PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE
ESCUELA DE INGENIERÍA

OPTIMAL EXERCISE POLICY FOR AMERICAN OPTIONS UNDER GENERAL MARKOVIAN DYNAMICS

CAMILO IGNACIO ABURTO MARCHANT

Thesis submitted to the Office of Research and Graduate Studies
in partial fulfillment of the requirements for the degree of
Master of Science in Engineering

Advisor:
GONZALO CORTÁZAR

Santiago de Chile, April 2016

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Members of the Committee:
GONZALO CORTÁZAR
TOMÁS REYES
LORENZO NARANJO
HERNÁN SANTA MARÍA

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To my parents,

Renato and Angélica.

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ABSTRACT

In this thesis a fast and accurate iterative algorithm to solve for the early exercise boundary and price American options is proposed. The algorithm is based on a fixed-point iteration derived from the early exercise representation of American options and addresses the problem of solving for the early exercise boundary under general Markovian pricing models. This thesis extends the work done by Dattas (2015) on the application of the algorithm to price American options under Heston's Stochastic Volatility framework. The algorithm is tested using a set of nested specifications including the Stochastic Volatility with Contemporaneous Jumps pricing model by Duffie, Pan, and Singleton (2000), and we consider the Least-Square Monte Carlo (LSM) method by Longstaff and Schwartz (2001) as a benchmark. The method shows to be stable and robust, converging to the early exercise boundary for every setting tested. When compared to the LSM, we find that our algorithm discovers a more accurate early exercise boundary and leads to better empirical exercise decisions when tested with historical S&P500 index data. Furthermore, our algorithm exhibits higher efficiency as it lends itself to parallel programming.

Keywords: American Options, Parallel Computing, Stochastic Volatility, Jump-Diffusion, Fixed-Point Iteration.

RESUMEN

En esta tesis se propone un algoritmo iterativo rápido y preciso para resolver la frontera de ejercicio óptimo y valorizar opciones Americanas. El algoritmo se basa en una iteración de punto fijo derivada de la representación de ejercicio óptimo de opciones Americanas y aborda el problema de resolver su frontera de ejercicio óptimo bajo modelos Markovianos generales. Esta tesis extiende el trabajo realizado por Dattas (2015) en la aplicación del algoritmo para la valorización de opciones Americanas bajo el modelo de Volatilidad Estocástica de Heston. El algoritmo es puesto a prueba usando un conjunto de especificaciones anidadas incluyendo el modelo de Volatilidad Estocástica con Saltos Contemporáneos de Duffie et al. (2000) y el método Least-Squares Monte Carlo de Longstaff and Schwartz (2001) es utilizado como referencia. El método es estable y robusto, y converge a la frontera de ejercicio óptimo bajo todos los escenarios sometidos a prueba. Cuando se compara con el LSM, se halla que el algoritmo descubre fronteras de ejercicio óptimo más precisas y que permite tomar mejores decisiones de ejercicio cuando se prueba utilizando datos históricos del índice S&P500. Además, el algoritmo exhibe una mayor eficiencia pues se presta para realizar programación paralela de sus cómputos.

Palabras Claves: Commodities; Modelos Multifactoriales; Volatilidad Estocástica; Derivados; Valorización de Activos.

1. ARTICLE BACKGROUND

1.1. Introduction

Options are financial instruments consisting in a contract that allows its holder to exercise the right to buy (Call option) or sell (Put option) a defined underlying asset at a specified strike price within a fixed time frame. While a European option gives the right to exercise at maturity only, an American option can be exercised at any time before maturity. The early-exercise feature of the American option results in complex pricing methods due to the lack of closed-form valuation solutions.

The fact that no closed-form solution to the American option pricing problem exists has encouraged researchers to come up with numerical methods to price American options focusing on both the pricing precision and efficiency. In this context, the concept of the early exercise boundary arises. This boundary consists of a set of critical prices that will trigger optimal early-exercise when they are attained by the underlying asset. The early exercise boundary is of great importance, as it is needed to determine the optimal hold and exercise policy, which can be used to value this type of options.

Kim (1990), Jacka (1991), and Carr, Jarrow, and Myneni (1992), based on different approaches, derived an integral equation to determine the early exercise boundary for this type of contracts. The authors showed that the American option price is equal to the corresponding European price plus an early exercise premium, which depends on the early exercise boundary. They provide a quasi-analytic expression for the early exercise premium when the underlying follows a lognormal diffusion process. Numerical techniques can then applied in order to compute the early exercise premium and price these options.

However, lognormal diffusion has a series of limitations as consequence of the simplifying assumptions of the model. Heston (1993) proposes assuming stochastic instead of constant volatility to improve the pricing. introducing the Stochastic Volatility (SV) pricing model. Later was found by Bates (1996) that a model incorporating jump diffusion features to the stochastic volatility dynamic lead to even better results. Despite the significant progress, empirical evidence indicates that these models are still incapable of fully capturing all features of assets returns (Bakshi and Chen (1997), Bates (2000), and Pan (2002)). Numerous approaches have been considered to develop alternative asset return models in order to address this issue, being the incorporation of jumps in volatility by Duffie et al. (2000) one of the most relevant.

In this thesis, it is proved that the Fixed Point Iteration algorithm (FPI), initially proposed in Medina (2013) for the lognormal diffusion and then adapted in Dattas (2015) to the Heston Stochastic volatility model, can be further extended to a general Markovian pricing framework by solving for the early exercise boundary via Monte Carlo simulations. The Simulated Fixed-Point Iteration (S-FPI) proves to be stable, accurate and efficient, and shows to be capable of generating a smoother policy compared to the Least-Squares Monte Carlo (LSM) method, resulting in an efficient method in terms of pricing.

The rest of the thesis is organized as follows. Section 1.2 defines the main hypothesis. Section 1.3 defines the main objectives of this work. Section 1.4 outlines the methodology considered for this work. Section 1.5 introduces the theoretical framework and numerical methodologies related to American option pricing. Section 1.5 presents the perspectives for future research. Chapter II contains the main article of this thesis, written in colaboration with Gonzalo Cortázar and Lorenzo Naranjo. Within this chapter, Section 2.1 introduces the article. Section 2.2 outlines the problem of solving for the early exercise boundary of an American option under general Markovian dynamics. Section 2.3 presents the Simulated Fixed-Point Iteration method. In Section 2.4 we implement the S-FPI method and explore the impact of variance jumps on the early exercise boundary. In Section 2.5 we study the performance of our algorithm using historical data of the

S&P500 index. In Section 2.6 we extend our analysis by solving for the optimal exercise boundary under a Lévy process. Concluding remarks are presented in Section 2.7.

1.2. Hypothesis

The hypothesis of this work is that the iterative method presented in Medina (2013) and then adapted in Dattas (2015) can be further extended to obtain a novel, fast, simple, and precise procedure to solve for the early exercise boundary and price American options under general Markovian specifications. Under this setup, the algorithm should exhibit the flexibility to compute the early exercise boundary for highly complex models, such as the Stochastic Volatility and Contemporaneous Jumps model by Duffie et al. (2000), and still be coherent with the findings of Dattas (2015) for more standard models, such as the Stochastic Volatility model by Heston (1993).

1.3. Main Objectives

The main objective of this thesis is to present, explain and implement the Simulated Fixed-Point Iteration (S-FPI) algorithm as an extension of the Fixed Point Iteration algorithm (FPI) to a general Markovian pricing layout. This should be achieved by illustrating its competency to solve for the early exercise boundary and price American options explicitly under the Stochastic Volatility and Contemporaneous Jumps by Duffie et al. (2000), as this specification is considered to provide well-fitting American option prices and to be one of the most complex specifications amongst general Markovian configurations available today in the literature. In this context, the thesis has three specific objectives: First, this work intends to establish a theoretical framework that supports the utilization and the main features of the methodology. The second objective is to introduce the Simulated Fixed Point Iteration method for a general Markovian pricing framework. The third objective is to implement the S-FPI to obtain the early exercise boundary and price for the American option under the Stochastic Volatility pricing model, and illustrate that the method (i) robustly converges to the early exercise boundary, (ii) outperforms the Least-Square Monte Carlo method both in accuracy and efficiency and (iii) provides results coherent with the findings in Dattas (2015). Finally, the fourth objective is to demonstrate the full capabilities of the S-FPI method by computing the early exercise boundary and prices for a set of nested models including the Stochastic Volatility and Contemporaneous Jumps pricing model, assessing the pricing accuracy of the algorithm as well as its competency on providing an optimal exercise policy when tested against historical data.

1.4. Methodology

The source code for the Simulated Fixed-Point Iteration and the corresponding benchmarks is implemented using MATLAB 2013b running on a 2.50GHz Intel Core i7-4710HQ with 16GB RAM, and additional CUDA implementations are carried out on a NVIDIA GeForce GTX 860M. The Fixed-Point Iteration method is implemented according to Dattas (2015) and the Least-Squares Monte Carlo (LSM) method is implemented according to Longstaff and Schwartz (2001). Root Mean Square Error (RMSE), Root Mean Square Relative Error (RMSRE) and Mean Relative Error (MRE) were measured against the true values computed by the LSM method.

All early exercise boundaries and prices are computed using parameter values estimated in Eraker (2004). Real data employed to measure the performance of the early exercise boundary is obtained from historical S&P500 index and VXO index quotes, for a three year time-period starting on January 1, 1987.

1.5. Literature Review

1.5.1. Numerical methods for Pricing the American Option

Given the nature of American options, financial theorists and researchers have given much attention to developing numerical methods to price them. Numerical methods usually depend on the dynamics considered for the underlying asset, and become more complex as more stochastic factors are considered.

The simplest case is given by lognormal diffusion by Black and Scholes (1973). There are numerous studies in the literature that propose methods to price American options for the lognormal diffusion. Brennan and Schwartz (1977) were the first to solve numerically a partial differential equation (PDE) to price American options. Another popular method that discretizes the time space and the asset price is the binomial method of Cox, Ross, and Rubinstein (1979). Both methods are still widely used because of their simplicity. Longstaff and Schwartz (2001) developed a novel method to value options by simulation that determines the conditional expected payoff by least-squares. A different approach to improve the speed at the expense of precision are the quasi-analytical approximation methods (Barone-Adesi and Whaley (1987), Ju and Zhong (1999)). Other methods involve using Richardson extrapolation in order to improve the accuracy of the computations (Geske and Johnson (1984)) or use quadrature formulas in order to price the option (Sullivan (2000), Kallast and Kivinukk (2003)).

In order to better adjust to the implied volatility smiles found in market prices, stochastic volatility models are later introduced. This models feature an additional stochastic process to model the underlying's volatility. Several methodologies have been proposed to price American options under stochastic volatility, although less frequently encountered than for the lognormal diffusion case. The approaches considered are diverse and include pricing by solving the resulting PDE (Cryer (1971), Brandt and Cryer

(1983), Clarke and Parrot (1999), Osterlee (2003), Ikonen and Toivanen (2004),Zvan, Forsyth, and Vetzal (1998), Ikoneb and Toivanen (007a), Ikonen and Toivanen (007b), Chockalingam and Muthuraman (2011)), by simulation (Longstaff and Schwartz (2001), Andersen and Broadie (Andersen and Broadie), Broadie and Glasserman (2004)) and by numerical integration (Tzavalis and Wang (2003), Adolfsson, Chiarella, Ziogas, and Ziveyi (2013)).

