PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE
ESCUELA DE INGENIERÍA

STRAIGHT-FORWARD AND ROBUST NEW METHOD TO PRICE AMERICAN OPTIONS BY SIMULATION: A PROBABILITY DISTRIBUTION FUNCTION APPROACH

NICOLÁS EDGARDO DIETZ PARR

Thesis submitted to the Office of Research and Graduate Studies
in partial fulfillment of the requirements for the degree of
Master of Science in Engineering

Advisor:
GONZALO CORTAZAR

Santiago de Chile, December 2016

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Gratefully to my parents and siblings

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ABSTRACT

In this thesis a straight-forward and robust new method to price European as well as American options is proposed. The algorithm uses the characteristics of each stochastic process implemented to describe the behavior of the underlying asset to compare the immediate exercise value with the value of postpone the exercise decision based on a conditional probability distribution function. This conditional probability distribution function is used to determine the risk-adjusted probabilities assigned to the expected payoffs. The algorithm is tested using a general Markovian pricing framework under the Black & Scholes dynamic and under the stochastic volatility pricing model proposed by Heston (1993). The method can be seen as a generalization of the Binary Tree model, since it has similar characteristics to Binomial Trees methods regarding the simplicity of the valuation procedure and the flexibility to price both type of options. Also, the method shows to be robust, converging towards the True Value defined for each option value and with a low level of error. Furthermore, we developed the theoretical mathematical framework to price more complex stochastic jump-diffusion processes under the proposed new method, including the stochastic volatility with stochastic jumps pricing model proposed by Duffie, Pan, and Singleton (2000).

Keywords: European Options, American Options, Conditional Probability Distribution Function, Immediate Exercise Value, Value of Postpone, Risk-Adjusted Probabilities, Black & Scholes Model, Heston Model.

RESUMEN

En esta tesis se propone un nuevo método, sencillo y robusto, para determinar el valor tanto de opciones Europeas como Americanas. El algoritmo usa las características de cada proceso estocástico implementado para describir el comportamiento del activo subyacente comparando el valor inmediato de ejercicio con el valor de posponer la decisión de ejercicio, basandose en una función de distribución de probabilidad condicional. Esta función de distribución de probabilidad condicional es utilizada para determinar las probabilidades ajustadas por riesgo asignadas a los pagos esperados. El algoritmo es testado usando un marco general de procesos Markovianos bajo la dinámica de Black & Scholes y bajo el modelo de precios de volatilidad estocástica propuesto por Heston (1993). El método puede ser visto como una generalización del modelo Binario de Árbol, ya que tiene características similares a métodos de Árboles Binomiales respecto a la simplicidad del procesos de valorización y a la flexibilidad para asignar precios a ambos tipos de opciones. También, el método muestra ser robusto, convergiendo al Valor Verdadero definido para cada opción y con un bajo nivel de error. Además, hemos desarrollamos el marco teórico matemático para determinar el valor de opciones bajo procesos de difusión más complejos utilizando el nuevo método propuesto, incluyendo el modelo de volatilidad estocástica con saltos estocásticos propuesto por Duffie, Pan y Singleton (2000).

Palabras Claves: Opciones Europeas, Opciones Americanas, Función de Distribución Condicional, Valor Inmediato de Ejercicio, Valor de Posponer, Probabilidades Ajustadas por Riesgo, Modelo de Black & Scholes, Modelo de Heston.

1. ARTICLE BACKGROUND

1.1. Introduction

In the option pricing theory, American options have been widely studied because of the challenge they present in the valuation and determination of the optimal exercise policy. Through the years, we have seen in the literature several studies that propose methods to price American options, starting with Brennan and Schwartz (1977), who were the first ones to price American options through a partial differential equation solving method (PDE).

Cox, Ross, and Rubinstein (1979) proposed a binomial method in which they discretized the time space and the price of the asset. Although, these methods are widely used because of their simplicity, they are not easily adapted to price options that follow more complex stochastic processes, for example, the stochastic volatility and jump-diffusion models, which incorporate other risk factors.

Furthermore, Longstaff and Schwartz (2001) developed the world known least-squares method to value options by simulation, which determines the conditional expected payoff by regressing the realized payoff against the price of the asset. In this case, the least square approach is used with the cross-sectional information in the simulation to compute the expectation function. This method is more flexible and easily adapted to more complex stochastic processes, but its main constrain lies on that fact that there is uncertainty on the election of the appropriate number of variables and polynomial to be used in the least-square regression.

Notwithstanding we have seen even further advances in the development of option pricing models, including the fixed-point iteration methods to solve the early-exercise boundary first proposed by Kim (1990), we have also seen that models are becoming increasingly complex over the years due to the advance that has been made on the implementation of additional risk factors to improve the simulation of the behavior of the underlying variables.

In contrast to the problem described above, we present a straight-forward and robust new method to price American options, which uses the characteristics of each stochastic process implemented to describe the behavior of the underlying asset to compare the immediate exercise value with the value of postpone the exercise decision, throughout a conditional expectation function. In this context, we use a conditional probability distribution function as the expectation function to determine the risk-adjusted probabilities assigned to the expected payoffs, which is not subject to an implementation decision, since this function corresponds to the implied distribution function of the process under which the variable is simulated.

Moreover, the pricing approach presented in this paper can be seen as a generalization of the Binary Tree model, since it has similar characteristics to Binomial Trees methods regarding the simplicity of the valuation procedure and the flexibility to price American-style as well as European-style feature options. However, it incorporates the adaptability to price options under any stochastic process that can be simulated, since a conditional probability distribution function can always be computed when the Euler discretized simulation approach is used to simulate the behavior of the underlying variable. The reason is that the probability distribution function depends uniquely on the process chosen to describe the behavior of the variable and the conditional value of the variable in the previous time step. Hence, the probability distribution function will be normally or mixture of normally distributed due to the discretization of the time space.

The rest of the document is organized as follows. In Section 2, we introduce the theoretical framework of the proposed new model and present a numerical example to price American options under our methodology. Section 3 presents numerical results computed we our model using Black & Scholes and Heston processes. In Section 4 we perform a convergence analysis to test the scope and limits of the new pricing approach. In Section 5 we present final results and conclusions.

1.2. Hypothesis

The hypothesis of this work is that it is possible to formulate a option pricing method, for European-style as well as American-style feature options, with the capability of being adaptable to price options under any stochastic process by using the characteristics of each stochastic process implemented to describe the behavior of the underlying asset, while maintaining the simplicity of its valuation procedure.

1.3. Main Objectives

The main objective of this thesis is develop of a straight-forward and robust new method to price European as well as American options, which uses the characteristics of each stochastic process implemented to describe the behavior of the underlying asset through the implementation of a conditional probability distribution function. In this context, the thesis has three specific objectives: First, this work intends to establish a theoretical framework that supports the utilization and the main features of the methodology. The second objective is to introduce the SFR Method for a general Markovian pricing framework under the Black & Scholes dynamic and under the stochastic volatility pricing model proposed by Heston (1993). Finally, the third objective is to demonstrate the convergence of the method to the True Value defined for each option value and to determine research topics to further study the benefits and constrains of the new pricing mechanism.

1.4. Methodology

The source code for the SFR Method and the corresponding benchmarks are implemented using MATLAB 2013b running on a 2.50GHz Intel Core i5 8GB RAM. The Black &Scholes closed-form solution is implemented according to Black and Scholes (1973), Binomal Tree is implemented according to Cox, Ross, and Rubinstein (1979), Heston closed-form solution is implemented according to Heston (1993) and the Least-Squares Monte Carlo (LSM) method is implemented according to Longstaff and Schwartz (2001).

Mean Absolute Errors (MAE) are measured against the True Values computed by the Black &Scholes closed-form solution, Binomal Tree, Heston closed-form solution and LSM.

2. DESCRIPTION OF THE NEW PRICING METHOD

2.1. General Model

Let S denote the underlying asset and S_t its value on a given time t. Accordingly, the dynamics of the asset are characterized by the following Markov process under the risk neutral pricing measure:

$$\frac{dS_t}{S_t} = (r(Y_t) - q(Y_t))dt + \sigma^S(Y_t)dW_t^S + \xi(Y_t)dJ_t \tag{2.1}$$

$$dY_t = \mu^Y(Y_t)dt + \sigma^Y(Y_t)dW_t^Y + \psi(Y_t)dJ_t \tag{2.2}$$

where Y_t is a Markov process in a space state $D \subset \mathbb{R}^n$, W^S and W^Y are standard Brownian motions in $\mathbb{R}$ and $\mathbb{R}^n$, respectively and J_t is a Poisson process in $\mathbb{R}^{n+1}$. In (2.1), $r : D \to \mathbb{R}$ denotes the instantaneous risk-free rate, $q : D \to \mathbb{R}$ denotes the instantaneous dividend yield, $\sigma^S : D \to \mathbb{R}$ denotes the instantaneous volatility and $\xi : D \to \mathbb{R}^{n+1}$ denotes the jump scale of J. In (2.2), $\mu^Y : D \to \mathbb{R}^n$, $\sigma^Y : D \to \mathbb{R}^{n\times n}$ and $\psi : D \to \mathbb{R}^{n+1}$. Given the general process described in this section, we focus our attention on presenting the simulation approach used on the new Straight-Forward and Robust Method (SFR Method).

2.2. Simulation Approach

We adopted the Euler discretized simulation approach to construct the paths of the underlying asset through the time horizon. The technique applied consists on generating discrete paths of the variables S_t and Y_t, which we denote by $\widehat{S}_t$ and $\widehat{Y}_t$, respectively. The dynamics of these variables are given by a discrete version of (2.1) and (2.2), which are presented as follows:

$$\widehat{S}_{t_{i+1}} = \widehat{S}_{t_i} + \widehat{S}_{t_i}(r(\widehat{Y}_{t_i}) - q(\widehat{Y}_{t_i}))\Delta t + \widehat{S}_{t_i}\sigma^S(\widehat{Y}_{t_i})\Delta W_{t_i}^S + \widehat{S}_{t_i}\xi(\widehat{Y}_{t_i})\Delta J_{t_i} \tag{2.3}$$

$$\widehat{Y}_{t_{i+1}} = \widehat{Y}_{t_i} + \mu^Y(\widehat{Y}_{t_i})\Delta t + \sigma^Y(\widehat{Y}_{t_i})\Delta W_{t_i}^V + \psi(\widehat{Y}_{t_i})\Delta J_{t_i} \tag{2.4}$$

The simulation starts with the initial values $(\widehat{S}_0, \widehat{Y}_0)$ and continues according to (2.3) and (2.4) until it reaches the time horizon given for the option being value.

2.3. Description of the Pricing Procedure for the SFR Method

American options are exercisable at any time and the option holder has to compare the immediate exercise payoff with the value to postpone the exercise decision to the next time step, which is also known as the expectation value. In Longstaff and Schwartz (2001), the conditional expectation value is determined by an expectation function. In their case, a least square approach was used with the cross-sectional information in the simulation to compute the expectation function.

In this paper, we present a new approach to compute the conditional expectation function, which uses the specific characteristics behind each process assumed to describe the behavior of the underlying variable. In this context, we use a conditional probability function as the expectation function to determine the risk-adjusted probabilities assigned to the expected payoffs. This conditional probability function is computed using the probability distribution of the process chosen to simulate the underlying variable, conditional on the value of the variable at the time of comparison $(t-1)$. A more detailed explanation is presented as follows:

(i) Using the Euler discretized simulation approach, we simulate a certain number $i : 1 \to I$ of random paths of the underlying variable for the complete discretized time period, which we will denominated $S_{i,t}$.

