The hp-BEM with quasi-uniform meshes for the electric field integral equation on polyhedral surfaces: A priori error analysis

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The hp-BEM with quasi-uniform meshes for the electric field integral equation on polyhedral surfaces: a priori error analysis *

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Abstract

This paper presents an a priori error analysis of the hp-version of the boundary element method for the electric field integral equation on a piecewise plane (open or closed) Lipschitz surface. We use $\mathbf{H}(\mathrm{div})$-conforming discretisations with Raviart-Thomas elements on a sequence of quasi-uniform meshes of triangles and/or parallelograms. Assuming the regularity of the solution to the electric field integral equation in terms of Sobolev spaces of tangential vector fields, we prove an a priori error estimate of the method in the energy norm. This estimate proves the expected rate of convergence with respect to the mesh parameter h and the polynomial degree p.

Key words: hp-version with quasi-uniform meshes, boundary element method, electric field integral equation, time-harmonic electro-magnetic scattering, a priori error estimate
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1 Introduction

With this paper we continue the analysis of high-order boundary element methods (BEM) for the electric field integral equation (EFIE) started in [4, 6]. Our BEM is based on discretisations of the variational formulation of the EFIE (called Rumsey's principle) with an $\mathbf{H}(\mathrm{div})$-conforming family of boundary elements. This approach is referred to as the natural boundary element method for the EFIE. In [4] we analysed the natural p-BEM for the EFIE on a plane open surface with polygonal boundary. We proved convergence of the p-version with Raviart-Thomas (RT) parallelogram elements and derived an a priori error estimate which takes into account the strong singular behaviour of the solution at edges and corners of the surface. In our previous paper [6] we considered the EFIE on a piecewise plane (open or closed) Lipschitz surface Γ and proved quasi-optimal convergence of the natural hp-BEM with quasi-uniform meshes of triangles and quadrilaterals. In the present note we perform an a priori error analysis of that method on

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affine meshes under the assumption that the regularity of the exact solution is given in Sobolev spaces of tangential vector fields on Γ. As the main result we prove an a priori error estimate in the energy norm (Theorem 2.2). The estimate appears to be optimal with respect to the mesh size h and the polynomial degree p as the convergence rates in both h and p^{-1} are $r + 1/2$ for p large enough. This corresponds to the expected rate which is the Sobolev regularity order r of the exact solution minus the Sobolev order $-1/2$ of the energy norm.

While in the h-version the degrees of approximating polynomials are fixed (usually at a low level) and convergence is achieved by refining the mesh, the p-version keeps the mesh fixed and improves approximations by increasing polynomial degrees. The hp-version combines both mesh refinement and increase of polynomial degrees. For boundary integral equations governing the Laplace equation optimal hp-convergence rates for singular problems (and quasi-uniform meshes) are proved in [5, 3]. The analysis of optimal hp-BEM convergence rates for the EFIE with singular solutions is an open problem and under investigation. In this paper we deal with the case of solutions with Sobolev regularity. We also note that the hp-BEM with geometrically graded meshes (yielding an exponential rate of convergence) has been studied in [23], again for the Laplace equation and for hypersingular and weakly singular integral operators. For the EFIE its analysis is an open problem.

An a priori error analysis of the natural h-BEM for the EFIE was performed in [25] for polyhedral surfaces and in [11] for open Lipschitz surfaces (see also [15] for a survey of results and techniques). In particular, an optimal h-convergence rate of the method for given Sobolev regularity of the solution and given polynomial degree has been proved in [11] (being proved on open surfaces, this result extends to polyhedral surfaces as well). If the BEM for the EFIE converges quasi-optimally, then a priori error analysis reduces to an approximation problem within the energy space, which is either $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ or $\tilde{\mathbf{H}}_0^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ depending on whether the surface Γ is closed or open. The main tool in the approximation analysis is an appropriate $\mathbf{H}(\mathrm{div})$-conforming interpolation operator. Whereas the standard $\mathbf{H}(\mathrm{div})$-conforming RT interpolation operator works well for the h-version (cf. [11]), it does not provide an optimal result for high order methods. The main reason for this is the lack of stability (with respect to polynomial degrees) of this operator for low-regular vector fields, which always appear when dealing with the EFIE on non-smooth surfaces. Furthermore, existing techniques to prove p-estimates for the error of RT interpolation work only on quadrilateral elements, and their extension to triangular elements does not seem feasible.

An alternative to the classical RT interpolation operator is a corresponding projection based interpolation operator. Such operators were first introduced in [19] to analyse high-order finite element approximations with $\mathbf{H}(\mathrm{curl})$-conforming edge elements for Maxwell's equations in two dimensions. However, applying a simple rotation argument the results of [19] can be formulated in the $\mathbf{H}(\mathrm{div})$-conforming setting, which is intrinsic to natural boundary element discretisations of the EFIE (see also [7]). The projection based interpolation operators are stable with respect to polynomial degrees, they work equally well on both triangular and quadrilateral elements and also for low-regular fields. That is why these operators have become an efficient tool in the analysis of high-order methods (see [9, 8, 24] for the finite element methods and [4, 6] for the

BEM). The a priori error analysis in this paper relies on such an operator as well. In particular, we demonstrate that employing the $\mathbf{H}(\mathrm{div})$-conforming projection based interpolation operator (rather than the classical RT interpolation operator) one obtains an optimal error estimate for the *hp*-BEM with quasi-uniform meshes.

The paper is organised as follows. In the next section we formulate the EFIE (in a variational form) and define the *hp*-version of the BEM with quasi-uniform meshes. We also formulate the main result (Theorem 2.2), which states an a priori error estimate of the approximation method. Section 3 gives necessary preliminaries: first, in §3.1 we introduce the needed notation and recall definitions of Sobolev spaces of tangential vector fields; then, in §3.2 we sketch the definition of the $\mathbf{H}(\mathrm{div})$-conforming projection based interpolation operator on the reference element and prove a new property of this operator related to approximations of normal traces on the element's edges (Lemma 3.3). In Section 4 we study approximating properties of the discrete (boundary element) space $\mathbf{X}_{hp}$ in the energy space $\mathbf{X}$ of the EFIE, and prove that the orthogonal projection onto $\mathbf{X}_{hp}$ with respect to the norm in $\mathbf{X}$ satisfies an optimal error estimate in both h and p.

Throughout the paper, C denotes a generic positive constant which is independent of h, p and involved functions.

2 Formulation of the problem and the main result

We consider the EFIE for which we have proved quasi-optimal convergence of the *hp*-BEM with quasi-uniform meshes [6]. In this paper we provide an a priori error estimate. To this end let us recall the model problem, its *hp*-discretisation and the involved spaces.

Let Γ denote a piecewise plane (open or closed) Lipschitz surface in $\mathbb{R}^3$. In the case of an open surface we additionally assume that Γ is orientable. Let us introduce Rumsey's formulation of the electric field integral equation on Γ. For a given wave number $k > 0$ and a scalar function v (resp., tangential vector field $\mathbf{v}$) we define the single layer operator Ψ_k (resp., $\boldsymbol{\Psi}_k$) by

$$\Psi_k v(x) = \frac{1}{4\pi} \int_\Gamma v(y) \frac{e^{ik|x-y|}}{|x-y|}\, dS_y, \qquad x \in \mathbb{R}^3 \backslash \Gamma$$

$$\Big(\text{resp.,} \quad \boldsymbol{\Psi}_k \mathbf{v}(x) = \frac{1}{4\pi} \int_\Gamma \mathbf{v}(y) \frac{e^{ik|x-y|}}{|x-y|}\, dS_y, \qquad x \in \mathbb{R}^3 \backslash \Gamma\Big).$$

Let $\mathbf{L}^2_t(\Gamma)$ be the space of two-dimensional, tangential, square integrable vector fields on Γ. By ∇_Γ (resp., div_Γ) we denote the surface gradient (resp., surface divergence) acting on scalar functions (resp., tangential vector fields) on Γ. We will need the following space:

$$\mathbf{X} = \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) := \{\mathbf{u} \in \mathbf{H}^{-1/2}_{\|}(\Gamma);\ \mathrm{div}_\Gamma\, \mathbf{u} \in H^{-1/2}(\Gamma)\}$$

if Γ is a closed surface, and

$$\begin{aligned} \mathbf{X} = \tilde{\mathbf{H}}^{-1/2}_0(\mathrm{div}_\Gamma, \Gamma) \quad := \quad & \{\mathbf{u} \in \tilde{\mathbf{H}}^{-1/2}_{\|}(\Gamma);\ \mathrm{div}_\Gamma\, \mathbf{u} \in \tilde{H}^{-1/2}(\Gamma) \text{ and} \\ & \langle \mathbf{u}, \nabla_\Gamma v\rangle + \langle \mathrm{div}_\Gamma\, \mathbf{u}, v\rangle = 0 \quad \text{for all } \ v \in C^\infty(\Gamma)\} \end{aligned}$$

if Γ is an open surface. In the latter definition the brackets $\langle \cdot, \cdot \rangle$ denote dualities associated with $\mathbf{H}_{\|}^{1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, respectively. For definitions of the space $C^\infty(\Gamma)$ and the Sobolev spaces on Γ we refer to §3.1 below. Throughout, we use boldface symbols for vector fields. The spaces (or sets) of vector fields are also denoted in boldface (e.g., $\mathbf{H}^s(\Gamma) = (H^s(\Gamma))^3$).

Let $\mathbf{X}'$ be the dual space of $\mathbf{X}$ (with $\mathbf{L}_t^2(\Gamma)$ as pivot space). Now, for a given tangential vector field $\mathbf{f} \in \mathbf{X}'$ ($\mathbf{f}$ represents the excitation by an incident wave), Rumsey's formulation reads as: *find a complex tangential field* $\mathbf{u} \in \mathbf{X}$ *such that*

$$a(\mathbf{u}, \mathbf{v}) := \langle \gamma_{\mathrm{tr}}(\Psi_k \mathrm{div}_\Gamma\, \mathbf{u}), \mathrm{div}_\Gamma\, \mathbf{v} \rangle - k^2 \langle \pi_\tau(\mathbf{\Psi}_k \mathbf{u}), \mathbf{v} \rangle = \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{X}. \tag{2.1}$$

Here γ_{tr} is the standard trace operator, and π_τ denotes the tangential components trace mapping (see §3.1 for the definition). To ensure the uniqueness of the solution to (2.1) in the case of the closed surface Γ we always assume that k^2 is not an electrical eigenvalue of the interior problem.

For the approximate solution of (2.1) we apply the *hp*-version of the BEM based on Galerkin discretisations with Raviart-Thomas spaces on quasi-uniform meshes. In what follows, $h > 0$ and $p \ge 1$ will always specify the mesh parameter and a polynomial degree, respectively. For any $\Omega \subset \mathbb{R}^n$ we will denote $\rho_\Omega = \sup\{\mathrm{diam}(B);\ B \text{ is a ball in } \Omega\}$. Furthermore, throughout the paper, K is either the equilateral reference triangle $T = \{x_2 > 0,\ x_2 < x_1\sqrt{3},\ x_2 < (1 - x_1)\sqrt{3}\}$ or the reference square $Q = (0, 1)^2$. A generic side of K will be denoted by ℓ.