Later was found that a model incorporating jump diffusion features to the stochastic volatility dynamic lead to even better results. Such model is known as the stochastic volatility jump diffusion (SVJ) model by Bates (1996). Several methods have been proposed for numerical valuation of options under the SVJ model. A finite element approach is considered by Ballestra and Sgarra (2010), Miglio and Sgarra (2011), Rambeerich, Tangman, Lollchund, and Bhuruth (2013). Toivanen (2010) uses a linear complementarity problem to obtain the partial integro-differential equation resulting from the option pricing and then solve it by means of finite-differences.

1.5.2. Solving for the Early Exercise Boundary

The pricing of American options using the early exercise boundary is a method widely use in the literature. Karatzas and Shreve (1998) or Barone-Adesi (2005) provide a historical review. Among the most relevant works in this field, is worth noting the contributions by Kim (1990), where integral equation that provides an explicit solution to the early exercise premium when the underlying asset follows a lognormal diffusion process is derived. This early exercise premium representation is later used by Kallast and Kivinukk (2003) to propose a robust numerical method for its computation. In Broadie and Detemple (1996) volatility is allowed to be stochastic and a nonparametric approach is used in order to estimate American call prices and exercise boundaries. The method of Kallast and Kivinukk (2003) is extended by Chiarella and Ziogas (2005) to solve for the early exercise boundary for a model incorporing stochastic volatility. This method is then further extended on Chiarella and Ziogas (2009) to allow both stochastic volatility and price jumps. Kim's representation of the early exercise premium is again used by Cortazar, Dattas, Medina, and Naranjo (2015) to derive the Fixed-Point Iteration method (FPI). The proposed methodology consists in an iterative algorithm that solves for the early exercise boundary of the American option under constant and stochastic volatility specifications.

Notwithstanding, most works available in the literature lack flexibility as they address the computation of the early exercise boundary for a given pricing model and are limited to the stochastic volatility and price jumps pricing model. Considering that empirical evidence found by Bakshi and Chen (1997), Bates (2000), and Pan (2002) indicates that the conditional volatility of returns rapidly increases under periods of financial crisis and stochastic volatility and price jump features are incapable of fully capturing dynamics present in equity index returns, numerous approaches have been considered to develop alternative asset return models in order to address this issue. The incorporation of jumps in volatility proposed by Duffie et al. (2000) has proven to be effective in addressing this issue. Even so, there has been relatively little research related to American option early exercise boundary estimation under this model, mainly due to its high analytical and computational complexity.

1.6. Further Research

The Simulated Fixed Point Iteration algorithm defined in this thesis shows some promising features for future research given its capacity of discovering smooth and accurate early exercise boundaries explicitly for general Markovian pricing models. Given the flexibility of the algorithm, future research could focus on studying results for early exercise boundaries and prices for fixed-income oriented pricing models, for example, bonds dynamics considering a stochastic interest rate model such as the Cox-Ingersoll-Ros model.

On the other hand, promising results obtained in this work regarding the algorithms competency on discovering an optimal exercise policy for the S&P500 index motivates future research for other equity indexes as well as other kind of underlying such as bonds, stock, commodities and currencies.

Finally, since this algorithm is simulation-based, future research could be advocated to implement more sophisticated simulation technics in order to improve its speed, thus increasing its competitiveness against other numerical methods that address the problem of solving for the early exercise boundary for particular specifications.

2. OPTIMAL EXERCISE POLICY FOR AMERICAN OPTIONS UNDER GENERAL MARKOVIAN DYNAMICS

2.1. Introduction

We propose a simple and robust numerical method to calculate the optimal exercise policy of an American option under general Markovian dynamics. Our approach is based on a Newton-Kantorovich fixed-point iteration that is easy to compute and exhibits fast global convergence. In the paper we solve numerically for optimal exercise policies for several models that exhibit stochastic volatility and jumps. The results show that our method is stable, robust, and converges accurately for all the models that we test. We also show that when using real data, our algorithm uncovers a more profitable exercise policy than the widely used least-square Monte Carlo (LSM) method of Longstaff and Schwartz (2001). Of course, the LSM approach was developed to yield accurate pricing of American options and not to uncover optimal-exercise policies. We believe that our paper complements this literature well by providing a general method to estimate the early-exercise policy of such financial contracts.

The optimal stopping problem was analyzed by McKean (1965) and Merton (1973), among others, as a free boundary problem where optimal exercise is triggered by an early-exercise boundary (EEB) that defines the optimal exercise policy for the option holder. Once the EEB is computed, it can be employed to price the option by estimating the early-exercise premium as proposed by Kim (1990), or via Monte Carlo simulation, among other approaches.

The estimation of the EEB for American options is of great interest for both applied and academic purposes. In addition to giving the option's "fair" price -relevant for trading and research on these contracts-, the EEB provides valuable information regarding the early-exercise feature of American options. For example, profitable investment strategies can be obtained from knowing the boundary, as it maximizes the profits from early-exercise. Moreover, having a method to estimate the EEB allows us to study how different model specifications affect early-exercise decisions for these kind of financial contracts.

There is an important literature that has studied the pricing of American options using the notion of an EEB.[1] Kim (1990) derives an integral equation that provides an explicit solution to the early-exercise premium when the underlying asset follows a lognormal diffusion process. Kallast and Kivinukk (2003) propose a robust numerical method to solve for the EEB using Kim's early exercise premium representation. Broadie and Detemple (1996) allow for stochastic volatility and use a nonparametric approach to approximate American option prices and EEBs. Chiarella and Ziogas (2005) extend the method of Kallast and Kivinukk (2003) to solve for the EEB in a stochastic volatility model, while Chiarella and Ziogas (2009) price American options in a model that allows for only price jumps. Chockalingam and Muthuraman (2011) use a transformation procedure and solve for the EEB and the price of an American option under different stochastic volatility models. Finally, Cortazar et al. (2015) use Kim's (1990) representation of the early-exercise premium to derive a fixed-point iteration that solves for the EEB in Heston's (1993) stochastic volatility model.

Despite this significant progress, existing algorithms are in general model specific and lack the flexibility to deal with more complex configurations, such as stochastic volatility and jumps, for example. Moreover, empirical evidence indicates that the conditional volatility of returns rapidly increases under stress scenarios, making stochastic volatility and jumps in price incapable of fully capturing the leptokurtic features found in equity index returns (Bakshi and Chen (1997), Bates (2000), and Pan (2002)). To

[1]See Karatzas and Shreve (1998) or Barone-Adesi (2005) for a comprehensive review of the literature.

account for these effects, Duffie et al. (2000) propose the SVCJ model that incorporates stochastic volatility, and jumps in prices and volatility. This model nests the well-known stochastic volatility (SV) model of Heston (1993), and its extension (SVJ) that allows for stochastic volatility and jumps only in returns. To the best of our knowledge, besides the LSM method, no other algorithm in the literature can solve for the EEB in the SVCJ model.

In this paper, we propose the Simulated Fixed-Point Iteration method (S-FPI) that can compute the EEB for a broad range of underlying specifications, such as the SVCJ model. We test our algorithm through numerical experiments for a set of nested models including the SVCJ, and for a type of infinite activity Lévy process, namely the Variance Gamma process introduced in Madan, Carr, and Chang (1998). We then use the S-FPI estimated EEB to price American put options and to define exercise strategies using real data on the S&P500 index. As a benchmark for the S-FPI we consider a simple adaptation of the LSM method of Longstaff and Schwartz (2001). We find that our algorithm is able to estimate more accurate EEBs. Furthermore, we find that the estimated S-FPI early-exercise rule obtains larger average profits from exercising American options than the LSM method when tested using historical data. Finally, we find that the method is stable, robust and is well suited for parallel calculations and programming, substantially increasing the speed of execution.

The remainder of the paper is structured as follows. Section 2 outlines the problem of solving for the EEB under general Markovian dynamics. Section 3 presents the LSM benchmark. Section 4 introduces the S-FPI method and characterizes its convergence. In Section 5 we implement the S-FPI method and explore the impact of variance jumps on the EEB. Section 6 studies the performance of our algorithm using historical data of the S&P500 index. In Section 7 we extend our analysis by solving for the EEB under the variance-gamma process. Concluding remarks are presented in Section 8.

2.2. The Early Exercise Boundary

Consider a continuous-time economy in which is defined a complete probability space $(\Omega, \mathscr{F}, \mathcal{P})$ and a filtration $\mathbb{F} = \{\mathscr{F}_t, t \geq 0\}$ satisfaying the usual conditions (see, e.g., Protter (2005)). Let $X = \{S, Y\}$ be a general Markovian system where S corresponds to the underlying asset and Y denotes a vector of state variables living in a space state $D \subset \mathbb{R}^n$. We denote by S_t and Y_t the values of S and Y on time t, respectively.

In the paper we solve examples of American options in which the stochastic process of the underlying asset price S exhibits constant returns to scale. This type of models first appeared in Merton (1973), and is used in most European and American option pricing models. As pointed out by **?**, using such a model rules out the so-called level illusion, in the sense that whether the S&P 500 index is at 500 or 1000 should be irrelevant for the distribution of stock market returns.

We consider an American put option written on S with maturity T and exercise price K. Let r denote the risk free rate, which may depend on Y, and $R_t = \int_0^t r(Y_u)du$. Let us define $x^+ = \max(0, x)$. The price P of a put option is then given by

$$P(t, S_t, Y_t) = \sup_{\tau \in [t,T]} \mathbb{E}\left[e^{-R_\tau}(K - S_\tau)^+ | \mathscr{F}_t\right], \tag{2.1}$$

where the supremum is taken over all stopping times τ taking values in $[t, T]$.

For put options, early exercise is triggered whenever $S \leq S^*$, i.e when the exercise payoff $K - S^*$ exceeds the value of keeping the option alive, known as the *value of continuation*, which we denote by $F = F(t, S^*, Y)$. The *exercise region* is defined by the set $\{S \leq S^*\}$ and the early-exercise boundary (EEB) denotes its frontier that separates the *continuation region* from the exercise region. The continuity of the option premium with respect to S implies that the value of continuation is equal to the exercise payoff when the EEB is reached.

It follows that, for any given pair (t, Y), this boundary is characterized as the critical spot price $S^c = S^c(t, Y)$ such that $K - S^c = F(t, S^c, Y)$, with boundary condition $S^c(T, Y_T) = K$.[2] Therefore, exercising according to the EEB is optimal as it maximizes the option's premium. In the analysis we will restrict our attention to models that generate EEBs that are bounded and continuous on $[0, T] \times D$.

