]

5.1 Introduction

Let $b \geq 2$ be an integer. Every real number $x \in \mathbb{R}$ can be written in *base* b as

$$x = \sum_{n=1}^{\infty} \frac{\epsilon_n(x)}{b^n}$$

where $\epsilon_n(x) \in \{0, 1, \ldots, b-1\}$. This representation is unique except for a countable number of points. A number $x \in \mathbb{R}$ is $b-$*normal* (in the weak sense) if the frequency of appearance of every digit is equal to $1/b$. That is, for every $d \in \{0, 1, \ldots, b-1\}$

$$\lim_{n\to\infty} \frac{1}{n} \# \{i \in \{1, \ldots, n\} : \epsilon_i(x) = d\} = \frac{1}{b},$$

where $\#A$ denotes the cardinality of the set A. A classical result by Borel [Bor] states that Lebesgue almost every number is $b-$normal with respect to every base b. It is well known that the base b expansion is closely related to the following dynamical system, $T_b : [0,1] \mapsto [0,1]$, defined by

$$T_b(x) := bx \mod 1 = bx - [bx] = \{bx\}$$

where $[x]$ denotes the integer part of x and $\{x\}$ its fractional part. Indeed, $\epsilon_n(x) = [bT_b^{n-1}x]$. The Lebesgue measure, that we denote by λ, is invariant and ergodic for every map T_b. Therefore, Borel's normal number theorem is a simple consequence of Birkhoff's ergodic theorem. The frequency of the digit d in the base-b expansion of the point $x \in [0,1]$ is given by

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_{[d]}(T_b^i x),$$

where $[d] := \{x \in [0,1] : \epsilon_1(x) = d\}$ and $\mathbb{1}_{[d]}$ is the characteristic function of $[d]$. It directly follows from Birkhoff's theorem that for Lebesgue almost every $x \in [0,1]$ this limit equals the Lebesgue measure of $[d]$, which is $1/b$.

In 1869, Cantor [Can] generalized the notion of b-expansion in the following direction. Let $B = \{b_n\}_{n\geq 1}$ be a sequence of integers each of which is greater than 2. Cantor showed that every real number $x \in [0,1)$ can be written as infinite series of the form

$$x = \sum_{n=1}^{\infty} \frac{c_n}{b_1 b_2 \cdots b_n},$$

with $c_n \in \{0, 1, \ldots, b_n - 1\}$. Observe that if for every $n \in \mathbb{N}$ we have $b_n = b$ then we recover the base b-expansion. As in the case of base $b-$expansion, the Cantor series is related to a dynamical system. However, in this case it is a non-autonomous system. Indeed, consider the maps defined in $[0, 1]$ by $T_{b_n}(x) = \{b_n x\}$. The iteration is defined by

$$T_B^n(x) = T_{b_n} \circ T_{b_{n-1}} \circ \cdots \circ T_{b_1}(x).$$

The dynamics is, therefore, obtained applying different maps T_{b_i} at prescribed times. Note that, as in the case of the base $b-$expansion, we have $c_n = [b_n T_B^{n-1}]$. Unfortunately, there is no analog of Birkhoff's ergodic theorem for non-autonomous systems. Therefore, the question for normality in this setting has to be addressed with different methods. It was actually shown by Renyi [Rén] that Lebesgue almost every number is normal for $B = \{b_n\}_{n \geq 1}$ if and only if $\sum_{n=1}^{\infty} 1/b_n = \infty$. More recently, constructions and properties of normal numbers for Cantor series have been studied by Mance [Man2].

In this note we introduce a new numerical system, the so called Q-Lüroth system. In analogy to the Cantor expansion, the role played by the base $b-$expansion is played by the so called $k-$Lüroth expansion (see section 5.2). Again, the associated dynamical system is a non-autonomous system. However, in this case each interval map has countably many branches and infinite entropy. This lack of compactness yields several complications that have to be addressed in order to first, define a notion of normality and second to prove that Lebesgue almost every point is indeed normal. In this paper we provide such a definition and prove the analog of Borel's result in this setting.

Let $Q = \{q_n\}_{n \geq 1}$ be a sequence of positive real numbers. Consider the family of interval maps $L_{q_n} : [0, 1) \to [0, 1)$ defined by

$$L_{q_n}(x) := x\frac{(n + q_n)(n + q_n + 1)}{q_n} - (n + q_n), \text{ if } x \in \left[\frac{q_n}{n + q_n + 1}, \frac{q_n}{n + q_n}\right), n \in \mathbb{N}_0$$

and $L_{q_n}(0) := 0$. The maps $\{L_{q_n}\}_{n\geq 1}$ will be called the family of Q-Lüroth maps. The associated non-autonomous dynamical system is defined by $L_Q : [0,1) \to [0,1)$

$$L_Q^n(x) := L_{q_n} \circ L_{q_{n-1}} \circ \cdots \circ L_{q_1}(x).$$

Denote by $E := \{x \in [0,1) : \text{there exists } n \in \mathbb{N} \text{ such that } L_Q^n(x) = 0\}$. As in the the case of the $b-$expansion or in the Cantor series expansion, to every real number $x \in [0,1) \setminus E$ it corresponds a unique infinite sequence of positive integers $[a_1(x), \ldots, a_n(x), \ldots]_Q$ that determines its $Q-$Lüroth expansion (details are provided in section 5.2). In definition 5.3.1 we propose a notion of normal number in this setting. Our main result in this setting is

Theorem 5.1.1. *Let $Q = \{q_n\}_{n\geq 1}$ be a sequence of positive real numbers. Then we have that Lebesgue almost every real number in $[0,1]$ is normal with respect to the Q-Lüroth expansion if and only if for all $a \geq 1$, the series $\sum_n \lambda(I_a^{q_n})$ is divergent.*

This result extends previously known normality results on Cantor series expansions to the (non-compact) setting Q-Lüroth expansions.

In a complementary direction, we also study the Hausdorff dimension of the set of non-normal numbers in Q-Lüroth expansions. We prove,

Theorem 5.1.2. *The set of non-normal numbers in the Q-Lüroth expansion has Hausdorff dimension equal to one.*

To prove this theorem, we use tools from dimension theory and thermodynamic formalism put in the setting of non-autonomous dynamics. The idea of relating this two theories goes back to the work of Bowen [Bow] in the late 1970s. The setting we consider is, however, very different in that non-autonomous dynamical systems are considered. In this setting the work of Rempe-Gillen and Urbański [RGU] will be of use. Ir is well known that the set of badly-approximable numbers has Hausdorff dimension equal to one. Our proof of Theorem 5.1.2 provides a version of that result in the setting of Q-Lüroth expansions.

5.2 Lüroth expansions

5.2.1 Classical Lüroth expansions

The concept of Lüroth expansions was introduced in the 1883 work of Lüroth [Lür], when he proved that every irrational $x \in (0,1]$ has a unique infinite expansion in the form

$$
\begin{aligned}
x = \frac{1}{a_1} + \frac{1}{a_1\,(a_1-1)\,a_2} + \ldots + \frac{1}{a_1\,(a_1-1)\cdots a_{n-1}\,(a_{n-1}-1)\,a_n} + \cdots \\
= \sum_{n=1}^{\infty} \frac{1}{a_1\,(a_1-1)\cdots a_{n-1}\,(a_{n-1}-1)\,a_n}
\end{aligned}
$$

where $a_n \geq 2$, for all $n \geq 1$. This expansion is closely related to the dynamics (see [DK]) of the function $L : [0,1) \to [0,1)$ defined by

$$
L(x) := \begin{cases} n(n+1)x - n & \text{if } x \in \left[\frac{1}{n+1}, \frac{1}{n}\right), n \in \mathbb{N} \\ 0 & \text{if } x = 0. \end{cases}
$$

The classical Lüroth expansion can be generalized in the following direction (see [KMS1]). Let $k > 0$ and consider the map $L_k : [0,1) \to [0,1)$ defined by

$$
L_k(x) := \begin{cases} x\frac{(n+k)(n+k+1)}{k} - (n+k) & \text{if } x \in \left[\frac{k}{n+k+1}, \frac{k}{n+k}\right), n \in \mathbb{N}_0 \\ 0 & \text{if } x = 0. \end{cases}
$$

Note that $L_1 = L$. Observe that each map L_k induces a partition $\{I_n^k\}_{n \geq 0}$ of $[0,1)$, where

$$
I_n^k := \left[\frac{k}{n+k+1}, \frac{k}{n+k}\right).
$$

For every $k > 0$ the map L_k has countably many branches and infinite entropy. Moreover, for each $k > 0$ a $k-$Lüroth expansion can defined such that the map L_k acts as the shift on it (see [KMS1]).

Proposition 5.2.1. *Let $k > 0$. Every $x \in [0,1)$ such that $L_k^n(x) \neq 0$ for all $n \geq 1$ can be expanded in a infinite k-Lüroth expansion, that is*

$$x = \sum_{n=1}^{\infty} \frac{k^n}{(a_1 + k)(a_1 + k - 1) \cdots (a_{n-1} + k)(a_{n-1} + k - 1)(a_n + k)}$$

where $a_n = [L_k^{n-1}(x)] + 1$, for all $n \geq 1$.

We denote the $k-$Lüroth expansion of $x \in [0,1)$ by $x = [a_1(x), \ldots, a_n(x), \ldots]_k$. Recall that a probability measure μ in $[0,1]$ is invariant for the map $T : [0,1] \to [0,1]$ if for every Borel set $A \subset [0,1]$ we have $\mu(A) = \mu(T^{-1}A)$. Moreover, we say that an invariant measure μ is ergodic if for every set with the property that $A = T^{-1}A$ we have that $\mu(A) = 0$ or $\mu(A) = 1$, see [Wal]. Denote by λ the Lebesgue measure on the interval $[0,1]$. It was shown in [KMS1, Lemma 2.4] that:

Proposition 5.2.2. *For every $k > 0$, the map L_k is an ergodic transformation with respect to the Lebesgue measure.*

The following is a natural definition of normal numbers with respect to k-Lüroth expansions.

Definition 5.2.1. Let $x \in [0,1]$, $a > 1$ and $k > 0$. Denote by

$$N_n^k(a,x) := \#\{1 \leq i \leq n : a_i(x) = a\}.$$

We say that $x \in [0,1]$ is normal with respect to the k-Lüroth expansion, if for every integer $a > 1$, we have

$$\lim_{n \to \infty} \frac{N_k(a,x)}{n\lambda(I_a^k)} = 1.$$

Note that definition 5.2.1 is analogous to the definition of normal number in the continued fraction expansion. The following result is a direct consequence of the ergodicity of the

Lebesgue measure with respect to L_k and Birkhoff's ergodic theorem.

Proposition 5.2.3. *For every $k > 0$, Lebesgue almost every point $x \in [0,1]$ is normal with respect to the k-Lüroth expansion.*

5.2.2 Q-Lüroth expansions

In this subsection we introduce a new numerical expansion based on the $k-$Lüroth maps. Let $Q = \{q_n\}_{n\geq 1}$ be a sequence of integers $q_n \geq 2$. Consider the family of Lüroth maps $\{L_{q_n}\}_{n\geq 1}$, that we will denote by L_Q. This family of transformations induces a non-autonomous dynamical system $([0,1), L_Q)$. Indeed, the time evolution of the system is defined by composing the maps L_{q_n} in the prescribed order given by the sequence $Q = \{q_n\}_{n\geq 1}$. In other words, for all $n \geq 1$, we define:

$$L_Q^n := L_{q_n} \circ L_{q_{n-1}} \circ \cdots \circ L_{q_1}.$$

The *orbit* of $x \in [0,1]$ is the sequence $\{L_{q_n}(x)\}_{n\geq 1}$.

Proposition 5.2.4. *Let $x \in [0,1)$ and suppose that $L_Q^{n+1}(x) \neq 0$ for all $n \geq 1$. Then,*

$$x = \sum_{i=1}^{n} \frac{q_1 q_2 \cdots q_i}{(a_1 - 1 + q_1)(a_1 + q_1) \cdots (a_i - 1 + q_i)} + \frac{q_1 q_2 \cdots q_n L_Q^n(x)}{(a_1 - 1 + q_1)(a_1 + q_1) \cdots (a_n - 1 + q_n)(a_n + q_n)}$$

where a_n is the unique positive integer satisfying

$$L_Q^{n-1}(x) \in \left[\frac{q_n}{a_n + q_n}, \frac{q_n}{a_n - 1 + q_n}\right) \iff a_n = \left[\frac{q_n}{L_Q^{n-1}(x)} - q_n\right] + 1.$$

The proof of Proposition 5.2.4 is analogous to the corresponding results for classical Lüroth expansions. For further details, see [DK, pages 88-89].

Theorem 5.2.5. *Each $x \in [0,1)$ such that $L_Q^n(x) \neq 0$ for all $n \geq 0$, can be expanded uniquely in a infinite series of the form*

$$x = \sum_{n=1}^{\infty} \frac{q_1 q_2 \cdots q_n}{(a_1 - 1 + q_1)(a_1 + q_1) \cdots (a_n - 1 + q_n)} \tag{5.2.1}$$

where $a_n = \left[\frac{q_n}{L_Q^{n-1}(x)} - q_n \right] + 1.$

Proof. By proposition 5.2.4, there exists a unique sequence $(a_n)_{n \geq 1} \subset \mathbb{N}$ such that

$$x = \sum_{i=1}^{n} \frac{q_1 q_2 \cdots q_i}{(a_1 - 1 + q_1)(a_1 + q_1) \cdots (a_i - 1 + q_i)} + \tag{5.2.2}$$

$$\frac{q_1 q_2 \cdots q_n L_Q^n(x)}{(a_1 - 1 + q_1)(a_1 + q_1) \cdots (a_n - 1 + q_n)(a_n + q_n)} \tag{5.2.3}$$

for all $n \geq 1$. Let S_n be the sum involved in 5.2.3. We will prove that $S_n \to x$ when $n \to \infty$. In fact

$$|x - S_n| = \frac{q_1 \ldots q_n L_Q^n(x)}{(a_1 - 1 + q_1)(a_1 + q_1) \cdots (a_n - 1 + q_n)(a_n + q_n)} \leq \frac{1}{2^n} \to 0,$$

which prove that

$$x = \sum_{n=1}^{\infty} \frac{q_1 q_2 \cdots q_n}{(a_1 - 1 + q_1)(a_1 + q_1) \cdots (a_n - 1 + q_n)}.$$

□

For $x \in [0,1)$ we denote by $x = [a_1(x), a_2(x), \ldots]_Q$ the $Q-$Lüroth expansion of x and call the numbers $a_i(x)$ *digits*. We stress that the expansion can be either finite (see Proposition 5.2.4) or infinite (see Theorem 5.2.5).

