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ICT integration in mathematics initial teacher training and its impact on visualization: the case of GeoGebra

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ABSTRACT
This case study investigates the impact of the integration of information and communications technology (ICT) in mathematics visualization skills and initial teacher education programmes. It reports on the influence GeoGebra dynamic software use has on promoting mathematical learning at secondary school and on its impact on teachers' conceptions about teaching and learning mathematics. This paper describes how GeoGebra-based dynamic applets – designed and used in an exploratory manner – promote mathematical processes such as conjectures. It also refers to the changes prospective teachers experience regarding the relevance visual dynamic representations acquire in teaching mathematics. This study observes a shift in school routines when incorporating technology into the mathematics classroom. Visualization appears as a basic competence associated to key mathematical processes. Implications of an early integration of ICT in mathematics initial teacher training and its impact on developing technological pedagogical content knowledge (TPCK) are drawn.

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1. Introduction

Several teacher education studies inquire about the competencies and knowledge prospective teachers need in order to integrate technologies into mathematics education. For example, based on the notion of pedagogical content knowledge (PCK) developed by Shulman [1], *Technological pedagogical content knowledge* (TPCK) was proposed as the interconnection and intersection of technological knowledge (TK), content knowledge (CK) and pedagogical knowledge (PK) [2,3].

However, the progress in technology development has not been accompanied yet by an appropriate and rigorous training of the teachers who are going to use those tools with their students [4]. Consequently, many teachers are not fully prepared to integrate new digital technologies into the mathematics curriculum.

Nevertheless, one of the most popular and most widely used dynamic mathematical software packages is GeoGebra, a free and multi-platform software for all levels of education that brings together geometry, algebra, calculus and statistics in one user-friendly interactive application [5]. The programs Cabri II Plus and Geometer's Sketchpad have been the most widely used programs in Europe and the USA [6], but in the last 5 years or

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so GeoGebra has begun to feature more prominently in mathematics teacher conference presentations [7], due to its free access, availability to teachers and students and effectiveness.

Moreover, it appears to be an easy-to-use software that can be operated intuitively and does not require advanced skills to get started [8]. It has been found that, compared to the traditional methodology, the use of GeoGebra as a didactic approach when teaching Analytic Geometry improves performance in mathematization, increases motivation, self-awareness and involvement of students in the learning process [9], and that it allows students to visualize problems and avoid algebraic obstacles [10]. In general, the use of dynamic representations promotes geometric thinking and provides visual, algebraic and conceptual support for the majority of students.

One may understand visualization as the process of representing mathematical concepts, principles or problems, manually or using the computer [11]. In particular, Zazkis et al. [12] propose that visualization lies in the connection that an individual makes between an internal construction and an image (be it mental, drawn or digital). Visualization is a process that facilitates problem solving; according to Zimmerman [13], students who use representations and graphs as part of their problem-solving strategies tend to be more successful in obtaining results.

In this context, we will address two interesting dimensions regarding the use of GeoGebra: the impact and relevance visualization acquires when using GeoGebra in the mathematics classroom and its contribution in developing TPCK during initial teacher training programmes. These dimensions have shaped the purpose of the study we present in this article: to describe the benefits of introducing GeoGebra in developing visualization skills for learning mathematics and to characterize the TPCK of prospective teachers during their formative phase.

2. Theoretical framework

In order to address these two dimensions, we will examine the main features of GeoGebra as a cognitive tool for facilitating mathematical visualizations and learning processes as well as their integration into initial teacher training programmes.

2.1. Dynamic visualizations

Within the spectrum of software that is available to support the teaching and learning of mathematics, two prominent forms are computer algebra systems (CAS) and dynamic geometry software (DGS). GeoGebra combines DGS and CAS [14]. GeoGebra presents a single piece of software that can be used to successfully explore algebra and geometry with the added benefit of being entirely free of cost [5]. This very effective combination is possible because GeoGebra brings both forms together synchronically: DGS and CAS. The first allows teachers and students to work with points, vectors, segments, lines, and conic sections, among others, and the latter is used to directly enter equations and functions algebraically, with the possibility of modifying them dynamically.

The impact of visual mediators in the teaching and learning of mathematics is so powerful that these tools have changed the nature of school routines; in particular, the use of GeoGebra enables a shift in focus from spending time producing graphs to spending time interpreting and understanding graphs, and systematically exploring these representations, giving a more central role to the formulation of mathematical conjectures [15]. The reorientation of classroom activities from procedural assignments to exploratory and analytical tasks allows for a better understanding of a particular mathematical object of learning. The use of GeoGebra appears to be fundamental because it 'both builds on the human sense of perception and action and links this to the "making sense" of the symbolic operations' [16, p.110].

Technology-enriched learning environments empower students by enhancing their ability to explore, re-construct (or re-invent) and explain mathematical concepts by promoting connections between graphic representations and formal definitions. Thus, the applet can be regarded as a cognitive tool because it evokes known mathematical concepts, while at the same time supporting the development of mathematical ideas via student–applet interactive utilization schemes [17]. Learning from dynamic visual representations has confirmed that an effective visualization has a great impact on learning because it allows students to interact with the content and manipulate the application, which promotes connections of symbolic and iconic representations with formal definitions [18].

The theory of variation – an expansion of phenomenography, which explores several ways of experiencing and understanding phenomena – offers a mathematics pedagogical model based on variation, which reflects a process of mathematical knowledge acquisition. A significant feature of variation theory is its strong focus on the object of learning, and thus the way in which it is represented becomes the most powerful factor for learning. Discernment of critical features occurs under systematic interaction between a learner and the thing to be learnt, and variation is the agent that generates such interaction [19]. This approach introduces four patterns of variation (and invariance) in the learning environment as a pedagogical tool; *contrast* is to discern whether something satisfies a certain condition or not, seeking to distinguish an object from other previously known things; *separation* is the awareness of critical features of this object, acquired by purposely varying some aspects of the phenomenon; *fusion* simultaneously integrates key dimensions of variation; and *generalization* allows the establishment of decontextualized invariant patterns. Generating these patterns of variation and allowing students to experience them enables them to visualize and discern the main features of the object of learning, enhancing their understanding of it by focusing on conceptual rather than procedural tasks [20]. GeoGebra's dynamic nature is fundamental to this approach, given its 'ability to visually make explicit the implicit dynamism of *thinking about* mathematical, in particular geometrical, concepts' [21, p.197].

