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A DIRECT COUPLING OF LOCAL DISCONTINUOUS GALERKIN AND BOUNDARY ELEMENT METHODS

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Abstract. The coupling of local discontinuous Galerkin (LDG) and boundary element methods (BEM), which has been developed recently to solve linear and nonlinear exterior transmission problems, employs a mortar-type auxiliary unknown to deal with the weak continuity of the traces at the interface boundary. As a consequence, the main features of LDG and BEM are maintained and hence the coupled approach benefits from the advantages of both methods. In this paper we propose and analyze a simplified procedure that avoids the mortar variable by employing LDG subspaces whose functions are continuous on the coupling boundary. The continuity can be implemented either directly or indirectly via the use of Lagrangian multipliers. In this way, the normal derivative becomes the only boundary unknown, and hence the total number of unknown functions is reduced by two. We prove the stability of the new discrete scheme and derive an a priori error estimate in the energy norm. A numerical example confirming the theoretical result is provided. The analysis is also extended to the case of nonlinear problems and to the coupling with other discontinuous Galerkin methods.

1. Introduction

The coupling of local discontinuous Galerkin and boundary element methods, as applied to linear exterior boundary value problems in the plane, has been introduced and analyzed for the first time in [18]. The model problem there is the Poisson equation in an annular domain coupled with the Laplace equation in the surrounding unbounded exterior region. The corresponding extension to a class of nonlinear-linear exterior transmission problems, which is also motivated by previous applications of the LDG method to some nonlinear problems in heat conduction and fluid mechanics (see, e.g. [7], [8], and [23]), was developed recently in [9], [10], and [11]. In these works, the authors consider a nonlinear elliptic equation in divergence form in an annular region coupled with discontinuous transmission conditions on the interface boundary and the Poisson equation in the exterior unbounded domain. In both the linear and nonlinear cases the technique employed resembles the usual coupling of finite element and boundary element methods, but

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the corresponding analysis becomes quite different. In particular, in order to deal with the weak continuity of the traces at the coupling boundary, a mortar-type auxiliary unknown representing an interior approximation of the normal derivative needs to be defined. Hence, different mesh sizes on that boundary and special relationships between them are required. In addition, the continuity and ellipticity estimates of the bilinear form involved hold with different mesh-dependent norms, and Strang-type a priori error estimates instead of the usual Céa ones are obtained.

In the present paper we simplify the approach from [18] and develop a direct procedure for the coupling of LDG and BEM which does not make use of any mortar unknown. Instead, it employs a finite element subspace of functions that are required to be continuous only on the coupling boundary Γ. Consequently, in this paper, the normal derivative becomes the only boundary unknown and then the total number of unknown functions is reduced by two. The continuity of LDG functions on Γ can be implemented directly by considering appropriate LDG subspaces. However, one can maintain the full flexibility of the LDG method by implementing the continuity condition via a Lagrangian multiplier. In this way standard LDG and BEM implementations are sufficient for the coupling procedure.

In order to introduce the model problem let Ω_0 be a simply connected and bounded domain in $\mathbb{R}^2$ with polygonal boundary Γ_0. Then, given $f \in L^2(\mathbb{R}^2 \setminus \bar{\Omega}_0)$ with compact support, we consider the exterior Dirichlet problem:

$$\begin{gathered} -\Delta u = f \quad \text{in} \quad \mathbb{R}^2 \setminus \bar{\Omega}_0, \quad u = 0 \quad \text{on} \quad \Gamma_0\,, \\ u(\boldsymbol{x}) = \mathcal{O}(1) \quad \text{as} \quad |\boldsymbol{x}| \to \infty\,. \end{gathered} \tag{1.1}$$

Next, let Γ be a closed polygonal curve such that the support of f is inside the annular domain Ω enclosed by Γ_0 and Γ. We assume that this support does not intersect Γ. Then (1.1) can be written as the Poisson equation in Ω:

$$-\Delta u = f \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \Gamma_0\,, \tag{1.2}$$

and the Laplace equation in the exterior domain $\Omega_e := \mathbb{R}^2 \setminus (\bar{\Omega}_0 \cup \bar{\Omega})$:

$$-\Delta u_e = 0 \quad \text{in} \quad \Omega_e, \quad u_e(\boldsymbol{x}) = \mathcal{O}(1) \quad \text{as} \quad |\boldsymbol{x}| \to \infty\,, \tag{1.3}$$

coupled by the transmission conditions:

$$u = u_e \quad \text{on}\ \Gamma \quad \text{and} \quad \partial_{\boldsymbol{\nu}} u = \partial_{\boldsymbol{\nu}} u_e \quad \text{on}\ \Gamma\,. \tag{1.4}$$

Here, $\partial_{\boldsymbol{\nu}} u$ denotes the normal derivative of u with normal vector pointing outside Ω. The purpose of this work is to solve numerically (1.1) by means of a new LDG-BEM coupling which, similarly to [18], consists of applying the LDG method to (1.2) and the BEM to (1.3).

The remainder of this work is organized as follows. In Section 2 we introduce the boundary integral equation formulation in Ω_e, define the LDG method in Ω, and establish the resulting coupled LDG-BEM approach. Next, in Section 3 we prove the unique solvability and stability of our discrete scheme. The associated a priori error analysis is provided in Section 4. Then, in Section 5 we describe a Lagrange multiplier based implementation of the coupled scheme which maintains the discontinuous character of the LDG method. The good performance of this scheme is illustrated with a simple numerical example, which also confirms the theoretical rate of convergence of the method for the regular case $u \in H^2(\Omega)$. In Section 6 we extend our analysis to the class of nonlinear problems studied in [9],

[10], and [11]. Finally, in Section 7 we discuss some aspects of the coupling of BEM with other discontinuous Galerkin methods.

Throughout this paper, c and C denote positive constants, independent of the parameters and functions involved, which may take different values at different occurrences. Given any linear space V, the corresponding vector-valued space $V \times V$ endowed with the product norm will be denoted by $\mathbf{V}$. If $\mathcal{O}$ is an open set, its closure, or a polygonal curve, and $s \in \mathbb{R}$, then $|\cdot|_{s,\mathcal{O}}$ and $\|\cdot\|_{s,\mathcal{O}}$ denote the seminorm and norm in the Sobolev space $H^s(\mathcal{O})$. In particular, the norms of $H^s(\Gamma)$ are denoted by $\|\cdot\|_{s,\Gamma}$. Also, $\langle\cdot,\cdot\rangle$ denotes both the $L^2(\Gamma)$ inner product and its extension to the duality pairing of $H^{-s}(\Gamma) \times H^s(\Gamma)$.

2. The coupled LDG-BEM approach

2.1. **The boundary integral formulation in the exterior domain.** We use Green's representation formula for u_e in Ω_e,

$$u_e(\mathbf{x}) = \int_\Gamma \partial_{\boldsymbol{\nu}(\boldsymbol{y})} E(\mathbf{x},\boldsymbol{y})\, u(\boldsymbol{y})\, ds_{\boldsymbol{y}} - \int_\Gamma E(\mathbf{x},\boldsymbol{y})\, \lambda(\boldsymbol{y})\, ds_{\boldsymbol{y}} + c \qquad \forall\, \mathbf{x} \in \Omega_e\,, \tag{2.1}$$

where $E(\boldsymbol{x},\boldsymbol{y}) := -\dfrac{1}{2\pi}\log|\boldsymbol{x}-\boldsymbol{y}|$ is the fundamental solution of the Laplacian in $\mathbb{R}^2$, $\lambda = \partial_{\boldsymbol{\nu}} u$, and c is a constant. Note that we made use of the transmission conditions (1.4). It is well known that (2.1) gives rise to the following system of boundary integral equations (see, e.g. [24]):

$$\begin{aligned} \mathcal{W}u - (\tfrac{1}{2}\mathcal{I} - \mathcal{K}')\lambda &= -\lambda \qquad \text{on} \quad \Gamma\,, \\ (\tfrac{1}{2}\mathcal{I} - \mathcal{K})u + \mathcal{V}\lambda \ + \ c &= 0 \qquad \text{on} \quad \Gamma\,, \end{aligned} \tag{2.2}$$

where $\mathcal{V}$, $\mathcal{K}$, $\mathcal{K}'$, and $\mathcal{W}$ are the boundary integral operators associated with the single, double, adjoint of the double, and hypersingular layer potentials, respectively. We recall from [14] that $\mathcal{V} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$, $\mathcal{K} : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$, $\mathcal{K}' : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, and $\mathcal{W} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ are bounded linear operators, and that they are defined as follows:

$$\begin{aligned}
\mathcal{V}\mu(\mathbf{x}) &:= \int_\Gamma E(\mathbf{x},\boldsymbol{y})\, \mu(\boldsymbol{y})\, ds_{\boldsymbol{y}} && \forall\,(a.e.)\,\mathbf{x} \in \Gamma\,,\ \forall\, \mu \in H^{-1/2}(\Gamma)\,, \\
\mathcal{K}\psi(\mathbf{x}) &:= \int_\Gamma \partial_{\boldsymbol{\nu}(\boldsymbol{y})} E(\mathbf{x},\boldsymbol{y})\, \psi(\boldsymbol{y})\, ds_{\boldsymbol{y}} && \forall\,(a.e.)\,\mathbf{x} \in \Gamma\,,\ \forall\, \psi \in H^{1/2}(\Gamma)\,, \\
\mathcal{K}'\mu(\mathbf{x}) &:= \int_\Gamma \partial_{\boldsymbol{\nu}(\mathbf{x})} E(\mathbf{x},\boldsymbol{y})\, \mu(\boldsymbol{y})\, ds_{\boldsymbol{y}} && \forall\,(a.e.)\,\mathbf{x} \in \Gamma\,,\ \forall\, \mu \in H^{-1/2}(\Gamma)\,, \\
\mathcal{W}\psi(\mathbf{x}) &:= -\partial_{\boldsymbol{\nu}(\mathbf{x})} \int_\Gamma \partial_{\boldsymbol{\nu}(\boldsymbol{y})} E(\mathbf{x},\boldsymbol{y})\, \psi(\boldsymbol{y})\, ds_{\boldsymbol{y}} && \forall\,(a.e.)\,\mathbf{x} \in \Gamma\,,\ \forall\, \psi \in H^{1/2}(\Gamma)\,.
\end{aligned}$$

Here, $\partial_{\boldsymbol{\nu}(\mathbf{x})}$ stands for the normal derivative operator at $\mathbf{x} \in \Gamma$.

Next, according to the behavior of u at infinity (cf. (1.1)), we observe that λ belongs to $H_0^{-1/2}(\Gamma)$ where

$$H_0^{-1/2}(\Gamma) := \{\mu \in H^{-1/2}(\Gamma) : \quad \langle \mu, 1\rangle = 0\}\,.$$

According to the decomposition $H^{1/2}(\Gamma) = H_0^{1/2}(\Gamma) \oplus \mathbb{R}$, with

$$H_0^{1/2}(\Gamma) := \{\psi \in H^{1/2}(\Gamma) : \quad \langle 1, \psi\rangle = 0\}\,,$$

we define

$$\|\psi\|_{1/2,\Gamma,0} := \left\|\psi - \text{length}(\Gamma)^{-1}\textstyle\int_\Gamma \psi\right\|_{1/2,\Gamma}. \tag{2.3}$$

Equivalently, $\|\cdot\|_{1/2,\Gamma,0}$ denotes the quotient space norm

$$\|\psi\|_{1/2,\Gamma,0} := \inf_{c\in\mathbb{R}} \|\psi + c\|_{1/2,\Gamma} \qquad \forall\, \psi \in H^{1/2}(\Gamma)\,.$$

The analysis of (2.2) and its discrete counterpart below will depend on the symmetry of $\mathcal{W}$ and the ellipticity of $\mathcal{V}$ and $\mathcal{W}$:

$$\begin{array}{rcll} \langle \mathcal{W}\varphi, \psi\rangle & = & \langle \mathcal{W}\psi, \varphi\rangle & \forall\, \varphi,\, \psi \in H^{1/2}(\Gamma)\,, \\ \langle \mu, \mathcal{V}\mu\rangle & \geq & C\,\|\mu\|^2_{-1/2,\Gamma} & \forall\, \mu \in H_0^{-1/2}(\Gamma)\,, \\ \langle \mathcal{W}\psi, \psi\rangle & \geq & C\,\|\psi\|^2_{1/2,\Gamma,0} & \forall\, \psi \in H^{1/2}(\Gamma)\,, \end{array} \tag{2.4}$$

2.2. **The LDG formulation in the interior domain.** The setting and analysis of the LDG formulation in Ω require several notations, definitions, and assumptions that we recall from [18]. Let $\mathcal{T}_h$ be a shape regular triangulation of $\bar{\Omega}$ (with possible hanging nodes) made up of straight triangles K with diameter h_K and unit outward normal to ∂K given by $\boldsymbol{\nu}_K$. As usual, the index h denotes $h := \max\limits_{K\in\mathcal{T}_h} h_K$. Then, the edges of $\mathcal{T}_h$ are defined as follows. An *interior edge of $\mathcal{T}_h$ is the* (nonempty) interior of $\partial K \cap \partial K'$ where K and K' are two adjacent elements of $\mathcal{T}_h$. Similarly, a *boundary edge* of $\mathcal{T}_h$ is the (nonempty) interior of $\partial K \cap \Gamma_0$ or $\partial K \cap \Gamma$ where K is an element of $\mathcal{T}_h$ which has an edge on Γ_0 or Γ. For each edge e, h_e represents its length. In addition, we define $\mathcal{E}(K)$:=$\{$edges of $K\}$, $\mathcal{E}_h^{\text{int}}$: set of interior edges (counted only once), $\mathcal{E}_h^{\Gamma}$: set of edges on Γ, $\mathcal{E}_h^{\Gamma_0}$: set of edges on Γ_0, and I_h: interior grid generated by the triangulation, that is $I_h := \bigcup\{e : \; e \in \mathcal{E}_h^{\text{int}}\}$. Also, we let Γ_h and Γ_h^0 be the induced meshes on the boundaries Γ and Γ_0, whose lists of edges are $\mathcal{E}_h^{\Gamma}$ and $\mathcal{E}_h^{\Gamma_0}$, respectively.