Lemma 5.2.6. *The set of $x \in [0,1)$ having a finite Q-Lüroth expansion is countable.*

Proof. Recall that such subset of $[0,1)$ is denoted by E. The lemma follows since

$$E = \bigcup_{n\geq 1} \{x \in [0,1) : L_Q^n(x) = 0, L_Q^j(x) \neq 0, \text{ for } j < n\}$$

and the fact that each equation $L_{q_n}(x) = 0$ has countably many solutions. $\square$

Definition 5.2.2. Given integers $a_1, a_2, \ldots, a_n \geq 1$ we define $\Delta_Q^{(n)}(a_1, a_2, \ldots, a_n)$ as the subset of the interval $[0,1)$ containing every number $x \in [0,1)$ whose first n digits in its $Q-$Lüroth expansion are $(a_1, a_2, \ldots, a_n)$. In other words

$$\Delta_Q^{(n)}(a_1, a_2, \ldots, a_n) := \{x \in [0,1) : a_1(x) = a_1, a_2(x) = a_2, \ldots, a_n(x) = a_n\}.$$

Proposition 5.2.7. *Let $(a_1, \ldots, a_n)$ be a word of length n with $a_i \in \mathbb{N}$ for every $i \in \{1, \ldots, n\}$. Denote by*

$$S_n(a_1, \ldots, a_n) := \sum_{i=1}^{n} \frac{q_1 q_2 \cdots q_i}{(a_1 - 1 + q_1)(a_1 + q_1) \cdots (a_i - 1 + q_i)}.$$

Then $\Delta_Q^{(n)}(a_1, a_2, \ldots, a_n)$ is the subinterval

$$\left[S_n(a_1, \ldots, a_n), S_n(a_1, \ldots, a_n) + \frac{q_1 q_2 \cdots q_n}{(a_1 - 1 + q_1)(a_1 + q_1) \cdots (a_n - 1 + q_n)(a_n + q_n)}\right).$$

Proof. By Proposition 5.2.4, we have that $x \in \Delta_Q(a_1, \ldots, a_n)$ if and only if

$$x = S_n(a_1, \ldots, a_n) + \frac{q_1 q_2 \cdots q_n L_Q^n(x)}{(a_1 - 1 + q_1)(a_1 + q_1) \cdots (a_n - 1 + q_n)(a_n + q_n)}. \tag{5.2.4}$$

Since the n-th composition $L_Q^n : [0,1) \to [0,1)$ is onto (each L_{q_i} is onto $[0,1)$), we observe that equality (5.2.4) is equivalent to x belongs to

$$\left[S_n(a_1,\ldots,a_n), S_n(a_1,\ldots,a_n)+\frac{q_1q_2\cdots q_n}{(a_1-1+q_1)(a_1+q_1)\cdots(a_n-1+q_n)(a_n+q_n)}\right).$$

$\square$

As a consequence, each cylinder $\Delta_Q^{(n)}(a_1,a_2,\ldots,a_n)$ is a subinterval of $[0,1]$ with Lebesgue measure equal to

$$\lambda(\Delta_Q^{(n)}(a_1,a_2,\ldots,a_n))=\frac{q_1q_2\cdots q_n}{(a_1-1+q_1)(a_1+q_1)\cdots(a_n-1+q_n)(a_n+q_n)}.$$

5.3 Normality in $Q-$Lüroth expansions

The notion of normality for Q-Lüroth expansions is captured by the following definition. It generalizes the normality notion for $k-$Lüroth expansions (see definition 5.2.1). Recall that λ is the Lebesgue measure.

Definition 5.3.1. Let $x \in [0,1)$ and $a \in \mathbb{N}\setminus\{1\}$. For $Q=\{q_n\}_{\geq 1}$ let

$$N_n^Q(a,x)=\#\{1\leq i\leq n : a_i(x)=a\}.$$

The number $x\in[0,1]$ is *normal* with respect to the Q-Lüroth expansion, if for every $a\in\mathbb{N}$,

$$\lim_{n\to\infty}\frac{N(a,x)}{\sum_{i=1}^n \lambda(I_a^{q_n})}=1$$

Our main result can be thought of as a Borel normal number theorem for Q-Lüroth expansions.

Theorem 5.3.1. *Let $Q=\{q_n\}_{n\geq 1}$ be a sequence of positive real numbers. Then we have that*

Lebesgue almost every real number in $[0,1]$ *is normal with respect to the* Q*-Lüroth expansion if and only if for all* $a \geq 1$*, the series* $\sum_n \lambda(I_a^{q_n})$ *is divergent.*

Remark 5.3.1. The proof of this result is based on work by Erdös [Rén] about normality in Cantor Series. See also [Rév, page 152] and [Man1]. In the original work of Erdös, the corresponding assumption on the series $\sum_n \lambda(I_a^{q_n})$ is the divergence of the series $\sum \frac{1}{q_n}$. Observe that, if $q_n \nrightarrow 0$ when $n \to \infty$, the divergence of $\sum_n \lambda(I_a^{q_n})$ (for all $a \geq 1$) is equivalent to the divergence of $\sum \frac{1}{q_n}$. On the other hand, it is possible to have $q_n \to 0$ and that $\sum \lambda(I_a^{q_n})$ be convergent. Actually, $q_n = 1/n^2$ implies that $\lambda(I_a^{q_n}) \sim 1/n^2$.

5.3.1 Preliminaries from Probability Theory

We consider the probability space given by $([0,1], \mathcal{B}, \lambda)$, where $\mathcal{B}$ is the sigma-algebra of Borel sets in $[0,1]$ and λ denotes the Lebesgue measure on $[0,1]$.

Definition 5.3.2. We consider the following objects,

1. A measurable function $X : [0,1] \to \mathbb{R}$ is called a *random variable*.

2. A random variable is called *discrete* if the image of $[0,1]$ under X is a countable subset of $\mathbb{R}$.

Definition 5.3.3. We say that a sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is *independent* if and only if

$$\lambda(X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k) = \prod_{i=1}^{k} \lambda(X_i = x_i)$$

for every x_i that belongs to the range of X_i and for every $k \in \mathbb{N}$.

Example. Given $x \in [0,1]$ we define the random variable $a_n(x) = a_n$, i.e. the function a_n gives the n-th digit of x in the Q-Lüroth expansion. We observe that the random variables

$\{a_n\}$ are independent. Indeed,

$$\begin{aligned}
\lambda(a_1(x) = a_1, \ldots, a_n(x) = a_n) &= \lambda(\Delta(a_1, a_2, \ldots, a_n)) \\
&= \frac{q_1 q_2 \cdots q_n}{(a_1 - 1 + q_1)(a_1 + q_1) \cdots (a_n - 1 + q_n)(a_n + q_n)} \\
&= \prod_{i=1}^{n} \lambda(a_i(x) = a_i)
\end{aligned}$$

Definition 5.3.4. The mean of a random variable X is defined by

$$\mathbb{E}(X) := \int X d\lambda$$

and the variance of X is defined by

$$\mathrm{Var}(X) := \mathbb{E}(X^2) - (\mathbb{E}(X))^2.$$

The following theorem, known as the *Law of iterated Logarithm* (see [Gal, page 49], [Rév, page 69]), will be the main tool to proof our main result.

Theorem 5.3.2. *Let $\{X_i\}$ be a sequence of independent random variables. Assume that there exists a constant $c > 0$ such that $|X_i| < c$ for all $i \in \mathbb{N}$ and that $s_n := \sum_{i=1}^{n} \mathrm{Var}(X_i) \to \infty$ when $n \to \infty$. Then, with probability one,*

$$\limsup_{n \to \infty} \frac{\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \mathbb{E}[X_i]}{\sqrt{s_n \log \log s_n}} = 1.$$

5.3.2 Proof of Theorem 5.3.1

Assume that the series $\sum \lambda(I_a^{q_n}) = +\infty$ for all $a \geq 1$. For $a, i \in \mathbb{N}$ we consider the random variable $\xi_{i,a}(x) = \chi_{\{a_i(x)=a\}}$. We note that

$$\lambda(\xi_{i,a}(x) = 1) = \lambda(a_i(x) = a) = \lambda(\Delta(a)) = \frac{q_i}{(a-1+q_i)(a+q_i)}$$

and

$$\begin{aligned}\lambda(\xi_{1,a_1}(x) = 1, \xi_{2,a_2}(x) = 1, \dots, \xi_{n,a_n}(x) = 1) &= \lambda(\Delta(a_1, a_2, \dots, a_n)) \\ &= \prod_{i=1}^{n} \frac{q_i}{(a_i-1+q_i)(a_i+q_i)} \\ &= \prod_{i=1}^{n} \lambda(\xi_{i,a_i}(x) = 1).\end{aligned}$$

Thus, the sequence of random variables $\Phi = \{\xi_{i,a}\}_{i\in\mathbb{N}}$ is independent. We will verify the hypothesis of Theorem 5.3.2. The sequence Φ is uniformly bounded. On the other hand,

$$\mathbb{E}(\xi_{i,a}) = \frac{q_i}{(a-1+q_i)(a+q_i)}; \qquad \sum_{i=1}^{n} \mathbb{E}(\xi_{i,a}) = \sum_{i=1}^{n} \lambda(I_a^{q_i})$$

and

$$\operatorname{Var}(\xi_{i,a}) = \mathbb{E}(\xi_{i,a}^2) - \mathbb{E}^2(\xi_{i,a}) = \frac{q_i}{(a-1+q_i)(a+q_i)}\left(1 - \frac{q_i}{(a-1+q_i)(a+q_i)}\right).$$

Now, we will prove that, when $n \to \infty$, we have

$$s_n = \sum_{i=1}^{n} \frac{q_i}{(a-1+q_i)(a+q_i)}\left(1 - \frac{q_i}{(a-1+q_i)(a+q_i)}\right) \to \infty.$$

In fact, if $q_n \to \infty$ we observe that the divergence of $\sum \lambda(I_a^{q_n})$ for all $a \geq 1$ implies that $s_n \to \infty$ by comparing both series. The same reasoning holds when $q_n \to 0$. In the case of q_n has a convergent subsequence $q_{n_k} \to l \notin \{0, \infty\}$, then we have that

$$\lambda(I_a^{q_{n_k}})(1 - \lambda(I_a^{q_{n_k}})) \to \frac{l}{(a-1+l)(a+l)} \left(1 - \frac{l}{(a-1+l)(a+l)}\right) \neq 0.$$

In consequence

$$\lim_{n\to\infty} s_n \geq \sum_k \lambda(I_a^{q_{n_k}})(1 - \lambda(I_a^{q_{n_k}})) = \infty.$$

If we write $N(a,x) = \sum_{i=1}^n \xi_{i,a}(x)$, then by Theorem 5.3.2 we have, for Lebesgue-almost all $x \in [0,1]$,

$$1 = \lim_{n\to\infty} \frac{N_n(a,x) - \sum_{i=1}^n \lambda(I_a^{q_i})}{s_n \log\log s_n} = \lim_{n\to\infty} \frac{\sum_{i=1}^n \lambda(I_a^{q_i})}{\sqrt{s_n \log\log s_n}} \left(\frac{N_n(a,x)}{\sum_{i=1}^n \lambda(I_a^{q_i})} - 1\right).$$

Let

$$b_i = \frac{q_i}{(a-1+q_i)(a+q_i)}.$$

To finishing the proof of the first implicance, it is sufficient to show that

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n \lambda(I_a^{q_i})}{\sqrt{s_n \log\log s_n}} = \lim_{n\to\infty} \frac{\sum_{i=1}^n b_i}{\sqrt{\sum_{i=1}^n b_i(1-b_i) \log\log \sum_{i=1}^n b_i(1-b_i)}} = \infty,$$

which is true since the divergence of $\sum b_i$, the inequality

$$\frac{\sum_{i=1}^n b_i}{\sqrt{\sum_{i=1}^n b_i(1-b_i) \log\log \sum_{i=1}^n b_i}} \geq \frac{\sum_{i=1}^n b_i}{\sqrt{\sum_{i=1}^n b_i \log\log \sum_{i=1}^n b_i}}$$

and $x/\sqrt{x \log\log x} \to \infty$ when $n \to \infty$. We conclude that, for Lebesgue-almost every $x \in [0,1]$ we have

$$\lim_{n\to\infty} \frac{N_n(a,x)}{\sum_{i=1}^n \lambda(I_a^{q_i})} - 1 = 0,$$

for all $a \geq 1$.

To prove the other direction, suppose that there exists $a \geq 1$ such that the series $\sum_n \lambda(I_a^{q_n})$ is convergent. We will prove that the set

$$\Omega_a := \{x : \text{for all } n \geq 1, a_n(x) \neq a\}$$

has positive Lebesgue measure. Let $N \in \mathbb{N}$. Note that, by independence of the random variables $a_n(x)$ (see Example 5.3.1), we have

$$\begin{aligned}
\lambda(\Omega) &= \lim_{N\to\infty} \lambda\left(\bigcap_{n=1}^{N}\{x : a_n \neq a\}\right) \\
&= \lim_{N\to\infty} \prod_{n=1}^{N} \lambda(\{x : a_n \neq a\}) \\
&= \lim_{N\to\infty} \prod_{n=1}^{N} (1 - \lambda(\{x : a_n = a\})) \\
&= \lim_{N\to\infty} \prod_{n=1}^{N} (1 - \lambda(I_a^{q_n}))
\end{aligned}$$

which converges to some positive number because $\sum_n \lambda(I_a^{q_n}) < \infty$.

5.4 Hausdorff dimension of non-normal numbers

In the main result of the previous section, Theorem 5.3.1, we proved that under some mild assumptions the set of normal numbers is large from the measure theoretic point of view. Indeed, it has full Lebesgue measure. Consequently, the set of non-normal numbers has zero Lebesgue measure. The purpose of this section is to show that, despite the above, from the point of view of dimension theory the set of non-normal numbers is as large as possible. In Theorem 5.4.3 we prove that the set of non-normal numbers has Hausdorff dimension equal to one. Actually, we prove that the set of numbers for which its $Q-$Lüroth expansion only has finitely many digits has Hausdorff dimension one. More precisely, for $N \in \mathbb{N}$ we consider the set

$$A_N := \{x = [a_1(x)a_2(x)\dots]_Q \in [0,1] : a_i(x) \le N \text{ for every } i \in \mathbb{N}\},$$

and go on to prove that $\lim_{N\to\infty} \dim_H A_N = 1$, where $\dim_H$ denotes the Hausdorff dimension. This result should be compared to the fact that the set of real numbers having a continued fraction with finitely many digits also has Hausdorff dimension equal to one.

5.4.1 Hausdorff Dimension

We start by recalling the definition of Hausdorff dimension, see [Fal, Chapter 2] for further details. Given a subset $G \subset \mathbb{R}$, we say that a countable family of sets $\{U_n\}_{n\ge1}$ is a δ-cover of G if $G \subset \bigcup_n U_n$ and every set U_n has diameter at most δ. Given $s > 0$, we define

$$H^s(G) := \lim_{\delta\to0} H^s_\delta(G),$$

where

$$H^s_\delta(G) := \inf\left\{\sum_{n=1}^{\infty} |U_n|^s : \{U_n\}_{n\ge1} \text{ is a } \delta\text{-cover of } G\right\}.$$

The *Hausdorff dimension* of the set G is defined by

$$\dim_H(G) := \inf\{s > 0 : H^s(G) = 0\}.$$

5.4.2 Non-autonomous iterated function systems

In this subsection we will consider the Q-Lüroth series from the *iterated function systems* (IFS) point of view.

Definition 5.4.1. A non-autonomous iterated function system Φ on $[0,1]$ is a sequence

$\Phi := \{\Phi^{(n)}\}_{n\in\mathbb{N}}$, where

$$\Phi^{(n)} = \{\phi_j^{(n)} : [0,1] \to [0,1]\}_{j\in J^{(n)}}$$

is a collection of contractions of $[0,1]$, and, for all $n \in \mathbb{N}$, $J^{(n)}$ is an index set (finite or countably infinite).