By definition, the value of continuation corresponds to the expectation of the remaining discounted cash flows, implying that $F(t, S_t, Y_t) = \sup_{\tau \in (t,T]} \mathbb{E}\left[e^{-R_\tau}(K - S_\tau)^+ | \mathscr{F}_t\right]$, where now the supremum is taken over all stopping times greater than t. The above equation, together with the definition of the EEB implies that,

$$K - S^c(t, Y_t) = \sup_{\tau \in (t,T]} \mathbb{E}\left[e^{-R_\tau}(K - S_\tau)^+ | S_t = S^c(t, Y_t), Y_t\right], \qquad \forall t, Y_t \in [0, T] \times D, \tag{2.2}$$

where the left hand-side in Eq. (2.2) is the early-exercise payoff and the right-hand side is the value of continuation. The above expression not only characterizes S^c but also gives an intuitive characterization for the optimal stopping time, which is the time when the underlying asset price reaches the critical value S^c.

2.3. An LSM-based Approach to Solve for the EEB

Solving for S^c in Eq. (2.2) is not straightforward as it requires to estimate the value of continuation. Longstaff and Schwartz (2001) propose the least-squares Monte Carlo (LSM) algorithm to approximate the price of American options by successive computations of the value of continuation throughout the life of the option.

In this section we adapt the LSM method in order to obtain the EEB, and use this exercise rule to price a set of American put options via Monte Carlo simulation. We will use this adaptation of the LSM method as a benchmark to test our methodology. Since

[2]Note that the value of $\lim_{t \to T} S^c(t, Y_t)$ might be smaller than K if the asset pays a dividend yield large enough.

the algorithm proposed in this paper is a generalization of the FPI algorithm of Cortazar et al. (2015), we first compare the EEB obtained through LSM to the one computed using the FPI method when the underlying asset follows an SV process, allowing us to identify potential weaknesses of the LSM approach as a method to estimate EEBs.

Option prices obtained using the modified LSM method are then compared to the ones computed using the traditional LSM and the FPI method. We find significant differences in the EEB and prices with respect to these benchmarks, illustrating the difficulties of solving for the EEB using the LSM approach.

2.3.1. Description of the Algorithm

The LSM framework starts by assuming the option can only be exercised at discrete time points $0 \leq t_1 \leq t_2 \leq \ldots \leq t_N = T$. If the option is exercised at maturity, the value of the option will simply be that of the payoff at maturity. For a given sample path (S, Y) at time t_i, the payoff from immediate exercise is known. If the option is not exercised at time t_i, the continuation value is then the risk-neutral expectation of the remaining cash flows, which we denote as $C(t, S, Y)$, discounted at the risk-free rate. Hence, at time t_i, the value of continuation $F(t_i, S_{t_i}, Y_{t_i})$, is given by

$$F(t_i, S_{t_i}, Y_{t_i}) = \mathbb{E}\left[\sum_{j=i+1}^{N} \exp\left(-\int_{t_i}^{t_j} r_t dt\right) C(t_j, S_{t_j}, Y_{t_j}) \middle| \mathscr{F}_{t_i}\right] \tag{2.3}$$

where r_t is the riskless interest-rate and the expectation is conditional on the information available up to time t_i. With this setup, the problem reduces to evaluating the conditional expected payoff $F(t_i, S_{t_i}, Y_{t_i})$ at every time step t_i, for every path S, and comparing it with the immediate payoff.

The LSM method assumes that the unknown functional form of $F(t_i, S_{t_i}, Y_{t_i})$ in Eq. (2.3) can be represented as a linear combination of $\mathscr{F}_{t_i}$-measurable functions. We choose the basis functions as simple powers of the state variables $P_{k,l}(S, Y) = S^k Y^l$. With this specification, $F(t_i, S_{t_i}, Y_{t_i})$ can be approximated by

$$F(t_i, S_{t_i}, Y_{t_i}) \approx \sum_{k+l \leq D} a_{k,l}^{(i)} P_{k,l}(S_{t_i}, Y_{t_i}),$$

where the coefficients $a_{k,l}^{(i)}$ are estimated through ordinary least-squares and D is a given integer, i.e. the value of $F(t_i, S_{t_i}, Y_{t_i})$ is approximated by regressing the discounted payoffs onto the basis functions for the paths where the option is in-the-money. By additionally solving Eq. (2.2) on every step, the LSM method can uncover the EEB. Since knowing F on a neighborhood of S^c is required for solving this equation, we adapt the traditional LSM approach so its simulations start from a range of initial spot values ranging from $S = 0, \ldots, K$. We denote this extension of the LSM method by LSM-EEB.

2.3.2. Application

We apply the previous algorithm to find the EEB for a 1-month American put option written on the S&P500 index with a strike price of $K = 100$. We assume that the underlying dynamics of the spot price follow the stochastic volatility model of Heston (1993). Parameter values are the ones estimated in Eraker (2004).

Figure 1 plots the EEB computed using the LSM-EEB algorithm. The estimated surface is predominantly smooth, although singularities arise for low variance levels near maturity. The early-exercise rule is fairly monotone in t and v. The computed exercise policy is aggresive, as it dictates early exercise only when the underlying asset drops significantly below the strike price. This pattern is persistent throughout the life of the option, and only changes when approaching expiration.

Figure 2 compares LSM-EEB's cross-sections for different spot variances ($v = 0.01$, 0.05, 0.5, 1), to the ones obtained with the FPI method proposed by Cortazar et al. (2015). We observe significant differences between the two boundaries. The FPI boundary is smoother, specially near maturity, and triggers early exercise for greater values of the underlying than the LSM-EEB rule. The difference between both EEBs suggests that the LSM-EEB method estimates a different early-exercise policy.

We can further measure the accuracy of the LSM-EEB by pricing a set of put options on the S&P500 index and comparing the results against prices given by the traditional LSM and the FPI method. Prices for the LSM-EEB are computed estimating Eq. (2.1) via Monte Carlo simulation. Table 1 displays the results for different moneyness and spot-variance levels[3]. Overall, there are significant differences between the LSM-EEB and the FPI methods. Root-mean-square errors (RMSE) and root-mean-square relative errors (RMSRE) for the LSM-EEB method are roughly 15 and 20 times greater than the ones obtained with the FPI approach, respectively, indicating that the boundary discovered by the LSM-EEB is quite sub-optimal. Additionally, the negative Mean Relative Error (MRE) indicates that the LSM-EEB underestimates prices with respect to the benchmark.

The differences between the EEBs obtained with the LSM-EEB and the FPI methods are due to the inaccurate estimation of the continuation value F in the LSM-EEB approach. Indeed, the solution for Eq. (2.2) will be reasonable as long as the estimation of $F(t, S^c, Y)$ is accurate. For this condition to hold, the simulation performed by the LSM method must provide information about future cash flows for initial spot prices close to S^c. However, since the simulation is governed by previously chosen dynamics and parameter values, it is difficult to assure that simulated spot prices will hit a neighborhood of S^c, even when the simulation starts from a wide range of values, as we have set. Correcting this issue is not straightforward as it would require to perform multiple simulations throughout the boundary to assure initial values are always close to S^c.

[3]We have selected in-the-money options as they better illustrate the influence of the EEB in pricing.

In the next section we propose a simple and fast alternative to the LSM method, that extends the FPI method for applying it to more general stochastic processes, capable of discovering a smooth and accurate EEB.

2.4. The Simulated Fixed-Point Iteration Method

The Fixed-Point Iteration method (FPI) has been applied to price American options with constant and stochastic volatility by Cortazar et al. (2015). The method has several advantages when solving for the EEB, as it yields accurate exercise rules and displays superior performance over several commonly used algorithms used to price American options. In this section we extend the method proposed by Cortazar et al. (2015) to price American options under general Markovian dynamics.

The FPI method solves for the EEB iteratively by rewriting Eq. (2.2) as a fixed-point. When the underlying asset follows a SV diffusion, integral expressions are available for computing the right hand side of Eq. (2.2) in analytic form. Numerical techniques can then be employed for solving it efficiently. Nevertheless, the addition of jumps to the underlying dynamics makes the FPI approach unfeasible, turning simulation-based algorithms into an attractive alternative. We propose to generalize the approach of Cortazar et al. (2015) in order to solve for the EEB under general Markovian dynamics. We designate our approach the Simulated Fixed-Point Iteration (S-FPI) method.

Let us denote the optimal stopping time by τ^c, and formally define

$$\tau^c(t, Y_t; S^c) = \begin{cases} u & \text{if u =argmin}_{v\in[t,T]}\{S_v < S^c(v, Y_v)\} \text{ exists,} \\ \infty & \text{if not.} \end{cases} \tag{2.4}$$

Using the above expression, we can re-write equation (2.1) as

$$P(t, S_t, Y_t; S^c) = \mathbb{E}\left[e^{-R_{\tau^c}}(K - S_{\tau^c})^+ | \mathscr{F}_t\right]. \tag{2.5}$$

The following proposition characterizes the delta and gamma of the option.

PROPOSITION 1. For the American put we have that

$$\Delta_P = -S_t^{-1}\mathbb{E}\left[e^{-R_{\tau^c}} S_{\tau^c} \mathbb{1}_{\{K \geq S_{\tau^c}\}} | \mathscr{F}_t\right], \tag{2.6}$$

and $\Gamma_P > 0$.

PROOF. See the appendix. □

Furthermore, we note that the value of the American put in (2.5) stays constant if just one point of the EEB changes. Formally, if we index the set $[0,T] \times D$ by α and denote by S^c_α a point of the EEB, then it must be the case that

$$\frac{\partial P(t, S_t, Y_t; S^c)}{\partial S^c_\alpha} = 0, \qquad \forall \alpha \in [0,T] \times D. \tag{2.7}$$

Hence, for $\alpha = (t, Y_t)$, we also have that:

$$\frac{\partial P(t, S^c_\alpha, Y_t; S^c)}{\partial S^c_\alpha} = \left.\frac{\partial P(t, S_t, Y_t; S^c)}{\partial S_t}\right|_{S_t = S^c_\alpha} + \left.\frac{\partial P(t, S_t, Y_t; S^c)}{\partial S^c_\alpha}\right|_{S_t = S^c_\alpha} = \Delta_P(t, S^c_\alpha, Y_t). \tag{2.8}$$

We consider now the operator $\Phi : \mathcal{C}_b([0,T] \times D) \to \mathcal{C}_b([0,T] \times D)$ defined point-wise as

$$\Phi(S^c)(t, Y_t) = S^c(t, Y_t) - K + \mathbb{E}\left[e^{-R_{\tau^c}}(K - S_{\tau^c})^+ | S_t = S^c(t, Y_t), Y_t\right], \forall (t, Y_t) \in [0,T] \times D, \tag{2.9}$$

where $\mathcal{C}_b([0,T] \times D)$ denotes the space of bounded continuous functions on $[0,T] \times D$ endowed with the supremum norm. The previous analysis shows that the Fréchet derivative Φ' of Φ is given point-wise by

$$\Phi'(S^c)(t, Y_t) = 1 + \Delta_P(t, S^c(t, Y_t), Y_t), \qquad \forall (t, Y_t) \in [0,T] \times D,$$

and more importantly, it's inverse $(\Phi')^{-1}$ is given point-wise by

$$[\Phi'(S^c)]^{-1}(t, Y_t) = \frac{1}{1 + \Delta_P(t, S^c(t, Y_t), Y_t)}, \qquad \forall (t, Y_t) \in [0, T] \times D.$$