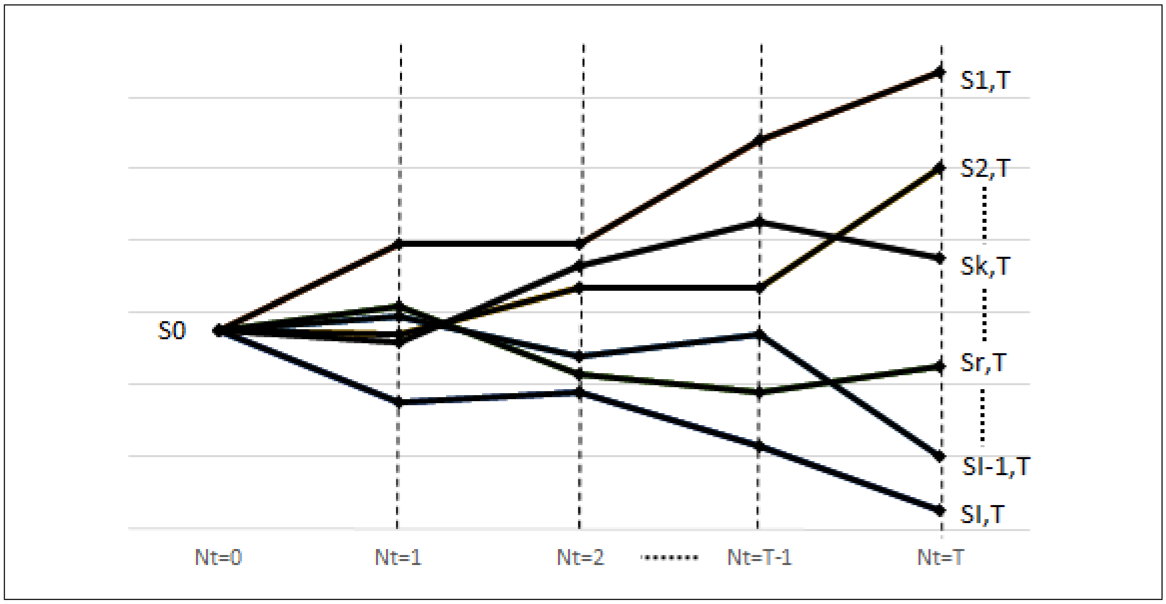

Figure 2.1. Simulated paths for the complete time period T

(ii) Using the information given by the simulation, we compute for each path i the value of the associated expected payoff at time T ($CF_{i,T}$), as $Max(S_{i,T} - K)$ for a call option, or $Max(K - S_{i,T})$ for a put option, with a strike price of K, which in this case corresponds to the value of immediately exercising the option at its maturity.

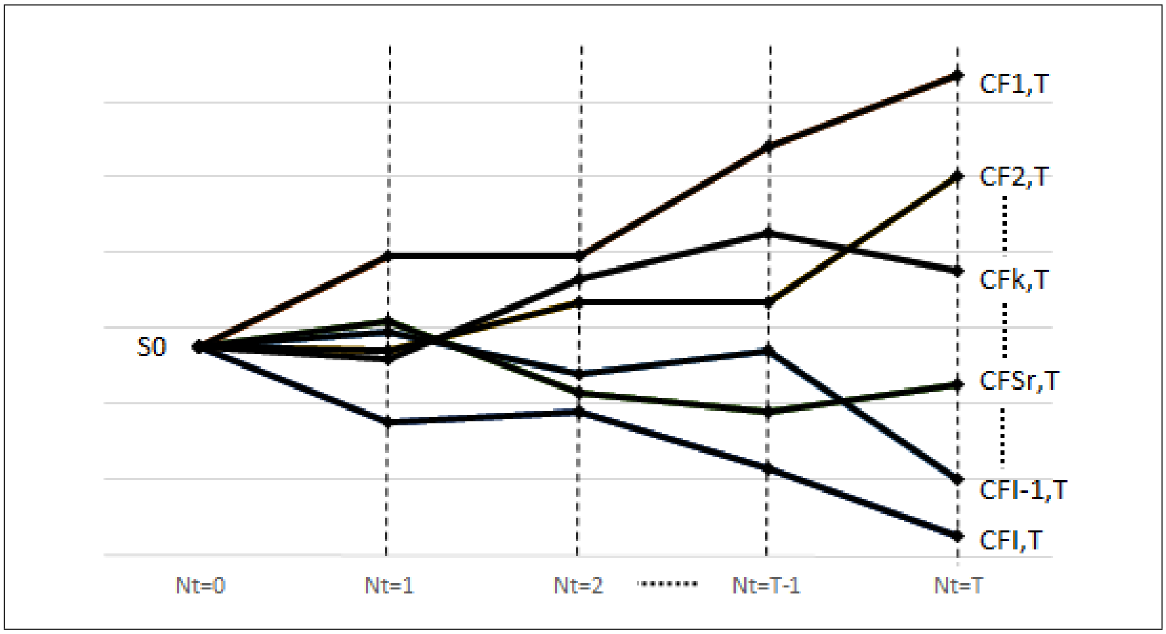

Figure 2.2. Expected cash flows at time T

(iii) For each path i at time $t-1$, we compute the expected cash flows ($CF_{i,t-1}$) by comparing the value of postponing the exercise of the option at time $t-1$ with the value of immediate exercise it (we described in point 2 how to compute the immediate exercise value). As we explained before, to determine the value of postponing we use a conditional probability function as the expectation function to define the risk-adjusted probabilities assigned to each expected payoffs $CF_{i,t}$. Hence, for each path i at time $t-1$, we define a conditional probability function, $F^i(X|S_{i,t-1})$, which is conditional on the value of the path i of the variable at time $t-1$.

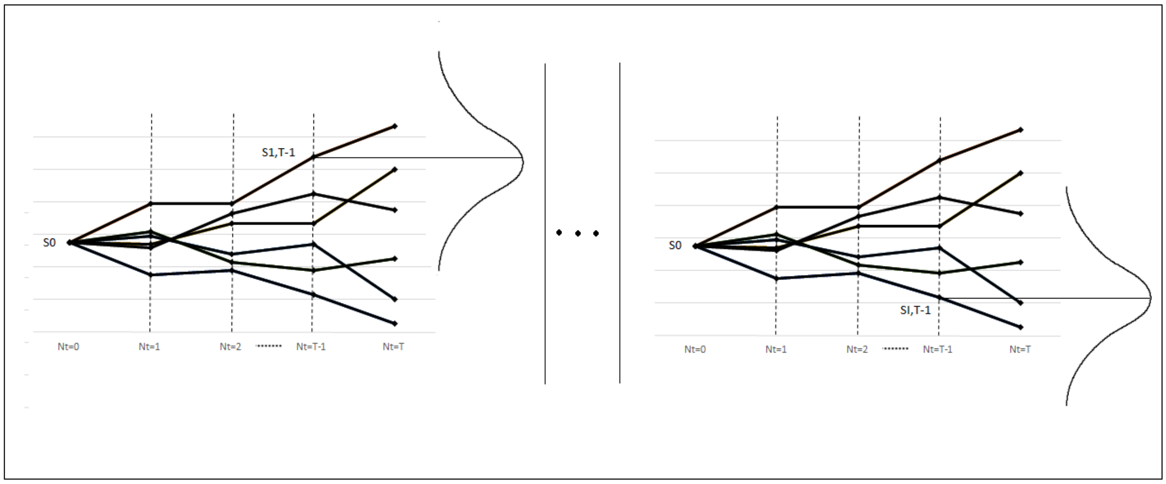

Figure 2.3. Definition of the expectation function as a conditional risk-adjusted probability function, $F^i(X|S_{i,t-1})$, for each path i at $t-1$

(iv) Once we define the probability function $F^i(X|S_{i,t-1})$, we compute the risk-adjusted probabilities of occurrence for each expected cash flow $CF_{i,t}$, which we will denominate $Pr[CF_{i,t}|S_{i,t-1}]$ for $j : 1 \rightarrow I$. Hence, for each path i at time $t-1$ we proceed as follows:

(a) First, we evaluate each simulated value, $S_{i,t}$, on the conditional expectation function calculated in point 3, obtaining for each $S_{i,t}$ and for each associated cash flow, $CF_{i,t}$, a conditional risk-adjusted cumulative probability, which we will denominate $P_j = P[X <= S_{i,t}|S_{i,t-1}]$. In Figure 2.4, we illustrate how to compute a conditional cumulative probability, using as an example path 1 at time $t-1$.

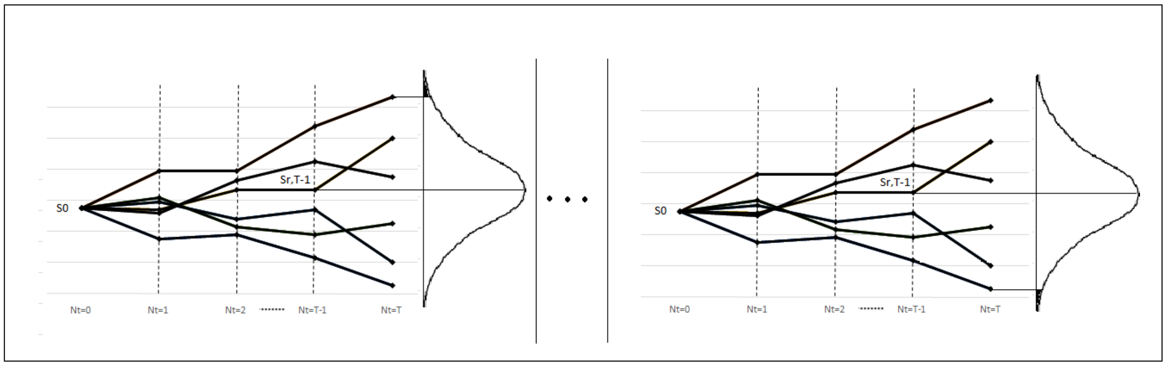

Figure 2.4. Illustrative example of how to compute the conditional cumulative probabilities for path 1 at time $t-1$

(b) In second place, after we have obtained the cumulative probabilities P_j associated to each $S_{i,t}$ and to its corresponding $CF_{i,j}$, we determine the risk-adjusted probabilities of occurrence p_j. To determine these probabilities, we decided to assign to each expected cash flow, $CF_{i,t}$, half of the difference of the cumulative probabilities between that expected cash flow and its predecessor ($CF_{i-1,t}$) and successor ($CF_{i+1,t}$). This process of assigning the risk-adjusted probabilities is not model-specific, since its subject to implementation decision. Mathematically, this process of assigning the probabilities of occurrence can be described as:

(i) For each P_j from $j = 2$ to $j = j-1$, compute the risk-adjusted probabilities p_j as

$$p_j = \frac{1}{2}(P_j - P_{j-1}) + \frac{1}{2}(P_{j+1} - P_j)$$

(ii) For $P_{j=1}$ compute the risk-adjusted probability $p_{j=1}$ as

$$p_{j=1} = P_{j=1} + \frac{1}{2}(P_{j=2} - P_{j=1})$$

(iii) For $P_{j=I}$ compute the risk-adjusted probability $p_{j=I}$ as

$$p_{j=I} = \frac{1}{2}(P_{j=I} - P_{j=I-1}) + P_{j=I}$$

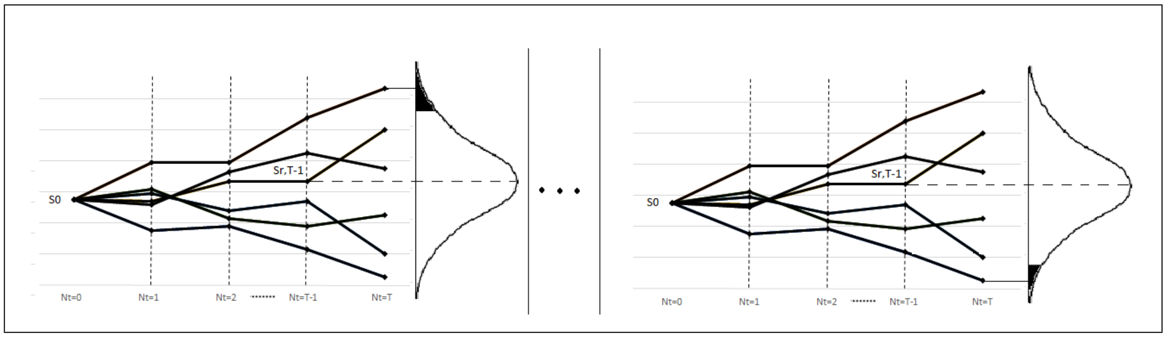

Figure 2.5. Illustrative definition of the risk-adjusted probabilities of occurrence p_j for path 1 at time $t-1$

(v) After completing step 4, we compute for each path i at time $t-1$ the discounted cash flows by multiplying the risk-adjusted probabilities, p_j, by its corresponding cash flow, $CF_{i,t}$, and then discount them at the risk free rate. Once we have summed up all discounted expected cash flows for each path i at time $t-1$, we can compute the value of postponing the exercise decision with the value of immediate exercise.