Given $n \geq 1$, we denote by $J^n = \prod_{m=1}^{n} J^{(m)}$. To any element $w = w_1 w_2 \cdots w_n \in J^n$ we associate the function

$$\varphi_w^n = \varphi_{w_1}^{(1)} \circ \varphi_{w_2}^{(2)} \circ \cdots \circ \varphi_{w_n}^{(n)}.$$

The limit set of Φ is defined by

$$\mathcal{J}(\Phi) := \bigcap_{n=1}^{\infty} \bigcup_{w\in J^n} \varphi_w^n([0,1]).$$

Definition 5.4.2. The Q-Lüroth non-autonoumous IFS is given by the sequence $\Theta = \{\Theta^{(n)}\}_{n\in\mathbb{N}}$, where $\Theta^{(n)} = \{\theta_j^n : [0,1] \to [0,1]\}_{j\in\mathbb{N}}$ and $\theta_j^n(x)$ is the inverse branch of $L_{q_n} : [0,1] \to [0,1]$ restricted to $I_j^{q_n}$, namely:

$$\theta_j^n(x) := x\frac{q_n}{(j+q_n)(j+q_n+1)} + \frac{1}{j+q_n+1}. \tag{5.4.1}$$

for all $j, n \geq 1$.

There exists a well established theory that relates thermodynamic formalism with the Hausdorff dimension of attractors (see [MU2]). Recently, a thermodynamic formalism has been developed by Rempe-Gillen and Urbański [RGU] in this (non-autonomous) setting with the purpose of studying the dimension theory of non-autonomous IFS. Given a differentiable function $f : [0,1] \to [0,1]$, we denote by $Df(x)$ its derivative at x and let $\|Df\| := \sup_{x\in[0,1]} |f(x)|$.

Definition 5.4.3. For any $t \geq 0$ and $n \in \mathbb{N}$, the lower pressure function is defined by

$$\underline{P}^{\Phi}(t) := \liminf_{n\to\infty} \frac{1}{n} \log \sum_{w\in I^n} \|D\varphi_w^n\|^t.$$

Remark 5.4.1. When finite, the lower pressure function is strictly decreasing, see [RGU, Lemma 2.6].

Remark 5.4.2. For the Q-Lüroth non-autonomous IFS, the pressure function is given by

$$\begin{aligned}
\underline{P}^{\Theta}(t) &= \liminf_{n\to\infty} \frac{1}{n} \log \sum_{\substack{w\in\mathbb{N}^n \\ w=(w_1,\ldots,w_n)}} \prod_{i=1}^{n} \frac{q_i^t}{(w_i+q_i)^t(w_i+q_i-1)^t} \\
&= \liminf_{n\to\infty} \frac{1}{n} \log \prod_{i=1}^{n} \sum_{w_i=1}^{\infty} \frac{q_i^t}{(w_i+q_i)^t(w_i+q_i-1)^t} \\
&= \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log \sum_{w_i=1}^{\infty} \frac{q_i^t}{(w_i+q_i)^t(w_i+q_i-1)^t} \\
&= \liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} P_{q_i}(t)
\end{aligned}$$

where, for $k > 0$,

$$P_k(t) := P(-t\log|L_k'|) = \log \sum_{n=1}^{\infty} \frac{k^t}{(n+k)^t(n+k-1)^t}$$

denotes the pressure function for the k-Lüroth map $L_k : [0,1) \to [0,1)$ (see [Con2, BI, MU2, Sar1]). The above shows that the non-autonomous pressure can be understood as an average of the autonomous ones. This has several consequences, for example, if $q_n \to q$ when $n \to \infty$, we have that $\underline{P}^{\Theta}(t) = P_q(t)$. Observe that $\underline{P}^{\Theta}(t) < \infty$ if and only if $t < 1/2$.

The following approximation by compact non-autonomous systems of the pressure will be our main technical device in the proof of Theorem 5.4.3, but it is also of independent interest.

Proposition 5.4.1. *Let $\Phi = \{\Phi^{(n)}\}_{n\in\mathbb{N}}$ be a non-autonomous IFS. Suppose that, for all $n \geq 1$, the index set $J^{(n)} = \mathbb{N}$. Then the pressure function $\underline{P}^{\Phi}(t)$ satisfies the following approximation property:*

$$\underline{P}^{\Phi}(t) := \lim_{N\to\infty} \underline{P}^{\Lambda_N}(t)$$

where

$$\underline{P}^{\Lambda_N}(t) = \liminf_{n\to\infty} \frac{1}{n} \log \sum_{w\in\Lambda_N^n} \|D\varphi_w^n\|^t$$

and $\Lambda_N^n := \prod_{j=1}^n \{1, 2, \ldots, N\}$, for all $n \geq 1$.

Proof. We note that the sequence $\{P^{\Lambda_N}(t)\}_{N\geq 1}$ is increasing in N and bounded above by $\underline{P}^{\Phi}(t)$. Moreover

$$\lim_{N\to\infty} \underline{P}^{\Lambda_N}(t) = \lim_{N\to\infty} \lim_{n\to\infty} \inf_{m\geq n} \frac{1}{m} \log \sum_{w\in\Lambda_N^m} \|D\varphi_w^m\|^t =: \lim_{N\to\infty} \lim_{n\to\infty} a_{Nn}$$

and a_{Nn} is increasing in N (when we fix n), and increasing in n (when we fix N). Since $P_N^{\Lambda_N}(t)$ is convergent, we can change the order of the limits (see [BB, Theorem 7.3]) to obtain

$$\begin{aligned}\lim_{N\to\infty} \underline{P}^{\Lambda_N}(t) &= \lim_{n\to\infty} \lim_{N\to\infty} \inf_{m\geq n} \frac{1}{m} \log \sum_{w\in\Lambda_N^m} \|D\varphi_w^m\|^t \\ &= \lim_{n\to\infty} \inf_{m\geq n} \frac{1}{m} \lim_{N\to\infty} \log \sum_{w\in\Lambda_N^m} \|D\varphi_w^m\|^t \\ &= \underline{P}^{\Phi}(t).\end{aligned}$$

□

The next result was obtained by Rempe-Gillen and Urbański [RGU, Theorem 1.1]. It is a non-autonomous version of the so called Bowen-formula that relates the pressure with the Hausdorff dimension of the attractor.

Theorem 5.4.2. *Suppose that Φ is a non-autonomous iterated function system of sub-exponential growth, that is*

$$\lim_{n\to\infty}\frac{1}{n}\log\#J^{(n)}=0.$$

Then,

$$\dim_H(\mathcal{J}^{\Phi})=\sup\{t\geq 0:\underline{P}(t)>0\}=\inf\{t\geq 0:\underline{P}(t)<0\}$$

Remark 5.4.3. In particular, Theorem 5.4.2 holds for any $\Phi=\{\Phi^{(n)}\}_{J^{(n)}}$ having a uniform bound for the size of the index sets $J^{(n)}$, for any $n\geq 1$.

We can now prove the main result of this section.

Theorem 5.4.3. *The set of non-normal numbers in the Q-Lüroth expansion has Hausdorff dimension equal to one.*

Proof. Let Ω be the subset of non-normal numbers. Observe that if we define A_N as the set of $x\in[0,1]$ whose digits in the Q-Lüroth expansion are bounded above by N, then $A_N\subset\Omega$, for all $N\geq 1$. Moreover A_N is the limit set of the Q-Lüroth non-autonomous IFS by restricting all the alphabets to be $\Lambda_N=\{1,\ldots,N\}$. In particular, from Theorem 5.4.2, we have that $P^{\Lambda_N}(t_N)=0$ if and only if $t_N=\dim_H(A_N)$. Define $t:=\lim_{N\to\infty}t_N$ and note that

$$t=\dim_H\bigcup_{N=1}^{\infty}A_N\leq\dim_H\Omega\leq 1.$$

We claim that $t=1$. Assume by way of contradiction that $t<1$. Since $t_N\leq t$ and the pressure functions are strictly decreasing, we have that $P^{\Lambda_N}(t)\leq 0$. Therefore, by Proposition 5.4.1, $P^{\Theta}(t)\leq 0$. This is a contradiction with the fact that $P^{\Theta}(t)>0$ when $t<1$. $\square$

Chapter 6

The Dimension of non-differentiability points of conjugacies between Gauss-like maps

In this chapter[1]*, we consider a family of interval maps* $\{T_k\}_{k>0}$ *which generalize the Gauss map on continued fractions. Any map* T_k *can be modeled by a fullshift on countable symbols. Using that fact, we construct a topological conjugacy between any two maps of the family and study the derivative of this conjugacy. In particular, we will prove that it is a singular map on the interval; that is, a non-constant function with derivative zero, Lebesgue a.e. point. From a fractal analysis point of view, we will calculate the exact value of the Hausdorff dimension of the set where the derivative does not exist and the set where the derivative is equal to infinity. It is important to remark that we will use strongly tools from thermodynamic formalism on non-compact spaces applied to dimension theory.*

[1]Joint work with Thomas Jordan

6.1 Introduction

In 1770 Lagrange [Lag] proved a theorem on continued fractions, which states that real roots of quadratic polynomials with integer coefficients, corresponds to numbers with eventually periodic continued fraction expansion. This property motived to Minkowski [Min] who, at the beginning of the 20th century, was interested in mapping quadratic surds of [0,1] into the non-dyadic rational numbers and for that constructed the so-called *Minkowski's question mark function* denoted by $Q : [0,1] \to [0,1]$. This function is defined on rationals by

$$Q(0) = 0, \qquad Q(1) = 1, \qquad Q\left(\frac{p+p'}{q+q'}\right) = \frac{Q(p/q) + Q(p'/q')}{2}$$

and can be extended to irrationals by continuity. The question mark function has been a matter of study of several mathematicians. For example, in 1938 Denjoy [Den] proved that Q can be expressed as

$$Q([x_1, x_2, \ldots]) = -2\sum_{k=1}^{\infty}(-1)^k 2^{-\sum_{i=1}^{k} x_i}$$

where $[x_1, x_2, \ldots]$ denotes the classic expansion in continued fractions.

On the other hand, Salem [Sal] proved that Q is a strictly increasing function and that it is *singular* with respect to the Lebesgue measure, that is, Q' exists and it is equal to 0, Lebesgue-almost everywhere in $[0,1]$. Therefore, Q is an example of a *slippery devil's staircase function*, a concept that was introduced firstly by Gutzwiller and Mandelbrot in 1988 [GM] and that now refer to strictly increasing and singular functions. Moreover, questions about different values of the derivative of Q can be posed. In fact, the problem of finding points with derivative non zero was solved by Paradis, Viader [PVB] and by Kesseböhmer, Stratmann [KS1]. The authors proved that, when the derivative of Q exists, then $Q'(x) \in \{0, \infty\}$. This result cause the following partition of the interval $[0,1] = D_0 \cup D_\infty \cup D_\sim$, where

$$D_0 := \{x \in [0,1] : Q'(x) = 0\}, \quad D_\infty := \{x \in [0,1] : Q'(x) = \infty\}$$

and

$$D_\sim := [0,1] \setminus (D_0 \cup D_\infty).$$

In [KS1], the authors were interested in the fractal analysis of Q'. They asked how large are the sets where the derivative is different to zero, with respect to the Hausdorff dimension. In particular, they proved that

$$0.875 < \dim_H D_\infty = \dim_H D_\sim < 1.$$

Thermodynamic formalism tools associated to Stern-Brocot partitions of the interval $[0,1]$ were used in the proof (see [KS2]). In addition, it is important to note that in [KS2] the authors completed the multifractal analysis of Lyapunov exponents for the Gauss map.

From the dynamical systems point of view, we remark the fact that the question mark function is the topological conjugation between the Farey map (F) and the Tent map (T): $T \circ Q = Q \circ F$, (see [KS1] for more details). This observation allows us to ask about the derivative of conjugations between two dynamical systems. In the trivial case when the dynamics are the same, the identity map is a topological conjugacy. Therefore the derivative of the conjugacy is always equal to one. Thus, conjugacies in general can be thought of as perturbations of the identity. Actually, in [JMS] the impact of pointwise perturbations of conjugacies into the Hausdorff dimensions of the points where the derivative is non-zero was studied. See also [Mun] where the authors considered the fractal analysis of conjugations between generalized Lüroth maps.

In this article, we consider a family of Markov expanding maps defined on $[0,1]$. More precisely, let k be a positive number. We define the map $T_k : [0,1] \to [0,1]$ by $T_k(0) := 0$ and, if $x \in (0,1]$

$$T_k(x) = \frac{k(1-x)}{x} - \left[\frac{k(1-x)}{x}\right].$$

We remark that this family belongs to a more general class of maps which are defined in [HM]. Moreover, since they generalize the Gauss map on continued fractions, each map T_k gives a more general class of continued fractions (see Subsection 2.2 and [HM] for further details).

The Markov property allows to code each T_k by the full-shift on countable symbols. This gives a conjugacy π_{k_1,k_2} between any two maps T_{k_1}, T_{k_2}, $k_1 \neq k_2$. The aim of this article is to study the derivative of π_{k_1,k_2}. In first place, we will prove that it is a singular function. Also, and in a dimension theory direction, we will interested in how large are the sets

$$\mathcal{D}_\infty := \{x \in [0,1] : \pi'_{k_1,k_2}(x) = \infty\}$$

and

$$\mathcal{D}_\infty := \{x \in [0,1] : \pi'_{k_1,k_2}(x) \text{ does not exists}\},$$

in terms of the Hausdorff dimension. The main result of this article will be to prove the following theorem.

Theorem. *Let k_1, k_2 be two positive numbers. Then the sets $\mathcal{D}_\infty, \mathcal{D}_\sim$ defined as above have the following Hausdorff dimensions:*

$$1/2 < \dim_H(\mathcal{D}_\infty) = \dim_H(\mathcal{D}_\sim) = \delta_0 < 1$$

where

$$\delta_0 := \sup\{\delta \in (1/2,1] : \textit{ for all } q \in \mathbb{R}, P(q\psi - \delta \log |T'_{k_2}|) > 0\}.$$

Here $P(\cdot)$ denotes the pressure function associated to T_{k_2}. The proof of this theorem involves applications of thermodynamic formalism. In particular, the behavior of the pressure function of maps T_k will be one of the main tools.

Notation. If $f, g : \mathbb{R} \to \mathbb{R}$ are two functions, we denote by $f \ll g$ when there exists a positive constant C (independent of x) such that $f \leq Cg$. Also, we write $f \asymp g$ when we have $f \ll g$ and $g \ll f$.

6.2 Preliminaries on continued fraction maps

This section is devoted to define a family $\{T_k\}_{k>0}$ of maps that generalize the classical Gauss map. We also collect some ergodic properties of T_k. At the end of this section, we will review the continued fraction expansions defined by T_k and will prove some useful results for our purposes. We mainly follow [HM].

6.2.1 Definitions and dynamic properties

Let k be a positive number. We define the map $T_k : [0,1] \to [0,1]$ by $T_k(0) := 0$ and, if $x \in (0,1]$

$$T_k(x) = \frac{k(1-x)}{x} - \left[\frac{k(1-x)}{x}\right]$$

where $[w]$ denotes the integer part of the real number w. As the Gauss map, T_k has a T_k-invariant measure absolutely continuous with respect to Lebesgue. In [Haa], the authors proved the following theorem.