Let $\mathcal{T} = \{\Delta_h\}$ be a family of meshes $\Delta_h = \{\Gamma_j;\ j = 1, \ldots, J\}$ on Γ, where the elements Γ_j are open triangles or parallelograms such that $\bar{\Gamma} = \cup_{j=1}^J \bar{\Gamma}_j$, and the intersection of any two elements $\bar{\Gamma}_j$, $\bar{\Gamma}_k$ $(j \ne k)$ is either a common vertex, an entire side, or empty.

We denote $h_j = \mathrm{diam}(\Gamma_j)$ for any $\Gamma_j \in \Delta_h$. The elements are assumed to be shape regular, i.e., there exists a positive constant C independent of $h = \max_j h_j$ such that for any $\Gamma_j \in \Delta_h$ and arbitrary $\Delta_h \in \mathcal{T}$ there holds $h_j \le C\, \rho_{\Gamma_j}$. Furthermore, any element Γ_j is the image of the corresponding reference element K under an affine mapping T_j, more precisely

$$\bar{\Gamma}_j = T_j(\bar{K}), \quad \mathbf{x} = T_j(\boldsymbol{\xi}),\ \mathbf{x} = (x_1, x_2) \in \bar{\Gamma}_j,\ \boldsymbol{\xi} = (\xi_1, \xi_2) \in \bar{K}.$$

The Jacobian matrix of T_j is denoted by DT_j and its determinant $J_j := \det(DT_j)$ satisfies the relation $|J_j| \simeq h_j^2$.

We consider a family $\mathcal{T}$ of quasi-uniform meshes Δ_h on Γ in the sense that there exists a positive constant C independent of h such that for any $\Gamma_j \in \Delta_h$ and arbitrary $\Delta_h \in \mathcal{T}$ there holds $h \le C\, h_j$.

The mapping T_j introduced above is used to associate the scalar function u defined on the real element Γ_j with the function $\hat{u}$ defined on the reference element K:

$$u = \hat{u} \circ T_j^{-1} \ \text{ on } \Gamma_j \qquad \text{and} \qquad \hat{u} = u \circ T_j \ \text{ on } K.$$

Any vector-valued function $\hat{\mathbf{v}}$ defined on K is transformed to the function $\mathbf{v}$ on Γ_j by using the Piola transformation:

$$\mathbf{v} = \mathcal{M}_j(\hat{\mathbf{v}}) = \tfrac{1}{J_j} DT_j \hat{\mathbf{v}} \circ T_j^{-1}, \quad \hat{\mathbf{v}} = \mathcal{M}_j^{-1}(\mathbf{v}) = J_j DT_j^{-1} \mathbf{v} \circ T_j. \tag{2.2}$$

Let us introduce the needed polynomial sets. By $\mathcal{P}_p(I)$ we denote the set of polynomials of degree $\leq p$ on an interval $I \subset \mathbb{R}$, and $\mathcal{P}_p^0(I)$ denotes the subset of $\mathcal{P}_p(I)$ which consists of polynomials vanishing at the end points of I. In particular, these two sets will be used for an edge $\ell \subset \partial K$.

Further, $\mathcal{P}_p^1(T)$ denotes the set of polynomials on T of total degree $\leq p$, and $\mathcal{P}^2_{p_1,p_2}(Q)$ is the set of polynomials on Q of degree $\leq p_1$ in ξ_1 and degree $\leq p_2$ in ξ_2. For $p_1 = p_2 = p$ we denote $\mathcal{P}_p^2(Q) = \mathcal{P}^2_{p,p}(Q)$, and we will use the unified notation $\mathcal{P}_p(K)$, which refers to $\mathcal{P}_p^1(T)$ if $K = T$ and to $\mathcal{P}_p^2(Q)$ if $K = Q$. The corresponding set of polynomial (scalar) bubble functions on K is denoted by $\mathcal{P}_p^0(K)$.

Let us denote by $\boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K)$ the RT-space of order $p \geq 1$ on the reference element K (see, e.g., [10, 28]), i.e.,

$$\boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K) = (\mathcal{P}_{p-1}(K))^2 \oplus \boldsymbol{\xi}\mathcal{P}_{p-1}(K) = \begin{cases} (\mathcal{P}^1_{p-1}(T))^2 \oplus \boldsymbol{\xi}\mathcal{P}^1_{p-1}(T) & \text{if } \ K = T, \\ \mathcal{P}^2_{p,p-1}(Q) \times \mathcal{P}^2_{p-1,p}(Q) & \text{if } \ K = Q. \end{cases}$$

The subset of $\boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K)$ which consists of vector-valued polynomials with vanishing normal trace on the boundary ∂K (vector bubble-functions) will be denoted by $\boldsymbol{\mathcal{P}}_p^{\mathrm{RT},0}(K)$.

Then using transformations (2.2), we set

$$\mathbf{X}_{hp} := \{\mathbf{v} \in \mathbf{X}^0;\ \mathcal{M}_j^{-1}(\mathbf{v}|_{\Gamma_j}) \in \boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K),\ j = 1, \ldots, J\}, \tag{2.3}$$

where the space $\mathbf{X}^0 \subset \mathbf{X}$ is defined in §3.1 ($\mathbf{X}^0 = \mathbf{H}(\mathrm{div}_\Gamma, \Gamma)$ if Γ is closed and $\mathbf{X}^0 = \mathbf{H}_0(\mathrm{div}_\Gamma, \Gamma)$ if Γ is an open surface). We will denote by $N = N(h,p)$ the dimension of the discrete space $\mathbf{X}_{hp}$. One has $N \simeq h^{-2}$ for fixed p and $N \simeq p^2$ for fixed h.

The hp-version of the Galerkin BEM for the EFIE reads as: *Find* $\mathbf{u}_{hp} \in \mathbf{X}_{hp}$ *such that*

$$a(\mathbf{u}_{hp}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{X}_{hp}. \tag{2.4}$$

First, let us formulate the result which states the unique solvability of (2.4) and quasi-optimal convergence of the hp-version of the BEM for the EFIE.

Theorem 2.1 [6, Theorem 2.1] *There exists $N_0 \geq 1$ such that for any $\mathbf{f} \in \mathbf{X}'$ and for arbitrary mesh-degree combination satisfying $N(h,p) \geq N_0$ the discrete problem* (2.4) *is uniquely solvable and the hp-version of the Galerkin BEM generated by RT-elements converges quasi-optimally, i.e.,*

$$\|\mathbf{u} - \mathbf{u}_{hp}\|_{\mathbf{X}} \leq C \inf\{\|\mathbf{u} - \mathbf{v}\|_{\mathbf{X}};\ \mathbf{v} \in \mathbf{X}_{hp}\}. \tag{2.5}$$

Here, $\mathbf{u} \in \mathbf{X}$ is the solution of (2.1), *$\mathbf{u}_{hp} \in \mathbf{X}_{hp}$ is the solution of* (2.4), *$\|\cdot\|_{\mathbf{X}}$ denotes the norm in $\mathbf{X}$, and $C > 0$ is a constant independent of h and p.*

The following theorem is the main result of this paper (its formulation involves the space $\mathbf{X}^r$ and the norm $\|\cdot\|_{\mathbf{X}^r}$ which are defined in §3.1).

Theorem 2.2 *Let* $\mathbf{u} \in \mathbf{X}$ *and* $\mathbf{u}_{hp} \in \mathbf{X}_{hp}$ *be the solutions of* (2.1) *and* (2.4)*, respectively. Then there exists a real number* r *(*$r > 0$ *if* Γ *is closed, and* $r \in (-\frac{1}{2}, 0)$ *if* Γ *is an open surface) such that* $\mathbf{u} \in \mathbf{X}^r$ *and the following a priori error estimate holds*

$$\|\mathbf{u} - \mathbf{u}_{hp}\|_{\mathbf{X}} \leq C\, h^{1/2+\min\{r,p\}}\, p^{-(r+1/2)}\, \|\mathbf{u}\|_{\mathbf{X}^r} \tag{2.6}$$

with a positive constant C *independent of* h *and* p*.*

Proof. The assertion regarding the regularity of the solution $\mathbf{u}$ to the EFIE is a direct consequence of the regularity results in [17, Section 4.4] (see also [4, Appendix A]). Due to the quasi-optimal convergence (2.5) of the *hp*-BEM, the error estimate in (2.6) then immediately follows from the approximation result of Theorem 4.1 below. $\square$

Remark 2.1 *If* Γ *is a closed Lipschitz polyhedral surface (*$\Gamma = \partial\Omega$*) and the EFIE represents a boundary value problem for the time-harmonic Maxwell's equations in the exterior domain* $\mathbb{R}^3 \backslash \Omega$*, then one can be more specific about the regularity of the solution* $\mathbf{u}$ *of* (2.1)*. In fact, the presence of re-entrant edges is inevitable for the exterior of any polyhedral domain. Therefore, using the results for Maxwell singularities (see* [17]*), we conclude that for a closed polyhedral surface* Γ *there exists* $r \in (0, \frac{1}{2})$ *such that* $\mathbf{u} \in \mathbf{X}^r$*, and* $\mathbf{u} \notin \mathbf{X}^{1/2}$*.*

Remark 2.2 *The convergence rate of the hp-BEM for the EFIE is limited by the low regularity of the solution to the corresponding Maxwell's equations, especially for problems with screens which represent the least regular case. To obtain sharp a priori error estimates, a refined approximation analysis of singularities inherent to the solution of the EFIE is needed (see* [4]*). However, when the regularity of the solution to the EFIE is stated in terms of Sobolev spaces of tangential vector fields on* Γ*, the result of Theorem* 2.2 *is optimal with respect to both* h *and* p*.*

3 Preliminaries

3.1 Functional spaces, norms, and inner products

In our previous paper [6] we recalled definitions of the full range of Sobolev spaces necessary for the convergence analysis of the BEM for the EFIE (see §3.1 therein). This included Sobolev spaces on a Lipschitz domain $\Omega \subset \mathbb{R}^n$ and Sobolev spaces of scalar functions and tangential vector fields on a piecewise smooth (open or closed) Lipschitz surface $\Gamma \subset \mathbb{R}^3$. We have also defined basic differential operators on Γ. In the present paper we will use the same notation as in [6] for all differential operators, Sobolev spaces and their norms. For convenience of the reader, let us repeat some essential definitions, in particular, those for Sobolev spaces of tangential vector fields. Furthermore, we will introduce some more spaces and norms, which are indispensable for the error analysis of the *hp*-BEM.

Let Ω be a Lipschitz domain in $\mathbb{R}^n$. We use a traditional notation for the Sobolev spaces $H^s(\Omega)$ $(s \geq -1)$, $H^s_0(\Omega)$ $(s \in (0,1])$, and $\tilde{H}^s(\Omega)$ $(s \in [-1,1])$ with their standard norms (cf., [26]). In particular, for $s \in (0,1)$, the spaces $\tilde{H}^s(\Omega)$ are defined by using the real K-method of interpolation, and for $s \in [-1,0)$, the spaces $H^s(\Omega)$, $\tilde{H}^s(\Omega)$ and their norms are defined by duality with $L^2(\Omega) = H^0(\Omega) = \tilde{H}^0(\Omega)$ as pivot space. The norm and inner product in $L^2(\Omega)$ will be denoted as $\|\cdot\|_{0,\Omega}$ and $\langle\cdot,\cdot\rangle_{0,\Omega}$, respectively.