2.2. ICT in initial teacher education programmes

TPCK is a type of knowledge that goes beyond the convergence of its disciplinary, pedagogical and technological components. It relates to the changes in teaching and learning that occur because of technology use, and teachers' need to choose a certain resource for a specific task, considering its possibilities and limitations [2].

Several teacher education programmes have shown positive outcomes when integrating ICT, not only in relation to curricular content but also associated with affective aspects of learning. On the one hand, it facilitates transitions from exclusive to inclusive mathematical definitions; on the other, its integration generates positive attitudes during this formative stage and with future students, as a mechanism to facilitate learning and as a way of appealing to students' interests [22]. Gómez-Chacón and Joglar [4] point out that prospective teachers who use technology to solve mathematical problems do not spontaneously develop a method to integrate these tools in their future practice. Thus, they propose to address technology incorporation into the mathematics classroom from an integral perspective, considering all of its components: cognitive, didactic, technical and affective. Confidence, motivation and expertise in its instrumental use are central aspects for a successful future integration of ICT in the school context, which reinforces the need to incorporate the use of GeoGebra in this initial formative phase.

The affective component is focused on the establishment of a professional mathematics teacher identity, which involves a biographic process of developing a personal relationship between technology and its influence on shaping the teacher's role [4].

Prospective teachers note that their vision of how learning takes place in schools changes when integrating technology, emphasizing a type of learning promoted by discovery in contrast to a transmission-oriented teaching, shifting from an inherited paradigm to new identities born within this technological scenario. The role of the teacher changes when using technology in the classroom, where she/he has to become a technology mediator and a new teaching contract with the students' needs to be established [4].

In addition to this affective dimension, GeoGebra is an effective tool for fostering processes associated with visualizations, where mathematical assignments based on applets support processes associated with visualization as conjectures and explorations [15]. From a mathematical experimental perspective, according to Borwein [23], ICT is used with several purposes, such as gaining insight and intuition, discovering new patterns and relationships, testing and rejecting conjectures, verifying analytically derived results, generating graphic representations of mathematical principles, giving counterexamples, exploring a possible result to see if it merits a formal proof, calculating and finally proving, all of which is used in initial teacher training. Sinclair et al. identified the first four listed purposes for ICT use in their research [24]. In addition, Carranza [25] states that integrating geometrical dynamic environments, in particular GeoGebra, in mathematics initial teacher training programmes promotes the acquisition of significant, operational and structured mathematical knowledge, which allows prospective teachers to easily navigate between symbolic, numeric, graphic and analytical representation systems.

3. Methodology

This study takes place within the Pontificia Universidad Católica de Chile's Teacher Education Programme, which prepares students with a professional degree to become teachers in a one-year period. The goal of project PUC1201[1] was to achieve a deep curricular renovation of this programme through a systematic integration of ICTs, focusing on the main school curriculum subjects: Spanish language, Mathematics, Sciences and History. Specifically, for mathematics prospective teachers, the programme contemplates two subject mathematics education courses during this year. Beginning in 2014, an ICT line was included in these courses using GeoGebra as the main teaching/learning resource. The pedagogical strategies and practices learned by using GeoGebra are to be implemented by all pre-service teachers in their school teaching practices. During the first semester course – Teaching and Learning Mathematics I – all four prospective teachers had the opportunity to address and revise several mathematical contents covering some units of the secondary curriculum (ages 14–18) with technology, using digital worksheets designed to be solved in a GeoGebra environment. At the end of each worksheet, pre-service teachers were asked to assess the instrument. This experience allowed them to manage all features and tools this software package provides, and to get to know and test many concrete curricular applications. Afterwards, prospective teachers designed an applet to implement in their teaching practice context, and completed the project with a reflection on its elaboration and implementation. Additionally, during the second semester course – Teaching and Learning Mathematics II – each prospective teacher designed and implemented pedagogical activities structured with worksheets (similar to the ones they solved during the first semester), which students had to complete using GeoGebra, focusing on student exploration by manipulating technology. This type of activity required computers for students to work with, and therefore it took place in the computer laboratory, where school students worked in pairs.

We chose a case study as a design methodology because this approach allowed us to address our research purposes from a comprehensive perspective. Out of the four pre-service teachers, we selected prospective teacher Simón as the subject of our study because he meets the conditions to inquire about our research focal points: he excelled as a BA student in mathematics and was already very knowledgeable about other mathematics software, which allowed him to learn the use of GeoGebra very quickly. But most importantly, Simón experienced a major shift during his training process, not only regarding his vision about teaching and learning mathematics but also in his performance. These features became clear in his work and assessment solving the worksheets, where he exhibited solid mathematical knowledge and a very high-quality reflection level in his comments. These characteristics were consolidated in Simón's applet construction and design, which are described in Section 4.

We selected one out of four cases, because Simón's trajectory is the most representative formative experience that reflects the reality of the other three teacher training processes.

The data collection process was organized around our two research dimensions: the development of visualization skills for learning mathematics and the development of TPCK during initial teacher training.

In order to analyze the first dimension, we collected two types of data: we started by producing a documental content analysis of the applet designed by Simón for the course Teaching and Learning Mathematics I during the first semester of 2014. Afterwards, we carried out a non-participant observation, recording Simón's practice experience while implementing an applet designed for the course Teaching and Learning Mathematics II, during the second semester of 2014.

As mentioned, Simón designed and implemented pedagogical activities structured with worksheets as well. This third activity focused on visualizing systems of linear equations. Students were instructed to type these equations in their algebraic form, find and interpret their solutions guided by the digital worksheet's questions and sequence. A second phase included transforming word problems into linear equations systems by translating them to algebraic language, and solving them using the software.