Theorem 6.2.1. *Let $k > 0$. The transformation $T_k : [0,1] \to [0,1]$ preserve the measure μ_k defined on Borel subsets $A \subset [0,1]$ by*

$$\mu^k(A) = \int_A \frac{c_k}{x+k} dx$$

where $c_k = \left(\log \frac{k+1}{k}\right)^{-1}$. *Moreover, it is an ergodic measure for* T_k.

The Markov structure of T_k can be described in a similar way. The sequence of intervals

$$\{I^k(n)\}_{n\in\mathbb{N}_0} := \left\{\left(\frac{k}{n+k+1}, \frac{k}{n+k}\right]\right\}_{n\in\mathbb{N}_0}$$

is a Markov partition for the map T_k. Observe that $x \in I^k(n)$ if and only if $[\frac{k(1-x)}{x}] = n$. Let $n \geq 1$. To each finite sequence $\{a_1, \ldots, a_n\} \subset \mathbb{N}_0$, we associate the n-th cylinder by

$$I_n^k(a_1, \ldots, a_n) := \bigcap_{i=1}^{n} T_k^{-(i-1)} I^k(a_i).$$

Note that $x \in I_n^k(a_1, \ldots, a_n)$ if and only if $T^{i-1}(x) \in I_1^k(a_i)$, for all $i \in \{1, \ldots, n\}$. As in the classical setting of Gauss map, cylinders at level n gives a partition of the interval $[0,1]$ modulo a countable set. Given $n \geq 1$, let $\mathbb{Q}_k^{(n)}$ be the set

$$\mathbb{Q}_k^{(n)} := \{x \in [0,1] : T_k^m(x) = 0 \text{ for some } m \leq n\}.$$

Proposition 6.2.2. *For each* $n \geq 1$

$$[0,1] = \left(\bigcup_{a_1,\ldots,a_n\in\mathbb{N}_0^n} I_n^k(a_1, a_2, \ldots, a_n)\right) \cup \mathbb{Q}_k^{(n)}.$$

Proof. [HM, p. 2855, Proposition 2]. □

Definition 6.2.1. We will call the set of *k-rational numbers* to the union

$$\mathbb{Q}_k := \bigcup_{n=1}^{\infty} \mathbb{Q}_k^n.$$

The complement of $\mathbb{Q}_k$ in $[0,1]$ is called the set of *k-irrational numbers.*

Now, we pass to define the k-continued fractions. Also, we will see how the maps T_k are the key in the algorithm to find an expansion in k-continued fractions of some $x \in [0,1]$.

Definition 6.2.2. Let $n \in \mathbb{N}$. If $\{a_1, \dots, a_n\} \subset \mathbb{N}_0$ is a finite sequence, then we define the *finite k-continued fraction* expansion by

$$[a_1, a_2, \dots, a_n]_k := \cfrac{k}{a_1 + k + \cfrac{k}{a_2 + k + \cfrac{k}{\dots + \cfrac{k}{a_n + k}}}} =: \frac{p_n^k}{q_n^k}.$$

We say that $a_1, a_2, \dots, a_n$ are the *digits* of the k-continued fraction. The set of possible digits is $\mathbb{N}_0$.

Observe that p_n^k/q_n^k depend on $a_1, a_2, \dots, a_n$ although it is not evident from the notation. In that follow, and if the context is clear, we will write just p_n/q_n instead of p_n^k/q_n^k. We will define an infinite k-continued fraction.

Definition 6.2.3. If $\{a_1, a_n, \dots\} \subset \mathbb{N}_0$ is a infinite sequence, then we define (formally) the *infinite k-continued fraction* expansion by

$$[a_1, a_2, \dots]_k := \cfrac{k}{a_1 + k + \cfrac{k}{a_2 + k + \cfrac{k}{\ddots}}}.$$

Proposition 6.2.3. *Each k-irrational x has a unique, infinite expansion in k-continued* *fractions. We have* $x = [a_1, a_2, \ldots]_k$ *if and only if*

$$x \in I_n^k(a_1, a_2, \ldots, a_n) \text{ for all } n \geq 1,$$

if and only if

$$T_k^{n-1}(x) \in I_1^k(a_n) \text{ for all } n \geq 1.$$

Proof. [HM, p. 2856, Proposition 3]. □

Remark 6.2.1.

1. This proposition implies that the cylinders are given by

$$I_n^k(a_1, a_2, \ldots, a_n) = \{x \in [0,1] : a_1(x) = a_1, a_2(x) = a_2, \ldots, a_n(x) = a_n\}$$

   where $a_i(x)$ denotes the i-th digit of x in the k-continued fraction expansion.

2. The definition 6.2.3 now is well-defined, an infinite continued fraction represent a real number in $[0,1]$, since

$$[a_1, a_2, \ldots]_k = \lim_{n\to\infty} [a_1, a_2, \ldots, a_n]_k = \lim_{n\to\infty} \frac{p_n}{q_n}.$$

   and, for all $n \in \mathbb{N}, p_n/q_n$ is a endpoint of the closed interval $\overline{I_n^k(a_1 a_2 \ldots a_n)}$ which is a sequence of nested closed sets [HM, p. 2856].

3. From Proposition 6.2.3 we can deduce that the digits are given by $a_n = [A_k(T_k^{n-1}(x))]$, for all $n \geq 1$. Observe also that allows us to write a k-irrational x as a limit of k-rationals

$$x = \lim_{n\to\infty} \frac{p_n}{q_n}.$$

Definition 6.2.4. For each $n \geq 1$, the rationals $\frac{p_n}{q_n}$ are called the *convergents* of the k-continued fraction.

Convergents have similar properties that in the case of classical continued fractions. The following proposition summarise some of them which will be useful for our purposes.

Proposition 6.2.4. *Let $x = [a_1, a_2, \ldots]$ be a k-irrational. Then, the following properties related to convergents are satisfied:*

1. $p_n = \frac{1}{\sqrt{k}}((a_n + k)p_{n-1} + \sqrt{k}p_{n-2}), \qquad n \geq 2$
2. $q_n = \frac{1}{\sqrt{k}}((a_n + k)q_{n-1} + \sqrt{k}q_{n-2}), \qquad n \geq 2$
3. $|p_n q_{n-1} - q_n p_{n-1}| = \frac{1}{\sqrt{k}}$
4. $q_n > \sqrt{k}q_{n-1}$

Proof. This follow from Lemma 2; Propositions 4 and 5 of [HM] . □

We finish this section with a useful result for our purposes.

Lemma 6.2.5. *For any sequence of digits $(x_n)_{n\geq 1} \subset \mathbb{N}_0$ we have that*

$$\frac{I^k_{n+1}(x_1, \ldots, x_n, x_{n+1})}{I^k_n(x_1, \ldots, x_n)} \asymp \frac{1}{(x_{n+1} + k)^2}$$

where the constants involved depend only on k.

Proof. We know that

$$I^k_{n+1}(x_1, \ldots, x_{n+1}) = \frac{1}{q^k_{n+1}(\sqrt{k}q^k_{n+1} + q^k_n)}.$$

Using the equation for q_n from Lemma (6.2.4), we have that

$$I^k_n(x_1, \ldots, x_{n+1}) = \frac{\sqrt{k}}{(x_{n+1} + k)^2 q_n^2 \left(1 + \frac{\sqrt{k}q_{n-1}}{(x_{n+1}+k)q_n}\right)\left(1 + \frac{1}{x_{n+1}+k} + \frac{\sqrt{k}q_{n-1}}{(x_{n+1}+k)q_n}\right)}.$$

Then

$$\begin{aligned}\frac{I^k_{n+1}(x_1,\ldots,x_n,x_{n+1})}{I^k_n(x_1,\ldots,x_n)} &= \frac{\sqrt{k}q_n(\sqrt{k}q_n+q_{n-1})}{(x_{n+1}+k)^2q_n^2\left(1+\frac{\sqrt{k}q_{n-1}}{(x_{n+1}+k)q_n}\right)\left(1+\frac{1}{x_{n+1}+k}+\frac{\sqrt{k}q_{n-1}}{(x_{n+1}+k)q_n}\right)}\\ &\leq \frac{\sqrt{k}q_n^2\left(\sqrt{k}+\frac{1}{\sqrt{k}}\right)}{(x_{n+1}+k)^2q_n^2} = \frac{k+1}{(x_{n+1}+k)^2}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{I^k_{n+1}(x_1,\ldots,x_n,x_{n+1})}{I^k_n(x_1,\ldots,x_n)} &\geq \frac{kq_n}{(x_{n+1}+k)^2q_n\left(1+\frac{\sqrt{k}}{(x_{n+1}+k)\sqrt{k}}\right)\left(1+\frac{1}{x_{n+1}+k}+\frac{\sqrt{k}}{(x_{n+1}+k)\sqrt{k}}\right)}\\ &= \frac{k}{(x_{n+1}+k)^2\left(1+\frac{1}{x_{n+1}+k}\right)\left(1+\frac{2}{x_{n+1}+k}\right)^2}\\ &\geq \frac{k}{(x_{n+1}+k)^2\left(1+\frac{1}{k}\right)\left(1+\frac{2}{k}\right)^2}\end{aligned}$$

which ends the proof.

□

6.2.2 Symbolic model

The Markov structure of T_k implies that the dynamic associated can be coded by a full-shift on countable symbols. More precisely, let

$$\Sigma := \left\{(x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{N}_0 \text{ for every } n \in \mathbb{N}\right\},$$

and the shift map $\sigma : \Sigma \to \Sigma$ defined by $\sigma(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$. We call the pair (Σ, σ) the full-shift on countable symbols. The set

$$C_{a_1\ldots a_n} := \{(x_n)_n \in \Sigma : x_1 = a_1 \ldots x_n = a_n\}$$

is called a *symbolic cylinder* of length n. The space Σ endowed with the topology generated by the cylinder sets is a non-compact space. This fact is one of the main difficulties that need to be addressed to develop the theory. The map

$$\pi_k : \Sigma \to [0,1] \setminus \bigcup_{n\in\mathbb{N}} T_k^{-n}(0)$$
$$(x_1, x_2, \ldots) \mapsto [x_1, x_2, \ldots]_k.$$

is a topological conjugacy between the full-shift and T_k.

Remark 6.2.2. We observe that every cylinder is the projection of a symbolic cylinder $C_{a_1,\ldots,a_n}$, that is $I_n^k(a_1,\ldots,a_n) = \pi_k(C_{a_1,\ldots,a_n})$. Note that

$$\bigcup_{n\in\mathbb{N}} T_k^{-n}(0) = \mathbb{Q}_k.$$

6.2.3 Conjugacies

In this subsection we will consider the main objects in this article. Let k_1, k_2 be two different positive numbers. Observe that this define a topological conjugation between the systems T_{k_2} and T_{k_1} given by

$$\pi_{k_1,k_2} : ([0,1] \setminus \mathbb{Q}_{k_2}, T_{k_2}) \to ([0,1] \setminus \mathbb{Q}_{k_1}, T_{k_1})$$
$$x \mapsto \pi_{k_1} \circ \pi_{k_2}^{-1}(x).$$

Note that, in terms of continued fractions expansions, the action of π_{k_1,k_2} is given by

$$\pi_{k_1,k_2}([a_1, a_2, \ldots]_{k_2}) = [a_1, a_2, \ldots]_{k_1}.$$

In particular, $\pi_{k_1,k_2}(I_n^{k_2}(a_1, a_2, \ldots, a_n)) = I_n^{k_1}(a_1, a_2, \ldots, a_n)$.

6.2.4 The EMR class

In this subsection we will consider the class of EMR (Expanding-Markov-Rényi) interval maps. This class of maps was considered by Pollicott and Weiss in [PW] to study multifractal analysis of Lyapunov exponents for the Gauss map (see Section 6.3 for further details).

Definition 6.2.5. A map $T : I \to I$ is an *EMR map*, if there exists a countable family $\{I_i\}_i$ of closed intervals (with disjoint interiors $\text{int}(I_n)$), with $I_n \subset I$ for every $i \in \mathbb{N}$, satisfying

1. If $I_n = [a_n, b_n]$, then a_n, b_n are decreasing sequences, $b_1 = 1$, and $b_n \to 0$.
2. The map is C^2 on $\bigcup_{i=1}^{\infty} \text{int}(I_i)$.
3. (Expansiveness) There exists a constant $\alpha > 1$ and $N \in \mathbb{N}$ such that for every $x \in \bigcup_{i=1}^{\infty} \text{int}(I_i)$, we have $|(T^N)'(x)| > \alpha$.
4. (Markov) The sequence $\{\text{int}(I_n)\}_{n \geq 1}$ is a Markov partition for T.
5. (Rényi) There exists a positive number $K > 0$ such that

$$\sup_{n \in \mathbb{N}} \sup_{x,y,z \in I_n} \frac{|T''(x)|}{|T'(y)||T'(z)|} \leq K.$$

Remark 6.2.3. It is not difficult to prove that T_k is an EMR map, for each $k > 0$. The Rényi condition can be verified with constant $M = \frac{(k+1)^3}{k^3}$. The expansiveness condition (b) can be proved with the second iterate of T_k. Let $n \geq 0$ such that $T_k(x) \in I_1^k(n)$. Note that $T_k''(x) = 2k/x$ which is always positive and therefore T_k' is increasing. Thus

$$(T^2(x))' = T_k'(T_k(x))T_k'(x) > \frac{k}{(k/(n+k+1))^2} \cdot \frac{k}{x^2} > (n+k+1)^2 \geq (k+1)^2.$$

We finish this subsection recalling that the Rényi condition in Definition 6.2.5 of an EMR map, gives the following behaviour of the derivatives of iterates on cylinders of level n: there exists a constant C such that, for any $n \geq 1$ and any $x \in I_n^k(a_1, \ldots, a_n)$, then

$$C^{-1} \leq \frac{|(T^n)'(x)|}{|(T^n)'(y)|} \leq C$$

for any $y \in I_n^k(a_1, \ldots, a_n)$. This property is known as *Bounded Distortion.*

6.3 Thermodynamic Formalism and Dimension Theory

Thermodynamical formalism is a set of tools and methods brought into hyperbolic dynamics with great success in the early seventies from statistical physics. It allows for the selection of relevant measures from the, sometimes very large, set of invariant measures. It has been used as tool in the dimension theory of dynamical systems at least since the work of Bowen in the 70s [Bow].

6.3.1 Thermodynamic formalism on the full-shift on countable symbols

Thermodynamic formalism for dynamical systems defined in non-compact spaces has been studied and developed over the last 20 years. The particular case of the full-shift on countable many symbols (Σ, σ) has been very well studied, see [BS, MU2, Sar2]. In this section we recall the main definitions and results.

Definition 6.3.1. We say that a potential φ is *weakly Hölder* if there exists $\theta \in (0, 1)$ such that for all $n \geq 1$, we have

$$\sup \{|\varphi(x) - \varphi(y)| : x, y \in \Sigma, x_i = y_i \text{ for } i = 1, \ldots, n\} \leq C\theta^n$$

for some positive constant C independent of n.