It is known that the standard norms $\|\cdot\|_{H^s(\Omega)}$ and $\|\cdot\|_{\tilde{H}^{-s}(\Omega)}$ are not scalable for $s \in (0,1]$ under affine transformations of Ω onto the reference domain (element). However, we will need a scalable norm in the space $\tilde{H}^{-1}$ on a generic interval $\ell \subset \mathbb{R}^1$. Scalable families of norms in H^s and in $\tilde{H}^{-s}$ for $s \in [-1,1]$ were introduced in [20]. Following [20] we define

$$\|f\|^2_{H^1_h(\ell)} = (\mathrm{meas}(\ell))^{-2}\|u\|^2_{0,\ell} + |u|^2_{H^1(\ell)}.$$

Then the norm $\|\cdot\|_{\tilde{H}^{-1}_h(\ell)}$ is defined by duality:

$$\|f\|_{\tilde{H}^{-1}_h(\ell)} = \sup_{0\neq\varphi\in H^1_h(\ell)} \frac{|\langle f,\varphi\rangle_{0,\ell}|}{\|\varphi\|_{H^1_h(\ell)}}. \tag{3.1}$$

Let ℓ be the image of the reference interval $\hat{\ell}$ under an affine transformation M, i.e., $\ell = M(\hat{\ell})$, and let $\mathrm{meas}(\ell) \simeq h$. Denote $\hat{f} = f \circ M$. Then the norms $\|\cdot\|_{H^1_h(\ell)}$ and $\|\cdot\|_{\tilde{H}^{-1}_h(\ell)}$ are scalable (see [20, Lemma 3.1]), i.e.,

$$\|f\|_{H^1_h(\ell)} \simeq h^{-1/2}\|\hat{f}\|_{H^1(\hat{\ell})} \quad \text{and} \quad \|f\|_{\tilde{H}^{-1}_h(\ell)} \simeq h^{3/2}\|\hat{f}\|_{\tilde{H}^{-1}(\hat{\ell})} \tag{3.2}$$

for any $\hat{f} \in H^1(\hat{\ell})$ and $\hat{f} \in \tilde{H}^{-1}(\hat{\ell})$, respectively, and both equivalences are uniform for $h > 0$.

An important fact related to the norm $\|\cdot\|_{\tilde{H}^{-1}_h}$ is that it enjoys the localisation property, provided that the function has zero average on each sub-domain (see [20, Lemma 3.2]). In particular, we will need the following result.

Lemma 3.1 *Assume that the boundary $\partial\Omega$ of the polygonal domain $\Omega \subset \mathbb{R}^2$ is partitioned into N segments ℓ_j $(j = 1,\ldots,N)$. Then, for all $f \in H^{-1}(\partial\Omega)$ with $f|_{\ell_j} \in \tilde{H}^{-1}(\ell_j)$ and $\int_{\ell_j} f\,d\sigma = 0$ $(j = 1,\ldots,N)$, there holds*

$$\|f\|^2_{H^{-1}(\partial\Omega)} \le C\sum_{j=1}^{N} \|f|_{\ell_j}\|^2_{\tilde{H}^{-1}_h(\ell_j)}$$

with a positive constant C independent of f and N.

Now, let Γ be a piecewise smooth (open or closed) Lipschitz surface in $\mathbb{R}^3$. We will assume that Γ has plane faces $\Gamma^{(i)}$ $(i = 1,\ldots,\mathcal{I}$; without loss of generality it is assumed that $\mathcal{I} > 1)$ and straight edges $e_{ij} = \bar{\Gamma}^{(i)} \cap \bar{\Gamma}^{(j)} \neq \emptyset$ $(i \neq j)$. If Γ is a closed surface, we will denote by Ω the Lipschitz polyhedron bounded by Γ, i.e., $\Gamma = \partial\Omega$. If Γ is an open surface, we additionally assume that Γ is orientable. In this case, we first introduce a piecewise plane closed Lipschitz surface $\tilde{\Gamma}$ which contains Γ, and then denote by Ω the Lipschitz polyhedron bounded by $\tilde{\Gamma}$, i.e.,

$\tilde{\Gamma} = \partial\Omega$. For each face $\Gamma^{(i)} \subset \Gamma$ there exists a constant unit normal vector $\boldsymbol{\nu}_i$, which is an outer normal vector to Ω. These vectors are then blended into a unit normal vector $\boldsymbol{\nu}$ defined almost everywhere on Γ. For each pair of indices $i, j = 1, \ldots, \mathcal{I}$ such that $\bar{\Gamma}^{(i)} \cap \bar{\Gamma}^{(j)} = e_{ij}$ we consider unit vectors $\boldsymbol{\tau}_{ij}$, $\boldsymbol{\tau}_i^{(j)}$, and $\boldsymbol{\tau}_j^{(i)}$ such that $\boldsymbol{\tau}_{ij} \| e_{ij}$, $\boldsymbol{\tau}_i^{(j)} = \boldsymbol{\tau}_{ij} \times \boldsymbol{\nu}_i$, and $\boldsymbol{\tau}_j^{(i)} = \boldsymbol{\tau}_{ij} \times \boldsymbol{\nu}_j$. Since each $\Gamma^{(i)}$ can be identified with a bounded subset in $\mathbb{R}^2$, the pair $(\boldsymbol{\tau}_i^{(j)}, \boldsymbol{\tau}_{ij})$ is an orthonormal basis of the plane generated by $\Gamma^{(i)}$.

Let Γ be a closed surface. Then $\Gamma = \partial\Omega$ is locally the graph of a Lipschitz function. Since the Sobolev spaces H^s for $|s| \leq 1$ are invariant under Lipschitz (i.e., $C^{0,1}$) coordinate transformations, the spaces $H^s(\Gamma)$ with $|s| \leq 1$ are defined in the usual way via a partition of unity subordinate to a finite family of local coordinate patches (see [27]). Due to this definition, the properties of Sobolev spaces on Lipschitz domains in $\mathbb{R}^n$ carry over to Sobolev spaces on Lipschitz surfaces. If Γ is an open surface, then the Sobolev spaces $H^s(\Gamma)$, $\tilde{H}^s(\Gamma)$ for $|s| \leq 1$ and $H_0^s(\Gamma)$ for $0 < s \leq 1$ are constructed in terms of the Sobolev spaces $H^s(\tilde{\Gamma})$ on a closed Lipschitz surface $\tilde{\Gamma} \supset \Gamma$ (see [27]). Note that the spaces $H^s(\Gamma^{(i)})$ and $\tilde{H}^s(\Gamma^{(i)})$ on each face $\Gamma^{(i)}$ are well-defined for any $s \geq -1$.

We will denote by γ_{tr} the standard trace operator, $\gamma_{\mathrm{tr}}(u) = u|_\Gamma$, $u \in C^\infty(\bar{\Omega})$. For $s \in (0,1)$ (resp., $s > 1$), γ_{tr} has a unique extension to a continuous operator $H^{s+1/2}(\Omega) \to H^s(\Gamma)$ (resp., $H^{s+1/2}(\Omega) \to H^1(\Gamma)$), see [16, 11]. We will use the notation $C^\infty(\Gamma) = \gamma_{\mathrm{tr}}(C^\infty(\bar{\Omega}))$.

Using the introduced Sobolev spaces of scalar functions, we define:

$$\mathbf{H}^s(\Omega) = (H^s(\Omega))^3, \quad \mathbf{H}^s(\Gamma^{(i)}) = (H^s(\Gamma^{(i)}))^2, \quad \tilde{\mathbf{H}}^s(\Gamma^{(i)}) = (\tilde{H}^s(\Gamma^{(i)}))^2 \quad (1 \leq i \leq \mathcal{I})$$

for $s \geq -1$, and

$$\mathbf{H}^s(\Gamma) = (H^s(\Gamma))^3 \quad \text{for } s \in [-1, 1].$$

The norms and inner products in all these spaces are defined component-wise and usual conventions $\mathbf{H}^0(\Omega) = \mathbf{L}^2(\Omega)$, $\mathbf{H}^0(\Gamma) = \mathbf{L}^2(\Gamma)$, $\mathbf{H}^0(\Gamma^{(i)}) = \tilde{\mathbf{H}}^0(\Gamma^{(i)}) = \mathbf{L}^2(\Gamma^{(i)})$ hold.

Now let us introduce the Sobolev spaces of tangential vector fields defined on Γ (see [12, 13, 14]). We start with the space

$$\mathbf{L}_t^2(\Gamma) := \{\mathbf{u} \in \mathbf{L}^2(\Gamma);\ \mathbf{u} \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma\},$$

which will be identified with the space of two-dimensional, tangential, square integrable vector fields. The norm and inner product in this space will be denoted by $\|\cdot\|_{0,\Gamma}$ and $\langle \cdot, \cdot \rangle_{0,\Gamma}$, respectively. The similarity of this notation with the one for scalar functions should not lead to any confusion. Then we define (hereafter, $\mathbf{u}_i$ denotes the restriction of $\mathbf{u}$ to the face $\Gamma^{(i)}$):

$$\mathbf{H}^s_-(\Gamma) := \{\mathbf{u} \in \mathbf{L}_t^2(\Gamma);\ \mathbf{u}_i \in \mathbf{H}^s(\Gamma^{(i)}),\ \ 1 \leq i \leq \mathcal{I}\}, \quad s \geq 0,$$

$$\|\mathbf{u}\|_{\mathbf{H}^s_-(\Gamma)} := \left(\sum_{i=1}^{\mathcal{I}} \|\mathbf{u}_i\|^2_{\mathbf{H}^s(\Gamma^{(i)})} \right)^{\frac{1}{2}}.$$

Let γ_{tr} be the trace operator (now acting on vector fields), $\gamma_{\mathrm{tr}}(\mathbf{u}) = \mathbf{u}|_\Gamma$, $\gamma_{\mathrm{tr}} : \mathbf{H}^{s+1/2}(\Omega) \to \mathbf{H}^s(\Gamma)$ for $s \in (0,1)$, and let $\gamma_{\mathrm{tr}}^{-1}$ be one of its right inverses. We will use the "tangential components trace" mapping $\pi_\tau : (C^\infty(\bar\Omega))^3 \to \mathbf{L}^2_t(\Gamma)$ and the "tangential trace" mapping γ_τ : $(C^\infty(\bar\Omega))^3 \to \mathbf{L}^2_t(\Gamma)$, which are defined as $\mathbf{u} \mapsto \boldsymbol{\nu} \times (\mathbf{u} \times \boldsymbol{\nu})|_\Gamma$ and $\mathbf{u} \times \boldsymbol{\nu}|_\Gamma$, respectively. We will also use the notation π_τ (resp., γ_τ) for the composite operator $\pi_\tau \circ \gamma_{\mathrm{tr}}^{-1}$ (resp., $\gamma_\tau \circ \gamma_{\mathrm{tr}}^{-1}$), which acts on traces. Then we define the spaces

$$\mathbf{H}^{1/2}_{\|}(\Gamma) := \pi_\tau(\mathbf{H}^{1/2}(\Gamma)), \qquad \mathbf{H}^{1/2}_{\perp}(\Gamma) := \gamma_\tau(\mathbf{H}^{1/2}(\Gamma)),$$

endowed with their operator norms

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{H}^{1/2}_{\|}(\Gamma)} &:= \inf_{\boldsymbol{\phi}\in\mathbf{H}^{1/2}(\Gamma)} \{\|\boldsymbol{\phi}\|_{\mathbf{H}^{1/2}(\Gamma)};\ \pi_\tau(\boldsymbol{\phi}) = \mathbf{u}\}, \\ \|\mathbf{u}\|_{\mathbf{H}^{1/2}_{\perp}(\Gamma)} &:= \inf_{\boldsymbol{\phi}\in\mathbf{H}^{1/2}(\Gamma)} \{\|\boldsymbol{\phi}\|_{\mathbf{H}^{1/2}(\Gamma)};\ \gamma_\tau(\boldsymbol{\phi}) = \mathbf{u}\}. \end{aligned}$$

It has been shown in [12] that the space $\mathbf{H}^{1/2}_{\|}(\Gamma)$ (resp., $\mathbf{H}^{1/2}_{\perp}(\Gamma)$) can be characterised as the space of tangential vector fields belonging to $\mathbf{H}^{1/2}_{-}(\Gamma)$ and satisfying an appropriate "weak continuity" condition for the tangential (resp., normal) component across each edge e_{ij} of Γ.

