Non-Abelian group quantization and quantum kinematic invariants of some noncompact Lie groups

J. Krause
Facultad de Física, Pontificia Universidad Católica de Chile, Casilla 6177, Santiago 22, Chile

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The formalism of non-Abelian group quantization is briefly revisited, within the regular representation of noncompact Lie groups. It is shown that some of such r-dimensional groups have a set of r basic quantum-kinematic invariants, which substantially differ from the traditional invariants. The relation of the traditional invariants of the Lie algebra with the new quantum-kinematic invariants is also briefly examined. This paper contains two miscellaneous examples of quantum-kinematic invariant operators.

I. INTRODUCTION

One of the most fundamental problems in the representation theory of Lie groups (or Lie algebras) is the determination of the invariant operators of a group.[1] The purpose of this paper is to bring this fundamental problem under a new perspective. In fact, here we shall not follow the traditional approach to this subject;[2] instead, to this end, in this paper we adopt the new standpoint offered by the *group quantization* formalism of non-Abelian quantum kinematics, which we have recently introduced in the literature.[3-5] (The concept of "quantizing a Lie group" means that all its essential parameters are treated as q numbers. This concept will be analyzed in the sequel.)

The invariant operators of Lie group theory have been traditionally obtained from the exclusive viewpoint of the Lie algebra, since they have been defined hitherto as those functions of the generators that commute with all the generators of a given representation. Henceforth, for the sake of briefness, we shall refer to this familiar notion of invariant operators as the *traditional invariants* of Lie group theory. In this sense, we have to point out that the set of invariants considered in the present line of research (i.e., *quantum-kinematic invariants*) are of a conceptually new type, because they are not only invariant functions that depend exclusively on the generators, and because the new set of invariants contains all the traditional invariants as a proper subset of a rather special kind. Thus the quantum-kinematic theory of invariants (which is introduced in this paper) entails an interesting generalization of the traditional theory.

The importance of Lie group invariant operators is well known, both from the mathematical point of view, as well as for their physical applications.[6] The problem of their explicit construction for semisimple Lie algebras was first considered by Casimir,[7] and the traditional invariants of semisimple Lie groups were determined long ago.[8] On the contrary, traditional invariants of nonsemisimple groups have been determined only for a small number of cases, and their physical meaning have been also discussed in the literature.[9]

Maybe the best method for obtaining the traditional invariants of a Lie group consists in using the *associate adjoint realization* of the corresponding Lie algebra,[10] which reduces the problem to that of solving a system of linear homogeneous first-order partial differential equations [cf. Eq. (4.3), below]. As is well known, the solutions of these equations lead to the most general set of traditional invariants of the group.[11] This method is completely general indeed. However, the results obtained in this way are both remarkably successful and faintly distressing because, for some Lie groups, these equations have no polynomial solution (and the Lie algebra has no Casimir operator, *sensu stricto*); for other Lie groups, the equations have only transcendental solutions (yielding invariant operators that do not belong to the enveloping algebra). Moreover, it also happens that for some Lie groups the equations provided by the associate adjoint Lie algebra have no solution (and therefore the group has no traditional invariant at all).

As we shall see presently, one arrives at completely different results if one uses the "group quantization" method, for then it turns out that *every* r-dimensional Lie group has a *set of r basic quantum-kinematic operators that commute with all the generators* of the group. Moreover, it can be proven that once a Lie group has been "quantized," its basic kinematic invariant operators arise in a rather natural manner (even in those extreme cases where the group has no traditional invariant at all). In this paper we prove this fact for a special kind of noncompact Lie groups. Although this feature is valid also for other kinds of Lie groups, quantum kinematics of Lie groups, in general, sets a rather difficult issue. (We postpone the consideration of the general formalism of quantum kinematics to some forthcoming papers.)

The organization of this paper is as follows. Section II contains a brief review of group quantization and presents some features of non-Abelian quantum kinematics of noncompact Lie groups of a special kind. The basic quantum-kinematic invariants are introduced in Sec. III, and some of their properties are examined. The relation of the traditional Lie algebra invariant theory with the new kind of quantum-kinematic invariants is next discussed in Sec. IV. In Sec. V we present two rather simple (albeit interesting) instances of quantum kinematic invariant operators. Finally, Sec. VI contains some concluding remarks. This paper includes an Appendix that serves the purpose of introducing our notation and some required basic concepts.

II. GROUP QUANTIZATION AND NON-ABELIAN QUANTUM KINEMATICS

We begin this work with a brief review of the main concepts leading to group quantization and non-Abelian quan-

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tum kinematics of noncompact Lie groups. It is our intention to describe here only those features that are directly relevant for the discussion of the quantum kinematic invariants. In particular, here we shall *not* dwell on the physical meaning of the formalism.[4,5]

Henceforth, G denotes a noncompact, connected and simply connected, r-dimensional Lie group (as, for instance, the universal covering group of a noncompact finite-dimensional Lie group). Furthermore, we shall also *assume* that there exists a coordinate patch $q = (q^1,...,q^r)$, which covers the *whole* group manifold $M(G)$ and mantains everywhere a one-to-one correspondence with the elements of G; i.e., the coordinates q^a, $a = 1,...,r$, are real and provide a set of r essential parameters of G. This is a strong condition, to be sure. However, as a matter of fact, this assumption holds for a large class of Lie groups that are physically relevant. Indeed, most Lie groups of physical interest are of a type known as "linear Lie group," in the sense that they have at least one faithful finite-dimensional representation.[12] It can be shown that the *whole* of a connected linear Lie group of dimension r can be parametrized by r real numbers $q^1,...,q^r$, which form a connected set in $\mathscr{R}^r$. (Let us recall that a linear Lie group is said to be "connected" if it possesses only one connected component.) Of course, there is *no* requirement in general that this *global parametrization* of G be faithful. Nevertheless, there are many instances of noncompact, connected and simply connected linear Lie groups (of physical relevance) for which the global parametrization provides a one-to-one faithful mapping.[12] For the sake of simplicity, and in order to concentrate ourselves on the issue of *existence* of a *new* kind of quantum kinematic invariants, in this paper we shall deal exclusively with Lie groups that satisfy this strong condition.