In this article, we present evidence of the first two interventions only because these data reflect better how visualization is enhanced by the use of technology in the mathematics classroom. Therefore, this evidence focuses on teacher–students interactions where pre-service teacher Simón is the one controlling GeoGebra.

For the TPCK development dimension, we applied two surveys, both at the end of the courses Teaching and Learning Mathematics I and II, respectively. These questionnaires inquired about his experience using GeoGebra and his ideas about teaching mathematics using this tool.

4. Data analysis

4.1. Visualization dimension: additional elements in teaching mathematics using ICT

First of all, we analyzed the applet 'Euclid's theorem' developed by Simón for the course Teaching and Learning Mathematics I during the first semester of 2014; later, we examined the interactions that occur during a practice class taught by Simón using the applet 'Systems of linear equations: relative position' developed for the course Teaching and Learning Mathematics II during the second semester of 2014.

4.1.1. 'Euclid's theorem' applet

The following analysis mainly describes the dynamic features of the 'Euclid's theorem' applet and the way in which they promote several mathematical thinking processes when introduced into the teaching unit.

The referred theorem is as follows:

> In the words of Euclid: If in a right-angled triangle a perpendicular be drawn from the right angle to the base, the triangles adjoining the perpendicular are similar both to the whole and to one another.
>
> (The Elements: Book VI: Proposition 8)

The construction and design chosen by Simón seek to ground the theorem's understanding sequentially (Stages 1–7) until the content becomes explicit (Figure 1). The GeoGebra-based applet under analysis represents a right-angle triangle ABC, whose height relative to the hypotenuse is drawn, conforming two minor triangles (Stage 1). By using the *Show/Hide Object Tool*, all interior angles are highlighted using different colours for congruent acute angles, and a square symbol for the right-angle. Likewise, the congruent angles of the two minor triangles are highlighted with the corresponding colour. By means of *sliders*, both minor triangles are rotated around vertices A and B, respectively, resulting in their horizontal layout next to the original triangle ABC (Stage 2). Since the interior angles of the three triangles appear highlighted, students conjectured about their similarity based on the AA (angle–angle) criterion, learnt by the students in the previous unit. Vertex C, which by construction corresponds to a *point on a conic* (semi-circumference), *moves* along the path revealing that all triangles conformed by drawing the height relative to the hypotenuse are similar, for *any* right-angle triangle (*generalization*) (Stage 3). Again, by using the *Show/Hide Object Tool* this time the corresponding sides of the minor triangles are highlighted. Students know the proportionality property that corresponding sides of similar figures verify, which allows them to *deduce* the right-angle triangle altitude theorem, based on their ratios (Stage 4). Equally, Stages 5 and 6 of Figure 1 illustrate the use of the *Show/Hide*

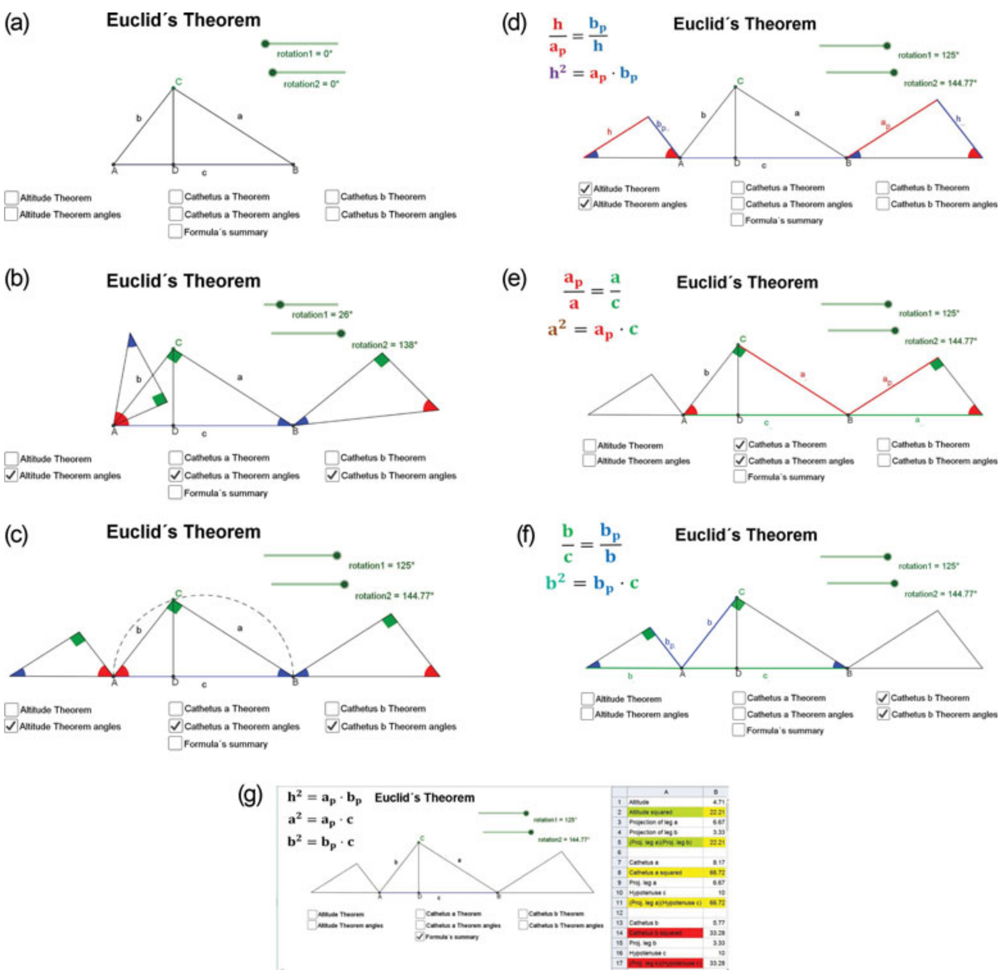


Figure 1. 'Euclid's theorem' applet sequence. (a) Stage 1; (b) Stage 2; (c) Stage 3; (d) Stage 4; (e) Stage 5; (f) Stage 6; (g) Stage 7.