(vi) Finally, to obtain the value of the option, we have to repeat the process describe above until we have analyzed all early exercise decisions of each path i at each time period t.

The main constraint of the method is the challenge of determining the cumulative probability distribution function for each path i at time $t-1$ and to evaluate it for each path i at time t. However, it can be anticipated that the conditional probability distribution function will be normal or mixture of normal distributions. The former is a consequence of the approach being used, since we know from the simulation that each value of S_t and Y_t from equations (2.1) and (2.2) will remain constant until the next discrete period, because of the Euler discretized simulation approach.

2.4. Pricing The American Option: A Numerical Example

As it was explained in Subsection 2.3, the conditional expected value of continuing has to be determined and compared to the value of immediate exercise to identify the optimal exercise strategy for an American option. In our approach this is done by computing the conditional expectation function with the characteristics of the distribution function of the process used to simulate the behavior of the underlying variable, conditional on the value of the variable in the previous time step. To illustrate how the SFR Model operates we present the following numerical example.

Consider an American put option on an asset that does not pay dividends, has an initial value of 100 and volatility of 20%. The put option has a strike price of 120 and a maturity of 2 years, which is divided into 4 time periods. The risk free rate is 4%. Suppose the dynamics for the risky asset S_t under the Black & Scholes model is given by the following SDE:

$$\frac{dS_t}{S_t} = 0.04 * dt + 0.2 * dW_t \tag{2.5}$$

In this case, we simulate 6 (3 plus 3 antithetic) random paths for the asset to illustrate the mechanism of the algorithm, which are shown in Table 2.1.

Table 2.1. Simulated asset paths

Path	t=0	t=1	t=2	t=3	t=4
$S_{1,t}$	100	104.3973	95.3886	99.0492	93.6654
$S_{2,t}$	100	123.6339	124.4971	152.6123	172.2063
$S_{3,t}$	100	96.8789	117.8335	130.2552	109.2978
$S_{4,t}$	100	99.6027	112.1818	112.3640	122.9660
$S_{5,t}$	100	80.3661	83.0196	67.5921	61.6176
$S_{6,t}$	100	107.1211	88.2360	82.4638	99.0304

Following the approach explained in Subsection 2.3, we need to compute the expected cash flow matrix at each time step, conditional on not exercising the option, comparing the value of continuing with the immediate exercise value. The cash flow realized at time 4 are presented in Table 2.2.

Table 2.2. Cash flows at time 4

Path	t=1	t=2	t=3	t=4
$S_{1,t}$	-	-	-	26.3364
$S_{2,t}$	-	-	-	0
$S_{3,t}$	-	-	-	10.7022
$S_{4,t}$	-	-	-	0
$S_{5,t}$	-	-	-	58.3824
$S_{6,t}$	-	-	-	20.9696

For each path at time 3 ($S_{i,t=3}$), the holder of the option has to decide whether to immediately exercise the option if it is in-the-money (paths 1, 4, 5 and 6) or to hold the option until its maturity date at time 4. Hence, for each path i at time $t-1$ we compare the value of continuing with the value of immediate exercise at the same period, which we proceed as follows:

(i) Decreasingly sort all cash flows $CF_{i,t}$ and asset values $S_{i,t}$ at time t, obtaining $\widehat{CF}_{i,t}$ and $\widehat{S}_{i,t}$. In Table 2.3 we present each sorted asset value at time 4 with its respectively cash flow.

Table 2.3. Sorted cash flows

$\widehat{S}_{i,t=4}$	$\widehat{CF}_{i,t=4}$
61.6176	58.3824
93.6654	26.3346
99.0304	20.9696
109.2978	10.7022
122.9660	0
172.2063	0

(ii) Compute the conditional expectation function as a normally distributed function, conditional on the value of the path i at time $t-1$, as

$$N(X <= x|S_{i,t-1}) = N(\frac{x - S_{i,t-1} * (1 + r * dt)}{S_{i,t-1} * \sigma * \sqrt{dt}})$$

In Table 2.4 we present the conditional expectation function conditional on the value of each path i at time 3.

Table 2.4. Conditional Expectation Function

$S_{i,t=3}$	Expectation Function Conditional on $S_{i,t=3}$
99.0492	$N(\frac{x-101.0302}{14.0077})$
152.6123	$N(\frac{x-155.6646}{21.5826})$
130.2552	$N(\frac{x-132.8603}{18.4209})$
112.3640	$N(\frac{x-114.6113}{15.8907})$
67.5921	$N(\frac{x-68.9439}{9.5590})$
82.4638	$N(\frac{x-84.1131}{11.6621})$

(iii) Evaluate each asset value $\widehat{S}_{i,t}$ on the conditional expectation function calculated in 2, obtaining for each $\widehat{S}_{i,t}$ a conditional cumulative normally distributed probability, $N_j = N(X <= \widehat{S}_{i,t}|S_{i,t-1})$ with $j : 1 \to I$. In Table 2.5 we evaluate each asset value $\widehat{S}_{i,t=4}$ on the conditional expectation function conditional on each $S_{i,t=3}$.

Table 2.5. Cumulative normally distributed probabilities for each expectation function conditional on each $S_{i,t=3}$

$\widehat{CF}_{i,t=4}$	$N(X\|S_{1,t=3})$	$N(X\|S_{2,t=3})$	$N(X\|S_{3,t=3})$	$N(X\|S_{4,t=3})$	$N(X\|S_{5,t=3})$	$N(X\|S_{6,t=3})$
58.3824	0.0024	0.0000	0.0001	0.0004	0.2217	0.0269
26.3346	0.2995	0.0020	0.0167	0.0937	0.9951	0.7936
20.9696	0.4432	0.0043	0.0331	0.1634	0.9992	0.8996
10.7022	0.7225	0.0158	0.1004	0.3690	1.0000	0.9846
0	0.9413	0.0649	0.2956	0.7005	1.0000	0.9996
0	1.0000	0.7783	0.9837	0.9999	1.0000	1.000

(iv) For each cumulative probability N_j associated to each $\widehat{S}_{i,t}$, determine a risk-adjusted probability of occurrence p_j:

(a) For each N_j from $j = 2$ to $j = i-1$, compute the risk-adjusted probabilities p_j as

$$p_j = \frac{1}{2}(N_j - N_{j-1}) + \frac{1}{2}(N_{j+1} - N_j)$$

(b) For $N_{j=1}$ compute the risk-adjusted probability $p_{j=1}$ as

$$p_{j=1} = N_{j=1} + \frac{1}{2}(N_{j=2} - N_{j=1})$$

(c) For $N_{j=I}$ compute the risk-adjusted probability $pj = I$ as

$$p_{j=I} = \frac{1}{2}(N_{j=I} - N_{j=I-1}) + N_{j=I}$$

In Table 2.6, we present the risk-adjusted probabilities of occurrence of each cash flow $\widehat{CF}_{i,t=4}$ for each conditional expectation function.

Table 2.6. Risk-adjusted probabilities of occureance for each expectation function conditional on each $S_{i,t=3}$

$\widehat{CF}_{i,t=4}$	$P(X\|S_{1,t=3})$	$P(X\|S_{2,t=3})$	$P(X\|S_{3,t=3})$	$P(X\|S_{4,t=3})$	$P(X\|S_{5,t=3})$	$P(X\|S_{6,t=3})$
58.3824	0.1510	0.0010	0.0084	0.0417	0.6084	0.4103
26.3346	0.2204	0.0022	0.0165	0.0815	0.3887	0.4364
20.9696	0.2115	0.0069	0.0419	0.1377	0.0024	0.0955
10.7022	0.2490	0.0303	0.1312	0.2685	0.0004	0.0500
0	0.1388	0.3812	0.4416	0.3154	0.0000	0.0077
0	0.0293	0.5784	0.3604	0.1498	0.0000	0.0002

(v) Finally, we compute the value of continuing at time $t-1$ as a weighted average sum, multiplying each risk-adjusted probability p_i by the associated cash flow $\widehat{CF}_{i,t}$, discounted at the risk free rate r as

$$\sum p_i * \widehat{CF}_{i,t} * exp(-r)$$

In Table 2.7 we show the calculation of the value of continuing at time 4 for each path $S_{i,t=3}$.

Table 2.7. Value of continuing to time 4 for each path $S_{i,t=3}$

Illustrative Calculation of the Continuing Value
(58.38 * 0.15 + 26.33 * 0.22 + 20.97 * 0.21 + 10.70 * 0.25 + 0 * 0.14 + 0 * 0.03)*$\exp(-r)$
(58.38 * 0.00 + 26.33 * 0.00 + 20.97 * 0.01 + 10.70 * 0.03 + 0 * 0.38 + 0 * 0.58)*$\exp(-r)$
(58.38 * 0.01 + 26.33 * 0.02 + 20.97 * 0.04 + 10.70 * 0.13 + 0 * 0.44 + 0 * 0.36)*$\exp(-r)$
(58.38 * 0.04 + 26.33 * 0.08 + 20.97 * 0.14 + 10.70 * 0.27 + 0 * 0.32 + 0 * 0.15)*$\exp(-r)$
(58.38 * 0.61 + 26.33 * 0.08 + 20.97 * 0.00 + 10.70 * 0.00 + 0 * 0.00 + 0 * 0.00)*$\exp(-r)$
(58.38 * 0.41 + 26.33 * 0.44 + 20.97 * 0.10 + 10.70 * 0.05 + 0 * 0.01 + 0 * 0.00)*$\exp(-r)$

After following the procedure explained before, we compare the value of continuing with the immediate exercise value for each path at time 3.

Table 2.8. Comparison between continuing value and immidate exercise value at time 3

Path	Exercise	Continuing
$S_{1,t=3}$	20.9508	21.28888
$S_{2,t=3}$	0	0.5738
$S_{3,t=3}$	0	3.1431
$S_{4,t=3}$	7.6360	10.4442
$S_{5,t=3}$	52.4079	44.9066
$S_{6,t=3}$	37.5362	37.2278

The comparison implies that it is optimal to exercise the option in the case of path 5 and 6, and to not exercise it in all other cases, which is represented in the cash flow matrix of Table 2.9.

Table 2.9. Cash flows at time 3

Path	t=1	t=2	t=3	t=4
$S_{1,t}$	-	-	21.2888	26.3364
$S_{2,t}$	-	-	0.5738	0
$S_{3,t}$	-	-	3.1431	10.7022
$S_{4,t}$	-	-	10.4442	0
$S_{5,t}$	-	-	52.4079	58.3824
$S_{6,t}$	-	-	37.5362	20.9696

Proceeding using the same approach, we compared the value of continuing with the immediate exercise value for each path at time 2.

Table 2.10. Comparison between continuing value and immidate exercise value at time 2

Path	Exercise	Continuing
$S_{1,t=2}$	24.6114	23.8399
$S_{2,t=2}$	0	7.1486
$S_{3,t=2}$	2.1665	9.6897
$S_{4,t=2}$	7.8182	12.4195
$S_{5,t=2}$	36.9804	34.2954
$S_{6,t=2}$	31.7640	29.8193

The comparison implies that it is optimal to exercise the option in the case of path 1, 5 and 6, and to not exercise it in all other cases, which is represented in the cash flow matrix of Table 2.11.

Table 2.11. Cash flows at time 2

Path	t=1	t=2	t=3	t=4
$S_{1,t}$	-	24.6114	21.2888	26.3364
$S_{2,t}$	-	7.1486	0.5738	0
$S_{3,t}$	-	9.6897	3.1431	10.7022
$S_{4,t}$	-	12.4195	10.4442	0
$S_{5,t}$	-	36.9804	52.4079	58.3824
$S_{6,t}$	-	31.7640	37.5362	20.9696

Then, we compared the value of continuing with the immediate exercise value for each path at time 1.