In the sequel we shall write $\bar{q} = \bar{q}(q)$ to denote that point in $M(G)$ which labels the inverse element of the element corresponding to q, and $e = (e^1,...,e^r) \in M(G)$ to denote the labels of the identity element. Of course, $M(G)$ carries an analytic mapping, g: $M(G) \times M(G) \to M(G)$, that is endowed with the group property of G. Hence, in this parametrization of G one has a well-defined set of r group-multiplication functions, $g^a(q';q) = q''^a \in M(G)$, which realize the group law in $M(G)$.[13]

Now, in order to "quantize" the group G we have to associate its essential parameters q^a with a set of commuting Hermitian operators Q^a, which act within the carrier space of a relevant representation and may be interpreted as generalized "position" operators of the group manifold.[3] It is clear that the best (if not the only) way of achieving this endeavor is by means of the regular representation of G. Hence, let us consider the quantum-kinematic formalism obtained from both (the *left* and *right*) *regular representations* of G (cf. the Appendix of this paper for a unified formalism of the two regular representations that shall be assumed as theoretical framework in what follows).

Within the common Hilbert space $H(G)$ that carries both regular representations [cf. the Appendix, Eq. (A9)], we next "quantize" the group in the standard way. That is, we define the following spectral integrals over the group manifold:

$$Q^a = \int d^rq|q\rangle q^a\langle q| = \int d\mu_L(q)|q\rangle_L q^a\langle q|_L = \int d\mu_R(q)|q\rangle_R q^a\langle q|_R \qquad (2.1)$$

[cf. Eqs. (A5) and (A8)]; i.e., we set $Q^a = Q^a_L = Q^a_R$. Certainly, the Q's are *generalized position operators* of $M(G)$, acting in $H(G)$; in fact, one has

$$Q^a|q\rangle_L = q^a|q\rangle_L, \quad Q^a|q\rangle_R = q^a|q\rangle_R, \qquad (2.2)$$

and moreover

$$[Q^a,Q^b] = O, \quad a,b = 1,...,r. \qquad (2.3)$$

Hence, the Q's provide a complete set of commuting Hermitian operators in $H(G)$.

Of course, the generators L_a and R_a of the regular representations of G are Hermitian operators in their own right [cf. Eq. (A14)], which are given by

$$U_L(e + \delta q) = I - (i/\hbar)\delta q^a L_a, \qquad (2.4)$$

$$U_R(e + \delta q) = I - (i/\hbar)\delta q^a R_a. \qquad (2.5)$$

(The constant $\hbar$ will have no relevance in this paper, and is kept here only to recall the familiar and suggestive expressions of quantum mechanics.) Therefore, in the "Q representation" of quantum kinematics these generators become realized as follows:

$${}_L\langle q|L_a|\psi\rangle = -i\hbar X_a(q)\psi_L(q), \qquad (2.6)$$

$${}_R\langle q|R_a|\psi\rangle = -i\hbar Y_a(q)\psi_R(q) \qquad (2.7)$$

[cf. Eqs. (A1) and (A12)]; or, for that matter, we can also write:

$$L_a|q\rangle_L = i\hbar X_a(q)|q\rangle_L, \quad R_a|q\rangle_R = i\hbar Y_a(q)|q\rangle_R. \qquad (2.8)$$

It is interesting to consider the active transformation laws that bring the operators Q^a, L_a, and R_a, from the "Schrödinger picture" into the "Heisenberg picture" of quantum kinematics. After some manipulations, one gets:

$$U^\dagger_L(q)Q^aU_L(q) = g^a(q;Q), \qquad (2.9)$$

$$U^\dagger_L(q)L_aU_L(q) = A^{\ b}_a(q)L_b, \qquad (2.10)$$

and

$$U^\dagger_R(q)Q^aU_R(q) = g^a(Q;q), \qquad (2.11)$$

$$U^\dagger_R(q)R^aU_R(q) = \bar{A}^{\ a}_b(q)R_b, \qquad (2.12)$$

where $g^a(q;Q)$ and $g^a(Q;q)$ are defined by their spectral integrals, namely,

$$g^a(q;Q) = \int d\mu_L(q')|q'\rangle_L g^a(q;q')\langle q'|_L, \qquad (2.13)$$

$$g^a(Q;q) = \int d\mu_R(q')|q'\rangle_R g^a(q';q)\langle q'|_R, \qquad (2.14)$$

and where we have written

$$A^{\ b}_a(q) = R^{\ c}_a(q)\bar{L}^{\ b}_c(q), \quad \bar{A}^{\ b}_a(q) = L^{\ c}_a(q)\bar{R}^{\ b}_c(q). \qquad (2.15)$$

[As a matter of fact, these 'mixed' transport matrices in $M(G)$ correspond to the *adjoint representation* of G; for it can be shown that

$$A^{\ c}_a(q')A^{\ b}_c(q) = A^{\ b}_a[g(q';q)], \qquad (2.16)$$

and also

$$X_a(q)A^c_b(q)=f^d_{ab}A^c_d(q), \tag{2.17}$$

$$Y_a(q)A^c_b(q)=f^c_{ad}A^d_b(q), \tag{2.18}$$

wherefrom $A^c_{a,b}(e)=f^c_{ba}$ follows. Therefore, one has

$$A^b_a(e+\delta q)=\delta^b_a+\delta q^c f^b_{ca}, \tag{2.19}$$

as required.] Hence, Eqs. (2.10) and (2.12) mean that L_a and R_a transform as covariant vector operators with respect to the adjoint representation of G (which is well known, to be sure[2]). On the other hand, Eqs. (2.9) and (2.11) (although rather natural) to our best knowledge are new.[3] They state that the generalized position operators of the group transform covariantly upon the group law.

We are now ready to consider one of the most important results of the group quantization approach to quantum kinematics. In fact, if we evaluate the transformation laws (2.9)–(2.12) in a small neighborhood of the identity we get a set of new interesting *commutation relations*, which can be used to obtain the *quantum-kinematic algebras* of the formalism. Thus from Eqs. (2.9) and (2.10) we obtain:

$$[Q^a,L_b]=i\hbar R^a_b(Q), \tag{2.20}$$

$$[L_a,L_b]=-i\hbar f^c_{ab}L_c, \tag{2.21}$$

where

$$R^b_a(Q)=\int d\mu_L(q)|q\rangle_L R^b_a(q)\langle q|_L. \tag{2.22}$$

In the same way, from Eqs. (2.11) and (2.12), it follows:

$$[Q^a,R_b]=i\hbar L^a_b(Q), \tag{2.23}$$

$$[R_a,R_b]=i\hbar f^c_{ab}R_c, \tag{2.24}$$

with

$$L^a_b(Q)=\int d\mu_R(q)|q\rangle_R L^a_b(q)\langle q|_R. \tag{2.25}$$

Certainly, Eqs. (2.21) and (2.24) correspond to the Lie algebra obeyed by the generators [cf. Eq. (A2)]. On the other hand, the "canonical" commutation relations shown in Eqs. (2.20) and (2.23) are characteristic of a non-Abelian (noncompact) Lie group like G. They are interesting, for they correspond to *generalized Heisenberg commutation relations* that are obeyed by the position operators and the *non-Abelian momenta* represented by the generators.[3] Indeed, it is clear that if the group is Abelian (and the parameters are canonical), one recovers the conventional Heisenberg–Weyl algebra within the present formalism, for in that case (and only in that case) one has $L_a=R_a=P_a$, $L^a_b(q)=R^a_b(q)\equiv\delta^a_b$, and $f^c_{ab}=0$, for $a,b,c,=1,...,r$, and for all $q\in M(G)$. Hence, the commutators presented in Eqs. (2.20)–(2.25) entail a generalization of the Heisenberg commutation relations of ordinary quantum mechanics (for the case of a non-Abelian G), in a very natural fashion indeed. Such a result is, of course, of potential interest for physics.[3-5]

We next easily obtain the *quantum kinematic algebras* of G. In fact, from Eq. (2.20) one has

$$[F(Q),L_a]=i\hbar X_a(Q)F(Q), \tag{2.26}$$

where $X_a(q)F(q)=R^b_a(q)F_{,b}(q)$, and therefore Eq. (2.17) yields immediately:

$$[A^c_b(Q),L_a]=i\hbar f^d_{ab}A^c_d(Q). \tag{2.27}$$

In the same way, from Eqs. (2.23) and (2.18), one gets

$$[A^c_b(Q),R_a]=i\hbar f^c_{ad}A^d_b(Q). \tag{2.28}$$

Thus we see that the commutation relations (2.20)–(2.21) and (2.23)–(2.24) close separately to form two finite-dimensional algebras. Equations (2.21) and (2.27) define the left quantum kinematic algebra, while Eqs. (2.24) and (2.28) exhibit the right quantum kinematic algebra of G. It is clear what are exactly the generators (and the dimension) of these algebras. Of course, since

$$[A^b_a(Q),A^d_c(Q)]=0 \tag{2.29}$$

and

$$[L_a,R_b]=0, \tag{2.30}$$

for $a,b,c,d=1,...,r$ [cf. Eqs. (2.3) and (A3)], one has also a *larger* quantum kinematic algebra of G, which is given by Eqs. (2.21), (2.24), (2.27), (2.28), (2.29), and (2.30). The discussion of the structure of these algebras is not difficult; albeit, it shall be given elsewhere.

Finally, we observe that, from the "crossed" action of the representative operators [i.e., $U_L(q)|q'\rangle_R$ and $U_R(q)|q'\rangle_L$, as given in Eqs. (A16) and (A17)], it follows:

$$L_a|q\rangle_R=i\hbar[X_a(q)-\tfrac{1}{2}f^b_{ab}]|q\rangle_R, \tag{2.31}$$

$$R_a|q\rangle_L=i\hbar[Y_a(q)+\tfrac{1}{2}f^b_{ab}]|q\rangle_L, \tag{2.32}$$

which yield the realizations (i.e., the Q representation) of the left (right) generators in the right (left) regular representation of G, respectively. (Note that $f^b_{ab}=f^1_{a1}+\cdots+f^r_{ar}$.)

To end this short review, we would also like to remark that, from a mathematical point of view, the importance of non-Abelian quantum kinematics stems from the fact that the sets $\{Q^1,...,Q^r;\ L_1,...,L_r\}$ and $\{Q^1,...,Q^r;\ R_1,...,R_r\}$ [and not just the set of generators $\{L_1,...,L_r\}$, nor $\{R_1,...,R_r\}$] are the *irreducible sets* of Hermitian operators that characterize the carrier Hilbert space $H(G)$.[14]

III. QUANTUM-KINEMATIC INVARIANTS OF *G*

We now proceed to study those functions of the position operators and the generators that commute with all the generators of (one of) the regular representations of G. (For the sake of concreteness, here we choose the left regular representation.) It is clear that such functions correspond to the most general invariants of G.

First, we recall that the left and right representative operators commute:

$$U_L(q)U_R(q')=U_R(q')U_L(q), \tag{3.1}$$

for all $q,q'\in M(G)$. [In the present context, this fact can be shown by means of Eqs. (A10) and (A11).] Hence, a glance at Eqs. (A3), (2.8), and (2.15) recommend us to define the following set of r linearly independent operators in $H(G)$:

$$\hat{R}_a(Q;L)=\bar{A}^b_a(Q)L_b. \tag{3.2}$$

Indeed, by means of Eqs. (2.10) and (2.16), it can be shown

$$U_L^\dagger(q)\hat{R}_a(Q;L)U_L(q) = \hat{R}_a(Q;L); \tag{3.3}$$

or else, using Eqs. (2.17) and (2.23) one can prove that

$$[\hat{R}_a(Q;L),L_b] = 0, \quad a,b = 1,...,r, \tag{3.4}$$

holds, as required. Thus these are invariant operators of the left regular representation. Furthermore, in the Q representation (of the left quantum-kinematic formalism) these operators become realized as follows:

$$_L\langle q|\hat{R}_a(Q;L)|\psi\rangle = -i\hbar Y_a(q)\psi_L(q), \tag{3.5}$$

which also exhibits their invariant property [i.e., Eqs. (2.6) and (A3)].

However, the $\hat{R}$'s are not Hermitian, in general, since from Eqs. (2.17) and (2.20), one obtains $\hat{R}_a^\dagger = \hat{R}_a - i\hbar f^c_{bc}\bar{A}^b_a(Q)$. Certainly, this means that the operators $f^c_{bc}\bar{A}^b_a(Q)$ must be multiples of the identity [since they commute with all the operators of the irreducible set $\{Q;L\}$]. This is the case, indeed, because from the Jacobi identity obeyed by the structure constants it follows $f^d_{cd}f^c_{ab} = 0$ (for $a,b = 1,...,r$) and therefore, after some manipulations, Eq. (2.17) yields $f^c_{bc}\bar{A}^b_a(q) \equiv f^b_{ab}$. Hence, we can always define a set of r Hermitian invariant operators in the left regular representation of G, which are given by

$$R_a(Q;L) = \bar{A}^b_a(Q)L_b - \tfrac{1}{2}i\hbar f^b_{ab}. \tag{3.6}$$

Furthermore, using Eq. (3.5), it is an easy matter to obtain

$$R_a(Q;L)|q\rangle_L = i\hbar[Y_a(q) + \tfrac{1}{2}f^b_{ab}]|q\rangle_L, \tag{3.7}$$

which immediately identifies these operators as the *generators of the right regular representation* [recall Eq. (2.32)], acting as *invariant operators* within the left regular representation of G.