For $s > \frac{1}{2}$ we set

$$\begin{aligned} \mathbf{H}^s_{\|}(\Gamma) &:= \{\mathbf{u} \in \mathbf{H}^s_{-}(\Gamma);\ \mathbf{u}_i \cdot \boldsymbol{\tau}_{ij} = \mathbf{u}_j \cdot \boldsymbol{\tau}_{ij} \quad \text{at each } e_{ij}\}, \\ \mathbf{H}^s_{\perp}(\Gamma) &:= \{\mathbf{u} \in \mathbf{H}^s_{-}(\Gamma);\ \mathbf{u}_i \cdot \boldsymbol{\tau}^{(j)}_i = \mathbf{u}_j \cdot \boldsymbol{\tau}^{(i)}_j \quad \text{at each } e_{ij}\}. \end{aligned}$$

For any $s > \frac{1}{2}$ the spaces $\mathbf{H}^s_{\|}(\Gamma)$ and $\mathbf{H}^s_{\perp}(\Gamma)$ are closed subspaces of $\mathbf{H}^s_{-}(\Gamma)$. Finally, for $s \in [0, \frac{1}{2})$ we set

$$\mathbf{H}^s_{\|}(\Gamma) = \mathbf{H}^s_{\perp}(\Gamma) := \mathbf{H}^s_{-}(\Gamma).$$

If Γ is an open surface, then we also need to define subspaces of $\mathbf{H}^s_{\|}(\Gamma)$ and $\mathbf{H}^s_{\perp}(\Gamma)$ incorporating boundary conditions on $\partial\Gamma$ (for tangential and normal components, respectively). In this case, for a given function $\mathbf{u}$ on Γ, we will denote by $\tilde{\mathbf{u}}$ the extension of $\mathbf{u}$ by zero onto a closed Lipschitz polyhedral surface $\tilde\Gamma \supset \Gamma$. Then we define the spaces

$$\begin{aligned} \tilde{\mathbf{H}}^s_{\|}(\Gamma) &:= \{\mathbf{u} \in \mathbf{H}^s_{\|}(\Gamma);\ \tilde{\mathbf{u}} \in \mathbf{H}^s_{\|}(\tilde\Gamma)\}, \quad s \ge 0, \\ \tilde{\mathbf{H}}^s_{\perp}(\Gamma) &:= \{\mathbf{u} \in \mathbf{H}^s_{\perp}(\Gamma);\ \tilde{\mathbf{u}} \in \mathbf{H}^s_{\perp}(\tilde\Gamma)\}, \quad s \ge 0, \end{aligned}$$

which are furnished with the norms

$$\|\mathbf{u}\|_{\tilde{\mathbf{H}}^s_{\|}(\Gamma)} := \|\tilde{\mathbf{u}}\|_{\mathbf{H}^s_{\|}(\tilde\Gamma)}, \quad \|\mathbf{u}\|_{\tilde{\mathbf{H}}^s_{\perp}(\Gamma)} := \|\tilde{\mathbf{u}}\|_{\mathbf{H}^s_{\perp}(\tilde\Gamma)}, \quad s \ge 0.$$

When considering open and closed surfaces at the same time we use the notation $\tilde{\mathbf{H}}^s_{\|}(\Gamma)$, $\tilde{\mathbf{H}}^s_{\perp}(\Gamma)$, etc. also for closed surfaces by assuming that $\tilde{\mathbf{H}}^s_{\|}(\Gamma) = \mathbf{H}^s_{\|}(\Gamma)$, $\tilde{\mathbf{H}}^s_{\perp}(\Gamma) = \mathbf{H}^s_{\perp}(\Gamma)$, etc. in this case. This in particular applies to the following definition of dual spaces. For $s \in [-1, 0)$, the spaces $\mathbf{H}^s_\|(\Gamma)$, $\tilde{\mathbf{H}}^s_\|(\Gamma)$, $\mathbf{H}^s_\perp(\Gamma)$, and $\tilde{\mathbf{H}}^s_\perp(\Gamma)$ are defined as the dual spaces of $\tilde{\mathbf{H}}^{-s}_\|(\Gamma)$, $\mathbf{H}^{-s}_\|(\Gamma)$, $\tilde{\mathbf{H}}^{-s}_\perp(\Gamma)$, and $\mathbf{H}^{-s}_\perp(\Gamma)$, respectively (with $\mathbf{L}^2_t(\Gamma)$ as pivot space). They are equipped with their natural (dual) norms. Moreover, for any $s \in (-\frac{1}{2}, \frac{1}{2})$ there holds (cf. [21])

$$\tilde{\mathbf{H}}^s_\|(\Gamma) = \mathbf{H}^s_\|(\Gamma) = \tilde{\mathbf{H}}^s_\perp(\Gamma) = \mathbf{H}^s_\perp(\Gamma).$$

Using the above spaces of tangential vector fields, one can define basic differential operators on Γ. The tangential gradient, $\nabla_\Gamma : H^1(\Gamma) \to \mathbf{L}^2_t(\Gamma)$, and the tangential vector curl, $\mathbf{curl}_\Gamma : H^1(\Gamma) \to \mathbf{L}^2_t(\Gamma)$, are defined in the usual way by localisation to each face $\Gamma^{(i)}$. The adjoint operator of $-\nabla_\Gamma$ is the surface divergence denoted by div_Γ (we refer to [12, 13] for more details regarding definitions and properties of differential operators on both closed and open surfaces).

Now we can introduce the spaces which appear when dealing with the EFIE on Γ. First, we set

$$\mathbf{H}^s(\mathrm{div}_\Gamma, \Gamma) := \{\mathbf{u} \in \mathbf{H}^s_\|(\Gamma);\ \mathrm{div}_\Gamma\, \mathbf{u} \in H^s(\Gamma)\}, \quad s \in [-1/2, 1/2]$$

and

$$\mathbf{H}^s_-(\mathrm{div}_\Gamma, \Gamma) := \{\mathbf{u} \in \mathbf{H}^s_-(\Gamma);\ \mathrm{div}_\Gamma\, \mathbf{u} \in H^s_-(\Gamma)\}, \quad s \geq 0$$

for Γ being either a closed or an open surface. Here, the space $H^s_-(\Gamma)$ is defined similarly to the space $\mathbf{H}^s_-(\Gamma)$ in a piecewise fashion:

$$H^s_-(\Gamma) := \{u \in L^2(\Gamma);\ u|_{\Gamma^{(i)}} \in H^s(\Gamma^{(i)}),\ i = 1, \ldots, \mathcal{I}\}, \quad s \geq 0,$$

$$\|u\|^2_{H^s_-(\Gamma)} := \sum_{i=1}^{\mathcal{I}} \|u|_{\Gamma^{(i)}}\|^2_{H^s(\Gamma^{(i)})}.$$

If Γ is an open surface, then we will also use the space

$$\tilde{\mathbf{H}}^s(\mathrm{div}_\Gamma, \Gamma) := \{\mathbf{u} \in \tilde{\mathbf{H}}^s_\|(\Gamma);\ \mathrm{div}_\Gamma\, \mathbf{u} \in \tilde{H}^s(\Gamma)\}, \quad s \in [-1/2, 0].$$

The spaces $\mathbf{H}^s(\mathrm{div}_\Gamma, \Gamma)$, $\mathbf{H}^s_-(\mathrm{div}_\Gamma, \Gamma)$, and $\tilde{\mathbf{H}}^s(\mathrm{div}_\Gamma, \Gamma)$ are equipped with their graph norms $\|\cdot\|_{\mathbf{H}^s(\mathrm{div}_\Gamma,\Gamma)}$, $\|\cdot\|_{\mathbf{H}^s_-(\mathrm{div}_\Gamma,\Gamma)}$, and $\|\cdot\|_{\tilde{\mathbf{H}}^s(\mathrm{div}_\Gamma,\Gamma)}$, respectively. If $s = 0$, then we will drop the superscript and for open surfaces also the tilde in the above notation, $\mathbf{H}^0(\mathrm{div}_\Gamma, \Gamma) = \mathbf{H}^0_-(\mathrm{div}_\Gamma, \Gamma) = \tilde{\mathbf{H}}^0(\mathrm{div}_\Gamma, \Gamma) = \mathbf{H}(\mathrm{div}_\Gamma, \Gamma)$.

On open surfaces, one needs the spaces incorporating homogeneous boundary conditions for the trace of the normal component on $\partial\Gamma$. By $\tilde{\mathbf{H}}^s_0(\mathrm{div}_\Gamma, \Gamma)$ with $s \in [-\frac{1}{2}, 0]$ we denote the subspace of elements $\mathbf{u} \in \tilde{\mathbf{H}}^s(\mathrm{div}_\Gamma, \Gamma)$ such that for all $v \in C^\infty(\Gamma)$ there holds

$$\langle \mathbf{u}, \nabla_\Gamma v \rangle + \langle \mathrm{div}_\Gamma\, \mathbf{u}, v \rangle = 0,$$

where brackets $\langle \cdot, \cdot \rangle$ denote the corresponding dualities. Similarly as above, we will drop the superscript and the tilde if $s = 0$. We note that $\tilde{\mathbf{H}}^s_0(\mathrm{div}_\Gamma, \Gamma)$ is a closed subspace of $\tilde{\mathbf{H}}^s(\mathrm{div}_\Gamma, \Gamma)$ for $s \in [-\frac{1}{2}, 0]$.

The notation above related to open surfaces will be used also for two-dimensional domains (in particular, for single faces of Γ and for reference elements). When we need to join the notation for open and closed surfaces, we will write

$$\mathbf{X} = \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma), \quad \mathbf{X}^s = \begin{cases} \mathbf{H}^s(\mathrm{div}_\Gamma, \Gamma) & \text{for } s \in [-\frac{1}{2}, 0), \\ \mathbf{H}(\mathrm{div}_\Gamma, \Gamma) & \text{for } s = 0, \\ \mathbf{H}^s_-(\mathrm{div}_\Gamma, \Gamma) & \text{for } s > 0 \end{cases}$$

if Γ is a closed surface, and

$$\mathbf{X} = \tilde{\mathbf{H}}_0^{-1/2}(\mathrm{div}_\Gamma, \Gamma), \quad \mathbf{X}^s = \begin{cases} \tilde{\mathbf{H}}_0^s(\mathrm{div}_\Gamma, \Gamma) & \text{for } s \in [-\frac{1}{2}, 0), \\ \mathbf{H}_0(\mathrm{div}_\Gamma, \Gamma) & \text{for } s = 0, \\ \mathbf{H}^s_-(\mathrm{div}_\Gamma, \Gamma) \cap \mathbf{H}_0(\mathrm{div}_\Gamma, \Gamma) & \text{for } s > 0 \end{cases}$$

if Γ is an open surface. In all these cases the norms will be denoted as $\|\cdot\|_{\mathbf{X}}$ and $\|\cdot\|_{\mathbf{X}^s}$.

