Essays on Simulation Methods

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Essays on Simulation Methods

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To my parents
Andrés and Adriana,
and to my sister Natalia,
and my niece Ariana.

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Abstract

Essays on Simulation Methods

by

Andrés Fernando Villegas Rodríguez

Doctor in Ciencias de la Agricultura

Professor Eugenio Bobenrieth H., Chair

This dissertation consists of two essays in which I use simulation methods to study the structural parameters estimates from econometric models considering the complexity of water and commodity markets. In the first, I study the efficiency of several companies using non parametric methods, and next employing bootstrapping techniques for statistical inference. In the second, I analyze the performance of different econometric methods in the context of storage models, using Monte Carlo experiments on heuristics representations comparing the distribution of the estimate parameters using different descriptive statistics.

In the first chapter, I implement the double bootstrap to non-parametric Data Envelopment Analysis with the purpose to estimate the efficiency of Chilean water and sewerage companies. The relevance of applied this bootstrap technique, is that allows statistical inferences that cannot be drawn directly from such non-parametric model. This feature is important in the framework of water utilities performance com-

parisons since it is well-known that several exogenous variables influence the water utilities efficiency. My results show that the ranking of water companies changes notably whether efficiency scores are computed applying conventional or double-bootstrap DEA models. Moreover, I found that the percentage of non-revenue water and customer density are factors that influencing the efficiency of Chilean water and sewerage companies

In the second chapter, I design a Monte Carlo experiment in the context of storage model to compare finite sample performance of the Simulated Methods of Moments estimator of Duffie and Singleton (1993), the Indirect Inference estimator of Gourieroux et al. (1993), the Efficient Method of Moments estimator of Gallant and Tauchen (1996), the Pseudo Maximum Likelihood estimator (PML) of Deaton and Laroque (1995), The Conditional Maximum Likelihood estimator of Cafiero et al. (2015) and the Unconditional Maximum Likelihood of Gouel and Legrand (2017). My results suggest that for parameterizations that imply low average storage and frequent stock-outs, the PML estimator for small sample presents low bias and is more efficient than Simulations estimators. However, for parameterizations that imply a more significant role of storage, the Simulations estimators present bias that decrease with sample size increase, while the PML estimator biases do not disappear but instead tend to stabilize. I prove theoretically and numerically that Maximum Likelihood estimator is consistent and achieves better small sample performance than the others.

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Chapter 1

Measuring and Comparing the Efficiency of Water Companies: A Double Bootstrap Approach[1]

[1]This chapter was published to Environmental Science and Pollution Research, March 2018, Volume 25, Issue 9, pp 8432-8440. Cite as: Molinos-Senante, M., Donoso, G., Sala-Garrido, R. et al. Environ Sci Pollut Res (2018) 25: 8432.