Of course, if one is interested in the right regular representation, one can invert Eq. (3.6) to read

$$L_a(Q;R) = A^b_a(Q)R_b + \tfrac{1}{2}i\hbar f^b_{ab}. \tag{3.8}$$

These are the generators of the left regular representation acting as invariant operators within the right regular representation.

Clearly, the clue of this nice feature lies in Eq. (A3), which is a well-known fact. However, we would like to underline that without recourse to group quantization it is not possible to take advantage of Eq. (A3), because in order to cast the right (left) generators as invariant operators of the left (right) regular representation, one needs to express them as functions of the left (right) generators and of the position operators of the group (which, thus, has to be quantized).

IV. TRADITIONAL INVARIANTS OF THE LIE ALGEBRA

According to the previous discussion, it is evident that a function $F(Q)$, which does depend only on the Q's, cannot yield an invariant operator of G. On the other hand, the traditional invariants of G are defined (within the left representation) as those functions $F(L)$ which do depend only on the generators, and which are such that

$$[F(L),L_a] = 0, \quad a = 1,...,r. \tag{4.1}$$

Hence, the interesting question arises concerning the possible relations between the quantum-kinematic invariants of G and its traditional invariant operators.

In order to tackle this problem let us assume that there is a function $F(\hat{R})$ with the following property

$$F(\bar{A}(Q)\cdot L) \equiv f(L); \tag{4.2}$$

i.e., F is such a function of the $\hat{R}$'s that its implicit dependence on the Q's cancels out. (It is clear that such function would correspond to a traditional invariant of G.) We then introduce an auxiliary space $X = \{x\}$, with coordinates $x = (x_1,...,x_r) \in X$, and we consider the following c-number functions: $\rho_a(q;x) = \bar{A}^b_a(q)x_b$, defined on $M(G)\times X$. We now demand for a function $F(\rho)$ such that $F[\bar{A}(q)\cdot x] = f(x)$; thus, we demand $\partial F(\rho)/\partial q^a \equiv 0$, for $a = 1,...,r$. If we evaluate this condition at the identity point $q = e$ of $M(G)$, after some straightforward steps, we obtain:

$$f^c_{ab}x_c\frac{\partial F(x)}{\partial x_b} = 0, \quad a = 1,...,r, \tag{4.3}$$

which we recognize as precisely the first-order partial differential equations of the associate adjoint realization of the Lie algebra, which one has to solve in order to obtain the traditional invariants of G.[11]

Conversely, if $F(x)$ is a solution to the system of partial differential equations given in (4.3) [and therefore $F(L)$ is a traditional invariant of the Lie algebra], let us introduce the change of variables $x_a \to \rho_a = \bar{A}^b_a(q)x_b$ into that equation; so we get, formally:

$$f^c_{ab}\rho_c\frac{\partial F(\rho)}{\partial \rho_b} \equiv f^c_{ab}\bar{A}^d_c(q)x_d F_{,b}(\bar{A}(q)\cdot x) = 0. \tag{4.4}$$

Then, taking into account that Eq. (2.18) can be cast in the equivalent form

$$Y_a(q)\bar{A}^c_b(q) = -f^d_{ab}\bar{A}^c_d(q), \tag{4.5}$$

from Eq. (4.4) we obtain

$$Y_a(q)F(\bar{A}(q)\cdot x) = 0, \tag{4.6}$$

which means $\partial F(\bar{A}(q)\cdot x)/\partial q^a = 0$, for $a = 1,...,r$, everywhere in $M(G)$; i.e., we get $F(\bar{A}(q)\cdot x) = f(x)$, and thus: $F(\hat{R}) = f(L)$.

Hence, let us epitomize: All the traditional invariants $F(L)$ of G in $H(G)$ correspond to a special kind of functions of the quantum kinematic invariants $\hat{R}_a$, for which $F(\hat{R}) = f(L)$ holds. (This result can be easily extended to the Hermitian R's). Thus we have proven that a noncompact Lie group like G has infinitely more invariant operators than those which have been traditionally considered hitherto, since every function $F(R)$ is certainly an invariant operator of the left regular representation and, furthermore, every traditional invariant function $f(L)$ of this representation is necessarily a function of the R's.

V. TWO MISCELLANEOUS EXAMPLES

With the aim of exhibiting the explicit form adopted by the quantum-kinematic invariants, in this section we present two instances, which also serve to illustrate some particular points of the previous formalism. Our examples are taken from elementary Lie group theory. For the sake of brevity, our discussion is very sketchy.

Let us consider some features of the left regular representation of the following noncompact Lie groups: (a) the affine group of the real line, and (b) the inhomogeneous restricted Lorentz group of two-dimensional Minkowski space-time. In the space alloted here, we content ourselves with those formal results of quantum kinematics that throw some light on the new invariants of these groups. Hence, besides some few remarks, we shall not delve into the complete solution of the outcoming "quantum models" and their physical interpretation.

A. The affine group of the real line

It is well known that only one algebraically indecomposable real Lie algebra of dimension $r = 2$ exists. The commutation rule is $[Y_1, Y_2] = Y_2$; it is solvable. It has *no traditional invariant*. [Indeed, the only solution to Eqs. (4.3), in this case, is $F = \text{const.}$] This Lie algebra corresponds to the affine group of the line $\{-\infty < t < +\infty\}$; namely,

$$t' = e^{q^1} t + q^2, \tag{5.1}$$

where the group manifold is given by $M = \{-\infty < q^a < +\infty; a = 1,2\}$, and the identity point is $e = (0,0)$. In fact, the group law reads

$$q''^1 = g^1(q';q) = q'^1 + q^1,$$
$$q''^2 = g^2(q';q) = q'^2 + e^{q'^1} q^2; \tag{5.2}$$

and therefore one obtains:

$$R_a^{\ b}(q) = \begin{bmatrix} 1 & q^2 \\ 0 & 1 \end{bmatrix}, \quad L_a^{\ b}(q) = \begin{bmatrix} 1 & 0 \\ 0 & e^{q^1} \end{bmatrix}, \tag{5.3}$$