As in Cortazar et al. (2015), we use the Newton-Kantorovich method to solve for the EEB as follows. Starting from an initial guess $S^{c(0)} \in \mathcal{C}_b([0, T] \times D)$ of the whole early exercise boundary, a new approximation $S^{c(1)} \in \mathcal{C}_b([0, T] \times D)$ can be obtained as follows:

$$S^{c(1)} = S^{c(0)} - [\Phi'(S^{c(0)})]^{-1}\Phi(S^{c(0)}).$$

Hence, given an approximation of the whole early exercise boundary $S^{c(k)} \in \mathcal{C}_b([0, T] \times D)$ after k iterations, a new approximation $S^{c(k+1)} \in \mathcal{C}_b([0, T] \times D)$ can be found:

$$S^{c(k+1)} = S^{c(k)} - [\Phi'(S^{c(k)})]^{-1}\Phi(S^{c(k)}).$$

We now operationalize the method by noting that for a given pair (t, Y_t) we have that

$$S^{c(k+1)}(t, Y_t) = K\frac{V(t, Y_t; S^{c(k)})}{U(t, Y_t; S^{c(k)})}, \tag{2.10}$$

where the two functions in the fraction are defined as

$$\begin{aligned} U(t, Y_t; S^c) &= 1 - S^c(t, Y_t)^{-1}\mathbb{E}\left[e^{-R_{\tau^c}} S_{\tau^c} \mathbb{1}_{\{K \geq S_{\tau^c}\}} | S_t = S^c(t, Y_t), Y_t\right], && (2.11) \\ V(t, Y_t; S^c) &= 1 - \mathbb{E}\left[e^{-R_{\tau^c}} \mathbb{1}_{\{K \geq S_{\tau^c}\}} | S_t = S^c(t, Y_t), Y_t\right]. && (2.12) \end{aligned}$$

It is interesting to note that (2.10) implies the fixed-point:

$$S^c(t, Y_t) = K\frac{V(t, Y_t; S^c)}{U(t, Y_t; S^c)}, \tag{2.13}$$

that can be obtained directly from (2.2). In other words, the fixed-point iteration implied by the early-exercise optimality conditions turns-out to be a Newton-Kantorovich iteration, which among other things, converges faster then a normal fixed-point iteration.

Furthermore, the method is guaranteed to converge. Say we discretize time such that $\mathcal{T} = [0 = t_0, t_1, \ldots, t_N = T]$ denotes the set of possible time values, and that N is large enough for the time-discretization to yield accurate results. First, we have that $S^c(T, Y_T) = K$. Second, the iteration converges for $t = t_{N-1}$ since $\Phi(S^c)(t_{N-1}, Y_{t_{N-1}})$ is decreasing and convex in $S^c(t_{N-1}, Y_{t_{N-1}})$ according to Proposition 1. This occurs for all $Y_{t_{N-1}} \in D$. Finally, assuming that we have the EEB up to time t_{h+1}, the method will also converge to $t = t_h$ since $\Phi(S^c)(t_h, Y_{t_h})$ is decreasing and convex in $S^c(t_h, Y_{t_h})$. This also occurs for all $Y_{t_h} \in D$. We collect the previous remarks in the following proposition.

PROPOSITION 2. The iteration defined as:

$$S^{c(k+1)}(t, Y_t) = K \frac{V(t, Y_t; S^{c(k)})}{U(t, Y_t; S^{c(k)})}, \quad \forall (t, Y_t) \in \mathcal{T} \times D$$

is equivalent to a Newton-Kantorovich iteration and converges globally to the solution of

$$S^c(t, Y_t) - K + \mathbb{E}\left[e^{-R_{\tau^c}}(K - S_{\tau^c})^+ | S_t = S^c(t, Y_t), Y_t\right] = 0, \qquad \forall (t, Y_t) \in \mathcal{T} \times D.$$

Note that, at each step k, the new approximation $S^{c(k+1)}(t_1, y_1)$ for a given $t = t_1$ and $Y_t = y_1$ is computed independently from the new approximation $S^{c(k+1)}(t_2, y_2)$, corresponding to $t = t_2$ and $Y_t = y_2$. This feature of the fixed-point iteration is convenient from a numerical point of view, since it allows to compute the values of $S^{c(n)}$ at each point of $[0, T] \times D$ in parallel.

In Cortazar et al. (2015), the computation of expectations in (2.11) and (2.12) relies on quasi-analytical expressions for the early-exercise premium. In order to develop a more flexible method for computing the EEB under jump-diffusion and even more general models, we solve these expectations via Monte Carlo simulations.

Let us consider a discrete version of $[0, T] \times D$ denoted by $\mathscr{P}$. We denote the discrete version of S^c by $B = \{B_p\}_{p \in \mathscr{P}}$, so that B and S^c have matching values on $\mathscr{P}$. We extend this discrete version to $[0, T] \times D$ by linear interpolation. The algorithm starts with an initial estimation given by $B^{(0)} = K$ and approximations are refined through

$$B_p^{(k+1)} = K\frac{V(p; B^{(k)})}{U(p; B^{(k)})}, \qquad p \in \mathscr{P}. \tag{2.14}$$

Let us denote by $\widehat{S}$, $\widehat{Y}$ the discretely-simulated paths for S and Y, respectively. Paths $\widehat{S}$ and $\widehat{Y}$ follow dynamics determined by the continuous-time specification of the system X. Discrete paths are simulated using an Euler approximation with time-step of length h and a number of time-steps N. We denote the simulation's i-th time-step by $t_i = i \cdot h$ for $1 \leq i \leq N$. For each iteration k of the algorithm and each $p = (t, y) \in \mathscr{P}$, the simulation starts with a pair of initial values $(\widehat{S}_0, \widehat{Y}_0) = (B_p^{(k)}, y)$ and stops either when the option expires, or the simulated path $\widehat{S}$ crosses the current early exercise boundary $B^{(k)}$. Note that this early exercise decision is taken individually for each trajectory, leading to improvements in efficiency when the simulations are carried out in parallel.

We define the stopping time estimator $\widehat{\tau}$ and the stopped process estimator $\widehat{S}_{\widehat{\tau}}$ as follows:

$$\begin{aligned} \widehat{\tau} &= \begin{cases} t_l & \text{if } l = \min\{i | \widehat{S}_{t_i} < B^{(k)}(t_i, \widehat{Y}_{t_i})\} \text{ exists,} \\ \infty & \text{if not.} \end{cases} \\ \widehat{S_{\widehat{\tau}}} &= \begin{cases} \widehat{S}_{t_l} & \text{if } l = \min\{i | \widehat{S}_{t_i} < B^{(k)}(h_i, \widehat{Y}_{t_i})\} \text{ exists,} \\ 0 & \text{if not.} \end{cases} \end{aligned} \tag{2.15}$$

We consider the following estimators of the operators U and V:

$$\widehat{U}\left(p; B^{(k)}\right) = 1 - \left(B_p^{(k)} M\right)^{-1} \sum_{m=1}^{M} e^{-R_{\widehat{\tau}^{(m)}}} \widehat{S_{\widehat{\tau}}}^{(m)} \mathbb{1}_{\{K \geq \widehat{S_{\widehat{\tau}}}^{(m)}\}} \tag{2.16}$$

$$\widehat{V}\left(p; B^{(k)}\right) = 1 - M^{-1} \sum_{m=1}^{M} e^{-R_{\widehat{\tau}^{(m)}}} \mathbb{1}_{\{K \geq \widehat{S_{\widehat{\tau}}}^{(m)}\}}, \tag{2.17}$$

where M represents the number of simulation trajectories and $\left(\tau^{(m)}, \widehat{S}_{\widehat{t}}^{(m)}\right)$ denote the stopping time and stopped process estimators for trajectory m, respectively.

In each iteration of the algorithm, we go over each node $p \in \mathscr{P}$ individually until the improvement on the estimation of B_p is no longer significant. Given a tolerance $\epsilon > 0$, and $p \in \mathscr{P}$, we define k_p as the smallest integer such the following condition is attained:

$$\left| \frac{B_p^{(k_p)} - B_p^{(k_p-1)}}{K} \right| < \epsilon \tag{2.18}$$

Once the iteration k_p is reached, we define $B_p^{(k)} = B_p^{(k_p)}, \forall k > k_p$. Thus, the algorithm stops independently for each $p \in \mathscr{P}$.

We price the American option via Monte Carlo simulation using the estimated EEB. Paths are simulated for the underlying asset and its stochastic factors, starting on the spot values (S_0, Y_0) and use the EEB to decide which and when paths must be exercised. Cash flows yielded by exercising the option are then discounted and averaged to price the option as in (2.5). Following the scheme described previously in this section, we employ the estimators $\widehat{\tau}$ and $\widehat{S}_{\widehat{\tau}}$ defined above to introduce the price estimate:

$$\widehat{P}(t; B) = M^{-1} \sum_{m=1}^{M} e^{-R_{\widehat{\tau}^{(m)}}} \left(K - \widehat{S_{\widehat{\tau}}}^{(m)} \right)^+ . \tag{2.19}$$

2.5. Numerical Implementation

To demonstrate the performance of the S-FPI algorithm outlined in Sec. 2.4 we implement the method to obtain the EEBs for three nested models: stochastic volatility (SV), stochastic volatility with jumps (SVJ) and stochastic volatility with contemporaneous jumps (SVCJ). We start by introducing the three models and their properties, and then turn to the numerical implementation and results. We compare the exercise rules across the different models, and discuss their most relevant differences. We test the pricing accuracy of the S-FPI method using the LSM method as a benchmark, and find that the S-FPI provide accurate prices for all models considered.

2.5.1. Underlying Dynamics

We consider three nested models in our analysis: the SV, SVJ and SVCJ model. Under the risk-neutral measure, general dynamics of stock prices for these three specifications are given by

$$\frac{dS_t}{S_t} = (r - q - \lambda\eta)dt + \sqrt{v_t}dW_t^S + Z_t^S dN_t^S \tag{2.20}$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t^v + Z_t^v dN_t^v, \tag{2.21}$$

where S_t is the price process and v_t is the variance process. The riskless instantaneous rate of return is given by r, and q corresponds to the instantaneuos dividend yield. The parameters θ and κ measure the long-term level of variance and the speed of reversion, respectively. The parameter σ is known as the ”volatility-of-volatility”. The Brownian increments, dW^S and dW^v are correlated and $\mathbb{E}[dW_t^S dW_t^v] = \rho dt$. The jump term has a jump-size component Z_t and a component given by a Poisson counting process N_t with intensity λ. Furthermore, $\lambda\eta$ correspond to the jump compensator term under the equivalent measure. Jump sizes Z_t^S and Z_t^v can be correlated and depend on the specification considered for the model.