Table 2.12. Comparison between continuing value and immidate exercise value at time 1

Path	Exercise	Continuing
$S_{1,t=1}$	15.6027	17.8243
$S_{2,t=1}$	0	10.4462
$S_{3,t=1}$	23.1211	22.1491
$S_{4,t=1}$	20.3973	20.5138
$S_{5,t=1}$	39.6339	32.1422
$S_{6,t=1}$	12.8789	16.4287

The comparison implies that it is optimal to exercise the option in the case of path 3 and 5, and to not exercise it in all other cases, which is represented in the cash flow matrix of Table 2.13.

Table 2.13. Cash flows at time 1

Path	t=1	t=2	t=3	t=4
$S_{1,t}$	17.8243	24.6114	21.2888	26.3364
$S_{2,t}$	10.4462	7.1486	0.5738	0
$S_{3,t}$	23.1211	9.6897	3.1431	10.7022
$S_{4,t}$	20.5138	12.4195	10.4442	0
$S_{5,t}$	39.6339	36.9804	52.4079	58.3824
$S_{6,t}$	16.4287	31.7640	37.5362	20.9696

Finally, we discount and average the value of the expected cash flows at time 1 to obtain the value of the American put option, corresponding to 20.9057. The same simulation sample and method can be used to compute the value of an European put option, in which case in each iteration and for each path, we do not compute neither compare the immediate exercise value with the value of postponing. In this case, the European put option value corresponds to 18.0712[1].

We tested the SFR Method under two different dynamics for the risky asset satisfying (2.1) and (2.2), which we present in the Section 3.

[1]European and American option values for the Black & Scholes closed-formed solution and a Binomial Tree proposed by Cox et al. (1979) with 17,000 time steps, using the same parameters used to illustrate the SFR algorithm are 18.0004 and 21.1506, respectively.

3. NUMERICAL RESULTS

In this section we present our findings regarding the calculation of American, as well as, European option prices by the SFR Method under two different dynamics for a variable satisfying (2.1) y (2.2), including the Black & Scholes and Heston models.

3.1. The Black & Scholes Model

The dynamics for the risky asset S_t under the Black & Scholes model introduced in Black and Scholes (1973) is given by the following SDE:

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t \tag{3.1}$$

In (3.1), r corresponds to the instantaneous risk-free rate, q corresponds to the instantaneous dividend rate, σ corresponds to the instantaneous volatility and W_t is a standard Brownian process. In the Black & Scholes model, r, q and σ are considered constant. Following the simulation approach discussed earlier, the Euler discretized process followed by the underlying asset is characterized by the following equation:

$$S_{t+1} = S_t + (r - q)S_t \Delta t + \sigma S_t \Delta W_t \tag{3.2}$$

For the purposed of the SFR Method, we need to know the distribution of the underling asset at time $t+1$ under the Black & Scholes process, conditional on the value of the asset at time t ($F[S_{t+1}|S_t]$). It follows from (3.2) that the probability distribution of the asset is:

$$\frac{S_{t+1} - S_t}{S_t}|S_t \sim N((r - q)\Delta t, \sigma^2 \Delta t) \tag{3.3}$$

The Black & Scholes closed-form solution is the first comparison analysis that was tested. In this case, we compute several option values using the well-known equation introduced in Black & Scholes (1973), which we compare to the values obtained by the SFR Method used for European options. We compute at-the-money option prices considering the values 80, 100 and 120 for the underlying asset using the pricing method described previously.

Also, we use 100 time steps (Nt), while the number of paths (Ns) range from 1000 to 100.000, which is presented as $SFR(Nt/Ns)$.

Another important feature of the simulation approach is that we computed 50 option prices for each value of the underlying asset, which were averaged and used to obtain diverse accuracy parameters (i.e. Mean Absolute Errors and standard deviations), which we then used in Section 4 to test the convergence of the method and its deviation respect to the True Value. The parameters used to test the method are provided in Table 3.1.

Table 3.1. Parameter values

Parameter	Value
T	0.50
r	0.04
q	0.04
σ	0.20

In Table 3.2, we present the European put prices computed by the SFR Method and Black & Scholes closed-form solution. We also present the standard deviation and the mean absolute error for each option value in relation to the True Value given by the European closed-form solution.

Table 3.2. At-the-money European put option values under the Black & Scholes process

	S		
	80	100	120
Closed-Form	4.4205	5.5256	6.6307
SFR(100/2,500)	4.4200	5.5249	6.6314
Std Dev.	0.0050	0.0060	0.0072
MAE	0.0039	0.0048	0.0057
SFR(100/5,000)	4.4205	5.5252	6.6305
Std Dev.	0.0026	0.0031	0.0035
MAE	0.0021	0.0025	0.0029
SFR(100/10,000)	4.4205	5.5259	6.6308
Std Dev.	0.0013	0.0012	0.0016
MAE	0.0010	0.0009	0.0009
SFR(100/20,000)	4.4208	5.5259	6.6313
Std Dev.	0.0006	0.0007	0.0009
MAE	0.0005	0.0006	0.0009
SFR(100/50,000)	4.4209	5.5261	6.6313
Std Dev.	0.0002	0.0003	0.0003
MAE	0.0004	0.0005	0.0007
SFR(100/100,000)	4.4208	5.5261	6.6313
Std Dev.	0.0001	0.0001	0.0002
MAE	0.0003	0.0005	0.0006

We observe that for European options the convergence of the algorithm is obtained using over 20,000 paths, which is similar to what we observe in the next table for American options.

In Table 3.3, we present the American put prices computed by the SFR Method, as well as the standard deviation and the mean absolute error for each option price in relation to the True Value given by a Binomial Tree with 17,000 time steps.

Table 3.3. At-the-money American put prices under the Black & Scholes model

	S		
	80	100	120
Binomial Tree	4.4371	5.5464	6.6557
SFR(100/2,500)	4.4478	5.5604	6.6739
Std Dev.	0.0049	0.0061	0.0105
MAE	0.0107	0.0140	0.0182
SFR(100/5,000)	4.4418	5.5524	6.6630
Std Dev.	0.0026	0.0031	0.0040
MAE	0.0047	0.0060	0.0073
SFR(100/10,000)	4.4390	5.5489	6.6586
Std Dev.	0.0009	0.0013	0.0014
MAE	0.0019	0.0025	0.0029
SFR(100/20,000)	4.4381	5.5477	6.6573
Std Dev.	0.0005	0.0008	0.0009
MAE	0.0010	0.0013	0.0016
SFR(100/50,000)	4.4376	5.5470	6.6564
Std Dev.	0.0002	0.0002	0.0003
MAE	0.0005	0.0006	0.0006
SFR(100/100,000)	4.4374	5.5468	6.6561
Std Dev.	0.0001	0.0001	0.0001
MAE	0.0003	0.0004	0.0004

The results presented in Tables 3.2 and 3.3, respectively, show that even with a low number of simulated paths, the SFR Method computes results with low MAEs and standard deviations and as we increase the number of simulated paths used to compute the value of the option, the prices obtained with the SFR Method converge to the True Value. Moreover, in Section 4.1 we present a convergence analysis of the results presented in this section. Finally, in Section A we present further results for in and out-of-the-money option prices using the SFR Method under the Black & Scholes process.

3.2. The Heston Model

The dynamics for the risky asset S_t are given by a Geometric Brownian Motion, while the squared volatility follows the dynamics proposed by Heston (1973). Thus, the model is characterized by the following SDEs system:

$$dS_t = (r-q)S_t dt + \sqrt{V_t} S_t dW_t^S \tag{3.4}$$

$$dV_t = \kappa(\theta - V_t)dt + \xi\sqrt{V_t}dW_t^V \tag{3.5}$$

$$dW_t^S dW_t^V = \rho dt \tag{3.6}$$

V_t represents the instantaneous squared volatility, W^S and W^V correspond to standard Brownian processes with correlation of ρ. Furthermore, θ is the long-run mean for V_t, κ is the rate of mean reversion and ξ is the instantaneous variance of V_t. Following the simulation approach discussed earlier, the Euler discretized process followed by the underlying variable is characterized by the following equation:

$$S_{t+1} = S_t + (r-q)S_t\Delta t + \sqrt{V_t} S_t \Delta W_t^S \tag{3.7}$$

$$dV_t = \kappa(\theta - V_t)dt + \xi\sqrt{V_t}dW_t^V \tag{3.8}$$

$$dW_t^S dW_t^V = \rho\Delta t \tag{3.9}$$

For the purposed of the SFR Method, we need to know the distribution of the underling asset at time $t+1$ under the Heston process, conditional on the value of the asset at time t and the value of the instantaneous squared volatility at time t ($F[S_{t+1}|S_t, V_t]$). It follows from equations (3.7) to (3.9) that the probability distribution of the asset is:

$$\frac{S_{t+1} - S_t}{S_t}|S_t, V_t \sim N((r-q)\Delta t, V_t\Delta t) \tag{3.10}$$

Following Selection 3.1, we compute several European option values using the closed-form solution introduced in Heston (1993), which we compare to the values obtained by the SFR Method used for European option. We compute at-the-money option prices considering the values 8, 10 and 12 for the underlying asset using the pricing method described previously. We also follow the same simulation approach and result analysis described in Selection 3.1. The parameters used to test the method are provided in Table 3.4.

Table 3.4. Parameter values for Heston model

Parameter	Value	Parameter	Value
T	0.25	θ	0.16
r	0.1	ξ	0.90
q	0.00	ρ	0.10
κ	5.00	V_0	0.25

Moreover, we used a numerical integration technique know as Simpsons quadrature rule to evaluate the Heston integral. Adapted from Moodley (2005), the Simpsons quadrature rule for the numerical integration of a function p(x) is:

$$\int_a^b p(x)dx \approx \frac{b-a}{6}[p(a) + 4p(\frac{a+b}{2}) + p(b)]$$

$$Error = -\frac{1}{90}(\frac{b-a}{2})^5 p^{(4)}(\xi)$$

$$\xi \epsilon (a, b)$$

The accuracy of the technique depends on te number of subintervals, which is quantified by the error measure. The Matlab function $quad$ uses the Simpsons quadrature rule and produces a result that has an error of less than 10^{-6} and can be used to compute American options under the Heston dynamic.

In Table 3.5, we present the European put prices computed by the SFR Method and Heston closed-form solution. We also present the standard deviation and the mean absolute error for each option value in relation to the True Value given by the European closed-form solution.

We observe that for European options the convergence of the algorithm is obtained using over 20,000 paths, which is similar to what we observe in the next table for American options as well as what we observed for European options.

Table 3.5. At-the-money European put option values under the Heston model

	S		
	8	10	12
Closed-Form	0.6158	0.7697	0.9236
SFR(100/2,500)	0.6160	0.7700	0.9242
Std Dev.	0.0012	0.0013	0.0019
MAE	0.0010	0.0010	0.0016
SFR(100/5,000)	0.6162	0.7701	0.9242
Std Dev.	0.0007	0.0009	0.0011
MAE	0.0007	0.0008	0.0010
SFR(100/10,000)	0.6162	0.7703	0.9244
Std Dev.	0.0006	0.0007	0.0009
MAE	0.0006	0.0007	0.0008
SFR(100/20,000)	0.6162	0.7702	0.9243
Std Dev.	0.0003	0.0004	0.0005
MAE	0.0004	0.0005	0.0007
SFR(100/50,000)	0.6163	0.7703	0.9244
Std Dev.	0.0002	0.0003	0.0004
MAE	0.0005	0.0006	0.0008
SFR(100/100,000)	0.6162	0.7703	0.9243
Std Dev.	0.0001	0.0002	0.0002
MAE	0.0004	0.0006	0.0007

In Table 3.6, we present the American put prices computed by the SFR Method, as well as the standard deviation and the mean absolute error for each option price in relation to the True Value given by the Longstaff and Schwartz method (LSM) computed under the Heston process[1]. We used the Euler discretized simulation approach and computed 50 option prices for each value of the underlying asset, using 100 time steps and 150,000 paths, which we averaged to compute the option prices under the LSM.

[1]We implemented the LSM using a Languerre polynomial with two variables $X_1(x) = 1 - x$ and $X_2(x) = \frac{1}{2} * (x^2 - 4 * x + 2)$, respectively.

