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Citation: Journal of Mathematical Physics **33**, 1571 (1992); doi: 10.1063/1.529681
View online: https://doi.org/10.1063/1.529681
View Table of Contents: http://aip.scitation.org/toc/jmp/33/5
Published by the American Institute of Physics

SL(3,R) as the group of symmetry transformations for all one-dimensional linear systems. III. Equivalent Lagrangian formalisms

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(Received 16 November 1990; accepted for publication 13 January 1992)

The SL(3,R) theory of projective transformations of the plane is applied to the Lagrangians of all one-dimensional Newtonian linear systems. Noether and non-Noether equivalent Lagrangians, as well as the associated Noether and non-Noether constants of motion, are thus obtained in a completely general and systematic way. Complete unification is achieved by this group-theoretic approach to Lagrangians of one-dimensional linear systems.

I. INTRODUCTION

In this paper we are interested in the point symmetry properties exhibited by the Lagrangian function of one-dimensional linear systems. It is our aim to give a unified treatment of the similarity properties of such Lagrangians taking advantage of the fact that SL(3,R) is the maximal group of point symmetry transformations for *all* linear inhomogeneous ordinary differential equations of the second order, in one real dependent variable.[1] To this end, we shall use the group elements themselves[2] (instead of the Lie algebra generators[3]) to uncover the symmetry group of the Lagrangians, and thus calculating several equivalent Lagrangians, for any given one-dimensional linear system.

During the last years there has been a considerable progress in the study of symmetries and invariants in classical mechanics,[4] which stems from different concepts of what is the basic formulation for studying the symmetries of mechanical systems in general.[5]

We wish to remark that our main interest on this issue stems from its usefulness in quantum kinematics.[6] In fact, in Ref. 2 we have obtained a technique for calculating the specific *finite realizations* of SL(3,R) for any given one-dimensional linear Newtonian system, which is the starting point for quantizing these systems through their symmetry properties.[6] We have been unable to find this technique in the current literature.

Hence, as an important complement of our previous work, in this paper we shall adopt SL(3,R) as a group of space-time automorphisms that interconverts one admissible worldline of a linear system into another, in order to study the different equivalent Lagrangian functions one obtains from the action of the group. In this fashion, we get complete generalization and unification of a great amount of work, that has been previously performed on the symmetry properties of Lagrangians for one-dimensional linear systems.[7] Applications of the present formalism to quantum kinematics [namely, SL(3,R) quantum kinematics] shall be given in a forthcoming paper. Nevertheless, we wish to publish this subject separately, because we deem it interesting by itself from the standpoint of classical mechanics.

The organization of this paper is as follows. In Sec. II we present a brief review of the general theory of point transformations in Lagrangian mechanics, and of the point symmetry properties of the Lagrangian function, which shall be needed in the sequel. Section III contains the applications to the SL(3,R) theory of equivalent Lagrangians for a one-dimensional Newtonian free particle. Finally, in Sec. IV, we generalize the previous formalism to the case of SL(3,R)-equivalent Lagrangians for *all* one-dimensional linear Newtonian systems. Noether and non-Noether Lagrangians (and the associated Noether and non-Noether constants of motion) are thus calculated in a systematic way.

II. GENERAL POINT-TRANSFORMATION THEORY FOR ONE-DIMENSIONAL LAGRANGIAN SYSTEMS REVISITED

We begin this work recalling some basic notions of Lagrangian mechanics, which shall be needed in the sequel. Our discussion is rather sketchy and deals only with one-dimensional systems. Here we present (without proof) some useful features of the transformation theory of Lagrangian mechanics. For details, see Ref. 7.

First, let us remind the important class of *null Lagrangians*, which have the property that every conceivable worldline renders their action integrals stationary for all variations that vanish on the extremes. Therefore, null Lagrangians do not yield genuine equations of motion. It is well known[8] that a function $L_N(t,x,\dot{x})$ is a null Lagrangian if, and only if, it can be expressed as a total time derivative of a function $G(t,x)$. This concept gives rise to the notion of *g-equivalent* (i.e., gauge-equivalent) Lagrangians, which are those Lagrangians that are related by a gauge transformation of the form

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$$\widetilde{L}(t,x,\dot{x}) = KL(t,x,\dot{x}) + \dot{G}(t,x), \tag{2.1}$$

where K is an arbitrary constant and G is an arbitrary gauge function. It follows that $L(t,x,\dot{x})$ and $\widetilde{L}(t,x,\dot{x})$ are g-equivalent Lagrangians if, and only if, their variational derivatives are proportional functions. Clearly, the set $\mathbf{G} = \{(I,K,G)\}$ of all gauge transformations is a non-Abelian group, with the following combination rule $(I,K_1,G_1)(I,K_2,G_2) = (I,K_1K_2,K_1G_2 + G_1)$, where I denotes the identity and indicates that the variables (t,x) have not been transformed.

