Magnetic corrections to the pion electromagnetic form factor

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Abstract

In this work, we investigate the effect of a magnetic field background to the pion electromagnetic form factor. We face this problem through the Finite Energy Sum Rule (FESR) program, where a suitable current correlation function, built of Quantum Chromodynamics (QCD) degrees of freedom, is used to establish a map to the hadronic world and then extracting then the form factor. The magnetic field effects are encoded in the perturbative QCD side through the fermionic propagator in the presence of a magnetic field background, known as Schwinger propagator. We analyze the strong and weak magnetic field limits. For the weak field limit, the current correlator can be written as an expansion in powers of eB. We restricted the calculation to first order in eB leading to anomalous results which must be improved. However, for the strong field limit, we applied the Landau level expansion of the Schwinger propagator and consider up to the first Landau level leading to a proper FESR. The numerical results show that a strong magnetic field increases the pion form factor several times. For example, for a fixed magnetic field of $eB = 1$ GeV^2 the pion form factor can be four times larger. This result affects directly the electron-pion scattering cross section which is also connected to the Sullivan process, leading to potential effects of the magnetic field on collider experiments.

To my family

Contents

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Introduction

Large magnetic fields are created in nature, either by experiments conducted by us or by nature itself. In a non-central or peripheral heavy-ion collision a dominant magnetic field is generated in the direction perpendicular to the interaction plane. In the Large Hadron Collider (LHC) at the European Organization for Nuclear Research (CERN) the intensity of these magnetic fields can reach large values such as $eB \sim 15m_\pi^2 \sim 0.3$ GeV^2 and in the Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory (BNL), magnetic fields can be near $eB \sim m_\pi^2 \sim 0.02$ GeV^2 [1] [2], since their magnitudes depends mainly on the collision energy. Astrophysical objects, such as magnetars, possess natural conditions to generate large magnetic fields. For example, in anomalous X-ray pulsars and soft gamma-ray repeaters, we can encounter large magnetic fields reaching values up to $eB \sim 10^{18}$ $G \sim 0.02$ GeV^2 [3] [4]. If the lifetime of such magnetic fields is longer than the other relevant scales of the physical processes under study, we can treat them as a constant background field.

Besides, magnetic field backgrounds are known for their significant effects on observable phenomena such as magnetic catalysis or the chiral separation effect [5]. The former, for example, happens due to the dimensional reduction on the dynamics of the system leading to a spontaneous chiral symmetry breaking. Therefore, the features of an external magnetic field in quantum field theories have motivated the Quantum Chromodynamics (QCD) community, from lattice QCD to QCD Sum Rules (QCDSR), and also using the approach of effective lagrangians like Nambu-Jona-Lasinio model, linear and non-linear sigma models, etc, to study how this background field affects hadron physics. Vast progress has been made to understand the effect of the magnetic field on hadronic parameters, from the two fronts QCDSR and LQCD as, for example, the deconfinement phenomenological parameter s_0, magnetic field dependence of quark masses, the condensates and the pion decay constant f_π and finally the QCD phase diagram [6] [7] [8] [9] [10] [11]. But the list could go on.

This motivates us to the main focus of this thesis. The behaviour of the pion electromagnetic form factor F_π under the effects of a magnetic field background. We rely on the Finite Energy Sum Rules (FESR) approach [12] since the lowest dimensional sum rule concentrates on the high-energy region (perturbative region) of QCD and establishes a direct connection to the pion form factor. The magnetic contribution is given by the Schwinger propagator [13] introduced in a suitable three-point interpolating current correlation function, built of QCD degrees of freedom. This correlator via FESR allows one to connect the magnetic dependence on the perturbative QCD world with the hadronic parameter F_π. At first sight, there is no need to consider a magnetic dependence on the hadronic world but recent works have determined the appearance of new decay modes of the pion when a magnetic field is acting on the system [9] [14]. We found that for certain regions of finite magnetic field strength, the electromagnetic pion form factor is considerably larger than the pion form factor in absence of an external magnetic field. For example, for momentum transfer $Q^2 = 1$ GeV^2 and magnetic fields near $Q^2 \lesssim eB$ the pion form factor could reach up to four times the zero magnetic field form factor. This affects directly, for example, the pion-electron scattering cross section since it is proportional to the form factor.

This thesis is organized as follows. In Chapter 1 we give a short introduction to Quantum Chromodynamics (QCD). Then we explain the building blocks to understand the QCD Sum Rules program and we end the chapter with the standard calculation of the pion electromagnetic form factor in the vacuum, i.e. at zero magnetic field. In Chapter 2 we follow the same steps as Schwinger did to calculate the fermion propagator in a magnetic field background. In Chapter 3 we use the Schwinger propagator in the QCD three-point function which connects to the pion form factor. We take both, strong and weak magnetic field limits. In Chapter 4 we present the method to extract the pion form factor via Finite Energy Sum Rules with magnetic corrections and show the numerical results. Finally, we end with a brief discussion of the obtained results and give our conclusions.

Chapter 1

Quantum Chromodynamics and Sum Rules

1.1 Quantum Chromodynamics

The theory of strong interactions, known as Quantum Chromodynamics (QCD) [15] [16] [17], is a Yang-Mills quantum field theory with a local symmetry group $SU(N_c)$, where N_c is the number of colour degrees of freedom. Experimental evidence, as the pion decay rate through the $\pi^0 \to 2\gamma$ channel, the ratio R in electron-positron annihilation, etc, has shown that $N_c = 3$ [18]. The Lagrangian was first introduced in [19] by Gell-Mann, Fritzsch and Leutwyler, after 't Hooft demonstrated that massless and massive Yang-Mills theories (the latter, after spontaneous symmetry breaking) were renormalizable [20] [21]. It includes Dirac spinor fields ψ describing the matter fields called quarks and gauge fields G called gluons and is given by,

$$\mathcal{L}_{QCD} = i\bar{\psi}_i^q(\delta_{ij}\partial\!\!\!/ + ig G\!\!\!/_a\lambda_{ij}^a)\psi_j^q - m_q\bar{\psi}_i^q\psi_i^q - \frac{1}{4}G_{\mu\nu}^a G_a^{\mu\nu} + \mathcal{L}_{GF} + \mathcal{L}_{FP}, \tag{1.1}$$

where $i, j = 1, 2, 3$, λ_a are the $SU(3)$ generators, thus $a = 1, 2, ..., 8$, $G_{a\mu}$ is the gluon field. Also we included the gauge fixing term and the Faddeev-Popov ghost terms. The gluon field strength tensor is given by,

$$G_{\mu\nu}^a \equiv \partial_\mu G_\nu^a - \partial_\nu G_\mu^a - g f_{abc} G_\mu^b G_\nu^c. \tag{1.2}$$

Where f_{abc} are the structure constants of $SU(3)$. Additionally, there are six different types (flavours) of quarks in nature which are represented by the index q. This index runs according to $q = u, d, s, c, b, t$. Besides the principal gauge symmetry of the theory, we have different kinds of global symmetries depending on quark masses consideration. First, in the light-quark sector, the theory has a $SU(2)_L \times SU(2)_R$ global symmetry called chiral symmetry. This group acts only in the flavor indices and is parametrized as,

$$\begin{aligned} \psi_\alpha \to \psi_\alpha &= e^{-i\theta_q\tau^q}\psi_\alpha \\ \psi_\alpha \to \psi_\alpha &= e^{-i\gamma_5\beta_q\tau^q}\psi_\alpha. \end{aligned} \tag{1.3}$$

Where $\alpha = u, d$. This transformations, via Noether's theorem, lead to the gauge invariant currents

$$\begin{aligned} V^\mu &= \overline{\psi}\gamma^\mu\tau\psi \\ A^\mu &= \overline{\psi}\gamma^\mu\gamma_5\tau\psi \end{aligned} \tag{1.4}$$

which are known as vector and axial-vector current. From current algebra, it has been obtained [22] that $\partial_\mu V^\mu \propto (m_d - m_u)$ whereas for the axial current we have $\partial_\mu A^\mu \propto (m_u + m_d)$. Notice that with these relations one can see that these symmetries are exact symmetries when $m_u = m_d = 0$, and are explicitly broken symmetries at non-vanishing masses. Nevertheless, due to the small values of the masses these symmetries are considered as approximate symmetries. However, the $SU(2)_L \times SU(2)_R$ is spontaneously broken by the QCD vacuum condensates $\langle \overline{u}u \rangle$ and $\langle \overline{d}d \rangle$ leading to the $SU(2)_V$ isospin group. Since the chiral symmetry is spontaneously broken, it has a realization à la Nambu-Goldstone. The generators of the $SU(2)_A$ group do not annihilate the vacuum, giving rise to massive pseudo-Goldstone bosons associated to the triplet of pions (π^+, π^-, π^0). A consequence of the spontaneously breaking of the axial symmetry is that the matrix element of the associated axial current between the vacuum and a pion state is non-vanishing,

$$\langle 0 \left| j_{A\mu}(0) \right| \pi(p) \rangle = i\sqrt{2} f_\pi p_\mu, \tag{1.5}$$

where f_π is the pion decay constant and it is defined through this relation. One can also consider as an approximate symmetry the $SU(3)_L \times SU(3)_R$ group by adding up the strange quark s. This symmetry spontaneously breaks down to $SU(3)_V$ through the u, d and s condensates, leading to eight pseudo-Goldstone bosons: three pions π, four kaons K and the η particle. This symmetry is also related to the eightfold way introduced by Gell-Mann in 1961 [23] which leds to the discovery of the Ω^- baryon[1]. Besides the chiral symmetry, the Lagrangian is invariant under global $U(1)_B \times U(1)_A$ called baryon and axial symmetries. The $U(1)_B$ symmetry corresponds to the baryon number and is conserved even at non-vanishing quark masses. The axial symmetry, even in the massless limit is actually not conserved due to quantum effects known as anomalies.

QCD is an asymptotically free theory. This means that perturbative methods from quantum field theory can be applied to high-energy processes. The dependence of the running coupling $\alpha_\mathrm{s}(\mu) = g^2(\mu)/(4\pi)$ on the renormalization energy scale μ is given by the β-function,

$$\mu \frac{d}{d\mu} g^2(\mu) = \beta\left[g(\mu)\right]. \tag{1.6}$$

Assuming a small enough g, this β-function can be solved perturbatively. At one-loop approximation, the Renormalization Group Equation (RGE) is,

$$\mu \frac{d}{d\mu} \alpha_s = -\frac{\alpha_s^2}{2\pi} b_1 \tag{1.7}$$

where b_1 is defined as the one-loop contribution and depends on the number of quark flavors [24] [25],

$$b_1 = \left(11 - \frac{2}{3} N_f\right). \tag{1.8}$$

As long as $N_f < 17$, is adequately small for b_1 to be positive. It is conventional to identify the typical momentum transfer of a particular process with the physical scale

[1] There is the BBC Horizon documentary of 1964 in YouTube where Richard Feynman, Juval Ne'eman and Murray Gell-Mann explain this discovery.

$\mu = Q$. Thus, the solution can be written as,

$$\alpha_s(Q^2) = \frac{2\pi}{b_1} \frac{1}{\ln \frac{Q^2}{\Lambda^2_{\mathrm{QCD}}}}, \tag{1.9}$$

where Λ_{QCD} is a dimensionful integration constant fixed by experimental data of α_s and corresponds to the Landau pole of the theory, i.e. the value where α_s diverges. Notice that α_s decreases at high-energies. This property is called asymptotic freedom. That is why perturbative QCD can describe high-energy processes, viewing the quarks inside the Hadrons as a collection of free quarks interacting with other matter fields through gauge bosons. In Deep Inelastic Scattering, the process $e^- p^+ \to e^- \ X$ can be viewed as an electromagnetic scattering of an electron and a single quark inside the proton, i.e. a high-virtual photon ejects the interacting quark from the proton forming a *jet* of hadrons. The remnant spectator quarks breaks apart giving rise to a new *jet*, these two *jets* are denoted by X.

However, at low energies, comparable to Λ_{QCD}, the running coupling constant diverges. Hence pure perturbation theory in this energy region is meaningless. At such scales the hadronic degrees of freedom become dominant. Therefore, non-perturbative methods should be developed. One of these methods is the QCD Sum Rules approach, where non-perturbative effects are parametrized in terms of gauge invariant vacuum condensates.

1.2 Sum Rules

There exist several approaches to understand the low-energy phenomenology of QCD as for example, Lattice QCD or effective field theories. We will rely this work on the Sum Rules (QCDSR) method, introduced by Shifman, Vainshtein and Zakharov [26] in the late seventies. The QCDSR framework is built on two pillars: the first one is to exploit the analytic properties of several interpolating current correlation functions; the second one is the operator product expansion (OPE) of these correlators, which allows to go beyond perturbation theory, where the non-perturbative parts are parametrized as vacuum expectation values of gauge-invariant operators, the so-called condensates. These two building blocks allow to establish a mapping between both limits of the strong interactions, the high-energy limit where the degrees of freedom are the quarks and gluons and the low-energy limit where the relevant degrees of freedom are the hadrons, obtaining then information about Hadronic parameters like masses, decay constants, form factors, etc. Different mappings can be constructed, each one gives rise to a certain type of Sum Rule.

To illustrate the method [27] [12] let us consider, for definiteness, a general two-point current correlation function,

$$\Pi\left(q^2\right) = i \int d^4x e^{iqx} \langle 0|T(J(x)J(0))|0\rangle. \tag{1.10}$$

Where $J(x)$ is a color singlet local currents built up from QCD degrees of freedom, i.e. quark and gluon fields, and combinations of Lorentz structures γ_μ, γ_5, $g_{\mu\nu}$, etc. Its graphical representation is shown in Figure 1.1. The distinction between the perturbative

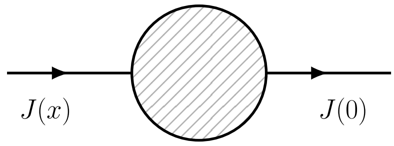


Figure 1.1: Two-point current correlation function. The dashed region represents all the possible perturbative and non-perturbative contributions.

and non-perturbative physics is encoded in the OPE as,

$$\Pi\left(q^2\right)\big|_{\mathrm{QCD}} = C_0\widehat{I} + \sum_{N=1} \frac{C_{2N}\left(q^2\right)}{\left(-q^2\right)^N}\left\langle \widehat{O}_{2N}\right\rangle, \tag{1.11}$$

where $\langle \hat{O}_{2N}\rangle \equiv \langle 0|\hat{O}_n|0\rangle$. As discussed earlier, the confinement of quarks and gluons is introduced through the condensates which are the vacuum expectation values of the operators $\hat{O}_{2N}$ in the full QCD vacuum. These operators are ordered in terms of increasing mass dimension. Also, the terms $C_n(q^2)$ are the Wilson coefficients and depend on Lorentz indices and the quantum numbers of the condensates and the currents. Each term in the expansion fall off by inverse powers of $-q^{2N}$. The Wilson coefficients are evaluated perturbatively and $C_I\langle I\rangle$ is the purely perturbative contribution since I is the identity operator. Since there is no evidence for an operator $\langle\hat{O}\rangle$ with dimension 2 in experimental data and also it can not be constructed as a gauge invariant quantity [28], the sum in (1.11) starts from dimension $d \equiv 2N = 4$.

The connection with the Hadronic world that we use in this work was proposed by Shankar [29]. When considering the complex and real axes of the squared energy plane (s-plane) expanded by $q^2 = s$, the physical Hadronic spectrum emerges as poles and cuts related to the multiparticle production along the positive real axes in the complex s-plane where, by hypothesis, the hadronic representation of the correlator lives $\Pi(s) \to \Pi(s)_{HAD}$. While the QCD spectral function $\Pi(s) \to \Pi(s)_{QCD}$ is still valid within a circle of radius s_0 in the s-plane avoiding the positive real axis, as shown in Figure 1.2 . Since there are no other singularities in the complex s-plane we can use Cauchy's theorem leading to,

$$\int_0^{s_0} \frac{1}{\pi}\,\mathrm{Im}\,\Pi(s)_{\mathrm{HAD}} W(s) ds = -\frac{1}{2\pi i}\oint_{C(|s_0|)} \Pi(s)_{\mathrm{QCD}} W(s) ds \tag{1.12}$$

where $W(s)$ is an integration kernel. The choose of this $W(s)$ is what differentiates the different types of Sum Rules. The equation above is a mathematical manifestation of the quark-hadron duality. We are going to work with $W(s) = s^N$ with $N \geq 0$ which is called Finite Energy Sum Rule (FESR). A feature of FESRs is that it focus on the high-energy region of hadronic observables, in contrast to other Sum Rules. It also produces decoupled equations for each condensate.