(where "a" labels the raws and "b" labels the columns), wherefrom the Lie vector fields in M follow, i.e.,

$$X_1(q) = \partial_1 + q^2 \partial_2, \quad X_2(q) = \partial_2, \tag{5.4}$$

$$Y_1(q) = \partial_1, \quad Y_2(q) = e^{q^1} \partial_2. \tag{5.5}$$

Thus one gets the Lie algebra:

$$[X_1, X_2] = -X_2, \quad [Y_1, Y_2] = Y_2, \tag{5.6}$$

and moreover, one also has

$$[X_a, Y_b] = 0, \quad a,b = 1,2. \tag{5.7}$$

Hence, the fundamental quantum kinematic commutation relations of the affine group are given by

$$[L_1, L_2] = i\hbar L_2, \tag{5.8}$$

$$[Q^1, L_1] = i\hbar, \quad [Q^2, L_1] = i\hbar Q^2,$$
$$[Q^1, L_2] = 0, \quad [Q^2, L_2] = i\hbar, \tag{5.9}$$

and

$$[R_1, R_2] = -i\hbar R_2, \tag{5.10}$$

$$[Q^1, R_1] = i\hbar, \quad [Q^2, R_1] = 0,$$
$$[Q^1, R_2] = 0, \quad [Q^2, R_2] = i\hbar e^{Q^1}, \tag{5.11}$$

where we have defined

$$Q^a = \mu_0 \int\int dq^1\, dq^2 e^{-q^1} |q^1 q^2\rangle_L q^a \langle q^1 q^2|_L$$
$$= \mu_0 \int\int dq^1\, dq^2 |q^1 q^2\rangle_R q^a \langle q^1 q^2|_R, \tag{5.12}$$

for $a = 1,2$. (Note that this group is not unimodular.)

From Eqs. (5.3) one also gets the adjoint representation; i.e.,

$$A_a^{\ b}(q) = \begin{bmatrix} 1 & q^2 e^{-q^1} \\ 0 & e^{-q^1} \end{bmatrix},$$
$$\bar{A}_a^{\ b}(q) = \begin{bmatrix} 1 & -q^2 \\ 0 & e^{q^1} \end{bmatrix}, \tag{5.13}$$

[which matches well with the group law (5.2), as the reader can see]. Therefore, in the left regular representation, one gets immediately the following *two* basic quantum-kinematic Hermitian invariant operators:

$$R_1(Q;L) = L_1 - Q^2 L_2 + \tfrac{1}{2} i\hbar, \tag{5.14}$$

$$R_2(Q;L) = e^{Q^1} L_2, \tag{5.15}$$

which yield:

$${}_L\langle q^1 q^2 | R_1(Q;L) | \psi\rangle = -i\hbar(\partial_1 - \tfrac{1}{2}) \psi_L(q^1, q^2), \tag{5.16}$$

$${}_L\langle q^1 q^2 | R_2(Q;L) | \psi\rangle = -i\hbar e^{q^1} \partial_2 \psi_L(q^1, q^2). \tag{5.17}$$

These are Hermitian operators with respect to the left invariant Hurwitz measure, indeed. Furthermore, by means of (5.8)–(5.9) the reader can check that $[R_a(Q;L), L_b] = 0$ holds for $a,b = 1,2$. It is also a simple exercise to prove that, as a consequence of Eqs. (5.8) and (5.9), the operators defined in Eqs. (5.14) and (5.15) satisfy (5.10)–(5.11) (as it must be). Moreover, from Eqs. (5.13), one easily obtains the quantum kinematic algebras of this group.

Hence, here we have an example of a Lie group whose Lie algebra admits *no* traditional invariant, which, however, has *two* ($r = 2$) basic quantum-kinematic invariants.

B. The restricted Poincaré group in two-dimensional space-time

We next consider the group P_2 of Poincaré transformations in two-dimensional Minkowski space-time:

$$x'^0 = \gamma(q^2)(x^0 - q^2 x^1) + q^0,$$
$$x'^1 = \gamma(q^2)(x^0 - q^2 x^0) + q^1, \tag{5.18}$$

where $\gamma(q^2) = [1 - (q^2)^2]^{-1/2}$. (We set $c = 1$.) The group manifold is given by $M = \{-\infty < q^0 < +\infty, -\infty < q^1 < +\infty, -1 < q^2 < +1\}$, and $e = (0,0,0)$. The group law reads:

$$q''^0 = g^0(q';q) = q'^0 + \gamma(q'^2)(q^0 - q'^2 q^1),$$
$$q''^1 = g^1(q';q) = q'^1 + \gamma(q'^2)(q^1 - q'^2 q^0), \tag{5.19}$$
$$q''^0 = g^0(q';q) = (q'^2 + q^2)(1 + q'^2 q^2)^{-1}.$$

Hence, one has

$$R_a^{\ b}(q) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -q^1 & -q^0 & \gamma^{-2} \end{bmatrix},$$
$$L_a^{\ b}(q) = \begin{bmatrix} \gamma & -\gamma q^2 & 0 \\ -\gamma q^2 & \gamma & 0 \\ 0 & 0 & \gamma^{-2} \end{bmatrix}, \tag{5.20}$$

wherefrom one gets the generators in $M(P_2)$:

$$X_0 = \partial_0, \quad X_1 = \partial_1,$$
$$X_2 = -q^1 \partial_0 - q^0 \partial_1 + \gamma^{-2} \partial_2, \tag{5.21}$$

$$Y_0 = \gamma(\partial_0 - q^2\partial_1),$$
$$Y_1 = \gamma(\partial_1 - q^2\partial_0), \quad Y_2 = \gamma^{-2}\partial_2, \tag{5.22}$$
which satisfy the well-known Lie algebra:
$$[X_0,X_1] = 0, \quad [X_0,X_2] = -X_1, [X_1,X_2] = -X_0, \tag{5.23}$$
$$[Y_0,Y_1] = 0, \quad [Y_0,Y_2] = Y_1, \quad [Y_1,Y_2] = Y_0, \tag{5.24}$$
and
$$[X_a,Y_b] = 0, \quad a,b = 0,1,2. \tag{5.25}$$
The group P_2 is unimodular [i.e., $R(q) = L(q) = \gamma^{-2}$], so one defines the (left and right) Hurwitz measure:
$$d\mu(q) = \mu_0 \gamma^2(q^2) dq^0\, dq^1\, dq^2. \tag{5.26}$$
In this way, the position operators of P_2 are given by
$$Q^a = \mu_0 \int\int dq^0\, dq^1 \times \int_{-1}^{1} dq^2\, \gamma^2(q^2) |q^0q^1q^2\rangle q^a \langle q^0q^1q^2|, \tag{5.27}$$
for $a = 0,1,2$, where
$$\langle q'^0q'^1q'^2|q^0q^1q^2\rangle = \mu_0^{-1}\gamma^{-2}(q^2)\delta(q'^0 - q^0)\delta(q'^1 - q^1)\delta(q'^2 - q^2), \tag{5.28}$$
and where (in this particular case) we have defined $|q^0q^1q^2\rangle_L = |q^0q^1q^2\rangle_R = |q^0q^1q^2\rangle$, once and for all. In fact, the adjoint representation is defined by
$$\overline{A}{}_a^{\,b}(q) = \begin{bmatrix} \gamma & -\gamma q^2 & 0 \\ -\gamma q^2 & \gamma & 0 \\ q^1 & q^0 & 1 \end{bmatrix} = A_a^{\,b}(\bar{q}), \tag{5.29}$$
so that $A(q) = R(q)\overline{L}(q) = 1$ follows.