Since g-equivalent Lagrangians provide the same equations of motion, it follows that the Lagrangian function for a given system is only determined to within a g class of equivalent Lagrangians. In fact, it is well known that the Lagrangian function that describes a given mechanical system is, in general, not unique. In this sense, we have to recall that there is also another concept of "dynamically equivalent" Lagrangians, which is not as trivial as g-equivalence. This is the notion of *s-equivalent* (i.e., solution-equivalent) Lagrangians.[9] One says that two Lagrangians, $L(t,x,\dot{x})$ and $\overline{L}(t,x,\dot{x})$, are s-equivalent when the manifolds of all solutions to the Euler–Lagrange equations obtained from them coincide. Note that it is not necessary for the Euler–Lagrange equations obtained from both Lagrangians to be exactly the same, so that situations more interesting than a mere gauge transformation may arise.

The problem of finding the most general relation between s-equivalent Lagrangians was solved by Currie and Saletan.[10] It can be shown that $L(t,x,\dot{x})$ and $\overline{L}(t,x,\dot{x})$ are s-equivalent Lagrangians if, and only if,[11]

$$\frac{\delta\overline{L}}{\delta x} = \Lambda(t,x,\dot{x})\,\frac{\delta L}{\delta x}, \tag{2.2}$$

where $\Lambda(t,x,\dot{x}) = (\lambda^2\overline{L}/\lambda\dot{x}^2)/(\lambda^2 L/\lambda\dot{x}^2)$ is a constant of motion. Moreover, given L and any constant of motion $\Lambda(t,x,\dot{x})$, Currie and Saletan provide a way to construct $\overline{L}$ explicitly as a *functional* of L (viz., the so called *fouling transformation* of L). One sees, therefore, that the class of s-equivalent Lagrangians includes the class of g-equivalent Lagrangians as a particular case. One must also note that these transformations of Lagrangian mechanics are not committed with any transformation of space-time variables, as are those transformations we shall discuss presently.

Now, since the basic formalism of Lagrangian mechanics is invariant under a group of transformations that do also affect time (besides the spatial coordinates), in Lagrangian theory one has to consider the *configuration space-time* as the fundamental differentiable manifold of the system.[12] Hence, let us introduce a smooth transformation of variables $(t,x)\rightarrow(t',x')$; say

$$t' = T(t,x), \quad x' = S(t,x), \tag{2.3}$$

where the functions T and S are of continuity class C^μ (with $\mu > 2$) in some open connected region $\mathbf{R}\subset\{(t,x,)\}$, and globally invertible on $\mathbf{R}$. Let us also write

$$t = \overline{T}(t',x'), \quad x = \overline{S}(t',x') \tag{2.4}$$

to denote the corresponding inverse transformation of variables. Henceforth, all admissible point transformations are assumed to meet these conditions, and all our considerations will have a local character, since we shall always assume that $(t,x)\in\mathbf{R}$.

In this paper we interpret $\mathbf{R}$ as a coordinate patch; namely, Eqs. (2.3) and (2.4) will be thought of as a local transformation of coordinates in configuration space-time (rather than as an active mapping of configuration events). Accordingly, one can give the following definition: given a Lagrangian $L(t,x,\dot{x})$ and a local diffeomorphism D [as stated in Eqs. (2.3)], then every arbitrary gauge transformation (I,K,G), induces a new Lagrangian function $L'(t',x',\dot{x})$ that is given by

$$\dot{T}L'(t',x',\dot{x}') = KL(t,x,\dot{x}) + \dot{G}(t,x). \tag{2.5}$$

We refer to Eqs. (2.3) and (2.5) as a *Lagrangian transformation.*[7] In this fashion, one has that the set $\mathbf{L} = \{(D,K,G)\}$ of all Lagrangian transformations constitutes a group under the following law of combination:

$$(D_1,K_1,G_1)(D_2,K_2,G_2) = (D_1D_2,K_1K_2,K_1G_2 + G_1),$$

where D_1D_2 is the composite diffeomorphism, which one obtains in the usual way. This group is called the *Lagrange group.* It has the direct product structure $\mathbf{L} = \mathbf{D}\otimes\mathbf{G}$, where $\mathbf{D} = \{D\}$ is the group of all space-time diffeomorphisms and $\mathbf{G} = \{(I,K,G)\}$ is the non-Abelian group of all gauge transformations.

One then proves that, under a Lagrangian transformation (D,K,G), the variational derivative $\delta L/\delta x$ of L transforms according to the equation

$$\frac{\delta L'}{\delta x'} = KJ^{-1}\,\frac{\delta L}{\delta x}, \tag{2.6}$$

where $J = T_tS_x - T_xS_t \neq 0$ is the Jacobian of the space-time diffeomorphism D given in Eqs. (2.3). Hence, the general covariance of the formalism of Lagrange upon Lagrangian transformations follows.

The previous discussion gives rise to the concept of "curve-equivalent" Lagrangians. Namely, two given Lagrangians are said to be *c-equivalent* when there exists a *Lagrangian transformation* that transforms one Lagrangian function into the other. We note that the new coordinates (t',x') are moving relative to the old coordinates, in general, which can be tantamount to a substantial change in the dynamics, so that *c-equivalent Lagrangians are, in general, not s-equivalent.*