1.3 Pion form factor from QCDSR

Until now we have explained the Sum Rules method and its theoretical formulation but we have not seen any application. As explained in the previous section, different classes of Sum Rules can be built depending on the purpose. We can extract any kind of Hadronic

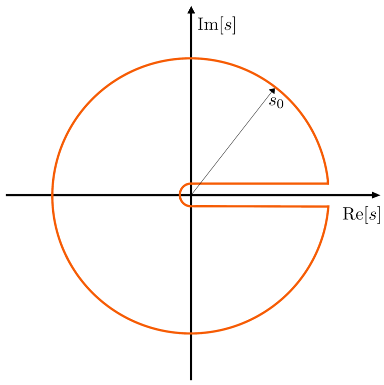


Figure 1.2: Contour in the complex s-plane used when applying the Cauchy's theorem.

parameters like quark masses, decay constants and form factors [30] [31]. The pion electromagnetic form factor is the hadronic observable that we are going to investigate in this work. It has been calculated in the vacuum in references [32] [33]. Using QCDSRs, the thermal behavior of the pion electromagnetic form factor has also been discussed in the literature [34] within the finite temperature Sum Rules framework [35]. We will follow closely the derivation of the pion electromagnetic form factor in the vacuum from [34] and then this approach will be extended to finite magnetic field background. From now on, the three-point current correlation function we are going to consider is the following,

$$\Pi_{\mu\nu\lambda}\left(p,p',q\right) = i^2 \int d^4x d^4y \; e^{i(p'x-qy)} \langle 0|T\left\{j^{\dagger}_{A\mu}(x), j^{el}_{\nu}(y), j_{A\lambda}(0)\right\}|0\rangle, \tag{1.13}$$

where $q = p' - p$ is the momentum transfer, $j_{A\nu}(x)$ is the axial current and j^{em}_{λ} is the electromagnetic current which are given by,

$$\begin{aligned} j_{A\nu}(x) &=: \bar{u}(x)\gamma_{\mu}\gamma_5 d(x) : \\ j^{el}_{\mu}(x) &=: q_u\bar{u}(x)\gamma_{\mu}u(x) + q_d\bar{d}(x)\gamma_{\mu}d(x) : . \end{aligned} \tag{1.14}$$

Where $q_u = \frac{2}{3}e$ and $q_d = -\frac{1}{3}e$ are the charges of the quarks. In what follows we consider only the light-quark sector u and d. The current correlator satisfies a double dispersion relation,

$$\Pi_{\mu\nu\lambda}\left(p,p',Q^2\right) = \frac{1}{\pi^2}\int_0^{\infty} ds \int_0^{\infty} ds' \frac{\mathrm{Im}\,\Pi_{\mu\nu\lambda}\left(s,s',Q^2\right)}{\left(s+p^2\right)\left(s'+p'^2\right)}, \tag{1.15}$$

where $Q^2 \equiv -q^2$. Recall that the imaginary part of Π can be evaluated using,

$$\mathrm{Disc}\,\Pi = (-2i)^2\,\mathrm{Im}\,\Pi. \tag{1.16}$$

The Perturbative QCD (PQCD) part of the Sum Rule program is given by the lowest order, i.e. one-loop, diagrams in Figure 1.3 and are determined through the QCD Feynman rules. This cumbersome calculation can be found in literature [32] [33], it is commonly calculated by taking the double discontinuity of the amplitude using Cutkosky's rules. We will not include radiative corrections to the triangle diagram. A single Lorentz tensor

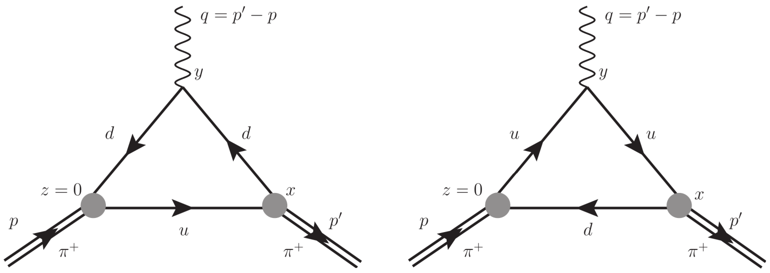


Figure 1.3: One-loop diagrams contributing to the QCD three-point function.

structure has to be selected to establish a mapping with the hadronic spectral function. Therefore, the QCD spectral function at the structure $P_\mu P_\nu P_\lambda$ was found to be,

$$\rho_{\mu\nu\lambda}\left(s,s',Q^2\right)\big|_{QCD} = \frac{3Q^4}{2\pi^2}\lambda^{-7/2}\left[3\lambda\left(x+Q^2\right)\left(x+2Q^2\right)-\lambda^2-5Q^2\left(x+Q^2\right)^3\right], \tag{1.17}$$

where,

$$\lambda\left(s,s',Q^2\right) = \left(s+s'+Q^2\right)^2 - 4ss'. \tag{1.18}$$

Here is useful to use the momentum average and momentum transfer variables given by $P = (p'+p)/2$ and Q. The contact with the hadronic world is made by writing the spectral function $\rho_{\mu\nu\lambda}$ in terms of the hadronic degrees of freedom. We can saturate the dispersion relation with the ground state followed by some threshold energy s_0, which is the beginning of the hadronic continuum modelled by QCD,

$$\begin{aligned}\rho_{\mu\nu\lambda}\left(s,s',Q^2\right)\big|_{HAD} =& < 0\left|j_{A\nu}(0)\right|\pi^+\left(p'\right) >< \pi^+\left(p'\right)\left|j^{el}_\lambda(0)\right|\pi^+(p) > \\ & \times < \pi^+(p)\left|j_{A\mu}(0)\right|0 > \delta(s)\delta\left(s'\right) + \text{ continuum}, \\ =& 4f_\pi^2 F_\pi\left(Q^2\right)\left[P_\mu P_\nu P_\lambda + \frac{1}{2}P_\lambda\left(P_\mu q_v - P_\nu q_\mu\right) - \frac{1}{4}q_\mu q_\nu P_\lambda\right] \\ & + \rho_{\mu\nu\lambda}\left(s,s',Q^2\right)\big|_{QCD}\left[1-\theta\left(s_0-s-s'\right)\right]\end{aligned} \tag{1.19}$$

where $f_\pi \simeq 93\ MeV$ and $F_\pi\left(Q^2\right)$ is the pion electromagnetic form factor. Now, we use the quark-hadron duality in equation (1.12). The lowest dimensional FESR leads to,

$$\int_0^{s_o}\int_0^{s_o} ds ds' \,\mathrm{Im}\,\Pi\left(s,s',Q^2\right)\big|_{HAD} = \int_0^{s_o}\int_0^{s_o} ds ds' \,\mathrm{Im}\,\Pi\left(s,s',Q^2\right)\big|_{QCD} \tag{1.20}$$

To construct the Sum Rule we have to insert equations (1.17) and (1.19) in (1.20). The integration region is defined by the θ-function in (1.19). We have to choose a suitable integration region in which the contribution of the hadronic continuum (modelled by QCD) cancels with the pure QCD contribution in that region. It turns out that this region, which we called Ω, is well given by figure 1.4. Therefore the FESR gives,

$$4f_\pi^2 F_\pi(Q^2)\int_\Omega ds ds'\delta(s)\delta(s') = \int_\Omega ds ds'\ \rho(s,s',Q^2)\big|_{QCD}. \tag{1.21}$$

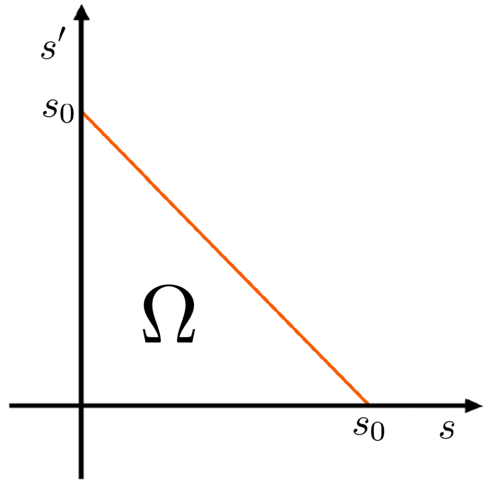


Figure 1.4: Integration region in the (s, s') plane after the subtraction of the continuum.

Therefore we can extract the form factor,

$$F_\pi(Q^2) = \frac{1}{4f_\pi^2}\int_\Omega ds ds'\ \rho(s,s',Q^2)\big|_{QCD}\,, \tag{1.22}$$

which yields to,

$$F_\pi\left(Q^2\right) = \frac{s_0}{16\pi^2 f_\pi^2}\frac{1}{\left(1+Q^2/2s_0\right)^2}. \tag{1.23}$$

Here we take $s_0 \simeq 1$ and $f_\pi \simeq 93\ MeV$ [18]. It is expected that this result fits the experimental data when $Q^2 \geq 1\ GeV^2$, since in lower energies the condensates need to be considered. Additionally, the electromagnetic form factor (1.23) is in good agreement with old and recent experimental data [36] [37] in a $Q^2 \simeq 1-4\ GeV^2$ region. Therefore we are going to present our later results within this energy interval. The behavior of $F_\pi(Q^2)$ is shown in Figure 1.5.

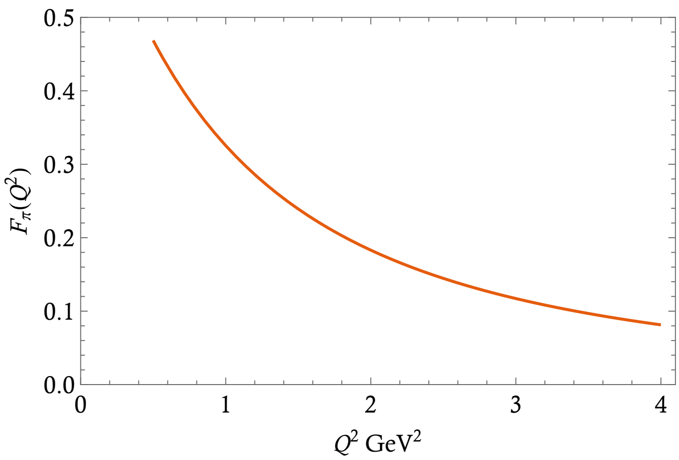


Figure 1.5: Electromagnetic pion form factor given by the equation (1.23). Where we followed [34].

Chapter 2

The Schwinger propagator

In the previous chapter, we introduced the pion electromagnetic form factor. It was calculated via QCDSR where in the QCD side only the one-loop diagrams shown above are needed. As we discussed at the beginning of this thesis, we want to extend this calculation by considering a constant magnetic field background of intensity eB, obtaining in this way the dependence on eB of the pion electromagnetic form factor

To start with this extension we can study the propagator of the electron field coupled to an external magnetic field. Then, this propagator can be replaced in each internal fermionic line of a diagram associated to a particular process, in the presence of a magnetic background. The exact electron propagator for this configuration was found by J. Schwinger [13] in the Fock-Schwinger [38] proper time formalism. In this chapter, we will follow closely the references [39] and [40] for the derivation of the fermionic propagator.

2.1 Proper time formalism for the fermionic propagator

The Dirac field equation in presence of an external field A_μ is written as,

$$(i\partial\!\!\!/ + q_f A\!\!\!/ - m_f)\psi = 0. \tag{2.1}$$

The Green's function for this field equation satisfy,

$$(i\partial\!\!\!/ + q_f A\!\!\!/ - m_f)G\left(x, x'\right) = \delta\left(x - x'\right). \tag{2.2}$$

We can introduce a one-particle Hilbert space spanned by $|x\rangle$ and then define the Green's function as an operator matrix element by $G\left(x, x'\right) = \langle x'|\, \hat{G}\, |x\rangle$ so that the equation (2.3) can be written as,

$$(\hat{\Pi\!\!\!/} - m_f)\hat{G} = 1, \tag{2.3}$$

where $\hat{\Pi}_\mu = \hat{P}_\mu + q_f A_\mu(\hat{x})$ is the conjugated momentum operator. This equation can be solved by,

$$\hat{G} = \frac{1}{\hat{\Pi\!\!\!/} - m} = (\hat{\Pi\!\!\!/} + m)\frac{1}{\hat{\Pi\!\!\!/}^2 - m^2}. \tag{2.4}$$

Now using Schwinger's parametrization technique, based on the identity,

$$\frac{1}{A} = -i\int_0^\infty ds e^{isA}, \tag{2.5}$$

we write the equation (2.4) as,

$$\hat{G} = \frac{1}{\hat{\not{\Pi}} - m} = -i\int_0^\infty ds(\hat{\not{\Pi}} + m)\exp\left[-i\left(m^2 - \hat{\not{\Pi}}^2\right)s\right]. \tag{2.6}$$

We define the effective Hamiltonian in this system as,

$$H \equiv -(\not{\Pi})^2 = -\Pi^2 - \frac{1}{2}q_f\sigma_{\mu\nu}F^{\mu\nu}, \tag{2.7}$$

and also $\hat{U}(s) = e^{-i\hat{H}s}$ which can be interpreted as a unitary time-evolution operator where an state $|x\rangle$ evolves as,

$$|x;s\rangle \equiv \hat{U}(s)|x;0\rangle. \tag{2.8}$$

With the above definitions, equation (2.6) can be written as,

$$\begin{aligned} G\left(x,x'\right) =& -i\int_0^\infty dse^{-im^2s}\left\langle x'\right|(\hat{\not{\Pi}}+m)\hat{U}(s)\left|x\right\rangle, \\ =& -i\int_0^\infty dse^{-im^2s}\left[\gamma^\mu\left\langle x';0\right|\hat{\Pi}_\mu(0)\left|x;s\right\rangle + m\left\langle x';0|x;s\right\rangle\right]. \end{aligned} \tag{2.9}$$

Here $\hat{\Pi}(s)$ operates on $|x;s\rangle$ and $\hat{\Pi}(0)$ operates on $|x;0\rangle$. We have interpreted s as a time variable known as Schwinger proper time. Recall that the commutation relations for this cannonical system are,

$$\begin{aligned} \left[\hat{\Pi}_\mu, \hat{x}_\nu\right] &= ig_{\mu\nu}, \\ \left[\hat{\Pi}_\mu, \hat{\Pi}_\nu\right] &= ieF_{\mu\nu}, \end{aligned} \tag{2.10}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ denotes the field-strength tensor of the gauge field A_μ and also we assume that it is constant. The evolution of the operators $\hat{x}_\mu$ and $\hat{\Pi}_\mu$ is generated by the Hamiltonian via Heisenberg equations of motion,

$$\frac{d\hat{x}_\mu}{ds} = -i\left[\hat{x}_\mu, \hat{H}\right] = 2\hat{\Pi}_\mu, \tag{2.11}$$

$$\frac{d\hat{\Pi}_\mu}{ds} = -i\left[\hat{\Pi}_\mu, \hat{H}\right] = -2eF_{\mu\nu}\hat{\Pi}^\nu. \tag{2.12}$$

Using the definition (2.8) we can find,

$$i\partial_s\left\langle x'(0)|x(s)\right\rangle = \left\langle x'(0)\right|\hat{H}\left|x(s)\right\rangle, \tag{2.13}$$

$$(i\partial_\mu + eA_\mu(x))\left\langle x'(0)|x(s)\right\rangle = \left\langle x'(0)\right|\hat{\Pi}_\mu(s)\left|x(s)\right\rangle, \tag{2.14}$$

$$\left(-i\partial'_\mu + eA_\mu\left(x'\right)\right)\left\langle x'(0)|x(s)\right\rangle = \left\langle x'(0)\right|\hat{\Pi}_\mu(0)\left|x(s)\right\rangle \tag{2.15}$$

with $\langle x'(0)|x(s)\rangle \to \delta^4\left(x - x'\right)$ as $s \to 0$. From now on, we drop the Lorentz indices and write the matrix and vector structures with boldface type and also we will drop the circumflexes on operators. Now we solve the equations (2.11) and (2.12),

$$\mathbf{\Pi}(s) = e^{-2eFs}\mathbf{\Pi}(0), \tag{2.16}$$

$$\mathbf{x}(s) - \mathbf{x}(0) = \left(1 - e^{-2e\mathbf{F}s}\right)(e\mathbf{F})^{-1}\mathbf{\Pi}(0). \tag{2.17}$$

Thus we can write the Hamiltonian in terms of $\mathbf{x}(s)$ and $\mathbf{x}(0)$,

$$\begin{aligned}\hat{H} &= -\mathbf{\Pi}^2 - \frac{1}{2}e\sigma\mathbf{F} \\ &= (\mathbf{x}(s) - \mathbf{x}(0))\mathbf{K}(\mathbf{x}(s) - \mathbf{x}(0)) - \frac{1}{2}e\sigma\mathbf{F}, \end{aligned} \tag{2.18}$$

where,

$$\mathbf{K} \equiv \frac{1}{4}(e\mathbf{F})^2 \sinh^{-2} e\mathbf{F}s,$$

finding,

$$[\mathbf{x}(s), \mathbf{x}(0)] = i\left(1 - e^{-2e\mathbf{F}s}\right)(e\mathbf{F})^{-1}.$$

In order to solve equation (2.13) we write,

$$\left\langle x'(0)\right| \hat{H} \left|x(s)\right\rangle = -\frac{1}{2}e\sigma\mathbf{F} - \left(\mathbf{x} - \mathbf{x}'\right)\mathbf{K}\left(\mathbf{x} - \mathbf{x}'\right) - \frac{i}{2}\,\mathrm{tr}(e\mathbf{F}\coth e\mathbf{F}s)\left\langle x'(0)|x(s)\right\rangle. \tag{2.19}$$

Now the operators $\mathbf{x}'$ and $\mathbf{x}$ turn to position vectors. Sice (2.13) is now just a diferential equation, the general solution is,

$$\begin{aligned}\left\langle x'(0)|x(s)\right\rangle =& C\left(x, x'\right)s^{-2}\exp\left[-\frac{1}{2}\,\mathrm{tr}\ln\left[(e\mathbf{F}s)^{-1}\sinh(e\mathbf{F}s)\right]\right] \\ &\times\exp\left[-\frac{i}{4}\left(\mathbf{x} - \mathbf{x}'\right)e\mathbf{F}\coth(e\mathbf{F}s)\left(\mathbf{x} - \mathbf{x}'\right) + \frac{i}{2}e\sigma_{\mu\nu}F^{\mu\nu}s\right]\end{aligned} \tag{2.20}$$

To determine the factor $C(x, x')$ we rewrite the right-hand side of the remaining equations (2.14) and (2.15),

$$\left\langle x'(0)|\mathbf{\Pi}(s)|x(s)\right\rangle = \frac{1}{2}[e\mathbf{F}\coth(e\mathbf{F}s) - e\mathbf{F}]\left(\mathbf{x} - \mathbf{x}'\right)\left\langle x'(0)|x(s)\right\rangle, \tag{2.21}$$

$$\left\langle x'(0)|\mathbf{\Pi}(0)|x(s)\right\rangle = \frac{1}{2}[e\mathbf{F}\coth(e\mathbf{F}s) + e\mathbf{F}]\left(\mathbf{x} - \mathbf{x}'\right)\left\langle x'(0)|x(s)\right\rangle. \tag{2.22}$$

Replacing (2.20) in the left-hand side of equations (2.14) and (2.15) we get,

$$\left[i\partial_\mu + eA_\mu(x) - \frac{1}{2}eF_{\mu\nu}\left(x' - x\right)^\nu\right]C\left(x, x'\right) = 0, \tag{2.23}$$

$$\left[-i\partial'_\mu + eA_\mu\left(x'\right) + \frac{1}{2}eF_{\mu\nu}\left(x' - x\right)^\nu\right]C\left(x, x'\right) = 0, \tag{2.24}$$

whose solution is given by,

$$C\left(x, x'\right) = C\exp\left[ie\int_{x'}^{x} d\xi^\mu\left(A_\mu + \frac{1}{2}F_{\mu\nu}\left(\xi - x'\right)^\nu\right)\right] \tag{2.25}$$