Table 3.6. At-the-money American put prices under the Heston model

	S		
	8	10	12
Heston LSM	0.6364	0.7960	0.9556
SFR(100/2,500)	0.6373	0.7966	0.9560
Std Dev.	0.0011	0.0012	0.0018
MAE	0.0046	0.0006	0.0070
SFR(100/5,000)	0.6370	0.7963	0.9555
Std Dev.	0.0007	0.0008	0.0010
MAE	0.0043	0.0003	0.0065
SFR(100/10,000)	0.6369	0.7961	0.9554
Std Dev.	0.0005	0.0007	0.0008
MAE	0.0042	0.0001	0.0064
SFR(100/20,000)	0.6368	0.7960	0.9552
Std Dev.	0.0004	0.0004	0.0005
MAE	0.0004	0.0003	0.0004
SFR(100/50,000)	0.6369	0.7961	0.9553
Std Dev.	0.0002	0.0003	0.0004
MAE	0.0005	0.0002	0.0004
SFR(100/100,000)	0.6368	0.7960	0.9552
Std Dev.	0.0001	0.0002	0.0002
MAE	0.0004	0.0001	0.0004

As in Selection 3.1, the results presented in Tables 3.5 and 3.6, respectively, show that even with a low number of simulated paths, the SFR Method computes results with low MAEs and standard deviations and as we increase the number of simulated paths, the prices obtained with the SFR Method converge to the True Value. Finally, in Section

B we present further results for in and out-of-the-money option prices using the SFR Method under the Heston process.

4. ANALYSIS OF THE RESULTS

4.1. Convergence Analysis

We perform a numerical convergence analysis to test the convergence of the method and the number of paths and time steps needed to obtain different levels of errors and volatility values. We compute put options values with the SFR Method using a fixed value for the underlying asset ($S = 80$), increasing the number of time steps and paths from 25 to 400 and from 100 to 5,000, respectively. We also use a risk free rate of 0.04, dividend rate of 0.04, variance rate of 0.2, strike price of 100 and maturity of 0.5.

First, we present the evolution of the mean absolute error and standard deviation obtained by the SFR Method for an European put option, increasing the number of paths for certain values of time steps. We use the Black & Scholes closed-form solution as benchmark to compute the MAE of each iteration.

In Figure 4.1, we observe that as the number of paths increase the MAE decreases, given different values for the number of time steps. This shows that the convergence of method, and thereby the value of the option, depends primary on the number of paths and secondary by the number of time steps.

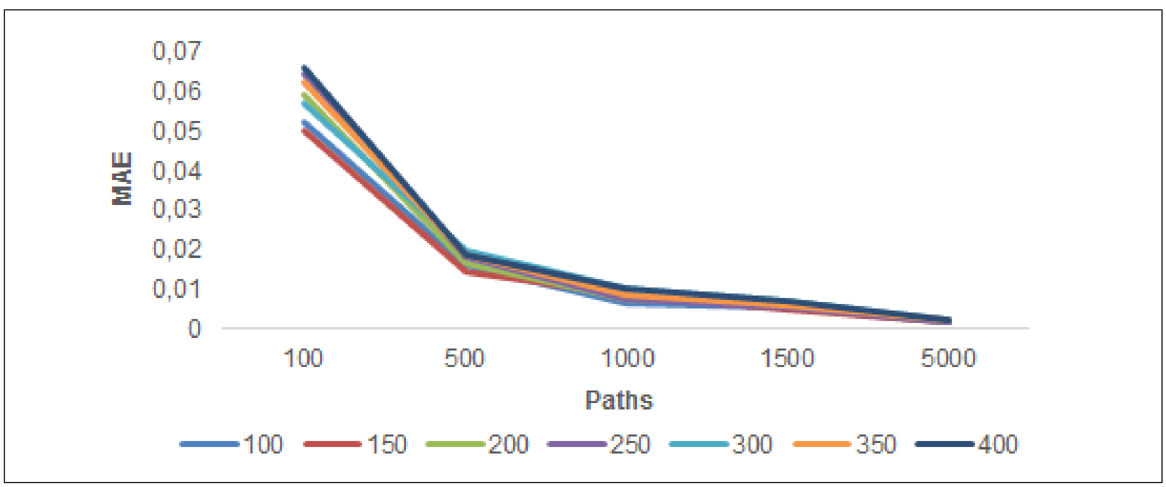

Figure 4.1. MAE vs number of paths for an European put option for different time steps

Figure 4.2 shows the variation of the standard deviation as the number of paths increase. It can be seen that the standard deviation behaves similar to the MAE when we change the number of paths.

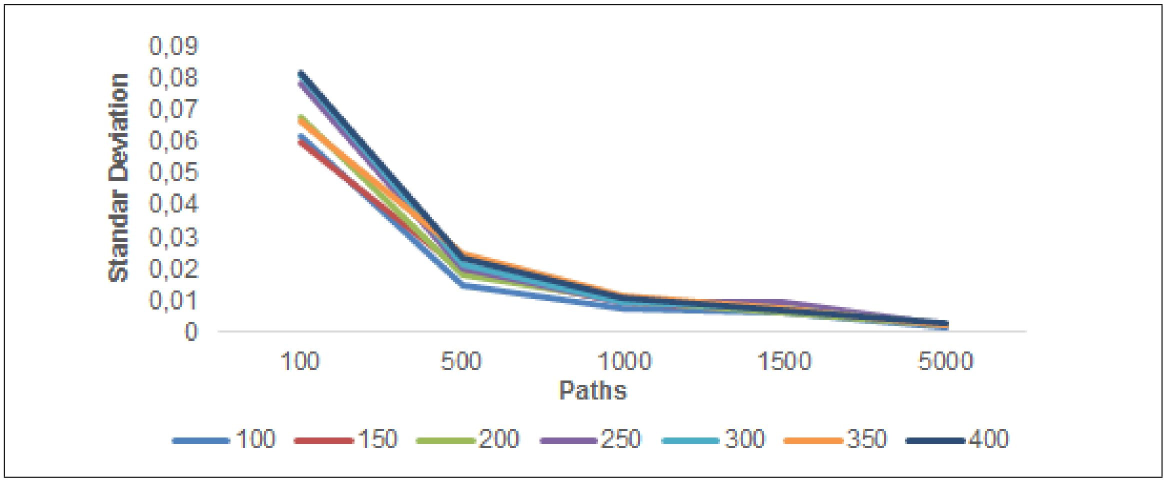

Figure 4.2. Standard deviation vs number of paths for an European put option for different time steps

We also compute the average value of the metrics discussed above for each series of paths (given a certain path number, we varied the number of time steps), which are presented in Table 4.1.

Table 4.1. Average MAE and standard deviation for each series of paths

Paths	MAE	Standard Deviation
100	0.0577	0.1407
500	0.0170	0.0434
1000	0.0083	0.0101
1500	0.0055	0.0064
5000	0.0021	0.0023

It can be observed that the MAE and standard deviation decrease as the number of paths increase, reaching for this analysis an average value of 0.0021 and 0.0023, respectively.

Moreover, we present the evolution of the MAE and standard deviation computed using the SFR Method for an American put option, increasing the number of paths for certain values of time steps. We use a Binomal Tree with 17,000 time steps as benchmark to compute the MAE of each iteration.

Figure 4.3 shows that the value of the MAE decreases as the number of paths increase, which is similar to what we presented for an European option.

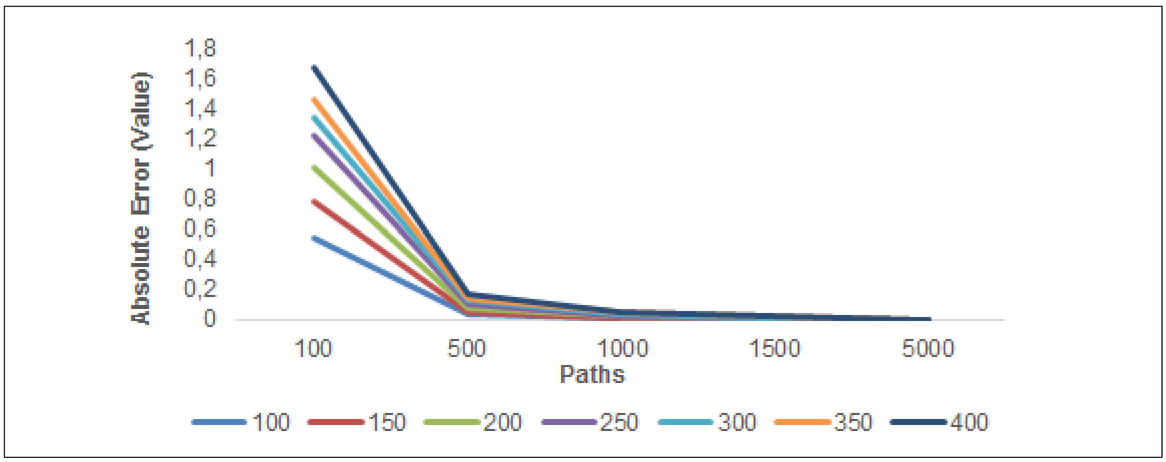

Figure 4.3. MAE vs number of paths for an American put option for different time steps

Figure 4.4 shows the variation of the standard deviation as the number of paths increases. It can be seen that the standard deviation has a similar behavior to the MAE when we change the number of paths. The same movement is observed for an European option.

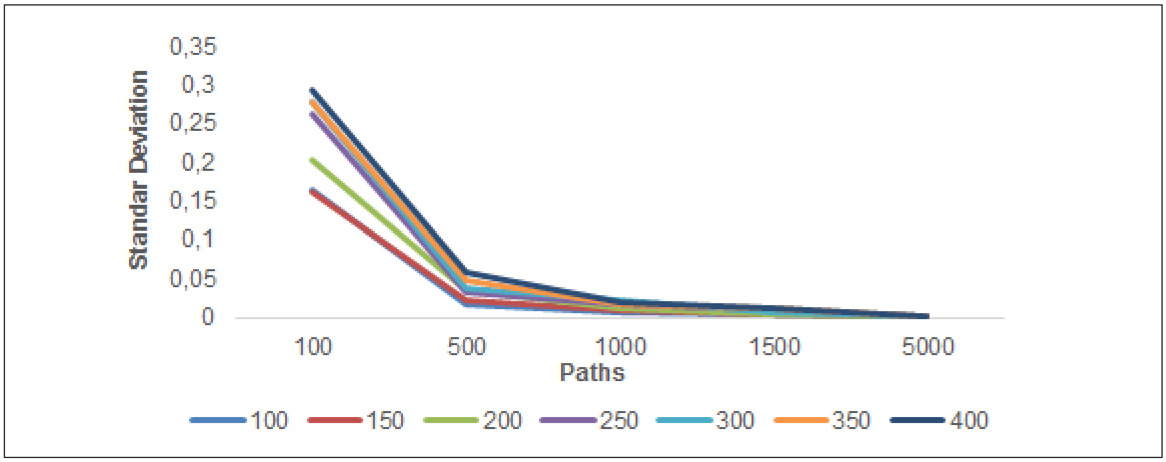

Figure 4.4. Standard deviation vs number of paths for an American put option for different time steps

Finally, we also compute the average value of the metrics discussed above for each series of paths (given a certain path number, we varied the number of time steps), which are presented the following table.

Table 4.2. MAE and standard deviation for each series of paths

Paths	MAE	Standard Deviation
100	1.03507	0.2233
500	0.0892	0.0334
1000	0.0250	0.0142
1500	0.0100	0.0067
5000	0.0024	0.0023

MAE and standard deviation decrease as the number of paths increases, reaching for this analysis a value of 0.0024 and 0.0023, respectively.
Moreover, we compute further analysis to further understand the convergence of the SFR Method. In this context, we perform a runtime analysis for different option prices to

test the time needed to obtain certain levels of errors, which we present in Section 4.2. Additionally, we compute different option prices applying Itos Lemma to understand the convergence and accuracy of the SFR Method when we assumed a log-normal distribution for the underlying asset, which we present in Appendix C.

4.2. Runtime and MAE Analysis

We present a runtime analysis for different option results to understand the numbers of path and time steps needed to obtain a certain level of error and to test the computational requirements of the method. We perform analysis using the SFR Method under the Black & Scholes process, where we averaged 50 iterations of each option value. We use for each option a value of 80 for the underlying asset, strike price of 100, risk free rate of 0.04, divided rate of 0.04, variance rate of 0.2 and maturity of 0.5.