In this fashion, we easily obtain the Lie algebra of P_2, which reads:
$$[L_0,L_1] = 0, \quad [L_0,L_2] = i\hbar L_1, \quad [L_1,L_2] = i\hbar L_0, \tag{5.30}$$
and the generalized canonical commutators of the left quantum kinematics of P_2 are given by
$$\begin{aligned} &[Q^0,L_0] = i\hbar, \quad [Q^1,L_0] = 0, \quad [Q^2,L_0] = 0, \\ &[Q^0,L_1] = 0, \quad [Q^1,L_1] = i\hbar, \quad [Q^2,L_1] = 0, \\ &[Q^0,L_2] = -i\hbar Q^1, \quad [Q^1,L_2] = -i\hbar Q^0, \\ &[Q^2,L_2] = -\hbar\gamma^{-2}(Q^2). \end{aligned} \tag{5.31}$$
It is well known that the Lie algebra (5.30) of P_2 has just *one* invariant operator, which is given by the Casimir operator
$$S = L_0^2 - L_1^2. \tag{5.32}$$
Clearly, in the physical interpretation of the formalism, this invariant operator yields the two-dimensional Klein–Gordon equation; say,
$$\langle q^0q^1q^2|S|\psi_m\rangle = -\hbar^2(\partial_0^2 - \partial_1^2)\psi_m(q^0,q^1,q^2) = m^2\psi_m(q^0,q^1,q^2), \tag{5.33}$$
for $S|\psi_m\rangle = m^2|\psi_m\rangle$ [which corresponds to a superselection rule in $H(P_2)$]. The important point to remark is that one does not go too far with the quantum kinematic model for P_2 if one pays attention exclusively to its Lie algebra, since in this traditional approach one obtains the theory of the Klein–Gordon equation, and nothing else.

However, the complete realm of the quantum-kinematic theory of P_2 is much broader than that of the traditional theory, because by means of the adjoint representation (5.29) one obtains *three* basic kinematic invariants, instead of only *one*. In effect, within the left regular representation of P_2, these are given as follows:
$$R_0(Q;L) = \gamma(Q^2)(L_0 - Q^2L_1), \tag{5.34}$$
$$R_1(Q;L) = \gamma(Q^2)(L_1 - Q^2L_0), \tag{5.35}$$
$$R_2(Q;L) = Q^1L_0 + Q^0L_1 + L_2. \tag{5.36}$$
Therefore, introducing the "Q-representation" of quantum kinematics (i.e., settling a "wave mechanical" formalism on the group manifold of P_2), one gets the following realizations of these invariant operators:
$$\langle q^0q^1q^2|R_0(Q;L)|\psi\rangle = -i\hbar\gamma(q^2)(\partial_0 - q^2\partial_1)\psi(q^0,q^1,q^2), \tag{5.37}$$
$$\langle q^0q^1q^2|R_1(Q;L)|\psi\rangle = -i\hbar\gamma(q^2)(\partial_1 - q^2\partial_0)\psi(q^0,q^1,q^2), \tag{5.38}$$
$$\langle q^0q^1q^2|R_2(Q;L)|\psi\rangle = -i\hbar\gamma^{-2}(q^2)\partial_2\psi(q^0,q^1,q^2). \tag{5.39}$$
Of course, these operators satisfy the *right* Lie algebra of P_2. Moreover, one has
$$S = L_0^2 - L_1^2 \equiv R_0^2 - R_1^2. \tag{5.40}$$
Hence, one can "diagonalize" this scheme in the following two ways: (a) using the fact that $[R_0,R_1] = 0$, or else (b) using $[S,R_2] = 0$. One thus reduces the left regular representation of P_2 (by means of the corresponding superselection rules)[4,5] into "physically" meaningful Hilbert subspaces, which one hopes to interpret properly.

Interesting enough, the quantum kinematic invariants of P_2 are *first-order differential operators*, and furthermore the formalism of P_2 quantum-kinematics is automatically *relativistic*. This subject seems worthy of further discussion; work is in progress concerning this most important quantum kinematic model.

VI. CONCLUDING REMARKS

Quantum kinematic treatment of Lie groups was initiated by Weyl, many years ago.[15] Weyl's most outstanding achievement, concerning this issue, was his discussion of Heisenberg's kinematics as an Abelian group of unitary transformations. It is well known, however, that Weyl's quantization scheme, in general, is not flexible enough for the needs of physics,[16] because it only contains the fundamental commutation relations in an implicit fashion.[17] In fact, the quantum kinematic formalism sketched in this paper was suggested by Weyl's approach to the group of space translations, and constitutes a direct generalization thereof. As we have seen, group quantization affords explicit "canonical" commutation relations, and the quantum kinematic algebra, which can be evaluated systematically by purely group theoretic methods. It may also afford a new perspective for the discussion of the fundamental invariants of a Lie group, that is much broader than the traditional standpoint, and may have some huge physical meaning.

We deem these results important because the problem of quantizing a system that is primarily described by non-Abelian dynamical variables sets an intriguing question in quantum theory.[17] This problem appears in several areas of contemporary physics. For instance, it arises in the recent trend of elementary particle physics, where several SU(n) and other gauge groups play a fundamental role.[18] The same problem arises in the formulations of lattice gauge theories,[19] where quantum kinematics of non-Abelian variables is an indispensable device to circumvent conventional perturbation theory[18] in the search for the appropriate description of quark confinement.[20,21] Non-Abelian quantization sets also a fundamental problem in connection with some attempts to generalize the theory of minimum-uncertainty states to variables other than Cartesian.[22-24] (In this sense, it would be also interesting to try to use the group quantization technique, the quantum kinematic algebra and the set of basic kinematic invariants, within the recent Lie group theory of generalized coherent states.[25]) In fact, this seems to be an unsolved problem, and we think that the quantum kinematic approach may offer a way out to these important questions.