Since the integrand has zero curl, this line integral is independent of the path which conects x' to x. The constant C is found to be $C = -i/(4\pi)^2$. Finally, with the equations (2.9), (2.22) and (2.25), the fermion propagator in presence of an external field takes the form,

$$G\left(x, x'\right) = \Phi\left(x, x'\right)\mathcal{G}\left(x, x'\right) \tag{2.26}$$

where,

$$\begin{aligned}&\mathcal{G}\left(x,x'\right)\equiv\\&-(4\pi)^{-2}\int_0^\infty\frac{ds}{s^2}\left[m+\frac{1}{2}\gamma\cdot\left(e\mathbf{F}\coth(e\mathbf{F}s)+e\mathbf{F}\right)\left(\mathbf{x}-\mathbf{x}'\right)\right]\\&\times\exp\left(-im^2s+\frac{i}{2}e\sigma_{\mu\nu}F^{\mu\nu}s\right)\\&\times\exp\left[-\frac{1}{2}\operatorname{tr}\ln\left[(e\mathbf{F}s)^{-1}\sinh(e\mathbf{F}s)\right]-\frac{i}{4}\left(\mathbf{x}-\mathbf{x}'\right)\left(eF\coth(e\mathbf{F}s)\right)\left(\mathbf{x}-\mathbf{x}'\right)\right]\end{aligned}\tag{2.27}$$

and,

$$\Phi\left(x,x'\right)\equiv\exp\left\{ie\int_{x'}^{x}d\xi^\mu\left[A_\mu+\frac{1}{2}F_{\mu\nu}\left(\xi-x'\right)^\nu\right]\right\}.\tag{2.28}$$

This last equation is the so called Schwinger phase factor, which breaks the translation invariance. For two point correlators, the phase factor can be shown to be removed by a gauge transformation. However, in our case, a three-point correlator it will play an important role. If the magnetic field is chosen to be along the $\hat{z}$ direction, we can use the parallel and perpendicular notation for four-vectors,

$$a_\parallel=(a_0,0,0,a_3)\,,\quad a_\perp=(0,a_1,a_2,0)\,,$$

with the properties,

$$\begin{aligned}(a\cdot b)_\parallel&=a_0b_0-a_3b_3\\(a\cdot b)_\perp&=a_1b_1+a_2b_2,\end{aligned}\tag{2.29}$$

also a squared four-vector can be written as,

$$a^2=a_\parallel^2-a_\perp^2=a_0^2-\vec{a}^2.\tag{2.30}$$

The metric tensor in this notation is considered as,

$$\begin{aligned}g_{\mu\nu}&=g_{\mu\nu}^\parallel-g_{\mu\nu}^\perp,\\g_{\mu\nu}^\parallel&=\mathrm{diag}(1,0,0,-1),\\g_{\mu\nu}^\perp&=\mathrm{diag}(0,1,1,0).\end{aligned}\tag{2.31}$$

Now the propagator $\mathcal{G}\left(x,x'\right)$ in (2.27) is written as,

$$\begin{aligned}\mathcal{G}(x)=&-(4\pi)^{-2}\int_0^\infty\frac{ds}{s^2}\frac{eBs}{\sin(eBs)}\exp\left(-im^2s+ieBs\sigma_3\right)\\&\times\exp\left[-\frac{i}{4s}\left(x_\parallel^2-eBs\cot(eBs)x_\perp^2\right)\right]\\&\times\left[m+\frac{1}{2s}\left(\gamma\cdot x_\parallel-\frac{eBs}{\sin(eBs)}\right.\right.\\&\left.\left.\times\exp\left(-ieBs\sigma_3\right)\gamma\cdot x_\perp\right)\right].\end{aligned}\tag{2.32}$$

It is also useful to write the propagator in the momentum space,

$$G\left(x,x'\right)=\Phi\left(x,x'\right)\int\frac{d^4p}{(2\pi)^4}e^{-ip(x-x')}S(p)\tag{2.33}$$

where $S(p)$ is defined in the next section.

2.2 Landau Level expansion

The fermionic propagator found in the previous section, through the Schwinger proper time formalism, is not quite appropriate for practical calculations. Therefore, we will use a representation where the propagator is written as a sum over Landau Levels. In the massless limit it is given by,

$$iS(p) = ie^{-p_\perp^2/|q_f B|} \sum_{n=0}^{+\infty} (-1)^n \frac{D_n\left(q_f B, p\right)}{p_\parallel^2 - 2n\left|q_f B\right|} \tag{2.34}$$

with,

$$D_n\left(q_f B, p\right) = \not{p}_\parallel \left[2\mathcal{O}_f^- L_n^0\left(\frac{2p_\perp^2}{|q_f B|}\right) - 2\mathcal{O}_f^+ L_{n-1}^0\left(\frac{2p_\perp^2}{|q_f B|}\right)\right] + 4\not{p}_\perp L_{n-1}^1\left(\frac{2p_\perp^2}{|q_f B|}\right)$$

where $L_n \equiv L_n^0$ and $L_{-1}^\alpha = 0$ are the associated Laguerre polynomials. Also $\mathcal{O}_f^\pm = \left(1 \pm i\gamma^1\gamma^2 \operatorname{sign}\left(q_f B\right)\right)/2$ are projection operators. For the details of this derivation please see reference [39]. For the next calculations, the following properties are useful,

$$O^\pm \gamma_\mu O^\pm = O^\pm \gamma_\mu^\parallel \tag{2.35}$$

$$O^\pm \gamma_\mu O^\mp = O^\pm \gamma_\mu^\perp \tag{2.36}$$

$$\left[\gamma_\mu^\parallel, O^\pm\right] = 0 \tag{2.37}$$

$$O^\pm \gamma_\mu^\perp = \gamma_\mu^\perp O^\mp \tag{2.38}$$

where we omitted the sub-index f.

2.2.1 Strong field limit

In the limit where the magnetic field is the largest energy scale in the system, the level with $n = 0$ in (2.34) might be the most important contribution[1]. Thus, the fermionic Schwinger propagator in the lowest Landau level (LLL) is given by,

$$i\mathcal{S}(p) = i2e^{-\frac{p_\perp^2}{|q_f B|}} \frac{\not{p}_\parallel}{p_\parallel^2} O_f^- \tag{2.39}$$

An important feature of this approximation is that it demonstrates a dimensional reduction where the parallel and perpendicular momentum decouple from each other, since the motion of the charged particles is restricted to directions perpendicular to the magnetic field.

2.2.2 Weak field limit

When the magnetic field is considerably smaller than the momenta. The Schwinger propagator can be expanded in powers of $q_f B$ [39]. Thus, up to order $q_f B$, the propagator is,

$$i\mathcal{S}(p) = i\frac{\not{p}}{p^2} - \frac{\gamma_1\gamma_2\left(\gamma \cdot p_\parallel\right)}{\left(p^2\right)^2} q_f B \tag{2.40}$$

[1] This is not generally true. In fact, in this study higher Landau levels are needed to be considered.

2.3 Bosonic propagator

The derivation of the boson propagator in a constant magnetic field background with the Schwinger's proper time formalism is analogous as in section 2.1, and is found in the literature [41]. We present it here for completeness. For the strong magnetic field limit the bosonic propagator in the lowest Landau level is given by,

$$iD^B(k) \overset{qB\to\infty}{\longrightarrow} 2i\frac{e^{-k_\perp^2/qB}}{k_\parallel^2 - qB - m^2} \tag{2.41}$$

For the weak magnetic field limit is given by,

$$iD^B(k) \overset{qB\to 0}{\longrightarrow} \frac{i}{k_\parallel^2 - k_\perp^2 - m^2}\left\{1 - \frac{(qB)^2}{\left(k_\parallel^2 - k_\perp^2 - m^2\right)^2} - \frac{2(qB)^2\left(k_\perp^2\right)}{\left(k_\parallel^2 - k_\perp^2 - m^2\right)^3}\right\}. \tag{2.42}$$

2.4 Comments on the phase factor

Since we are going to calculate three-point functions of the form (3.1) with Schwinger propagators in its internal lines, we will have to deal with products of Schwinger phase factors. First, note that a single phase factor (2.28) can be reduced if we chose the integration path from x' to x to be a straigh line,

$$\xi^\mu = x'^\mu + t(x-x')^\mu \tag{2.43}$$

leaving (2.28) as,

$$\Phi\left(x,x'\right) \equiv \exp\left\{ie\int_0^1 dt\left[A_\mu(x-x')^\mu + \frac{1}{2}F_{\mu\nu}\left(x-x'\right)^\nu\left(x-x'\right)^\mu\right]\right\}, \tag{2.44}$$

since $F_{\mu\nu}$ is antisymmetric the last term in the exponential vanish. Thus,

$$\Phi\left(x,x'\right) \equiv \exp\left\{ie\int_0^1 dtA_\mu(x-x')^\mu\right\}. \tag{2.45}$$

The constant magnetic field background along the $\hat{z}$ axis can be produced by the vector potential,

$$A_\mu(x) = \frac{B}{2}(0,-x_2,x_1,0). \tag{2.46}$$

Now, if we make a gauge transformation of the form,

$$A_\mu(\xi) \to A'_\mu(\xi) = A_\mu + \frac{\partial}{\partial\xi^\mu}\Lambda(\xi), \tag{2.47}$$

where $\Lambda(\xi) = \frac{B}{2}(x'_2\xi_1 - x'_1\xi_2)$, the integral in (2.44) vanishes, thus the single phase factor is equal to one. Therefore, when we are dealing with products of phases where we can choose this configuration, the whole phase factor can be *gauged away*. That is, when we have a single phase factor or a product of two factors that coincide at a given space-time point. However, in the magnetic corrections to the three-current correlation function (3.1) at one loop, we face a product of three phase factors which gives a non-trivial contribution.

To illustrate this feature we rewrite a single phase factor which goes from P to Q with the boundary dependence isolated as,

$$\Phi\left(P,Q\right)=\exp\left[ie\int_Q^P dx^\mu\left(A_\mu+\frac{1}{2}F_{\mu\nu}x^\nu\right)\right]\exp\left[-i\frac{e}{2}F_{\mu\nu}\int_Q^P dx^\mu Q^\nu\right] \tag{2.48}$$

So that when we link together the three phase factors, the only terms that survive are,

$$\begin{aligned}&\Phi\left(P,Q\right)\Phi\left(Q,R\right)\Phi\left(R,P\right)=\\&\qquad\exp\left[-i\frac{e}{2}F_{\mu\nu}\left(\int_Q^P dx^\mu Q^\nu+\int_R^Q dx^\mu R^\nu+\int_P^R dx^\mu P^\nu\right)\right]\end{aligned} \tag{2.49}$$

leading to,

$$\Phi\left(P,Q\right)\Phi\left(Q,R\right)\Phi\left(R,P\right)=\exp\left[-i\frac{e}{2}F_{\mu\nu}(R^\mu-P^\mu)(P^\nu-Q^\nu)\right]. \tag{2.50}$$

This non-trivial Schwinger phase contribution will affect directly in amplitude of a three-point process, where the Schwinger propagator is involved. Nevertheless, this term is treated differently in each magnetic field limit.

Chapter 3

Magnetic corrections to the PQCD term

Now that we have introduced the fermion propagator in a magnetic field background. In this chapter, we will describe the implementation of the Schwinger propagator in the three-point function, since this corresponds to the perturbative QCD side to construct the Sum Rule in Chapter 4. We will go through both limits of the magnetic field defined in the previous chapter. In the weak magnetic field limit we will use the expansion on powers of eB of the Schwinger propagators and write the QCD three-point function up to the linear terms in eB. For the strong magnetic field limit we will consider up to the first Landau level contributions to the Schwinger propagator.

3.1 Strong magnetic field limit

3.1.1 Lowest Landau Level

Consider the three-point correlation function from equation (1.13). To obtain the two Feynman diagrams contributing to this correlator (Figure 1.3), we replace the electromagnetic and axial interpolating currents and use Wick's theorem (see Appendix B). These two contributions are given by,

$$\begin{aligned}\Pi_{\mu\nu\lambda}\left(p,p',q\right) = -i^2 N_c \int d^4x d^4y\ e^{i(p'x-qy)} \\ \{q_u \mathrm{Tr}[\gamma_5\gamma_\mu iS^u(x-y)\gamma_\nu iS^u(y-0)\gamma_\lambda\gamma_5 iS^d(0-x)] \\ +q_d \mathrm{Tr}[\gamma_\lambda\gamma_5 iS^d(0-y)\gamma_\nu iS^d(y-x)\gamma_5\gamma_\mu iS^u(x-0)]\}.\end{aligned} \tag{3.1}$$

We can rewrite the second trace in a suitable form by inserting the charge conjugation operator identity $CC^{-1} = 1$ and making use of the trace cyclical property. Also, the Schwinger propagator in both weak and strong magnetic field limit transforms as $CiS(p)C^{-1} = -iS(p)^T$ under charge conjugation and it satisfies the anti-commutation relation $\{\gamma_5, iS(p)\} = 0$. For the details in this procedure, please see Appendix B. Thus, we can start from the three-point function,

$$\begin{aligned}\Pi_{\mu\nu\lambda}\left(p,p',q\right) = -i^2 N_c \int \frac{d^4l}{(2\pi)^4}\frac{d^4t}{(2\pi)^4}\frac{d^4k}{(2\pi)^4} d^4x d^4y \; e^{i(p'x-qy)}e^{-il(x-y)}e^{-ity}e^{ikx} \\ \{q_d\Omega_d^C \mathrm{Tr}[\gamma_\mu i\mathcal{S}^d(l)\gamma_\nu i\mathcal{S}^d(t)\gamma_\lambda i\mathcal{S}^u(k)] \\ -q_u\Omega_u \mathrm{Tr}[\gamma_\mu i\mathcal{S}^u(l)\gamma_\nu i\mathcal{S}^u(t)\gamma_\lambda i\mathcal{S}^d(k)]\}.\end{aligned} \tag{3.2}$$

The strong magnetic field effects are introduced through the Landau Level expansion of the Schwinger propagator (equation (2.34)). For every term in the expansion there is a product of three Schwinger phases that does not vanish, as we discussed in section 2.4. These products are computed through the equation (2.50). We can always take $z=0$, getting,

$$\Omega_u = \Phi^u(x,y)\Phi^u(y,0)\Phi^d(0,x) = \exp\left\{-i\frac{|q_uB|}{2}\epsilon_{ij}x^iy^j\right\}, \tag{3.3}$$

$$\Omega_d^C = \Phi^u(x,0)\Phi^d(0,y)\Phi^d(y,x) = \exp\left\{-i\frac{|q_dB|}{2}\epsilon_{ij}x^iy^j\right\}. \tag{3.4}$$

First, we will work with the Lowest Landau Level (LLL) term since it is simpler and holds all the mathematical properties which appear when considering higher terms in the Landau Level expansion. Therefore, we take,

$$i\mathcal{S}(p) = i2e^{-\frac{p_\perp^2}{|q_fB|}}\frac{\not{p}_\parallel}{p_\parallel^2}O_f^-. \tag{3.5}$$

We can rewrite the traces in (3.2) carefully, using the properties introduced in (2.35)-(2.38). The term $\mathrm{sign}(q_fB)$ within $O_f^\pm$ changes the sign depending on the charge of the fermion, so keep in mind that $O_{q_d}^\pm = O^\mp$. Thus, the traces are,

$$\begin{aligned}\mathrm{Tr}[\gamma_\mu \not{l}_\parallel O_u^- \gamma_\nu \not{t}_\parallel O_u^- \gamma_\lambda \not{k}_\parallel O_d^-] = \mathrm{Tr}[O^+\gamma_\mu^\perp \not{l}_\parallel \gamma_\nu \not{t}_\parallel \gamma_\lambda^\perp \not{k}_\parallel], \\ \mathrm{Tr}[\gamma_\lambda \not{t}_\parallel O_d^- \gamma_\nu \not{l}_\parallel O_d^- \gamma_\mu \not{k}_\parallel O_u^-] = \mathrm{Tr}[O^-\gamma_\mu^\perp \not{l}_\parallel \gamma_\nu \not{t}_\parallel \gamma_\lambda^\perp \not{k}_\parallel].\end{aligned} \tag{3.6}$$

Splitting the three-point function into two contributions, one proportional to q_u and the other one to q_d,

$$\Pi_{\mu\nu\lambda}\left(p,p',q\right) = \Pi^u_{\mu\nu\lambda}\left(p,p',q\right) + \Pi^d_{\mu\nu\lambda}\left(p,p',q\right), \tag{3.7}$$

we end up with the following expressions,

$$\begin{aligned}&\Pi^u_{\mu\nu\lambda}\left(p,p',q\right) \\ &= i^2(2i)^3 q_u N_c \int d^4x d^4y \int \frac{d^4l}{(2\pi)^4}\frac{d^4t}{(2\pi)^4}\frac{d^4k}{(2\pi)^4} e^{-ix(l-p'-k)}e^{-iy(t+q-l)} \\ &\quad e^{-i\frac{|q_uB|}{2}\epsilon_{ij}x^iy^j} e^{-\frac{l_\perp^2}{|q_uB|}} e^{-\frac{t_\perp^2}{|q_uB|}} e^{-\frac{k_\perp^2}{|q_dB|}} \frac{1}{t_\parallel^2 k_\parallel^2 l_\parallel^2} \mathrm{Tr}[O^+\gamma_\mu^\perp \not{l}_\parallel \gamma_\nu \not{t}_\parallel \gamma_\lambda^\perp \not{k}_\parallel],\end{aligned} \tag{3.8}$$

and,

$$
\begin{aligned}
&\Pi^{d}_{\mu\nu\lambda}\left(p,p',q\right)\\
&=-\,i^2(2i)^3 q_d N_c \int d^4x d^4y \int \frac{d^4l}{(2\pi)^4}\frac{d^4t}{(2\pi)^4}\frac{d^4k}{(2\pi)^4} e^{-ix(l-p'-k)}e^{-iy(t+q-l)}\\
&\quad e^{-i\frac{|q_dB|}{2}\epsilon_{ij}x^iy^j} e^{-\frac{l_\perp^2}{|q_dB|}} e^{-\frac{t_\perp^2}{|q_dB|}} e^{-\frac{k_\perp^2}{|q_uB|}} \frac{1}{t_\parallel^2 k_\parallel^2 l_\parallel^2}\mathrm{Tr}[O^-\gamma_\mu^\perp \not{l}_\parallel \gamma_\nu \not{t}_\parallel \gamma_\lambda^\perp \not{k}_\parallel].
\end{aligned}
\tag{3.9}
$$

The evaluation of the multidimensional integral in the above equation is straightforward but lengthy. The procedure that we apply is explained in the next section, but the idea is the following. First, we integrate the space variables obtaining Dirac delta functions. The argument of perpendicular delta functions, gets modified due to the non-vanishing Schwinger phase, making an unusual shift in the perpendicular momenta. The perpendicular momenta integrals are solved easily as Gaussian integrals. To evaluate the parallel momenta integrals we use the Feynman parametrization technique. Since we are computing a triangle loop we end up with two Feynman parameters integrals. We use the FeynCalc Mathematica package [42] which was made for symbolic evaluation of Feynman diagrams and algebraic calculations in quantum field theory. The main features that we exploit from this package are the Lorentz index contraction and Dirac γ-matrices manipulation.