Also, we run the SFR Method using two computational implementations. In first place, we use a regular CPU computer with Intel i5 core processing system, which results we compare with the ones obtained from a Cluster implementation with 200 cores. Hence, for each of the implementations, runtime values were computed as the average runtime per iteration, so that total runtime can be computed as the multiplication of the average runtime by the number of iterations. Moreover, we compute MAE for European and American option prices as described in previous sections, using as benchmark Black & Scholes closed-form solution and a Binomal Tree with 17,000 time steps, respectively. The results obtained are presented in Table 4.3

Table 4.3 shows the speedup and the levels of errors when we use the two different computational implementations. As we observed in Sections 3 and 4.1, we reach convergence increasing the number of paths, but the amount of time required to price the option increases as well. However, the amount of time required to price the option decreases significantly when we use the Cluster implementation. Even though the Cluster implementation is faster than the regular CPU, the algorithm could run faster if we implement it using an optimized parallel computing language.

Table 4.3. Option prices under the Black & Scholes model computed using the SFR Method for regular CPU and Cluster implementations

Method (Nt/Ns)	Runtime CPU (sec)	European MAE	American MAE	Runtime Cluster (sec)
SFR(50/100)	0.95	0.0539	0.3029	1.44
SFR(50/500)	2.36	0.0139	0.0131	2.64
SFR(100/1,000)	12.47	0.0081	0.0060	4.28
SFR(100/5,000)	207.66	0.0020	0.0032	20.28
SFR(100/10,000)	782.50	0.0027	0.0036	46.49
SFR(100/20,000)	2,859.24	0.0023	0.0029	252.95
SFR(100/50,000)	-	0.0015	0.0029	1,213.09
SFR(100/100,000)	-	0.0014	0.0029	4,179.07
SFR(300/20,000)	-	0.0002	0.0014	765.48
SFR(300/50,000)	-	0.0001	0.0015	3,711.36
LSM(150/50,000)	2.13	-	0.0114	-
LSM(250/100,000)	8.04	-	0.0031	-

On the other hand, we can further reduce the pricing errors by increasing the number of time steps, decreasing the error from 0.0015 to 0.0001, using the same number of paths (50,000 paths), but increasing the number of time steps from 100 to 300. However, to increase the number of time steps and paths used, a regular CPU and even a Cluster implementation are not sufficient given the time required to price the option using more than 5,000 paths and 100 time steps, requiring a parallel computing implementation approach to price options in times closer to other methods commonly used.

Finally, we observe that even when the time required to price options under the SFR

Method is longer than using LSM method[1], the use of new computational implementation methods to speedup the algorithm (i.e. parallel computing) could make it competitive regarding this metric. On the other hand, we also observe that the level of error using the same number of time steps and paths is lower, which could be an advantage over the LSM method, but this feature has to be further studied.

[1]We implemented the LSM using a Languerre polynomial with two variables $X_1(x) = 1 - x$ and $X_2(x) = \frac{1}{2} * (x^2 - 4 * x + 2)$, respectively.

5. FURTHER RESEARCH TOPICS

In this section, we present research topics for further studies and testing of the SFR Method, including the development of the theoretical mathematical approach for three additional processes to simulate the underlying asset, which include more complex stochastic jump-diffusion processes.
Other potential lines of investigation and research not covered in this paper are the implementation of an optimized parallelization of the SFR algorithm in GPU devices and/or the use of different probability-assignation algorithms for the risk-adjusted probabilities assigned to the expected cash flow.

5.1. The Merton Model

We present the mathematical approach to implement the SFR Method under the jump-diffusion process proposed by Merton (1976). The dynamics for the risky asset S_t is characterized by the following SDE:

$$\frac{dS_t}{S_t} = (r - q - \lambda\bar{\mu})dt + \sigma dW_t + (Y - 1)dJ_t \tag{5.1}$$

In (5.1), r corresponds to the instantaneous risk-free rate, q corresponds to the instantaneous dividend rate, σ corresponds to the instantaneous volatility. W_t is the Brownian process and J_t is the Poisson process of intensity λ, which are statically independent. In the case of Y, it corresponds to the amplitude of the jump and $\bar{\mu}$ is the expected change in the value of the underlying asset if a jump occurs. Following the simulation approach discussed earlier, the Euler discretized process followed by the underlying asset, where there are N_j jumps between $t-1$ and t, is characterized by the following equation:

$$S_{t+1} = S_t + (r - q - \lambda\bar{\mu})S_t\Delta t + \sigma S_t \Delta W_t + \sum_{j=1}^{N_t}(Y_j - 1)S_t \tag{5.2}$$

Furthermore, 5.2 can be written as:

$$ds_{t+1} = \ln(\frac{S_{t+1}}{S_t}) = \{(r - q - \frac{\sigma^2}{2} - \lambda\bar{\mu})\Delta t + \sigma\Delta W_t + \ln(\prod_{j=1}^{N_t} Y_j)\} \tag{5.3}$$

Following Merton's model, Y_j is a lognormal random variable with mean μ_Y and variance σ_Y^2. Moreover, $\bar{\mu} = E\{Y_j - 1\}$ and μ_Y and $\bar{\mu}$ area related by the following expression:

$$\bar{\mu} = \exp(\mu_Y + \frac{1}{2}\sigma_Y^2) - 1$$

For the purpose of the SFR Method, it is needed to know the distribution of the underling asset at time $t+1$ under the Merton dynamic, conditional on the value of the asset at time t ($F[S_{t+1}|S_t]$). To obtain the conditional probability distribution function, we use the same argument as in Zhou (1997), where $F_{t+1}^Q(X|S_t)$ is the probability that the event $\{S_t \leq X\}$ occurs conditional on the value of S_t, under a risk-adjusted probability measure Q:

$$F_t^Q(X|S_t) = Q(S_{t+1} \leq X|S_t) \tag{5.4}$$

If N_j are the number of jumps between t and $t+1$ and the risk-adjusted probability can be written as:

$$Q(S_{t+1} \leq X|S_t) = Q(ln(S_{t+1}) \leq ln(X)|ln(S_t)) \tag{5.5}$$

Then, the risk-adjusted probability conditional on the number of jumps occurred between the time step can be written as:

$$F_{t+1}^Q(X|S_t) = \sum_{i=0}^{Nj} Q(ln(S_{t+1}) \leq ln(X)|ln(S_t), N_j = i) * Pr(Nj = i) \tag{5.6}$$

In this case, the conditional distribution function is the sum of multiplications of Normal and Poisson distributions. It follows from (5.6) that the conditional probability distribution of the asset, $F_{t+1}^Q(X|S_t)$, is:

$$F_{t+1}^Q(X|S_t) = \sum_{i=0}^{Nj} N(\frac{ln(\frac{S_t}{K}) + (r - q - \frac{\sigma^2}{2} - \lambda\bar{\mu} + \frac{i*\mu_Y}{\Delta t})\Delta t}{\sqrt{\sigma^2\Delta t + i*\sigma_Y^2}}) * \frac{exp(-\lambda\Delta t)(\lambda\Delta t)^i}{i!} \tag{5.7}$$

Finally, it follows from (5.7) that the probability distribution of the asset is:

$$ln(S_{t+1})|ln(S_t) \sim \sum_{i=0}^{Nj} N(ln(S_t) + (r - q - \frac{\sigma^2}{2} - \lambda\bar{\mu} + \frac{i\mu_Y}{\Delta t})\Delta t, \sigma^2\Delta t + i\sigma_Y^2)Pr(N_j = i) \tag{5.8}$$

$$Pr(N_j = i) = \frac{exp(-\lambda\Delta t)(\lambda\Delta t)^i}{i!} \tag{5.9}$$

5.2. The SVJ Model

We present the mathematical approach to implement the SFR Method under a the stochastic volatility with jumps (SVJ) model, where the SDE for S_t is given by the jump-diffusion process proposed in section 5.1 and the stochastic volatility process proposed in section 3.2. Thus, the dynamics for S_t are given by the SDEs system:

$$\frac{dS_t}{S_t} = (r - q - \lambda\bar{\mu})dt + \sqrt{V_t}dW_t^S + (Y-1)dJ_t \tag{5.10}$$

$$dV_t = \kappa(\theta - V_t)dt + \xi\sqrt{V_t}dW_t^V \tag{5.11}$$

$$dW_t^S dW_t^V = \rho dt \tag{5.12}$$

Here, W^S and W^V corresponds to Brownian processes and J_t is the Poisson process of intensity λ, both statically independent of each other. All other parameters are the same that those presented in previous sections. Following the simulation approach discussed earlier, the Euler discretized processes followed by the underlying asset, where there are N_j jumps between $t-1$ and t, is characterized by the following equation:

$$S_{t+1} = S_t + (r - q - \lambda\bar{\mu})S_t\Delta t + \sqrt{V_t}\Delta W_t^S + \sum_{j=1}^{N_t}(Y_j - 1)S_t \tag{5.13}$$

$$dV_t = \kappa(\theta - V_t)dt + \xi\sqrt{V_t}dW_t^V \tag{5.14}$$

$$dW_t^S dW_t^V = \rho\Delta t \tag{5.15}$$

As presented in Section 5.1, Y_j is a lognormal random variable with mean μ_Y and variance σ_Y^2. Furthermore, $\bar{\mu} = E\{Y-1\}$ and μ_Y and $\bar{\mu}$ are related by the following expression:

$$\bar{\mu} = \exp(\mu_Y + \frac{1}{2}\sigma_Y^2) - 1$$

For the purposed of the SFR Method, it is needed to know the distribution of the underling asset at time $t+1$, conditional on the value of the asset at time t and the square volatility at time t, under the Heston and Merton dynamic ($F[S_{t+1}|S_t, V_t]$). To obtain the conditional probability distribution function, we used the same argument as in Zhou (1997) and in Section 5.1, where $F^Q_{t+1}(X|S_t, V_t)$ is the probability that the event $\{S_t \leq X\}$ occurs conditional on the value of S_t and V_t, under a risk-adjusted probability measure Q:

$$F^Q_t(X|S_t, V_t) = Q(S_{t+1} \leq X|S_t, V_t) \tag{5.16}$$

If N_j are the number of jumps between t and $t+1$ and the risk-adjusted probability can be written as:

$$Q(S_{t+1} \leq X|S_t, V_t) = Q(ln(S_{t+1}) \leq ln(X)|ln(S_t), V_t) \tag{5.17}$$

Then, the risk-adjusted probability conditional on the number of jumps occurred between the time step can be written as:

$$F^Q_{t+1}(X|S_t, V_t) = \sum_{i=0}^{Nj} Q(ln(S_{t+1}) \leq ln(X)|ln(S_t), V_t, N_j = i) * Pr(Nj = i) \tag{5.18}$$

In this case, the conditional distribution function is the sum of N_j multiplication of Normal and Poisson distributions. It follows from (5.18) that the conditional probability distribution of the asset, $F^Q_{t+1}(X|S_t, V_t)$, is:

$$F^Q_{t+1}(X|S_t, V_t) = \sum_{i=0}^{Nj} N\left(\frac{ln(\frac{S_t}{K}) + (r - q - \frac{V_t}{2} - \lambda\bar{\mu} + \frac{i*\mu_Y}{\Delta t})\Delta t}{\sqrt{V_t\Delta t + i * \sigma^2_Y}}\right) * \frac{exp(-\lambda\Delta t)(\lambda\Delta t)^i}{i!} \tag{5.19}$$

Finally, it follows from (5.19) that the probability distribution of the asset is:

$$ln(S_{t+1})|ln(S_t), V_t \sim$$

$$\sum_{i=0}^{Nj} \frac{exp(-\lambda\Delta t)(\lambda\Delta t)^i}{i!} * N(ln(S_t) + (r - q - \frac{V_t}{2} - \lambda\bar{\mu} + \frac{i\mu_Y}{\Delta t})\Delta t, V_t\Delta t + i\sigma^2_Y) \tag{5.20}$$