In what follows we assume that $\mathcal{T}_h$ is a *locally quasi-uniform* mesh, i.e., there exists $l > 1$, independent of the meshsize h, such that $l^{-1} \;\leq\; \frac{h_K}{h_{K'}} \;\leq\; l$ for each pair $K,\, K' \in \mathcal{T}_h$ sharing an interior edge. We notice that the hypotheses on the triangulation imply that the cardinality of $\mathcal{E}(K)$ is uniformly bounded, and that

$$h_e \leq h_K \leq C\, l\, h_e, \qquad \forall e \in \mathcal{E}(K). \tag{2.5}$$

Now we consider integers $m \geq 1$ and $r \geq m-1 \geq 0$, and define the finite element spaces

$$V_h := \prod_{K\in\mathcal{T}_h} P_m(K) \quad \text{and} \quad \boldsymbol{\Sigma}_h := \prod_{K\in\mathcal{T}_h} \mathbf{P}_r(K)\,. \tag{2.6}$$

Hereafter, given an integer $k \geq 0$ and a domain $S \subseteq \mathbb{R}^2$, $P_k(S)$ denotes the space of polynomials of degree at most k on S. For each $v := \{v_K\}_{K\in\mathcal{T}_h} \in V_h$ and $\boldsymbol{\tau} := \{\boldsymbol{\tau}_K\}_{K\in\mathcal{T}_h} \in \boldsymbol{\Sigma}_h$, the components v_K and $\boldsymbol{\tau}_K$ coincide with the restrictions $v|_K$ and $\boldsymbol{\tau}|_K$, when v and $\boldsymbol{\tau}$ are identified as elements in $L^2(\Omega)$ and $\mathbf{L}^2(\Omega)$, respectively. Further, when no confusion arises, we omit the subscript K and just write v and $\boldsymbol{\tau}$.

Next, consider the broken Sobolev spaces

$$H^s(\mathcal{T}_h) := \prod_{K\in\mathcal{T}_h} H^s(K), \qquad (s > 1/2)$$

as well as the spaces on the skeleton of the triangulation

$$L^2(I_h) := \prod_{e\in\mathcal{E}_h^{\rm int}} L^2(e)\,, \qquad P_0(I_h) := \prod_{e\in\mathcal{E}_h^{\rm int}} P_0(e)$$

and

$$P_0(I_h\cup\Gamma_h^0) := \prod_{e\in\mathcal{E}_h^{\rm int}\cup\mathcal{E}_h^{\Gamma_0}} P_0(e)\,.$$

An analogue remark to the one given before, concerning components and restrictions of the elements in V_h and $\mathbf{\Sigma}_h$, is valid here for each of the product spaces above. Also, we will not use any symbol for the trace on edges, provided it is clear from which side of an interior edge we are taking the trace. Hence, given $v\in H^1(\mathcal{T}_h)$, we define the *averages* $\{v\}\in L^2(I_h)$ and *jumps* $[\![v]\!]\in\mathbf{L}^2(I_h)$ on the interior grid I_h by

$$\{v\}_e := \tfrac{1}{2}(v_K+v_{K'}) \quad \text{and} \quad [\![v]\!]_e := v_K\boldsymbol{\nu}_K + v_{K'}\boldsymbol{\nu}_{K'} \quad \forall\, e\in\mathcal{E}(K)\cap\mathcal{E}(K')\,.$$

Similarly, for vector-valued functions $\boldsymbol{\tau}\in\mathbf{H}^1(\mathcal{T}_h)$, we define $\{\boldsymbol{\tau}\}\in\mathbf{L}^2(I_h)$ and $[\![\boldsymbol{\tau}]\!]\in L^2(I_h)$ by

$$\{\boldsymbol{\tau}\}_e := \tfrac{1}{2}(\boldsymbol{\tau}_K+\boldsymbol{\tau}_{K'}) \quad \text{and} \quad [\![\boldsymbol{\tau}]\!]_e := \boldsymbol{\tau}_K\cdot\boldsymbol{\nu}_K + \boldsymbol{\tau}_{K'}\cdot\boldsymbol{\nu}_{K'} \quad \forall\, e\in\mathcal{E}(K)\cap\mathcal{E}(K')\,.$$

In addition, let $\alpha\in P_0(I_h\cup\Gamma_h^0)$ and $\boldsymbol{\beta}\in\mathbf{P}_0(I_h)$ be given functions and assume that there exist C, c_0, $c_1>0$, independent of the grid, such that

$$\max_{e\in\mathcal{E}_h^{\rm int}} |\boldsymbol{\beta}_e| \le C \quad \text{and} \quad 0<c_0\le h_{\mathcal{E}}\,\alpha\le c_1\,, \tag{2.7}$$

where $h_{\mathcal{E}}\in P_0(I_h\cup\Gamma_h^0)$ is defined by $h_{\mathcal{E}}|_e := h_e \quad \forall\, e\in\mathcal{E}_h^{\rm int}\cup\mathcal{E}_h^{\Gamma_0}$.

We are now in a position to introduce the LDG scheme for the interior problem (1.2). As usual, we first define the gradient $\boldsymbol{\sigma}:=\nabla u$ in Ω as an additional unknown where u is the exact solution of (1.2)–(1.3). Then, let $\lambda_h\in L^2(\Gamma)$ be a discrete approximation (to be defined below) of the normal derivative λ, and proceeding as in [12, 18] we arrive at the following global LDG formulation: Find $(\boldsymbol{\sigma}_h,u_h)\in\mathbf{\Sigma}_h\times V_h$ such that

$$\begin{aligned} &\int_\Omega \boldsymbol{\sigma}_h\cdot\boldsymbol{\tau} - \left\{\int_\Omega \nabla_h u_h\cdot\boldsymbol{\tau} - S(u_h,\boldsymbol{\tau})\right\} = 0 && \forall\,\boldsymbol{\tau}\in\mathbf{\Sigma}_h\,,\\ &\left\{\int_\Omega \nabla_h v\cdot\boldsymbol{\sigma}_h - S(v,\boldsymbol{\sigma}_h)\right\} + \boldsymbol{\alpha}(u_h,v) = \int_\Omega f\,v + \int_\Gamma \lambda_h\,v && \forall\, v\in V_h\,, \end{aligned} \tag{2.8}$$

where ∇_h stands for the piecewise defined gradient, and $S: H^1(\mathcal{T}_h)\times\mathbf{H}^1(\mathcal{T}_h)\to\mathbb{R}$ and $\boldsymbol{\alpha}: H^1(\mathcal{T}_h)\times H^1(\mathcal{T}_h)\to\mathbb{R}$ are the bilinear forms defined by

$$S(w,\boldsymbol{\tau}) := \int_{I_h} [\![w]\!]\cdot(\{\boldsymbol{\tau}\} - [\![\boldsymbol{\tau}]\!]\,\boldsymbol{\beta}) + \int_{\Gamma_0} w\,(\boldsymbol{\tau}\cdot\boldsymbol{\nu}) \qquad \forall\,(w,\boldsymbol{\tau})\in H^1(\mathcal{T}_h)\times\mathbf{H}^1(\mathcal{T}_h)\,, \tag{2.9}$$

and

$$\boldsymbol{\alpha}(w,v) := \int_{I_h} \alpha\,[\![w]\!]\cdot[\![v]\!] + \int_{\Gamma_0} \alpha\,w\,v \qquad \forall\,(w,v)\in H^1(\mathcal{T}_h)\times H^1(\mathcal{T}_h)\,, \tag{2.10}$$

with the traces of w, v, and $\boldsymbol{\tau}$ on Γ_0 being defined elementwise.

2.3. **The coupled LDG-BEM scheme.** We now establish the coupled LDG-BEM scheme by combining a discrete form of (2.2) with the LDG formulation (2.8). This requires a subspace for λ_h and an approximant u_h of u which is continuous on Γ. For the discrete space approximating λ we take, for simplicity, the partition of Γ induced by $\mathcal{T}_h$ and introduce

(2.11)
$$X_h := \prod_{e\in\mathcal{E}_h^\Gamma} P_{m-1}(e) \quad \text{and} \quad X_h^0 := \{\mu_h \in X_h : \int_\Gamma \mu_h = 0\} = X_h \cap H_0^{-1/2}(\Gamma).$$

Then, we consider the subspace $\tilde{V}_h$ of V_h defined by

$$\tilde{V}_h := \{v_h \in V_h : \quad v_h|_\Gamma \in C(\Gamma)\}.$$

Here, the trace $v_h|_\Gamma$ for $v_h \in V_h$ is defined in a piecewise manner on the edges of Γ_h and the condition $v_h|_\Gamma \in C(\Gamma)$ means that the function composed by the piecewise traces is continuous on Γ. Hence, substituting λ_h in (2.8) by a discrete version of the first equation in (2.2), in which u is replaced by its approximant u_h, and adding also a discrete formulation of the second equation in (2.2), we obtain the following coupled LDG-BEM scheme: Find $(\boldsymbol{\sigma}_h, u_h, \lambda_h) \in \boldsymbol{\Sigma}_h \times \tilde{V}_h \times X_h^0$ such that

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau} - \boldsymbol{\rho}(u_h, \boldsymbol{\tau}) &= 0\,, \\ \boldsymbol{\rho}(v, \boldsymbol{\sigma}_h) + \boldsymbol{\alpha}\,(u_h, v) + \langle \mathcal{W} u_h, v\rangle - \langle (\tfrac{1}{2}\mathcal{I} - \mathcal{K}')\lambda_h, v\rangle &= \int_\Omega f\, v\,, \\ \langle \mu, (\tfrac{1}{2}\mathcal{I} - \mathcal{K})u_h\rangle + \langle \mu, \mathcal{V}\lambda_h\rangle &= 0 \end{aligned} \tag{2.12}$$

for all $(\boldsymbol{\tau}, v, \mu) \in \boldsymbol{\Sigma}_h \times \tilde{V}_h \times X_h^0$, where $\boldsymbol{\rho} : H^1(\mathcal{T}_h) \times \mathbf{H}^1(\mathcal{T}_h) \to \mathbb{R}$ is the bilinear form defined by

$$\boldsymbol{\rho}(v, \boldsymbol{\tau}) := \int_\Omega \nabla_h v \cdot \boldsymbol{\tau} - S(v, \boldsymbol{\tau}) \qquad \forall\,(v, \boldsymbol{\tau}) \in H^1(\mathcal{T}_h) \times \mathbf{H}^1(\mathcal{T}_h)\,. \tag{2.13}$$

This coupled LDG-BEM scheme has the usual form of the traditional coupling of finite and boundary elements (see [13, 20]): diagonal operators are symmetric and off-diagonal operators form a skew symmetric matrix. The complete system can be made symmetric (although indefinite) by changing the sign of the second equation. Also, notice that occurrences of u_h as well as $v \in \tilde{V}_h$ inside the duality bracket and under the action of integral operators include the use of the piecewise trace which belongs to $H^{1/2}(\Gamma)$.

In order to compare the formulation (2.12) with the one from [18] we recall that the latter is given as follows: Find $(\boldsymbol{\sigma}_h, u_h, \lambda_{\tilde{h}}, \varphi_{\hat{h}}, \gamma_{\hat{h}}) \in \boldsymbol{\Sigma}_h \times V_h \times X_{\tilde{h}}^0 \times Y_{\hat{h}}^0 \times Z_{\hat{h}}^0$

such that

$$
\begin{aligned}
\int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau} - \boldsymbol{\rho}(u_h, \boldsymbol{\tau}) &= 0\,,\\
\boldsymbol{\rho}(v, \boldsymbol{\sigma}_h) + \boldsymbol{\alpha}\,(u_h, v) - \langle \lambda_{\tilde{h}}, v\rangle &= \int_\Omega f\, v\,,\\
\langle \xi, u_h\rangle - \langle \xi, \varphi_{\hat{h}}\rangle &= 0\,,\\
\langle \lambda_{\tilde{h}}, \psi\rangle + \langle \mathcal{W}\varphi_{\hat{h}}, \psi\rangle - \langle (\frac{1}{2}\mathcal{I} - \mathcal{K}')\gamma_{\hat{h}}, \psi\rangle &= 0\,,\\
\langle \mu, (\frac{1}{2}\mathcal{I} - \mathcal{K})\varphi_{\hat{h}}\rangle + \langle \mu, \mathcal{V}\gamma_{\hat{h}}\rangle &= 0
\end{aligned}
\tag{2.14}
$$

for all $(\boldsymbol{\tau}, v, \xi, \psi, \mu) \in \boldsymbol{\Sigma}_h \times V_h \times X^0_{\tilde{h}} \times Y^0_{\hat{h}} \times Z^0_{\hat{h}}$, where $X^0_{\tilde{h}} \subseteq L^2(\Gamma) \cap H^{-1/2}_0(\Gamma)$, $Y^0_{\hat{h}} \subseteq C(\Gamma) \cap H^{1/2}_0(\Gamma)$, and $Z^0_{\hat{h}} \subseteq L^2(\Gamma) \cap H^{-1/2}_0(\Gamma)$ are boundary element subspaces, with independent meshsizes $\tilde{h}$ and $\hat{h}$, for the mortar-type auxiliary unknown $\lambda_{\tilde{h}}$ gluing the LDG and BEM modules, and for the Cauchy data $\varphi_{\hat{h}}$ and $\gamma_{\hat{h}}$, respectively. We observe that the computational implementation of (2.14) can be easily obtained by incorporating individual codes for each module, which constitutes the main advantage of this formulation, whereas the lower number of unknowns involved is the main strength of the present approach (2.12).