The SV model, initially proposed by Heston (1993), is obtained by setting $N_t^S = N_t^v = 0$. The SVJ model is an extension to the SV model that allows for jumps to occur in spot prices but not in the variance, i.e. $N_t^v = 0$. In this model, jump sizes are distributed

$$Z_t^S \sim N(\mu_S, \sigma_S^2).$$

Finally, the SVCJ model allows for jumps in prices and volatility, where both jumps are driven by the same Poisson process, i.e. $N^S = N^v$. This allows jump sizes to be correlated, and we have that

$$Z_t^v \sim \exp(\mu_v),$$

$$Z_t^S | Z_t^v \sim N(\mu_S + \rho_J Z_t^v \sigma_S^2).$$

2.5.2. Numerical Results

We test our algorithm against the FPI method for the SV specification, and against the LSM-EEB method for more general specifications. For the SV specification, we find that S-FPI and FPI deliver similar results both in the pricing and EEB estimation. We use our algorithm to explore further properties of the early exercise rules for the SV, SVJ and SVCJ models, and assess the efficiency of the S-FPI and LSM-EEB methods in solving for the EEB under these specifications. Finally, we study the pricing accuracy of our algorithm with respect to prices provided by the LSM.

In the case of the three nested specifications given by the SV, SVJ and SVCJ models, the state space D defined in Sec. 2.2 corresponds to the range of the stochastic process v. Thus, we define the mesh $\mathscr{P}$ as a discrete version of $[0,T] \times [0,V]$ consisting in N_T time-nodes and N_V variance-nodes, where we set $V = 2.50$. EEBs computed in this section are estimated using $N_T = 12$ and $N_V = 12$, and simulations are carried out with 64 time-steps per month -roughly two steps per day- and $M = 4$ million trajectories for all maturities. Pricing is carried out using simulations of $M = 1$ million trajectories and 64 time-steps per month. The source code for all methods is implemented in MATLAB running on an Intel Core i7-4710HQ 16GB with 2.50GHz and a NVIDIA GeForce GTX 860M. All simulations are executed in parallel using CUDA kernels designed with this purpose.

In the numerical experiments we use the parameters reported by Eraker (2004), who uses data on S&P500 prices and options from 1987 to 1990 for his estimation, a period of time that includes the stock market crash of October 1987. Table 2 presents the annualized parameters estimated by Eraker (2004). Throughout this section we consider a strike price $K = 100$ for every contract.

As in Sec. 2.3, let us consider a 1-month put option and the SV specification to compare the S-FPI, FPI and LSM-EEB early exercise boundaries. Figure 3 shows the exercise rule estimated by the S-FPI. The resulting exercise rule is a smooth boundary, strictly increasing on t and decreasing on v. Figure 4 compares S-FPI, FPI and LSM-EEB boundaries' cross-sections for spot variances $v = 0.01, 0.05, 0.5,$ and 1. The figure reveals that the S-FPI boundary closely approximates the one obtained by the FPI for the SV specification. Our boundary presents a small positive bias with respect to its quasi-analytical counterpart. This bias is due to the domain discretization and tends to zero as the mesh $\mathscr{P}$ is refined. Hence, the S-FPI and FPI methods feature a much more conservative policy than the one discovered by LSM-EEB. This feature persists until maturity, where the LSM-EEB exhibits an abrupt convergence to the strike value.

Table 3 shows prices obtained with each algorithm for a 1-month put option using the SV specification for different moneyness and spot variance levels. The results show that the S-FPI algorithm is accurate in pricing the option contracts. The S-FPI algorithm presents smaller RMSE and RRMSE values than the LSM-EEB for all settings, specially for low variances, which is consistent with the differences presented in Fig. 4. Note that S-FPI prices are greater than LSM-EEB prices in every setting, empirically confirming that a better exercise rule leads to a higher option value.

Figure 5 compares the optimal surface discovered by the S-FPI for a 1-month put option under the SV, SVJ and SVCJ models. Cross-sections are computed for spot variances $v = 0.01, 0.05, 0.5,$ and 1. The figure reveals that the cross-sections corresponding to the SV and SVJ models describe more conservative policies than the one found for the SVCJ model, particularly for small variances. The fundamental reason lies on the incorporation of variance jump events on the SVCJ model. Such events can lead to large negative returns and increases in variance. Thus, the SVCJ model requires a smaller critical price for the early exercise of the option. This implicitly accounts for the higher probability of strong devaluation of the underlying asset, a main feature of the crisis period that is incorporated in the SVCJ parameter estimates. On the other hand, as the spot variance increases, the exercises boundaries under the SV and SVJ model tend to resemble the SVCJ policy. In these high variance scenarios, the mean-reversion to long-term variance dominates the impact of expected jumps on the variance process. Therefore, higher spot variances imply lower jump impact on the policy, dramatically decreasing the gap between the three models' exercise rules.

Table 4 exhibits both algorithms' runtimes for 1, 3, 6 and 12-month put options under SV, SVJ and SVCJ specifications. For this experiment we have considered a smaller setup, since computation of the EEB using the LSM-EEB method is highly demanding in terms of memory consumption, specially for longer contracts. We set $N_T = N_V = 7$ and $M = 1$ million simulated trajectories. Results show the S-FPI and LSM-EEB exhibit comparable runtimes for relatively small sized problems. As the size of the problem increases, the advantages of the S-FPI become more apparent. On one hand, backward regressions performed by the LSM require to store the whole trajectory for every simulated path, resulting in high memory usage and a rapidly scaling computational cost when the parallel computing capacity is limited. On the other hand, the S-FPI only needs to keep track of the most recent price for every trajectory, and only until it crosses the exercise boundary or reaches expiration. The difference between both algorithms is fully appreciated for the longest maturity, where the S-FPI is up to 50% faster than the LSM-EEB.

2.5.3. Pricing Accuracy of the S-FPI Method

The purpose of this section is to establish the accuracy of the S-FPI pricing methodology described in Sec. 2.4. In order to achieve this, we use our algorithm to price put options for several maturities, levels of moneyness and spot variance, and compare the results to those obtained with the LSM methodology. We price options using EEBs with refining $\mathscr{P}$ to measure the convergence and find that our prices tend to the ones discovered by the LSM as the mesh grid dimension increases.

In the interest of covering a wide range of scenarios, we consider 1, 3 and 6-month put contracts, with moneyness ranging from 100% to 120%, and spot variance ranging from 5% to 100%. We refine the grids by increasing the number of time-nodes and variance-nodes, set to $N_T = 6, 8, 12$ and $N_V = N_T$. The experiment is carried out for the SV, SVJ and SVCJ specifications. All pricing simulations were carried out using $M = 1$ million paths and 64 time-steps per month.

Table 5 displays the overall pricing errors for each maturity and specification considered. The table presents the RMSE, RMSRE and MRE for every configuration. We first note that the RMSE tends to decrease as we refine the grid, accompanied by an increase on the MRE. This implies that our prices increase as the mesh becomes finer, which is consistent with maximizing the option value. In this sense, we see pricing convergence as the grid is refined.

Our experiments show that the method converges for all three models. However, we note that the rate of convergence is higher for the SV specification than for the other models, where the rate of convergence seems to moderately decrease as we add jumps to prices and the variance, to a level such that the RMSE for the SVJ and SVCJ specifications slightly increases when reaching the finer mesh for some maturities. Note that the MRE is positive in these scenarios, indicating that our prices are greater than those provided by the LSM. Since pricing bias for the S-FPI is due to discrete simulation and the LSM shares this bias, positive MRE is unlikely to be due to S-FPI pricing overestimation. It is possible that the LSM may underestimate prices when jump dynamics are considered on the specification.

2.6. Optimal Exercise Policy for the S&P500 Index

In this section we analyze whether the EEB estimated by the S-FPI method leads to higher payoffs than the LSM-EEB when exercising the options. We consider the particular case of options written on the S&P500 in 1987, and compute the cash flows that these contracts would yield to an option holder when they are exercised according to each one of the exercise rules.

As before, we use the SV, SVJ and SVCJ specifications in our analysis. We start by generating paths for the underlying asset and its variance by sampling values from the S&P500 index and VXO index daily historical quotes, respectively. For consistency with the methodology employed in Eraker (2004), we set a testing period ranging from January 1, 1987 to December 31, 1990. We consider 1, 3, 6 and 12-month put contracts. Thus, we sample paths of $d = 21, 63, 126$ and 250 days from the historical data. For each term, the first trajectory will consist of the first d values of the index. Then, we move one day forward and repeat to obtain the second trajectory, and so on. For each path, we compute the cash flow according to the S-FPI and LSM-EEB exercise rules separately, and then we discount and average the payments. Intuitively, this is equivalent to entering an option contract for a given maturity on each day of the period. Each of these contracts is then exercised according to the computed optimal policies.

As in previous experiments, we set the moneyness ranging from 1.0 to 1.2 and scale the price trajectories to achieve the required moneyness. We perform a two-tailed paired t-test between the S-FPI and LSM-EEB payoffs for every setting in order to establish which of them present statistically significant results. Table 6 presents the results of the experiment. The analysis reveals that the S-FPI method finds a better exercise policy than the LSM-EEB algorithm. Performance of the S-FPI algorithm is similar across models. At the 1% of statistical significance, discounted payoffs are on average higher by 8,3% for the SVCJ model, 9,9% for the SVJ model and 9,5% for the SV model. Considering only statistically significant scenarios, the S-FPI method turns out to yield better results on 12 out of 13 settings for the SVCJ model and in 14 out of 15 for both the SVJ and the SV models.

2.7. Extension: Lévy Processes

Although the analysis has focused so far on the SVCJ family of models, the flexibility of the S-FPI method allows us to test even more general alternatives for asset price dynamics, such as infinite activity pure jump Lévy Processes. A number of authors have proposed the use of infinite activity pure jump Lévy processes to model the dynamics of asset prices (Eberlein, Keller, and Prause (1998), Barndorff-Nielsen and Shepard (2001) and Madan et al. (1998), Geman, Madan, and Yor (2001), Hirsa and Madan (2004)). Geman et al. (2001) argue that such processes are the norm when it is recognized that time changes with martingale components are involved in describing the price evolution. At an empirical level, Carr and Hirsa (2003) recognize that the infinite activity of such Lévy processes efectively synthesizes the role of a diffusion component. We implement the S-FPI methodology for a Lévy process in order to discover its early exercise boundary.

We choose the Variance Gamma (VG) process introduced in Madan et al. (1998) to illustrate the performance of our algorithm under an infinite activity Lévy process. The VG stock price process has no continuous martingale component. It is an example of a pure jump process having an infinite number of jumps in any interval of time. The process may be presented in a variety of ways and is often described as a time changed Brownian motion with drift.

Let $b(t;\theta,\sigma) = \theta t + \sigma W_t$, be a Brownian motion with constant drift rate θ and volatility σ, where W_t is standard Brownian motion. Now define the gamma process $\gamma_t(1,\nu)$ with independent gamma increments over intervals of length h with mean h and variance rate νh. The three parameter VG process $X(t; \sigma, \theta, \nu)$ is defined by

$$X_t(\sigma, \theta, \nu) = b(\gamma_t(1, \nu), \theta, \sigma). \tag{2.22}$$

The VG dynamics of the stock price mirrors that of a geometric Brownian motion for a stock paying a continuous dividend yield of q in an economy with a constant continuously compounded interest rate of r. The risk neutral drift rate for the stock price is $r - q$ and the forward stock price is modeled as the exponential of a VG process normalized by its expectation. The VG risk neutral process for the stock price is given by

$$S_t = S_0 e^{(r-q)t + X_t + \omega t} \tag{2.23}$$

where $\omega = \nu^{-1} \ln(1 - \theta\nu - \sigma^2\nu/2)$. Numerical estimation of the EEB is performed as in previous sections. We consider parameters found on Hirsa and Madan (2004) which are calibrated using S&P500 options for 1999.