Object Tool to highlight the corresponding sides of the original and the minor triangles, which facilitates cathetus a and b theorem's deduction. It should be pointed out that these interactive elements were constructed individually, which allowed their sides to be highlighted first, then showing their ratios and the corresponding theorem, giving students the time and opportunity to suggest proportional magnitudes and to calculate and obtain the formulas before they appear explicitly in the applet as confirmation. Finally, the *Spreadsheet View* is activated, showing all lengths of sides, projections and height, as well as all products involved in the theorems. It was possible to confirm numerically that all magnitudes verify all theorem equalities for continually changing values, while vertex C moved along the semi-circumference (Stage 7). Readers can access the following link to visualize this applet dynamically: https://youtu.be/NqmczKRU62Q.

As shown in the previous analysis, the 'Euclid's theorem' GeoGebra-based applet, which includes several dynamic and interactive elements (sliders, point on object, show/hide object tool and the synchronic spreadsheet view), stimulates certain mathematical

Table 1. Dynamic elements and their relationship to visualization-associated processes.

Dynamic elements
Slider (interactive)
Point on object (moves on its perimeter)
Show/hide object tool (action object)
Spreadsheet (shows changing values synchronically)

processes such as conjecture, deduction and verification. The development of these processes appears to be facilitated by visualization, a basic and immediate skill that promotes better understanding, access and mathematical achievements for more students. Learning does not appear restricted only to students capable of algebraically communicating ideas and concepts, but it extends to those less skilled with mathematical abstract language, enabling them to acquire new knowledge by means of this visual platform.

Table 1 resumes all dynamic features of Figure 1 'Euclid's theorem' GeoGebra-based applet and its associated processes, which in combination can be understood as the visualization competence.

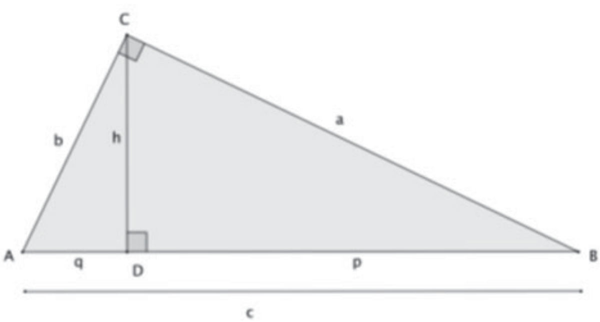
In general, textbooks offer a summary box of Euclid's theorem similar to the one seen in Figure 2, where all three theorems appear both in common and algebraic language, stating the relationships between all segments involved.

Traditionally, Chilean teachers have based their lesson plans, sequences and examples on the available textbook provided by the Ministry of Education, and therefore it is safe to assume that this is the most common approach with which Chilean students are confronted when learning these theorems. Subsequently, the focus is put on their direct and mechanical application in similar situations. Naturally, the main difficulties students face

Every right triangle satisfies the following:

- The square of the height upon the hypotenuse equals the product of the projections of both catheti on the hypotenuse.
- The square of a cathetus equals the product of the hypotenuse and the projection of the cathetus on the hypotenuse.

Therefore,



$$h^2 = p \cdot q$$
$$a^2 = p \cdot c$$
$$b^2 = q \cdot c$$

The first statement of the Euclidean theorem is known as the altitude theorem, and the second, as the cathetus theorem.

Figure 2. Mathematics textbook summary box of Euclidś theorem [26, p.111].

when using this method are visualizing all triangles' similarity and successfully establishing the proportionality between corresponding sides. Unlike the textbooks approach, Simón's GeoGebra-based applet allows this content to be presented in a constructive visual way, focusing the analysis on establishing relationships based on previously learnt concepts that were easily visually recognized by students. Distinguishing similar triangles becomes a simple task when a dynamic resource like GeoGebra is available and used in this manner.

It is precisely the possibility of highlighting all interior angles simultaneously, so that both minor triangles emerge from the original one through rotation that allows their individualization without losing sight of the similarity criterion AA that connects them.

Additionally, the resulting spatial arrangement of all three triangles allows for an easy identification of the corresponding sides – coloured accordingly to the angle they face – and to pose without difficulty the proportionality they verify. With this teaching and learning approach, students are required to connect earlier geometry contents to the current ones, in order to build new knowledge. Hence, learners understand the mathematical grounds that underlie all theorems and are able to apply them to a wider set of problems and situations.

4.1.2. 'Systems of linear equations: relative position' applet

The GeoGebra-based applet 'Systems of linear equations: relative position', designed during the course Teaching and Learning Mathematics II, was implemented in a 10th grade, in an Algebra class about linear equations systems, in the context of the first pedagogical unit during the second semester.

Prospective teacher Simón conducts the class by projecting the images obtained from GeoGebra on data show, starting it by reviewing (Figure 3) the role that the (blue) straight-line equation parameters play – slope and intercept – interactively using the applet. Continuing with the sequence, Simón activates the second straight-line equation (red) to make it visible in the graphic view next to the blue straight line. He asks students for the criteria these parameters should meet for both straight lines (blue and red), to become parallel. The students propose several conjectures and hypotheses. For each case, Simón adjusts the parameters so students are able to visualize the relations they proposed.

The following transcripts (Stages 1–7) correspond to the interactions between Simón and the students during the exploration of the first relative position examined: parallel lines.

Simón: So, how should both slope parameters behave in order to get parallel lines?
Gonzalo: Negative and positive.
Simón: Negative and positive? Let´s see.
That *m* and this *m* …the one belonging to the red line. Let´s keep it positive … and the other one negative …
They are still intersecting, ¿right?
Gonzalo: Yes.
Simón: Which criteria should both *m* meet in order …?
Raimundo: Both should be positive.
Simón: Both positive? Let´s see.
Carlos: Yes, because in that case both would pass through the same number.
Simón: There, we have both positive, right?
Raimundo: Ah, they still intersect.
Simón: one is 3,4 and the other one is 2,2. Both positive, right?
Raimundo: Then, both should be negative!
Carlos: That´s the same, you´re only changing the sign.
Simón: Both negative …now we have both negative, 1,9 in one slope and the other one is 1,4, negative.
Luciano: Teacher, we have to translate them, just horizontally and vertically.
Raimundo: Translation.