Let $[\cdot,\cdot]_\theta$ ($\theta \in [0,1]$) denote the standard interpolation (see [26, 2]). We quote the following interpolation result from [11] (see Theorem 4.12 therein).

Lemma 3.2 *There exists an $s_0 \in (0, \frac{1}{2}]$ such that for any $s \in [-\frac{1}{2}, s_0)$ there holds $[\mathbf{X}, \mathbf{X}^s]_\theta = \mathbf{X}^{(1/2+s)\theta-1/2}$.*

3.2 Interpolation operators

As it was mentioned in the introduction, our analysis of hp-approximations essentially relies on the properties of the $\mathbf{H}(\mathrm{div})$-conforming projection based interpolation operator. Let us briefly sketch the definition of this operator on the reference element K (see [19] for details). In this sub-section we use standard differential operators ∇, $\mathbf{curl}$ and div acting on 2D scalar functions and vector fields, respectively.

Given a vector field $\mathbf{u} \in \mathbf{H}^r(K) \cap \mathbf{H}(\mathrm{div}, K)$ with $r > 0$, the interpolant $\mathbf{u}^p = \Pi_p^{\mathrm{div}}\mathbf{u} \in \boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K)$ is defined as the sum of three terms:

$$\mathbf{u}^p = \mathbf{u}_1 + \mathbf{u}_2^p + \mathbf{u}_3^p.$$

Here, $\mathbf{u}_1$ is the lowest order interpolant

$$\mathbf{u}_1 = \sum_{\ell \subset \partial K} \Big(\int_\ell \mathbf{u} \cdot \hat{\mathbf{n}}\, d\sigma \Big) \boldsymbol{\phi}_\ell,$$

where $\hat{\mathbf{n}}$ denotes the outward normal unit vector to ∂K, and $\boldsymbol{\phi}_\ell$ are the standard basis functions (associated with edges ℓ) for $\boldsymbol{\mathcal{P}}_1^{\mathrm{RT}}(K)$. For any edge $\ell \subset \partial K$ one has

$$\int_\ell (\mathbf{u} - \mathbf{u}_1) \cdot \hat{\mathbf{n}}\, d\sigma = 0. \tag{3.3}$$

Hence, there exists a scalar function ψ, defined on ∂K, such that

$$\frac{\partial \psi}{\partial \sigma} = (\mathbf{u} - \mathbf{u}_1) \cdot \hat{\mathbf{n}}, \quad \psi = 0 \;\text{ at all vertices.} \tag{3.4}$$

Then, for each edge ℓ, the restriction $\psi|_\ell$ is projected onto the set of polynomials $\mathcal{P}^0_{p+1}(\ell)$

$$\psi_{2,\ell} \in \mathcal{P}^0_{p+1}(\ell): \quad \langle \psi|_\ell - \psi_{2,\ell}, \varphi \rangle_{\tilde{H}^{1/2}(\ell)} = 0 \quad \forall\, \varphi \in \mathcal{P}^0_{p+1}(\ell). \tag{3.5}$$

Extending $\psi_{2,\ell}$ by zero from ℓ onto ∂K (and keeping its notation) and using the polynomial extension from the boundary, we define $\psi^\ell_{2,p+1} \in \mathcal{P}_{p+1}(K)$ such that $\psi^\ell_{2,p+1}|_{\partial K} = \psi_{2,\ell}$. Then we set

$$\mathbf{u}^p_2 = \sum_{\ell \subset \partial K} \mathbf{u}^p_{2,\ell} \in \boldsymbol{\mathcal{P}}^{\mathrm{RT}}_p(K), \quad \text{where } \; \mathbf{u}^p_{2,\ell} = \mathbf{curl}\, \psi^\ell_{2,p+1}.$$

The interior interpolant $\mathbf{u}^p_3$ is a vector bubble function that solves the constrained minimization problem

$$\begin{aligned}
&\mathbf{u}^p_3 \in \boldsymbol{\mathcal{P}}^{\mathrm{RT},0}_p(K): \\
&\|\mathrm{div}(\mathbf{u} - (\mathbf{u}_1 + \mathbf{u}^p_2 + \mathbf{u}^p_3))\|_{0,K} \to \min, \\
&\langle \mathbf{u} - (\mathbf{u}_1 + \mathbf{u}^p_2 + \mathbf{u}^p_3), \mathbf{curl}\,\phi \rangle_{0,K} = 0 \quad \forall \phi \in \mathcal{P}^0_{p+1}(K).
\end{aligned}$$

Remark 3.1 *The Sobolev space $\tilde{H}^{1/2}$ and the corresponding norm were defined in* [6] *using the real K-method of interpolation (see* [26]*). However, the expression for the $\tilde{H}^{1/2}$-inner product in* (3.5) *is based on another (equivalent) norm in $\tilde{H}^{1/2}$. Without loss of generality, let us assume that $\ell = I = (0,1)$ is the reference interval. Then one has (see, e.g.,* [19]*)*

$$\|\phi\|_{\tilde{H}^{1/2}(I)} \simeq \|\nabla \widetilde{\phi^\circ}\|_{0,T},$$

where T is the reference triangle, ϕ° denotes the extension of the function $\phi \in \tilde{H}^{1/2}(I)$ by zero onto $\partial T \supset I$, and $\widetilde{\phi^\circ}$ is the harmonic lift of $\phi^\circ \in H^{1/2}(\partial T)$. Then, applying the parallelogram law and integrating by parts, we find the expression of the corresponding inner product (cf. [19]*):*

$$\begin{aligned}
\langle \phi, \psi \rangle_{\tilde{H}^{1/2}(I)} &= \langle \nabla \widetilde{\phi^\circ}, \nabla \widetilde{\psi^\circ} \rangle_{0,T} = \left\langle \frac{\partial \widetilde{\phi^\circ}}{\partial \hat{\mathbf{n}}}, \widetilde{\psi^\circ} \right\rangle_{0,\partial T} = \\
&= \left\langle \frac{\partial \widetilde{\phi^\circ}}{\partial \hat{\mathbf{n}}}, \psi \right\rangle_{0,I} = \left\langle \phi, \frac{\partial \widetilde{\psi^\circ}}{\partial \hat{\mathbf{n}}} \right\rangle_{0,I} \quad \forall \phi, \psi \in \tilde{H}^{1/2}(I).
\end{aligned}$$

This expression can also be written as

$$\langle \phi, \psi \rangle_{\tilde{H}^{1/2}(I)} = \Big\langle \phi, \mathcal{DN}_T(\psi) \Big\rangle_{0,I} = \Big\langle \mathcal{DN}_T(\phi), \psi \Big\rangle_{0,I} \quad \forall \phi, \psi \in \tilde{H}^{1/2}(I), \tag{3.6}$$

where $\mathcal{DN}_T : \tilde{H}^{1/2}(I) \to H^{-1/2}(I)$ denotes the Dirichlet-to-Neumann operator associated with the triangle T. Exactly this inner product is employed in (3.5)*.*

For $r > 0$ the operator $\Pi_p^{\rm div} : \mathbf{H}^r(K) \cap \mathbf{H}(\mathrm{div}, K) \to \mathbf{H}(\mathrm{div}, K)$ is well defined and bounded, with corresponding operator norm being independent of the polynomial degree p (cf. [19, Propositions 2]). Moreover, $\Pi_p^{\rm div}$ preserves polynomial vector fields in $\boldsymbol{\mathcal{P}}_p^{\rm RT}(K)$, and there holds the following estimate for the interpolation error (see [7, Theorem 5.1])

$$\|\mathbf{u} - \Pi_p^{\rm div}\, \mathbf{u}\|_{\mathbf{H}(\mathrm{div},K)} \le C\, p^{-r}\, \|\mathbf{u}\|_{\mathbf{H}^r(\mathrm{div},K)}, \tag{3.7}$$

provided that $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, K) := \{\mathbf{u} \in \mathbf{H}^r(K);\ \mathrm{div}\,\mathbf{u} \in H^r(K)\}$ for $r > 0$.

We will need the following auxiliary result regarding the operator $\Pi_p^{\rm div}$.

Lemma 3.3 *Let $\mathbf{u} \in \mathbf{H}^r(K) \cap \mathbf{H}(\mathrm{div}, K)$ with $r > 0$, and let $\mathbf{u}^p = \Pi_p^{\rm div}\mathbf{u} \in \boldsymbol{\mathcal{P}}_p^{\rm RT}(K)$. Then for any edge $\ell \subset \partial K$ there holds*

$$\|(\mathbf{u} - \mathbf{u}^p) \cdot \hat{\mathbf{n}}\|_{\tilde{H}^{-1}(\ell)} \le C\, p^{-1/2}\, \|(\mathbf{u} - \mathbf{u}^p) \cdot \hat{\mathbf{n}}\|_{H^{-1/2}(\partial K)} \le C\, p^{-1/2}\, \|\mathbf{u} - \mathbf{u}^p\|_{H(\mathrm{div},K)}. \tag{3.8}$$

Proof. Let us fix an edge $\ell \subset \partial K$. Using the definition of the interpolant $\mathbf{u}^p$ we have

$$\|(\mathbf{u} - \mathbf{u}^p) \cdot \hat{\mathbf{n}}\|_{\tilde{H}^{-1}(\ell)} = \|(\mathbf{u} - \mathbf{u}_1) \cdot \hat{\mathbf{n}} - \mathbf{u}_2^p \cdot \hat{\mathbf{n}}\|_{\tilde{H}^{-1}(\ell)} = \left\| \frac{\partial \psi}{\partial \sigma} - \frac{\partial \psi_{2,\ell}}{\partial \sigma} \right\|_{\tilde{H}^{-1}(\ell)}. \tag{3.9}$$

Since the derivative with respect to the arc length $\frac{\partial}{\partial \sigma}$ is a bounded operator mapping $L^2(\ell)$ to $\tilde{H}^{-1}(\ell)$ (see, e.g., [22, Lemma 3]), we deduce from (3.9)

$$\|(\mathbf{u} - \mathbf{u}^p) \cdot \hat{\mathbf{n}}\|_{\tilde{H}^{-1}(\ell)} \le C\, \|\psi|_\ell - \psi_{2,\ell}\|_{0,\ell}. \tag{3.10}$$

Let $\tilde{T}$ be an equilateral triangle having ℓ as one of its edges (if $K = T$ then $\tilde{T} = T$). Denoting $\phi := \psi|_\ell - \psi_{2,\ell}$ and using the expression for the $\tilde{H}^{1/2}(\ell)$-inner product as in (3.6), we can rewrite the orthogonality relation in (3.5) as

$$\langle \phi, \phi_p \rangle_{\tilde{H}^{1/2}(\ell)} = \left\langle \phi, \mathcal{DN}_{\tilde{T}}(\phi_p) \right\rangle_{0,\ell} = 0 \qquad \forall\, \phi_p \in \mathcal{P}_{p+1}^0(\ell). \tag{3.11}$$

Now, let us consider an auxiliary mixed boundary value problem on $\tilde{T}$: find $\Phi \in H^1(\tilde{T})$ such that

$$-\Delta \Phi = 0 \ \text{ in } \tilde{T}, \quad \frac{\partial \Phi}{\partial \hat{\mathbf{n}}} = \phi \ \text{ on } \ell, \quad \Phi = 0 \ \text{ on } \partial\tilde{T} \backslash \ell.$$