We estimate the EEB for a 6-month put contract with exercise price $K = 100$. The resulting boundary is displayed on Fig. 6. The computed EEB using the S-FPI algorithm under the Variance Gamma specification is smooth and increasing in t. In unreported results, we use this boundary to price the American put and compare the results with prices obtained using the LSM. As in previous sections, we find that prices computed by the S-FPI approximate their true value as the boundary is refined, reasserting the accuracy of the algorithm for computing the EEB and pricing American options for more exotic configurations, as general Lévy processes.

2.8. Concluding Remarks

In this paper we propose the Simulated Fixed-Point Iteration method (S-FPI), a simple and robust numerical approach to calculate the optimal exercise policy of an American option under general Markovian dynamics. Our approach is based on a Newton-Kantorovich fixed-point iteration that is easy to compute and exhibits global fast convergence. We solve numerically for optimal exercise policies for several models that exhibit stochastic volatility and jumps.

We test our algorithm through numerical experiments using the SV, SVJ, and SVCJ models, and for a type of infinite activity Lévy process, namely the Variance Gamma model introduced by Madan et al. (1998). In these experiments, we assess the accuracy of the S-FPI method by comparing the estimated EEBs and prices computed by the algorithm with those obtained by (i) the FPI method of Cortazar et al. (2015) for the SV model, and (ii) the LSM method of Longstaff and Schwartz (2001).

The results show that our method is stable, robust, converges accurately for all the models that we test, and is well suited for parallel calculations and programming, substantially increasing the speed of execution. In particular, our analysis reveals that the results obtained using the S-FPI and FPI algorithms are remarkably similar, validating our methodology for the SV specification. On the other hand, we find that the LSM methodology presents a series of shortcomings when used to solve for the EEB, in particular in the presence of jump dynamics for the underlying's price and variance, something that our method is able to handle well.

We also show that when using real data, our algorithm uncovers a more profitable exercise policy than the widely used least-square Monte Carlo (LSM) method of Longstaff and Schwartz (2001). Of course, the LSM approach was developed to yield accurate pricing of American options and not to uncover optimal-exercise policies.

We believe that our method complements well the existing literature on American option pricing by providing a simple and robust method to estimate the EEB in a wide variety of interesting and challenging models.

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APPENDIX

A. PROOFS

PROOF OF PROPOSITION 1. Let $V(S,K) = \mathbb{E}\left[e^{-R_{\tau^c}}(K - S_{\tau^c})^+ | S_t = S, Y_t\right]$ and note that $V(S,K)$ is homogeneous of order one, i.e. $V(\lambda S, \lambda K) = \lambda V(S,K)$ since S exhibits constant returns to scale. We can then apply Euler's theorem to find

$$SV_S + KV_K = V, \tag{A.1}$$

which can be written as

$$\Delta_P = V_S = \frac{V - KV_K}{S}.$$

Moreover,

$$\begin{aligned}
V_K &= \frac{\partial}{\partial K}\mathbb{E}\left[e^{-R_{\tau^c}}(K - S_{\tau^c})^+ | S_t = S, Y_t\right], \\
&= \frac{\partial}{\partial K}\int_0^T \mathbb{E}\left[e^{-R_{\tau^c}}(K - S_{\tau^c})^+ | S_t = S, Y_t, \tau^c = u\right] f_{\tau^c}(u)du, \\
&= \frac{\partial}{\partial K}\int_0^T \mathbb{E}\left[e^{-R_u}(K - S_u)^+ | S_t = S, Y_t\right] f_{\tau^c}(u)du, \\
&= \int_0^T e^{-R_u}\left(\frac{\partial}{\partial K}\int_0^K (K - v) f_{S_u}(v)dv\right) f_{\tau^c}(u)du, \\
&= \int_0^T e^{-R_u}\left(\int_0^K f_{S_u}(v)dv\right) f_{\tau^c}(u)du, \\
&= \mathbb{E}\left[e^{-R_{\tau^c}}\mathbb{1}_{\{K \geq S_{\tau^c}\}} | S_t = S, Y_t\right],
\end{aligned}$$

where we make use of Leibniz's rule in the fifth line, and $f_{\tau^c}(u)$ and $f_{S_u}(v)$ denote the $\mathscr{F}_t$-conditional density functions of τ^c and S_u, respectively.

Replacing $V = \mathbb{E}\left[e^{-R_{\tau^c}}K\mathbb{1}_{\{K \geq S_{\tau^c}\}} | S_t = S, Y_t\right] - \mathbb{E}\left[e^{-R_{\tau^c}}S_{\tau^c}\mathbb{1}_{\{K \geq S_{\tau^c}\}} | S_t = S, Y_t\right]$ and $V_K = \mathbb{E}\left[e^{-R_{\tau^c}}\mathbb{1}_{\{K \geq S_{\tau^c}\}} | S_t = S, Y_t\right]$ in the above expression yields (2.6). Furthermore, differentiating (A.1) with respect S and K, and canceling out the cross-derivative terms yields

$$f_{SS} = \left(\frac{K}{S}\right)^2 f_{KK}.$$

We note that as K increases, the measure of the set $\{K \geq S_{\tau^c}\}$ is larger, implying that F_K also increases. Hence we can conclude that $f_{KK} > 0$, which proves that $\Gamma_P = f_{SS} > 0$. $\square$

B. FIGURES

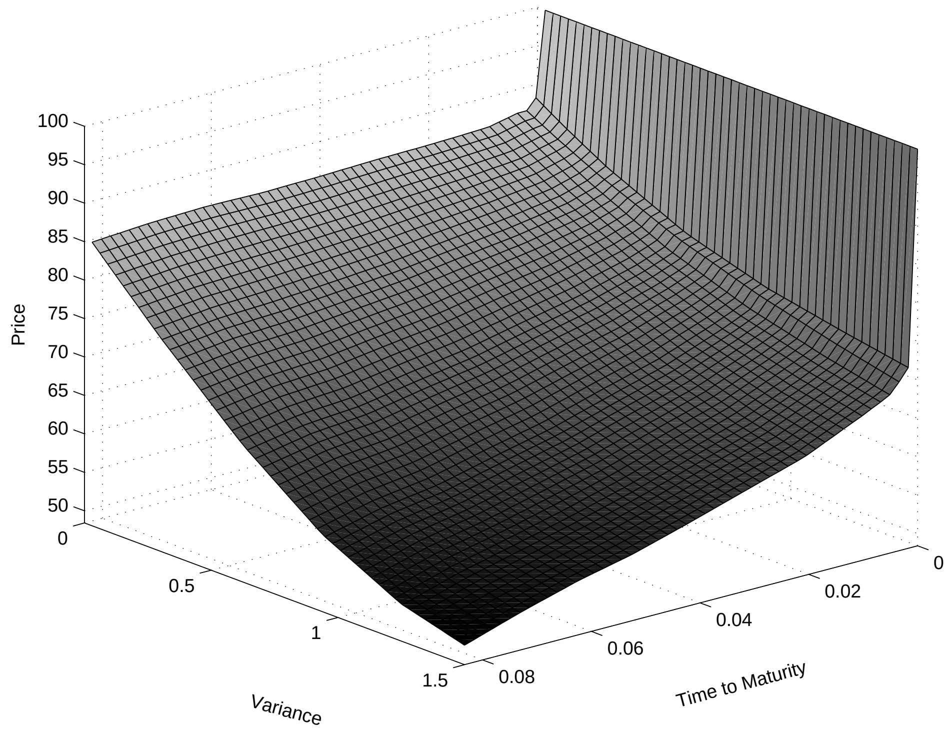

Figure 1. Early-exercise boundary for a 1-month American put contract computed with the LSM-EEB Algorithm under the stochastic volatility specification.

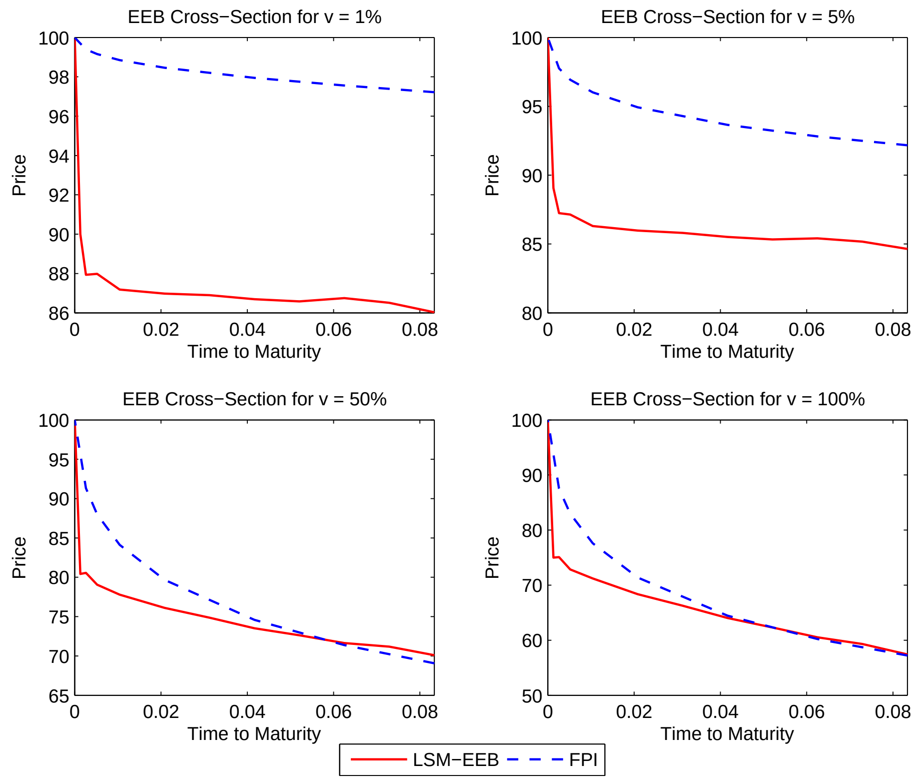

Figure 2. Early-exercise boundary cross-sections for a 1-month American put contract computed with the LSM-EEB and FPI algorithms under the stochastic volatility specification. Spot variance ranges from 1% to 100%.