Finally, we would also like to remark that, in two previous papers, the quantum kinematic invariants of the Euclidean group of the plane,[4] and those of the Galilei group in two-dimensional space-time,[5] were used as superselection rules in order to reduce the (left) regular ray representation of these groups into physically meaningful Hilbert subspaces. In this way, the usual quantum model of the simple harmonic oscillator,[4] and that of the Galilean free particle,[5] respectively, were deduced by purely group-theoretic manipulations, based solely on the Newtonian symmetry groups of these systems. Let us here underline that the well-known *space-time propagators* of these systems were also obtained by means of Hurwitz-invariant integrals over the (respective) group manifolds, instead of path integrals. [Certainly, in those previous papers we have quantized the central extension of the corresponding groups by U(1). Nevertheless, the extension of the quantum-kinematic invariant formalism from the "true" (i.e., vector) regular representation to the "projective" (i.e., ray) regular representation is a straightforward matter.]

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APPENDIX: LEFT AND RIGHT REGULAR REPRESENTATIONS REVISITED

Of course, the regular representation of Lie groups is a well-known subject.[1] The only aim here is to introduce the notation used in this paper, presenting a unified formalism for the simultaneous description of both (the *left* and the *right*) regular representations, in a rather simple fashion. To this end, we shall use the standard Dirac notation (i.e., "kets" and "bras") of quantum mechanics. Here, G denotes an r-dimensional Lie group (endowed with the properties already introduced in Sec. II), and $M(G)$ denotes the group manifold.[26]

As is well known, one defines Lie's (*right* and *left*) vector fields as follows:

$$X_a(q) \equiv R_a^{\,b}(q)\partial_b, \quad Y_a(q) \equiv L_a^{\,b}(q)\partial_b, \tag{A1}$$

where $R_a^{\,b}$ and $L_a^{\,b}$ are the (right and left) *transport matrices* for contravariant vectors in $M(G)$ that are obtained from $g^a(q';q)$ in the usual "classical" fashion;[27] i.e., $R_a^{\,b}(q) = \partial'_a g^b(q';q)|_{q'=e}$, and $L_a^{\,b}(q) = \partial'_a g^b(q;q')|_{q'=e}$. In this paper we also need the inverse transport matrices in $M(G)$, which are defined by $\overline{R}_a^{\,b}(q) = \partial'_a g^b(q';\overline{q})|_{q'=q}$, and $\overline{L}_a^{\,b}(q) = \partial'_a g^b(\overline{q};q')|_{q'=q}$. [Clearly, one has: $R_a^{\,b}(e) = L_a^{\,b}(e) = \delta_a^b$, and $\overline{R}_a^{\,c}(q) R_c^{\,b}(q) = \overline{L}_a^{\,c}(q) L_c^{\,b}(q) = \delta_a^b$.] The Lie operators satisfy the Lie algebra:

$$[X_a(q), X_b(q)] = f_{ab}^c X_c(q),$$
$$[Y_a(q), Y_b(q)] = -f_{ab}^c Y_c(q), \tag{A2}$$

where the structure constants are given by $f_{ab}^c = R_{b,a}^{\,c}(e) - R_{a,b}^{\,c}(e) = L_{a,b}^{\,c}(e) - L_{b,a}^{\,c}(e)$. It is interesting to recall that

$$[X_a(q), Y_b(q)] = 0, \tag{A3}$$

$a,b = 1,\ldots,r$, holds for all $q \in M(G)$, since this is the cornerstone for building the quantum kinematic invariants of G (cf. Sec. III).

Now, as basic carrier space *of both* regular representations let us first introduce the Hilbert space $H(G) = L^2(G)$ for all complex-valued functions $\psi(q)$ defined in $M(G)$, and such that $\langle\psi|\psi\rangle < \infty$, with $d^rq = dq^1 \cdots dq^r$, and where the integration is over $M(G)$. We then consider a set of continuous kets $|q\rangle = |q^1,\ldots,q^r\rangle \in H(G)$ (*rigged*), which are in one-to-one correspondence with the points $q \in M(G)$, and constitute an orthogonal complete basis of $H(G)$; i.e.

$$\langle q|q'\rangle = \delta^{(r)}(q - q'), \tag{A4}$$

$$\int d^rq |q\rangle\langle q| = I, \tag{A5}$$

where I stands for the identity operator in $H(G)$. Thus we set $\psi(q) = \langle q|\psi\rangle$, for $|\psi\rangle \in H(G)$, as usual. Using this rigged Hilbert space, it is clear that the formalism becomes much simpler if one introduces the *invariant measures* on $M(G)$, instead of the coordinate volume element d^rq. So, let us define the following *ad hoc* kets in $H(G)$ (*rigged*):

$$|q\rangle_L = [\mu_0^{-1} L(q)]^{1/2} |q\rangle,$$
$$|q\rangle_R = [\mu_0^{-1} R(q)]^{1/2} |q\rangle, \tag{A6}$$

where $L(q)$ and $R(q)$ are the determinants of the corresponding transport matrices, and μ_0 is an arbitrary normalization constant. These yield

$$_L\langle q|q'\rangle_L = \mu_0^{-1} L(q) \delta^{(r)}(q - q'),$$

$$_R\langle q|q'\rangle_R = \mu_0^{-1} R(q)\delta^{(r)}(q-q') \tag{A7}$$

and

$$I = \int d\mu_L(q)|q\rangle_L\langle q|_L = \int d\mu_R(q)|q\rangle_R\langle q|_R. \tag{A8}$$

Here we have used the *Hurwitz invariant measures*[14] on $M(G)$; i.e. $d\mu_L(q) \equiv \mu_0 \overline{L}(q) d^r q$, and $d\mu_R(q) \equiv \mu_0 \overline{R}(q) d^r q$. (In order to simplify the notation, in this paper we assume $\mu_L = \mu_R = \mu_0$, but this choice is not strictly necessary.[28])

Thus we have

$$\langle\psi|\psi\rangle = \int d\mu_L(q)|\psi_L(q)|^2 = \int d\mu_R(q)|\psi_R(q)|^2, \tag{A9}$$

since, clearly, we define $\psi_L(q) = {}_L\langle q|\psi\rangle$ and $\psi_R(q) = {}_R\langle q|\psi\rangle$, for all $|\psi\rangle \in H(G)$. Hence, as an important remark, it follows (even if one takes $\mu_L \neq \mu_R$), that one and the same Hilbert space carries both (the left and the right) regular representations of G, because there is a one-to-one mapping $\psi_L(q) \leftrightarrow \psi_R(q)$, for all $|\psi\rangle \in H(G)$; namely: $\psi_R(q) = \sqrt{A(q)}\,\psi_L(q)$ where $A(q)$ is the determinant of the matrix of the adjoint representation of G [cf. Eqs. (2.15) and (2.19)].