Recalling that $\phi \in \tilde{H}^{1/2}(\ell) \subset H^{1/2}(\ell)$ and using the regularity theory for elliptic problems in non-smooth domains (see, e.g., [18, 21]), we conclude that $\Phi \in H^2(\tilde{T})$ and $\|\Phi\|_{H^2(\tilde{T})} \le C\, \|\phi\|_{H^{1/2}(\ell)}$. Therefore, applying the trace theorem for a single edge $\ell \subset \tilde{T}$ (see [21, Theorem 1.4.2]) we prove that $\Phi|_\ell \in H^{3/2}(\ell)$ and

$$\|\Phi|_\ell\|_{H^{3/2}(\ell)} \le C\, \|\Phi\|_{H^2(\tilde{T})} \le C\, \|\phi\|_{H^{1/2}(\ell)}. \tag{3.12}$$

On the other hand, $\phi = \partial\Phi/\partial\hat{\mathbf{n}} = \mathcal{DN}_{\tilde{T}}(\Phi|_\ell)$ and using (3.11) we obtain for any $\phi_p \in \mathcal{P}_p^0(\ell)$

$$\begin{aligned} \|\phi\|_{0,\ell}^2 &= \langle \phi, \phi \rangle_{0,\ell} = \Big\langle \phi, \mathcal{DN}_{\tilde{T}}(\Phi|_\ell) \Big\rangle_{0,\ell} = \Big\langle \phi, \mathcal{DN}_{\tilde{T}}(\Phi|_\ell - \phi_p) \Big\rangle_{0,\ell} \\ &\le \|\phi\|_{0,\ell} \, \Big\| \mathcal{DN}_{\tilde{T}}(\Phi|_\ell - \phi_p) \Big\|_{0,\ell}. \end{aligned}$$

Hence, using the fact that the operator $\mathcal{DN}_{\tilde{T}}$ is continuous as a mapping $H_0^1(\ell) \to L^2(\ell)$, we find

$$\|\phi\|_{0,\ell} \le C \, |\Phi|_\ell - \phi_p|_{H^1(\ell)} \qquad \forall\, \phi_p \in \mathcal{P}_p^0(\ell). \tag{3.13}$$

To estimate the semi-norm in (3.13) we apply the standard p-approximation result in 1D (see, e.g., [1, Lemma 3.2]): there exists $\phi_p \in \mathcal{P}_p^0(\ell)$ such that

$$\|\Phi|_\ell - \phi_p\|_{H^1(\ell)} \le C\, p^{-1/2} \, \|\Phi|_\ell\|_{H^{3/2}(\ell)}. \tag{3.14}$$

Putting together inequalities (3.12)–(3.14) and recalling our notation for the function ϕ, we obtain

$$\|\psi|_\ell - \psi_{2,\ell}\|_{0,\ell} \le C\, p^{-1/2} \, \|\psi|_\ell - \psi_{2,\ell}\|_{H^{1/2}(\ell)}. \tag{3.15}$$

Since

$$(\mathbf{u} - \mathbf{u}^p) \cdot \hat{\mathbf{n}}|_{\partial K} = (\mathbf{u} - \mathbf{u}_1 - \mathbf{u}_2^p) \cdot \hat{\mathbf{n}}|_{\partial K} = \frac{\partial}{\partial \sigma}\Big(\psi - \sum_{\ell \subset \partial K} \psi_{2,\ell}\Big) \in H_*^{-1/2}(\partial K)$$

with $H_*^{-1/2}(\partial K) := \{u \in H^{-1/2}(\partial K);\ \langle u, 1\rangle_{0,\partial K} = 0\}$, and the tangential derivative defines an isomorphism $\frac{\partial}{\partial\sigma} :\ H^{1/2}(\partial K)/\mathbb{R} \to H_*^{-1/2}(\partial K)$ (see [19, Lemma 2]), we prove that

$$\|\psi|_\ell - \psi_{2,\ell}\|_{H^{1/2}(\ell)} \le C \, \Big\|\psi - \sum_{\ell \subset \partial K} \psi_{2,\ell}\Big\|_{H^{1/2}(\partial K)} \le C \, \|(\mathbf{u} - \mathbf{u}^p) \cdot \hat{\mathbf{n}}\|_{H^{-1/2}(\partial K)}. \tag{3.16}$$

The first inequality in (3.8) then immediately follows from estimates (3.10), (3.15), and (3.16). The second inequality in (3.8) is true due to the continuity of the normal trace operator $\mathbf{v} \to \mathbf{v} \cdot \hat{\mathbf{n}}|_{\partial K}$ as a mapping $\mathbf{H}(\mathrm{div}, K) \to H^{-1/2}(\partial K)$. □

Now, let us consider our Lipschitz surface Γ discretised by the quasi-uniform mesh. Using the Piola transform $\mathcal{M}_j$ for each element Γ_j and applying the p-interpolation operator Π_p^{div} on the reference elements, one can define the "global" $\mathbf{H}(\mathrm{div})$-conforming hp-interpolation operator $\Pi_{hp}^{\mathrm{div}} :\ \mathbf{H}^r_-(\Gamma) \cap \mathbf{H}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{X}_{hp}$ ($r > 0$) such that for $\mathbf{u}^{hp} := \Pi_{hp}^{\mathrm{div}}\mathbf{u}$ there holds

$$\mathcal{M}_j^{-1}(\mathbf{u}^{hp}|_{\Gamma_j}) = \Pi_p^{\mathrm{div}}\Big(\mathcal{M}_j^{-1}(\mathbf{u}|_{\Gamma_j})\Big).$$

Since the projection-based interpolation operator Π_p^{div} preserves polynomial vector fields and provides $\mathbf{H}(\mathrm{div})$-conforming approximations, the p-interpolation error estimate (3.7) on the reference element extends to the corresponding hp-estimate in a standard way by using the Bramble-Hilbert argument and scaling. This result is formulated in the following theorem.

Theorem 3.1 *Let* $\mathbf{u} \in \mathbf{H}^r_-(\mathrm{div}_\Gamma, \Gamma)$, $r > 0$. *Then there exists a positive constant* C *independent of* h, p, *and* $\mathbf{u}$ *such that*

$$\|\mathbf{u} - \Pi^{\mathrm{div}}_{hp}\mathbf{u}\|_{\mathbf{H}(\mathrm{div}_\Gamma,\Gamma)} \le C\, h^{\min\{r,p\}}\, p^{-r}\, \|\mathbf{u}\|_{\mathbf{H}^r_-(\mathrm{div}_\Gamma,\Gamma)}.$$

4 Approximating properties of $\mathbf{X}_{hp}$ in $\mathbf{X}$

The main purpose of this section is to prove that the orthogonal projection $P_{hp} : \mathbf{X} \to \mathbf{X}_{hp}$ with respect to the norm in $\mathbf{X}$ satisfies an optimal error estimate (when the error is measured in the norm of $\mathbf{X}$). The standard approach to tackle such kind of problems is to use the duality argument. Following this approach on Γ in our setting would require that for $s \in (0, \frac{1}{2}]$ the space $\mathbf{H}^{-s}(\mathrm{div}_\Gamma, \Gamma)$ (or, $\tilde{\mathbf{H}}^{-s}(\mathrm{div}_\Gamma, \Gamma)$ if Γ is an open surface) is the dual space of $\mathbf{H}^s(\mathrm{div}_\Gamma, \Gamma)$ with respect to the $\mathbf{H}(\mathrm{div}_\Gamma, \Gamma)$-inner product. However, this fact is true only for smooth surfaces. On the other hand, the duality argument can be applied face by face. This idea was suggested by Buffa & Christiansen in [11] and was successfully exploited by these authors in the context of the h-version of the BEM for the EFIE. We will demonstrate below that, when employing the projection based $\mathbf{H}(\mathrm{div})$-conforming interpolation operator, this approach provides an optimal hp-approximation result in the energy norm of the EFIE.

Let $\tilde{\Gamma}$ be a single face of Γ. To simplify the presentation (in particular, to avoid imposing boundary conditions on the edges of $\partial\tilde{\Gamma}$ which also belong to $\partial\Gamma$ in the case that $\partial\Gamma \ne \emptyset$), we will assume that Γ is a (closed) Lipschitz polyhedral surface. All arguments extend to the case of an open piecewise plane Lipschitz surface in a straightforward way (cf. [11]). For the sake of simplicity of notation we will also omit the subscript $\tilde{\Gamma}$ for differential operators over this face, e.g., we will write div for $\mathrm{div}_{\tilde{\Gamma}}$.

Let $\mathbf{H}^r(\mathrm{div}, \tilde{\Gamma}) := \{\mathbf{u} \in \mathbf{H}^r(\tilde{\Gamma});\ \mathrm{div}\,\mathbf{u} \in H^r(\tilde{\Gamma})\}$ for $r \ge 0$. We will also denote by $\mathbf{X}_{hp}(\tilde{\Gamma})$ the restriction of $\mathbf{X}_{hp}$ to $\tilde{\Gamma}$. Then, for $r > 0$ we introduce the operator $\mathcal{Q}_{hp} : \mathbf{H}^r(\mathrm{div}, \tilde{\Gamma}) \to \mathbf{X}_{hp}(\tilde{\Gamma})$ as follows: given $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, \tilde{\Gamma})$, we define $\mathcal{Q}_{hp}\mathbf{u} \in \mathbf{X}_{hp}(\tilde{\Gamma})$ such that

$$\begin{aligned} (\mathbf{u} - \mathcal{Q}_{hp}\mathbf{u}, \mathbf{v})_{\mathbf{H}(\mathrm{div},\tilde{\Gamma})} = 0 \qquad & \forall\, \mathbf{v} \in \mathbf{X}_{hp}(\tilde{\Gamma}) \cap \mathbf{H}_0(\mathrm{div}, \tilde{\Gamma}), \\ \mathcal{Q}_{hp}\mathbf{u} \cdot \tilde{\mathbf{n}} = \Pi^{\mathrm{div}}_{hp}\mathbf{u} \cdot \tilde{\mathbf{n}} \qquad & \text{on } \partial\tilde{\Gamma}. \end{aligned} \tag{4.1}$$

Here, $(\cdot,\cdot)_{\mathbf{H}(\mathrm{div},\tilde{\Gamma})}$ denotes the $\mathbf{H}(\mathrm{div}, \tilde{\Gamma})$-inner product, and $\tilde{\mathbf{n}}$ is the unit outward normal vector to $\partial\tilde{\Gamma}$. It follows immediately from (4.1) that

$$\|\mathbf{u} - \mathcal{Q}_{hp}\mathbf{u}\|_{\mathbf{H}(\mathrm{div},\tilde{\Gamma})} \le \|\mathbf{u} - \Pi^{\mathrm{div}}_{hp}\mathbf{u}\|_{\mathbf{H}(\mathrm{div},\tilde{\Gamma})}. \tag{4.2}$$

Lemma 4.1 *For* $r > 0$ *let* $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, \tilde{\Gamma})$. *Then there holds*

$$\|\mathbf{u} - \mathcal{Q}_{hp}\mathbf{u}\|_{\tilde{\mathbf{H}}^{-1/2}(\mathrm{div},\tilde{\Gamma})} \le C \left(\frac{h}{p}\right)^{1/2} \|\mathbf{u} - \Pi^{\mathrm{div}}_{hp}\mathbf{u}\|_{\mathbf{H}(\mathrm{div},\tilde{\Gamma})}, \tag{4.3}$$

where the constant C *is independent of* h *and* p.