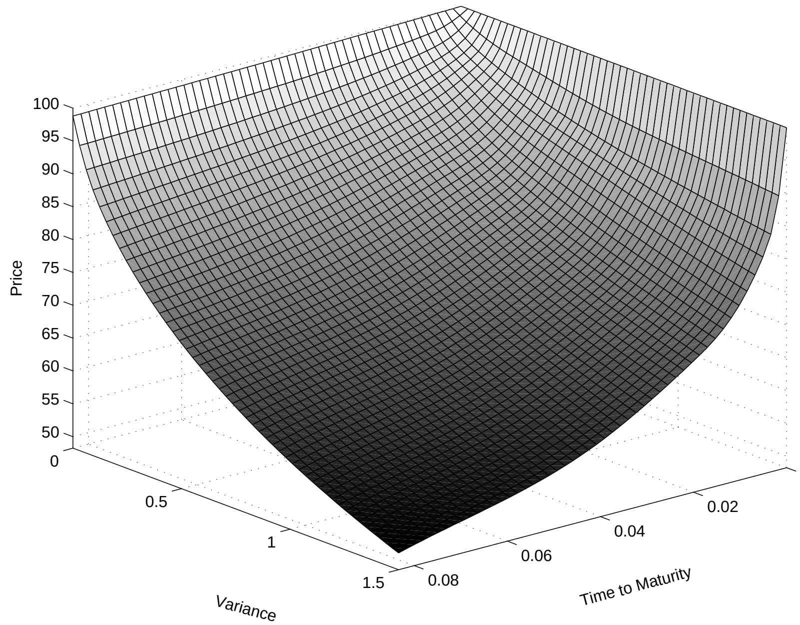

Figure 3. Early-exercise boundary for a 1-month American put contract computed with the S-FPI algorithm under the stochastic volatility specification.

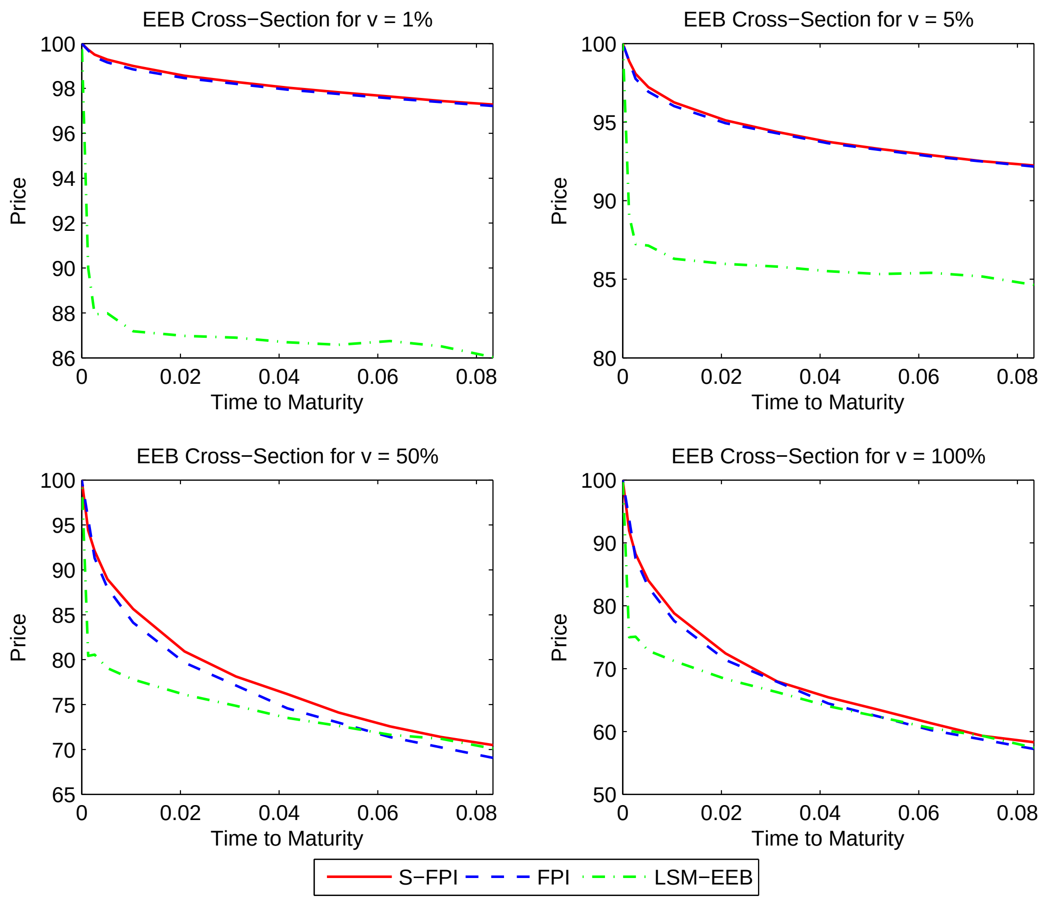

Figure 4. Early-exercise boundary cross-sections for a 1-month American put contract computed with the S-FPI, FPI and LSM-EEB algorithms. Spot variance values range from 1% to 100%.

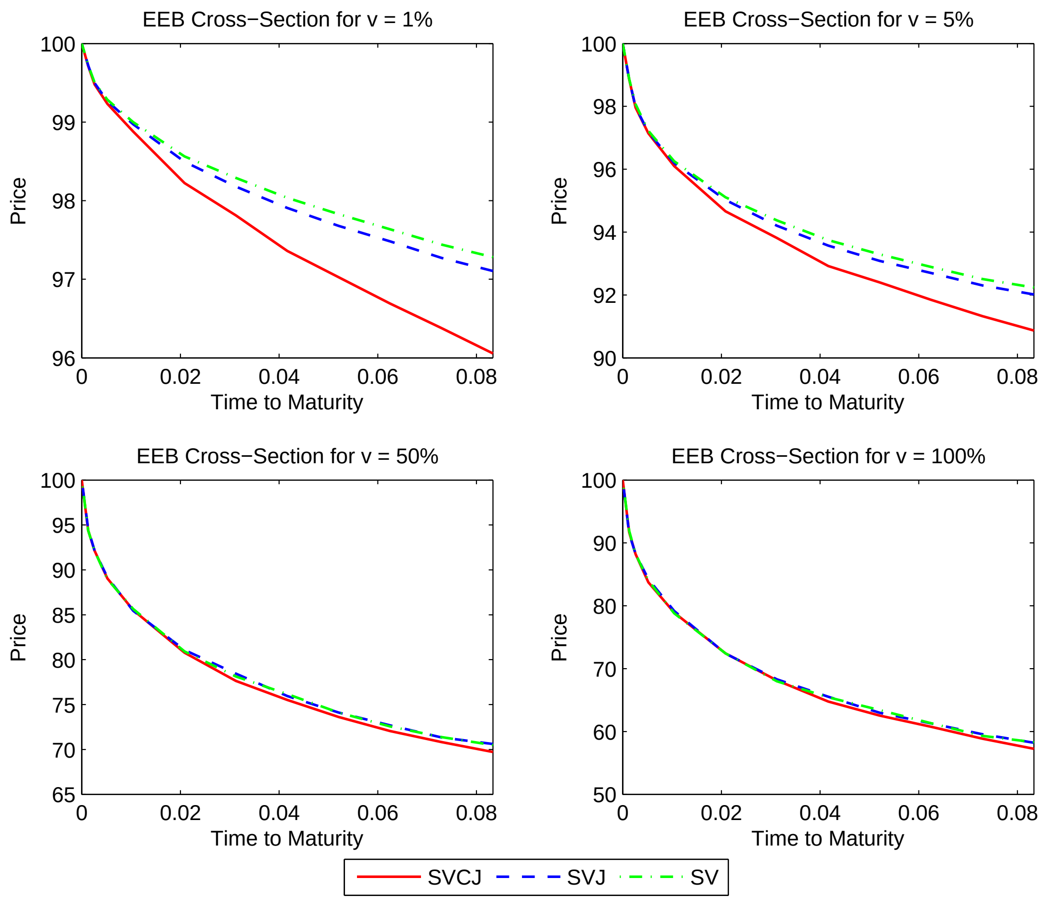

Figure 5. Cross-sections for the early exercise boundary for a 1-month American put contract computed with the S-FPI algorithm. Spot variance values range from 1% to 100%.

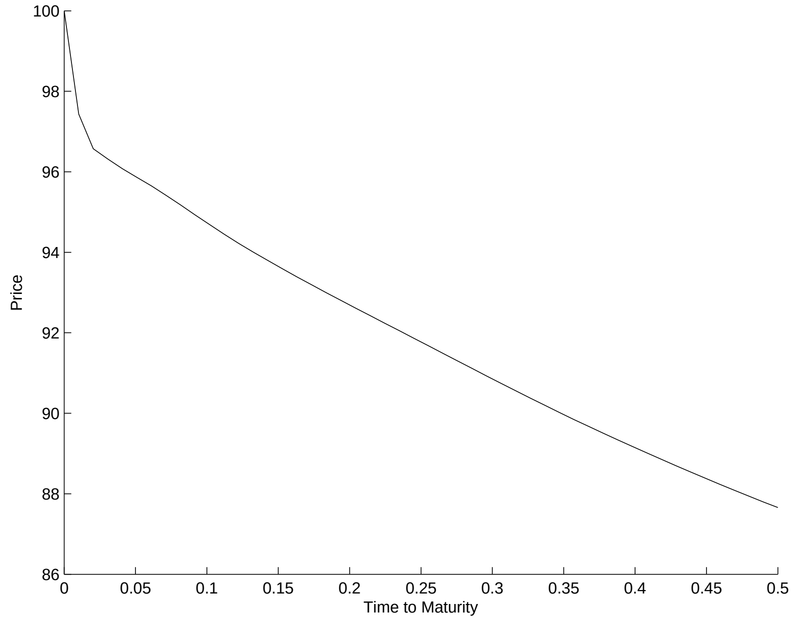

Figure 6. Early-exercise boundary for a 6-month American put contract computed with the S-FPI algorithm under the Variance Gamma pricing model.

C. TABLES

Table 1. American put prices and performance statistics for the LSM-EEB and FPI pricing method.

		Moneyness							
	Method	1,00	1,05	1,10	1,15	1,20	RMSE	RMSRE	MRE
$v_0 = 5\%$	True Value	2,29750	5,12600	9,08230	13,03430	16,65750			
	LSM-EEB	2,26024	4,98590	8,70622	12,74490	16,65659	*0,22190*	*0,02537*	*-0,02144*
	FPI	2,29747	5,12784	9,08309	13,04134	16,66509	*0,00471*	*0,00036*	*0,00029*
$v_0 = 10\%$	True Value	3,28150	5,91180	9,28460	13,03560	16,65750			
	LSM-EEB	3,24972	5,83788	9,14412	12,82872	16,58662	*0,12168*	*0,01224*	*-0,01149*
	FPI	3,27963	5,91242	9,29123	13,04566	16,66562	*0,00656*	*0,00058*	*0,00030*
$v_0 = 50\%$	True Value	7,50970	9,8428	12,39630	15,07930	17,85710			
	LSM-EEB	7,49237	9,82513	12,37982	15,08132	17,85211	*0,01351*	*0,00144*	*-0,00112*
	FPI	7,49773	9,83257	12,38965	15,09199	17,86556	*0,01024*	*0,00098*	*-0,00037*
$v_0 = 100\%$	True Value	10,68160	12,89490	15,21520	17,59890	19,9921			
	LSM-EEB	10,66845	12,88986	15,20523	17,58657	20,00142	*0,01036*	*0,00075*	*-0,00050*
	FPI	10,66655	12,88345	15,20414	17,58830	19,99943	*0,01137*	*0,00087*	*-0,00065*
Overall	LSM-EEB						*0,12682*	*0,01411*	*-0,00864*
	FPI						*0,00865*	*0,00074*	*-0,00011*

Note - The table reports American put prices and performance statistics for the LSM-EEB and FPI methods under the stochastic volatility option pricing model for a 1-month contract. Column (1) reports the spot variance level considered. Column (2) reports the method used, where true value makes reference to the put prices computed using the Least Square Monte Carlo Method by Longstaff and Schwartz (2001). Columns (3) to (7) report put prices for each initial moneyness level and $T = 1$ month. Columns (8), (9) and (10) refer to the root mean square error (RMSE), root mean square relative error (RMSRE) and mean relative error (MRE), respectively. Last three rows present the RMSE, RMSRE and MRE accross all variance levels presented in this table.