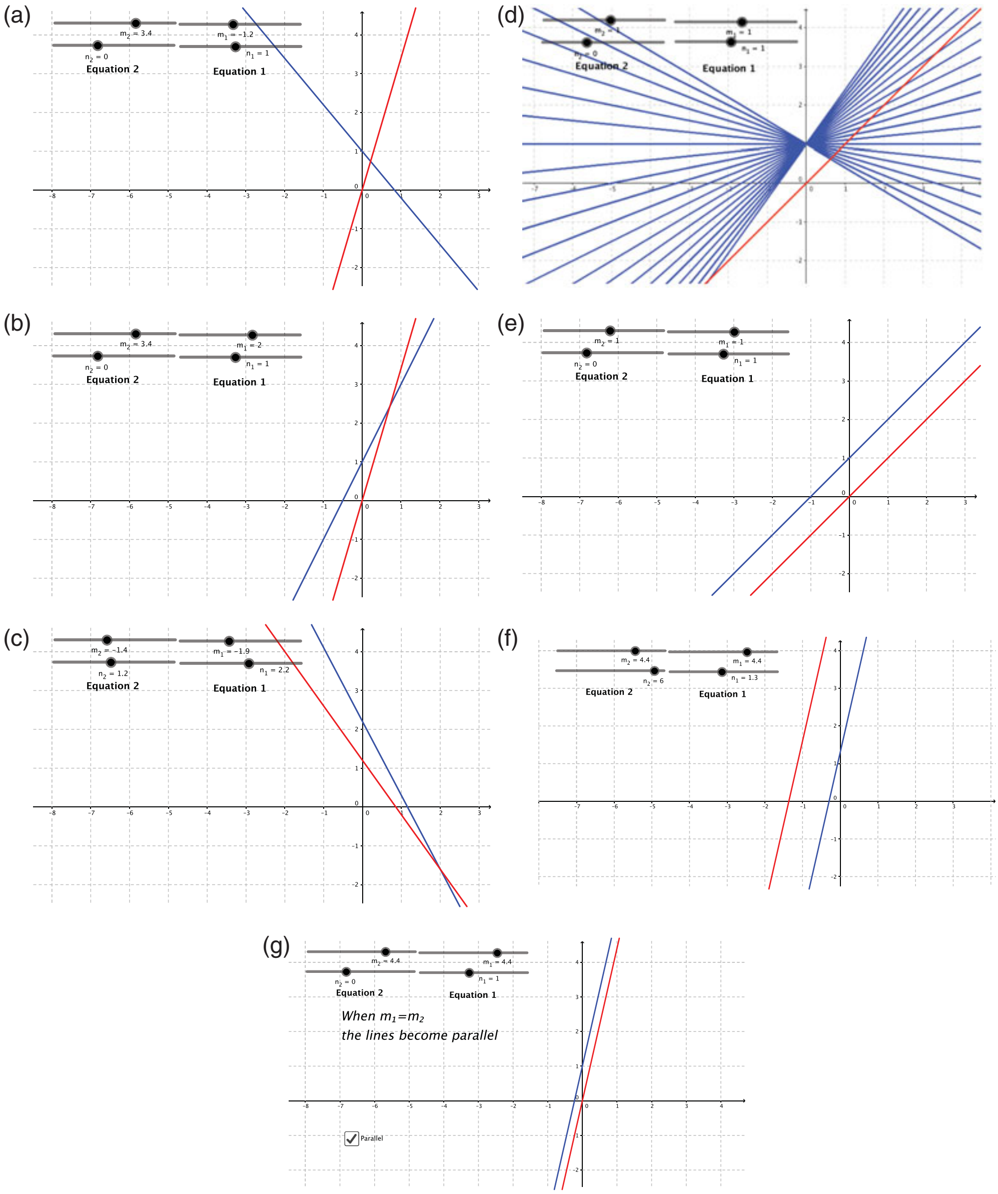


Figure 3. (Colour online) 'Systems of linear equations: relative position' applet sequence. (a) Stage 1; (b) Stage 2; (c) Stage 3; (d) Stage 4; (d) Stage 4; (e) Stage 5; (f) Stage 6; (g) Stage 7.

From the viewpoint of visualization-associated processes, we observe that when Simón asked the question for the first time, Gonzalo answered proposing one positive and one negative slope. In our visualization model, we will label this proposal a *hypothesis*, because this student's answer is not based on the data or the graph, but on a fixed idea that the searched relationship is based on the signs of the slopes. Simón manipulates the sliders to represent Gonzalo's proposal (Stage 1: $m_1 = -1{,}2$ negative and $m_2 = 3{,}4$ positive), and it is Simón who informs the classroom that the lines are still intersecting, and therefore not parallel. Then, he reformulates the question 'Which criteria should both m meet in order …?' but he is interrupted by Raimundo who suggests both slopes to be positive. In this case, we switch from a hypothesis to a *conjecture* because we assume the student incorporates the data and the information provided by the software, to elaborate a new proposal of a conjectural nature.

Again, Simón adjusts the sliders to represent this second case (Stage 2: $m_1 = 2{,}2$ and $m_2 = 3{,}4$ positive), but instead of being him who interprets the graph, it is Raimundo who explicitly notices that the lines still meet in a point, rejecting his own conjecture. Nevertheless, the student insists on a third case in which both slopes should take negative values, but before Simón could change the sliders and test this new suggestion, his classmate Carlos rejects the idea stating that it is the same case as before, only with negative numbers. Only to confirm visually that the parallel condition does not occur, Simón represents it using the applet (Stage 3: $m_1 = -1{,}9$ and $m_2 = -1{,}4$ negative). It is interesting to emphasize the growing participation level of students during the exploratory activity: at the beginning, Simón is the protagonist in managing the class and providing arguments for rejecting the initial hypothesis, but as the sliders are changed testing new ideas, the students play more of a leading role in examining and refuting the following conjectures.