1.1 Introduction

Sustainable urban water management involves the efficient technical and economic use of resources. In this context, it is essential to improve the efficiency of water utilities because it allows to reduce costs that could facilitate an increase in investments to improve customer service quality (Guerrini et al. 2015). Moreover, in many countries, water and sewerage industry provides services under monopoly regimes which implies that operators have no incentives towards efficiency. Hence, benchmarking is of strategic importance to regulate water companies (Molinos-Senante and Sala-Garrido 2017). Because of its usefulness, over the last few years, efficiency assessment in the water industry has attracted considerable attention by researchers, water companies, and regulators (Romano and Guerrini 2011). From a methodological point of view, most studies have adopted a production frontier approach (Worthington 2014) which can be estimated using parametric approaches such as stochastic frontier analysis (Saal et al. 2007) or non-parametric methods such as data envelopment analysis (DEA) (Molinos-Senante and SalaGarrido 2016). DEA has three primary positive features that have favored its use to evaluate the efficiency of water companies, namely, (i) it does not require a priori assumptions about the functional relationship between inputs and outputs; (ii) it allows for the computation of efficiency of units that use multiple inputs to produce multiple outputs; and (iii) the weights to aggregate inputs and outputs are generated endogenously (Cooper et al. 2007). Despite these advantages, DEA has two major drawbacks which are related to its deterministic nature (Ananda 2014). Firstly, DEA models assume that there is no noise, nor outliers in the sample. However, for robust efficiency assessment, it is essential to detect atypical observations (De Witte and Marques 2010a). As a second disadvantage, statistical inferences cannot be drawn from conventional DEA analysis (Badin et al. 2014). This limitation is especially relevant in the framework of water utility performance comparisons since it is well-known that several exogenous variables influence water utility efficiency (Molinos-Senante et al. 2015). To overcome this limitation, Cazals et al. (2002) proposed the order-m method, which is a partial frontier method that only uses a portion of the sample to compute efficiency scores. In the framework of water utilities, this approach was used to incorporate environmental variables into efficiency assessment (De Witte and Marques 2010b). In spite of the advantages of this method, it also had some difficulties (Daraio and Simar 2006); specifically, the selection of the value for “m” was challenging because it affected the efficiency scores (Da Cruz and Marques 2014). Alternatively to the order-m method, Simar and Wilson (2007) proposed a double-bootstrap procedure that enables statistical inferences and hypothesis testing in DEA models. In other words, reliable results are obtained with this approach since it estimates bias-corrected efficiency scores and also identifies the determinants of efficiency (Benito-López et al. 2011). Despite its usefulness, to the best of our knowledge, only three previous studies (De Witte and Marques 2010c; Ananda 2014; See 2015) employed a double-bootstrap DEA approach to estimate bias-corrected efficiency scores and to explore the sources of efficiency in water utilities. The three empirical applications focused on water companies that only provide drinking water services (WoCs), i.e., sewerage and wastewater treatment services were not considered in their assessment. Hence, the study scope and the selection of the inputs and outputs to assess the efficiency were adapted for water only companies and not for water and sewerage companies (WaSCs). This issue is very relevant because water companies might be affected by economies of scope (Guerrini et al. 2013). Moreover, an efficiency assessment focused only on WoCs ignores the potential cost savings associated to sewerage services, i.e., economies of scope, which are not negligible (Carvalho and Marques 2014). Against this background, the objectives of this paper are twofold. The first one is to evaluate the bias-corrected efficiency of a sample of WaSCs by applying the double-bootstrap DEA method proposed by Simar and Wilson (2007). With this approach, we obtain more reliable evidence with respect to results obtained using traditional DEA models. The second objective is to identify the determinants of efficiency in WaSCs. The empirical application focuses on the Chilean water and sewerage industry for 2014. This paper contributes to the current strand of literature by computing for the first-time bias-corrected efficiency scores and by identifying factors affecting efficiency of WaSCs. To the authors' knowledge, this is the first study that applies a double-bootstrap DEA procedure to assess the efficiency of a sample of WaSCs. The Chilean water and sewerage industry is a paradigmatic case study since it has long been a pioneer in the privatization of water and sewerage services. The level of coverage and quality of water and sewerage services in Latin America has been defined as moderate; therefore, many emerging economies facing the challenge of improving water and sewerage services can learn important lessons from the Chilean case. Actually, recent studies (Molinos-Senante et al. 2015, 2016; Molinos-Senante and Sala-Garrido, 2016, 2017) have evaluated the performance of Chilean water utilities. However, none of these previous studies identify the factors influencing the efficiency of WaSCs using a reliable approach such as bootstrap method. From a policy perspective, the methodology and results of this study provide evidences that are of great interest both for WaSCs' managers and regulators. On the one hand, the estimation of bias-corrected efficiency scores provides a more reliable performance comparison of the WaSCs. This issue is essential for the water regulator to promote competition between WaSCs contributing to reduce monopoly problems and also to set suitable water tariffs. On the other hand, the identification of factors affecting efficiency scores is essential to support decisions, contributing to the improvement of longterm sustainability of the urban water cycle. This paper illustrates that implementing a consistent and reliable methodology is vital to increase the relevance of benchmarking tools. Moreover, it provides evidence of the linkages between environmental, social, and economic issues in the framework of water companies' performance.

Against this background, the objectives of this chapter are twofold. The first one is to evaluate the bias-corrected efficiency of a sample of WaSCs by applying the double-bootstrap DEA method proposed by Simar and Wilson (2007). With this approach, I obtain more reliable evidence with respect to results obtained using traditional DEA models. The second objective is to identify the determinants of efficiency in WaSCs. The empirical application focuses on the Chilean water and sewerage industry for 2014. The rest of the chapter is organized as follows. Section 1.2 describe the regulatory model and water tariffs setting process in Chile. Section 1.3 define the methodology of double-bootstrap DEA and details the sample description. Section 1.4 presents the results and discussion, and section 1.5 concludes.