Space integrals

Note that the Schwinger phase term contributes with an exponential that mixes just the perpendicular space variables, hence we have to integrate them separately. If we split the spatial integration measure into $d^4x\, d^4y \to d^2x_\perp d^2x_\parallel d^2y_\perp d^2y_\parallel$ we can easily integrate the parallel part,

$$
\begin{aligned}
&\int d^4xd^4y\; e^{i(p'x-qy)}e^{-il(x-y)}e^{-ity}e^{ikx}e^{-i\frac{|q_uB|}{2}\epsilon_{ij}x^iy^j}\\
&\quad=\int d^2x_\perp d^2y_\perp (2\pi)^4\delta^{(2)}((l-k-p')_\parallel)\delta^{(2)}((t+q-l)_\parallel)\\
&\qquad\times e^{-ix_\perp(l-k-p')_\perp}e^{-iy_\perp(t+q-l)_\perp}e^{-i\frac{|q_uB|}{2}\epsilon_{ij}x^iy^j}.
\end{aligned}
\tag{3.10}
$$

Now we integrate the $y_\perp$ terms making a Dirac delta function for $x_\perp$. Recall that the subindex $\perp$ refers to the two components $i=1,2$, so we write the perpendicular structures in the above equation as,

$$
\begin{aligned}
&\int d^2x_\perp d^2y_\perp e^{-ix_\perp(l-k-p')_\perp}e^{-iy_\perp(t+q-l)_\perp}e^{-i\frac{|q_uB|}{2}\epsilon_{ij}x^iy^j}\\
&\quad=\prod_{i=1}^{2}\int dx_i e^{-ix_i(l-k-p')_i}\delta^{(2)}((t+q-l)_j+\frac{|q_uB|}{2}\epsilon_{ij}x^i)(2\pi)^2.
\end{aligned}
\tag{3.11}
$$

The property $\epsilon_{ij}\epsilon_{in}=\delta_{jn}$ of the Levi-Civita symbol allows us write the Dirac delta function as,

$$
\delta^{(2)}((t+q-l)_j+\frac{|q_uB|}{2}\epsilon_{ij}x^i)=\frac{2}{|q_uB|}\delta^{(2)}(\frac{2\epsilon_{ij}}{|q_uB|}(t+q-l)_j+x_i),
\tag{3.12}
$$

so with the last two equations the perpendicular space integral is,

$$\int d^2x_\perp d^2y_\perp e^{-ix_\perp(l-k-p')_\perp}e^{-iy_\perp(t+q-l)_\perp}e^{-i\frac{|q_u|}{2}B\epsilon_{ij}x^iy^j} \\ = \left(\frac{4\pi}{|q_uB|}\right)^2 e^{i\frac{2\epsilon_{ij}}{|q_uB|}(t+q-l)_j(l-k-p')_i}. \tag{3.13}$$

Therefore, the total space integral for the three-point function is given by,

$$(2\pi)^4\left(\frac{4\pi}{|q_uB|}\right)^2\delta^{(2)}((l-k-p')_\parallel)\delta^{(2)}((t+q-l)_\parallel)e^{i\frac{2\epsilon_{ij}}{|q_uB|}(t+q-l)_j(l-k-p')_i}, \tag{3.14}$$

leading to,

$$\begin{aligned}\Pi^u_{\mu\nu\lambda}(p,p',q) =& i^2(2i)^3q_uN_c(2\pi)^4\left(\frac{4\pi}{|q_uB|}\right)^2\int\frac{d^4l}{(2\pi)^4}\frac{d^4t}{(2\pi)^4}\frac{d^4k}{(2\pi)^4} \\ &\times\delta^{(2)}((l-k-p')_\parallel)\delta^{(2)}((t+q-l)_\parallel)e^{i\frac{2\epsilon_{ij}}{|q_uB|}(t+q-l)_j(l-k-p')_i} \\ &\times e^{-\frac{l_\perp^2}{|q_uB|}}e^{-\frac{t_\perp^2}{|q_uB|}}e^{-\frac{k_\perp^2}{|q_dB|}}\frac{1}{t_\parallel^2k_\parallel^2l_\parallel^2}\mathrm{Tr}[O^+\gamma_\mu^\perp \not{l}_\parallel\gamma_\nu\not{t}_\parallel\gamma_\lambda^\perp\not{k}_\parallel].\end{aligned} \tag{3.15}$$

The evaluation of the space integrals for the $\Pi^d_{\mu\nu\lambda}(p,p',q)$ contribution is the same, but with the change $q_u \to q_d$.

Momentum integrals

For the momentum integration, we also split the momentum variables into perpendicular and parallel components. We can integrate the perpendicular part of the three-point function (3.15) by a sequential square completion in the exponentials. Thus, these integrals become Gaussian and can be solved easily. First, we complete the square in the structure given by,

$$e^{i\frac{2\epsilon_{ij}}{|q_uB|}(t+q-l)_j(l-k-p')_i}e^{-\frac{l_\perp^2}{|q_uB|}} = e^{-\frac{1}{|q_uB|}\widetilde{l}_\perp^2-\frac{1}{|q_uB|}(t-k-p)_\perp^2-\frac{2i}{|q_uB|}\epsilon_{ij}(t+q)_j(k+p')_i}, \tag{3.16}$$

where we did the change of variable,

$$\widetilde{l}_i = l_i - i\epsilon_{ij}(t-k-p)_j, \tag{3.17}$$

so that we can integrate the variable $\widetilde{l}$ as a Gaussian integral and so on. The remaining perpendicular structure is

$$e^{-\frac{1}{|q_uB|}(t-k-p)_\perp^2-\frac{2i}{|q_uB|}\epsilon_{ij}(t+q)_j(k+p')_i-\frac{t_\perp^2}{|q_uB|}-\frac{k_\perp^2}{|q_dB|}}, \tag{3.18}$$

completing the square for $t_\perp$ in the exponential we have,

$$(3.18) = e^{-\frac{2}{|q_uB|}\widetilde{t}_\perp^2+\frac{1}{2|q_uB|}(i\epsilon_{ij}(k+p')_i-(k+p)_j)^2-\frac{(k+p)_\perp^2}{|q_uB|}-\frac{2i}{|q_uB|}\epsilon_{ij}q_j(k+p')_i}, \tag{3.19}$$

with the change of variable,

$$\widetilde{t}_j = t_j + \frac{1}{2}(i\epsilon_{ij}(k+p')_i - (k+p)_j). \tag{3.20}$$

The last perpendicular structure is,

$$e^{-\frac{k_\perp^2}{|q_dB|}+\frac{1}{2|q_uB|}(i\epsilon_{ij}(k+p')_i-(k+p)_j)^2-\frac{(k+p)_\perp^2}{|q_uB|}-\frac{2i}{|q_uB|}\epsilon_{ij}q_j(k+p')_i}, \tag{3.21}$$

again, completing the square for $k_\perp$ we get,

$$(3.21) = e^{-\frac{|q_dB|+|q_uB|}{|q_dB||q_uB|}\widetilde{k}_\perp^2+\frac{1}{4}\frac{|q_dB|}{|q_uB|}\frac{(p'_i+p_i+i\epsilon_{ij}q_j)^2}{(|q_dB|+|q_uB|)})-\frac{1}{2|q_uB|}(p'^2_\perp+p_\perp^2-4i\epsilon_{ij}p'_ip_j)}, \tag{3.22}$$

with the last change of variable,

$$\widetilde{k}_i = k_i + \frac{|q_dB|}{2(|q_dB|+|q_uB|)}(p'_i+p_i+i\epsilon_{ij}q_j). \tag{3.23}$$

Now we can integrate the perpendicular momentum variables since there are no other perpendicular dependencies in the three-point function. We get,

$$\begin{aligned} e^{f_u(p_\perp,p'_\perp)}\int d^2\widetilde{l}_\perp d^2\widetilde{t}_\perp d^2\widetilde{k}_\perp\ e^{-\frac{\widetilde{l}_\perp^2}{|q_uB|}}e^{-\frac{2}{|q_uB|}\widetilde{t}_\perp^2}e^{-\frac{|q_dB|+|q_uB|}{|q_dB||q_uB|}\widetilde{k}_\perp^2} \\ = e^{f_u(p_\perp,p'_\perp)}\frac{\pi^3}{2}\frac{|q_uB|^3|q_dB|}{|q_uB|+|q_dB|}, \end{aligned} \tag{3.24}$$

where we defined the remnant structure,

$$f_u(p_\perp,p'_\perp) \equiv \frac{1}{4}\frac{|q_dB|}{|q_uB|}\frac{(p'_i+p_i+i\epsilon_{ij}q_j)^2}{(|q_dB|+|q_uB|)}) - \frac{1}{2|q_uB|}(p'^2_\perp+p^2_\perp-4i\epsilon_{ij}p'_ip_j), \tag{3.25}$$

which goes out from the integral. The integration of the parallel Dirac delta functions in (3.15) leads to $l_\parallel = k_\parallel + p'_\parallel$ and $t_\parallel = k_\parallel + p_\parallel$. Thus, the three-point function is given by,

$$\begin{aligned} \Pi^u_{\mu\nu\lambda}(p,p',q) = i^2(2i)^3q_uN_c\left(\frac{4\pi}{|q_uB|}\right)^2\frac{\pi^3}{2}\frac{1}{(2\pi)^8}\frac{|q_uB|^3|q_dB|}{|q_uB|+|q_dB|}e^{f(p_\perp,p'_\perp)} \\ \underbrace{\int d^2k_\parallel\frac{1}{(k_\parallel+p_\parallel)^2(k_\parallel+p'_\parallel)^2k_\parallel^2}\mathrm{Tr}[O^+\gamma^\perp_\mu(\not{k}_\parallel+\not{p}'_\parallel)\gamma_\nu(\not{k}_\parallel+\not{p}_\parallel)\gamma^\perp_\lambda\not{k}_\parallel]}_{\mathbf{I_1}}. \end{aligned} \tag{3.26}$$

Following the same steps but changing $q_u \leftrightarrow q_d$ we get a similar expression for the $\Pi^d_{\mu\nu\lambda}$ contribution,

$$\begin{aligned} \Pi^d_{\mu\nu\lambda}(p,p',q) = -i^2(2i)^3q_dN_c\left(\frac{4\pi}{|q_dB|}\right)^2\frac{\pi^3}{2}\frac{1}{(2\pi)^8}\frac{|q_dB|^3|q_uB|}{|q_uB|+|q_dB|}e^{f_d(p_\perp,p'_\perp)} \\ \underbrace{\int d^2k_\parallel\frac{1}{(k_\parallel+p_\parallel)^2(k_\parallel+p'_\parallel)^2k_\parallel^2}\mathrm{Tr}[O^+\gamma^\parallel_\nu(\not{k}_\parallel+\not{p}'_\parallel)\gamma_\mu\not{k}_\parallel\gamma^\perp_\lambda(\not{k}_\parallel+\not{p}_\parallel)]}_{\mathbf{I_2}}, \end{aligned} \tag{3.27}$$

with $f_d(p_\perp, p'_\perp)$ defined as,

$$f_d(p_\perp, p'_\perp) \equiv \frac{1}{4}\frac{|q_u B|}{|q_d B|}\frac{(p'_i + p_i + i\epsilon_{ij}q_j)^2}{(|q_d B| + |q_u B|)}) - \frac{1}{2|q_d B|}(p'^2_\perp + p^2_\perp - 4i\epsilon_{ij}p'_i p_j). \tag{3.28}$$

In equations (3.26) and (3.27) we defined $\mathbf{I}_1$ and $\mathbf{I}_2$. The next step is to evaluate these parallel momentum integrals which contain the relevant tensor structures due to the Dirac traces. For this we use the FeynCalc Mathematica package that we mentioned before. The procedure is the following. First, we rewrite the $\mathbf{I}_1$ and $\mathbf{I}_2$ integrals, with the Feynman parametrization defined in the Appendix C.1. Then we evaluate the Dirac trace with the shift on the momentum k_i defined in equation (C.2) for both contributions. The surviving Lorentz structures we get are combinations of $g_{\mu\nu}$, $\epsilon^\perp_{\mu\nu} = g_{1\mu}g_{2\nu} - g_{2\mu}g_{1\nu}$, p_μ and p'_μ. We have to integrate all the resulting terms of these traces, but only terms of even order in l will survive because of the symmetry of the integrals. Thus, we collect the quadratic and zero-order terms using the following identities [16],

$$\int \frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2 - \Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}}\frac{\Gamma\left(n - \frac{d}{2}\right)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}}, \tag{3.29}$$

$$\int \frac{d^d\ell}{(2\pi)^d}\frac{\ell^2}{(\ell^2 - \Delta)^n} = \frac{(-1)^{n-1} i}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma\left(n - \frac{d}{2} - 1\right)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1}, \tag{3.30}$$

$$\int \frac{d^d\ell}{(2\pi)^d}\frac{\ell^\mu \ell^\nu}{(\ell^2 - \Delta)^n} = \frac{(-1)^{n-1} i}{(4\pi)^{d/2}}\frac{g^{\mu\nu}}{2}\frac{\Gamma\left(n - \frac{d}{2} - 1\right)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1}. \tag{3.31}$$

Since we are taking $d = 2$, these integrals are not divergent. Therefore, simplifying terms in equations (3.26) and (3.27), we can write,

$$\begin{aligned}\Pi^d_{\mu\nu\lambda}\left(p, p', q\right) &= \frac{ie^2}{18\pi^3}Be^{f_d(p_\perp, p'_\perp)}\mathbf{I}_2,\\ \Pi^u_{\mu\nu\lambda}\left(p, p', q\right) &= \frac{ie^2}{9\pi^3}Be^{f_u(p_\perp, p'_\perp)}\mathbf{I}_1,\end{aligned} \tag{3.32}$$

where we considered $q_u = \frac{2}{3}e$, $q_d = -\frac{1}{3}e$ and $N_c = 3$. Thus, according to (3.7) the three-point function becomes,

$$\Pi_{\mu\nu\lambda}\left(p, p', q\right) = \frac{ie^2}{18\pi^3}B(e^{f_d(p_\perp, p'_\perp)}\mathbf{I}_2 + 2e^{f_u(p_\perp, p'_\perp)}\mathbf{I}_1). \tag{3.33}$$

Now, we change variables to $q = p' - p$ and $P = \frac{p'+p}{2}$, where q is the transferred four-momentum and P is the momentum average. We consider the symmetric three-momentum configuration where $-p = (E, -\vec{p})$ and $p' = (E, \vec{p})$, so that,

$$\begin{aligned}q &= (2E, 0)\\ P &= (0, \vec{p})\end{aligned}. \tag{3.34}$$

With this configuration $q_\perp = 0$ and $P_\parallel$ has a non-vanishing component only in the $\hat{z}$ direction. The functions (3.25) and (3.28) become,

$$f_u(P_\perp) = f_d(P_\perp) = -\frac{1}{|eB|}P^2_\perp \tag{3.35}$$

Finally, the three-point function can be expanded in its tensor structures given by,

$$\Pi_{\mu\nu\lambda}\left(p,p',q\right)=\epsilon^{\perp}_{\mu\lambda}P^{\parallel}_{\nu}\Pi^{\epsilon}_{P}+\epsilon^{\perp}_{\mu\lambda}q^{\parallel}_{\nu}\Pi^{\epsilon}_{q}+g^{\perp}_{\mu\lambda}P^{\parallel}_{\nu}\Pi^{\perp}_{P}+g^{\perp}_{\mu\lambda}q^{\parallel}_{\nu}\Pi^{\perp}_{q}, \tag{3.36}$$

where we defined Π^{ϵ}_{P}, Π^{ϵ}_{q}, $\Pi^{\perp}_{P}$ and $\Pi^{\perp}_{q}$ being the one loop QCD form factors. These functions depend on the squared parallel momenta and also contain the Feynman parameters integrals. They have the following form,

$$\Pi^{\epsilon}_{P}=\frac{4i|eB|}{9\pi^2}e^{\frac{-P_{\perp}^2}{|eB|}}\int_0^1 dx_1\int_0^{1-x_1}dx_2$$
$$\frac{(x_1+x_2)(4P_{\parallel}^2(x_1+x_2-1)^2+q_{\parallel}^2((x_1)^2-2x_1(x_2+1)+(x_2-1)^2))}{(4P_{\parallel}^2((x_1)^2+x_1(2x_2-1)+(x_2-1)x_2)+q_{\parallel}^2((x_1)^2-x_1(2x_2+1)+(x_2-1)x_2))^2},$$

where the functions Π^{ϵ}_{q}, $\Pi^{\perp}_{P}$ and $\Pi^{\perp}_{q}$ are given in the Appendix D, equations (D.1)-(D.3).