5.3. The SVCJ Model

We present the mathematical approach to implement the SFR Method under the stochastic volatility with co-jumps (SVCJ) model first introduced in Duffie, Pan, and Singleton (2000), which is similar to the model described in section 5.2, but in this case the model includes jump in the stochastic volatility process:

$$\begin{aligned} \frac{dS_t}{S_t} &= (r - q - \lambda\bar{\mu})dt + \sqrt{V_t}dW_t^S + (Y-1)dJ_t & (5.21)\\ dV_t &= \kappa(\theta - V_t)dt + \xi\sqrt{V_t}dW_t^V + XdJ_t & (5.22)\end{aligned}$$

In (5.21) and (5.22), W^S and W^V correspond to Brownian processes, J_t is the Poisson process of intensity λ respectively, statically independent with the Brownian processes, Y is a log-normal variable corresponding to the size of the jump in the underlying asset and X is an exponential variable corresponding to the size of the jump for the variance with mean μ_X. Jumps in the underlying asset and the variance occur simultaneously, and are correlated by a factor ρ_j. Given X, Y is a log-normal variable with mean $\mu_Y + \rho_j X$ and variance σ_Y^2, such as $\bar{\mu}$ and μ_Y are related by the following expression:

$$\bar{\mu} = \frac{\exp(\mu_Y + \frac{1}{2}\sigma_Y^2)}{1 - \rho_j\mu_X} - 1$$

Following the simulation approach discussed earlier, the Euler discretized processes followed by the underlying asset, where there are N_j jumps between $t-1$ and t, is characterized by the following equations:

$$S_{t+1} = S_t + (r - q - \lambda\bar{\mu})S_t\Delta t + \sqrt{V_t}\Delta W_t^S + \sum_{j=1}^{N_t}(Y_j - 1)S_t \qquad (5.23)$$

$$dV_t = \kappa(\theta - V_t)dt + \xi\sqrt{V_t}dW_t^V + \sum_{j=1}^{N_t}X_j \qquad (5.24)$$

For the purposed of the SFR Method, it is needed to know the distribution of the underling asset at time $t+1$, conditional on the value of the asset at time t and the square volatility at time t, under the Heston and Merton dynamic ($F[S_{t+1}|S_t, V_t]$). To obtain the conditional probability distribution function, we used the same argument as in Zhou (1997) and in Section 5.2, where $F^Q_{t+1}(X|S_t, V_t)$ is the probability that the event $\{S_t \leq X\}$ occurs conditional on the value of S_t and V_t, under a risk-adjusted probability measure Q:

$$F^Q_{t+1}(X|S_t, V_t) = \sum_{i=0}^{Nj} Q(ln(S_{t+1}) \leq ln(X)|ln(S_t), V_t, N_j = i) * Pr(Nj = i) \quad (5.25)$$

In this case, the conditional distribution function is the sum of N_j multiplication of Normal and Poisson distributions. It follows from (5.25) that the conditional probability distribution of the asset, $F^Q_{t+1}(X|S_t, V_t)$, is:

$$\sum_{i=0}^{Nj} N(\frac{ln(\frac{S_t}{K}) + (r - q - \frac{V_t}{2} - \lambda\bar{\mu} + \frac{i*(\mu_Y + \rho_j\mu_X)}{\Delta t})\Delta t}{\sqrt{V_t\Delta t + i * \sigma_Y^2}}) * \frac{exp(-\lambda\Delta t)(\lambda\Delta t)^i}{i!} \quad (5.26)$$

Finally, it follows from (5.26) that the probability distribution of the asset is:

$$ln(S_{t+1})|ln(S_t), V_t \sim$$

$$\sum_{i=0}^{Nj} \frac{exp(-\lambda\Delta t)(\lambda\Delta t)^i}{i!} * N(ln(S_t) + (r - q - \frac{V_t}{2} - \lambda\bar{\mu} + \frac{i * (\mu_Y + \rho_j\mu_X)}{\Delta t})\Delta t, V_t\Delta t + i\sigma_Y^2) \quad (5.27)$$

6. CONCLUSIONS

In this thesis, we have presented a straight-forward and robust new method, the SFR Method, to price American options using the characteristics implied on the conditional probability distribution function (expectation function) of each stochastic process implemented to describe the behavior of the underlying asset that is being simulated. Throughout the implementation of this conditional expectation function we have obtained a new method to determine risk-adjusted probabilities to be assigned to the expected payoffs, which is not subject to an implementation decision as oppose to other option pricing models (i.e. Longstaff and Schwartz method).

Furthermore, we have proved that the SFR Method can be used as a generalized version of the Binary Tree model, since as we have shown, it has the flexibility to price American-style as well as European-style feature options and the simplicity on the valuation procedure. Moreover, we have shown that the model incorporates the adaptability to price options under different stochastic processes, including Black & Scholes, Heston, Merton models as well as other more complex jump-diffusion processes.

We tested the new method under two different dynamics for the underlying asset, which was presented in Section 3. For the simulation approach, we computed 50 option prices for each value of the underlying asset, which we averaged and used to obtain MAE and standard deviations, which we used to test the convergence of the method and the deviation respect to the True Value of reference. First, we implemented the SFR Method under the Black & Scholes model, calculating at-the-money European and American put option prices and compared them against their True Value, which we obtained using Black & Scholes closed-form solution and a Binomal Tree with 17,000 time steps, respectively. We obtained MAE and standard deviation of 0.0003 and 0.0001 in the lower case, respectively for European as well as American options. We also observed that the convergence of the algorithm was obtained using over 20,000 paths.

In second place, we implemented the SFR Method under the Heston model, calculating at-the-money European and American put prices and compared them against their True

Value, which we obtained using Heston closed-form solution and the LSM method under the Heston process, respectively. We obtained MAE and standard deviation of 0.0004 and 0.0001 in the lower case, respectively for European options and of 0.0001 and 0.0001 in the lower case, respectively for American options. Under the Heston process, the convergence of the algorithm was obtained with equal number of paths.

Moreover, we performed a series of additional analysis of the results obtained with the SFR Method, which we presented in Section 4. We performed the analysis averaging 50 option prices computed with the SFR Method under the Black & Scholes model for different number of paths.

In the first place, we performed a convergence analysis, where we concluded that as the number of paths increase, the MAE and standard deviation decreases independent of the number of time steps used,showing that the convergence of the method depends primary on the number of paths and secondary by the number of time steps. Also, we observed that using a number of 5,000 paths per iteration for different time steps, we obtained an average value for the MAE and standard deviation of 0.0021 and 0.0023, respectively for European options, and an average value of 0.0024 and 0.0023, respectively for American options.

In second place, we performed a runtime analysis to understand the number of paths and time steps needed to obtain a certain level of error and to test the computational requirements of the method. We concluded that as we increase the number of paths, the amount of time required to price the option increases as well, reaching 2,859 seconds for 100 time steps and 20,000 paths (number of paths required for the model to converge). The time can be reduced by implementing the algorithm in a cluster computer, reaching 252 second for the same number of paths and time steps. Also, we observed that the MAE was further reduced by increasing the number of time steps, decreasing from 0.0015 to 0.0001 for European options, but increasing the number of time steps from 100 to 300.

Finally, we have presented further research topics to continue studying and testing the SFR Method, including the development of the theoretical mathematical approach for the Merton, SVJ and SVCJ processes to be implemented under the SFR pricing model;

and also we have identified other potential lines of investigation and research not cover in this work, which could improve the time constrain and the error metrics studied in this thesis through the implementation of the SFR algorithm in parallel computing language and the used of a different probability-assignation algorithm for the risk-adjusted probabilities assigned to the expected cash flows, respectively.

REFERENCES

Black, F., & Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of political economy*.

Cox, J., Ross, S., & Rubinstein, M. (1979). Option pricing: A simplified approach. *Journal of financial Economics*, *7*(1), 229-263.

Duffie, D., Pan, J., & Singleton, K. (2000). Transform analysis and asset pricing for affine jump-diffusions. *Econometrica*, *68*(6), 1343-1376.

Heston, S. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, *6*(2), 327-343.

Kim, I. (1990). The analytic valuation of american options. *Review of Financial Studies*, *3*(4), 547-572.

Longstaff, F., & Schwartz, E. (2001). Valuing american options by simulation: a simple least-squares approach. *The Review of Financial Studies*, *14*(1), 113-147.

Merton, R. (1976). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, *3*(1-2), 125-144.

Moodley, N. (2005). The heston model: A practical approach with matlab code. *University of the Witwatersrand*.

Zhou, C. (1997). A jump-diffusion approach to modeling credit risk and valuing defaultable securities. *Available at SSRN 39800*.

APPENDIX

A. FURTHER RESULTS UNDER THE BLACK & SCHOLES MODEL

We present further European option results computed with the SFR Method under the Black & Scholes process in Table A.1. We used the Euler discretized simulation approach to compute and average 50 iterations of each option value presented in the table. We also used the following parameters: K = 100, T = 0.50, r = 0.04, q = 0.04, and σ = 0.20. Moreover, we calculate the MAE and standard deviation as described in previous sections, using as benchmark Black & Scholes closed-form solution.

Table A.1. European put prices under the Black & Scholes model

	S				
Method(Nt/Nw)	80	90	100	110	120
Closed-Form	19.9070	11.5393	5.5256	2.1675	0.7061
SFR(100/1,500)	19.9041	11.5366	5.5250	2.1700	0.7062
Std Dev.	0.0069	0.0086	0.0106	0.0104	0.0127
MAE	0.0063	0.0067	0.0085	0.0091	0.0106
SFR(100/5,000)	19.9056	11.5376	5.5250	2.1696	0.7084
Std Dev.	0.0020	0.0027	0.0041	0.0033	0.0029
MAE	0.0019	0.0026	0.0032	0.0031	0.0029
SFR(100/10,000)	19.9055	11.5375	5.5256	2.1697	0.7085
Std Dev.	0.0011	0.0010	0.0013	0.0017	0.0018
MAE	0.0016	0.0019	0.0010	0.0023	0.0024
SFR(100/50,000)	19.9056	11.5376	5.5261	2.1701	0.7088
Std Dev.	0.0002	0.0002	0.0003	0.0003	0.0003
MAE	0.0014	0.0017	0.0005	0.0026	0.0027
SFR(100/100,000)	19.9056	11.5376	5.5261	2.1701	0.7088
Std Dev.	0.0001	0.0001	0.0001	0.0001	0.0002
MAE	0.0014	0.0017	0.0005	0.0026	0.0027

In Table A.2, we present further American option results computed with the SFR Method under the Black & Scholes process, using the same parameters used for the European option results describe above. Moreover, we calculate the MAE and standard deviation as described in previous sections, using as benchmark a Binomal Tree with 17,000 time steps.

Table A.2. American put prices under the Black & Scholes model

	S				
Method(Nt/Nw)	80	90	100	110	120
Binomial Tree	20.1448	11.6147	5.5466	2.1726	0.7073
SFR(100/1,500)	20.1476	11.6298	5.5742	2.2061	0.7415
Std Dev.	0.0038	0.0091	0.0118	0.0128	0.0165
MAE	0.0028	0.0151	0.026	0.0335	0.0342
SFR(100/5,000)	20.1426	11.6146	5.5526	2.1826	0.7176
Std Dev.	0.0010	0.0018	0.0033	0.0040	0.0031
MAE	0.0022	0.0001	0.0060	0.0100	0.0103
SFR(100/10,000)	20.1423	11.6131	5.5488	2.1782	0.7135
Std Dev.	0.0005	0.0006	0.0012	0.0015	0.0017
MAE	0.0025	0.0016	0.0022	0.0056	0.0062
SFR(100/50,000)	20.1419	11.6122	5.5470	2.1756	0.7106
Std Dev.	0.0001	0.0001	0.0002	0.0003	0.0004
MAE	0.0029	0.0025	0.0004	0.0030	0.0033
SFR(100/100,000)	20.1419	11.6121	5.5468	2.1753	0.7102
Std Dev.	0.0001	0.0001	0.0001	0.0002	0.0002
MAE	0.0029	0.0026	0.0004	0.0027	0.0029

B. FURTHER RESULTS UNDER THE HESTON MODEL

We present further results computed with the SFR Method under the Heston process in Table B.1. We used the Euler discretized simulation approach to compute and average 50 iterations of each option value presented in the table. We also used the following parameters: K = 100, T = 0.25, θ = 0.16, r = 0.10, ξ = 0.90, q = 0.00, ρ = 0.10, κ = 5.00 and V_0 = 0.25. Moreover, we calculate the MAE and standard deviation as described in previous sections, using as benchmark Heston closed-form solution.