Now, for the solvability and stability of (2.12) we need an equivalent reduced formulation which is taken from [18]. To this end let $\mathbf{S}_h : H^1(\mathcal{T}_h) \to \boldsymbol{\Sigma}_h$ be the linear operator associated with the bilinear form S restricted to $H^1(\mathcal{T}_h) \times \boldsymbol{\Sigma}_h$. That is, given $w \in H^1(\mathcal{T}_h)$, $\mathbf{S}_h(w)$ is the unique element in $\boldsymbol{\Sigma}_h$ satisfying

$$
\int_\Omega \mathbf{S}_h(w) \cdot \boldsymbol{\tau} \;=\; S(w, \boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h\,. \tag{2.15}
$$

Next, let $B_h : H^1(\mathcal{T}_h) \times H^1(\mathcal{T}_h) \to \mathbb{R}$ be the bilinear form defined by

$$
B_h(w, v) := \boldsymbol{\alpha}(w, v) + \int_\Omega (\nabla_h w - \mathbf{S}_h(w)) \cdot (\nabla_h v - \mathbf{S}_h(v)) \qquad \forall\, w,\, v \,\in\, H^1(\mathcal{T}_h). \tag{2.16}
$$

The equivalence between (2.12) and a reduced problem involving B_h is established by the following lemma.

Lemma 2.1. *Let $(\boldsymbol{\sigma}_h, u_h, \lambda_h) \in \boldsymbol{\Sigma}_h \times \tilde{V}_h \times X^0_h$ be a solution of* (2.12)*. Then it holds that*

$$
\begin{aligned}
B_h(u_h, v) +\; \langle \mathcal{W} u_h, v\rangle - \langle (\frac{1}{2}\mathcal{I} - \mathcal{K}')\lambda_h, v\rangle &= \int_\Omega f\, v\,,\\
\langle \mu, (\frac{1}{2}\mathcal{I} - \mathcal{K}) u_h\rangle + \langle \mu, \mathcal{V}\lambda_h\rangle &= 0
\end{aligned}
\tag{2.17}
$$

for any $(v, \mu) \in \tilde{V}_h \times X^0_h$. Conversely, if $(u_h, \lambda_h) \in \tilde{V}_h \times X^0_h$ satisfies (2.17) *and $\boldsymbol{\sigma}_h := \nabla_h u_h - \mathbf{S}_h(u_h)$, then $(\boldsymbol{\sigma}_h, u_h, \lambda_h)$ is a solution of* (2.12)*. If $(u_h, \lambda_h) \in \tilde{V}_h \times X^0_h$ is the only solution of* (2.17)*, then $(\boldsymbol{\sigma}_h, u_h, \lambda_h)$, with $\boldsymbol{\sigma}_h$ defined as before, is the only solution of* (2.12)*.*

Proof. This result is analogous to Lemma 2.2 in [18] and is based on the fact that the first equation in (2.12) can be written as

$$
\int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau} \;-\; \int_\Omega (\, \nabla_h u_h - \mathbf{S}_h(u_h)\,) \cdot \boldsymbol{\tau} = 0 \qquad \forall\, \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h.
$$

The fact that $r \geq m-1$ guarantees that $\nabla_h u_h \in \mathbf{\Sigma}_h$, which yields $\boldsymbol{\sigma}_h = \nabla_h u_h - \mathbf{S}_h(u_h)$ and leads to the result. $\square$

3. Unique solvability and stability

In this section we prove the unique solvability and stability of (2.12) through the corresponding analysis of the equivalent reduced formulation (2.17). We first introduce seminorms

$$|v|_{1,h}^2 := \|\nabla_h v\|_{0,\Omega}^2 , \qquad |v|_*^2 := \|h_{\mathcal{E}}^{-1/2} [\![v]\!]\|_{0,I_h}^2 + \|h_{\mathcal{E}}^{-1/2} v\|_{0,\Gamma_0}^2 \quad \forall\, v \in H^1(\mathcal{T}_h) ,$$

and the norm

$$|\!|\!| v |\!|\!|_h^2 := |v|_{1,h}^2 + |v|_*^2 \qquad \forall\, v \in H^1(\mathcal{T}_h) .$$

Next, we let $\mathbf{B}_h$ denote the bilinear form defined by the left-hand side of (2.17), i.e.

$$\begin{aligned}\mathbf{B}_h(w,\eta;v,\mu) := {}& B_h(w,v) + \langle \mathcal{W} w, v\rangle \\ & - \langle (\frac{1}{2}\mathcal{I} - \mathcal{K}')\eta, v\rangle + \langle \mu, (\frac{1}{2}\mathcal{I} - \mathcal{K}) w\rangle + \langle \mu, \mathcal{V}\eta\rangle\end{aligned} \tag{3.1}$$

for

$$w, v \in H^1_{1/2}(\mathcal{T}_h) := \{ w \in H^1(\mathcal{T}_h) : \quad w|_\Gamma \in H^{1/2}(\Gamma)\}$$

and $\eta, \mu \in H_0^{-1/2}(\Gamma)$. Analogously as before, the trace $w|_\Gamma \in L^2(\Gamma)$ for $w \in H^1(\mathcal{T}_h)$ is defined first on each edge of Γ_h and the condition $w|_\Gamma \in H^{1/2}(\Gamma)$ means that the function composed by the piecewise traces is in $H^{1/2}(\Gamma)$.

The full discrete norm, defined for elements $(v,\mu) \in H^1_{1/2}(\mathcal{T}_h) \times H_0^{-1/2}(\Gamma)$ will be given by the expression

$$\|(v,\mu)\|_{h,\Gamma}^2 := |\!|\!| v |\!|\!|_h^2 \;+\; \|v\|_{1/2,\Gamma,0}^2 \;+\; \|\mu\|_{-1/2,\Gamma}^2 .$$

Essential ingredients of our analysis are the properties of the bilinear form $\mathbf{B}_h$ with respect to this norm.

Lemma 3.1. *There exist positive constants c, C, independent of h, such that*

$$|\mathbf{B}_h(w,\eta;v,\mu)| \leq c \|(w,\eta)\|_{h,\Gamma} \|(v,\mu)\|_{h,\Gamma} \quad \forall\,(w,\eta),(v,\mu) \in H^1_{1/2}(\mathcal{T}_h) \times H_0^{-1/2}(\Gamma) \tag{3.2}$$

and

$$\mathbf{B}_h(v,\mu;v,\mu) \;\geq\; C\, \|(v,\mu)\|_{h,\Gamma}^2 \quad \forall\,(v,\mu) \in H^1_{1/2}(\mathcal{T}_h) \times H_0^{-1/2}(\Gamma). \tag{3.3}$$

Proof. Recall first that by [18, Lemma 3.2], there exist positive constants c, C, independent of h, such that

$$|B_h(w,v)| \;\leq\; c\, |\!|\!| w |\!|\!|_h \, |\!|\!| v |\!|\!|_h \qquad \forall\, w,\, v \in H^1(\mathcal{T}_h) \tag{3.4}$$

and

$$B_h(v,v) \;\geq\; C\, |\!|\!| v |\!|\!|_h^2 \qquad \forall\, v \in H^1(\mathcal{T}_h) . \tag{3.5}$$

According to the properties of the operators $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{K}$ (cf. Section 2.1), noting that $\mathcal{W}1 = 0$ and $\mathcal{K}1 = -\frac{1}{2}$ on Γ, and using the decomposition $H^{1/2}(\Gamma) = H_0^{1/2}(\Gamma) \oplus \mathbb{R}$ and the definition of the seminorm $\|\cdot\|_{1/2,\Gamma,0}$ (cf. (2.3)), we find that

$$\begin{aligned} |\,\langle \mu, \mathcal{V}\eta\rangle\,| \;&\leq\; C\, \|\mu\|_{-1/2,\Gamma}\, \|\eta\|_{-1/2,\Gamma} \quad \forall\, \mu,\, \eta \in H_0^{-1/2}(\Gamma) , \\ |\,\langle \mathcal{W} w, v\rangle\,| \;&\leq\; C\, \|w\|_{1/2,\Gamma,0}\, \|v\|_{1/2,\Gamma,0} \quad \forall\, w,\, v \in H^1_{1/2}(\mathcal{T}_h) , \end{aligned}$$

and

$$|\langle\mu,(\frac{1}{2}\mathcal{I}-\mathcal{K})w\rangle| \;\leq\; C\,\|w\|_{1/2,\Gamma,0}\,\|\mu\|_{-1/2,\Gamma} \quad \forall\,(w,\mu)\,\in H^1_{1/2}(\mathcal{T}_h)\times H_0^{-1/2}(\Gamma)\,.$$

The inequalities above and (3.4) yield the continuity estimate (3.2) for $\mathbf{B}_h$. Next, we observe from the definition of $\mathbf{B}_h$ that

$$\mathbf{B}_h(v,\mu;v,\mu)=B_h(v,v)+\langle\mathcal{W}v,v\rangle+\langle\mu,\mathcal{V}\mu\rangle \quad \forall\,(v,\mu)\,\in\, H^1_{1/2}(\mathcal{T}_h)\times H_0^{-1/2}(\Gamma)\,,$$

and hence, (2.4) and (3.5) imply the ellipticity estimate (3.3) for $\mathbf{B}_h$. □

We are now in a position to prove the unique solvability and stability of (2.12).

Theorem 3.1. *The coupled* LDG-BEM *scheme* (2.12) *is uniquely solvable and the stability estimate below holds:*

$$\|\boldsymbol{\sigma}_h\|_{0,\Omega}\,+\,\|(u_h,\lambda_h)\|_{h,\Gamma}\;\leq\;C\,\|f\|_{0,\Omega}\,.$$

Proof. By Lemma 2.1 it suffices to study the system (2.17) instead of (2.12). Indeed, the ellipticity of $\mathbf{B}_h$ (cf. Lemma 3.1) implies the unique solvability of (2.17), and using additionally that $\|v\|_{0,\Omega}\;\leq\;C\,\|v\|_h\;\forall\,v\,\in\,V_h$ (see [1]), we deduce the stability estimate

$$\|(u_h,\lambda_h)\|_{h,\Gamma}\;\leq\;C\,\|f\|_{0,\Omega}\,.$$

By Lemma 2.1 we then conclude the unique solvability of (2.12). By equation (3.11) in [18] it holds that

$$\|\mathbf{S}_h(w)\|_{0,\Omega}\leq C|w|_* \qquad \forall\,w\,\in\,H^1(\mathcal{T}_h). \tag{3.6}$$

Therefore, making use of the relation $\boldsymbol{\sigma}_h\,=\,\nabla_h u_h\,-\,\mathbf{S}_h(u_h)$, we find that

$$\|\boldsymbol{\sigma}_h\|_{0,\Omega}\;\leq\;C\,\|u_h\|_h\;\leq\;C\,\|f\|_{0,\Omega}\,,$$

which finishes the proof of the theorem. □

4. A priori error analysis

In order to derive the a priori error estimate of the coupled scheme some technical results are needed. Because of (2.5) it is easy to see that

$$\|u\|_h^2\leq C\sum_{K\in\mathcal{T}_h}\Big(\|\nabla u\|_{0,K}^2+\|h_K^{-1/2}u\|_{0,\partial K}^2\Big) \qquad \forall u\in H^1(\mathcal{T}_h). \tag{4.1}$$

In what follows let $\hat{K}$ denote the reference triangle

$$\hat{K}\;:=\;\{\,(x_1,x_2):\;0\,<\,x_1\,<\,1,\;0\,<\,x_2\,<\,1-x_1\,\}\,.$$

For any $K\in\mathcal{T}_h$ we choose an invertible affine map $M_K:\hat{K}\to K$. As usual in the finite element literature, given $u:K\to\mathbb{R}$ we will denote $\hat{u}:=u\circ M_K:\hat{K}\to\mathbb{R}$.

We begin by recalling some local approximation properties. The following result rephrases [6, Lemma 4.1], which itself collects several results from [3, 4].

Lemma 4.1. *Given an integer $m \geq 1$, there exists an operator $\boldsymbol{\hat\pi} : L^2(\hat K) \to P_m(\hat K)$ such that for all $u \in H^k(K)$,*

$$\|\hat u - \boldsymbol{\hat\pi}\,\hat u\|_{q,\hat K} \;\leq\; \frac{C}{m^{k-q}}\,h^{\mu}\,\|u\|_{k,K},\;\; 0 \leq q \leq k\,, \tag{4.2}$$

$$\sup_{\hat x\in\hat K} |(\hat u - \boldsymbol{\hat\pi}\,\hat u)(\hat x)| \;\leq\; \frac{C}{m^{k-1}}\,h^{\mu}\,\|u\|_{k,K}, \quad k > 1, \tag{4.3}$$

$$\|\hat u - \boldsymbol{\hat\pi}\,\hat u\|_{s,\partial\hat K} \;\leq\; \frac{C}{m^{k-s-1/2}}\,h^{\mu}\,\|u\|_{k,K}, \quad k > 3/2, \quad s \in \{0,1\}\,, \tag{4.4}$$

where $\mu = \min\{k-1, m\}$. The constant C depends only on k.

The following lemma, whose proof below makes extensive use of the estimates (4.2)–(4.4), provides a global approximation property of the subspace $\tilde V_h$.

Lemma 4.2. *Assume that $u \in H^{1+\delta}(\Omega)$ for some $\delta > 1/2$. Then there exists $v_h \in \tilde V_h$ such that*

$$|\!|\!| u - v_h |\!|\!|_h \;+\; \|u - v_h\|_{1/2,\Gamma,0} \;\leq\; C\,h^{\min\{\delta,m\}}\,\|u\|_{1+\delta,\Omega}\,. \tag{4.5}$$

Here, $C > 0$ is a constant independent of h.

Proof. Let $\bar v_h \in V_h$ be constructed using locally the operator of Lemma 4.1. Namely, let $u_K := u|_K$ for each $K \in \mathcal{T}_h$ and, as usual, $\hat u_K := u_K \circ M_K$. Then we define $\bar v_h|_K := (\boldsymbol{\hat\pi}\hat u_K)\circ M_K^{-1}$. Taking into account the scaling properties of the norms and applying (4.2) and (4.4) we obtain

$$\begin{aligned} |u-\bar v_h|^2_{1,K} + \|h_K^{-1/2}(u-\bar v_h)\|^2_{0,\partial K} \quad &\leq \quad C\Big(|\hat u_K - \boldsymbol{\hat\pi}\hat u_K|^2_{1,K} + \|\hat u_K - \boldsymbol{\hat\pi}\hat u_K\|^2_{0,\partial\hat K}\Big) \\ &\leq \quad C' h^{2\min\{\delta,m\}}\|u\|^2_{1+\delta,K}, \end{aligned}$$

since $\delta > 1/2$. Adding the contributions of the different triangles and using (4.1) we have proved that

$$|\!|\!| u - \bar v_h |\!|\!|_h \leq C\,h^{\min\{\delta,m\}}\,\|u\|_{1+\delta,\Omega}. \tag{4.6}$$

We now correct the value of $\bar v_h$ *only on triangles with an edge on* Γ, in such a way that we construct $v_h \in \tilde V_h$ with the same order of approximation as $\bar v_h$.