We finally revisit one of the main concepts of this subject. In particular, one says that a coordinate transformation (2.3) corresponds to a *point symmetry* of a given Lagrangian function $L(t,x,\dot{x})$ if, and only if, there exists an element $(I,\kappa,\sigma)\in\mathbf{G}$, such that $\dot{T}L(t',x',\dot{x}') = \kappa L(t,x,\dot{x}) + \dot{\sigma}(t,x)$. Note that, according to this definition, one has $L(t,x,\dot{x})\rightarrow L(t',x',\dot{x}')$, so that the *form* of the function L does not change, notwithstanding the fact that, eventually, one may have $\kappa\neq 1$. Let us also remark that the *symmetry gauge function* $\sigma(t,x)$ as well as the *symmetry scaling constant* κ are no longer arbitrary. Of course, the set $\mathbf{S}(L) = \{(D_S,\kappa,\sigma)\}$ (where D_S denotes the *symmetry* diffeomorphism) constitutes the *point symmetry group* of the given Lagrangian L, upon the same product law already introduced for $\mathbf{L}$. In most instances, this group $\mathbf{S}(L)$ is a finite Lie group, which acts locally in the configuration space-time through the symmetry diffeomorphisms D_S that characterize L. As a matter of fact, one has the isomorphism $\mathbf{S}(L)\sim\{D_S\}$, because for each D_S the corresponding gauge factor (I,κ,σ), is *unique*, and furthermore the one-to-one relation $(D_S,\kappa,\sigma)\leftrightarrow D_S$ preserves the group structure of $\{D_S\}$. Of course, $\mathbf{G}$ is not a subgroup of $\mathbf{S}(L)$, since $(I,\kappa,\sigma)\notin\mathbf{S}(L)$ [unless $\kappa=1$ and $\sigma=0$]. It is a simple exercise to show the following results: (1) All *g*-equivalent Lagrangians have essentially the same point symmetries; (2) if D_S is a symmetry diffeomorphism of L, and D is an arbitrary diffeomorphism, then the conjugate diffeomorphism $D'_S = DD_SD^{-1}$ yields a point symmetry transformation for *all* those *c*-equivalent Lagrangians $L'(t',x',\dot{x}')$, obtained from L by the action of D and an arbitrary gauge transformation; and (3) a necessary condition for the *c*-equivalence of two given Lagrangians is that their point symmetry groups be isomorphic. This means that the action of the Lagrange group $\mathbf{L}$ on the manifold $\{L\}$ of all Lagrangians is not transitive, for there are many examples of Lagrangians with nonisomorphic point symmetry groups. Of course, the action of $\mathbf{L}$ is transitive within each *c*-class of Lagrangians; and, furthermore, each *c*-class is characterized by a well-defined Lie group, which becomes locally realized as the group $\mathbf{S}(L)\sim\{D_S\}$, for any L that belongs to the given *c*-class. However, it must be also emphasized that the isomorphism $\mathbf{S}(L')\sim\mathbf{S}(L)$ is *not* a sufficient condition to ensure the *c*-equivalence of L' and L. From a practical point of view, these features are telling us that it is not possible to transform a Lagrangian function into another (by means of a Lagrangian transformation) if they have essentially different point symmetry properties; i.e., Lagrangian transformations are neither trivial nor spurious.[7] These results can be extended rather naturally to Lagrangian field theories. Their interest for the Lagrangian theory of *interactions*, in general, is immediate.

III. SL(3,*R*)-EQUIVALENT LAGRANGIANS FOR THE ONE-DIMENSIONAL FREE PARTICLE

Let us now consider the standard Lagrangian of a free Newtonian system with one degree of freedom, i.e.,

$$L_0 = \tfrac{1}{2}\dot{x}^2, \tag{3.1}$$

under the scope of the transformation theory of Lagrangian mechanics. As we know, the *maximal point symmetry group* of the Euler–Lagrange equation,

$$\frac{\delta L_0}{\delta x} \equiv -\ddot{x} = 0, \tag{3.2}$$

becomes realized by the projective diffeomorphisms of the (t,x) plane:

$$\begin{aligned} t' &= (q^1t + q^2x + q^3)/(q^7t + q^8x + 1), \\ x' &= (q^4t + q^5x + q^6)/(q^7t + q^8x + 1), \end{aligned} \tag{3.3}$$

where the q's are the eight parameters of the group, provided $q^1q^5\neq 0$. [This group is locally isomorphic with $SL(3,R)$. We must here recall that *all* one-dimensional linear Newtonian systems have the same point symmetry group, namely $SL(3,R)$.][1]

On the other hand, it can be shown that the *maximal point symmetry group* $\mathbf{S}(L_0)$ of the standard Lagrangian (3.1) is given by the following six-parameter subgroup of projective transformations:

$$\begin{aligned} t' &= (q^1t + q^3)/(q^7t + 1), \\ x' &= (q^4t + q^5x + q^6)/(q^7t + 1), \end{aligned} \tag{3.4}$$

with the associated symmetry gauge transformation (I,κ,σ) being given by

$$\begin{aligned} \kappa &= \frac{(q^5)^2}{q^1 - q^3q^7}, \\ \sigma(t,x) &= \frac{[(q^4 - q^6q^7) - q^5q^7x]^2}{2q^7(q^3q^7 - q^1)(q^7t+1)} + \sigma_0, \end{aligned} \tag{3.5}$$

where σ_0 is a constant. (One easily adjusts σ_0 so that σ be well defined at $q^7=0$.) Indeed, it is a straightforward matter to verify that these Lagrange transformations [i.e., Eqs. (3.4) and (3.5)] are such that

$$\frac{dt'}{dt}\left[\frac{1}{2}\left(\frac{dx'}{dt'}\right)^2\right] = \kappa\left[\frac{1}{2}\dot{x}^2\right] + \dot{\sigma}(t,x), \tag{3.6}$$

and that every Lagrangian transformation that satisfies this equation is of the form stated in Eqs. (3.4) and (3.5). [It can be shown also that $\mathbf{S}(L_0)$ is committed with the restriction $\partial t'/\partial x=0$, when imposed on the point symmetries of Eq. (3.2).]