Proof. We follow the technique used by Buffa and Christiansen in the proof of Proposition 4.6 in [11] but rely on the properties of the $\mathbf{H}(\mathrm{div})$-conforming projection based interpolation operator Π^{div}_{hp}. For given $r > 0$ and $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, \tilde{\Gamma})$ let us consider the following problem: find $\mathbf{u}_0 \in \mathbf{H}(\mathrm{div}, \tilde{\Gamma})$ such that

$$
\begin{aligned}
(\mathbf{u} - \mathbf{u}_0, \mathbf{v})_{\mathbf{H}(\mathrm{div},\tilde{\Gamma})} &= 0 \qquad && \forall\, \mathbf{v} \in \mathbf{H}_0(\mathrm{div}, \tilde{\Gamma}),\\
\mathbf{u}_0 \cdot \tilde{\mathbf{n}} &= \Pi^{\mathrm{div}}_{hp} \mathbf{u} \cdot \tilde{\mathbf{n}} \qquad && \text{on } \partial\tilde{\Gamma}.
\end{aligned}
$$

Then, as an immediate consequence of Lemma 4.8 in [11], there holds

$$
\|\mathbf{u} - \mathbf{u}_0\|_{\tilde{\mathbf{H}}^{s+1/2}(\mathrm{div},\tilde{\Gamma})} \le C\, \|(\mathbf{u} - \Pi^{\mathrm{div}}_{hp}\mathbf{u}) \cdot \tilde{\mathbf{n}}\|_{H^s(\partial\tilde{\Gamma})}, \quad s = -1,\ -\tfrac{1}{2}. \tag{4.4}
$$

By the triangle inequality one has

$$
\|\mathbf{u} - \mathcal{Q}_{hp}\mathbf{u}\|_{\tilde{\mathbf{H}}^{-1/2}(\mathrm{div},\tilde{\Gamma})} \le \|\mathbf{u} - \mathbf{u}_0\|_{\tilde{\mathbf{H}}^{-1/2}(\mathrm{div},\tilde{\Gamma})} + \|\mathbf{u}_0 - \mathcal{Q}_{hp}\mathbf{u}\|_{\tilde{\mathbf{H}}^{-1/2}(\mathrm{div},\tilde{\Gamma})}. \tag{4.5}
$$

Let us estimate each term on the right-hand side of (4.5). For the first term we have by using (4.4) with $s = -1$

$$
\|\mathbf{u} - \mathbf{u}_0\|_{\tilde{\mathbf{H}}^{-1/2}(\mathrm{div},\tilde{\Gamma})} \le C\, \|(\mathbf{u} - \Pi^{\mathrm{div}}_{hp}\mathbf{u}) \cdot \tilde{\mathbf{n}}\|_{H^{-1}(\partial\tilde{\Gamma})}. \tag{4.6}
$$

Denote $\mathbf{u}^{hp} := \Pi^{\mathrm{div}}_{hp}\mathbf{u}$. Since $\int_{\ell_h} (\mathbf{u} - \mathbf{u}^{hp}) \cdot \tilde{\mathbf{n}}\, d\sigma = 0$ for any mesh edge $\ell_h \subset \partial\tilde{\Gamma}$, we can apply the localisation result of Lemma 3.1 to estimate

$$
\|(\mathbf{u} - \mathbf{u}^{hp}) \cdot \tilde{\mathbf{n}}\|^2_{H^{-1}(\partial\tilde{\Gamma})} \le C \sum_{\ell_h \subset \partial\tilde{\Gamma}} \|(\mathbf{u} - \mathbf{u}^{hp}) \cdot \tilde{\mathbf{n}}\|^2_{\tilde{H}^{-1}_h(\ell_h)}. \tag{4.7}
$$

Let us fix an edge $\ell_h \subset \partial\tilde{\Gamma}$. Then there exists an element $K_h \subset \tilde{\Gamma}$ such that $\bar{\ell}_h = \partial\tilde{\Gamma} \cap \partial K_h$. This element K_h is the image of the reference element K under an affine mapping M, and let $\mathcal{M}$ be the corresponding Piola transform. Then using (3.1), (3.2) and the standard property of the Piola transform (see [10, Lemma 1.5]), we have

$$
\begin{aligned}
\|(\mathbf{u} - \mathbf{u}^{hp}) \cdot \tilde{\mathbf{n}}\|_{\tilde{H}^{-1}_h(\ell_h)} &= \sup_{0 \ne \varphi \in H^1_h(\ell_h)} \frac{\langle (\mathbf{u} - \mathbf{u}^{hp}) \cdot \tilde{\mathbf{n}}, \varphi \rangle_{0,\ell_h}}{\|\varphi\|_{H^1_h(\ell_h)}}\\
&\simeq \sup_{0 \ne \hat{\varphi} \in H^1(\hat{\ell})} \frac{\langle (\hat{\mathbf{u}} - \Pi^{\mathrm{div}}_p \hat{\mathbf{u}}) \cdot \hat{\mathbf{n}}, \hat{\varphi} \rangle_{0,\hat{\ell}}}{h^{-1/2}\, \|\hat{\varphi}\|_{H^1(\hat{\ell})}}\\
&= C\, h^{1/2}\, \|(\hat{\mathbf{u}} - \Pi^{\mathrm{div}}_p \hat{\mathbf{u}}) \cdot \hat{\mathbf{n}}\|_{\tilde{H}^{-1}(\hat{\ell})},
\end{aligned}
$$

where $\hat{\ell} = M^{-1}(\ell_h) \subset \partial K$, $\hat{\mathbf{u}} = \mathcal{M}^{-1}(\mathbf{u}|_{K_h})$, $\hat{\varphi} = \varphi \circ M$, and $\hat{\mathbf{n}}$ denotes the unit outward normal vector to ∂K. Hence, applying Lemma 3.3 and using standard scaling properties of the Piola transform (see [10, Lemma 1.6]), we estimate

$$
\|(\mathbf{u} - \mathbf{u}^{hp}) \cdot \tilde{\mathbf{n}}\|_{\tilde{H}^{-1}_h(\ell_h)} \le C \left(\frac{h}{p}\right)^{1/2} \|\hat{\mathbf{u}} - \Pi^{\mathrm{div}}_p \hat{\mathbf{u}}\|_{\mathbf{H}(\mathrm{div},K)} \le C \left(\frac{h}{p}\right)^{1/2} \|\mathbf{u} - \mathbf{u}^{hp}\|_{\mathbf{H}(\mathrm{div},K_h)}. \tag{4.8}
$$

Combining inequalities (4.8) over all mesh edges $\ell_h \subset \partial\tilde{\Gamma}$ and recalling the notation for $\mathbf{u}^{hp}$, we deduce from (4.6) and (4.7):

$$\|\mathbf{u} - \mathbf{u}_0\|_{\tilde{\mathbf{H}}^{-1/2}(\mathrm{div},\tilde{\Gamma})} \le C \left(\frac{h}{p}\right)^{1/2} \|\mathbf{u} - \Pi_{hp}^{\mathrm{div}}\mathbf{u}\|_{\mathbf{H}(\mathrm{div},\tilde{\Gamma})}. \tag{4.9}$$

Now we focus on the second term on the right-hand side of (4.5) and prove that

$$\|\mathbf{u}_0 - \mathcal{Q}_{hp}\mathbf{u}\|_{\tilde{\mathbf{H}}^{-1/2}(\mathrm{div},\tilde{\Gamma})} \le C \left(\frac{h}{p}\right)^{1/2} \|\mathbf{u} - \Pi_{hp}^{\mathrm{div}}\mathbf{u}\|_{\mathbf{H}(\mathrm{div},\tilde{\Gamma})}. \tag{4.10}$$

Denote $\mathbf{X}_{\tilde{\Gamma}} := \tilde{\mathbf{H}}_0^{-1/2}(\mathrm{div},\tilde{\Gamma})$ and $\mathbf{H}_0^{1/2}(\mathrm{div},\tilde{\Gamma}) := \mathbf{H}^{1/2}(\mathrm{div},\tilde{\Gamma}) \cap \mathbf{H}_0(\mathrm{div},\tilde{\Gamma})$. Let $\mathbf{X}'_{\tilde{\Gamma}}$ be the dual space of $\mathbf{X}_{\tilde{\Gamma}}$. Then, by Lemma 4.7 in [11], the operator $I - \nabla(\mathrm{div}) : \mathbf{H}_0^{1/2}(\mathrm{div},\tilde{\Gamma}) \to \mathbf{X}'_{\tilde{\Gamma}}$ is an isomorphism. Moreover, it is easy to see that $\mathcal{Q}_{hp}\mathbf{u}_0 = \mathcal{Q}_{hp}\mathbf{u}$ and $\mathbf{u}_0 - \mathcal{Q}_{hp}\mathbf{u}_0 \in \mathbf{H}_0(\mathrm{div},\tilde{\Gamma}) \subset \mathbf{X}_{\tilde{\Gamma}}$. Therefore, we can estimate as follows:

$$\begin{aligned}
\|\mathbf{u}_0 - \mathcal{Q}_{hp}\mathbf{u}\|_{\tilde{\mathbf{H}}^{-1/2}(\mathrm{div},\tilde{\Gamma})} &= \|\mathbf{u}_0 - \mathcal{Q}_{hp}\mathbf{u}_0\|_{\tilde{\mathbf{H}}^{-1/2}(\mathrm{div},\tilde{\Gamma})} \\
&\le C \sup_{\mathbf{0}\ne\mathbf{v}\in\mathbf{X}'_{\tilde{\Gamma}}} \frac{\langle \mathbf{v}, \mathbf{u}_0 - \mathcal{Q}_{hp}\mathbf{u}_0\rangle_{\mathbf{X}'_{\tilde{\Gamma}},\mathbf{X}_{\tilde{\Gamma}}}}{\|\mathbf{v}\|_{\mathbf{X}'_{\tilde{\Gamma}}}} \\
&\le C \sup_{\mathbf{0}\ne\mathbf{w}\in\mathbf{H}_0^{1/2}(\mathrm{div},\tilde{\Gamma})} \frac{\langle \mathbf{w} - \nabla(\mathrm{div}\,\mathbf{w}), \mathbf{u}_0 - \mathcal{Q}_{hp}\mathbf{u}_0\rangle_{\mathbf{X}'_{\tilde{\Gamma}},\mathbf{X}_{\tilde{\Gamma}}}}{\|\mathbf{w}\|_{\mathbf{H}^{1/2}(\mathrm{div},\tilde{\Gamma})}} \\
&= C \sup_{\mathbf{0}\ne\mathbf{w}\in\mathbf{H}_0^{1/2}(\mathrm{div},\tilde{\Gamma})} \frac{(\mathbf{u}_0 - \mathcal{Q}_{hp}\mathbf{u}_0, \mathbf{w})_{\mathbf{H}(\mathrm{div},\tilde{\Gamma})}}{\|\mathbf{w}\|_{\mathbf{H}^{1/2}(\mathrm{div},\tilde{\Gamma})}} \\
&= C \sup_{\mathbf{0}\ne\mathbf{w}\in\mathbf{H}_0^{1/2}(\mathrm{div},\tilde{\Gamma})} \frac{(\mathbf{u}_0 - \mathcal{Q}_{hp}\mathbf{u}_0, \mathbf{w} - \Pi_{hp}^{\mathrm{div}}\mathbf{w})_{\mathbf{H}(\mathrm{div},\tilde{\Gamma})}}{\|\mathbf{w}\|_{\mathbf{H}^{1/2}(\mathrm{div},\tilde{\Gamma})}};
\end{aligned}$$

for the last step we used the definition of $\mathcal{Q}_{hp}$ (see (4.1) with $\mathbf{v} = \Pi_{hp}^{\mathrm{div}}\mathbf{w}$). Hence, using the Cauchy-Schwarz inequality and the interpolation error estimate of Theorem 3.1 (with $\Gamma = \tilde{\Gamma}$ and $r = \frac{1}{2}$) we prove that

$$\|\mathbf{u}_0 - \mathcal{Q}_{hp}\mathbf{u}\|_{\tilde{\mathbf{H}}^{-1/2}(\mathrm{div},\tilde{\Gamma})} \le C \left(\frac{h}{p}\right)^{1/2} \|\mathbf{u}_0 - \mathcal{Q}_{hp}\mathbf{u}\|_{\mathbf{H}(\mathrm{div},\tilde{\Gamma})}. \tag{4.11}$$