The technique is standard in finite element analysis. Let $\hat P_1 := (0,0)$, $\hat P_2 := (1,0)$ and $\hat P_3 := (0,1)$ be the three vertices of $\hat K$. Consider also the functions $\hat N_1(x_1,x_2) := 1-x_1-x_2$ and $\hat N_2(x_1,x_2) := x_2$. Consider the map $\hat C : \mathcal{C}(\hat K) \to P_1(\hat K)$ given by

$$\hat C\hat u := \hat u(\hat P_1)\hat N_1 + \hat u(\hat P_2)\hat N_2$$

which yields a linear interpolant of $\hat u$ on the edge connecting $\hat P_1$ and $\hat P_2$ ($\hat e$ in Figure 1) and makes $(\hat C\hat u)(\hat P_3) = 0$. We then correct $\boldsymbol{\hat\pi}$ in the following form:

$$\hat\Pi\hat u := \boldsymbol{\hat\pi}\hat u + \hat C(\hat u - \boldsymbol{\hat\pi}\hat u) = \boldsymbol{\hat\pi}\hat u - \hat C(\boldsymbol{\hat\pi}\hat u) + \hat C\hat u, \qquad \hat u \in \mathcal{C}(\hat K).$$

Notice that $(\hat\Pi\hat u)(\hat P_j) = \hat u(\hat P_j)$ for $j = 1$ and 2, whereas $(\hat\Pi\hat u)(\hat P_3) = (\boldsymbol{\hat\pi}\hat u)(\hat P_3)$. Using (4.2), (4.3) and (4.4) we can easily prove that if $u \in H^{1+\delta}(K)$, with $\delta > 1/2$, then

$$\begin{aligned} |\hat u - \hat\Pi\hat u|_{1,\hat K} + \|\hat u - \hat\Pi u\|_{0,\partial\hat K} &\leq |\hat u - \boldsymbol{\hat\pi} u|_{1,\hat K} + \|\hat u - \boldsymbol{\hat\pi}\hat u\|_{0,\partial\hat K} \\ &\quad + C\max_{j=1,2}|\hat u(\hat P_j) - (\boldsymbol{\hat\pi}\hat u)(\hat P_j)| \\ &\leq C h^{\min\{\delta,m\}}\|u\|_{1+\delta,K}. \end{aligned} \tag{4.7}$$

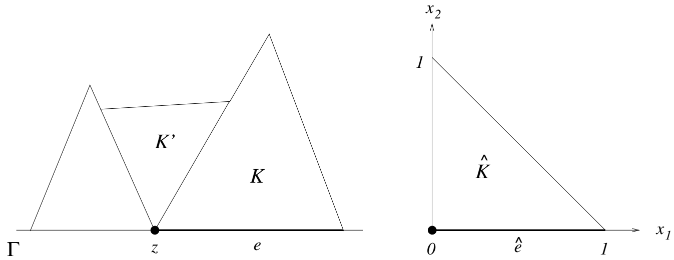

FIGURE 1. Adjusting to continuity at the boundary. The value on z of the approximation is changed for K but not for K'.

We then construct v_h as follows. If K does not have an edge on Γ, we take $v_h|_K := \bar{v}_h|_K$. If the edge $e \in \mathcal{E}(K)$ is contained in Γ, we take the map $M_K : \hat{K} \to K$ so that the side $\hat{e} := [0,1] \times \{0\}$ is mapped onto e. Then $v_h|_K := (\hat{\Pi}\hat{u}_K) \circ M_K^{-1}$. Notice that if $\boldsymbol{z}$ is one vertex of e, then $v_h(\boldsymbol{z}) = u(\boldsymbol{z})$. Therefore the restriction of v_h to the boundary is continuous (see Figure 1). The same arguments as we used for $\bar{v}_h$ together with (4.7) prove that

$$\|u - v_h\|_h \; \leq \; C\, h^{\min\{\delta,m\}}\, \|u\|_{1+\delta,\Omega}\,. \tag{4.8}$$

In order to conclude (4.5) it just remains to show that

$$\|u - v_h\|_{1/2,\Gamma,0} \; \leq \; C\, h^{\min\{\delta,m\}}\, \|u\|_{1+\delta,\Omega}\,. \tag{4.9}$$

Notice that by interpolation of norms

$$\begin{aligned} \|u - v_h\|_{1/2,\Gamma,0} &\leq \|u - v_h\|_{1/2,\Gamma} \\ &\leq \left(\sum_{e\in\mathcal{E}_h^\Gamma} \|u - v_h\|_{0,e}^2 \right)^{1/2} \left(\sum_{e\in\mathcal{E}_h^\Gamma} \|u - v_h\|_{1,e}^2 \right)^{1/2}. \end{aligned} \tag{4.10}$$

Moving to the boundary of the reference domain and again using (4.3) and (4.4) we easily prove that for $s \in \{0,1\}$,

$$\|u - v_h\|_{s,\partial K} \leq C h_K^{1/2-s} \|\hat{u}_K - \hat{\Pi}\hat{u}_K\|_{s,\partial\hat{K}} \leq C h^{\min\{\delta,m\}+1/2-s} \|u\|_{1+\delta,K}.$$

Since all the terms in the right–hand side of (4.10) can be bounded by the estimate above (take the triangle K such that $e \in \mathcal{E}(K)$), then (4.9) follows readily. $\square$

We note that defining $\bar{v}_h$ as the $L^2(\Omega)$-orthogonal projection of u onto V_h would also yield the estimate (4.6) (see Lemmas 4.2 and 4.4 in [18] for details). However, this choice of $\bar{v}_h$ does not allow the further construction of $v_h \in \tilde{V}_h$ satisfying the approximation property (4.5). This is the reason why we proceed differently and employ the local approximant provided by Lemma 4.1.

Next, we derive an approximation property for the subspace X_h^0. First let us clarify some notation. For $|t| \leq 1$ the spaces $H^t(\Gamma)$ are well defined in a classical way. Let $\{\Gamma_1, \ldots, \Gamma_N\}$ denote the edges of the polygon Γ. For $t \geq 0$, we define $H^t(\Gamma_j)$ as the space of functions that can be extended to a function in $H^t(\mathbb{R})$, after identification of Γ_j with an interval on the real line. This space is endowed with the image topology of the restriction operator. Finally, we denote $H^t_{\mathrm{prod}}(\Gamma) := \prod_j H^t(\Gamma_j)$ and denote its norm by $\|\cdot\|_{t,\mathrm{prod},\Gamma}$. Since the normal vector field is constant on each edge, it is easy to see that if $u \in H^{1+\delta}(\Omega)$, with $\delta > 1/2$, then $\lambda := \partial_{\boldsymbol{\nu}} u \in H^{\delta-1/2}_{\rm prod}(\Gamma)$ and

$$\|\lambda\|_{\delta-1/2,{\rm prod},\Gamma} \le C\, \|u\|_{1+\delta,\Omega}\,. \tag{4.11}$$

For the particular form of the precise image space of the trace and normal derivative operators on polygons see [19]. Note that for $0 < t \le 1$, $H^t(\Gamma)$ is a closed subspace of $H^t_{\rm prod}(\Gamma)$ and the injection is continuous.

Lemma 4.3. *Assume that $\lambda \in H_0^{-1/2}(\Gamma) \cap H^t_{\rm prod}(\Gamma)$ for some $t > 0$. Then there exists $\mu_h \in X_h^0$ such that*

$$\|\lambda - \mu_h\|_{-1/2,\Gamma} \;\le\; C\, h^{\min\{t,m\}+1/2}\, \|\lambda\|_{t,{\rm prod},\Gamma}\,. \tag{4.12}$$

Here, $C > 0$ is a constant independent of h.

Proof. Let μ_h be the best $L^2(\Gamma)$ approximation of λ on X_h. Since constant functions belong to X_h, it follows that if $\lambda \in H_0^{-1/2}(\Gamma) \cap L^2(\Gamma)$, then $\mu_h \in X_h^0$. Notice also that, being X_h a product of local spaces, μ_h is defined element by element. Using well-known arguments we can easily prove that

$$\|\lambda - \mu_h\|_{0,\Gamma} \le C h^{\min\{t,m\}} \|\lambda\|_{t,{\rm prod},\Gamma}.$$

Also, if $\xi \in H^1(\Gamma)$ and ξ_h is its best $L^2(\Gamma)$ approximation on X_h we have

$$|\langle \lambda - \mu_h, \xi\rangle| = |\langle \lambda - \mu_h, \xi - \xi_h\rangle| \le C h^{\min\{t,m\}+1} \|\lambda\|_{t,{\rm prod},\Gamma} \|\xi\|_{1,\Gamma}$$

and therefore

$$\|\lambda - \mu_h\|_{-1,\Gamma} \le C h^{\min\{t,m\}+1} \|\lambda\|_{t,{\rm prod},\Gamma}.$$

The result then follows by interpolation. □

The a priori error estimate for the coupled LDG-BEM scheme (2.12) can be established now.

Theorem 4.1. *Assume that $u \in H^{1+\delta}(\Omega)$ with $\delta > 1/2$. Then there exists $C > 0$, independent of h, such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \,+\, \|(u,\lambda) - (u_h,\lambda_h)\|_{h,\Gamma} \;\le\; C\, h^{\min\{\delta,m\}}\, \|u\|_{1+\delta,\Omega}\,. \tag{4.13}$$

Proof. It is not difficult to see that u and λ satisfy

$$\begin{aligned} B_h(u,v) + \langle \mathcal{W} u, v\rangle - \langle (\tfrac{1}{2}\mathcal{I} - \mathcal{K}')\lambda, v\rangle &= \int_\Omega f\, v\,, \\ \langle \mu, (\tfrac{1}{2}\mathcal{I} - \mathcal{K}) u\rangle + \langle \mu, \mathcal{V}\lambda\rangle &= 0 \end{aligned} \tag{4.14}$$

for any $(v,\mu) \in H^1_{1/2}(\mathcal{T}_h) \times H_0^{-1/2}(\Gamma)$. Using the bilinear form $\mathbf{B}_h$ (cf. (3.1)), the above means that

$$\mathbf{B}_h(u,\lambda;v,\mu) \;=\; \int_\Omega f\, v \qquad \forall\,(v,\mu) \,\in\, H^1_{1/2}(\mathcal{T}_h) \times H^{-1/2}(\Gamma)\,.$$

On the other hand, the discrete system (2.17) can be rewritten as

$$\mathbf{B}_h(u_h,\lambda_h;v,\mu) \;=\; \int_\Omega f\, v \qquad \forall\,(v,\mu) \,\in\, \tilde{V}_h \times X_h^0\,.$$

Hence, the ellipticity and continuity of the bilinear form $\mathbf{B}_h$ (cf. Lemma 3.1) imply the quasi-optimality

$$\|(u,\lambda) - (u_h,\lambda_h)\|_{h,\Gamma} \;\le\; C\, \|(u,\lambda) - (v_h,\mu_h)\|_{h,\Gamma} \qquad \forall\,(v_h,\mu_h) \,\in\, \tilde{V}_h \times X_h^0\,. \tag{4.15}$$

Also, since $\boldsymbol{\sigma} = \nabla u = \nabla u - \mathbf{S}_h(u)$ and $\boldsymbol{\sigma}_h = \nabla_h u_h - \mathbf{S}_h(u_h)$ (cf. Lemma 2.1), we obtain with (3.6) the upper bound

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \le C \, \| u - u_h \|_h \,, \tag{4.16}$$

which, together with (4.15), gives

$$\begin{aligned} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} + \|(u,\lambda) - (u_h,\lambda_h)\|_{h,\Gamma} \le C \, \|(u,\lambda) - (v_h,\mu_h)\|_{h,\Gamma} \\ \forall\, (v_h,\mu_h) \in \tilde{V}_h \times X_h^0 \,. \end{aligned} \tag{4.17}$$

Finally, applying the approximation properties from Lemmas 4.2 and 4.3 (with $t = \delta - 1/2$), using (4.11) in the latter one, and combining the resulting estimates with (4.17) we arrive at (4.13). This finishes the proof. □

We remark that the a priori error estimate (4.13) is independent of the polynomial degree r that defines the subspace $\mathbf{\Sigma}_h$ (cf. (2.6)). Hence, since the restriction $r \ge m-1$ is required only to deduce that $\boldsymbol{\sigma}_h = \nabla_h u_h - \mathbf{S}_h(u_h)$ (cf. Lemma 2.1), for practical computations it suffices to take $r = m-1$.

5. The coupled LDG-BEM scheme with Lagrangian multiplier

To implement the discrete scheme (2.12) one has to deal with the continuity condition of the space $\tilde{V}_h$. A direct implementation is possible without any difficulty. However, in order to maintain the full flexibility of the discontinuous method one can use a Lagrangian multiplier instead and work with V_h rather than $\tilde{V}_h$. The needed multiplier is simply a vector of constants. In addition, the zero mean value condition of the unknown $\lambda_h \in X_h^0$ can be dealt with similarly, whence the resulting formulation employs the subspace X_h instead of X_h^0. The description of this strategy and a simple numerical example illustrating its performance are provided in this section.