Definition 6.3.2. Let (Σ, σ) be the full-shift on a countable alphabet and $\varphi : \Sigma \to \mathbb{R}$ a weakly Hölder function. The *pressure* of φ is defined by

$$P(\varphi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n x = x} \exp\left(\sum_{i=0}^{n-1} \varphi(\sigma^i x) \right).$$

The limit exists, but it can be infinity [BS, MU2, Sar2]. The following theorem summarizes results proved in [MU2, Sar2].

Theorem 6.3.1. *Let $\varphi : \Sigma \to \mathbb{R}$ be a weakly Hölder function such that $P(\varphi) < \infty$. Then, there exists a critical value $t^* \in (0, 1]$ such that*

$$P(t\varphi) \text{ is } \begin{cases} \text{infinite}, & \text{if } t < t^* \\ \text{finite}, & \text{if } t > t^*. \end{cases}$$

Moreover, when $t > t^*$ the pressure function $t \mapsto P(t\varphi)$ is real analytic and strictly convex. Also, for any $t > t^*$ there exists an equilibrium measure μ_t, that is, a measure such that

$$P(t\varphi) = h(\mu_t) + t \int \varphi \mu_t,$$

where $h(\mu_t)$ denotes the entropy of the measure. We have also that the derivative of P is given by the following formula

$$\frac{dP}{dt} = \int \phi \, d\mu_t.$$

Finite symbols

We recall that the theory and properties of the pressure remain true when we consider the compact case, that is, when finite symbols are considered. In other words, if we denote Σ_N as the subset of Σ consisting in all sequences with symbols only in $\{0, 1, \ldots, N\}$, any potential $\phi : \Sigma \to \mathbb{R}$ can be restricted to Σ_N and we can define the pressure in a similar way as in Definition 6.3.2. See [Wal] for further details. Denote as $P_N(\varphi)$ the pressure of a potential

φ restricted to Σ_N. A remarkable property that relates the two definitions is the following approximation theorem which was proved in [Sar1].

Theorem 6.3.2. *If $\varphi : \sigma \to \mathbb{R}$ is a weakly Hölder potential, then*

$$P(\varphi) = \sup_{N \in \mathbb{N}} \{P_N(\varphi)\}.$$

6.3.2 Pressure for T_k maps

Recall that the Markov property of T_k allows to codify the map T by the conjugation $\pi_k : \Sigma \to [0,1] \setminus \mathbb{Q}_k$ defined on Σ. We will use this fact to define a pressure.

Definition 6.3.3. Let $\phi : [0,1] \setminus \mathbb{Q}_k \to \mathbb{R}$ such that $\phi \circ \pi_k : \Sigma \to \mathbb{R}$ is a weakly Hölder potential. Then the pressure of ϕ with respect to T_k is defined by

$$P_{T_k}(\varphi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{T_k^n x = x} \exp\left(\sum_{i=0}^{n-1} \varphi(T_k^i x) \right).$$

Henceforth, we will use $P(\phi)$ instead of $P_{T_k}(\phi)$. A classical example is when the potential is $\varphi = \log |T_k'|$. In [KS2, PW] the authors studied the thermodynamic formalism for Gauss map. They gave a complete description of the pressure function $t \mapsto P(-t \log |T'|)$ and used this analysis to describe the size (with respect the Hausdorff dimension) of the points having the same speed of approximations by rationals in the classical continued expansions. One of the main tools to our results will be to know how is the pressure function $t \mapsto P(-t \log |T_k'|)$. Observe that this function can be written explicitly in the following way

$$P(-\log |T_k'|) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{T_k^n x = x} |(T_k^n)'(x)|^{-t}.$$

Proposition 6.3.3. *Let $k > 0$. The pressure function $t \mapsto P(-t \log |T_k'|)$ is finite if $t > 1/2$ and it is equal to ∞ if $t < \frac{1}{2}$. When $t > 1/2$, $P(-t \log |T_k'|)$ is real analytic, strictly decreasing* *and strictly convex. Moreover* $P(-t\log|T_k'|)\to\infty$ *when* $t\to\frac{1}{2}^+$.

Proof. For each n, the Mean Value theorem guarantees the existence of $z\in I_1^k(n)$ such that

$$|T_k'(z)|=\frac{1}{|I_1^k(n)|},$$

then, for all $x\in I_1^k(n)$ we have

$$\frac{1}{C}\le\frac{|T_k'(x)|}{|I_1^k(n)|^{-1}}\le C.$$

Therefore

$$C^{-tn}\sum_{(j_1,\dots,j_n)\in\mathbb{N}_0^n}\prod_{i=1}^{n}|I_1^k(j_i)|^t\le\sum_{T_k^n x=x}\prod_{i=0}^{n-1}|T_k'(T_k^i x)|^{-t}\le C^{tn}\sum_{(j_1,\dots,j_n)\in\mathbb{N}_0^n}\prod_{i=1}^{n}|I_1^k(j_i)|^t$$

We note that each one of the sums at left and right are equal to $\left(\sum_{j=0}^{\infty}|I_1^k(j)|^t\right)^n$, which implies

$$-tn\log C+n\log\sum_{j=0}^{\infty}|I_1^k(j)|^t\le\log\sum_{T_k^n x=x}\prod_{i=0}^{n-1}|T_k'(T_k^i x)|^{-t}\le -tn\log C+n\log\sum_{j=0}^{\infty}|I_1^k(j)|^t$$

and

$$-t\log C+\log\sum_{j=0}^{\infty}|I_1^k(j)|^t\le P(-t\log|T_k'|)\le -t\log C+\log\sum_{j=0}^{\infty}|I_1^k(j)|^t \tag{6.3.1}$$

First two assumptions are given by the convergence of series involved in inequality (6.3.1). For the limit, we first note that by Fatou's lemma

$$\liminf_{n\to\infty}\sum_{j=0}^{\infty}\frac{1}{(j+k+1)^{\frac{1}{2}+\frac{1}{n}}(j+k)^{\frac{1}{2}+\frac{1}{n}}}\ge\sum_{j=0}^{\infty}\liminf_{n\to\infty}\frac{1}{(j+k+1)^{\frac{1}{2}+\frac{1}{n}}(j+k)^{\frac{1}{2}+\frac{1}{n}}}=\infty.$$

By Theorem 6.3.1 $P(-t\log|T_k'|)$ is a real analytic, strictly convex and strictly decreasing function on $(\frac{1}{2}, \infty)$ since

$$\frac{dP(-t\log|T_k'|)}{dt} = -\int \log|T_k'| d\mu_t^k < 0.$$

Using that and the inequality (6.3.1) we have that $P(-t\log|T_k'|) \to \infty$, when $t \to 1/2^+$. $\square$

6.3.3 Dimension Theory

We start recalling the definition of Hausdorff dimension, see [Fal, Chapter 2] for further details. Given a subset $G \subset \mathbb{R}$, we say that a countable family of sets $\{U_n\}_{n\geq 1}$ is a δ-cover of G if $G \subset \bigcup_n U_n$ and every set U_n has diameter at most δ. Given $s > 0$, we define

$$H^s(G) := \lim_{\delta\to 0} H^s_\delta(G),$$

where

$$H^s_\delta(G) := \inf\left\{\sum_{n=1}^{\infty} |U_n|^s : \{U_n\}_{n\geq 1} \text{ is a } \delta\text{-cover of } G\right\}.$$

The *Hausdorff dimension* of the set G is defined by

$$\dim_H(G) := \inf\{s > 0 : H^s(G) = 0\}.$$

Given a probability measure μ on $[0,1]$, we define the Hausdorff dimension of μ by

$$\dim_H \mu := \inf\{\dim_H(A) : \mu(A) = 1\}.$$

On the other hand, we define the Lyapunov exponent of the measure μ with respect a countable Markov map by

$$\lambda(\mu) = \int \log|T'| d\mu.$$

A useful result that link the last two notions is the following (see [MU2]).

Proposition 6.3.4 (Volume Lemma)**.** *Suppose that $T : [0,1] \to [0,1]$ is a countable Markov map and μ is an ergodic T-invariant probability measure on $[0,1]$ with finite entropy $h(\mu)$. Then*

$$\dim_H \mu = \frac{h(\mu)}{\lambda(\mu)}.$$

6.4 Statement and proofs of main results

6.4.1 About the derivative of π_{k_1,k_1}

In this subsection we will prove several results about the derivative of π_{k_1,k_2}. We first prove the same result proved by Salem, but in the case of maps π_{k_1,k_2}, that is, π_{k_1,k_2} are singular maps. After that, we will interested in to find a criteria which will help us to decide if a point x belongs to $\mathcal{D}_\infty$ or $\mathcal{D}_\sim$.

To prove the singularity of π_{k_1,k_2}, we use the following characterization. See [Leo, p. 107] for a proof.

Theorem 6.4.1. *Let $I \subset \mathbb{R}$ be an interval and let $u : I \to \mathbb{R}$ be a non constant function such that $u'(x)$ exists (possible infinite) for Leb-a.e. $x \in I$. Then u is a singular function if and only if there exists a Lebesgue measurable set $E \subset I$ such that the Leb$(I \setminus E) = 0$ and Leb$(u(E)) = 0$.*

Proposition 6.4.2. *The conjugation maps π_{k_1,k_2} are singular.*

Proof. For any borel set $B \subset [0,1]$, we define the measure

$$\nu(A) := \mu_{k_1}(\pi_{k_1,k_2}(B)).$$

We observe that ν is ergodic with respect to T_{k_2}. Since μ_{k_2} is also an ergodic measure for T_{k_2}, then they are either equal or mutually singular. Suppose that $\nu(B) = \mu_{k_2}(B)$ for every Borel subset B of the unit interval. In particular, taking $B = (\frac{k_1}{k_1+1}, 1)$ then $\mu_{k_1}(\frac{k_1}{k_1+1}, 1) = \mu_{k_2}(\frac{k_2}{k_2+1}, 1)$. This implies the equality

$$c_{k_1}\left(\log(k_1+1) - \log\frac{k_1}{k_1+1}\right) = c_{k_2}\left(\log(k_2+1) - \log\frac{k_2}{k_2+1}\right)$$

which is impossible because of the function defined for $x > 0$ given by $g(x) := c_x(\log(x+1) - \log(\frac{x}{x+1} + x))$ is strictly decreasing. Thus $\nu \neq \mu_{k_2}$ and therefore they are mutually singular. In consequence, there is a Borel measurable set B, such that

$$\mu_{k_1}(\pi_{k_1,k_2}(B)) = 0 \qquad \text{and} \qquad \mu_{k_2}(B) = 1.$$

The equivalence of those measures with the Lebesgue measure and the Theorem 6.4.1 allows to deduce that the function π_{k_1,k_2} is singular. □

In the following, we will denote by $I_n^k(x)$ the unique cylinder of level n which contains x in the dynamic of T_k.

Proposition 6.4.3. *Suppose that $\pi'_{k_1,k_2}(x)$ exists in a generalized sense, that is, $\pi'_{k_1,k_2}(x) \in [0, \infty]$. Then*

$$\pi'_{k_1,k_2}(x) = \lim_{n\to\infty} \frac{|\pi_{k_2,k_2}(I_n^{k_2}(x))|}{|I_n^{k_2}(x)|}$$

Proof. Let $n \geq 1$. Denote by l_n and r_n the left and right endpoints of the interval $I_n^{k_2}$. We note that there are two cases about the position of the point $(x, \pi_{k_1,k_2}(x))$ in regard to the line L joining the points $(l_n, \pi_{k_1,k_2}(l_n))$ and $(l_n, \pi_{k_1,k_2}(l_n))$:

1) *The point $(x, \pi_{k_1,k_2}(x))$ is above or on L.* Comparing the slopes of the segments, we

obtain

$$\frac{\pi_{k_1,k_2}(x)-\pi_{k_1,k_2}(l_n)}{x-l_n}\geq\frac{\pi_{k_1,k_2}(r_n)-\pi_{k_1,k_2}(l_n)}{r_n-l_n}\geq\frac{\pi_{k_1,k_2}(r_n)-\pi_{k_1,k_2}(x)}{r_n-x}.$$

2) *The point* $(x,\pi_{k_1,k_2}(x))$ *is below or on* L. Analogously, we obtain

$$\frac{\pi_{k_1,k_2}(x)-\pi_{k_1,k_2}(l_n)}{x-l_n}\leq\frac{\pi_{k_1,k_2}(r_n)-\pi_{k_1,k_2}(l_n)}{r_n-l_n}\leq\frac{\pi_{k_1,k_2}(r_n)-\pi_{k_1,k_2}(x)}{r_n-x}$$

The proposition is therefore deduced from the fact that

$$\lim_{n\to\infty}\frac{\pi_{k_1,k_2}(x)-\pi_{k_1,k_2}(l_n)}{x-l_n}=\lim_{n\to\infty}\frac{\pi_{k_1,k_2}(r_n)-\pi_{k_1,k_2}(x)}{r_n-x}=\pi'_{k_1,k_2}(x)$$

since the derivative exists or is equal to infinity. $\square$

Lemma 6.4.4. *Let* x *be a* k_2*-irrational number. Then, for all* $n\geq 1$

$$\frac{|\pi_{k_1,k_2}(I_n^{k_2}(x))|}{|I_n^{k_2}(x)|}\asymp e^{S_n\psi(x)}$$

where

$$\psi(x)=-\log|T'_{k_1}(\pi_{k_1,k_2}(x))|+\log|T'_{k_2}(x)|$$

Proof. Follows directly from the Bounded distortion property. $\square$

Proposition 6.4.5. *Let* x *be a* k_2*-irrational number. If the following conditions holds for* x*:*

$$\liminf_{n\to\infty}e^{S_n\psi(x)}=0\quad\textit{and}\quad\limsup_{n\to\infty}e^{S_n\psi(x)}=\infty\tag{6.4.1}$$

then $x\in\mathcal{D}_\sim$*.*

Proof. Suppose that the derivative exists in the generalized sense at x. By Proposition 6.4.3, we have

$$\pi'_{k_1,k_2}(x) = \lim_{n\to\infty} \frac{|\pi_{k_2,k_2}(I_n^{k_2}(x))|}{|I_n^{k_2}(x)|}.$$

Moreover, from the conditions (6.4.1) and Proposition 6.4.4 we obtain the following limits:

$$\limsup_{n\to\infty} \frac{|\pi_{k_2,k_2}(I_n^{k_2}(x))|}{|I_n^{k_2}(x)|} = \infty$$

and

$$\liminf_{n\to\infty} \frac{|\pi_{k_2,k_2}(I_n^{k_2}(x))|}{|I_n^{k_2}(x)|} = 0,$$

which is a contradiction with the existence of $\pi'_{k_1,k_2}(x)$.

□

Proposition 6.4.6. *Let* $x = [x_1, x_2, \ldots]_{k_2}, y = [y_1, y_2, \ldots]_{k_2} \in (0,1)$ *be two* k_2*-irrational numbers in* $[0,1]$*. Suppose that there exists* $n \geq 1$ *such that* $x_i = y_i$ *for all* $1 \leq i \leq n$ *and* $x_{n+1} \neq y_{n+1}$*. Then*

$$\frac{\pi_{k_1,k_2}(x) - \pi_{k_1,k_2}(y)}{x - y} \asymp e^{S_n\psi(x)}.$$

Proof. First we prove the inequality $\ll$. Suppose first that $x < y$. To ease the exposition, we will denote by $\pi := \pi_{k_1,k_2}$. The main idea of the proof is to find lower and upper bounds for $|x-y|$ and $|\pi(x) - \pi(y)|$ respectively, which will depend on the digits of x and y.