3.1.2 First Landau Level

As we saw in the last section, when considering just the Lowest Landau Level we obtain four kinds of tensor structures. Since we want to construct a QCDSR to extract the pion electromagnetic form factor, we have to compare the tensor structures in equation (3.36) to the ones which appear in the hadronic spectral function, defined in equation (1.19). Since any of the Lorentz structures of equation (3.36) appear also in the hadronic side, we will consider the contribution of the next Landau level of the Schwinger propagator (2.34). This in fact will give proper additional tensor structures to establish a QCDSR. Thus, up to the first Landau Level (1LL) the Schwinger propagator is given by,

$$iS(p)=i2e^{-p_{\perp}^2/|q_fB|}\left(\frac{\not{p}_{\parallel}}{p_{\parallel}^2}O_f^{-}-\frac{1}{p_{\parallel}^2-2|q_fB|}\left[\not{p}_{\parallel}O_f^{-}\left(1-\frac{2p_{\perp}^2}{|q_fB|}\right)-\not{p}_{\parallel}O_f^{+}+2\not{p}_{\perp}\right]\right), \tag{3.37}$$

Recall that we start from equation (3.2), so here we are also considering the splitting of the three-point function into $\Pi=\Pi_u+\Pi_d$. For the perpendicular momenta integrals, we use the method developed in equations (3.16) to (3.23), since it is still valid. However, the Gaussian integrals change, due to the linear and quadratic perpendicular momenta dependencies in the traces. To summarize, in Π_u we make the consecutive square completion for each momentum variable,

$$\int d^2l_{\perp}d^2t_{\perp}d^2k_{\perp}\;e^{i\frac{2\epsilon_{ij}}{|q_uB|}(t+q-l)_j(l-k-p')_i}e^{-\frac{l_{\perp}^2}{|q_uB|}-\frac{t_{\perp}^2}{|q_uB|}-\frac{k_{\perp}^2}{|q_dB|}}$$
$$=\int d^2\widetilde{l}_{\perp}d^2\widetilde{t}_{\perp}d^2\widetilde{k}_{\perp}\;e^{-\frac{\widetilde{l}_{\perp}^2}{|q_uB|}}e^{-\frac{2}{|q_uB|}\widetilde{t}_{\perp}^2}e^{-\frac{|q_dB|+|q_uB|}{|q_dB||q_uB|}\widetilde{k}_{\perp}^2},$$

where we made the following change of variable in sequential order,

$$\widetilde{l}_i=l_i-i\epsilon_{ij}(t-k-p)_j, \tag{3.38}$$
$$\widetilde{t}_j=t_j+\frac{1}{2}(i\epsilon_{ij}(k+p')_i-(k+p)_j), \tag{3.39}$$
$$\widetilde{k}_i=k_i+\frac{|q_dB|}{2(|q_dB|+|q_uB|)}(p'_i+p_i+i\epsilon_{ij}q_j). \tag{3.40}$$

We want the triangle to have one internal line with the 1LL and the other two with the LLL. Then, for Π_u we have,

$$
\begin{aligned}
&\mathrm{Tr}[\gamma_\mu i\mathcal{S}^u(l)\gamma_\nu i\mathcal{S}^u(t)\gamma_\lambda i\mathcal{S}^d(k)] \\
&= -\,\mathrm{Tr}[\gamma_\mu \frac{1}{l_\parallel^2 - 2|q_uB|}\left[\not{l}_\parallel O_u^-\left(1-\frac{2l_\perp^2}{|q_uB|}\right) - \not{l}_\parallel O_u^+ + 2\not{l}_\perp\right]\gamma_\nu \frac{\not{t}_\parallel}{t_\parallel^2}O_u^-\gamma_\lambda \frac{\not{k}_\parallel}{k_\parallel^2}O_d^-] && (3.41)\\
&- \mathrm{Tr}[\gamma_\mu \frac{\not{l}_\parallel}{l_\parallel^2}O_u^-\gamma_\nu \frac{1}{t_\parallel^2 - 2|q_uB|}\left[\not{t}_\parallel O_u^-\left(1-\frac{2t_\perp^2}{|q_uB|}\right) - \not{t}_\parallel O_u^+ + 2\not{t}_\perp\right]\gamma_\lambda \frac{\not{k}_\parallel}{k_\parallel^2}O_d^-] && (3.42)\\
&- \mathrm{Tr}[\gamma_\mu \frac{\not{l}_\parallel}{l_\parallel^2}O_u^-\gamma_\nu \frac{\not{t}_\parallel}{t_\parallel^2}O_u^-\gamma_\lambda \frac{1}{k_\parallel^2 - 2|q_dB|}\left[\not{k}_\parallel O_d^-\left(1-\frac{2k_\perp^2}{|q_dB|}\right) - \not{k}_\parallel O_d^+ + 2\not{k}_\perp\right]]. && (3.43)
\end{aligned}
$$

Expanding the traces and with the properties in (2.35)-(2.38) we rewrite each one above as,

$$
\begin{aligned}
(3.41) = \frac{1}{(l_\parallel^2 - 2|q_uB|)t_\parallel^2 k_\parallel^2}\Bigg[&\mathrm{Tr}[O^+\gamma_\mu \not{l}_\parallel \gamma_\nu \not{t}_\parallel \gamma_\lambda^\perp \not{k}_\parallel] + \frac{2l_\perp^2}{|q_uB|}\mathrm{Tr}[O^+\gamma_\mu^\perp \not{l}_\parallel \gamma_\nu \not{t}_\parallel \gamma_\lambda^\perp \not{k}_\parallel] \\
&-2\mathrm{Tr}[O^+\gamma_\mu \not{l}_\perp \gamma_\nu \not{t}_\parallel \gamma_\lambda^\perp \not{k}_\parallel]\Bigg],
\end{aligned}
\tag{3.44}
$$

$$
\begin{aligned}
(3.42) = \frac{1}{l_\parallel^2(t_\parallel^2 - 2|q_uB|)k_\parallel^2}\Bigg[&\mathrm{Tr}[O^+\gamma_\mu^\perp \not{l}_\parallel \gamma_\nu \not{t}_\parallel \gamma_\lambda \not{k}_\parallel] + \frac{2t_\perp^2}{|q_uB|}\mathrm{Tr}[O^+\gamma_\mu^\perp \not{l}_\parallel \gamma_\nu \not{t}_\parallel \gamma_\lambda^\perp \not{k}_\parallel] \\
&-2\mathrm{Tr}[O^+\gamma_\mu^\perp \not{l}_\parallel \gamma_\nu \not{t}_\perp \gamma_\lambda \not{k}_\parallel]\Bigg],
\end{aligned}
\tag{3.45}
$$

$$
\begin{aligned}
(3.43) = \frac{1}{l_\parallel^2 t_\parallel^2(k_\parallel^2 - 2|q_dB|)}\Bigg[&-\left(1-\frac{2k_\perp^2}{|q_dB|}\right)\mathrm{Tr}[O^+\gamma_\mu^\perp \not{l}_\parallel \gamma_\nu \not{t}_\parallel \gamma_\lambda^\perp \not{k}_\parallel] + \mathrm{Tr}[O^-\gamma_\mu^\parallel \not{l}_\parallel \gamma_\nu \not{t}_\parallel \gamma_\lambda^\parallel \not{k}_\parallel] \\
&-2\mathrm{Tr}[O^+\gamma_\mu^\perp \not{l}_\parallel \gamma_\nu \not{t}_\parallel \gamma_\lambda^\parallel \not{k}_\perp] + 2\mathrm{Tr}[O^-\gamma_\mu^\parallel \not{l}_\parallel \gamma_\nu \not{t}_\parallel \gamma_\lambda^\perp \not{k}_\perp]\Bigg].
\end{aligned}
\tag{3.46}
$$

Denoting each equations (3.44), (3.45) and (3.46) as $A_{\mu\nu\lambda}, B_{\mu\nu\lambda}$ and $C_{\mu\nu\lambda}$, we can write,

$$
\mathrm{Tr}[\gamma_\mu i\mathcal{S}^u(l)\gamma_\nu i\mathcal{S}^u(t)\gamma_\lambda i\mathcal{S}^d(k)] = A_{\mu\nu\lambda} + B_{\mu\nu\lambda} + C_{\mu\nu\lambda}. \tag{3.47}
$$

For the Π_d contribution we have to change $q_u \leftrightarrow q_d$ and $O^- \leftrightarrow O^+$ then redo the calculations. Therefore the total integral that we have to solve for the Π_u contribution is given by,

$$
\begin{aligned}
&\Pi^u_{\mu\nu\lambda}\left(p,p',q\right) = \\
&-i^2 N_c \frac{-q_u}{(2\pi)^8}\left(\frac{4\pi}{|q_uB|}\right)^2 e^{f_u(p_\perp,p'_\perp)}\int d^2l_\parallel d^2t_\parallel d^2k_\parallel\ \delta^{(2)}((l-k-p')_\parallel)\delta^{(2)}((t+q-l)_\parallel) \\
&d^2\widetilde{l}_\perp d^2\widetilde{t}_\perp d^2\widetilde{k}_\perp\ e^{-\frac{\widetilde{l}_\perp^2}{|q_uB|}}e^{-\frac{2}{|q_uB|}\widetilde{t}_\perp^2}e^{-\frac{|q_dB|+|q_uB|}{|q_dB||q_uB|}\widetilde{k}_\perp^2}\left(A_{\mu\nu\lambda}+B_{\mu\nu\lambda}+C_{\mu\nu\lambda}\right).
\end{aligned}
\tag{3.48}
$$

From now on, we use Mathematica to make every calculation, since the algebraic expressions for the integrals are extremely laborious and cumbersome. We will explain, however, the procedure. Note that the terms $A_{\mu\nu\lambda}, B_{\mu\nu\lambda}$ and $C_{\mu\nu\lambda}$ have linear and quadratic terms in a singly momentum variable. In each one, we have to make the corresponding transformation of equations (3.38), (3.39) and (3.40) so that we can integrate all variables as Gaussian integrals. Here we keep the integrals with zero and quadratic order in the momentum i.e. the even ones, since the odd integrals vanish. But each change of variable depends on the remaining ones and, therefore, we have to do it sequentially. Thus, for the perpendicular momenta integrals we use the identities,

$$\int_{-\infty}^{\infty} d^2p_{\parallel} p_{\parallel}^2 e^{-Ap_{\parallel}^2} = -i\frac{\pi}{A^2}, \quad \int_{-\infty}^{\infty} d^2p_{\parallel} e^{-Ap_{\parallel}^2} = -i\frac{\pi}{A},$$
$$\int_{-\infty}^{\infty} d^2p_{\perp} p_{\perp}^2 e^{-Ap_{\perp}^2} = -\frac{\pi}{A^2}, \quad \int_{-\infty}^{\infty} d^2p_{\perp} e^{-Ap_{\perp}^2} = -\frac{\pi}{A},$$

Having done the perpendicular part, we can proceed with the parallel integrals using the Feynman parametrization method. From the Dirac delta functions, we introduce $l_{\parallel} = k_{\parallel} + p'_{\parallel}$ and $t_{\parallel} = k_{\parallel} + p_{\parallel}$. We use the Feynman parametrizations for $A_{\mu\nu\lambda}$, $B_{\mu\nu\lambda}$ and $_{C}\mu\nu\lambda$ defined in Appendix C, equations (C.4)-(C.6). The first Landau Level contribution to the three-point function contains many new different combinations of $g_{\mu\nu}$, $\epsilon^{\perp}_{\mu\nu} = g_{1\mu}g_{2\nu} - g_{2\mu}g_{1\nu}$, q_μ and P_μ, recall the change of variables (3.34). Therefore, as we did in the previous section, we can expand Π in its constituents Lorentz structures and define for each one a function factor. Here we will concentrate on the following tensor structures,

$$\Pi_{\mu\nu\lambda}(P,q) = \Pi_1^{\parallel} P_{\lambda}^{\parallel} P_{\mu}^{\parallel} P_{\nu}^{\parallel} + \Pi_1^S \widetilde{P}_{\lambda} P_{\mu}^{\parallel} P_{\nu}^{\parallel}, \tag{3.49}$$

where we defined $\widetilde{P}_\mu \equiv \epsilon^{\perp}_{\mu\nu} P^\nu = (g_{1\mu}P_2 - g_{2\mu}P_1)$ and the QCD form factors $\Pi_1^{\parallel}$ and Π_1^S are defined in the Appendix D. The first term in the last equation is the parallel part of the same tensor structure that is conventionally used to extract the pion form factor at zero eB. The second tensor structure does not have an analogous structure in the hadronic side but we include it here for purposes that will become clear later in this work.

3.2 Weak magnetic field limit

3.2.1 First order eB-corrections

In chapter 2 we introduced the Schwinger propagator and its representation in both magnetic field limits. Now we want to evaluate the three-point correlation function for a weak magnetic field background. In the strong field limit all the internal fermionic propagators have to include magnetic corrections from the beginning with the LLL, and then all the combinations with one internal line being in the 1LL. In contrast with the weak magnetic field we can start from zero eB and then add contributions in powers of eB. The massless Schwinger propagator in the weak magnetic field limit and up to order eB is written as,

$$i\mathcal{S}(p) = i\frac{\not{p}}{p^2} - \frac{\gamma_1\gamma_2\left(\gamma\cdot p_\parallel\right)}{\left(p^2\right)^2}q_f B \tag{3.50}$$

We also start from equation (B.13). As we said in the previous chapter we can get this form for the correlator through the charge conjugation operator and the trace properties (see Appendix B),

$$\begin{aligned}\Pi_{\mu\nu\lambda}\left(p,p',q\right) = -i^2 N_c \int & \frac{d^4l}{(2\pi)^4}\frac{d^4t}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}d^4x d^4y\; e^{i(p'x-qy)}e^{-il(x-y)}e^{-ity}e^{ikx}\\ & \{q_d\Omega_d^C \mathrm{Tr}[\gamma_\mu i\mathcal{S}^d(l)\gamma_\nu i\mathcal{S}^d(t)\gamma_\lambda i\mathcal{S}^u(k)]\\ & -q_u\Omega_u \mathrm{Tr}[\gamma_\mu i\mathcal{S}^u(l)\gamma_\nu i\mathcal{S}^u(t)\gamma_\lambda i\mathcal{S}^d(k)]\}.\end{aligned}$$

Now we have to replace the propagator (3.50) in each trace of the equation above for the three-point function. Since we want to keep contributions just up to the first power in eB, each trace will split into four different traces. For the trace proportional to q_u we get,

$$\begin{aligned}\mathrm{Tr}\left[\gamma_\mu i\mathcal{S}^u(l)\gamma_\nu i\mathcal{S}^u(t)\gamma_\lambda i\mathcal{S}^d(k)\right] = \mathrm{Tr}\Bigg[&\gamma_\mu i\frac{\not{l}}{l^2}\gamma_\nu i\frac{\not{t}}{t^2}\gamma_\lambda i\frac{\not{k}}{k^2}\\ &-(q_uB)\gamma_\mu\frac{\gamma_1\gamma_2\not{l}_\parallel}{\left(l^2\right)^2}\gamma_\nu i\frac{\not{t}}{t^2}\gamma_\lambda i\frac{\not{k}}{k^2}\\ &-(q_uB)\gamma_\mu i\frac{\not{l}}{l^2}\gamma_\nu\frac{\gamma_1\gamma_2\not{t}_\parallel}{\left(t^2\right)^2}\gamma_\lambda i\frac{\not{k}}{k^2}\\ &-(q_dB)\;\gamma_\mu i\frac{\not{l}}{l^2}\gamma_\nu i\frac{\not{t}}{t^2}\gamma_\lambda\frac{\gamma_1\gamma_2\not{k}_\parallel}{\left(k^2\right)^2}\Bigg].\end{aligned} \tag{3.51}$$

We keep the zero-order term because we also have to expand the three-product Schwinger phase, which in this case for both contributions is given by,

$$\Omega_{u/d} = 1 - \frac{|q_{u/d}B|}{2}\epsilon_{ij}x^i y^j. \tag{3.52}$$

Equations (3.51) and (3.52) are also valid for the q_d contribution but with the change $u \leftrightarrow d$. Therefore, we insert these expressions into (3.51) and collect the terms up to order q_fB. If we replace the quark charge values $q_u = \frac{2}{3}e$ and $q_d = -\frac{1}{3}e$ the three-point correlation function is given by,

$$\Pi_{\mu\nu\lambda}\left(p,p',q\right) = -i^2 N_c \int \frac{d^4l}{(2\pi)^4}\frac{d^4t}{(2\pi)^4}\frac{d^4k}{(2\pi)^4} d^4x d^4y\ e^{i(p'x-qy)}e^{-il(x-y)}e^{-ity}e^{ikx}$$

$$\{-e\mathrm{Tr}\left[\gamma_\mu i\frac{\not{l}}{l^2}\gamma_\nu i\frac{\not{t}}{t^2}\gamma_\lambda i\frac{\not{k}}{k^2}\right] \tag{3.53}$$

$$+\frac{1}{3}|eB|\mathrm{Tr}\left[\gamma_\mu\frac{\gamma_1\gamma_2\not{l}_\parallel}{(l^2)^2}\gamma_\nu i\frac{\not{t}}{t^2}\gamma_\lambda i\frac{\not{k}}{k^2}\right] \tag{3.54}$$

$$+\frac{1}{3}|eB|\mathrm{Tr}\left[\gamma_\mu i\frac{\not{l}}{l^2}\gamma_\nu\frac{\gamma_1\gamma_2\not{t}_\parallel}{(t^2)^2}\gamma_\lambda i\frac{\not{k}}{k^2}\right] \tag{3.55}$$

$$+\frac{5}{18}|eB|\epsilon_{ij}x^iy^j\mathrm{Tr}\left[\gamma_\mu i\frac{\not{l}}{l^2}\gamma_\nu i\frac{\not{t}}{t^2}\gamma_\lambda i\frac{\not{k}}{k^2}\right]\} \tag{3.56}$$

The first trace is the zero-order in eB contribution which we introduced in section 1.3. In the last trace term we have a space dependence coming from the expansion of the Schwinger phase product. We can rewrite equation (3.56) to a proper form, as in [39], with the identity,

$$e^{-i(l-k-p')x}e^{-i(q+t-l)y}x^iy^j = \frac{\partial}{\partial k^i}\frac{\partial}{\partial t^j}e^{-i(l-k-p')x}e^{-i(q+t-l)y}. \tag{3.57}$$

After integration by parts, the partial derivatives act on the trace. This causes an additional splitting into several traces coming from (3.56) due to,

$$\frac{\partial}{\partial t^i}\frac{\gamma_\mu t^\mu}{t^\mu t_\mu} = \frac{\gamma_i}{t^2} - \frac{\not{t}}{(t^2)^2}2t_i.$$

Therefore, we rewrite equation (3.56) as,

$$\frac{5}{18}|eB|\int\frac{d^4k}{(2\pi)^4}\ \epsilon_{ij}\Bigg(\mathrm{Tr}\left[\gamma_\mu i\frac{\not{k}+\not{p}'}{(k+p')^2}\gamma_\nu i\frac{\gamma_j}{(k+p)^2}\gamma_\lambda i\frac{\gamma_i}{k^2}\right] \tag{3.58}$$

$$-2\mathrm{Tr}\left[\gamma_\mu i\frac{\not{k}+\not{p}'}{(k+p')^2}\gamma_\nu i\frac{\gamma_j}{(k+p)^2}\gamma_\lambda i\frac{\not{k}}{(k^2)^2}k_i\right] \tag{3.59}$$

$$-2\mathrm{Tr}\left[\gamma_\mu i\frac{\not{k}+\not{p}'}{(k+p')^2}\gamma_\nu i\frac{\not{k}+\not{p}}{((k+p)^2)^2}(k+p)_j\gamma_\lambda i\frac{\gamma_i}{k^2}\right] \tag{3.60}$$

$$+4\mathrm{Tr}\left[\gamma_\mu i\frac{\not{k}+\not{p}'}{(k+p')^2}\gamma_\nu i\frac{\not{k}+\not{p}}{((k+p)^2)^2}(k+p)_j\gamma_\lambda i\frac{\not{k}}{(k^2)^2}k_i\right]\Bigg) \tag{3.61}$$

The equations (3.54) and (3.55) contain parallel and perpendicular structures but with $\not{p}_\parallel = \gamma_\parallel^\mu p_\mu$ and $p_i = g_{i\mu}p^\mu$ where $i=1,2$, we can project the parallel and perpendicular dependence into the gamma matrix or the metric tensor. We also can easily integrate the spatial variables getting,

$$\int d^4xd^4ye^{-ix(l-k-p')}e^{-iy(t+q-l)} = (2\pi)^4\delta^{(4)}\left(l-k-p'\right)(2\pi)^4\delta^{(4)}(t+q-l). \tag{3.62}$$

Now, we follow the same procedure as in the previous section. We use the Feynman parametrizations defined in Appendix C.2. With the FeynCalc Mathematica package we evaluate the Dirac traces and collect the zero and quadratic order terms in the momentum integration variable. Thus, we can integrate the momentum with the identities defined in equations (3.29)-(3.31). We get all the tensor structures, being their coefficients the form factor functions. At this point, to be consistent, we also change to the variables $q = p' - p$ and $P = \frac{p'+p}{2}$ and choose the symmetric three-momentum configuration.