5.1. **The Lagrangian multiplier approach.**

We first notice that the bilinear form of the coupled system (2.12), which is given by

$$\begin{aligned} \mathbf{A}_h(\boldsymbol{\zeta}, w, \xi; \boldsymbol{\tau}, v, \mu) \;&:=\; \int_\Omega \boldsymbol{\zeta}\cdot\boldsymbol{\tau} \,-\, \boldsymbol{\rho}(w,\boldsymbol{\tau}) \,+\, \boldsymbol{\rho}(v,\boldsymbol{\zeta}) \,+\, \boldsymbol{\alpha}(w,v) \,+\, \langle \mathcal{W} w, v\rangle \\ &\quad - \langle (\tfrac{1}{2}\mathcal{I} - \mathcal{K}')\xi, v\rangle \,+\, \langle \mu, (\tfrac{1}{2}\mathcal{I} - \mathcal{K})w\rangle \,+\, \langle \mu, \mathcal{V}\xi\rangle \,, \end{aligned}$$

is not well defined on $\mathbf{\Sigma}_h \times V_h \times X_h$. For instance, the correct definition of the bilinear form $\langle \mathcal{W} w, v\rangle$ requires that $w|_\Gamma, v|_\Gamma \in H^{1/2}(\Gamma)$. This is in general not true for $w, v \in V_h$. Therefore, we consider instead the bilinear form

$$\begin{aligned} \tilde{\mathbf{A}}_h(\boldsymbol{\zeta}, w, \xi; \boldsymbol{\tau}, v, \mu) \;&:=\; \int_\Omega \boldsymbol{\zeta}\cdot\boldsymbol{\tau} \,-\, \boldsymbol{\rho}(w,\boldsymbol{\tau}) \,+\, \boldsymbol{\rho}(v,\boldsymbol{\zeta}) \,+\, \boldsymbol{\alpha}(w,v) \,+\, \langle \partial_h w, \mathcal{V}\partial_h v\rangle \\ &\quad - \langle (\tfrac{1}{2}\mathcal{I} - \mathcal{K}')\xi, v\rangle \,+\, \langle \mu, (\tfrac{1}{2}\mathcal{I} - \mathcal{K})w\rangle \,+\, \langle \mu, \mathcal{V}\xi\rangle \,. \end{aligned}$$

Here, $\partial_h w$ is defined piecewise by $\partial_h w|_e = (w|_e)'$ for any edge $e \in \Gamma_h$, and $(w|_e)'$ denotes the derivative of w on e with respect to the arc length. Note that $\partial_h w \in L^2(\Gamma)$ for any $w \in V_h$. Then the updated bilinear form $\langle \partial_h w, \mathcal{V}\partial_h v\rangle$ is well defined for $w, v \in V_h$ and it holds that

$$\langle \mathcal{W} w, v\rangle \,=\, \langle \partial_h w, \mathcal{V}\partial_h v\rangle \qquad \forall\, w, v \in \tilde{V}_h$$

(see [25]). Notice that the angled bracket in $\langle \mu, (\frac{1}{2}\mathcal{I} - \mathcal{K})w\rangle$ is simply the $L^2(\Gamma)$ product and the term $(\frac{1}{2}\mathcal{I}-\mathcal{K})w$ fails to be in $H^{1/2}(\Gamma)$ (it remains in $L^2(\Gamma)$ however), which is compensated, at this discrete level, by the fact that $\mu \in L^2(\Gamma)$. Therefore,

$$\mathbf{A}_h(\boldsymbol{\zeta}, w, \xi; \boldsymbol{\tau}, v, \mu) \,=\, \tilde{\mathbf{A}}_h(\boldsymbol{\zeta}, w, \xi; \boldsymbol{\tau}, v, \mu) \quad \forall\, (\boldsymbol{\zeta}, w, \xi),\, (\boldsymbol{\tau}, v, \mu) \,\in\, \boldsymbol{\Sigma}_h \times \tilde{V}_h \times X_h\,.$$

Now, let $\{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n\}$ denote the nodes of $\mathcal{T}_h$ on Γ which belong to at least two triangles, and let e_i^- and e_i^+ denote the two elements of Γ_h which have $\boldsymbol{z}_i$ as a common node. The continuity of u_h on Γ is enforced through the equation

$$\sum_{i=1}^{n} \Big(u_h|_{e_i^+}(\boldsymbol{z}_i) - u_h|_{e_i^-}(\boldsymbol{z}_i) \Big)\, y_i \,=\, 0 \qquad \forall\, (y_1, \ldots, y_n) \in \mathbb{R}^n\,.$$

Similarly, the zero mean value condition of λ_h is imposed as

$$y_{n+1} \int_\Gamma \lambda_h \,=\, 0 \qquad \forall\, y_{n+1} \,\in\, \mathbb{R}\,.$$

The above then suggests to define the bilinear form

$$b_h((v,\mu), \vec{\mathbf{y}}) := \sum_{i=1}^{n} \Big(v|_{e_i^+}(\boldsymbol{z}_i) - v|_{e_i^-}(\boldsymbol{z}_i) \Big)\, y_i \;+\; y_{n+1} \int_\Gamma \mu \tag{5.1}$$

for $(v,\mu) \in V_h \times X_h\,,\ \vec{\mathbf{y}} \,=\, (y_1, \ldots, y_{n+1}) \in \mathbb{R}^{n+1}$, and to consider the following LDG-BEM scheme with Lagrangian multiplier $\vec{\boldsymbol{x}}$: Find $(\boldsymbol{\sigma}_h, u_h, \lambda_h, \vec{\boldsymbol{x}}) \,\in\, \boldsymbol{\Sigma}_h \times V_h \times X_h \times \mathbb{R}^{n+1}$ such that

$$\begin{aligned} \tilde{\mathbf{A}}_h(\boldsymbol{\sigma}_h, u_h, \lambda_h; \boldsymbol{\tau}, v, \mu) + b_h((v,\mu), \vec{\boldsymbol{x}}) &= \int_\Omega f\, v\,, \\ b_h((u_h,\lambda_h), \vec{\mathbf{y}}) &= 0 \end{aligned} \tag{5.2}$$

for any $(\boldsymbol{\tau}, v, \mu, \vec{\mathbf{y}}) \,\in\, \boldsymbol{\Sigma}_h \times V_h \times X_h \times \mathbb{R}^{n+1}$. Then, we have the following result.

Theorem 5.1. *There exists a unique solution* $(\boldsymbol{\sigma}_h, u_h, \lambda_h, \vec{\boldsymbol{x}}) \,\in\, \boldsymbol{\Sigma}_h \times V_h \times X_h \times \mathbb{R}^{n+1}$ *of* (5.2) *and* $(\boldsymbol{\sigma}_h, u_h, \lambda_h)$ *solves* (2.12)*. In particular, the error estimate from Theorem* 4.1 *holds.*

Proof. It is immediate that there holds a (nonuniform) inf-sup condition for b_h:

$$\sup_{(v,\mu)\in V_h\times X_h} b_h((v,\mu), \vec{\mathbf{y}}) \,>\, 0 \qquad \forall\, \vec{\mathbf{y}} \,\in\, \mathbb{R}^{n+1}\,, \quad \vec{\mathbf{y}} \,\neq\, 0\,.$$

We also have that the discrete null space of b_h is given by

$$\tilde{V}_h \times X_h^0 \,=\, \{(v,\mu) \in V_h \times X_h : \quad b_h((v,\mu), \vec{\mathbf{y}}) \,=\, 0 \quad \forall\, \vec{\mathbf{y}} \,\in\, \mathbb{R}^{n+1}\}\,.$$

Therefore, Theorem 3.1 and the Babuška-Brezzi theory for discrete problems ensure the unique solvability of (5.2) and then $(\boldsymbol{\sigma}_h, u_h, \lambda_h) \in \boldsymbol{\Sigma}_h \times \tilde{V}_h \times X_h^0$ becomes the unique solution of (2.12), whence the error estimate of Theorem 4.1 holds. □

5.2. **A numerical example.** In this section we present a simple numerical example illustrating the performance of (5.2) with $m = 1$ and $r = m - 1 = 0$. This means, according to (2.6) and (2.11), that

$$\boldsymbol{\Sigma}_h \,:=\, \prod_{K\in\mathcal{T}_h} \mathbf{P}_0(K)\,, \quad V_h \,:=\, \prod_{K\in\mathcal{T}_h} P_1(K)\,, \quad \text{and} \quad X_h \,:=\, \prod_{e\in\mathcal{E}_h^\Gamma} P_0(e)\,.$$

In this case, as established by Theorem 4.1 with $\delta = 1$, for a continuous solution $u \in H^2(\Omega)$ there holds a rate of convergence of $O(h)$.

TABLE 1. Degrees of freedom, meshsizes, errors, and rates of convergence

N	h	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(u)$	$\mathtt{r}(u)$	$\mathtt{e}(\lambda)$	$\mathtt{r}(\lambda)$
54	0.50000	0.4053E+00	–	0.1714E+01	–	0.3269E+01	–
112	0.33333	0.2596E+00	1.098	0.1246E+01	0.786	0.2345E+01	0.819
190	0.25000	0.1871E+00	1.139	0.9307E+00	1.014	0.1794E+01	0.931
288	0.20000	0.1464E+00	1.099	0.7667E+00	0.869	0.1419E+01	1.051
406	0.16667	0.1203E+00	1.077	0.6530E+00	0.881	0.1171E+01	1.054
544	0.14286	0.1023E+00	1.051	0.5678E+00	0.907	0.9979E+00	1.038
702	0.12500	0.8898E-01	1.045	0.5021E+00	0.921	0.8699E+00	1.028
880	0.11111	0.7870E-01	1.042	0.4500E+00	0.930	0.7713E+00	1.021
1078	0.10000	0.7054E-01	1.039	0.4077E+00	0.937	0.6928E+00	1.019
1296	0.09091	0.6391E-01	1.036	0.3727E+00	0.942	0.6290E+00	1.014
1534	0.08333	0.5842E-01	1.032	0.3431E+00	0.951	0.5760E+00	1.011
1792	0.07692	0.5379E-01	1.032	0.3179E+00	0.953	0.5313E+00	1.009
2070	0.07143	0.4984E-01	1.030	0.2962E+00	0.955	0.4932E+00	1.005

We now describe the example. We consider a slightly simplified model with no interior region Ω_0, take $\Omega =]0,1[^2$, and choose the data so that the exact solution is given by

$$u(\mathbf{x}) \,=\, x_1^2 \,+\, x_2^2 \qquad \forall\, \mathbf{x} := (x_1, x_2)^{\mathtt{t}} \,\in\, \Omega$$

and

$$u_e(\mathbf{x}) \,=\, \frac{x_1 + x_2 - 1}{(x_1 - 0.5)^2 \,+\, (x_2 - 0.5)^2} \qquad \forall\, \mathbf{x} := (x_1, x_2)^{\mathtt{t}} \,\in\, \Omega_e\,.$$

Since u and u_e do not coincide on $\Gamma := \partial\Omega$, we need to allow for nonhomogeneous transmission conditions, which means replacing (1.4) by

$$(5.3)\quad u \,-\, u_e \,=\, g_0 \,\in\, H^{1/2}(\Gamma) \quad \text{on } \Gamma \quad \text{and} \quad \partial_{\boldsymbol{\nu}} u \,-\, \partial_{\boldsymbol{\nu}} u_e \,=\, g_1 \,\in\, L^2(\Gamma) \quad \text{on } \Gamma\,.$$

Note here that the smoother assumption on g_1 is required to be able to introduce the function $\lambda_h + g_1$ in the LDG module (2.8) as the suitable $L^2(\Gamma)$ approximation of the normal derivative $\partial_{\boldsymbol{\nu}} u$ on Γ. In this case without interior Dirichlet boundary Γ_0 one finds that $\int_\Gamma \lambda_h = 0$ is automatically satisfied (choose $(\boldsymbol{\tau}, v, \mu) = (0, 1, 0)$ in (5.2) and make use of the relation $\mathcal{K}1 = -1/2$). We therefore use, instead of the bilinear form b_h defined by (5.1), the reduced form

$$\tilde{b}_h((v,\mu), \vec{\mathbf{y}}) := \sum_{i=1}^{n} \Big(v|_{e_i^+}(\boldsymbol{z}_i) - v|_{e_i^-}(\boldsymbol{z}_i) \Big)\, y_i$$

for $(v,\mu) \in V_h \times X_h\,,\ \vec{\mathbf{y}} \,=\, (y_1, \ldots, y_n) \in \mathbb{R}^n$. As a consequence, the scheme (5.2) becomes: Find $(\boldsymbol{\sigma}_h, u_h, \lambda_h, \vec{\boldsymbol{x}}) \,\in\, \boldsymbol{\Sigma}_h \times V_h \times X_h \times \mathbb{R}^n$ such that

$$(5.4)\qquad \begin{aligned} \tilde{\mathbf{A}}_h(\boldsymbol{\sigma}_h, u_h, \lambda_h; \boldsymbol{\tau}, v, \mu) + \tilde{b}_h((v,\mu), \vec{\boldsymbol{x}}) &= F(v,\mu)\,, \\ \tilde{b}_h((u_h, \lambda_h), \vec{\mathbf{y}}) &= 0 \end{aligned}$$

for any $(\boldsymbol{\tau}, v, \mu, \vec{\mathbf{y}}) \,\in\, \boldsymbol{\Sigma}_h \times V_h \times X_h \times \mathbb{R}^n$, where

$$F(v,\mu) \,:=\, \int_\Omega f\, v \,+\, \int_\Gamma g_1\, v \,+\, \langle \partial_h g_0, \mathcal{V}\partial_h v\rangle \,+\, \langle \mu, (\frac{1}{2} - \mathcal{K}) g_0\rangle\,.$$

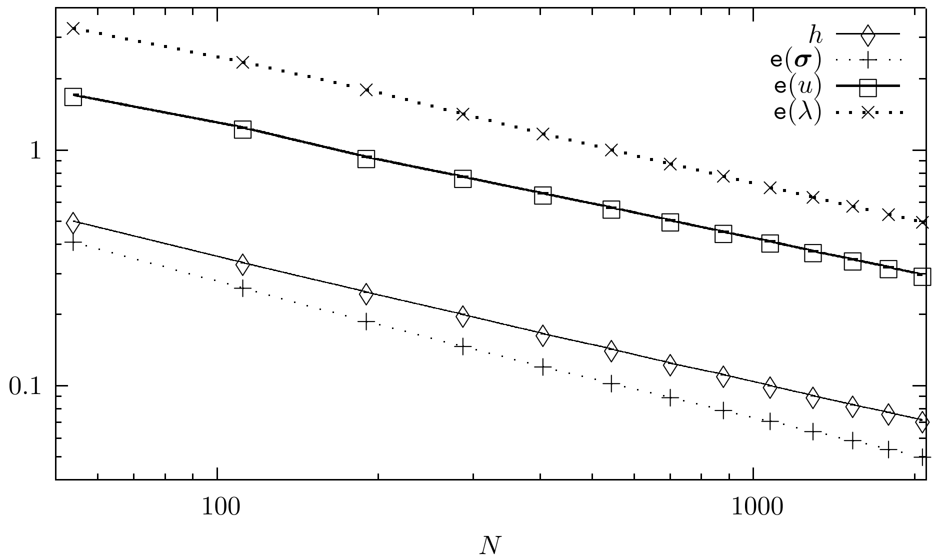

FIGURE 2. meshsize h and errors vs. degrees of freedom N

The numerical results shown below were obtained using a Fortran implementation. In what follows the variable N stands for the number of degrees of freedom defining $\boldsymbol{\Sigma}_h$, V_h, X_h, and $\mathbb{R}^n$, and the individual errors are denoted by

$$\mathsf{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega},$$

$$\mathsf{e}(u) := \left\{ |\!|\!| u - u_h |\!|\!|_h^2 \,+\, \|u - u_h\|_{0,\Gamma} |u - u_h|_{1,\Gamma} \right\}^{1/2},$$

$$\mathsf{e}(\lambda) = \|\lambda - \lambda_h\|_{0,\Gamma}.$$

By interpolation, $\|u - u_h\|_{0,\Gamma}|u - u_h|_{1,\Gamma}$ is an upper bound for $\|u - u_h\|^2_{1/2,\Gamma,0}$, and $\mathsf{e}(\lambda)$ is an upper bound for $\|\lambda - \lambda_h\|_{-1/2,\Gamma}$. Presenting the errors $\mathsf{e}(\boldsymbol{\sigma})$, $\mathsf{e}(u)$ and $\mathsf{e}(\lambda)$ is therefore sufficient to verify the a priori error estimate given by Theorem 4.1.