Hence, the interesting question arises as to the c-equivalent Lagrangians that are induced by the elements of *the full point symmetry group of the equation of motion*, according to their projective realizations (3.3). In order to tackle this question we better avoid arbitrary gauge transformations at this stage, since these would only render the forms of the outcoming Lagrangians less intelligible. After some manipulations we get the following set of (eight-parameter-dependent) Lagrangians for the description of a one-dimensional free particle:

$$L_q=\frac{[(q^5-q^6q^8)\dot{x}+(q^4q^8-q^5q^7)(x-t\dot{x})+(q^4-q^6q^7)]^2}{2(q^7t+q^8x+1)^2[(q^2-q^3q^8)\dot{x}+(q^1q^8-q^2q^7)(x-t\dot{x})+(q^1-q^3q^7)]} \tag{3.7}$$

(up to an arbitrary gauge transformation). The reader can verify that the Euler–Lagrange equation deduced from L_q yields precisely the equation of motion $\ddot{x}=0$, as it must be. Thus in Eq. (3.7) we have a family of c-equivalent Lagrangians, which are also s-equivalent.

Since $L_q(t,x,\dot{x})$ is a formidable Lagrangian function, in order to study g-equivalence we better consider separately the monoparametric subgroups of the projective group. Hence from Eq. (3.7) we obtain the set of eight *monoparametric s-equivalent Lagrangians* presented in Table I. They all correspond to a (one-dimensional) free particle system. We see (cf. Table I) that the Lagrangians L_1, L_3, L_4, L_5, L_6, and L_7 are g-equivalent with L_0, so that they have the same point symmetries. On the other hand, it can be shown that L_2 and L_8 are not g-equivalent Lagrangians, neither are they g-equivalent with L_0; so these are essentially *new* Lagrangians. However, it is easy to obtain

$$\frac{\delta L_2}{\delta x}=\Lambda_2(\dot{x})\ddot{x} \tag{3.8}$$

and

$$\frac{\delta L_8}{\delta x}=\Lambda_8(t,x,\dot{x})\ddot{x}, \tag{3.9}$$

where $\Lambda_2=(1+q^2\dot{x})^{-3}$ and $\Lambda_8=[1+q^8(x-t\dot{x})]^{-3}$ are two *non-Noether constants of motion.*[13,14]

TABLE I. Set of monoparametric s-equivalent Lagrangians induced by SL$(3,R)$ for a one-dimensional free particle. L_0 denotes the standard Lagrangian $\frac{1}{2}\dot{x}^2$.

$$L_1=\dot{x}^2/2q^1=(q^1)^{-1}L_0,$$
$$L_2=\dot{x}^2/2(1+q^2\dot{x}),$$
$$L_3=\tfrac{1}{2}\dot{x}^2=L_0,$$
$$L_4=\tfrac{1}{2}(\dot{x}+q^4)^2=L_0+\frac{d}{dt}[q^4(x+\tfrac{1}{2}q^4t)],$$
$$L_5=\tfrac{1}{2}(q^5\dot{x})^2=(q^5)^2L_0,$$
$$L_6=\tfrac{1}{2}\dot{x}^2=L_0$$
$$L_7=\frac{1}{2}\left[\frac{\dot{x}-q^7(x-t\dot{x})}{1+q^7t}\right]=L_0-\frac{d}{dt}\left[\frac{q^7x^2}{2(1+q^7t)}\right],$$
$$L_8=\frac{\dot{x}^2}{2(1+q^8x)[1+q^8(x-t\dot{x})]}.$$

IV. SL(3,R)-EQUIVALENT LAGRANGIANS FOR ALL ONE-DIMENSIONAL LINEAR SYSTEMS

We are now ready to generalize these results to *any* given one-dimensional Newtonian linear system. It can be shown that the most general diffeomorphism that locally reduces any given one-dimensional linear system, $\ddot{x}+f_2(t)\dot{x}+f_1(t)x=f_0(t)$, into a free particle $\ddot{x}'=0$, is given by the eight-parameter transformations

$$t'=[q^1\hat{t}(t)+q^2\hat{x}(t,x)+q^3]/[q^7\hat{t}(t)+q^8\hat{x}(t,x)+1], \tag{4.1}$$

$$x'=[q^4\hat{t}(t)+q^5\hat{x}(t,x)+q^6]/[q^7\hat{t}(t)+q^8\hat{x}(t,x)+1], \tag{4.2}$$

where one defines the functions $\hat{t}'$ and $\hat{x}'$ by means of the well-known Arnold transformation;[15] i.e.,

$$\hat{t}(t)=u_1(t)/u_2(t),\quad \hat{x}(t,x):=[x-u_p(t)]/u_2(t). \tag{4.3}$$

Here $u_p(t)$ denotes a particular solution of the original inhomogeneous equation while $u_1(t)$ and $u_2(t)$ are two linearly independent solutions of the corresponding homogeneous equation.