The purpose of the magnetic corrections calculations in the QCD side is to find a single structure which can be mapped to the hadronic world. But, there is no emergence of a tensor structure which also appears in the ones defined in the hadronic spectral function at zero eB, equation (1.19). Consequently, we cannot extract the form factor at the same tensor structure as in zero eB. Nevertheless, in the next chapter in section 4.2, we will see that we can establish a QCDSR to new tensor structures which arise through a redefinition of the hadronic spectral function at finite eB. With this in mind, from all the combinations of $g_{\mu\nu}$, $\epsilon^{\perp}_{\mu\nu} = g_{1\mu}g_{2\nu} - g_{2\mu}g_{1\nu}$, q_μ and P_μ that we got, we choose the following tensor structure,

$$\Pi^{W}_{\mu\nu\lambda}(P,q) = (eB)\Pi^{W}_{1}\widetilde{P}_\lambda P^{\parallel}_{\mu}P^{\parallel}_{\nu}. \tag{3.63}$$

Notice that we extract the factor eB out from Π^{W}_{1} since it is the only dependence. Recall that $\widetilde{P}_\mu \equiv \epsilon^{\perp}_{\mu\nu}P^{\nu}$. We defined Π^{W}_{1} in Appedix D, equation (D.6).

Chapter 4

Magnetic corrections to F_π

Let us summarize what we have obtained up to this point. In chapter 3 we evaluated the three-point current correlation function under the effects of a strong magnetic background. The influence of the magnetic field is encoded in the Schwinger propagator and its representation through a sum of Landau levels. At first glance, it is generally assumed that the lowest Landau level (LLL) is the dominant contribution in the limit of a strong magnetic field [39]. But recent works have shown that the LLL is not sufficient for certain particular processes and, therefore, other Landau levels need to be considered since they can be of the same order as the LLL [43]. For example in [44], they considered up to the first Landau level due to the high virtuality in the neutrino dispersion relation in external magnetic fields, improving the results of previous calculations. Furthermore, in equation (3.36) we demonstrated that the LLL contribution does not provide suitably Lorentz tensor structures to establish a QCDSR with the hadronic spectral function in the vacuum. Therefore we computed the first landau Level contribution leading to well-defined tensor structures which also appear in the hadronic side (see equation (1.19)).

At the end of chapter 3, we went through the weak magnetic field limit, where the Schwinger propagator can be written as an expansion in powers of $q_f B$. Here we just considered up to the first-order contribution, due to the analytical difficulties of the calculation. We also ended up with several Lorentz tensor structures contributing to the three-point function, but none of them appears also in the hadronic spectral function. However, these new kinds of tensor structures emerge in both limits of the QCD side, weak and strong magnetic field. The major difference between both limits is that in the strong magnetic field limit, when other Landau levels are considered, some tensor structures map to the hadronic side and some of them do not.

It turns out, that novel works [9] [14] have revealed that when studying the magnetic effects to pion decay, new decay constants arise, hence new channels have to be considered in the hadronic spectral function. These new contributions lead to hadronic tensor structures which also appear in the QCD side. We will study this extension to the hadronic spectral function in section 4.2.

4.1 Strong magnetic field FESR

In order to construct a FESR, in general, we need to let the external momentum variables p^2 and p'^2 of the triangle loop to be the s and s' variables. But we have a splitting in every Lorentz structure into parallel and perpendicular components due to the magnetic field. Consequently, we can first take the frame in which the perpendicular components vanishes $p'^2_\perp = p^2_\perp = 0$ leading to $s \equiv p^2_\parallel$ and $s' \equiv p'^2_\parallel$. Recall $\Pi^\parallel_1$ defined in equation (3.49) associated to the structure $P^\parallel_\mu P^\parallel_\nu P^\parallel_\lambda$,

$$\Pi^\parallel_1(eB, P^2_\perp, P^2_\parallel, q^2_\parallel) \equiv \Pi^\parallel_1 = \frac{1}{\pi^4|\mathrm{eB}|} e^{\frac{-P^2_\perp}{|eB|}} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \\ \frac{216i(x_1+x_2-1)^2(x_1+x_2)}{\left(8|eB|(x_1+x_2-1)-12P^2_\parallel(x_1+x_2)(x_1+x_2-1)+3q^2_\parallel\left(-x_1^2+2x_1x_2+x_1-x_2^2+x_2\right)\right)^2}. \tag{4.1}$$

To write the expression above in terms of the s, s' variables we change,

$$P^2_\parallel \to (Q^2_\parallel + (\sqrt{s} + \sqrt{s'})^2)/2, \tag{4.2}$$

which is valid in our frame of reference. Therefore, the QCD spectral function is given by,

$$\rho\left(eB, s, s', Q^2_\parallel\right)\Big|_{QCD} = \mathrm{Im}\,\Pi^\parallel_1\left(eB, s, s', Q^2_\parallel\right). \tag{4.3}$$

We follow the method introduced in section 1.3 to extract the electromagnetic form factor via FESR at $B = 0$. The lowest dimensional FESR defined in equation (1.20) leads to,

$$F_\pi(eB, Q^2_\parallel) = \frac{1}{4f^2_\pi} \int_\Omega ds ds' \; \rho(eB, s, s', Q^2_\parallel)\Big|_{QCD}, \tag{4.4}$$

Now we focus on technicalities to evaluate the four integrals involved, two on the s, s' variables and two on the Feynman parameters. We used Mathematica to calculate analytically one of the Feynman parameters integrals. Then we must use a numerical integration tool to estimate the rest since the analytical evaluation is not possible, as it was the case wherein $B = 0$. Here we also want to evaluate the s, s' integrals in the Ω triangle region on the (s, s')-plane (see Figure 1.4). To do so, we change $s' \to (s_0 - s)s'$ in order to get numerical integration limits.

An important effect that we have not discussed yet is that f_π gets corrections from the magnetic field background. Different works have calculated the behaviour of $f_\pi(eB)$, for example [8] [14] [9]. To introduce these results in our calculation we take the one from [14], which studied the masses and decay constants of charged pions through the Nambu–Jona-Lasino (NJL) model in intense magnetic fields. We extracted the $f_\pi(eB)$ data and also made an interpolation to get the value of f_π at slightly higher magnetic fields. The original data of [14] is shown in Figure 4.1. Therefore, the final form of the electromagnetic form factor is given by,

$$F_\pi(eB, Q^2_\parallel) = \frac{1}{4f^2_\pi} \int_0^{s_0} \int_0^{s_0} ds ds' \; (s_0 - s)\rho(eB, s, s', Q^2_\parallel)\Big|_{QCD}, \tag{4.5}$$

We used the Adaptive Monte Carlo Numerical integration method provided by Mathematica. In general, this method has a better performance than the crude Monte Carlo strategy. By partitioning the region in each dimension, this method concentrates on the subregion where the integrand has any kind of discontinuity. It can be adjusted by increasing the number of maximum recursions to tell the algorithm when to stop and the minimum number of recursions to guarantee that a narrow spike in the integrand is not

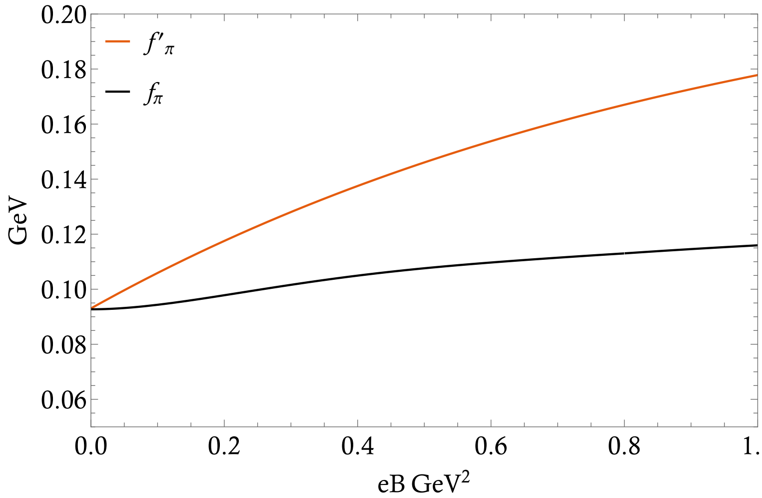


Figure 4.1: Magnetic field dependence of the pion decay constants f_π and f'_π.

missed. Also, the estimation improves when the number of sample points increments but also the execution time. All these features have to be tested to assure a right convergence of the integral. The numerical results for the equation (4.5) are shown in Figures **??** to 4.2, taking $s_0 = 1$ GeV^2. We get rid of the numerical noise by fitting an exponential model for each plot.

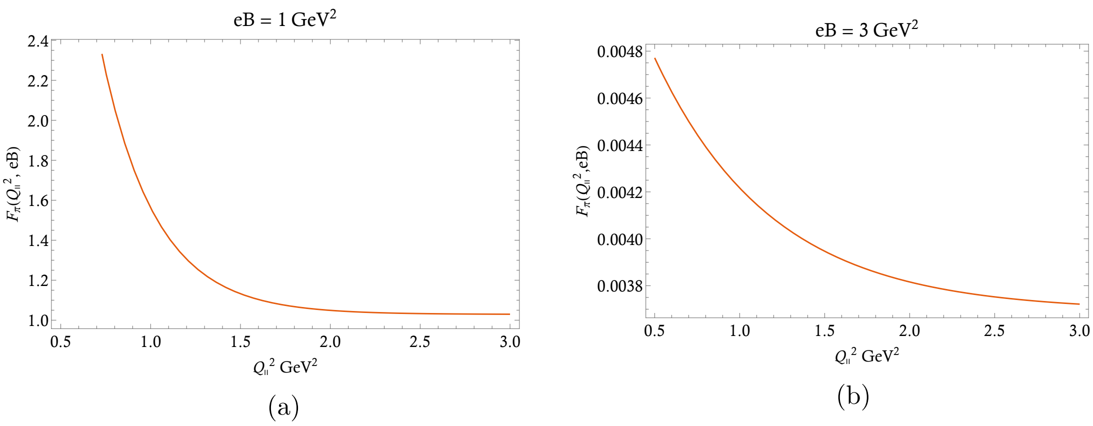
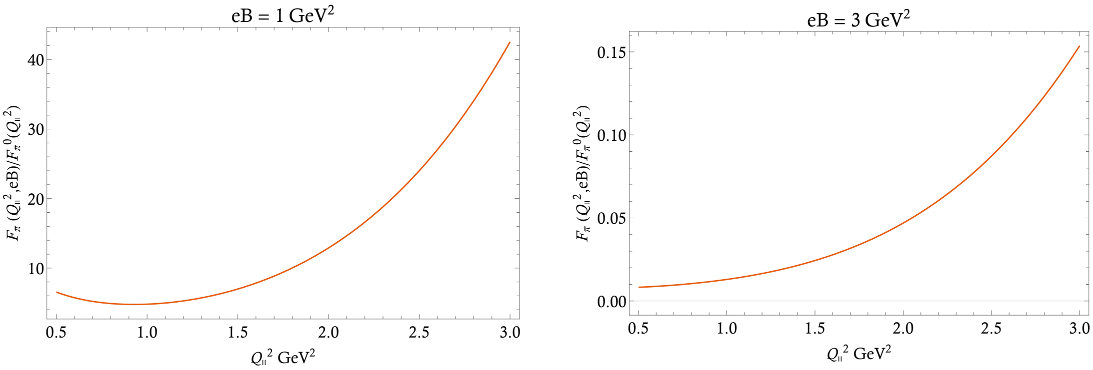


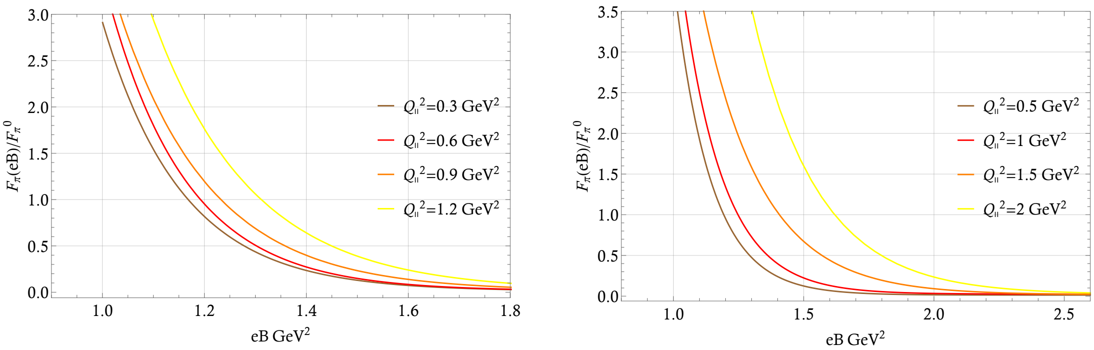
Figure 4.2: Pion Electromagnetic form factor under strong magnetic fields $eB = 1, 3$ GeV^2.

In Figures 4.2 and 4.3, we can see that the chosen value of the magnetic field has a huge impact on the magnitude of F_π. The ratio between the pion form factor with magnetic corrections and the vacuum form factor is shown in Figure 4.3. As we described in chapter 2, the magnetic field has to be the largest energy scale. A natural energy scale of a typical process involving F_π is the pion mass m_π. But since we took the chiral limit, the mass can not be a scaling parameter, due to numerical inconsistencies. Therefore, we establish the constraint $\{s_0, Q^2\} \lesssim eB$, so that we can identify the valid domain of our magnetic corrections. Following this constraint, in Figures 4.4 we notice that for each fixed $Q_\parallel^2$ the form factor is valid for magnetic fields such that $Q^2 \lesssim eB$ showing a significant increment of $F_\pi(eB)$ compared to the vacuum F_π^0 for some eB regions.



(a) (b)

Figure 4.3: Dependence on $Q_\parallel^2$ of the normalized pion electromagnetic form factor with fixed eB values.



(a) (b)

Figure 4.4: Dependence on eB of the normalized pion electromagnetic form factor with fixed $Q_\parallel^2$ values.

4.2 Hadronic spectral function redefinition

Recent works from Lattice QCD [9] and effective theories [14], generalized the hadronic matrix element of the axial current between the vacuum and a pion state under the influence of a magnetic field background. Since the $F_{\mu\nu}$ tensor can now be part of the tensor structure of the matrix element. The strength tensor can then be written in the convenient form $F_{\mu\nu} = \epsilon^{\perp}_{\mu\nu}B$, with the perpendicular antisymmetric tensor defined as $\epsilon^{\perp}_{\mu\nu} = g_{\mu 1}g_{\nu 2} - g_{\nu 1}g_{\mu 2}$. Thus we get,

$$\langle 0|J_{A\mu}|\pi^+(p)\rangle \equiv i\sqrt{2}f_\pi p_\mu + i\sqrt{2}f'_\pi eB\widetilde{p}_\mu, \tag{4.6}$$

where we defined $\widetilde{p}_\mu \equiv \epsilon^{\perp}_{\mu\nu}p^\nu$. Now we can calculate the hadronic spectral function with this new definition. We follow the same procedure as in section 1.3, using single pion states to saturate the spectral function followed by the continuum after the threshold s_0,

$$\begin{aligned}\rho_{\mu\nu\lambda}\left(s, s', Q^2\right)\big|_{HAD} = &< 0\left|j_{A\nu}(0)\right|\pi^+\left(p'\right) >< \pi^+\left(p'\right)\left|j^{el}_\lambda(0)\right|\pi^+(p) > \\ &\times < \pi^+(p)\left|j_{A\mu}(0)\right|0 > \delta(s)\delta\left(s'\right) + \text{ continuum }.\end{aligned} \tag{4.7}$$

Recall that $\langle\pi^+(p')|J^{el}_\mu|\pi^+(p)\rangle \to F_\pi(B,Q^2)(p_\mu + p'_\mu)$. Inserting equation (4.6) in (4.7) and changing to the variables P and q, the hadronic spectral function is given by,

$$\begin{aligned}&\rho_{\mu\nu\lambda}\left(s, s', Q^2\right)|_{HAD} \\ &\quad = f_\pi^2 F_\pi(Q^2)P^\nu(2P^\lambda - q^\lambda)(2P^\mu - q^\mu)\delta(s)\delta(s') \\ &\quad + 4(eB)^2F_\pi(B,Q^2)f'^2_\pi P^\nu\widetilde{P}^\lambda\widetilde{P}^\mu\delta(s)\delta(s') \\ &\quad + (eB)f_\pi f'_\pi F_\pi(B,Q^2)P^\nu\left(4P^\lambda\widetilde{P}^\mu + 4P^\mu\widetilde{P}^\lambda - 2q^\lambda\widetilde{P}^\mu - 2q^\mu\widetilde{P}^\lambda\right)\delta(s)\delta(s').\end{aligned} \tag{4.8}$$

Where we have omitted the continuum. These new kinds of tensor structures, like $\widetilde{P}^\mu P^\nu q^\lambda$, also emerge in the QCD three-point function with magnetic corrections introduced in chapter 3. In fact, we can construct new QCDSR with these structures. The difference now is that the new constant f'_π is needed. The dependence of this constant on the magnetic field is also extracted from [14] and is shown in Figure 4.1. We can establish the lowest dimensional FESR at the structure $\widetilde{P}_\lambda P^{\parallel}_\mu P^{\parallel}_\nu$ for both, strong magnetic field and weak magnetic field limit. The QCD one-loop form factors are defined trough equations (3.49) and (3.63). Then, following the same steps as in the previous section to extract the pion form factor, both Sum Rules are given by,

$$\widetilde{F}_\pi(eB, Q^2_\parallel) = \frac{1}{f_\pi f'_\pi eB}\int_\Omega dsds'\ \rho(eB, s, s', Q^2_\parallel)\Big|_{QCD}, \tag{4.9}$$

where we defined $\widetilde{F}_\pi$ to differentiate the one defined at the $P_\mu P_\nu P_\lambda$ structure (denoted by $\widetilde{F}_\pi$ (4.5)). The QCD spectral function in (4.9) can be either $\rho_{QCD} = \operatorname{Im}\Pi^S_1$ for strong magnetic field, or $\rho_{QCD} = (eB)\operatorname{Im}\Pi^W_1$ for weak magnetic field. Though, to be able to build these Sum Rules we have to extend the calculations to a finite $P^2_\perp$, since we are mapping at structures involving the perpendicular components of P. For the strong field limit, the numerical results show the same qualitative behavior as the form factor obtained in the previous section. However, early results for the extension to finite $P^2_\perp$ for the pion form factor defined in equation (4.5), show that the largest increasing values for $F_\pi(eB)$ are reached where the perpendicular momenta vanishes.