Also, we let $\mathtt{r}(\boldsymbol{\sigma})$, $\mathtt{r}(u)$, and $\mathtt{r}(\lambda)$ be the experimental rates of convergence given by

$$\mathtt{r}(\boldsymbol{\sigma}) := \frac{\log(\mathsf{e}(\boldsymbol{\sigma})/\mathsf{e}'(\boldsymbol{\sigma}))}{\log(h/h')},$$

$$\mathtt{r}(u) := \frac{\log(\mathsf{e}(u)/\mathsf{e}'(u))}{\log(h/h')},$$

$$\mathtt{r}(\lambda) := \frac{\log(\mathsf{e}(\lambda)/\mathsf{e}'(\lambda))}{\log(h/h')},$$

where h and h' denote two consecutive meshsizes with errors e and e'.

In Table 1 we present the convergence history of the example for a set of uniform triangulations of the computational domain $\Omega \cup \Gamma$. A rate of convergence $O(h)$ is attained by all the unknowns and this confirms the error estimate by Theorem 4.1. The dominant error is given by $\mathsf{e}(\lambda)$ which, being measured in the $L^2(\Gamma)$-norm, overestimates the error $\|\lambda - \lambda_h\|_{-1/2,\Gamma}$ but confirms the convergence like $O(h)$.

The experimental rates of convergence and the dominant component of the error can also be checked from Figure 2 where we display the meshsize h and the errors $\mathsf{e}(\boldsymbol{\sigma})$, $\mathsf{e}(u)$, and $\mathsf{e}(\lambda)$ vs. the degrees of freedom N.

We end this section by remarking that the same results are obtained by implementing the continuity on Γ for the functions in $\tilde{V}_h$ and then solving the original coupled LDG-BEM scheme (2.12) instead of (5.2) ((5.4) in this case). In addition, similar results and the same rate of convergence $O(h)$ are obtained with the mortar-based coupling scheme from [18]. In this respect we emphasize again that the present approach is computationally appealing since the number of unknown functions is reduced by two while only standard LDG (and BEM) discretizations are needed when using the Lagrangian multiplier.

6. Extension to nonlinear problems

In this section we extend the present LDG-BEM approach to the class of nonlinear exterior transmission problems studied in [9], [10], and [11]. In order to describe the model problem let Ω_0 be a simply connected and bounded domain in $\mathbb{R}^2$ with polygonal boundary Γ_0. Then, let Ω_1 be an annular and simply connected domain surrounded by Γ_0 and another polygonal boundary Γ_1. In addition, let $\mathbf{a} : \Omega_1 \times \mathbb{R}^2 \to \mathbb{R}^2$ be a nonlinear function satisfying the conditions specified in [7] (see also [9]) which, in particular, imply that the associated operator becomes Lipschitz continuous and strongly monotone. Thus, given $f \in L^2(\mathbb{R}^2 \setminus \bar{\Omega}_0)$ with compact support, $g_0 \in H^{1/2}(\Gamma_0)$, $g_1 \in H^{1/2}(\Gamma_1)$, and $g_2 \in L^2(\Gamma_1)$, we consider the nonlinear exterior transmission problem:

(6.1)
$$-\mathrm{div}\; \mathbf{a}(\cdot, \nabla u_1) \,=\, f \quad \text{in} \quad \Omega_1, \quad u_1 \,=\, g_0 \quad \text{on} \quad \Gamma_0,$$
$$-\Delta u_2 \,=\, f \quad \text{in} \quad \mathbb{R}^2 \setminus (\bar{\Omega}_0 \cup \bar{\Omega}_1), \quad u_1 - u_2 \,=\, g_1 \quad \text{on} \quad \Gamma_1,$$
$$\mathbf{a}(\cdot, \nabla u_1) \cdot \boldsymbol{\nu}_1 - \nabla u_2 \cdot \boldsymbol{\nu}_1 \,=\, g_2 \quad \text{on} \quad \Gamma_1, \quad \text{and} \quad u_2(\boldsymbol{x}) = \mathcal{O}(1) \quad \text{as} \quad |\boldsymbol{x}| \to \infty\,.$$

Here, $\boldsymbol{\nu}_1$ stands for the unit outward normal to Γ_1. This kind of problems appears in the computation of magnetic fields of electromagnetic devices (see, e.g. [21], [22]), in some subsonic flow and fluid mechanics problems (see, e.g. [16], [17]), and also in steady state heat conduction. For instance, in the latter case, one has $\mathbf{a}(\boldsymbol{x}, \nabla u(\boldsymbol{x})) = k(\boldsymbol{x}, \nabla u(\boldsymbol{x}))\, \nabla u$, where u is the temperature and $k : \Omega_1 \times \mathbb{R}^2 \to \mathbb{R}$ is the heat conductivity.

Next, we introduce a closed polygonal curve Γ such that its interior contains the support of f. Then, let Ω_2 be the annular domain bounded by Γ_1 and Γ and set $\Omega_e := \mathbb{R}^2 \setminus (\bar{\Omega}_0 \cup \bar{\Omega}_1 \cup \bar{\Omega}_2)$ (see Figure 3 below). It follows that (6.1) can be equivalently rewritten as the nonlinear boundary value problem in Ω_1:

$$-\mathrm{div}\, \mathbf{a}(\cdot, \nabla u_1) \,=\, f \quad \text{in} \quad \Omega_1, \quad u_1 \,=\, g_0 \quad \text{on} \quad \Gamma_0\,, \tag{6.2}$$

the Poisson equation in Ω_2:

$$-\Delta u_2 \,=\, f \quad \text{in} \quad \Omega_2\,, \tag{6.3}$$

and the Laplace equation in the exterior unbounded region Ω_e:

$$-\Delta u_2 \,=\, 0 \quad \text{in} \quad \Omega_e, \quad u_2(\boldsymbol{x}) = \mathcal{O}(1) \quad \text{as} \quad |\boldsymbol{x}| \to \infty\,, \tag{6.4}$$

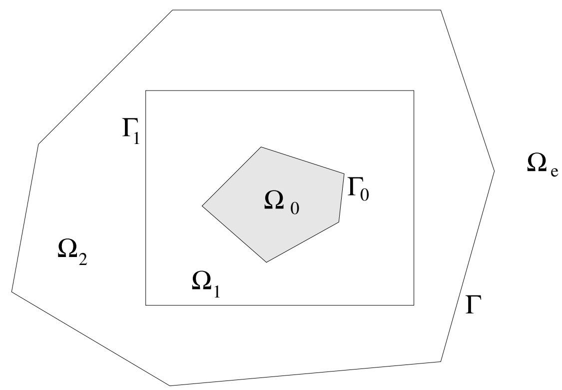

FIGURE 3. Geometry of the transmission problem.

coupled with the transmission conditions on Γ_1 and Γ, respectively,

$$u_1 - u_2 = g_1 \quad \text{and} \quad \mathbf{a}(\cdot, \nabla u_1) \cdot \boldsymbol{\nu}_1 - \nabla u_2 \cdot \boldsymbol{\nu}_1 = g_2 \quad \text{on} \quad \Gamma_1, \tag{6.5}$$

and

$$\lim_{\substack{\boldsymbol{x} \to \boldsymbol{x}_0 \\ \boldsymbol{x} \in \Omega_2}} u_2(\boldsymbol{x}) = \lim_{\substack{\boldsymbol{x} \to \boldsymbol{x}_0 \\ \boldsymbol{x} \in \Omega_e}} u_2(\boldsymbol{x}) \quad \text{and} \quad \lim_{\substack{\boldsymbol{x} \to \boldsymbol{x}_0 \\ \boldsymbol{x} \in \Omega_2}} \nabla u_2(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}_0) = \lim_{\substack{\boldsymbol{x} \to \boldsymbol{x}_0 \\ \boldsymbol{x} \in \Omega_e}} \nabla u_2(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}_0) \tag{6.6}$$

for almost all $\boldsymbol{x}_0 \in \Gamma$, where $\boldsymbol{\nu}(\boldsymbol{x}_0)$ denotes the unit outward normal to $\boldsymbol{x}_0$.

We now follow [18] and [9] and introduce the gradients $\boldsymbol{\theta}_1 := \nabla u_1$ in Ω_1 and $\boldsymbol{\theta}_2 := \nabla u_2$ in Ω_2, and the fluxes $\boldsymbol{\sigma}_1 := \mathbf{a}(\cdot, \boldsymbol{\theta}_1)$ in Ω_1 and $\boldsymbol{\sigma}_2 := \boldsymbol{\theta}_2$ in Ω_2, as additional unknowns. Also, as in Section 2, let $\lambda_h \in X_h^0$ be a discrete approximation of the normal derivative $\lambda := \partial_{\boldsymbol{\nu}} u_2$ on Γ, and proceeding in the usual way (see [9] for details). We arrive at the following global LDG formulation in $\Omega := \Omega_1 \cup \Gamma_1 \cup \Omega_2$: *Find* $(\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h, u_h) \in \boldsymbol{\Sigma}_h \times \boldsymbol{\Sigma}_h \times \tilde{V}_h$ *such that*

$$\begin{aligned}
\int_\Omega \bar{\mathbf{a}}(\cdot, \boldsymbol{\theta}_h) \cdot \boldsymbol{\zeta} - \int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\zeta} &= 0 && \forall\, \boldsymbol{\zeta} \in \boldsymbol{\Sigma}_h, \\
\int_\Omega \boldsymbol{\theta}_h \cdot \boldsymbol{\tau} - \left\{ \int_\Omega \nabla_h u_h \cdot \boldsymbol{\tau} - S(u_h, \boldsymbol{\tau}) \right\} &= G_h(\boldsymbol{\tau}) && \forall\, \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h, \\
\left\{ \int_\Omega \nabla_h v \cdot \boldsymbol{\sigma}_h - S(v, \boldsymbol{\sigma}_h) \right\} + \boldsymbol{\alpha}(u_h, v) &= F_h(v) + \int_\Gamma \lambda_h\, v && \forall\, v \in \tilde{V}_h,
\end{aligned} \tag{6.7}$$

where

$$\bar{\mathbf{a}}(\cdot, \boldsymbol{\zeta}) := \begin{cases} \mathbf{a}(\cdot, \boldsymbol{\zeta}) & \text{in } \Omega_1 \\ \boldsymbol{\zeta} & \text{in } \Omega_2 \end{cases} \qquad \forall\, \boldsymbol{\zeta} \in [L^2(\Omega)]^2,$$

and the bilinear forms $S : H^1(\mathcal{T}_h) \times \boldsymbol{L}^2(\Omega) \to \mathbb{R}$ and $\boldsymbol{\alpha} : H^1(\mathcal{T}_h) \times H^1(\mathcal{T}_h) \to \mathbb{R}$ as well as the linear operators $G_h : \boldsymbol{L}^2(\Omega) \to \mathbb{R}$ and $F_h : H^1(\mathcal{T}_h) \to \mathbb{R}$ are defined by

$$S(w, \boldsymbol{\tau}) := \int_{I_h} [\![w]\!] \cdot (\{\boldsymbol{\tau}\} - [\![\boldsymbol{\tau}]\!]\,\boldsymbol{\beta}) + \int_{\Gamma_0} w\,(\boldsymbol{\tau}_1 \cdot \boldsymbol{\nu}) + \int_{\Gamma_1} (w_1 - w_2)\,\boldsymbol{\tau}_1 \cdot \boldsymbol{\nu}_1\,,$$

$$\boldsymbol{\alpha}(w, v) := \int_{I_h} \alpha\,[\![w]\!] \cdot [\![v]\!] + \int_{\Gamma_0} \alpha\,w\,v + \int_{\Gamma_1} \alpha\,(w_1 - w_2)\,(v_1 - v_2)\,,$$

$$G_h(\boldsymbol{\tau}) := \int_{\Gamma_0} g_0\,\boldsymbol{\tau}_1 \cdot \boldsymbol{\nu} + \int_{\Gamma_1} g_1\,\boldsymbol{\tau}_1 \cdot \boldsymbol{\nu}_1\,,$$

and

$$F_h(v) := \int_\Omega f\,v + \int_{\Gamma_0} \alpha\,g_0\,v_1 + \int_{\Gamma_1} \alpha\,g_1\,(v_1 - v_2) + \int_{\Gamma_1} g_2\,v_2$$

for all $w\,, v \in H^1(\mathcal{T}_h)$, $\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)$, with $w_i := w|_{\Omega_i}$, $v_i := v|_{\Omega_i}$, and $\boldsymbol{\tau}_i := \boldsymbol{\tau}|_{\Omega_i}$, for each $i \in \{1, 2\}$. Hereafter, $\mathcal{T}_h = \mathcal{T}_{h,1} \cup \mathcal{T}_{h,2}$, where $\mathcal{T}_{h,1}$ and $\mathcal{T}_{h,2}$ are shape regular triangulations of $\bar{\Omega}_1$ and $\bar{\Omega}_2$, respectively, which satisfy the same properties and assumptions as indicated in Section 2.2.