We will use Lemma 6.4.4 and Lemma 6.2.5 repeatedly.

Case 1. Suppose that the cylinders of level $n+2$ are accumulating at the right side of $\pi(x)$. In particular, $x_{n+1} > y_{n+1} \geq 0$. Denote by $r_{n+1}^{k_1}$ the right end-point of the cylinder $I_{n+1}^{k_1}(x_1, \ldots, x_{n+1})$ (see Figure (6.1)).

The equality $|\pi(x) - \pi(y)| = (\pi(y) - r_{n+1}^{k_1}) + (r_{n+1}^{k_1} - \pi(x))$ will allow us to discriminate into two sub-cases which will help to find upper bounds for $|\pi(x) - \pi(y)|$.

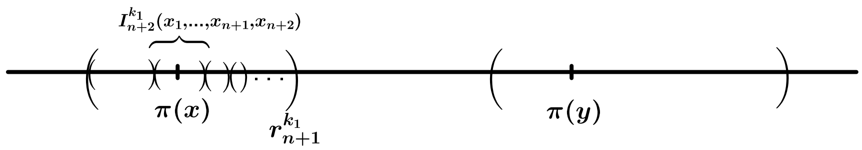

Figure 6.1: The biggest intervals are denoting the interior of the cylinders of level $n+1$ containing $\pi(x)$ and $\pi(y)$ respectively. The smallest intervals denote the cylinders of level $n+2$ which are accumulating at $r^{k_1}_{n+1}$.

- <u>*Sub-case 1.1.*</u> $\boldsymbol{\pi(y) - r^{k_1}_{n+1} \leq r^{k}_{n+1} - \pi(x)}$. Observe that

$$\begin{aligned}
|x-y| &\geq \sum_{j=x_{n+2}+1}^{\infty} |I^{k_2}_{n+2}(x_1,\dots,x_{n+1},j)| \\
&\gg |I^{k_2}_{n+1}(x_1,\dots,x_{n+1})| \sum_{j=x_{n+2}+1}^{\infty} \frac{1}{(j+k_2)^2} \\
&\geq |I^{k_2}_{n+1}(x_1,\dots,x_{n+1})| \int_{x_{n+2}+1}^{\infty} \frac{dt}{(t+k_2)^2} \\
&\geq |I^{k_2}_{n+1}(x_1,\dots,x_{n+1})| \frac{1}{x_{n+2}+1+k_2}.
\end{aligned}$$

Note that in this sub-case $|\pi(x)-\pi(y)| \leq 2(r^{k_1}_{n+1} - \pi(x))$. If $x_{n+2} \neq 0$, then

$$\begin{aligned}
|\pi(x)-\pi(y)| &\leq 2 \sum_{j=x_{n+2}}^{\infty} |I^{k_1}_{n+2}(x_1,\dots,x_{n+1},j)| \\
&\ll |I^{k_1}_{n}(x_1,\dots,x_n)| \frac{1}{(x_{n+1}+k_1)^2} \sum_{j=x_{n+2}}^{\infty} \frac{1}{(j+k_1)^2} \\
&\leq |I^{k_1}_{n}(x_1,\dots,x_n)| \frac{1}{(x_{n+1}+k_1)^2} \int_{x_{n+2}-1}^{\infty} \frac{dt}{(t+k_1)^2} \\
&= |I^{k_1}_{n}(x_1,\dots,x_n)| \frac{1}{(x_{n+1}+k_1)^2(x_{n+2}-1+k_1)}.
\end{aligned}$$

Then

$$\frac{|\pi(x)-\pi(y)|}{|x-y|} \ll \frac{|I_{n+1}^{k_1}(x_1,\ldots,x_{n+1})|}{|I_{n+1}^{k_2}(x_1,\ldots,x_{n+1})|}\frac{(x_{n+2}+1+k_2)}{(x_{n+2}-1+k_1)}$$
$$\ll e^{S_n\psi(x)}\frac{(x_{n+1}+k_2)^2(x_{n+2}+1+k_2)}{(x_{n+1}+k_1)^2(x_{n+2}-1+k_1)}.$$

If $x_{n+2}=0$, then $|\pi(x)-\pi(y)| \leq 2|I_{n+1}^{k_1}(x_1,\ldots,x_{n+1})|$ and

$$\frac{|\pi(x)-\pi(y)|}{|x-y|} \ll \frac{|I_{n+1}^{k_1}(x_1,\ldots,x_{n+1})|(k_2+1)}{|I_{n+1}^{k_2}(x_1,\ldots,x_{n+1})|} \ll e^{S_n\psi(x)}\frac{(x_{n+1}+k_2)^2}{(x_{n+1}+k_1)^2}.$$

- *Sub-case 1.2.* $\boldsymbol{\pi(y)-r_{n+1}^{k_1} \leq \pi(y)-r_{n+1}^{k_1}}$.

1.2.1 . In first place, we will suppose that there is at least one cylinder of level $n+1$ between $I_{n+1}^{k_1}(x_1,\ldots,x_n,x_{n+1})$ and $I_{n+1}^{k_1}(x_1,\ldots,x_n,y_{n+1})$. In particular $y_{n+1} \neq x_{n+1}+1$. Then

$$|x-y| \geq \sum_{j=y_{n+1}+1}^{x_{n+1}-1} |I_{n+1}^{k_2}(x_1,\ldots,x_n,j)|$$
$$\gg |I_n^{k_2}(x_1,\ldots,x_n)| \sum_{j=y_{n+1}+1}^{x_{n+1}-1} \frac{1}{(j+k_2)^2}$$
$$\geq |I_n^{k_2}(x_1,\ldots,x_n)| \int_{y_{n+1}+1}^{x_{n+1}} \frac{dt}{(t+k_2)^2}$$
$$= |I_n^{k_2}(x_1,\ldots,x_n)| \frac{x_{n+1}-y_{n+1}-1}{(y_{n+1}+1+k_2)(x_{n+1}+k_2)}$$

To find an upper bound for $|\pi(x)-\pi(y)|$, suppose first that $y_{n+1}\neq 0$. Then

$$\begin{aligned}
|\pi(x)-\pi(y)| &\leq 2\sum_{j=y_{n+1}}^{x_{n+1}-1}|I_{n+1}^{k_1}(x_1,\ldots,x_n,j)|\\
&\ll |I_n^{k_1}(x_1,\ldots,x_n)|\sum_{j=y_{n+1}}^{x_{n+1}-1}\frac{1}{(j+k_1)^2}\\
&\leq |I_n^{k_1}(x_1,\ldots,x_n)|\int_{y_{n+1}-1}^{x_{n+1}-1}\frac{dt}{(t+k_1)^2}\\
&= |I_n^{k_1}(x_1,\ldots,x_n)|\frac{x_{n+1}-y_{n+1}}{(y_{n+1}-1+k_1)(x_{n+1}-1+k_1)}
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{\pi(x)-\pi(y)}{x-y} &\ll \frac{|I_n^{k_1}(x_1,\ldots,x_n)|}{|I_n^{k_2}(x_1,\ldots,x_n)|}\frac{x_{n+1}-y_{n+1}}{x_{n+1}-y_{n+1}-1}\frac{(y_{n+1}+1+k_2)(x_{n+1}+1+k_2)}{(y_{n+1}-1+k_1)(x_{n+1}-1+k_1)}\\
&\ll e^{S_n\psi(x)}\frac{x_{n+1}-y_{n+1}}{x_{n+1}-y_{n+1}-1}\frac{(y_{n+1}+1+k_2)(x_{n+1}+1+k_2)}{(y_{n+1}-1+k_1)(x_{n+1}-1+k_1)}
\end{aligned}$$

Now, if we suppose that $y_{n+1}=0$, we have that

$$|x-y|\gg |I_n^{k_2}(x_1,\ldots,x_n)|\frac{x_{n+1}-1}{(1+k_2)(x_{n+1}+k_2)}$$

and $|\pi(x)-\pi(y)|\leq |I_n^{k_1}(x_1,\ldots,x_n)|$. Then

$$\frac{\pi(x)-\pi(y)}{x-y}\ll \frac{|I_n^{k_1}(x_1,\ldots,x_n)|}{|I_n^{k_2}(x_1,\ldots,x_n)|}\frac{(1+k_2)(x_{n+1}+k_2)}{x_{n+1}-1}.$$

1.2.2. Secondly, we will suppose that the cylinders of level $n+1$ of x and y respectively, are neighbour cylinders. In other words, $y_{n+1}=x_{n+1}+1$. Assume $y_{n+2}\neq 0$.

Then

$$
\begin{aligned}
|\pi(x)-\pi(y)| &\le 2(\pi(y)-r_{n+1}^{k_1}) \\
&\le 2|I_{n+1}^{k_1}(x_1,\ldots,x_n,y_{n+1})| \\
&\ll \frac{1}{(y_{n+1}+k_1)^2}|I_n^{k_1}(x_1,\ldots,x_n)|
\end{aligned}
$$

and

$$
\begin{aligned}
|x-y| &\ge |I_{n+2}^{k_1}(x_1,\ldots,x_n,y_{n+1},0)| \\
&\gg |I_{n+1}^{k_1}(x_1,\ldots,x_n,y_{n+1})| \\
&\gg \frac{1}{(y_{n+1}+k_2)^2}|I_n^{k_1}(x_1,\ldots,x_n)|
\end{aligned}
$$

and therefore

$$
\frac{\pi(x)-\pi(y)}{x-y} \ll e^{S_n\psi(x)}\frac{(y_{n+1}+k_2)^2}{(y_{n+1}+k_1)^2}.
$$

Now, suppose that $y_{n+2}=0$. In this case, we will pass to compare the distances $|\pi(x)-\pi(y)|$ and $|x-y|$ with lengths of cylinders of level $n+3$.

$$
\begin{aligned}
|x-y| &\ge \sum_{j=y_{n+3}+1}^{\infty}|I_{n+3}^{k_2}(x_1,\ldots,x_n,y_{n+1},0,j)| \\
&\gg |I_{n+2}^{k_2}(x_1,\ldots,x_n,y_{n+1},0)|\sum_{j=y_{n+3}+1}^{\infty}\frac{1}{(j+k_2)^2} \\
&\gg |I_{n+1}^{k_2}(x_1,\ldots,x_n,y_{n+1})|\int_{y_{n+3}+1}^{\infty}\frac{t}{(t+k_2)^2} \\
&\gg |I_n^{k_2}(x_1,\ldots,x_n)|\frac{1}{(y_{n+1}+k_2)^2(y_{n+3}+1+k_2)}.
\end{aligned}
$$

To find the upper bound for $|\pi(x)-\pi(y)|$, we will suppose two cases $y_{n+3}=0$

and $y_{n+3} \neq 0$. If $y_{n+3} \neq 0$,

$$\begin{aligned}
|\pi(x) - \pi(y)| \leq 2(\pi(y) - r_{n+1}) &\leq \sum_{j=y_{n+3}}^{\infty} |I_{n+3}^{k_1}(x_1, \ldots, x_n, y_{n+1}, 0, j)| \\
&\ll |I_{n+2}^{k_1}(x_1, \ldots, x_n, y_{n+1}, 0)| \sum_{j=y_{n+3}}^{\infty} \frac{1}{(j+k_1)^2} \\
&\ll |I_n^{k_1}(x_1, \ldots, x_n)| \frac{1}{(y_{n+1}+k_1)^2} \int_{y_{n+3}-1}^{\infty} \frac{dt}{(t+k_1)^2} \\
&= |I_n^{k_1}(x_1, \ldots, x_n)| \frac{1}{(y_{n+1}+k_1)^2 (y_{n+3}-1+k_1)}
\end{aligned}$$

then

$$\frac{\pi(x) - \pi(y)}{x - y} \ll e^{S_n \psi(x)} \frac{(y_{n+1}+k_2)^2 (y_{n+3}+k_2)}{(y_{n+1}+k_1)^2 (y_{n+3}-1+k_1)}.$$

When $y_{n+3} = 0$, we have

$$|x - y| \gg |I_n^{k_2}(x_1, \ldots, x_n)| \frac{1}{(y_{n+1}+k_2)^2}$$

and

$$\begin{aligned}
|\pi(x) - \pi(y)| &\leq 2|I_{n+3}(x_1, \ldots, x_n, y_{n+1}, 0, 0)| \\
&\ll |I_n^{k_1}(x_1, \ldots, x_n)| \frac{1}{(y_{n+1}+k_1)^2}
\end{aligned}$$

then

$$\frac{\pi(x) - \pi(y)}{x - y} \ll e^{S_n \psi(x)} \frac{(y_{n+1}+k_2)^2}{(y_{n+1}+k_1)^2}.$$

Case 2. Suppose that the cylinders of level $n+2$ are accumulating at the left side of $\pi(x)$. In particular, $0 \leq x_{n+1} < y_{n+1}$. Denote by $l_{n+1}^{k_1}$ the right end-point of the cylinder $I_{n+1}^{k_1}(x_1, \ldots, x_n, y_{n+1})$ (see Figure (6.2)).

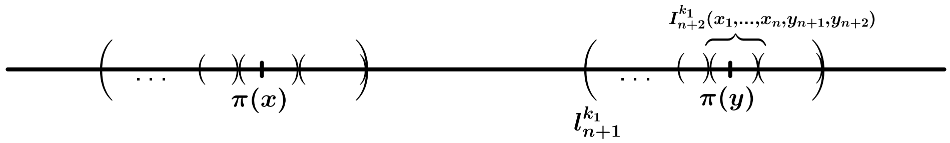

Figure 6.2: The biggest intervals are denoting the interior of the cylinders of level $n+1$ containing $\pi(x)$ and $\pi(y)$ respectively. The smallest intervals denote the cylinders of level $n+2$ which are accumulating at $l^{k_1}_{n+1}$.

- <u>*Subcase 2.1.*</u> $\boldsymbol{\pi(x) - l^{k_1}_{n+1} \leq l^{k}_{n+1} - \pi(y)}$.

$$\begin{aligned}
|x-y| &\geq \sum_{j=y_{n+2}+1}^{\infty} |I^{k_2}_{n+2}(x_1,\ldots,x_n,y_{n+1},j)| \\
&\gg |I^{k_2}_{n+1}(x_1,\ldots,x_n,y_{n+1})| \sum_{j=y_{n+2}+1}^{\infty} \frac{1}{(j+k_2)^2} \\
&\gg |I^{k_2}_{n}(x_1,\ldots,x_n)| \frac{1}{(y_{n+1}+k_2)^2(y_{n+2}+1+k_2)}
\end{aligned}$$

On the other hand, $|\pi(x)-\pi(y)| \leq 2(l^{k_1}_{n+1} - \pi(y))$. If $y_{n+1} \neq 0$

$$\begin{aligned}
|\pi(x)-\pi(y)| &\leq \sum_{j=y_{n+2}}^{\infty} |I^{k_1}_{n+2}(x_1,\ldots,x_n,y_{n+1},j)| \\
&\ll |I^{k_1}_{n}(x_1,\ldots,x_n)| \frac{1}{(y_{n+1}+k_1)^2} \int_{y_{n+2}-1}^{\infty} \frac{dt}{(t+k_1)^2} \\
&= |I^{k_1}_{n}(x_1,\ldots,x_n)| \frac{1}{(y_{n+1}+k_1)^2(y_{n+2}-1+k_1)}
\end{aligned}$$

Then

$$\frac{\pi(x)-\pi(y)}{x-y} \ll e^{S_n\psi(x)} \frac{(y_{n+1}+k_2)^2(y_{n+2}+1+k_2)}{(y_{n+1}+k_1)^2(y_{n+2}-1+k_1)}.$$

If $y_{n+2}=0$ then

$$|\pi(x)-\pi(y)| \leq 2|I_{n+1}(x_1,\ldots,x_n,y_{n+1})| \ll |I_n(x_1,\ldots,x_n)| \frac{1}{(y_{n+1}+k_1)^2}$$

and

$$|x-y| \gg |I_n^{k_1}(x_1,\dots,x_n)| \frac{1}{(y_{n+1}+k_2)^2(1+k_2)}.$$

In consequence

$$\frac{\pi(x)-\pi(y)}{x-y} \ll e^{S_n\psi(x)} \frac{(y_{n+1}+k_2)^2}{(y_{n+1}+k_1)^2}.$$

- <u>*Subcase 2.2.*</u> $\boldsymbol{\pi(y) - l_{n+1}^{k_1} \leq l_{n+1}^{k} - \pi(x)}$.