Student participation could be attributed to how easily and quickly all cases were represented using the applet. But less accurate and more time-consuming static resources (board and exercise book) could still be useful to graph all situations. In this sense, even though GeoGebra contributes to a fast and exact display of different linear equation systems, that is not the main benefit that a dynamic mathematical software provides. Moreover, it is evident that this type of use does not guarantee learners will find the conditions that verify parallelism – equal slopes.

That is why Simón's class management shift is so interesting, when he decides to integrate the automatic slider animation. Prospective teacher Simón suggests the learners to set the red straight-line slope $m_2 = 1$ and animate m_1, thus producing a rotation effect of the blue straight-line around its intercept, to discover the exact moment in which both lines become parallel.

Simón: Let´s pay attention to the following: I´ll set one, ok? I´ll set one slope at 1, ok? The red line equation which is equation 2, here …parameter *m* is set at 1 and the other *m* will change by animation.
Simón: When are both lines parallel?
Carlos: Teacher, both have to be the same ….ah … no.
Carlos: ¡See! They have to be equal values, or not? Both have to become equal.
Simón: The same value …
Carlos: Of *m*.
Simón: Of *m*.
Simón: Do you agree with this statement?
What happens if both equations have the same *m*?
Let´s see … I´ll pause this. Let´s set *m* to the same value as the other equation´s *m* which is 1. What happens to the lines? How do they look?
Carlos: Same, parallel.
Simón: Parallel.

The possibility of visualizing all cases enabled Carlos and the rest of the students to identify the correct criteria among parallel lines, because GeoGebra allows the learner to recognize a certain relative position (parallelism) while at the same time checking the parameter values that produce this graphic situation.

By means of this dynamic scheme (Stage 4), students surpassed their exclusively sign combination-based conjectures, since it was possible to continually observe the changes that the blue line underwent while its slope value was modified, visually finding the moment both lines became parallel (Stage 5).

From the analysis of the interactions between Simón and his students, it is possible to identify what could be the most essential benefit a dynamic software provides: building mathematical knowledge in the classroom based on a platform that enables them to discover patterns and regularities and identify relations and properties of diverse objects and mathematical phenomena displayed as mobile graphic scenes.

The dynamic nature of all described representations, so revealing and significant for students, is the core feature that makes the implemented learning strategy effective, inaccessible for teachers and students who only count with the traditional classroom resources.

The following episode shows how Simón questions students about the findings on parallel lines being valid for slopes different than 1. Particularly, Simón suggests to the students that these lines became parallel only because it is a particular case where both gradient parameters took the value 1. But as noticed from the transcription, learners do not doubt nor modify their conjecture; on the contrary, they confirm it reinforcing its *verification* (Stage 6). Probably, the dynamic visualization of all possible cases allowed them to deduce that when slopes take a different value, they will appear graphically more or less steep, but that the relation will remain the same, thus confirming the property. Still, Simón modifies the values of the slopes to 4,4 for the classroom to confirm that the relative postion criteria found are correct.

Simón: Maybe it only works for 1.
Raimundo: No, it works for all numbers.
Simón: Let´s set any number, for example 4,4 …and the other one 4,4 too. So, when do two straight lines have no interception?
Carlos: When *m* is the same as the other one.
Simón: When the value of *m* is the same as the other *m* value, ¿right? When the value … this number here. Which value?
Luciano: *m*'s.
Simón: But, what is *m*?
Carlos: The slope.

Finally, as a synthesis and summary, Simón gives students the opportunity of *generalizing* the relation that was found. For this purpose, he uses the show/hide object tool, an action object that allows to formalize in advance written relations and conclusions expected to be reached in class, which are hidden until students actually state them. Only after this, Simón makes them visible for students to register these findings in a organized way in their excercise books (Stage 7).

Simón: So, how can we summarize what we have seen until now? When are two straight lines parallel?
Carlos: When their slopes have the same value.
Simón: When m_1 equals m_2, slopes are the same and the lines become …
Luciano: Parallel.

As seen, the use of the dynamic GeoGebra-based applet regarding linear equations systems, introduced into the teaching unit, facilitates the development of mathematical

Table 2. Dynamic elements and their relationship to visualization-associated processes.

Question:
What are the criteria that straight line slopes should meet in order to become parallel?

Dynamic element	Process	Evidence
Slider (interactive): adjusts parameter values to the proposed case	Hypothesis	Students' proposal: an incorrect case
Slider (interactive): adjusts parameter values to the proposed case	Conjecture	Students' proposal: two incorrect cases
Slider (interactive): Simón sets one slope ($m_2 = 1$) and animates the other one m_1		Students' proposal: both slopes should be equal
Slider (interactive): adjusts parameter values to the same numbers	Verification	Students confirm their conjecture
Show/hide object tool (action object): summary of slope relationship for parallel lines	Generalization	Students' generalization, applet conclusion

processes trough visualization. The possibility of modifying both straight line parameters – slope and intercept – interactively, enhances students' formulation of hypotheses and conjectures, their verification and generalization. Being able to easily test all the proposed ideas, and rapidly reject the wrong ones, but most importantly – dynamically visualizing all different positions that diverse slopes generate – allowed students to discover significant regularities and relationships, benefiting from a meaningful and active learning.

Table 2 summarizes all of the applet's dynamic elements, their relationship to visualization-associated processes and the evidence that supports them.

4.2. TPCK development dimension: prospective teacher's reflections and his conception about the integration of ICT into mathematics teaching and learning

Both questionnaires that Simón completed at the end of each of the semester's Teaching and Learning Mathematics courses I and II aimed to assess his experience as a GeoGebra user; solving digital worksheets, designing and creating dynamic applets and implementing these visual tools during his school teaching practice.

The first survey focuses on collecting Simón's reflections in two different situations: as a user completing digital worksheets designed to be solved in a GeoGebra environment and as a pre-service teacher using a GeoGebra-based applet for teaching at school.