Hence, these diffeomorphisms [Eqs. (4.1)–(4.3)] allow one to transform the standard Lagrangian $L_0=\frac{1}{2}\dot{x}^2$ into a family of c-equivalent eight-parameter Lagrangians $L_q(t,x,\dot{x})$, for any given one-dimensional linear system. To this end, one uses the following Lagrangian transformation [cf. Eq. (3.6)]:

$$\frac{dt'}{dt}\left[\frac{1}{2}\left(\frac{dx'}{dt'}\right)^2\right]=L_q(q,x,\dot{x}), \tag{4.4}$$

where, for the sake of simplicity, we have omitted the arbitrary gauge transformation. [In general, we could write $KL_q(t,x,\dot{x})+\dot{G}(t,x)$ in the right-hand member (RHM) of Eq. (4.4), indeed.] After a direct calculation, one gets

$$L_q=\frac{[(q^4-q^6q^7)\,\dot{\hat{t}}+(q^5-q^6q^8)\,\dot{\hat{x}}+(q^4q^8-q^5q^7)(\,\dot{\hat{t}}\,\hat{x}-\hat{t}\,\dot{\hat{x}}\,)]^2}{2(q^7\hat{t}+q^8\hat{x}+1)^2[(q^1-q^3q^7)\,\dot{\hat{t}}+(q^2-q^3q^8)\,\dot{\hat{x}}+(q^1q^8-q^2q^7)(\,\dot{\hat{t}}\,\hat{x}-\hat{t}\,\dot{\hat{x}}\,)]} \tag{4.5}$$

[also see Eq. (3.7)].

First, it is important to note that Eq. (4.5) yields

$$\frac{\delta L_q}{\delta x}=\frac{\dot{\hat{t}}\,\Delta^2(q)}{u_2[(q^1-q^3q^7)\,\dot{\hat{t}}+(q^2-q^3q^8)\,\dot{\hat{x}}+(q^1q^8-q^2q^7(\,\dot{\hat{t}}\,\hat{x}-\hat{t}\,\dot{\hat{x}}\,)]^3}(\,\dot{\hat{t}}\,\ddot{\hat{x}}-\ddot{\hat{t}}\,\dot{\hat{x}}), \tag{4.6}$$

where

$$\Delta(q):=\begin{vmatrix} q^1 & q^2 & q^3\\ q^4 & q^5 & q^6\\ q^7 & q^8 & 1\end{vmatrix}\neq 0, \tag{4.7}$$

so that the induced equation of motion is given by

$$\dot{\hat{t}}\,\ddot{\hat{x}}-\ddot{\hat{t}}\,\dot{\hat{x}}=0. \tag{4.8}$$

Taking into account Eqs. (4.3), it is easy to check that this equation of motion corresponds precisely to $\ddot{x}+f_2(t)\dot{x}+f_1(t)x=f_0(t)$. Hence all Lagrangians $L_q(t,x,\dot{x})$ of the form given in Eq. (4.5) are s equivalent.

Second, it is also interesting to mention here that we can obtain the standard Lagrangian, namely

$$\overline{L}(t,x,\dot{x})=\tfrac{1}{2}[\dot{x}^2-f_1(t)x^2+2f_0(t)x]\exp\left(\int f_2(t)dt\right), \tag{4.9}$$

of the general linear system by means of a suitable gauge transformation; namely,

$$\overline{L}(t,x,\dot{x})=K_eL_e(t,x,\dot{x})+\dot{G}_e(t,x), \tag{4.10}$$

where L_e denotes the Lagrangian

$$L_e(t,x,\dot{x})=\dot{\hat{x}}^2(t,x)/2\,\dot{\hat{t}}\,(t), \tag{4.11}$$

with $K_e=-1$, and the gauge function G_e being defined as

$$G_e=\left[\frac{1}{2}(x-u_p)^2\frac{\dot{u}_2}{u_2}+x\dot{u}_p\right]\exp\left(\int f_2\,dt\right)-\frac{1}{2}\int(\dot{u}_p^2-f_1u_p^2)\exp\left(\int f_2\,dt\right)dt. \tag{4.12}$$

The Lagrangian function L_e corresponds to the choice $q^1=q^5=1$ and $q^2=q^3=q^4=q^6=q^7=q^8=0$ in Eq. (4.5); and thus it is obtained from $L_0=\frac{1}{2}\dot{x}^2$ by means of the Arnold diffeomorphism $t\rightarrow\hat{t}(t)$, $x\rightarrow\hat{x}(t,x)$ [cf. Eq. (4.3)].