<u>*2.2.1*</u> . In first place, we will suppose that there is at least one cylinder of level $n+1$ between $I_{n+1}^{k_1}(x_1,\dots,x_n,x_{n+1})$ and $I_{n+1}^{k_1}(x_1,\dots,x_n,y_{n+1})$. In particular $x_{n+1} \neq y_{n+1}+1$. Then

$$\begin{aligned}
|x-y| &\geq \sum_{j=x_{n+1}+1}^{y_{n+1}-1} |I_{n+1}^{k_2}(x_1,\dots,x_n,j)| \\
&\gg |I_n^{k_2}(x_1,\dots,x_n)| \sum_{j=x_{n+1}+1}^{y_{n+1}-1} \frac{1}{(j+k_2)^2} \\
&\geq |I_n^{k_2}(x_1,\dots,x_n)| \int_{x_{n+1}+1}^{y_{n+1}} \frac{dt}{(t+k_2)^2} \\
&= |I_n^{k_2}(x_1,\dots,x_n)| \frac{y_{n+1}-x_{n+1}-1}{(x_{n+1}+1+k_2)(y_{n+1}+k_2)}.
\end{aligned}$$

To find an upper bound for $|\pi(x)-\pi(y)|$, let suppose two cases: $x_{n+1} \neq 0$ and $x_{n+1}=0$. If $x_{n+1} \neq 0$ then

$$\begin{aligned}
|\pi(x)-\pi(x)| &\leq 2 \sum_{j=x_{n+1}}^{y_{n+1}-1} |I_{n+1}^{k_1}(x_1,\dots,x_n,j)| \\
&\ll |I_n^{k_1}(x_1,\dots,x_n)| \int_{x_{n+1}-1}^{y_{n+1}-1} \frac{dt}{(t+k_1)^2} \\
&= |I_n^{k_1}(x_1,\dots,x_n)| \frac{y_{n+1}-x_{n+1}}{(x_{n+1}-1+k_1)(y_{n+1}-1+k_1)}
\end{aligned}$$

therefore

$$\frac{\pi(x)-\pi(y)}{x-y} \ll e^{S_n\psi(x)} \frac{y_{n+1}-x_{n+1}}{y_{n+1}-x_{n+1}-1} \frac{(x_{n+1}+1+k_2)(x_{n+1}+k_2)}{(x_{n+1}-1+k_1)(y_{n+1}-1+k_1)}.$$

Now, if $x_{n+1}=0$ then

$$|x-y| \gg |I_n^{k_2}(x_1,\dots,x_n)| \frac{y_{n+1}-1}{(1+k_2)(y_{n+1}+k_2)}$$

and $|\pi(x)-\pi(y)| \leq |I_n^{k_1}(x_1,\dots,x_n)|$ and therefore

$$\frac{\pi(x)-\pi(y)}{x-y} \ll e^{S_n\psi(x)} \frac{(1+k_2)(y_{n+1}+k_2)}{y_{n+1}-1}$$

2.2.2 . Secondly, we will suppose that the cylinders of level $n+1$ of x and y respectively, are neighbour cylinders. In other words, $x_{n+1}=y_{n+1}+1$. Assume $x_{n+2}\neq 0$. Then

$$|\pi(x)-\pi(y)| \leq 2(l_{n+1}^{k_1}-\pi(x)) \leq 2|I_{n+1}^{k_1}(x_1,\dots,x_{n+1})| \ll |I_n^{k_1}(x_1,\dots,x_n)| \frac{1}{(x_{n+1}+k_1)^2}$$

and

$$\begin{aligned} |x-y| &\geq |I_{n+2}^{k_2}(x_1,\dots,x_{n+1},0)| \\ &\gg |I_n^{k_2}(x_1,\dots,x_n)| \frac{1}{(x_{n+1}+k_2)^2} \end{aligned}$$

Therefore-

$$\frac{\pi(x)-\pi(y)}{x-y} \ll e^{S_n\psi(x)} \frac{(x_{n+1}+k_2)^2}{(x_{n+1}+k_1)^2}.$$

When $x_{n+2}=0$ we pass to the level $n+3$.

$$
\begin{aligned}
|x-y| &\geq \sum_{j=x_{n+3}+1}^{\infty} |I_{n+3}^{k_2}(x_1,\ldots,x_{n+1},0,j)| \\
&\gg |I_n^{k_2}(x_1,\ldots,x_n)| \frac{1}{(x_{n+1}+k_2)^2} \sum_{j=x_{n+3}+1}^{\infty} \frac{1}{(j+k_2)^2} \\
&\geq |I_n^{k_2}(x_1,\ldots,x_n)| \frac{1}{(x_{n+1}+k_2)^2} \int_{x_{n+3}+1}^{\infty} \frac{dt}{(t+k_2)^2} \\
&= |I_n^{k_2}(x_1,\ldots,x_n)| \frac{1}{(x_{n+1}+k_2)^2(x_{n+3}+1+k_2)}.
\end{aligned}
$$

Now, to find an upper bound for $|\pi(x)-\pi(y)|$, we will suppose two cases $x_{n+3}\neq 0$ and $x_{n+3}=0$. If $x_{n+3}\neq 0$

$$
\begin{aligned}
|\pi(x)-\pi(y)| \leq 2(l_{n+1}^{k_1}-\pi(x)) &\leq 2\sum_{j=x_{n+3}}^{\infty} |I_{n+3}^{k_1}(x_1,\ldots,x_{n+1},0,j)| \\
&\ll |I_n^{k_1}(x_1,\ldots,x_n)| \frac{1}{(x_{n+1}+k_1)^2} \sum_{j=x_{n+3}}^{\infty} \frac{1}{(j+k_1)^2} \\
&\ll |I_n^{k_1}(x_1,\ldots,x_n)| \frac{1}{(x_{n+1}+k_1)^2} \int_{x_{n+3}-1}^{\infty} \frac{dt}{(t+k_1)^2} \\
&\ll |I_n^{k_1}(x_1,\ldots,x_n)| \frac{1}{(x_{n+1}+k_1)^2(x_{n+3}-1+k_1)}
\end{aligned}
$$

therefore

$$
\frac{\pi(x)-\pi(y)}{x-y} \ll e^{S_n\psi(x)} \frac{(x_{n+1}+k_2)^2(x_{n+3}+1+k_2)}{(x_{n+1}+k_1)^2(x_{n+3}-1+k_1)}.
$$

Observe that when $x_{n+3}=0$, we have

$$
|x-y| \gg |I_n^{k_2}(x_1,\ldots,x_n)| \frac{1}{(x_{n+1}+k_2)^2(1+k_2)}
$$

and

$$|\pi(x)-\pi(y)| \leq |I^{k_1}_{n+2}(x_1,\dots,x_{n+1},0)| \ll |I^{k_1}_n(x_1,\dots,x_n)|\frac{1}{(x_{n+1}+k_1)^2}$$

therefore

$$\frac{\pi(x)-\pi(y)}{x-y} \ll e^{S_n\psi(x)}\frac{(x_{n+1}+k_2)^2}{(x_{n+1}+k_1)^2}.$$

Observe that in any case we found a estimation of the form

$$\frac{\pi(x)-\pi(y)}{x-y} \ll e^{S_n\psi(x)}a_n$$

where a_n is depending on the digits of x, but, in any case, a_n is bounded independently of the behavior of the sequence of digits. The case $x>y$ is proved by a symmetric argument and the proof of the inequality $\gg$ is analogous. $\square$

A useful consequence of the last proposition is the following corollary.

Corollary 6.4.7. *Let x be a k_2-irrational number. We have that*

1. $x \in \mathcal{D}_\infty$ *if and only if* $\limsup_{n\to\infty} e^{S_n\psi(x)} = \infty$
2. $\{x : \lim_{n\to\infty} e^{S_n\psi(x)} = 0\} \subset \mathcal{D}_0$.

We finish this section proving that ψ is a bounded potential.

Lemma 6.4.8. *For all k_1, k_2 positive numbers, the potential*

$$\psi(x) = -\log|T'_{k_1}(\pi_{k_1,k_2}(x))| + \log|T'_{k_2}(x)|$$

is bounded in $(0,1]$*. Moreover*

$$\inf_{x\in(0,1]}\psi(x)\sup_{x\in(0,1]}\psi(x) < 0.$$

Proof. Recall that the partition of level 1 corresponding to T_{k_2} is

$$I_n^{k_2} = \left(\frac{k_2}{n+k_2+1}, \frac{k_2}{n+k_2} \right], \qquad n \geq 0.$$

We know also that $\pi_{k_1,k_2}(I_n^{k_2}) = I_n^{k_1}$. Let $n \geq 0$ and $x \in I_n^{k_2}$. Since

$$\log \frac{|T'_{k_2}(x)|}{|T'_{k_1} \circ \pi_{k_1,k_2}(x)|} = \log \frac{k_2 \pi^2_{k_1,k_2}(x)}{k_1 x^2}$$

then, for all $n \geq 0$

$$\log \frac{k_1(n+k_2)^2}{k_2(n+k_1+1)^2} \leq \log \frac{|T'_{k_2}(x)|}{|T'_{k_1} \circ \pi_{k_1,k_2}(x)|} \leq \log \frac{k_1(n+k_2+1)^2}{k_2(n+k_1)^2}. \tag{6.4.2}$$

Observing that both expressions bounding ψ in the last inequality converge to $\log \frac{k_1}{k_2}$ when n tends to ∞, we deduce that ψ is bounded in $(0,1]$. To obtain $\inf_{x\in(0,1]} \psi(x) < 0$ and $\sup_{x\in(0,1]} \psi(x) > 0$, assume $k_1 > k_2$. In this case we have $\psi(1) = \log \frac{k_2}{k_1} < 0$. Moreover, since the limit of the bounds in (6.4.2) is $\log \frac{k_1}{k_2} > 0$, in particular $\psi(x) > 0$ for x sufficiently close to zero. The case $k_1 < k_2$ is analogous. $\square$

6.4.2 Main result

Proposition 6.4.9. *Let δ be a number in $(1/2, 1]$. Define the function G_δ by the formula*

$$G_\delta(q) = P(q\psi - \delta \log |T'_{k_2}|) = P(q(-\log |T'_{k_1} \circ \pi_{k_1,k_2}| + \log |T'_{k_2}|) - \delta \log |T'_{k_2}|).$$

Then,

(1) for all $\delta \in (1/2, 1]$, $G_\delta(q)$ is finite for every $q \in \mathbb{R}$;

(2) for every $q \in \mathbb{R}$, the function $\delta \mapsto G_\delta(q)$ is strictly decrasing;

(3) $G_1(q) = 0$ *if and only if* $q = 0$ *or* $q = 1$*;*

(4) there exists $\delta \in (1/2, 1]$ *such that* $G_\delta(q) > 0$ *for all* $q \in \mathbb{R}$*.*

Proof. Let $\delta \in (0, 1/2]$. By Proposition 6.4.8, there exists real numbers a, b such that $ab < 0$ and $a \leq \psi(x) \leq b$, for all $x \in (0, 1]$. In consequence

$$aq + P(-\delta \log |T'_{k_2}|) \leq G_\delta(q) \leq bq + P(-\delta \log |T'_{k_2}|).$$

Since $P(-\delta \log |T'_{k_2}|)$ is finite for $\delta \in (0, 1/2]$, we obtain the finiteness of $G_\delta(q)$, for all $q \in \mathbb{R}$. This shows (1). Observe that (2) holds by Theorem 6.3.1 and Proposition 6.3.3. To prove (3), we note that

$$G_1(0) = P(-\log |T'_{k_2}|) = 0 \quad \text{and} \quad G_1(1) = P(-\log |T'_{k_1} \circ \pi_{k_1,k_2}|) = 0.$$

By convexity of G_1, we can have only two possibilities, namely, $G_1(q)$ is a constant function equal to zero, or, $G_1(q) = 0$ if and only if $q = 0, 1$. We will prove that $G_1(q) \not\equiv 0$. From the variational principle, we have that

$$\begin{aligned} G_1(q) &\geq h(\mu_{[\overline{0}]_{k_2}}) + \int q(-\log |T'_{k_1} \circ \pi_{k_1,k_2}| + \log |T'_{k_2}|) - \log |T'_{k_2}| d\mu_{[\overline{0}]_{k_2}} \\ &= q \log \frac{|T'_{k_2}([\overline{0}]_{k_2})|}{|T'_{k_1} \circ \pi_{k_1,k_2}([\overline{0}]_{k_2})|} - \log |T'_{k_2}([\overline{0}]_{k_2})| =: l(q) \end{aligned} \tag{6.4.3}$$

where

$$[\overline{0}]_k = \frac{-k + \sqrt{k^2 + 4k}}{2}$$

denotes the fixed point of T_{k_2} in $I_0^{k_2}$ and $\mu_{[\overline{0}]_{k_2}}$ denotes the Dirac measure supported at $[\overline{0}]_{k_2}$.

It is sufficient to prove that $l(q)$ is a non-horizontal line. The slope of $l(q)$ is given by

$$\log \frac{|T'_{k_2}([\overline{0}]_{k_2})|}{|T'_{k_1} \circ \pi_{k_1,k_2}([\overline{0}]_{k_2})|} = \log \frac{k_2(\sqrt{k_1^2+4k_1}-k_1)^2}{k_1(\sqrt{k_2^2+4k_2}-k_2)^2}.$$

We note that the slope is equal to zero, if and only if,

$$\frac{(\sqrt{k_1^2+4k_1}-k_1)^2}{k_1} = \frac{(\sqrt{k_2^2+4k_2}-k_2)^2}{k_2}$$

which is a contradiction because the function $x \mapsto \frac{(\sqrt{x^2+4x}-x)^2}{x}$ is strictly decreasing on $(0,\infty)$. Therefore (3) is proved. We pass now to prove (4). Let $\delta \in (1/2,1)$. Since $G_\delta(q)$ is strictly decreasing in δ and $G_1(0) = G_1(1)$, we deduce that, if $G_\delta(q) = 0$ then q must belong to the interval (0,1). Let $q \geq 0$. By Proposition 6.4.8 we have

$$aq + P(-\delta \log |T'_{k_2}|) \leq G_\delta(q) \leq bq + P(-\delta \log |T'_{k_2}|).$$

Let $\delta \in (1/2,1)$ such that

$$\frac{P(-\delta \log |T'_{k_2}|)}{-a} > 1$$

which exists since $P(-\delta \log |T'_{k_2}|) \to +\infty$ when $\delta \to 1/2^+$ and $-a > 0$. In consequence

$$G_\delta(q) > aq + P(-\delta) > 0$$

when $q \in (0,1)$. Therefore, $G_\delta(q) > 0$ for all $q \in \mathbb{R}$. □

Proposition 6.4.10. *Let δ_0 be defined by*

$$\delta_0 := \sup\{\delta \in (1/2,1] : \textit{ for all } q \in \mathbb{R}, P(q\psi - \delta \log |T'_{k_2}|) > 0\}$$

then $q \mapsto G_{\delta_0}(q)$ has a unique zero.