Simón remarks the usefulness of GeoGebra for learning certain mathematical concepts; in particular, he recognizes the development of the visualization competence when assessing one of the digital worksheets:

> **Simón:** GeoGebra allows to instantly visualize inequalities that before were only an algebraic expression. I believe that this tool enables to move rapidly from an algebraic type of analysis to a more geometrical approach to the problem.

Simón associates visualization with a shift in viewpoint, 'from an algebraic type of analysis to a more geometrical approach to the problem', identifying one of the main features of this competence.

His reflections about teaching at school using the GeoGebra-based applet 'Euclid's theorem' account for the relevance of articulating the use of GeoGebra with other resources during the learning sequence:

Simón: Since the content was Euclid´s theorems, I focused primarily on how to build this theorem based on triangle similarity. Therefore, I handed them concrete cardboard triangles, in order to easily recognize corresponding side ratios. Afterwards I generalized these relationships using a GeoGebra-based applet. Finally, I assigned some exercises to apply the theorem. The class´ idea was to replicate the geometrical construction of this theorem, as it appears in several specialized books, but using elements that are closer to the students, like cardboard triangles and the GeoGebra-based applet …the applet aimed to show students that the relationships they found examining particular triangles hold true for many triangles that satisfy certain conditions. We discovered these conditions by comparing the couple of different triangles each one had.

Even though Simón associates the applet with the generation of ratio relationships, in his remarks it is possible to observe how some of the mathematical processes associated with visualization are developed starting with the use of concrete materials – such as conjecture – until reconstructing the theorem by means of the applet, reflecting the sequence of mathematical processes that appeared during the activity.

Table 3 presents several extracts from Simón's answers to the second survey applied at the end of the course Teaching and Learning Mathematics II, which reveal his reflections on teaching and learning mathematics using GeoGebra during the course.

The first answer reveals that Simón recognizes that the importance of integrating ICT does not come from merely using it, but on how it is managed, risking hollow interventions where teaching and learning practices are kept the same, and emphasizing that real change occurs when students interact with the content by means of visualization and manipulation. Therefore, Simón associates the visualization competence to an adequate management of ICT tools for teaching mathematics.

When questioned about the benefits that ICT integration yields, Simón points out another attribute of visualization, namely that it promotes a better understanding when students have the possibility of manipulating the contents, which in turn has a positive impact on learning assessments. Additionally, he identifies the incidence ICT integration has on student motivation and attitudes towards learning.

The third question examines Simón's vision about mathematics from the perspective of his experience using GeoGebra. Simón acknowledges a change of perspective about the nature of mathematics shifting from an algebraic view to a visual one: 'The use of GeoGebra helped me to value and understand the importance visual representations have to explicitly state mathematical truths, which I personally developed and understood in an algebraic way, analytically but not visually'. Simón grants GeoGebra several attributes related to the development of the visualization competence, such as: 'displaying contents in a way that makes it easier to make connections among their multiple representations', and he even compares it to another software he had the chance of using during his mathematics undergraduate degree.

Finally, asked about the role that teachers play when using ICT, Simón states that when teaching with a projected applet in a regular classroom, the teacher's role does not change, but when students are working in the lab completing digital worksheets individually or in pairs, the teacher becomes a mediator between technology and learning, and students become the protagonists of the class.

Table 3. Extracts from Simón's answers.

Question	Extracts from Simón's answers
How does mathematical learning change when using ICT?	There can be major differences between a class that uses ICT and another one that does not have that experience, *but only if these resources are used correctly*. One can take students to the computer lab, but if they're going to do the same as in the regular classroom, the integration of these resources will have no impact on learning because of poor management of technological tools. Now, if we assume good management and expertise on the teacher´s side, the difference lies in the way students interact with the contents, the way they manipulate them, the way they visualize them, etc. All these factors end up having a positive effect on student learning.
What are the main benefits of integrating ICT into mathematics teaching and learning?	The first benefit is related to the way in which contents are presented to students. Presenting contents in a more dynamic way and closer to the platforms used by students tends to generate a better understanding of these concepts. This way, it is not the teacher who exhibits a static content, but it is they themselves who manipulate the information, arriving at new conclusions. This new way of approaching contents has a positive impact on the understanding students build and therefore it enhances assessments as well.
Of what nature are these benefits?	The second benefit is related to the attitude that students adopt when learning using a platform closer to their daily world. When using ICT one can observe the immediate interest that it generates in many students. They tend to pay more attention to what's being presented to them. A natural student interest arises. Hence, they are more willing to listen and understand new content. This new attitude brings about comments like 'This lecture was interesting and fun, all lectures should be like this one'.
How has the learning and use of GeoGebra impacted your ideas about mathematics?	The use of GeoGebra helped me to value and understand the importance of visual representations to explicitly state mathematical truths, which I personally developed and understood in an algebraic way, analytically but not visually. Visualization and the possibility of freely manipulating parameters allow for a more integral vision of mathematics, where contents are not isolated from each other, but displayed in a way that helps to make connections between their multiple representations. This richness of different representations is very useful to explain certain mathematical ideas and I owe it to GeoGebra, because before this experience I only knew MAPLE, a very analytical software, not visually driven and very impractical and inaccessible for new users.
Does the teacher's role change when using ICT in the classroom? Why?	The answer depends on the way in which we're using technology in the classroom. If the lecture is based on a dynamic applet in the regular class, this use does not change the nature of classroom routines, preserving the teacher's traditional role. But if we go to the lab where each student (or pair of students) completes a digital worksheet, the teacher, more than being a knowledge provider, becomes a guide, solving certain content doubts or helping with the software. This way, when students become protagonists of the class, the teacher shifts from a very active and central role and replaces it with a more secondary perspective, in which he/she guides and helps the students to build their own knowledge.

5. Discussion and conclusion

During the formative process, all four prospective mathematics teachers appropriated and familiarized themselves with GeoGebra tools and applications, enabling them to rediscover school mathematical content through technology and to introduce a more exploratory methodology into their school teaching practice. This was possible to observe in Simón's case, since he gradually connected the curricular applications revised to the visualization competence, establishing it as the major benefit of ICT integration into mathematics teaching and learning. The meaning and impact of this innovation is captured in the analyzed applets he designed and constructed during the year. By introducing dynamic elements and using colour interactively [18], he takes full advantage of this new visual platform, allowing more students to reach the expected learnings while developing the necessary mathematical processes.