One then introduces the *left* and *right* representative operators of the elements of G in $H(G)$. These can be defined as follows:

$$U_L(q) = \int d\mu_L(q')|g(q;q')\rangle_L\langle q'|_L, \tag{A10}$$

$$U_R(q) = \int d\mu_R(q')|g(q';q)\rangle_R\langle q'|_R, \tag{A11}$$

respectively. In this way, one has

$$U_L(q)|q'\rangle_L = |g(q;q')\rangle_L,$$
$$U_R(q)|q'\rangle_R = |g(q';q)\rangle_R. \tag{A12}$$

In fact, from these definitions one easily obtains:

$$U_L(q)U_L(q') = U_L[g(q;q')],$$
$$U_R(q)U_R(q') = U_R[g(q';q)] \tag{A13}$$

and

$$U_L^\dagger(q) = U_L(\bar{q}) = U_L^{-1}(q),$$
$$U_R^\dagger(q) = U_R(\bar{q}) = U_R^{-1}(q). \tag{A14}$$

Finally, it is evident that from Eqs. (A6) one gets

$$|q\rangle_R = [A(q)]^{1/2}|q\rangle_L, \quad |q\rangle_L = [\overline{A}(q)]^{1/2}|q\rangle_R, \tag{A15}$$

and thus one can readily calculate "mixed" expressions between the two regular representation formalisms, if needed. For instance, after some manipulations, one shows:

$$U_L(q)|q'\rangle_R = [\overline{A}(q)]^{1/2}|g(q;q')\rangle_R, \tag{A16}$$
$$U_R(q)|q'\rangle_L = [A(q)]^{1/2}|g(q';q)\rangle_L. \tag{A17}$$

[1] Cf., for instance, M. A. Naimark and A. I. Stern, *Theory of Group Representations* (Springer, New York, 1982).
[2] See, i.e., A. O. Barut and R. Raczka, *Theory of Group Representations and Applications* (PWN Polish Sci., Warzawa, 1977).
[3] J. Krause, J. Phys. A: Math. Gen. **18**, 1309 (1985).
[4] J. Krause, J. Math. Phys. **27**, 2922 (1986).
[5] J. Krause, J. Math. Phys. **29**, 393 (1988).
[6] Superselection rules and characterization of the states of a system, by means of the eigenvalues and eigenfunctions of invariant operators, have been known for a long time [E. P. Wigner, Ann. Math. **40**, 149 (1939)]. Also, it is well known that invariant operators of dynamical groups yield mass formulas [M. Gell-Mann, Phys. Rev. **125**, 1067 (1962), and S. Okubo, Progr. Theor. Phys. **16** 686 (1962)], energy spectra [V. Bargmann, Z. Phys. **99**, 576 (1936)], and other physical characterizations of several physical systems.
[7] H. Casimir, Proç. R. Acad. Amsterdam **34**, 844 (1931).
[8] G. Racah, Rend. Licei **8**, 108 (1950).
[9] Traditional invariants have been found, for instance, for the Poincaré group (Wigner, Ref. 6) and for the Galilei group [J. M. Levy-Leblond, in *Group Theory and its Applications*, edited by E. M. Loebl (Academic, New York, 1972)]. They have also been found for several groups containing the Poincaré group [P. Roman, J. J. Aghassi, and P. L. Huddleston, J. Math. Phys. **13**, 1852 (1972)], for some containing the Galilei group [L. Abellanos and L. M. Alonso, J. Math. Phys. **16**, 1580 (1975)], for the similitude groups of Minkowski space in four and three dimensions [J. Patera, P. Winternitz, and H. Zassenhaus, J. Math. Phys. **16**, 1615 (1975)], as well as for the $O(4,1)$ de Sitter group [J. Patera, P. Winternitz, and H. Zassenhaus, J. Math. Phys. **17**, 717 (1976], and probably for some few more nonsemisimple Lie groups.
[10] See, e.g., G. Rosen, *Formulations of Classical and Quantum Dynamical Theory* (Academic, New York, 1969).
[11] Cf., i.e., J. Patera, R. T. Sharp, P. Winternitz, and H. Zassenhaus, J. Math. Phys. **17**, 986 (1976), where this method was used to obtain the (traditional) invariants of all real Lie algebras of dimension $r \leqslant 5$, and of all real nilpotent algebras of dimension $r = 6$.
[12] See, for instance, J. E. Cornwell, *Group Theory in Physics* (Academic, London, 1984), 2 vols.
[13] The excellent lectures on Lie groups given by Giulio Racah, some 25 years ago, are enough for the understanding of this paper. See G. Racah, Ergeb. Exact. Naturwiss **37**, 28 (1965).
[14] See, i.e., L. Fonda and G. C. Ghirardi, *Symmetry Principles in Quantum Physics* (Marcel Dekker, New York, 1970).
[15] H. Weyl, *The Theory of Groups and Quantum Mechanics* (Dover, New York, 1931).
[16] I. Daubechies, J. Math. Phys. **24**, 1453 (1983).
[17] K. Yamada, Phys. Rev. D **25**, 3256 (1982).
[18] K. G. Wilson, Phys. Rev. D **10**, 2445 (1974).
[19] J. Kogut, Phys. Rep. **67**(1), 67 (1980).
[20] J. Kogut and L. Susskind, Phys. Rev. D **11**, 1128 (1977).
[21] M. Creutz, Phys. Rev. D **15**, 1128 (1977).
[22] P. Carruthers and M. M. Nieto, Rev. Mod. Phys. **40**, 411 (1968).
[23] M. M. Nieto, Phys. Rev. Lett. **18**, 182 (1967).
[24] M. M. Nieto *et al.*, Phys. Rev. D **23**, 927 (1981).
[25] A. Perelomov, *Generalized Coherent States and their Applications* (Springer-Verlag, Berlin, 1986).
[26] Cf., also, J. Krause, J. Math. Phys. **28**, 2268 (1987), for a summary of "non-Abelian calculus" on $M(G)$ which shall be needed in a later paper. (See, in particular, the Appendix of this reference.)
[27] L. P. Eisenhart, *Continuous Groups of Transformations* (Dover, New York, 1961).
[28] E. P. Wigner, *Group Theory* (Academic, New York, 1959).