Next, introducing the boundary integral formulation in Ω_e, exactly as in Section 2.1, substituting λ_h in (6.7) by a discrete version of the first equation in (2.2), in which u is replaced by its approximant u_h, and adding a discrete formulation of the second equation in (2.2), we obtain the following coupled LDG-BEM scheme: Find $(\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h, u_h, \lambda_h) \in \boldsymbol{\Sigma}_h \times \boldsymbol{\Sigma}_h \times \tilde{V}_h \times X_h^0$ such that

$$\begin{aligned} \int_\Omega \bar{\mathbf{a}}(\cdot, \boldsymbol{\theta}_h) \cdot \boldsymbol{\zeta} - \int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\zeta} &= 0\,, \\ \int_\Omega \boldsymbol{\theta}_h \cdot \boldsymbol{\tau} - \boldsymbol{\rho}(u_h, \boldsymbol{\tau}) &= G_h(\boldsymbol{\tau})\,, \\ \boldsymbol{\rho}(v, \boldsymbol{\sigma}_h) + \boldsymbol{\alpha}(u_h, v) + \langle \mathcal{W} u_h, v\rangle - \langle (\tfrac{1}{2}\mathcal{I} - \mathcal{K}')\lambda_h, v\rangle &= F_h(v)\,, \\ \langle \mu, (\tfrac{1}{2}\mathcal{I} - \mathcal{K})u_h\rangle + \langle \mu, \mathcal{V}\lambda_h\rangle &= 0 \end{aligned} \tag{6.8}$$

for all $(\boldsymbol{\zeta}, \boldsymbol{\tau}, v, \mu) \in \boldsymbol{\Sigma}_h \times \boldsymbol{\Sigma}_h \times \tilde{V}_h \times X_h^0$, where $\boldsymbol{\rho} : H^1(\mathcal{T}_h) \times \mathbf{H}^1(\mathcal{T}_h) \to \mathbb{R}$ is the analogue of the bilinear form defined by (2.13); that is,

$$\boldsymbol{\rho}(v, \boldsymbol{\tau}) := \int_\Omega \nabla_h v \cdot \boldsymbol{\tau} - S(v, \boldsymbol{\tau}) \qquad \forall\,(v, \boldsymbol{\tau}) \in H^1(\mathcal{T}_h) \times \mathbf{H}^1(\mathcal{T}_h)\,.$$

In what follows we proceed as in Section 2.3 (see also Section 2.4 of [18]) and derive an equivalent formulation to (6.8). We begin by defining a linear operator $\mathbf{S}_h : H^1(\mathcal{T}_h) \to \boldsymbol{\Sigma}_h$ as in (2.15), where, given $v \in H^1(\mathcal{T}_h)$, $\mathbf{S}_h(v)$ is the unique element in $\boldsymbol{\Sigma}_h$ such that

$$\int_\Omega \mathbf{S}_h(v) \cdot \boldsymbol{\tau} = S(v, \boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau} \in \boldsymbol{\Sigma}_h\,. \tag{6.9}$$

Next, let $\mathcal{G}_h$ be the unique element in $\boldsymbol{\Sigma}_h$ such that

$$\int_\Omega \mathcal{G}_h \cdot \boldsymbol{\tau} = G_h(\boldsymbol{\tau}) := \int_{\Gamma_0} g_0\,\boldsymbol{\tau}_1 \cdot \boldsymbol{\nu} + \int_{\Gamma_1} g_1\,\boldsymbol{\tau}_1 \cdot \boldsymbol{\nu}_1 \qquad \forall\,\boldsymbol{\tau} \in \boldsymbol{\Sigma}_h\,. \tag{6.10}$$

It is easy to see that $\mathcal{G}_h\big|_{\Omega_2} = \mathbf{0}$. From now on we set

$$u := \begin{cases} u_1 \text{ in } \Omega_1, \\ u_2 \text{ in } \Omega_2. \end{cases}$$

Then, if the solution of problem (6.1) satisfies $u_1 \in H^t(\Omega_1)$ and $u_2 \in H^s(\Omega_2)$, with $t, s > 1$, we find that $\mathbf{S}_h(u) = \mathcal{G}_h$. In addition, it follows from the first two equations in (6.8) that, whenever this system is solvable, it holds that

$$\boldsymbol{\theta}_h = \nabla_h u_h - \mathbf{S}_h(u_h) + \mathcal{G}_h \quad \text{and} \quad \boldsymbol{\sigma}_h = \Pi_{\boldsymbol{\Sigma}_h} \bar{\mathbf{a}}(\cdot, \boldsymbol{\theta}_h)\,, \tag{6.11}$$

where $\Pi_{\boldsymbol{\Sigma}_h}$ denotes the $\boldsymbol{L}^2(\Omega)$-orthogonal projection onto $\boldsymbol{\Sigma}_h$. We observe here, as in the proof of Lemma 2.1, that the fact that $r \geq m-1$ guarantees that $\nabla_h u_h \in \boldsymbol{\Sigma}_h$, which yields the above expression for $\boldsymbol{\theta}_h$. Then, replacing the unknown $\boldsymbol{\sigma}_h$ by

$$\Pi_{\boldsymbol{\Sigma}_h} \bar{\mathbf{a}}(\cdot, \nabla_h u_h - \mathbf{S}_h(u_h) + \mathcal{G}_h)$$

in the third equation of (6.8), we are led to the semilinear form $A_h : H^1(\mathcal{T}_h) \times H^1(\mathcal{T}_h) \to \mathbb{R}$ defined by

$$A_h(w, v) := \boldsymbol{\alpha}(w, v) + \int_\Omega \bar{\mathbf{a}}(\cdot, \nabla_h w - \mathbf{S}_h(w) + \mathcal{G}_h) \cdot (\nabla_h v - \mathbf{S}_h(v)) \quad \forall\, w, v \in H^1(\mathcal{T}_h)\,.$$

Moreover, we can establish the following equivalence result which constitutes the nonlinear analogue of Lemma 2.1.

Lemma 6.1. *Let $(\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h, u_h, \lambda_h) \in \boldsymbol{\Sigma}_h \times \boldsymbol{\Sigma}_h \times \tilde{V}_h \times X_h^0$ be a solution of* (6.8). *Then it holds that*

$$\begin{aligned} A_h(u_h, v) \;+\; \langle \mathcal{W} u_h, v\rangle \;-\; \langle (\tfrac{1}{2}\mathcal{I} - \mathcal{K}')\lambda_h, v\rangle &= F_h(v)\,, \\ \langle \mu, (\tfrac{1}{2}\mathcal{I} - \mathcal{K})u_h\rangle \;+\; \langle \mu, \mathcal{V}\lambda_h\rangle &= 0 \end{aligned} \tag{6.12}$$

for any $(v, \mu) \in \tilde{V}_h \times X_h^0$. Conversely, if $(u_h, \lambda_h) \in \tilde{V}_h \times X_h^0$ satisfies (6.12) *and $\boldsymbol{\theta}_h$ and $\boldsymbol{\sigma}_h$ are defined by* (6.11), *then $(\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h, u_h, \lambda_h)$ is a solution of* (6.8). *If $(u_h, \lambda_h) \in \tilde{V}_h \times X_h^0$ is the only solution of* (6.12), *then $(\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h, u_h, \lambda_h)$, with $\boldsymbol{\theta}_h$ and $\boldsymbol{\sigma}_h$ defined as indicated above, is the only solution of* (6.8).

Proof. It is similar to the proof of Lemma 2.1 (see also Lemma 2.2 in [18]) and is based on the identities (6.11). $\square$

We now introduce seminorms

$$|v|_{1,h}^2 := \|\nabla_h v\|_{0,\Omega}^2\,, \quad |v|_*^2 := \|h_{\mathcal{E}}^{-1/2} [\![v]\!]\|_{0,I_h}^2 + \|h_{\mathcal{E}}^{-1/2} v\|_{0,\Gamma_0}^2 + \|h_{\mathcal{E}}^{-1/2}(v_1 - v_2)\|_{0,\Gamma_1}^2 \quad \forall\, v \in H^1(\mathcal{T}_h),$$

and the norms

$$\begin{aligned} |\!|\!| v |\!|\!|_h^2 &:= |v|_{1,h}^2 + |v|_*^2 \qquad \forall\, v \in H^1(\mathcal{T}_h)\,, \\ \|(v, \mu)\|_{h,\Gamma}^2 &:= |\!|\!| v |\!|\!|_h^2 \;+\; \|v\|_{1/2,\Gamma,0}^2 \;+\; \|\mu\|_{-1/2,\Gamma}^2 \\ &\qquad \forall\, (v, \mu) \in H_{1/2}^1(\mathcal{T}_h) \times H_0^{-1/2}(\Gamma)\,. \end{aligned}$$

Next, let $\mathbf{A}_h$ be the semilinear form defined by the left-hand side of (6.12), i.e.,

$$\mathbf{A}_h(w, \eta; v, \mu) := A_h(w, v) + \langle \mathcal{W} w, v\rangle - \langle (\tfrac{1}{2}\mathcal{I} - \mathcal{K}')\eta, v\rangle + \langle \mu, (\tfrac{1}{2}\mathcal{I} - \mathcal{K})w\rangle + \langle \mu, \mathcal{V}\eta\rangle$$

for any (w,η), $(v,\mu) \in H^1_{1/2}(\mathcal{T}_h) \times H_0^{-1/2}(\Gamma)$. The following result shows that $\mathbf{A}_h$ is Lipschitz continuous and strongly monotone with respect to $\|\cdot\|_{h,\Gamma}$. This is crucial for the analysis of (6.12) (and hence of (6.8)).

Lemma 6.2. *There exist positive constants C_{LM} and C_{SM}, independent of h, such that*

$$|\mathbf{A}_h(w,\eta;z,\xi) - \mathbf{A}_h(v,\mu;z,\xi)| \le C_{LM}\, \|(w,\eta)-(v,\mu)\|_{h,\Gamma}\, \|(z,\xi)\|_{h,\Gamma} \tag{6.13}$$

and

$$\mathbf{A}_h(w,\eta;(w,\eta)-(v,\mu)) - \mathbf{A}_h(v,\mu;(w,\eta)-(v,\mu)) \ge C_{SM}\, \|(w,\eta)-(v,\mu)\|^2_{h,\Gamma} \tag{6.14}$$

for any (w,η), (v,μ), $(z,\xi) \in H^1_{1/2}(\mathcal{T}_h) \times H_0^{-1/2}(\Gamma)$.

Proof. The Lipschitz continuity and strong monotonicity of the semilinear form A_h with respect to the norm $|\!|\!|\cdot|\!|\!|_h$ are provided by Lemmas 4.1 and 4.2 in [7]. The estimates required for the remaining boundary integral terms of $\mathbf{A}_h$ follow exactly as in the proof of Lemma 3.1. We omit further details. □

The unique solvability of (6.8) is now established.

Theorem 6.1. *There exists a unique $(\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h, u_h, \lambda_h) \in \mathbf{\Sigma}_h \times \mathbf{\Sigma}_h \times \tilde{V}_h \times X_h^0$ solution to the coupled* LDG-BEM *scheme* (6.8)*. In addition, there exists $C > 0$, independent of h, such that*

$$\|\boldsymbol{\theta}_h\|_{0,\Omega} + \|\boldsymbol{\sigma}_h\|_{0,\Omega} + \|(u_h,\lambda_h)\|_{h,\Gamma} \le C\left\{\mathcal{N}(f,g_0,g_1,g_2) + \|\bar{\mathbf{a}}(\cdot,0)\|_{0,\Omega}\right\} \tag{6.15}$$

where

$$\mathcal{N}(f,g_0,g_1,g_2) := \left\{ ||f||^2_{0,\Omega} + ||\alpha^{1/2}\, g_0||^2_{0,\Gamma_0} + ||\alpha^{1/2}\, g_1||^2_{0,\Gamma_1} + ||\alpha^{1/2}\, g_2||^2_{0,\Gamma_1} \right\}^{1/2}.$$

Proof. By Lemma 6.1 it suffices to analyze the reduced system (6.12) instead of (6.8). It is clear that (6.12) can be equivalently formulated as: Find $(u_h,\lambda_h) \in \tilde{V}_h \times X_h^0$ such that

$$\mathbf{A}_h(u_h,\lambda_h;v,\mu) := F_h(v) \qquad \forall\,(v,\mu) \in \tilde{V}_h \times X_h^0 .$$

Now, proceeding as in the proof of Lemma 4.4 in [7], we find $C > 0$, independent of h, such that

$$|F_h(v)| \le C\, \mathcal{N}(f,g_0,g_1,g_2)\, |\!|\!|v|\!|\!|_h, \quad \forall\, v \in \tilde{V}_h . \tag{6.16}$$

Hence, Lemma 6.2 and a classical result of nonlinear functional analysis imply the unique solvability of (6.12). The rest of the proof follows very closely the proof of Theorem 3.2 in [9]. In fact, using again the strong monotonicity of $\mathbf{A}_h$, estimate (6.16), the fact that

$$\mathbf{A}_h((0,0),(v,\mu)) = A_h(0,v) = \int_\Omega \bar{\mathbf{a}}(\cdot,\mathcal{G}_h)\cdot(\nabla_h v - \boldsymbol{S}_h v) \qquad \forall\,(v,\mu) \in \tilde{V}_h \times X_h^0 ,$$

the boundedness of $\mathbf{S}_h$ (cf. (3.6)), and the Lipschitz continuity of the nonlinear operator induced by $\bar{\mathbf{a}}$, one deduces that

$$\|(u_h,\lambda_h)\|_{h,\Gamma} \le C\left\{\mathcal{N}(f,g_0,g_1,g_2) + \|\bar{\mathbf{a}}(\cdot,0)\|_{0,\Omega} + \|\mathcal{G}_h\|_{0,\Omega}\right\}. \tag{6.17}$$

Also, using the expressions for $\boldsymbol{\theta}_h$ and $\boldsymbol{\sigma}_h$ given by (6.11), and applying again the boundedness of $\mathbf{S}_h$ and the Lipschitz continuity of $\bar{\mathbf{a}}$, we obtain

(6.18)
$$\|\boldsymbol{\theta}_h\|_{0,\Omega} \le C\Big\{ \|\!|u_h\|\!|_h + \|\mathcal{G}_h\|_{0,\Omega} \Big\} \quad \text{and} \quad \|\boldsymbol{\sigma}_h\|_{0,\Omega} \le C\Big\{ \|\boldsymbol{\theta}_h\|_{0,\Omega} + \|\bar{\mathbf{a}}(\cdot,0)\|_{0,\Omega} \Big\}.$$

Then, it is easy to show, as in the proof of Lemma 3.4 in [7], that (cf. (6.10))

$$\|\mathcal{G}_h\|_{0,\Omega} \le C\Big\{ ||\alpha^{1/2}\, g_0||_{0,\Gamma_0} + ||\alpha^{1/2}\, g_1||_{0,\Gamma_1} \Big\}. \tag{6.19}$$

In this way, (6.15) follows directly from (6.17), (6.18), and (6.19), which ends the proof. □

Finally, we prove the a priori error estimate for the coupled LDG-BEM scheme (6.8).