Next, we tackle the problem of g equivalence. Let us then consider the six-parameter diffeomorphisms corresponding to the restriction

$$\frac{\partial t'}{\partial x}=0\Leftrightarrow q^2=q^8=0 \tag{4.13}$$

in Eqs. (4.1) and (4.2). These diffeomorphisms induce the following subset of s-equivalent Lagrangians:

$$L_q^{(N)}=L_q|_{q^2=q^8=0}=\frac{[(q^4-q^6q^7)\,\dot{\hat{t}}+q^5\dot{\hat{x}}-q^5q^7(\,\dot{\hat{t}}\,\hat{x}-\hat{t}\,\dot{\hat{x}}\,)]^2}{2(q^1-q^3q^7)(1+q^7t)^2\,\dot{\hat{t}}}. \tag{4.14}$$

Of course, in particular, the g-equivalent Lagrangians $\overline{L}$ and L_e belong to this family. Moreover, by a direct calculation it can be shown that

$$L_q^{(N)}(t,x,\dot{x})=K_q'L_e(t,x,\dot{x})+\dot{G}_q'(t,x), \tag{4.15}$$

where

$$K'_q = (q^5)^2/(q^1 - q^3q^7) \tag{4.16}$$

and

$$G'_q(t,x) = \begin{cases} (q^4/q^1)(q^5 + \tfrac{1}{2}q^4\hat{t}), & \text{if } q^7 = 0, \\ -\dfrac{[(q^4 - q^6q^7) - q^5q^7\hat{x}]^2}{2q^7(q^1 - q^3q^7)(1 + q^7t)}, & \text{if } q^7 \neq 0. \end{cases} \tag{4.17}$$

So we see that $\{L_q^{(N)}\}$ constitutes a family of g-equivalent Lagrangians. Conversely, if any two given Lagrangians of the form stated in Eq. (4.5) are g-equivalent, one necessarily has $q^2 = q^8 = 0$. This means that the set $\{L_q^{(N)}\}$ is the *maximal class of g-equivalent Lagrangians* for the one-dimensional Newtonian linear system.

We like to mention here an important fact concerning this issue. The *maximal point symmetry group* of the standard Lagrangian given in Eq. (4.9), as obtained by means of the Noether criterion of point symmetry of a given Lagrangian is given precisely by the *six-parameter* group of transformations one obtains from Eqs. (4.1) and (4.2) when one sets $q^2 = q^8 = 0$ into those equations. Hence the case $q^2 = q^8 = 0$ defines the *Noether point symmetry group* of the standard Lagrangian (4.9). We shall further discuss this matter elsewhere.

From the previous results we conclude that the necessary and sufficient condition for two given Lagrangians of the form (4.5) to be *not* g-equivalent is that one of them, at least, corresponds to $q^2 \neq 0$ or $q^8 \neq 0$. Thus, if we define the Lagrangians

$$L_q^{(*)}(t,x,\dot{x}) = L_q(t,x,\dot{x})\,|_{q^2\neq 0 \vee q^8 \neq 0}, \tag{4.18}$$

we have the following classification:

$$\{L_q\} = \{L_q^{(N)}\} \cup \{L_q^{(*)}\}, \quad \{L_q^{(N)}\} \cap \{L_q^{(*)}\} = \phi. \tag{4.19}$$

In this way, the point symmetry group SL(3,R) of the equation of motion induces a new class of Lagrangians, which are not g-equivalent to $L_q^{(N)}$. This class is generated by two basic *non-Noether Lagrangians*, i.e.,

$$L_2 = \dot{\hat{x}}^2/2(\dot{\hat{t}} + q^2\dot{\hat{x}}) \tag{4.20}$$

and

$$L_8 = \frac{\dot{\hat{x}}^2}{2(1 + q^8\hat{x})^2[\dot{\hat{t}} + q^8(\dot{\hat{t}}\hat{x} - \hat{t}\dot{\hat{x}})]}. \tag{4.21}$$

Finally, in view of the s-equivalence, and taking into account Eq. (2.2), we search for those constants of motion $\Lambda_q(t,x,\hat{x})$, such that

$$\frac{\delta L_q}{\delta x} = \Lambda_q(t,x,\dot{x})\,\frac{\delta L_e}{\delta x}. \tag{4.22}$$

It is easy to conclude that these are given by

$$\Lambda_q(t,x,\dot{x}) = \Delta^2(q)\,\dot{\hat{t}}^3[(q^1 - q^3q^7)\,\dot{\hat{t}} + (q^2 - q^3q^8)\,\dot{\hat{x}} + (q^1q^8 - q^2q^7)(\dot{\hat{t}}\hat{x} - \hat{t}\dot{\hat{x}})]^{-3}. \tag{4.23}$$

This equation shows neatly that the only nontrivial constants of motion are precisely the *non-Noether constants*:

$$\Lambda^2(t,x,\dot{x}) = [1 + q^2(\dot{\hat{x}}/\dot{\hat{t}}]^{-3} \tag{4.24}$$

and

$$\Lambda_8(t,x,\dot{x}) = [1 + q^8(\hat{x} - \hat{t}\dot{\hat{x}}/\dot{\hat{t}})]^{-3}. \tag{4.25}$$