Table B.1. European put prices under the Heston model

			- S		
Method(Nt/Nw)	8	9	10	11	12
Closed-Form	1.9773	1.2800	0.7697	0.4360	0.2373
SFR(100/1,500)	1.9763	1.2797	0.7697	0.4353	0.2391
Std Dev.	0.0033	0.0032	0.0022	0.0034	0.0034
MAE	0.0028	0.0024	0.0017	0.0026	0.0031
SFR(100/5,000)	1.9765	1.2804	0.7701	0.4365	0.2380
Std Dev.	0.0019	0.0019	0.0010	0.0017	0.0015
MAE	0.0016	0.0016	0.0008	0.0014	0.0014
SFR(100/10,000)	1.9767	1.2802	0.7703	0.4371	0.2385
Std Dev.	0.0013	0.0010	0.0007	0.0010	0.0012
MAE	0.0012	0.0008	0.0007	0.0012	0.0014
SFR(100/50,000)	1.9771	1.2802	0.7703	0.4370	0.2384
Std Dev.	0.0005	0.0005	0.0003	0.0005	0.0008
MAE	0.0004	0.0004	0.0006	0.0010	0.0012
SFR(100/100,000)	1.9771	1.2801	0.7703	0.4369	0.2384
Std Dev.	0.0004	0.0004	0.0002	0.0004	0.0008
MAE	0.0004	0.0003	0.0006	0.0009	0.0012

In Table B.2, we present further American option results computed with the SFR Method under the Heston process, using the same parameters used for the European option results described above. Moreover, we calculate the MAE and standard deviation as described in previous sections, using as benchmark the LSM method[1] computed under the Heston process.

Table B.2. American put prices under the Heston model

	S				
Method(Nt/Nw)	8	9	10	11	12
Heston LSM	2.0784	1.3336	0.7960	0.4483	0.2428
SFR(100/1,500)	2.0778	1.3337	0.7972	0.4505	0.2488
Std Dev.	0.0023	0.0032	0.0016	0.0033	0.0039
MAE	0.0006	0.0001	0.0002	0.0022	0.0060
SFR(100/5,000)	2.0778	1.3336	0.7962	0.4490	0.2442
Std Dev.	0.0015	0.0018	0.0008	0.0016	0.0016
MAE	0.0006	0.0000	0.0002	0.0007	0.0014
SFR(100/10,000)	2.0779	1.3334	0.7961	0.4492	0.2442
Std Dev.	0.0010	0.0010	0.0007	0.0010	0.0012
MAE	0.0005	0.0002	0.0001	0.0009	0.0014
SFR(100/50,000)	2.0782	1.3334	0.7961	0.4488	0.2436
Std Dev.	0.0004	0.0005	0.0003	0.0005	0.0006
MAE	0.0002	0.0002	0.0001	0.0005	0.0008
SFR(100/100,000)	2.0781	1.3333	0.7960	0.4487	0.2436
Std Dev.	0.0003	0.0004	0.0002	0.0004	0.0004
MAE	0.0003	0.0003	0.0000	0.0004	0.0008

[1]We implemented the LSM using a Languerre polynomial with two variables $X_1(x) = 1 - x$ and $X_2(x) = \frac{1}{2} * (x^2 - 4 * x + 2)$, respectively.

C. IMPLEMENTATION OF THE LOG-NORMAL APPROACH

In this section, we compute different option prices applying Itos Lemma to understand the convergence and accuracy of the SFR Method when we assumed a log-normal distribution for the underlying asset.
We reformulate Equation 2.1 and 2.2 of Section 2.1, applying Itos Lemma to obtain the following Euler discretized simulation process for the variables $\widehat{S}_{t_i+1}$ and $\widehat{Y}_{t+1}$:

$$\begin{aligned}
&\widehat{S}_{t_{i+1}} = \widehat{S}_{t_i} * exp\{(r(\widehat{Y}_{t_i}) - q(\widehat{Y}_{t_i}) - 1/2\,\sigma^S(\widehat{Y}_{t_i}))\Delta t + \sigma(\widehat{Y}_{t_i})\Delta W^S_{t_i} + \xi(\widehat{Y}_{t_i})\Delta J_{t_i}\} \\
&\widehat{Y}_{t_{i+1}} = \widehat{Y}_{t_i} + \mu^Y(\widehat{Y}_{t_i})\Delta t + \sigma^Y(\widehat{Y}_{t_i})\Delta W^V_{t_i} + \psi(\widehat{Y}_{t_i})\Delta J_{t_i}
\end{aligned} \tag{C.1}$$

First, we apply Itos Lemma to Black & Scholes SDE presented in Equation 3.1 of Section 3.1, obtaining the following SDE and probability distribution for the risky asset:

(i) SDE:

$$ds_{t+1} = \ln(\frac{S_{t+1}}{S_t}) = \{(r - q - \frac{\sigma^2}{2})\Delta t + \sigma\Delta W_t\} \tag{C.2}$$

(ii) Probability Distribution:

$$ds_{t+1} = \ln(\frac{S_{t+1}}{S_t})|S_t \sim N((r - q - \frac{\sigma^2}{2})\Delta t, \sigma^2\Delta t) \tag{C.3}$$

In second place, we apply Itos Lemma to Heston SDE presented in Equation 3.4, 3.5 and 3.6 of Section 3.2, obtaining the following SDE and probability distribution for the risky asset:

(i) SDE:

$$ds_{t+1} = \ln(\frac{S_{t+1}}{S_t}) = \{(r - q - \frac{\sigma^2}{2})\Delta t + \sqrt{V_t}\Delta W^S_t\} \tag{C.4}$$

$$dV_t = \kappa(\theta - V_t)dt + \xi\sqrt{V_t}dW^V_t \tag{C.5}$$

$$dW_t^S dW_t^V = \rho dt \tag{C.6}$$

(ii) Probability Distribution:

$$ds_{t+1} = \ln(\frac{S_{t+1}}{S_t})|S_t, V_t \sim N((r - q - \frac{\sigma^2}{2})\Delta t, V_t \Delta t) \tag{C.7}$$

We test the SFR Method under the two dynamics for the risky asset satisfying Equations C.2, and C.4, C.5 and C.6, which we present in Sections D and E, respectively.

D. RESULTS UNDER THE LOG-NORMAL BLACK & SCHOLES MODEL

We present European put option results computed with the SFR Method under the Black & Scholes process in Table D.1. We used the Euler discretized simulation approach described in Equations C.2 and C.3 of Appendix C to compute and average 50 iterations of each option value presented in the table. We also used the following parameters: K = 100, T = 0.50, r = 0.04, q = 0.04, and σ = 0.20. Moreover, we calculate the MAE and standard deviation as described in previous sections, using as benchmark Black & Scholes closed-form solution.

Table D.1. European put prices under the Black & Scholes model

			S		
Method(Nt/Nw)	80	90	100	110	120
Closed-Form	19.9070	11.5393	5.5256	2.1675	0.7061
SFR(100/20,000)	19.9081	11.5403	5.5261	2.1678	0.7063
Std Dev.	0.0005	0.0007	0.0008	0.0008	0.0009
MAE	0.0011	0.0010	0.0007	0.0007	0.0007
SFR(100/50,000)	19.9081	11.5403	5.5261	2.1678	0.7063
Std Dev.	0.0002	0.0002	0.0003	0.0003	0.0003
MAE	0.0011	0.0010	0.0005	0.0004	0.0003
SFR(100/100,000)	19.9081	11.5403	5.5261	2.1678	0.7063
Std Dev.	0.0001	0.0001	0.0001	0.0001	0.0002
MAE	0.0011	0.0010	0.0005	0.0003	0.0002

In Table D.2, we present American option results computed with the SFR Method under the Black & Scholes process, using the same simulation approach and parameters used for the European option results described above. Moreover, we calculate the MAE and standard deviation as described in previous sections, using as benchmark a Binomal Tree with 17,000 time steps.

Table D.2. American put prices under the Black & Scholes model

	S				
Method(Nt/Nw)	80	90	100	110	120
Binomial Tree	20.1448	11.6147	5.5466	2.1726	0.7073
SFR(100/20,000)	20.1432	11.6146	5.5478	2.1744	0.7093
Std Dev.	0.0002	0.0003	0.0008	0.0011	0.0012
MAE	0.0016	0.0003	0.0013	0.0018	0.0020
SFR(100/50,000)	20.1431	11.6142	5.5470	2.1732	0.7080
Std Dev.	0.0001	0.0001	0.0002	0.0003	0.0004
MAE	0.0017	0.0005	0.0006	0.0006	0.0007
SFR(100/100,000)	20.1431	11.6142	5.5468	2.1729	0.7077
Std Dev.	0.0001	0.0001	0.0001	0.0002	0.0002
MAE	0.0017	0.0005	0.0004	0.0003	0.0004

Results presented are similar to the ones computed with the Normal-distribution Euler discretized simulation approach. However, we can observe that MAE and standard deviation are lower using the Log-normal distribution, improving the MAE of the algorithm under the Black & Scholes process.

E. RESULTS UNDER THE LOG-NORMAL HESTON MODEL

We present results computed with the SFR Method under the Heston process in Table E.1. We use the Euler discretized simulation approach described in Equations C.4, C.5, C.6 and C.7 of Appendix C to compute and average 50 iterations of each option value presented in the table. We also used the following parameters: K = 100, T = 0.25, θ = 0.16, r = 0.10, ξ = 0.90, q = 0.00, ρ = 0.10, κ = 5.00 and V_0 = 0.25. Moreover, we calculate the MAE and standard deviation as described in previous sections, using as benchmark Heston closed-form solution.

Table E.1. European put prices under the Heston model

	S				
Method(Nt/Nw)	8	9	10	11	12
Closed-Form	1.9773	1.2800	0.7697	0.4360	0.2373
SFR(100/20,000)	1.9774	1.2803	0.7702	0.4363	0.2375
Std Dev.	0.0009	0.0008	0.0004	0.0004	0.0008
MAE	0.0008	0.0007	0.0006	0.0003	0.0007
SFR(100/50,000)	1.9776	1.2804	0.7703	0.4363	0.2375
Std Dev.	0.0005	0.0005	0.0003	0.0005	0.0006
MAE	0.0004	0.0005	0.0006	0.0005	0.0005
SFR(100/100,000)	1.9776	1.2804	0.7703	0.4363	0.2375
Std Dev.	0.0004	0.0004	0.0002	0.0004	0.0004
MAE	0.0004	0.0004	0.0006	0.0003	0.0004

In Table E.2, we present American option results computed with the SFR Method under the Heston process, using the same simulation approach and parameters used for the European option results described above. Moreover, we calculate the MAE and standard deviation as described in previous sections, using as benchmark the LSM method[2] computed under the Heston process.

Table E.2. American put prices under the Heston model

	S				
Method(Nt/Nw)	8	9	10	11	12
Heston LSM	2.0784	1.3336	0.7960	0.4483	0.2428
SFR(100/20,000)	2.0781	1.3331	0.7960	0.4479	0.2429
Std Dev.	0.0007	0.0008	0.0004	0.0003	0.0009
MAE	0.0006	0.0007	0.0003	0.0004	0.0007
SFR(100/50,000)	2.0782	1.3333	0.7961	0.4480	0.2427
Std Dev.	0.0004	0.0005	0.0003	0.0005	0.0006
MAE	0.0003	0.0005	0.0002	0.0005	0.0005
SFR(100/100,000)	2.0782	1.3332	0.7960	0.4479	0.2427
Std Dev.	0.0003	0.0004	0.0002	0.0004	0.0004
MAE	0.0003	0.0005	0.0001	0.0004	0.0003

Results presented are similar to the ones computed with the Normal-distribution Euler discretized simulation approach. However, we can observe that MAE and standard deviation are lower using the Log-normal distribution, improving the MAE of the algorithm under the Heston process.

[2]We implemented the LSM using a Languerre polynomial with two variables $X_1(x) = 1 - x$ and $X_2(x) = \frac{1}{2} * (x^2 - 4 * x + 2)$, respectively.