The norm on the right-hand side of (4.11) is estimated by applying the triangle inequality, (4.4) with $s = -\frac{1}{2}$, (4.2), and the continuity property of the normal trace operator:

$$\begin{aligned}
\|\mathbf{u}_0 - \mathcal{Q}_{hp}\mathbf{u}\|_{\mathbf{H}(\mathrm{div},\tilde{\Gamma})} &\le \|\mathbf{u}_0 - \mathbf{u}\|_{\mathbf{H}(\mathrm{div},\tilde{\Gamma})} + \|\mathbf{u} - \mathcal{Q}_{hp}\mathbf{u}\|_{\mathbf{H}(\mathrm{div},\tilde{\Gamma})} \\
&\le C\,\|(\mathbf{u} - \Pi_{hp}^{\mathrm{div}}\mathbf{u})\cdot\tilde{\mathbf{n}}\|_{H^{-1/2}(\partial\tilde{\Gamma})} + \|\mathbf{u} - \Pi_{hp}^{\mathrm{div}}\mathbf{u}\|_{\mathbf{H}(\mathrm{div},\tilde{\Gamma})} \\
&\le C\,\|\mathbf{u} - \Pi_{hp}^{\mathrm{div}}\mathbf{u}\|_{\mathbf{H}(\mathrm{div},\tilde{\Gamma})}.
\end{aligned} \tag{4.12}$$

The desired inequality in (4.10) then follows from (4.11) and (4.12).

To obtain (4.3) it remains to collect (4.9) and (4.10) in (4.5). This finishes the proof. $\square$

Now we can prove the main result of this section.

Theorem 4.1 *Let $P_{hp} : \mathbf{X} \to \mathbf{X}_{hp}$ be the orthogonal projection with respect to the norm in $\mathbf{X}$. If $\mathbf{u} \in \mathbf{X}^r$ with $r > -\frac{1}{2}$, then*

$$\|\mathbf{u} - P_{hp}\mathbf{u}\|_{\mathbf{X}} \le C\, h^{1/2+\min\{r,p\}}\, p^{-(r+1/2)}\, \|\mathbf{u}\|_{\mathbf{X}^r} \tag{4.13}$$

with a positive constant C independent of h, p, and $\mathbf{u}$.

Proof. First, let us assume that $r > 0$. We consider the discrete vector field $\mathbf{v} \in \mathbf{X}_{hp}$ such that $\mathbf{v}|_{\Gamma^{(i)}} = \mathcal{Q}_{hp}^{(i)}(\mathbf{u}|_{\Gamma^{(i)}})$ for each face $\Gamma^{(i)}$ of Γ (here, $\mathcal{Q}_{hp}^{(i)}$ denotes the operator defined as in (4.1) with respect to the face $\Gamma^{(i)}$). Due to the localisation properties of the norms in $\mathbf{X}$ and in $\mathbf{H}(\mathrm{div}_\Gamma, \Gamma)$, we have by Lemma 4.1

$$\begin{aligned}\|\mathbf{u} - P_{hp}\mathbf{u}\|_{\mathbf{X}} &\le \|\mathbf{u} - \mathbf{v}\|_{\mathbf{X}} \le C \sum_{i=1}^{\mathcal{I}} \|\mathbf{u}|_{\Gamma^{(i)}} - \mathcal{Q}_{hp}^{(i)}(\mathbf{u}|_{\Gamma^{(i)}})\|_{\tilde{\mathbf{H}}^{-1/2}(\mathrm{div}_{\Gamma^{(i)}}, \Gamma^{(i)})} \\ &\le C \left(\frac{h}{p}\right)^{1/2} \|\mathbf{u} - \Pi_{hp}^{\mathrm{div}}\mathbf{u}\|_{\mathbf{H}(\mathrm{div}_\Gamma, \Gamma)},\end{aligned}$$

and inequality (4.13) then follows from the error estimate of Theorem 3.1.

Now, let $r \in (-\frac{1}{2}, 0]$. Assume that $\mathbf{u} \in \mathbf{X}^s$ with some $s \in (0, s_0)$, where $s_0 \in (0, \frac{1}{2}]$ is the same as in Lemma 3.2. Then, using the first part of the proof, one has

$$\|\mathbf{u} - P_{hp}\mathbf{u}\|_{\mathbf{X}} \le C \left(\frac{h}{p}\right)^{1/2+s} \|\mathbf{u}\|_{\mathbf{X}^s}.$$

On the other hand, it is trivial that

$$\|\mathbf{u} - P_{hp}\mathbf{u}\|_{\mathbf{X}} \le \|\mathbf{u}\|_{\mathbf{X}}.$$

Therefore, applying the interpolation argument which relies on Lemma 3.2, we prove

$$\|\mathbf{u} - P_{hp}\mathbf{u}\|_{\mathbf{X}} \le C \left(\frac{h}{p}\right)^{1/2+r} \|\mathbf{u}\|_{\mathbf{X}^r} \quad \forall\, \mathbf{u} \in \mathbf{X}^s.$$

This estimate yields (4.13) due to the density of regular functions in $\mathbf{X}^r$, and the proof is finished. $\square$

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References

[1] I. Babuška and M. Suri, *The optimal convergence rate of the p-version of the finite element method*, SIAM J. Numer. Anal., 24 (1987), pp. 750–776.

[2] J. Bergh and J. Löfström, *Interpolation Spaces*, no. 223 in Grundlehren der mathematischen Wissenschaften, Springer-Verlag, Berlin, 1976.

[3] A. Bespalov and N. Heuer, *The hp-version of the boundary element method with quasi-uniform meshes for weakly singular operators on surfaces*, IMA J. Numer. Anal. DOI:10.1093/imanum/drn052.

[4] ——, *Natural p-BEM for the electric field integral equation on screens*, IMA J. Numer. Anal. DOI:10.1093/imanum/drn072.

[5] ——, *The hp-version of the boundary element method with quasi-uniform meshes in three dimensions*, ESAIM Math. Model. Numer. Anal., 42 (2008), pp. 821–849.

[6] ——, *On the convergence of the hp-BEM with quasi-uniform meshes for the electric field integral equation on polyhedral surfaces*, Preprint arXiv:0810.3590v1 [math.NA], 2008. http://arxiv.org/abs/0810.3590.

[7] ——, *Optimal error estimation for* $\mathbf{H}$*(curl)-conforming p-interpolation in two dimensions*, Preprint arXiv:0903.4453v1 [math.NA], 2009. http://arxiv.org/abs/0903.4453.

[8] D. Boffi, M. Costabel, M. Dauge, and L. Demkowicz, *Discrete compactness for the hp version of rectangular edge finite elements*, SIAM J. Numer. Anal., 44 (2006), pp. 979–1004.

[9] D. Boffi, L. Demkowicz, and M. Costabel, *Discrete compactness for the p and hp 2D edge finite elements*, Math. Models Methods Appl. Sci., 13 (2003), pp. 1673–1687.

[10] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, no. 15 in Springer Series in Computational Mathematics, Springer-Verlag, New York, 1991.

[11] A. Buffa and S. H. Christiansen, *The electric field integral equation on Lipschitz screens: definitions and numerical approximation*, Numer. Mat., 94 (2003), pp. 229–267.

[12] A. Buffa and P. Ciarlet, Jr., *On traces for functional spaces related to Maxwell's equations, Part I: An integration by parts formula in Lipschitz polyhedra*, Math. Methods Appl. Sci., 24 (2001), pp. 9–30.

[13] ——, *On traces for functional spaces related to Maxwell's equations, Part II: Hodge decompositions on the boundary of Lipschitz polyhedra and applications*, Math. Methods Appl. Sci., 24 (2001), pp. 31–48.

[14] A. Buffa, M. Costabel, and D. Sheen, *On traces for H(curl, Ω) in Lipschitz domains*, J. Math. Anal. Appl., 276 (2002), pp. 845–867.

[15] A. Buffa and R. Hiptmair, *Galerkin boundary element methods for electromagnetic scattering*, in Topics in computational wave propagation, vol. 31 of Lect. Notes Comput. Sci. Eng., Springer, Berlin, 2003, pp. 83–124.

[16] M. Costabel, *Boundary integral operators on Lipschitz domains: Elementary results*, SIAM J. Math. Anal., 19 (1988), pp. 613–626.

[17] M. Costabel and M. Dauge, *Singularities of electromagnetic fields in polyhedral domains*, Arch. Rational Mech. Anal., 151 (2000), pp. 221–276.

[18] M. Dauge, *Elliptic boundary value problems on corner domains*, vol. 1341 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, Heidelberg, New-York, 1988.

[19] L. Demkowicz and I. Babuška, *p interpolation error estimates for edge finite elements of variable order in two dimensions*, SIAM J. Numer. Anal., 41 (2003), pp. 1195–1208.

[20] V. J. Ervin and N. Heuer, *An adaptive boundary element method for the exterior Stokes problem in three dimensions*, IMA J. Numer. Anal., 26 (2006), pp. 297–325.

[21] P. Grisvard, *Singularities in Boundary Value Problems*, no. 22 in Research Notes in Applied Mathematics, Masson, Paris, 1992.

[22] M. Hahne and E. P. Stephan, *Schwarz iterations for the efficient solution of screen problems with boundary elements*, Computing, 55 (1996), pp. 61–85.

[23] N. Heuer, M. Maischak, and E. P. Stephan, *Exponential convergence of the hp-version for the boundary element method on open surfaces*, Numer. Math., 83 (1999), pp. 641–666.

[24] R. Hiptmair, *Discrete compactness for the p-version of tetrahedral edge elements*, Report 2008-31, Seminar for Applied Mathematics, ETH Zürich, Switzerland, 2008. http://arxiv.org/abs/0901.0761.

[25] R. Hiptmair and C. Schwab, *Natural boundary element methods for the electric field integral equation on polyhedra*, SIAM J. Numer. Anal., 40 (2002), pp. 66–86.

[26] J.-L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications I*, Springer-Verlag, New York, 1972.

[27] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, 2000.

[28] R. E. Roberts and J.-M. Thomas, *Mixed and hybrid methods*, in Handbook of Numerical Analysis. Vol. II, P. G. Ciarlet and J.-L. Lions, eds., Amsterdam, 1991, North-Holland, pp. 523–639.