For instance, for a forced harmonic oscillator

$$\ddot{x} + \omega^2 x = f_0 \sin \Omega t, \tag{4.26}$$

one obtains the following non-Noether Lagrangians:

$$L_2 = \frac{[\dot{x} + \omega x \tan \omega t + [f_0/(\Omega^2 - \omega^2)](\Omega \cos \Omega t + \omega \sin \Omega t \tan \omega t)]^2}{2\omega[1 + q^2\{x \sin \omega t + (\dot{x}/\omega)\cos \omega t + [f_0/(\Omega^2 - \omega^2)](\sin \Omega t \sin \omega t + (\Omega/\omega)\cos \Omega t \cos \omega t)\}]}, \tag{4.27}$$

$$L_8 = \frac{[\dot{x} + \omega x \tan \omega t + [f_0/(\Omega^2 - \omega^2)](\Omega \cos \Omega t + \omega \sin \Omega t \tan \omega t)]^2}{2\omega[1 + q^8(x + [f_0/(\Omega^2 - \omega^2)]\sin \Omega t)\sec \omega t]^2[1 + q^8(x \cos \omega t - (1/\omega)\dot{x} \sin \omega t) + [f_0/(\Omega^2 - \omega^2)](\sin \Omega t \cos \omega t - (\Omega/\omega)\cos \Omega t \sin \omega t)]}; \tag{4.28}$$

and the corresponding non-Noether constants of motion are obtained from

$$k_2 = x \sin \omega t + (1/\omega)\dot{x} \cos \omega t + f_0/(\Omega^2 - \omega^2) \times (\sin \Omega t \sin \omega t + (\Omega/\omega)\cos \Omega t \cos \omega t) \tag{4.29}$$

and

$$k_8 = x \cos \omega t - (1/\omega)\dot{x} \sin \omega t + [f_0/(\Omega^2 - \omega^2)] \times (\sin \Omega t \cos \omega t - (\Omega/\omega)\cos \Omega t \sin \omega t), \tag{4.30}$$

TABLE II. L_2 Lagrangians for some one-dimensional linear systems and their associated non-Noether constants of motion $\Lambda_2 = (1+q^2k_2)^{-3}$.

Linear System	Lagrangian L_2	Constant of Motion k_2
$\ddot{x} = -g$	$(\dot{x}+gt)^2/2[1+q^2(\dot{x}+gt)]$	$\dot{x}+gt$
$\ddot{x}+\lambda\dot{x} = -g$	$\dfrac{(\dot{x}+g/\lambda)^2}{2[-\lambda e^{-\lambda t}+q^2(\dot{x}+g/\lambda)]}$	$-(1/\lambda)(\dot{x}+g/\lambda)e^{\lambda t}$
$\ddot{x}+\omega^2 x = 0$	$\dfrac{(\dot{x}+\omega x\tan\omega t)^2}{2\omega[1+q^2(x\sin\omega t+(1/\omega)\dot{x}\cos\omega t)]}$	$x\sin\omega t+(1/\omega)\dot{x}\cos\omega t$
$\ddot{x}+2\lambda\dot{x}+\omega^2x=0$, $\Omega=(\omega^2-\lambda^2)^{1/2}$	$\dfrac{[\Omega x\tan\Omega t+(\dot{x}+\lambda x)]^2}{2\Omega[1+q^2\{x\sin\Omega t+(1/\Omega)(\dot{x}+\lambda x)\cos\Omega t\}e^{\lambda t}]}e^{2\gamma t}$	$[x\sin\Omega t+(1/\Omega)(\dot{x}+\lambda x)\cos\Omega t]e^{\lambda t}$
$\ddot{x}+t^{-1}\dot{x}-t^{-2}x=0$	$(t\dot{x}+x)^2/2[2t+q^2(t\dot{x}+x)]$	$\frac{1}{2}(\dot{x}+t^{-1}x)$

respectively, with $\Lambda_2 = (1+q^2k_2)^{-3}$ and $\Lambda_8 = (1+q^8k_8)^{-3}$.

In Tables II and III we present the non-Noether Lagrangians L_2 and L_8, and their corresponding constants of motion, for some miscellaneous linear systems.

CONCLUDING REMARKS

In this paper, the *maximal point symmetry group of the equations of motion* for a given class of mechanical systems (namely, one-dimensional linear Newtonian systems) has been used for obtaining systematically a family of *equivalent Lagrangians* for the description of these systems. The interest of this group-theoretic approach to the "inverse problem of the calculus of variations" stems from its systematic generality. In fact, once the maximal point symmetry group of a system is known to be an r-dimensional Lie group (which is the case, indeed, for all systems considered in ordinary Lagrangian mechanics) the method of Lagrange transformations introduced in this paper can be applied to obtain a set of r-equivalent (Noether and non-Noether) monoparametric Lagrangian functions for the given system. It is clear that by applying this same approach (i.e., by means of the point symmetries of the Euler–Lagrange equations and the use of Lagrange transformations) one can generalize our results to either nonlinear or multidimensional Lagrangian systems.

Of course, the nonlinear case can be faintly distressing since, as it is well known, differential equations of order higher than unity only exceptionally admit continuous groups of point symmetry transformations, and nonlinear differential equations, in general, exhibit some peculiar subsidiary constraints between symmetry and nonlinearity, which may produce a strong reduction in the number of essential parameters of the group.[16] (Indeed, there are instances such that the only point symmetry admitted by a nonlinear differential equation is the identity.) Plainly, this behavior reduces the number of admissible monoparametric equivalent Lagrangian functions for such systems.

Nevertheless, the interest of our results for the program of *quantization* in general (whether canonical, geo-

TABLE III. L_8 Lagrangians for some one-dimensional linear systems and their associated non-Noether constants of motion $\Lambda_8 = (1+q^8k_8)^3$.