For the weak field limit, the numerical result presents anomalous behaviour. We declare that this calculation must be improved. Even if the Sum Rule can be constructed, the eB dependence occurs only via the constants f_π and f'_π, since we restricted ourselves to linear order in eB for the three-point function. We believe that going further to quadratic order, would allow to get an expansion up to linear order in eB for F_π and also it might provide as well the conventional tensor structure $P_\mu P_\nu P_\lambda$ to extract the magnetic form factor.

Finally, we want to mention that besides the redefinition of the Axial matrix element (4.6) with its new constant f'_π, a new channel arises with the magnetic field corrections, the vector $\left\langle 0 \left| j^{el}_\mu \right| \pi^- \right\rangle$ channel [9] [14], which in $B = 0$ vanishes. Therefore, a future work for the magnetic corrections to F_π is to consider the V-A currents for the three-point function. From recent works [45], we can expect that the effect of this channel is small compared to the Axial contribution.

Discussion and Conclusions

This thesis was devoted to studying the magnetic corrections to the pion electromagnetic form factor $F_\pi(eB)$. We employed the Finite Energy Sum Rules framework to establish a mapping between a suitable interpolating QCD current correlator with the hadronic spectral function for the $\pi\pi\gamma$ vertex. The magnetic part of the calculation was done through the introduction of the Schwinger propagator in the QCD correlation function, considering both the strong as well as the weak field limits.

Restricting ourselves to the linear order in eB in the weak field limit, produced the lack of proper tensor structures for the construction of the Sum Rule. However, the redefinition of the hadronic spectral function made it possible to establish a Sum Rule with new kinds of tensor structures involving perpendicular parts of the external momenta. However, since both sides, hadronic and QCD, were proportional to eB the magnetic field dependence cancels out. The QCD contribution to the form factor happen to be trivial, leaving as the only magnetic dependence the hidden evolution of the decay constants f_π and f'_π. The numerical results were anomalous so we concluded that this result must be improved by adding higher orders in the expansion in eB.

For the strong field limit, we went further, taking the first Landau level contribution, since the lowest Landau level produced the same issues as in the weak field. The addition of this higher level made it possible to find the conventional tensor structure which is used in the literature to extract de pion form factor. We established a QCDSR and obtaining the magnetic dependence of the pion electromagnetic form factor, but restricted to the regions where $\{s_0, Q^2\} \lesssim eB$. We found that in the momentum transfer region around $Q^2 \simeq 1$ GeV^2 the corrected pion form factor is considerably larger than the zero-eB form factor, reaching values up to four times $F_\pi(eB = 0)$. And even for higher energies, for example, when $Q^2 \simeq 2$ GeV^2 it could be six times larger. Therefore our main result refers to the strong magnetic field limit. The increasing pion electromagnetic form factor affects directly the pion-positron scattering $e^+\pi^+ \to e^+\pi^+$ since the scattering amplitude is proportional to F_π. This scattering amplitude is related to the Sullivan process [46] which is used to extract information about hadron physics in collider experiments.

We mention some potential future directions to this work. The first one, is to extend to higher order in eB the weak magnetic field limit calculation. Second, to include the vector channel, presenting the finite magnetic field case, since it is not clear how this term will affect the results based exclusively on the axial channel. Finally, to employ the analytical methods used in [47] by summing over all Landau levels, and then take the appropriate strong and weak magnetic field limits.

Appendix A

Schwinger propagator under charge conjugation

The Schwinger propagator for a weak magnetic field intensity and in the massless limit is given by,

$$i\mathcal{S}(p) = i\frac{\not{p}}{p^2} - \frac{\gamma_1\gamma_2\left(\gamma\cdot p_\parallel\right)}{\left(p^2\right)^2}q_f B \tag{A.1}$$

to first order in eB. With the properties

$$C^{-1}\gamma_\mu C = -\gamma_\mu^T \tag{A.2}$$

$$C^{-1}\gamma_5 C = \gamma_5^T \tag{A.3}$$

$$C^{-1}q_f C = -q_f \tag{A.4}$$

The behavior of the propagator under charge conjugation is given by,

$$\begin{aligned}
C^{-1}i\mathbf{S}(x-y)C &= C^{-1}\Phi(x,y)C\int\frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}C^{-1}i\mathcal{S}(p)C \\
&= C^{-1}\Phi(x,y)C\int\frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}C^{-1}\left(i\frac{\not{p}}{p^2} - \frac{\gamma_1\gamma_2\left(\gamma\cdot p_\parallel\right)}{\left(p^2\right)^2}q_f B\right)C \\
&= C^{-1}\Phi(x,y)C\int\frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}\left(i\frac{-\not{p}^T}{p^2} - \frac{(-\gamma_1^T)(-\gamma_2^T)\left(-\not{p}_\parallel^T\right)}{\left(p^2\right)^2}(-q_f B)\right) \\
&= C^{-1}\Phi(x,y)C\int\frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}(-1)\left(i\frac{\not{p}^T}{p^2} + \frac{\left(\not{p}_\parallel\gamma_2\gamma_1\right)^T}{\left(p^2\right)^2}q_f B\right) \\
&= C^{-1}\Phi(x,y)C\int\frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}(-1)\left(i\frac{\not{p}}{p^2} + \frac{\not{p}_\parallel\gamma_2\gamma_1}{\left(p^2\right)^2}q_f B\right)^T .
\end{aligned}$$

With the identity $\gamma_1\gamma_2\not{p}_\parallel = -\not{p}_\parallel\gamma_2\gamma_1$ and the change $p\to -p$ we get the identity,

$$C^{-1}i\mathbf{S}(x-y)C = C^{-1}\Phi(x,y)C\int\frac{d^4p}{(2\pi)^4}e^{-ip(y-x)}i\mathcal{S}^T(p), \tag{A.5}$$

which also can be written in the momentum space as,

$$CiS(p)C^{-1} = -iS(p)^T. \tag{A.6}$$

Following the same method, one can demonstrate the identities (A.5) and (A.6) for the strong magnetic field limit of the Schwinger propagator.

In a triangle fermionic loop we face a three Schwinger phase product which we call Ω,

$$\Omega_f = \Phi(x,y)\Phi(y,z)\Phi(z,x)|_{z=0} = e^{-\frac{q_f}{2}B\epsilon_{ij}x^i y^j}. \tag{A.7}$$

We can operate C to this Schwinger phase product and get,

$$\Omega_f^C \equiv C^{-1}\Omega_f C = e^{\frac{q_f}{2}B\epsilon_{ij}x^i y^j} \tag{A.8}$$

which is equivalent to invert the direction of the momentum flow in the triangle.

The projection operators under charge conjugation transform as,

$$C^{-1}O_f^{\pm}C = (O_f^{\pm})^T. \tag{A.9}$$

This identity is useful to rewrite Dirac traces involving these projection operators.

Appendix B

Feynman diagrams from Wick's theorem in $\Pi_{\mu\nu\lambda}$

First we consider the three point function,

$$\Pi_{\mu\nu\lambda}\left(p,p',q\right) = i^2 \int d^4x d^4y \; e^{i(p'x-qy)} \langle 0| T\left\{ j^{\dagger}_{A\mu}(x), j^{el}_{\nu}(y), j_{A\lambda}(0)\right\} |0\rangle. \tag{B.1}$$

The interpolating electromagnetic and axial currents are given by,

$$\begin{aligned} j_{A\nu}(x) &=: \bar{u}(x)\gamma_\mu\gamma_5 d(x) : \\ j^{el}_{\mu}(x) &=: q_u\bar{u}(x)\gamma_\mu u(x) + q_d\bar{d}(x)\gamma_\mu d(x) : . \end{aligned} \tag{B.2}$$

If we replace the above currents in the equation (B.1),

$$\begin{aligned} \Pi_{\mu\nu\lambda}\left(p,p',q\right) = i^2 \int d^4x d^4y \; e^{i(p'x-qy)} \langle 0| : \bar{d}^a_i(x)(\gamma_5)_{ij}(\gamma_\mu)_{jk}u^a_k(x) : \\ : q_u\bar{u}^b_l(y)(\gamma_\nu)_{lm}u^b_m(y) + q_d\bar{d}^b_l(y)(\gamma_\nu)_{lm}d^b_m(y) :: \bar{u}^c_n(0)(\gamma_\lambda)_{no}(\gamma_5)_{op}d^c_p(0) : |0\rangle. \end{aligned} \tag{B.3}$$

We can split the three-point function in two contributions,

$$\Pi_{\mu\nu\lambda}\left(p,p',q\right) = i^2 \int d^4x d^4y \; e^{i(p'x-qy)} (\gamma_5)_{ij}(\gamma_\mu)_{jk}(\gamma_\nu)_{lm}(\gamma_\lambda)_{no}(\gamma_5)_{op}$$

$$\times \Big[q_u \langle 0|T\{: \bar{d}^a_i(x)u^a_k(x) :: \bar{u}^b_l(y)u^b_m(y) :: \bar{u}^c_n(0)d^c_p(0) :\}|0\rangle \tag{B.4}$$

$$+ q_d \langle 0|T\{: \bar{d}^a_i(x)u^a_k(x) :: \bar{d}^b_l(y)d^b_m(y) :: \bar{u}^c_n(0)d^c_p(0) :\}|0\rangle \Big]. \tag{B.5}$$

Now we apply Wick's theorem in both terms. The contribution proportional to q_u by (B.4) is given by,

$$\begin{aligned} q_u \langle 0|T\{: \bar{d}^a_i(x)u^a_k(x) :: \bar{u}^b_l(y)u^b_m(y) :: \bar{u}^c_n(0)d^c_p(0) :\}|0\rangle = \\ - q_u d^c_p(0)\bar{d}^a_i(x)u^a_k(x)\bar{u}^b_l(y)u^b_m(y)\bar{u}^c_n(0). \end{aligned} \tag{B.6}$$

Recall that each contraction is defined as,

$$Q^A_\alpha(x) \; \overline{Q}^B_\beta(y) = \langle 0|T\{Q^A_\alpha(x) \; \overline{Q}^B_\beta(y)\}|0\rangle = \delta_{AB} i S_{\alpha\beta}(x-y) \tag{B.7}$$

therefore, the RHS of the equation (B.6) is written as,

$$= -q_u \delta_{ca} i S^d_{pi}(0-x)\delta_{ab} i S^u_{kl}(x-y)\delta_{bc} i S^u_{mn}(y-0) \tag{B.8}$$

Rearranging all Dirac and color indices to build a trace, the color indices give a factor N_c. The contribution (B.4) is given by,

$$\begin{aligned}\Pi^{q_u}_{\mu\nu\lambda}\left(p,p',q\right) =& i^2 N_c \int d^4x d^4y \; e^{i(p'x-qy)} \\ & \{-q_u \mathrm{Tr}[\gamma_5\gamma_\mu i S^u(x-y)\gamma_\nu i S^u(y-0)\gamma_\lambda\gamma_5 i S^d(0-x)]\}.\end{aligned} \tag{B.9}$$

Now for the contribution (B.5) proportional to q_d,

$$\begin{aligned} & q_d\langle 0|T\{:\bar{d}^a_i(x)u^a_k(x)::\bar{d}^b_l(y)d^b_m(y)::\bar{u}^c_n(0)d^c_p(0):\}|0\rangle \\ &= -q_d d^b_m(y)\bar{d}^a_i(x)u^a_k(x)\bar{u}^c_n(0)d^c_p(0)\bar{d}^b_l(y) \\ &= -q_d\delta_{ba} i S^d_{mi}(y-x)\delta_{ac} i S^u_{kn}(x-0)\delta_{cb} i S^d_{pl}(0-y)\end{aligned} \tag{B.10}$$

Rearranging the terms, we get

$$\begin{aligned}\Pi^{q_d}_{\mu\nu\lambda}\left(p,p',q\right) = & i^2 N_c \int d^4x d^4y \; e^{i(p'x-qy)} \\ & \{-q_d \mathrm{Tr}[\gamma_5\gamma_\mu i S^u(x-0)\gamma_\lambda\gamma_5 i S^d(0-y)\gamma_\nu i S^d(y-x)]\}.\end{aligned} \tag{B.11}$$

Thus, the three current correlation function is given by,

$$\begin{aligned}\Pi_{\mu\nu\lambda}\left(p,p',q\right) = & -i^2 N_c \int d^4x d^4y \; e^{i(p'x-qy)} \\ & \{q_u \mathrm{Tr}[\gamma_5\gamma_\mu i S^u(x-y)\gamma_\nu i S^u(y-0)\gamma_\lambda\gamma_5 i S^d(0-x)] \\ & + q_d \mathrm{Tr}[\gamma_\lambda\gamma_5 i S^d(0-y)\gamma_\nu i S^d(y-x)\gamma_5\gamma_\mu i S^u(x-0)]\},\end{aligned} \tag{B.12}$$

which are the one-loop contributions shown in Figure 1.3.

We can rewrite the second trace, with the identities introduced in Appendix A, inserting $CC^{-1} = 1$ and making use of the trace cyclical property,

$$\begin{aligned} & q_d \mathrm{Tr}[\gamma_\lambda\gamma_5 i S^d(0-y)\gamma_\nu i S^d(y-x)\gamma_5\gamma_\mu i S^u(x-0)] \\ &= \mathrm{Tr}[C^{-1}\gamma_\lambda C C^{-1}\gamma_5 C C^{-1} i S^d(0-y) C C^{-1}\gamma_\nu C C^{-1} i S^d(y-x) C C^{-1}\gamma_5 C C^{-1}\gamma_\mu C C^{-1} i S^u(x-0) C] \\ &= q_d \mathrm{Tr}[(-\gamma^T_\lambda)\gamma^T_5 C^{-1} i S^d(0-y) C(-\gamma^T_\nu) C^{-1} i S^d(y-x) C\gamma^T_5(-\gamma^T_\mu) C^{-1} i S^u(x-0) C] \\ &= (-1) q_d \mathrm{Tr}[\gamma^T_\lambda\gamma^T_5 C^{-1} i S^d(0-y) C\gamma^T_\nu C^{-1} i S^d(y-x) C\gamma^T_5\gamma^T_\mu C^{-1} i S^u(x-0) C]\end{aligned}$$

so that we can write in the momentum space,

$$\begin{aligned} & q_d \mathrm{Tr}[\gamma_\lambda\gamma_5 i S^d(0-y)\gamma_\nu i S^d(y-x)\gamma_5\gamma_\mu i S^u(x-0)] \\ &= \int \frac{d^4l}{(2\pi)^4}\frac{d^4t}{(2\pi)^4}\frac{d^4k}{(2\pi)^4} e^{-il(x-y)} e^{-ity} e^{ikx}(-1) q_d \Omega^C_d \mathrm{Tr}[\gamma^T_\lambda\gamma^T_5 i\mathcal{S}^{dT}(t)\gamma^T_\nu i\mathcal{S}^{dT}(l)\gamma^T_5\gamma^T_\mu i\mathcal{S}^{uT}(k)] \\ &= \int \frac{d^4l}{(2\pi)^4}\frac{d^4t}{(2\pi)^4}\frac{d^4k}{(2\pi)^4} e^{-il(x-y)} e^{-ity} e^{ikx}(-1) q_d \Omega^C_d \mathrm{Tr}[i\mathcal{S}^u(k)\gamma_\mu\gamma_5 i\mathcal{S}^d(l)\gamma_\nu i\mathcal{S}^d(t)\gamma_5\gamma_\lambda]\end{aligned}$$

Then the three-point function is,

$$\begin{aligned}\Pi_{\mu\nu\lambda}\left(p,p',q\right) = -i^2 N_c \int \frac{d^4l}{(2\pi)^4}\frac{d^4t}{(2\pi)^4}\frac{d^4k}{(2\pi)^4} d^4x d^4y \; e^{i(p'x-qy)}e^{-il(x-y)}e^{-ity}e^{ikx}\\ \{q_u\Omega_u \mathrm{Tr}[\gamma_5\gamma_\mu i\mathcal{S}^u(l)\gamma_\nu i\mathcal{S}^u(t)\gamma_\lambda\gamma_5 i\mathcal{S}^d(k)]\\ -q_d\Omega_d^C \mathrm{Tr}[i\mathcal{S}^u(k)\gamma_\mu\gamma_5 i\mathcal{S}^d(l)\gamma_\nu i\mathcal{S}^d(t)\gamma_5\gamma_\lambda]\}.\end{aligned}$$

Getting rid of the γ_5 and using the cyclical property again, we end up with,

$$\begin{aligned}\Pi_{\mu\nu\lambda}\left(p,p',q\right) = -i^2 N_c \int \frac{d^4l}{(2\pi)^4}\frac{d^4t}{(2\pi)^4}\frac{d^4k}{(2\pi)^4} d^4x d^4y \; e^{i(p'x-qy)}e^{-il(x-y)}e^{-ity}e^{ikx}\\ \{q_d\Omega_d^C \mathrm{Tr}[\gamma_\mu i\mathcal{S}^d(l)\gamma_\nu i\mathcal{S}^d(t)\gamma_\lambda i\mathcal{S}^u(k)]\\ -q_u\Omega_u \mathrm{Tr}[\gamma_\mu i\mathcal{S}^u(l)\gamma_\nu i\mathcal{S}^u(t)\gamma_\lambda i\mathcal{S}^d(k)]\}\end{aligned} \tag{B.13}$$

