Eigenvalue Ratios for Sturm-Liouville Operators

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Abstract

In this paper we prove various optimal bounds for eigenvalue ratios for the Sturm-Liouville equation $-[p(x)\,y']' + q(x)y = \lambda w(x)y$ and certain specializations. Our results primarily concern the regular case with Dirichlet boundary conditions though various extensions and generalizations to other situations are possible. Our results here extend the result $\lambda_m/\lambda_1 \le m^2$ obtained in a previous paper for the one-dimensional Schrödinger equation, $-y'' + q(x)y = \lambda y$, on a finite interval with Dirichlet boundary conditions and nonnegative potential ($q \ge 0$). In particular, we obtain $\lambda_m/\lambda_1 \le Km^2/k$, where the constants k, K satisfy $0 < k \le p(x)\,w(x) \le K$ for all x. If $q \equiv 0$, lower bounds can also be obtained. Our methods involve a slight modification of the Prüfer variable techniques employed in the Schrödinger case. We also examine the consequences of our recent proof of the Payne-Pólya-Weinberger conjecture in the one-dimensional (Sturm-Liouville) setting. Finally, we compare our general bounds to the detailed analyses of Keller and of Mahar and Willner for the special case of the inhomogeneous stretched string.