Linear System	Lagrangian L_8	Constant of Motion k_8
$\ddot{x} = -g$	$\dfrac{(\dot{x}+gt)^2}{2[1+q^8(x+\frac{1}{2}gt^2)]^2[1+q^8(x-t\dot{x}-\frac{1}{2}gt^2)]}$	$x-t\dot{x}-\frac{1}{2}gt^2$
$\ddot{x}+\lambda\dot{x} = -g$	$\dfrac{(\dot{x}+g/\lambda)^2e^{\lambda t}}{-2\lambda[1+q^8(x+(g/\lambda)t)]^2[1+(q^8/\lambda)(\dot{x}+\lambda x+gt+g/\lambda)]}$	$(1/\lambda)(\dot{x}+\lambda x+gt+g/\lambda)$
$\ddot{x}+\omega^2x=0$	$\dfrac{(\dot{x}+\omega x\tan\omega t)^2}{2\omega[1+q^8x\sec\omega t]^2[1+q^8(x\cos\omega t-(1/\omega)\dot{x}\sin\omega t)]}$	$x\cos\omega t-(1/\omega)\dot{x}\sin\omega t$
$\ddot{x}+2\lambda\dot{x}+\omega^2x=0$, $\Omega=(\omega^2-\lambda^2)^{1/2}$	$\dfrac{[\Omega x\tan\Omega t+\dot{x}+\lambda x)]^2e^{2\lambda t}}{2\Omega[1+q^8xe^{\lambda t}\sec\Omega t]^2[1+q^8\{x\cos\Omega t-(1/\Omega)(\dot{x}+\lambda x)\sin\Omega t\}e^{\lambda t}]}$	$[x\cos\Omega t-(1/\Omega)(\dot{x}+\lambda x)\sin\Omega t]e^{\lambda t}$
$\ddot{x}+t^{-1}\dot{x}-t^{-2}x=0$	$\dfrac{(t\dot{x}+x)^2}{4t(1+q^8tx)^2[1+\frac{1}{2}q^8t(x-t\dot{x})]}$	$\frac{1}{2}t(x-t\dot{x})$

metric, kinematic, or otherwise) should be stressed, because every point transformation yields a canonical transformation, and to every Lagrangian there corresponds a particular Hamiltonian. This subject will be studied elsewhere.

ACKNOWLEDGMENTS

This work was supported in part by FONDECYT Grant No. 0629/91.

[1] It is well known indeed that the full point symmetry group of *all* such equations corresponds to SL(3,R). This fact figures in the current literature, [cf. L. V. Ovsiannikov, *Group Analysis of Differential Equations* (Academic, New York, 1982)], and moreover, it was well known to Lie himself [S. Lie; Vorlesungen über Differential-Gleichungen Mit Bekannten Infinitesimal Transformationen (B. G. Teubner, Leipzig, 1891; reprinted by Chelsea, New York, 1967); however, it seems to be not so well known to most physicists. For instance, the rediscovery 14 years ago [i.e., C. E. Wulfman and B. G. Wyborne, J. Phys. A: Math. Gen. **9**, 507 (1976)] that the Newtonian equation for the simple harmonic oscillator has point symmetry SL(3,R) was a surprise to physicists. Later on it was found that SL(3,R) is also the complete symmetry group of the one-dimensional *time-dependent* harmonic oscillator; cf., P. G. L. Leach, J. Math. Phys. **21**, 300 (1980). Furthermore, it has been shown that the full symmetry group of the time-dependent n-dimensional harmonic oscillator is SL($n+2$,R); cf., G. E. Prince and C. J. Eliezer, J. Phys. A: Math. Gen. **13**, 815 (1980). Also see M. Aguirre and K. Krause, J. Phys. A: Math. Gen. **20**, 3553 (1987), concerning the *finite* point realizations of SL(3,R) for the simple harmonic oscillator. By the way, the present paper belongs to a line of research that was initiated by us some few years ago; cf. M. Aguirre and J. Krause, J. Math. Phys. **25**, 210 (1984); and J. Math. Phys. **26**, 593 (1985).

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[6] Concerning quantum kinematics, see J. Krause, J. Phys. A: Math. Gen. **18**, 1309 (1985); J. Math. Phys. **27**, 2922 (1986); *ibid.* **29**, 393 (1988); *ibid.* **32**, 348 (1991).

[7] Also see M. Aguirre and J. Krause, Int. J. Theor. Phys. **30**, 495, 1461 (1991), and references quoted therein.

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[10] D. G. Currie and E. J. Saletan, J. Math. Phys. **7**, 967 (1966). This gives rise to the *inverse problem of the calculus of variations*, which consists in trying to find all Lagrangians that yield Euler–Lagrange equations that are equivalent to a given system of equations of motion. This problem was first solved for $n=1$ by Darboux. The case $n=2$ was treated much later by Douglas. A large amount of noteworthy work devoted to this fundamental problem has been published lately; see, for instance, Sarlet. For a recent approach, see Cariñena and Martínez (1989) and references therein. See G. Darboux, *Lecons Sur la Theorie Generale des Surfaces* (Gauthier-Villars, Paris, 1891); J. Douglas, Trans. Am. Math. Soc. **50**, 71 (1941); W. Sarlet, J. Phys. A: Math. Gen. **15**, 1503 (1982); and J. F. Cariñena, and E. Martínez, J. Phys. A: Math. Gen. **22**, 2659 (1989).

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[16] See, i.e., M. Aguirre and J. Krause, in Ref. 1.