Appendix C

Feynman parametrizations

C.1 Strong magnetic field parametrizations

Lowest Landau level

For the Feynman integrals in equations (3.26) and (3.27) we parametrize as,

$$\frac{1}{(k_\parallel + p_\parallel)^2 (k_\parallel + p'_\parallel)^2 k_\parallel^2} = \Gamma(3) \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{[l^2 - \Delta]^3}, \tag{C.1}$$

where,

$$l_i = k_i + (p' x_1 + p x_2)_i, \tag{C.2}$$

$$\Delta = p'^2_\parallel x_1 (x_1 - 1) + p^2_\parallel x_2 (x_2 - 1) + 2 p'_\parallel p_\parallel x_1 x_2. \tag{C.3}$$

First Landau level

For the Feynman integrals in equation (3.48) we use the following Feynman parametrizations: For $A_{\mu\nu\lambda}$,

$$\begin{aligned} &\frac{1}{[(k_\parallel + p'_\parallel)^2 - 2|q_u B|](k_\parallel + p_\parallel)^2 k_\parallel^2} = \Gamma(3) \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{[l^2 - \Delta]^3} \\ &l_i = k_i + (p' x_1 + p x_2)_i \\ &\Delta = p'^2_\parallel x_1 (x_1 - 1) + p^2_\parallel x_2 (x_2 - 1) + 2 p'_\parallel p_\parallel x_1 x_2 + 2|q_u B| x_1; \end{aligned} \tag{C.4}$$

for $B_{\mu\nu\lambda}$,

$$\begin{aligned} &\frac{1}{(k_\parallel + p'_\parallel)^2 [(k_\parallel + p_\parallel)^2 - 2|q_u B|] k_\parallel^2} = \Gamma(3) \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{[l^2 - \Delta]^3} \\ &l_i = k_i + (p' x_1 + p x_2)_i \\ &\Delta = p'^2_\parallel x_1 (x_1 - 1) + p^2_\parallel x_2 (x_2 - 1) + 2 p'_\parallel p_\parallel x_1 x_2 + 2|q_u B| x_2; \end{aligned} \tag{C.5}$$

and finally for $C_{\mu\nu\lambda}$,

$$\begin{aligned} &\frac{1}{(k_\parallel + p'_\parallel)^2 (k_\parallel + p_\parallel)^2 [k_\parallel^2 - 2|q_d B|]} = \Gamma(3) \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{[l^2 - \Delta]^3} \\ &l_i = k_i + (p' x_1 + p x_2)_i \\ &\Delta = p'^2_\parallel x_1 (x_1 - 1) + p^2_\parallel x_2 (x_2 - 1) + 2 p'_\parallel p_\parallel x_1 x_2 + 2|q_d B|(1 - x_1 - x_2). \end{aligned} \tag{C.6}$$

C.2 Weak magnetic field parametrizations

Integral in equation (3.54)

We use the Feynman parameters to rewrite the denominator as,

$$
\begin{aligned}
&\frac{1}{(k+p)^2\left((k+p')^2\right)^2 k^2} \\
&\qquad = \Gamma(4)\int_0^1 dx_1 x_1 \int_0^{1-x_1} dx_2 \frac{1}{[x_1(k+p')^2 + x_2(k+p)^2 + (1-x_1-x_2)k^2]^4} \\
&\qquad = \Gamma(4)\int_0^1 dx_1 x_1 \int_0^{1-x_1} dx_2 \frac{1}{[l^2-\Delta]^4}
\end{aligned}
$$

where,

$$
\begin{aligned}
\Delta &= x_1 p'^2(x_1-1) + x_2 p^2(x_2-1) + 2x_1x_2 pp' \\
l &= k + (x_1 p' + x_2 p)
\end{aligned}
$$

Therefore, with the change $k \to l - (x_1 p' + x_2 p)$ the trace will be,

$$
\begin{aligned}
&\mathrm{Tr}\left[\gamma_\mu\gamma_1\gamma_2(\not{k}_\parallel + \not{p}'_\parallel)\gamma_\nu(\not{k}+\not{p})\gamma_\lambda\not{k}\right] \\
&= \mathrm{Tr}\left[\gamma_\mu\gamma_1\gamma_2(\not{l}_\parallel - (x_1\not{p}'_\parallel + x_2\not{p}_\parallel) + \not{p}'_\parallel)\gamma_\nu(\not{l} - (x_1\not{p}' + x_2\not{p}) + \not{p})\gamma_\lambda(\not{l} - (x_1\not{p} + x_2\not{p}'))\right]
\end{aligned}
$$

Integral in equation (3.55)

We use the Feynman parameters to rewrite the denominator as,

$$
\begin{aligned}
&\frac{1}{(k+p')^2\left((k+p)^2\right)^2 k^2} \\
&= \Gamma(4)\int_0^1 dx_1 x_1 \int_0^{1-x_1} dx_2 \frac{1}{[x_1(k+p)^2 + x_2(k+p')^2 + (1-x_1-x_2)k^2]^4} \\
&= \Gamma(4)\int_0^1 dx_1 x_1 \int_0^{1-x_1} dx_2 \frac{1}{[l^2-\Delta]^4}
\end{aligned}
$$

where,

$$
\begin{aligned}
\Delta &= x_1 p^2(x_1-1) + x_2 p'^2(x_2-1) + 2x_1x_2 pp' \\
l &= k + (x_1 p + x_2 p')
\end{aligned}
$$

Therefore with the change $k \to l - (x_1 p + x_2 p')$ the trace will be,

$$
\begin{aligned}
&\mathrm{Tr}\left[\gamma_\mu(\not{k}+\not{p}')\gamma_\nu\gamma_1\gamma_2(\not{k}_\parallel + \not{p}_\parallel)\gamma_\lambda\not{k}\right] \\
&= \mathrm{Tr}\left[\gamma_\mu(\not{l} - (x_1\not{p} + x_2\not{p}') + \not{p}')\gamma_\nu\gamma_1\gamma_2(\not{l}_\parallel - (x_1\not{p}_\parallel + x_2\not{p}'_\parallel) + \not{p}_\parallel)\gamma_\lambda(\not{l} - (x_1\not{p} + x_2\not{p}'))\right]
\end{aligned}
$$

Integral in equation (3.58)

We use the Feynman parameters to rewrite the denominator as,

$$
\begin{aligned}
&\frac{1}{(k+p)^2\left(k+p'\right)^2k^2}\\
&\quad=\Gamma(3)\quad\int_0^1dx_1\int_0^{1-x_1}dx_2\frac{1}{\left[x_1(k+p)^2+x_2\left(k+p'\right)^2+\left(1-x_1-x_2\right)k^2\right]^3}\\
&\quad=\Gamma(3)\quad\int_0^1dx_1\int_0^{1-x_1}dx_2\frac{1}{\left[l^2-\Delta\right]^3}
\end{aligned}
$$

where,

$$
\begin{aligned}
\Delta&=x_1p^2(x_1-1)+x_2p'^2(x_2-1)+2x_1x_2pp'\\
l&=k+(x_1p+x_2p')
\end{aligned}
$$

Therefore with the change $k=l-(x_1p+x_2p')$ the trace will be,

$$
\mathrm{Tr}\left[\gamma_\mu i(\not{k}+\not{p}')\gamma_\nu i\gamma_j\gamma_\lambda i\gamma_i\right]=\mathrm{Tr}\left[\gamma_\mu i\left(\not{l}-(x_1\not{p}+x_2\not{p}')+\not{p}'\right)\gamma_\nu i\gamma_j\gamma_\lambda i\gamma_i\right]
$$

Integral in equation (3.59)

First we make the change of variable $k\to k+p'$ for our convenience, thus the trace and the denominator is

$$
\begin{aligned}
&-2\frac{5}{18}|eB|\int\frac{d^4k}{(2\pi)^4}\ \epsilon_{ij}\mathrm{Tr}\left[\gamma_\mu i\frac{\not{k}}{k^2}\gamma_\nu i\frac{\gamma_j}{(k-p'+p)^2}\gamma_\lambda i\frac{\not{k}-\not{p}'}{\left((k-p')^2\right)^2}(k-p')_i\right]\\
&=-2\frac{5}{18}|eB|\int\frac{d^4k}{(2\pi)^4}\ \epsilon_{ij}\mathrm{Tr}\left[\gamma_\mu i\not{k}\gamma_\nu i\gamma_j\gamma_\lambda i(\not{k}-\not{p}')(k-p')_i\right]\frac{1}{k^2(k-q)^2\left((k-p')^2\right)^2}.
\end{aligned}
$$

Then, we use the Feynman parametrization,

$$
\begin{aligned}
&\frac{1}{k^2(k-q)^2\left((k-p')^2\right)^2}\\
&\quad=\Gamma(4)\quad\int_0^1dx_1x_1\int_0^{1-x_1}dx_2\frac{1}{\left[x_1(k-p')^2+x_2\left(k-q\right)^2+\left(1-x_1-x_2\right)k^2\right]^4}\\
&\quad=\Gamma(4)\quad\int_0^1dx_1x_1\int_0^{1-x_1}dx_2\frac{1}{\left[l^2-\Delta\right]^4}
\end{aligned}
$$

where,

$$
\begin{aligned}
\Delta&=x_1p'^2(x_1-1)+x_2q^2(x_2-1)+2x_1x_2qp'\\
l&=k-(x_1p'+x_2q).
\end{aligned}
$$

Thus, the trace is,

$$
\begin{aligned}
&\mathrm{Tr}\left[\gamma_\mu i\not{k}\gamma_\nu i\gamma_j\gamma_\lambda i(\not{k}-\not{p}')(k-p')_i\right]\\
&\quad=\mathrm{Tr}\left[\gamma_\mu i(\not{l}+x_1\not{p}'+x_2\not{q})\gamma_\nu i\gamma_j\gamma_\lambda i(\not{l}+x_1\not{p}'+x_2\not{q}-\not{p}')(l+x_1p'+x_2q-p')_i\right]
\end{aligned}
$$

Integral in equation (3.60)

We use the Feynman parameters to rewrite the denominator as,

$$
\begin{aligned}
&\frac{1}{(k+p')^2\left((k+p)^2\right)^2 k^2} \\
&\qquad = \Gamma(4)\int_0^1 dx_1 x_1 \int_0^{1-x_1} dx_2 \frac{1}{[x_1(k+p)^2 + x_2(k+p')^2 + (1-x_1-x_2)k^2]^4} \\
&\qquad = \Gamma(4)\int_0^1 dx_1 x_1 \int_0^{1-x_1} dx_2 \frac{1}{[l^2-\Delta]^4}
\end{aligned}
$$

where,

$$
\begin{aligned}
\Delta &= x_1 p^2(x_1-1) + x_2 p'^2(x_2-1) + 2x_1x_2pp' \\
l &= k + (x_1p + x_2p')
\end{aligned}
$$

Therefore with the change $k \to l - (x_1p + x_2p')$ the trace will be,

$$
\begin{aligned}
&\mathrm{Tr}\left[\gamma_\mu i(\not{k}+\not{p}')\gamma_\nu i(\not{k}+\not{p})(k+p)_j\gamma_\lambda i\gamma_i\right] \\
&= \mathrm{Tr}\left[\gamma_\mu i(\not{l}-(x_1\not{p}+x_2\not{p}')+\not{p}')\gamma_\nu i(\not{l}-(x_1\not{p}+x_2\not{p}')+\not{p})(l-(x_1p+x_2p')+p)_j\gamma_\lambda i\gamma_i\right]
\end{aligned}
$$

Integral in equation (3.61)

First we make the change of variable $k \to k + p'$ for our convenience, thus the trace and the denominator is,

$$
\begin{aligned}
&\mathrm{Tr}\left[\gamma_\mu i\frac{\not{k}+\not{p}'}{(k+p')^2}\gamma_\nu i\frac{\not{k}+\not{p}}{\left((k+p)^2\right)^2}(k+p)_j\gamma_\lambda i\frac{\not{k}}{\left(k^2\right)^2}k_i\right] \\
&\quad = \mathrm{Tr}\left[\gamma_\mu i\not{k}\gamma_\nu i(\not{k}-\not{q})(k-q)_j\gamma_\lambda i(\not{k}-\not{p}')(k-p')_i\right]\frac{1}{(k)^2\left((k-q)^2\right)^2\left((k-p')^2\right)^2}.
\end{aligned}
$$

Then, we use the Feynman parametrization,

$$
\begin{aligned}
&\frac{1}{(k)^2\left((k-q)^2\right)^2\left((k-p')^2\right)^2} \\
&\qquad = \Gamma(5)\int_0^1 dx_1 x_1 \int_0^{1-x_1} dx_2 \frac{x_2}{[x_1(k-q)^2 + x_2(k-p')^2 + (1-x_1-x_2)k^2]^5} \\
&\qquad = \Gamma(5)\quad\int_0^1 dx_1 x_1 \int_0^{1-x_1} dx_2 \frac{x_2}{[l^2-\Delta]^5}
\end{aligned}
$$

where,

$$
\begin{aligned}
\Delta &= x_1 q^2(x_1-1) + x_2 p'^2(x_2-1) + 2x_1x_2qp' \\
l &= k - (x_1q + x_2p').
\end{aligned}
$$

Therefore, with the change $k = l + (x_1q + x_2p')$ the trace will be,

$$
\begin{aligned}
&\mathrm{Tr}\left[\gamma_\mu i\not{k}\gamma_\nu i(\not{k}-\not{q})(k-q)_j\gamma_\lambda i(\not{k}-\not{p}')(k-p')_i\right] \\
&\quad = \mathrm{Tr}\left[\gamma_\mu i(\not{l}+(x_1\not{q}+x_2\not{p}'))\gamma_\nu i(\not{l}+(x_1\not{q}+x_2\not{p}')-\not{q})\right. \\
&\quad \left.\times(l+(x_1q+x_2p')-q)_j\gamma_\lambda i(\not{l}+(x_1\not{q}+x_2\not{p}')-\not{p}')(l+(x_1q+x_2p')-p')_i\right]
\end{aligned}
$$

Appendix D

Form Factor functions

Lowest Landau level

$$\Pi_q^{\epsilon} = \frac{2i|eB|}{9\pi^2} e^{\frac{-P_\perp^2}{|eB|}} \int_0^1 dx_1 \int_0^{1-x_1} dx_2$$
$$\frac{(x_1 - x_2)(4P_\parallel^2(x_1 + x_2 - 1)^2 + q_\parallel^2((x_1)^2 - 2x_1(x_2 + 1) + (x_2 - 1)^2))}{(4P_\parallel^2((x_1)^2 + x_1(2x_2 - 1) + (x_2 - 1)x_2) + q_\parallel^2((x_1)^2 - x_1(2x_2 + 1) + (x_2 - 1)x_2))^2}, \tag{D.1}$$

$$\Pi_P^{\perp} = \frac{4i^2|eB|}{3\pi^2} e^{\frac{-P_\perp^2}{|eB|}} \int_0^1 dx_1 \int_0^{1-x_1} dx_2$$
$$\frac{(x_1 + x_2)(4P_\parallel^2(x_1 + x_2 - 1)^2 + q_\parallel^2((x_1)^2 - 2x_1(x_2 + 1) + (x_2 - 1)^2))}{(4P_\parallel^2((x_1)^2 + x_1(2x_2 - 1) + (x_2 - 1)x_2) + q_\parallel^2((x_1)^2 - x_1(2x_2 + 1) + (x_2 - 1)x_2))^2}, \tag{D.2}$$

$$\Pi_q^{\perp} = \frac{2i^2|eB|}{3\pi^2} e^{\frac{-P_\perp^2}{|eB|}} \int_0^1 dx_1 \int_0^{1-x_1} dx_2$$
$$\frac{(x_1 - x_2)(4P_\parallel^2(x_1 + x_2 - 1)^2 + q_\parallel^2((x_1)^2 - 2x_1(x_2 + 1) + (x_2 - 1)^2))}{(4P_\parallel^2((x_1)^2 + x_1(2x_2 - 1) + (x_2 - 1)x_2) + q_\parallel^2((x_1)^2 - x_1(2x_2 + 1) + (x_2 - 1)x_2))^2}, \tag{D.3}$$

First Landau level

$$\Pi_1^{\parallel} = \frac{1}{\pi^4|\mathrm{eB}|} e^{\frac{-P_\perp^2}{|eB|}} \int_0^1 dx_1 \int_0^{1-x_1} dx_2$$
$$\frac{216i(x_1 + x_2 - 1)^2(x_1 + x_2)}{\left(8|eB|(x_1 + x_2 - 1) - 12P_\parallel^2(x_1 + x_2)(x_1 + x_2 - 1) + 3q_\parallel^2\left(-x_1^2 + 2x_1x_2 + x_1 - x_2^2 + x_2\right)\right)^2}, \tag{D.4}$$

$$\Pi_1^S = \frac{1}{\pi^4|eB|}e^{\frac{-P_\perp^2}{|eB|}}\int_0^1 dx_1 \int_0^{1-x_1} dx_2$$
$$\frac{72(x_1+x_2-1)^2}{\left(8|eB|(x_1+x_2-1)-12P_\parallel^2(x_1+x_2-1)(x_1+x_2)+3q_\parallel^2\left(-x_1^2+2x_1x_2+x_1-x_2^2+x_2\right)\right)^2}. \tag{D.5}$$

Weak magnetic field

$$\Pi_1^W = \frac{16}{9}\pi^2\int_0^1 dx_1 x_1 \int_0^{1-x_1} dx_2$$
$$\left(\frac{20x_1(x_1-1)^2}{\left(4P_\parallel^2(x_1-1)x_1+q_\parallel^2\left(x_1^2+x_1(4x_2-1)+4(x_2-1)x_2\right)\right)^2}\right.$$
$$+\frac{5x_2(2x_2-1)}{\left(4P_\parallel^2(x_2-1)x_2+q_\parallel^2\left(4x_1^2+4x_1(x_2-1)+(x_2-1)x_2\right)\right)^2} \tag{D.6}$$
$$\left.-\frac{4i(x_1+x_2-1)^2((3+5i)x_1+(3+5i)x_2-5i)}{\left(4P_\parallel^2(x_1+x_2-1)(x_1+x_2)+q_\parallel^2\left(x_1^2-x_1(2x_2+1)+(x_2-1)x_2\right)\right)^2}\right)$$

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