Proof. First of all, observe that $G_{\delta_0}(q) \geq 0$ for all $q \in \mathbb{R}$. In fact, if $G_{\delta_0}(q^*) < 0$ for some $q^* \in \mathbb{R}$, then by continuity of $\delta \mapsto G_\delta(q^*)$, we have that $G_{\delta_0 - \epsilon}(q^*) < 0$ for some $\epsilon > 0$. Moreover, by definition of δ_0, there exists $\overline{\delta} > \delta_0 - \epsilon$ such that $G_{\overline{\delta}}(q) > 0$ for all $q \in \mathbb{R}$ which is a contradiction with the fact that $G_\delta(\cdot)$ is strictly decreasing in δ.

Now, we will prove that $G_{\delta_0}(q)$ has a unique zero. Let $n \in \mathbb{N}$ be large enough. By definition of δ_0 we have that $G_{\delta_0 + \frac{1}{n}}(q) = P(q\psi - (\delta_0 + \frac{1}{n}) \log |T'_{k_2}|)$ has at least one zero, for all n.

Assume that $G_{\delta_0 + \frac{1}{N}}(q)$ has a unique zero q_0, for some N. Then, by (2) of Proposition 6.4.9 we have that $G_{\delta_0 + \frac{1}{n}}(q) = 0$ if and only if $q = q_0$, for all $n \geq N$. By continuity in δ_0, taking $n \to \infty$, we have that $G_{\delta_0}(q_0) = 0$ which is unique again by (2) of Proposition 6.4.9 .

On the other hand, suppose that $G_{\delta_0 + \frac{1}{n}}(q)$ has two zeros for all n. Let $q_1^n < q_2^n$ be the zeros. Assume that the diameter of $[q_n^1, q_n^2]$ tends to zero and let $q_0 = \bigcap_n [q_n^1, q_n^2]$. Then $G_{\delta_0 + \frac{1}{n}}(q_0) \leq 0$, for all n. By continuity in δ_0, $G_{\delta_0}(q_0) \leq 0$. and moreover $G_{\delta_0}(q_0) = 0$ because G_{δ_0} is non-negative. Observe that if the diameter $[q_n^1, q_n^2]$ does not tend to zero, then we would have two points $q_1 \neq q_2$, both in $\bigcap_n [q_n^1, q_n^2]$, and satisfying $G_{\delta_0}(q_1) = G_{\delta_0}(q_2) = 0$ which is a contradiction with the strictly convexity of $G_\delta(q)$ in q, for any δ. $\square$

Corollary 6.4.11. *Let $\delta > 0$ be such that $G_\delta(q) > 0$ for all $q \in \mathbb{R}$. Then, there exists $N \in \mathbb{N}$ such that the function*

$$q \mapsto P_N(q\psi - \delta \log |T'_{k_2}|)$$

is positive for all $q \in \mathbb{R}$.

Proof. We will argue by contradiction. Suppose that for all $N \in \mathbb{N}$, the function $q \mapsto P_N(q\psi - \delta_0 \log |T'_{k_2}|)$ have at least one zero. By Theorem 6.3.2, we have that for all $q \in \mathbb{R}$,

$$\lim_{N \to \infty} P_N(q\psi - \delta_0 \log |T'_{k_2}|) = G_{\delta_0}(q).$$

In particular, and also by convexity of $P_N(q\psi - \delta_0 \log |T'_{k_2}|)$ in q, we have that $P_N(q\psi - \delta_0 \log |T'_{k_2}|)$ tends to infinity when $q \to \infty$. Consequently, for N large enough, $P_N(q\psi - \delta_0 \log |T'_{k_2}|)$ has at most two zeros. We observe that, if for some $N \in \mathbb{N}$, $P_N(q\psi - \delta_0 \log |T'_{k_2}|)$ has a unique zero called q_N, then $P_n(q\psi - \delta_0 \log |T'_{k_2}|)$ has a unique zero at q_N, for all $n \geq N$, in view of

$$P_n(q\psi - \delta_0 \log |T'_{k_2}|) \leq P_{n+1}(q\psi - \delta_0 \log |T'_{k_2}|) \tag{6.4.4}$$

for all $n \in \mathbb{N}$. Therefore, we can assume that $P_N(q\psi - \delta_0 \log |T'_{k_2}|)$ has two zeros: $q_1^N \leq q_2^N$. Let $J_N := [q_1^N, q_2^N]$. Because the inequality (6.4.4) holds for all $n \geq 1$, then $J_{N+1} \subset J_N$ for all N. This implies that the intersection $\bigcap_N J_N$ is non-empty. Let $q^* \in \bigcap_N J_N$. We deduce that $P_N(q\psi - \delta_0 \log |T'_{k_2}|) \leq 0$ for all N and, by the approximation property, $G_{\delta_0}(q^*) \leq 0$ which is a contradiction with Proposition 6.4.9. □

6.4.3 Lower bounds

Theorem 6.4.12. *We have that*

$$\dim_H(\mathcal{D}_\sim) \geq \delta_0 > 1/2,$$

where $\delta_0 = \sup\{\delta \in (\frac{1}{2}, 1] : P(q\psi - \delta \log |T'_{k_2}|) > 0, \textit{for all } q \in \mathbb{R}\}$.

Proof. Let $\delta \in (\frac{1}{2}, 1]$ such that $P(q\psi - \delta \log |T'_{k_2}|) > 0$ for all $q \in \mathbb{R}$. By Proposition 6.4.11, we know that there exists N such that

$$H(q) := P_N(q\psi - \delta \log |T'_{k_2}|) > 0$$

for all $q \in \mathbb{R}$. Also, by convexity and the fact that $H(q) \to \infty$ when $q \to \pm\infty$, there exists

$q_0 \in \mathbb{R}$ such that $H'(q_0) = 0$. In other words,

$$\int \psi d\mu_{q_0} = 0$$

which is equivalent to

$$\int \log |T'_{k_2}| d\mu_{q_0} = \int \log |T'_{k_1} \circ \pi_{k_1,k_2}| d\mu_{q_0}.$$

Moreover, since $H(q_0) > 0$ and using the variational principle, we obtain

$$h(\mu_{q_0}) + \int q\psi - \delta \log |T'_{k_2}| d\mu_{q_0} > 0$$

and therefore the bound

$$\frac{h(\mu_{q_0})}{\lambda(\mu_{q_0})} > \delta.$$

This implies, by Theorem 6.3.4

$$\dim_H \mu_{q_0} > \delta$$

and by [JMS, Lemma 4.7] and Proposition 6.4.5 we deduce that

$$\dim_H \mathcal{D}_\sim \geq \dim_H \{x : \liminf_{n\to\infty} e^{S_\psi(x)} = 0; \quad \limsup_{n\to\infty} e^{S_n\psi(x)} = \infty\} \geq \dim_H \mu_{q_0} > \delta.$$

In conclusion

$$\dim_H(\mathcal{D}_\sim) \geq \delta_0 > 1/2.$$

$\square$

Proposition 6.4.13. *We have that*

$$\dim_H(\{x : \limsup e^{S_n\psi(x)} = \infty\}) \geq \delta_0$$

and moreover

$$\dim_H \mathcal{D}_\infty \geq \delta_0.$$

Proof. Again, let q_0 the zero of the function $G_{\delta_0}(q)$ and let $q > q_0$. By the variational principle, for every $q > q_0$, there exist a unique equilibrium measure μ_q such that

$$G_{\delta_0}(q) = h(\mu_q) + \int q\psi - \delta_0 \log |T'_{k_2}| d\mu_q > 0.$$

Observe that

$$\dim \mu_q > -q \frac{\int \psi d\mu_q}{\lambda(\mu_q)} + \delta_0.$$

On the other hand, we note that, for μ_q-a.e. point $x \in [0,1]$ we have that

$$\frac{1}{n} S_n \psi(x) \to \int \psi d\mu_q > 0$$

and therefore, $e^{S_n\psi(x)} \to \infty$ when n tends to infinity. We deduce that, for all $q \geq q_0$

$$\dim_H(\{x : \limsup e^{S_n\psi(x)} = \infty\}) > -q \frac{\int \psi d\mu_q}{\lambda(\mu_q)} + \delta_0,$$

and in consequence

$$\dim_H(\{x : \limsup e^{S_n\psi(x)} = \infty\}) \geq \delta_0$$

since, when $q \to q_0$, $\int \psi d\mu_q \to 0$ and $\lambda(\mu_q)$ is uniformly bounded below by a positive constant. The second affirmation follows from the Corollary 6.4.7, part (a). $\square$

6.4.4 Upper bounds

Let $\eta > 0$ and define the set

$$A_\eta = \left\{ x : \limsup_{n\to\infty} e^{S_n\psi(x)} > \eta \right\}.$$

Proposition 6.4.14. *For all $\eta > 0$, we have that $\dim_H A_\eta \leq \delta_0$. Moreover $\dim \mathcal{D}_\infty \leq \delta_0$.*

Proof. In first place, we will prove that, for all $N \geq 1$, we have

$$A_\eta \subset \bigcup_{n\geq N} \left\{ I^{k_2}(x_1, x_2, \cdots, x_n) : e^{S_n\psi([\overline{x_1,x_2,\cdots,x_n}]_{k_2})} > \frac{\eta}{C_1C_2} \right\},$$

where C_1, C_2 are the constants involved in the bounded distortion property for T_{k_1} and T_{k_2} respectively. In fact, let $x \in A_\eta$. There exists $M \in \mathbb{N}$ such that $e^{S_M\psi(x)} > \eta$. By bounded distortion, we have that there exists $C_1, C_2 > 0$ such that

$$\frac{|(T_{k_1}^M)'(\pi_{k_1,k_2}([\overline{x_1, x_2, \cdots, x_M}]_{k_2}))|}{|(T_{k_1}^M)'(\pi_{k_1,k_2}(x))|} \asymp C_1$$

and

$$\frac{|(T_{k_2}^M)'(x)|}{|(T_{k_2}^M)'([\overline{x_1, x_2, \cdots, x_M}]_{k_2})|} \asymp C_2.$$

Then

$$\frac{e^{S_M(\log|T'_{k_2}|)(x)}}{e^{S_M(\log|T'_{k_2}|)([\overline{x_1,x_2,\cdots,x_M}]_{k_2})}} \asymp C_1$$

and

$$\frac{e^{S_M(-\log|T'_{k_1}\circ\pi_{k_1,k_2}|)([\overline{x_1,x_2,\cdots,x_M}]_{k_2})}}{e^{S_M(-\log|T'_{k_2}\circ\pi_{k_1,k_2}|)(x)}} \asymp C_1.$$

Multiplying both estimations, we get

$$\frac{e^{S_M\psi(x)}}{e^{S_M\psi([\overline{x_1,x_2,\ldots,x_M}])}} \asymp C_1C_2$$

and in particular

$$e^{S_M\psi([\overline{x_1,x_2,\ldots,x_M}])} > \frac{e^{S_M\psi(x)}}{C_1C_2} > \frac{\eta}{C_1C_2}.$$

Now, we will pass to show that $\dim_H A_\eta \leq \delta_0$. Let $\epsilon > 0$ and $r < 1/e < 1$ such that $|I^{k_2}(x_1, x_2, \ldots, x_n)| < r^n$ for all n. We will prove that the $(\delta_0+\epsilon)$-Hausdorff measure is equal to zero. In fact

$$\begin{aligned} H^{\delta_0+\epsilon}_{r^N}(A_\eta) &\leq \sum_{n\geq N}\sum_{I^{k_2}(x_1,\ldots,x_n)} |I^{k_2}(x_1,\ldots,x_n)|^{\delta_0+\epsilon} \\ &\leq \sum_{n\geq N} r^{n\epsilon} \sum_{I^{k_2}(x_1,\ldots,x_n)} |I^{k_2}(x_1,\ldots,x_n)|^{\delta_0} \\ &\leq C_2 \sum_{n\geq N} r^{n\epsilon} \sum_{I^{k_2}(x_1,\ldots,x_n)} e^{-\delta_0 S_n(\log|T'_{k_2}|)([\overline{x_1,\ldots,x_n}]_{k_2})} \\ &\leq C_2\left(\frac{C_1C_2}{\eta}\right)^{q_0} \sum_{n\geq N} r^{n\epsilon} \sum_{I^{k_2}(x_1,\ldots,x_n)} e^{q_0 S_n\psi([x_1,\ldots x_n]_{k_2})-\delta_0 S_n(\log|T'_{k_2}|)([\overline{x_1,\ldots,x_n}]_{k_2})} \qquad (6.4.5) \\ &\ll \sum_{n\geq N} r^{n\epsilon} \sum_{T^n_{k_2}(x)=x} e^{q_0 S_n\psi(x)-\delta_0 S_n(\log|T'_{k_2}|)(x)} \end{aligned}$$

where, in (6.4.5), $q_0 \in (0,1)$ denotes the zero of the function G_{δ_0}. By definition of the pressure function,

$$\sum_{T^n_{k_2}(x)=x} e^{q_0 S_n\psi(x)-\delta_0 S_n(\log|T'_{k_2}|)(x)} \leq e^{n\epsilon},$$

and thus we obtain that

$$H^{\delta_0+\epsilon}_{r^N}(A_\eta) \ll \sum_{n\geq N} (re)^{n\epsilon}.$$

Therefore $H^{\delta_0+\epsilon}_{r^N}(A_\eta) \to 0$ when $N \to \infty$. We deduce that $\dim_H(A_\eta) \leq \delta_0 + \epsilon$, for all $\epsilon > 0$. $\square$

Proposition 6.4.15. $\dim \mathcal{D}_\sim \leq \delta_0$.

Proof. Observe that, if x is such that $\limsup e^{S_n\psi(x)} = 0$ then $\lim e^{S_n\psi(x)} = 0$ and then, by

Corollary 6.4.7 part (b), we obtain that $x \in \mathcal{D}_0$. Therefore $\mathcal{D}_\sim \subset \cup_\eta A_\eta$ which implies that $\dim \mathcal{D}_\sim \leq \delta_0$. $\square$

Collecting all results of the last two subsections, we have proved the main theorem:

Theorem 6.4.16. *Let k_1, k_2 be two positive numbers. Then the sets $\mathcal{D}_\infty, \mathcal{D}_\sim$ have the following Hausdorff dimensions:*

$$1/2 < \dim_H(\mathcal{D}_\infty) = \dim_H(\mathcal{D}_\sim) = \delta_0 < 1.$$

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