We have seen in the practice class episodes' description how the ongoing stages of the analyzed applets triggered – through visualization – hypotheses and conjectures proposed by students, in order to facilitate the subsequent generalization of discovered patterns and regularities. Likewise, students had the possibility to verify these findings numerically using this technological resource [27].

Simón generated all variation patterns that Marton et al. [19] propose for a deep understanding of the object of learning in the applet's design. When analyzing the 'Euclid's theorem' applet from a phenomenographic point of view of variation patterns, we may notice that his proposal *contrasts* with the traditional-procedural way in which this content is normally approached. He starts by highlighting all interior angles to *separate* and individualize them through rotation. By using colour, he draws attention to the critical aspects that enable the establishment of the different theorems based on triangle similarity derived from angle congruence (invariance); then, the relations found are *generalized* for any right-angle triangle while vertex C travels through the semi-circumference it was built on; finally, he accomplishes a *fusion* of all its main features, presenting all theorems simultaneously and verifying their validity by numerical confirmations using the Spreadsheet View.

The content addressed in the 'Systems of linear equations: relative position' applet is dynamic in nature, since it examines different possible positions for two straight lines in a plane. Again, Simón generates all four variation patterns in his design, so students can distinguish invariant elements for each relative position examined. He starts by laying out only one straight line on the plane and studying its behaviour regarding its parameters' value, preparing learners for the subsequent analysis and *contrasting* it with the linear equation systems that are being introduced. Once the system of linear equations is represented and its solution has been found, he generates the variation pattern *separation*, changing the slope and/or intercept of one or both lines, so as to identify invariances and *generalize* each case. Afterwards, *fusion* requires a discernment of all critical aspects simultaneously (slope (m) and intercept (n) values), allowing him to determine which type of solution the system will present and associating it with its graphic representation.

In both cases, we notice how this methodological approach modified the nature of classroom routines [15], which traditionally address contents from a procedural standpoint – for example, using Euclid's theorems for obtaining an unknown length; graphing lines from a table of values; or solving linear systems of equations algebraically – to a more conceptual approach in which multiple representations of the object of learning are systematically examined to elucidate its critical aspects thus enriching its conceptualization [20].

Accordingly, we realize there is a close relation between visualization-associated processes and the patterns of variation proposed by Marton et al. [19]. On the one hand, we notice that the hypotheses and conjectures put forward appear connected to the variation pattern *separation*, which individually modifies the central aspects of the concept under scrutiny. Verification, on the other hand, is nurtured by the multiplicity of variation patterns that act as interaction agents between students and mathematical objects. Finally, generalization – considered as a visualization associated process – emerges in combination with the variation pattern *fusion*, which reveals the invariant structure of the complete phenomenon.

The preceding analysis shows how technology incorporation during initial teacher training programmes, in particular GeoGebra, shapes the development of prospective teachers' TPCK.

Since cohort 2014, we have been witnessing how the use of GeoGebra impacts practicing teacher's paradigms, perceptions and performance on how teaching and learning mathematics takes place, captures student attention and becomes significant at school. Prospective teachers' appreciation of the use of GeoGebra has been systematically positive and valued as one of the most important aspects of their formative process. We have observed how these pre-service teachers engage in a transformation involving not only their teacher practices but also their own role as educators. This trend has become stronger as we, their trainers, have understood this innovation as a fundamental and cross-curricular part of initial teacher training.

Due to Simón's varied experiences using GeoGebra, such as solving digital worksheets, designing applets, teaching using technology, and becoming a mediator between technology and learning, he has reconfigured his mathematics teaching conceptions from different dimensions. On the one hand, Simón has changed his ideas about mathematics themselves: the student profile that is admitted to the Teacher Education Programme receives a formal education in mathematics, and thus develops strict abstract and algebraic conceptions of the discipline. The experience of using GeoGebra tended to modify these conceptions towards a more visual approach to mathematics, generating an increasing acknowledgement of the importance of representations for learning.

On the other hand, Simón recognizes the importance of promoting the visualization competence in learning, which in turn favours processes such as conjecturing and generalizing. He also emphasizes the contribution of ICT in promoting for teachers a mediating role between technology and knowledge acquisition.

We consider these aspects to be part of the TPCK, and therefore they cannot be understood in a fragmented way; thus, in accordance with Gómez-Chacón and Joglar's [4] model, we agree on developing TPCK in combination with the courses that address the PCK; in particular, we have observed that introducing ICT into the mathematics education courses has profoundly contributed to the development of TPCK.

The fact that the Teacher Education Programme lasts only one year poses a challenge and strains the possibility of thoroughly impacting the PCK, as the time available is not enough to discuss each mathematical learning topic. Incorporating an ICT line into the mathematics education courses allowed this tension to be dealt with and contributed not only to an efficient use of technology for mathematics teaching and to promoting a more dialogical teacher role in the classroom, but also to the modification of conceptions about mathematics and its teaching.

As we describe in the study, this effort of introducing ICT into initial teacher training programmes in Chile is recent and, particularly for our programme at Pontificia Universidad Católica de Chile, it materialized as a curricular renovation project which aimed at a systematic integration of ICTs, which is still in the process of implementation.

Therefore, this study presents the initial research of a series of case studies and follow-up of our pre-service teachers as they become in-service teachers. Considering this temporal frame, we would like to characterize in our future studies how important the use of GeoGebra has remained in their daily teacher practice regarding three areas: attitudes of students towards the use of GeoGebra in the mathematics classroom, curricular mathematical contents addressed with GeoGebra, and its impact on learning and how to enhance its use as an exploratory tool.

Note

1. PUC 1201: Innovación en la Formación de Profesores: integración de competencias disciplinarias, pedagógicas y profesionales para la efectividad en las aulas [Innovation in teacher training: integrating disciplinary, pedagogical and professional competencies for effective teaching]. Spanish.

Disclosure statement

No potential conflict of interest was reported by the authors.

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