Theorem 6.2. *Define the additional continuous unknowns*

$$\boldsymbol{\theta} = \begin{cases} \boldsymbol{\theta}_1 := \nabla u_1 & \textit{in } \Omega_1, \\ \boldsymbol{\theta}_2 := \nabla u_2 & \textit{in } \Omega_2, \end{cases} \qquad \boldsymbol{\sigma} = \begin{cases} \boldsymbol{\sigma}_1 := \mathbf{a}(\cdot,\boldsymbol{\theta}_1) & \textit{in } \Omega_1, \\ \boldsymbol{\sigma}_2 := \boldsymbol{\theta}_2 & \textit{in } \Omega_2, \end{cases} \qquad \textit{and } \lambda = \partial_{\boldsymbol{\nu}} u_2 \quad \textit{on} \quad \Gamma.$$

Assume that there exist δ_1, $\delta_2 > 1/2$ such that $u_1 \in H^{1+\delta_1}(\Omega_1)$, $u_2 \in H^{1+\delta_2}(\Omega_2)$, and $\boldsymbol{\sigma}_1 \in [H^{\delta_1}(\Omega_1)]^2$. Then there exists $C > 0$, independent of h, such that

$$\begin{aligned} &\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_{0,\Omega} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} + \|(u,\lambda) - (u_h,\lambda_h)\|_{h,\Gamma} \\ &\le C\Big\{ h^{\min\{\delta_1,m\}}\, \|u_1\|_{1+\delta_1,\Omega_1} + h^{\min\{\delta_1,m\}}\, \|\boldsymbol{\sigma}_1\|_{\delta_1,\Omega_1} \\ &\qquad\qquad + h^{\min\{\delta_2,m\}}\, \|u_2\|_{1+\delta_2,\Omega_2} \Big\}. \end{aligned} \tag{6.20}$$

Proof. We observe, similarly as in the linear case (cf. Theorem 4.1), that $\lambda \in H^{\delta_2 - 1/2}(\Gamma)$ and $\|\lambda\|_{\delta_2-1/2,\Gamma} \le C\,\|u_2\|_{1+\delta_2,\Omega_2}$. Also, according to the definitions of the semilinear form $\mathbf{A}_h$ and the linear operator F_h, and taking into account the equations, the boundary conditions, and the transmission conditions satisfied by u, one can prove that u and λ satisfy

$$\mathbf{A}_h(u,\lambda;v,\mu) = F_h(v) \qquad \forall\,(v,\mu) \in H^1_{1/2}(\mathcal{T}_h) \times H^{-1/2}(\Gamma).$$

In addition, it is clear that the discrete system (6.12) renders

$$\mathbf{A}_h(u_h,\lambda_h;v,\mu) = F_h(v) \qquad \forall\,(v,\mu) \in \tilde{V}_h \times X_h^0.$$

Then, the Lipschitz continuity and strong monotonicity of $\mathbf{A}_h$ also yield the quasi-optimal estimate (4.15); that is,

$$\|(u,\lambda) - (u_h,\lambda_h)\|_{h,\Gamma} \le C\,\|(u,\lambda) - (v_h,\mu_h)\|_{h,\Gamma} \qquad \forall\,(v_h,\mu_h) \in \tilde{V}_h \times X_h^0. \tag{6.21}$$

Now, using that $\boldsymbol{\theta}_h = \nabla_h u_h - \mathbf{S}_h(u_h) + \mathcal{G}_h$ (cf. (6.11)), $\boldsymbol{\theta} = \nabla u$ in Ω, $\mathbf{S}_h(u) = \mathcal{G}_h$, and applying the boundedness of $\mathbf{S}_h$, we obtain

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_{0,\Omega} \le C\,\|\!|u - u_h\|\!|_h. \tag{6.22}$$

It remains to estimate $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega}$. Using that $\boldsymbol{\sigma}_h = \Pi_{\boldsymbol{\Sigma}_h}\, \bar{\mathbf{a}}(\cdot,\boldsymbol{\theta}_h)$ (cf. (6.11)) and $\boldsymbol{\sigma} = \bar{\mathbf{a}}(\cdot,\boldsymbol{\theta})$, and applying the triangle inequality and the Lipschitz continuity of the nonlinear operator induced by $\bar{\mathbf{a}}$, we deduce that

$$\begin{aligned} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} &\le \|\boldsymbol{\sigma} - \Pi_{\boldsymbol{\Sigma}_h}\boldsymbol{\sigma}\|_{0,\Omega} + \|\Pi_{\boldsymbol{\Sigma}_h}\big\{\bar{\mathbf{a}}(\cdot,\boldsymbol{\theta}) - \bar{\mathbf{a}}(\cdot,\boldsymbol{\theta}_h)\big\}\|_{0,\Omega} \\ &\le \|\boldsymbol{\sigma} - \Pi_{\boldsymbol{\Sigma}_h}\boldsymbol{\sigma}\|_{0,\Omega} + C\,\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_{0,\Omega}. \end{aligned} \tag{6.23}$$

Then, applying local approximation properties of piecewise polynomials (see, e.g. Lemma 4.2 in [18]), recalling from (2.6) that on $K \in \mathcal{T}_h$, $\Pi_{\mathbf{\Sigma}_h}$ reduces to the $\mathbf{L}^2(K)$-orthogonal projection onto $\mathbf{P}_r(K)$, which is denoted by Π_K^r, and noting that $r+1 \geq m$, we find that

(6.24)

$$\begin{aligned}\|\boldsymbol{\sigma} - \Pi_{\mathbf{\Sigma}_h}\boldsymbol{\sigma}\|_{0,\Omega_1} &= \sum_{K\in\mathcal{T}_{h,1}} \|\boldsymbol{\sigma}_1 - \Pi_K^r\boldsymbol{\sigma}_1\|_{0,K}^2 \\ &\leq C \sum_{K\in\mathcal{T}_{h,1}} h_K^{2\min\{\delta_1,r+1\}} \,||\boldsymbol{\sigma}_1||_{\delta_1,K}^2 \\ &\leq C\,h^{2\min\{\delta_1,r+1\}}\,\|\boldsymbol{\sigma}_1\|_{\delta_1,\Omega_1}^2 \leq C\,h^{2\min\{\delta_1,m\}}\,\|\boldsymbol{\sigma}_1\|_{\delta_1,\Omega_1}^2\end{aligned}$$

and

(6.25)

$$\begin{aligned}\|\boldsymbol{\sigma} - \Pi_{\mathbf{\Sigma}_h}\boldsymbol{\sigma}\|_{0,\Omega_2}^2 &= \sum_{K\in\mathcal{T}_{h,2}} \|\boldsymbol{\theta}_2 - \Pi_K^r\boldsymbol{\theta}_2\|_{0,K}^2 \\ &= \sum_{K\in\mathcal{T}_{h,2}} \|\nabla u_2 - \Pi_K^r\nabla u_2\|_{0,K}^2 \leq C \sum_{K\in\mathcal{T}_{h,2}} h_K^{2\min\{\delta_2,r+1\}}\,||\nabla u_2||_{\delta_2,K}^2 \\ &\leq C\,h^{2\min\{\delta_2,r+1\}}\,\|u_2\|_{1+\delta_2,\Omega_2}^2 \leq C\,h^{2\min\{\delta_2,m\}}\,\|u_2\|_{1+\delta_2,\Omega_2}^2 .\end{aligned}$$

In this way, the approximation properties from Lemmas 4.2 and 4.3 (with $t = \delta_2 - 1/2$), together with the bound $\|\lambda\|_{\delta_2-1/2,\Gamma} \leq C\,\|u_2\|_{1+\delta_2,\Omega_2}$, and inequalities (6.21), (6.22), (6.23), (6.24), and (6.25), imply the required a priori error estimate and finish the proof. □

We end this section by remarking, as we did for the linear case at the end of Section 4, that the a priori error estimate (6.20) is also independent of the polynomial degree r that defines the subspace $\mathbf{\Sigma}_h$ (cf. (2.6)). Therefore, since the restriction $r \geq m-1$ is required only to deduce that $\boldsymbol{\theta}_h = \nabla_h u_h - \mathbf{S}_h(u_h) + \mathcal{G}_h$ (cf. (6.11)), it suffices also to take $r = m-1$ in the present nonlinear case.

7. Coupling with other DG methods

As exposed, the theory on coupling the LDG method with a symmetric boundary element formulation demanding continuity on the trace for the discontinuous function can be extended to several other DG methods. We give here some very fast brushstrokes for three methods, all of which can be introduced more naturally in the primal form, i.e., expressed directly on the variable u_h, which is precisely the way we have approached the analysis. We remark, however, that for the extension to nonlinear problems given in the previous section, the formulation with two variables in the domain (the one with mixed flavor) has to be used.

The simplest adaptation is given by the method with discontinuous polynomials and jump penalization on the element interfaces, that can be traced back to Babuška and Zlámal (see [5]). In our notation it consists simply of erasing completely the bilinear form S or equivalently the gradient correction term $\mathbf{S}_h$ in all occurrences. In this case $\boldsymbol{\sigma}_h$ is simply $\nabla_h u_h$. The expression of the bilinear form B_h (see (2.16)) is greatly simplified and everything that has been said for this bilinear form (i.e., (3.4), (3.5) and (4.14)) still applies, so we can carry out our analysis to the very end.

The Interior Penalty method [15], [1] consists of taking the bilinear form

$$B_h(u,v) := \boldsymbol{\alpha}(u,v) + \int_\Omega \nabla_h u \cdot \nabla_h v - \int_{I_h} \big([\![u]\!] \cdot \{\nabla_h v\} + [\![v]\!] \cdot \{\nabla_h u\} \big) - \int_{\Gamma_0} \big(u\,\partial_{\boldsymbol{\nu}} v + v\,\partial_{\boldsymbol{\nu}} u \big)$$

and go on as usual. This is equivalent to taking $\boldsymbol{\beta} = 0$ in the LDG method and subtracting a term

$$\int_\Omega \mathbf{S}_h(u) \cdot \mathbf{S}_h(v)$$

from the bilinear form B_h. The corresponding flux is $\boldsymbol{\sigma}_h = \nabla_h u_h - \mathbf{S}_h(u_h)$. It is possible to unfold the system to write it in the expanded form reminiscent of (2.12). The difference is the fact that the second equation

$$\int_\Omega \nabla_h v \cdot \boldsymbol{\sigma}_h - S(v, \boldsymbol{\sigma}_h) + \boldsymbol{\alpha}\,(u_h, v) + \langle \mathcal{W} u_h, v\rangle - \langle (\frac{1}{2}\mathcal{I} - \mathcal{K}')\lambda_h, v\rangle = \int_\Omega f\,v$$

is substituted by

$$\int_\Omega \nabla_h v \cdot \boldsymbol{\sigma}_h - S(v, \nabla_h u_h) + \boldsymbol{\alpha}\,(u_h, v) + \langle \mathcal{W} u_h, v\rangle - \langle (\frac{1}{2}\mathcal{I} - \mathcal{K}')\lambda_h, v\rangle = \int_\Omega f\,v\,.$$

The global system loses in this way its symmetry (the symmetry of (2.12) is only apparent after changing the sign of the second equation) in this unfolded form, a symmetry that is recovered once the variable $\boldsymbol{\sigma}_h$ has been eliminated from the system. The only intricate part of the analysis refers to recovering the discrete ellipticity (3.5), which imposes some restrictions on the function that defines the bilinear form $\boldsymbol{\alpha}$; see [2] for the details. Apart from this, the remaining part of our analysis can be carried out without much difficulty.

Last but not least, NIPG also fits easily in this framework. The bilinear form in the primal formulation is

$$B_h(u,v) := \boldsymbol{\alpha}(u,v) + \int_\Omega \nabla_h u \cdot \nabla_h v - \int_{I_h} \big([\![u]\!] \cdot \{\nabla_h v\} - [\![v]\!] \cdot \{\nabla_h u\} \big) - \int_{\Gamma_0} \big(u\,\partial_{\boldsymbol{\nu}} v - v\,\partial_{\boldsymbol{\nu}} u \big),$$

(i.e., desymmetrizes the interface terms of IP) and makes the ellipticity estimate (3.5) a simple consequence, since the corresponding quadratic form is the same as in the first method we have just exposed. As in the previous case, it is possible to add $\boldsymbol{\sigma}_h$ as an unknown and unfold the system (cf. [2]).

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