Table 2. Parameter estimates for the SV, SVJ and SVCJ models from Eraker (2004).

	SV	SVJ	SVCJ
θ	0.0487	0.0416	0.0341
κ	2.2680	2.7720	2.7720
ρ	-0.5690	-0.5860	-0.5820
σ	0.5544	0.5116	0.4108
λ	-	0.5040	0.5040
μ_S	-	-0.020	-0.0751
σ_S	-	0.0663	0.0363
μ_v	-	-	1.6380
ρ_J	-	-	-0.6930

Table 3. American put prices and performance statistics for the S-FPI, FPI and LSM-EEB pricing methods.

	Method	Moneyness 1,00	1,05	1,10	1,15	1,20	RMSE	RMSRE	MRE
$v_0 = 5\%$	True Value	2,29750	5,1260	9,08230	13,0343	16,65750			
	LSM-EEB	2,26024	4,9859	8,70622	12,7449	16,65659	*0,22190*	*0,02537*	*-0,02144*
	FPI	2,29747	5,12784	9,08309	13,04134	16,66509	*0,00471*	*0,00036*	*0,00029*
	S-FPI	2,29738	5,13057	9,08203	13,03441	16,65760	*0,00205*	*0,00040*	*0,00016*
$v_0 = 10\%$	True Value	3,28150	5,91180	9,28460	13,03560	16,65750			
	LSM-EEB	3,24972	5,83788	9,14412	12,82872	16,58662	*0,12168*	*0,01224*	*-0,01149*
	FPI	3,27963	5,91242	9,29123	13,04566	16,66562	*0,00656*	*0,00058*	*0,00030*
	S-FPI	3,28075	5,91451	9,28964	13,03546	16,65762	*0,00258*	*0,00033*	*0,00015*
$v_0 = 50\%$	True Value	7,50970	9,84280	12,39630	15,07930	17,85710			
	LSM-EEB	7,49237	9,82513	12,37982	15,08132	17,85211	*0,01351*	*0,00144*	*-0,00112*
	FPI	7,49773	9,83257	12,38965	15,09199	17,86556	*0,01024*	*0,00098*	*-0,00037*
	S-FPI	7,50852	9,83966	12,39636	15,09897	17,87716	*0,01265*	*0,00079*	*0,00039*
$v_0 = 100\%$	True Value	10,68160	12,89490	15,21520	17,59890	19,99210			
	LSM-EEB	10,66845	12,88986	15,20523	17,58657	20,00142	*0,01036*	*0,00075*	*-0,00050*
	FPI	10,66655	12,88345	15,20414	17,58830	19,99943	*0,01137*	*0,00087*	*-0,00065*
	S-FPI	10,68164	12,89776	15,21667	17,59792	20,01147	*0,00879*	*0,00045*	*0,00025*
Overall	LSM-EEB						*0,12682*	*0,01411*	*-0,00864*
	FPI						*0,00865*	*0,00074*	*-0,00011*
	S-FPI						*0,00788*	*0,00052*	*0,00024*

Note - The table reports American put prices and performance statistics for the S-FPI, FPI and LSM-EEB methods under the stochastic volatility pricing model for a 1-month contract. Column (1) reports the spot variance level considered. Column (2) reports the method used, where true value makes reference to the put prices computed using the Least Square Monte Carlo Method by Longstaff and Schwartz (2001). Columns (3) to (7) report put prices for each initial moneyness level and $T = 1$ month. Columns (8), (9) and (10) refer to the root mean square error, root mean square relative error and mean relative error, respectively. Last three rows present the RMSE, RMSRE and MRE across all variance levels presented in this table. Parameter values correspond to the ones provided in Table 2.

Table 4. Computation runtimes for the S-FPI and LSM-EEB under the SV, SVJ and SVCJ.

		SV		SVJ		SVCJ	
T	N	S-FPI	LSM-EEB	S-FPI	LSM-EEB	S-FPI	LSM-EEB
1 Month	64	66	61	74	60	74	61
3 Months	192	121	174	146	175	155	176
6 Months	384	199	350	246	347	247	346
1 Year	768	337	682	399	685	394	690

Note - Column (1) reports the maturity of the contract, and column (2) reports the corresponding time-steps N for the simulation. Columns (3) through (8) report runtimes in seconds for each setup.

Table 5. Pricing accuracy of the S-FPI method.

		RMSE			RMSRE			MRE		
	$\mathscr{P}$	SV	SVJ	SVCJ	SV	SVJ	SVCJ	SV	SVJ	SVCJ
$T = 1$ month	(6, 6)	0,03279	0,03161	0,02360	0,68483%	0,66995%	0,42534%	-0,31481%	-0,3185%	-0,23694%
	(8, 8)	0,00656	0,00725	0,00864	0,08968%	0,10631%	0,09554%	-0,03349%	-0,04859%	-0,01976%
	(12, 12)	0,00586	0,00651	0,00940	0,04636%	0,06630%	0,12156%	0,00985%	-0,00891%	0,01057%
$T = 3$ months	(6, 6)	0,04954	0,04852	0,04754	0,67227%	0,67502%	0,44438%	-0,44912%	-0,43727%	-0,36219%
	(8, 8)	0,01309	0,01111	0,02056	0,11840%	0,10882%	0,15035%	-0,08536%	-0,07195%	-0,07976%
	(12, 12)	0,00966	0,00783	0,01997	0,08438%	0,07653%	0,15891%	-0,01763%	-0,00106%	-0,02375%
$T = 6$ months	(6, 6)	0,05052	0,05027	0,04542	0,44932%	0,50411%	0,27896%	-0,37198%	-0,39349%	-0,24513%
	(8, 8)	0,02650	0,02151	0,02449	0,16267%	0,12784%	0,14229%	-0,04004%	-0,04588%	0,01714%
	(12, 12)	0,02437	0,02104	0,02599	0,20155%	0,16163%	0,16842%	0,02439%	0,01603%	0,05960%

The table reports the pricing errors of the S-FPI method with respect to the true values computed using the Least Square Monte Carlo Mehotd by Longstaff and Schwartz (2001). Pricing errors are computed over a sample of different put contracts, with moneyness ranging from 100% to 120% and spot variance ranging from 5% to 100%. Prices are computed using the SV, SVJ and SVCJ pricing models. Column (1) reports maturity of the set of contracts considered. Column (2) reports the grid size used in the calculation. Grid size refers to (N_T, N_V). Columns (3)-(5) report the root mean square error for the three nested models. Columns (6)-(8) report the root mean square relative error for the three nested models. Columns (9)-(11) report the mean relative error for the three nested models.

Table 6. Perforance of S-FPI and LSM-EEB optimal exercise policies on real data for the S&P500 index under the SV, SVJ and SVCJ specifications.

SVCJ model

	Monthly			Quarterly			Semi-annually			Annually		
K/S	S-FPI	LSM-EEB	*difference*	S-FPI	LSM-EEB	*difference*	S-FPI	LSM-EEB	*difference*	S-FPI	LSM-EEB	*difference*
1,00	1,52703	1,63561	-0,10857	2,46367	2,79193	-0,32826***	4,29118	4,19219	0,09899	4,69029	4,18678	0,50351***
1,05	4,59342	4,45291	0,14052	4,79522	4,86647	-0,07125	6,2913	5,98674	0,30456***	6,25204	6,13423	0,1178
1,10	8,99198	8,37354	0,61844***	8,60798	7,69407	0,91391***	8,51065	8,48951	0,02114	8,85886	7,82258	1,03628***
1,15	13,01036	12,39214	0,61822***	12,78899	11,46364	1,32535***	12,40928	11,52383	0,88545***	11,53029	10,65809	0,8722***
1,20	16,60831	16,65425	-0,04593	16,36589	15,38757	0,97832***	16,37065	14,38059	1,99006***	15,47637	13,14431	2,33206***

SVJ model

	Monthly			Quarterly			Semi-annually			Annually		
K/S	S-FPI	LSM-EEB	*difference*	S-FPI	LSM-EEB	*difference*	S-FPI	LSM-EEB	*difference*	S-FPI	LSM-EEB	*difference*
1,00	1,53499	1,63536	-0,10037	2,30237	2,76088	-0,45852***	4,30313	4,25839	0,04474	5,25369	4,64165	0,61204***
1,05	4,78875	4,33822	0,45054***	4,99545	4,74137	0,25408	6,04093	6,03803	0,0029	7,12481	6,49661	0,6282***
1,10	8,9453	8,37729	0,56801***	8,95883	7,53447	1,42436***	9,21293	8,40456	0,80837***	9,72169	8,2148	1,50688***
1,15	12,97111	12,59638	0,37473***	12,72425	11,51703	1,20722***	12,73934	11,3788	1,36054***	13,28953	10,51719	2,77234***
1,20	16,63989	16,59329	0,0466	16,50557	16,0823	0,42327***	16,2833	14,70296	1,58034***	16,22956	14,30345	1,92611***

SV model

	Monthly			Quarterly			Semi-annually			Annually		
K/S	S-FPI	LSM-EEB	*difference*	S-FPI	LSM-EEB	*difference*	S-FPI	LSM-EEB	*difference*	S-FPI	LSM-EEB	*difference*
1,00	1,54353	1,63468	-0,09116	2,31289	2,78179	-0,4689***	4,33098	4,17122	0,15976***	5,1326	4,59978	0,53282***
1,05	4,78842	4,39234	0,39608***	4,98138	4,71046	0,27091	6,10428	6,07102	0,03326	7,01939	6,51806	0,50133***
1,10	8,95027	8,36562	0,58465***	8,96123	7,53951	1,42172***	9,23644	8,34405	0,8924***	9,85318	8,39535	1,45782***
1,15	12,96857	12,532	0,43657***	12,72069	11,43194	1,28875***	12,73463	11,35515	1,37949***	13,26989	10,49214	2,77775***
1,20	16,63439	16,60038	0,03402	16,48523	16,34354	0,14169	16,2784	14,91414	1,36426***	16,2362	14,56923	1,66697***

Note - The table reports average discounted payoffs yielded by exercing American put options written on the S&P500 index following S-FPI and LSM-EEB optimal exercise policies. Results are presented for monthly, quarterly, semi-annually and annually contracts, and for moneyness ranging from 100% to 120%. For each maturity, average discounted payoffs for the S-FPI and LSM-EEB are presented, along with their difference. We indicate statistical significance of this difference for the following levels of confidence, where p represents the two-tailed paired t-test p-value: (*) for $p < 10\%$, (**) for $p < 5\%$ and (***) for $p < 1\%$.