1.2 Water Tariff Setting Process

The process to set water tariffs is based on the definition of a hypothetical efficient firm, i.e., an “ideal firm” (Marques 2011). Under this approach, the performance of the “real” water company is compared with a virtual, efficient company known as the “model” company, which is considered to be the benchmark. It is a theoretical water company created by the regulator which satisfies the demand in a optimal manner taking into account prevailing norms and the geographical, demographic and technological restrictions that characterize the operation of the service (Gobierno de Chile 1988a, b, c). This model corresponds to a water company without assets, which must make the investment to provide water and sanitation services (WSS) and establish a development investment plan (Donoso 2017). The procedure to set water tariffs is shown in Fig 1.1. A year before the end of the tariff cycle, the regulator prepares terms of reference (ToR) for the tariff studies to be conducted by the water company as well as the regulator. Based on the estimation of the long-term costs of the hypothetical efficient water company, both the regulator and the water company

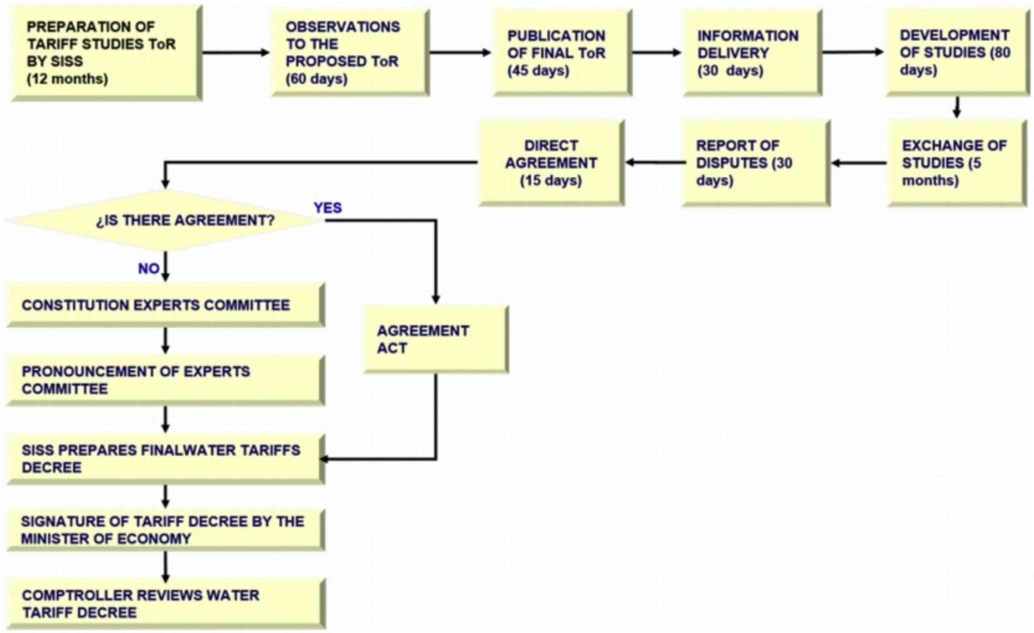

Figure 1.1: Water tariff setting procedure. (SISS 2015b).

proposed the water tariff to be charged by the regulated firm. If the parties cannot agree on the price, the disagreement is settled through and arbitration process.

The legal framework of the Chilean water and sanitation tariffs system defines four main principles to set water tariffs: (i) economic efficiency, (ii) water conservation incentives, (iii) equity, and (iv) affordability (Molinos-Senante and Donoso 2016). In this context , the objectives of the Chilean tariff model are to:

1. Finance the WSS operator's operating costs and maintenance and investment requirements so as to insure continuity of water supply and quality service.

2. Finance a minimum agreed operational margin that covers the private operator's capital opportunity cost.

3. Incentivize efficiency gains in the provision of WSS services.

4. Transmit efficiency gains to customers through tariff reductions.

5. Provide water value signals so that consumers internalize the scarcity value of water in their region.

To achieve these objectives, water and sanitation tariffs in Chile are based on a two-part structure, a fixed part ($) and a variable tariff $(\$/\mathrm{m}^3)$. The fixed charge is per connection and depends of the connection diameter and metering costs. The variable tariff is almost uniform since an extra charge for over-consumption is applied only in very exceptional cases. However, the variable component internalizes changes in seasonal demand by establishing a peak and non-peak charge. Thus, in summer months when water demand is high but water availability is low, a peak tariff is applied in contrast to the rest of the months. Hence, the difference in the provision costs of WWS during both time periods is covered. Formally, the tariff (τ) is set such that:

$$\tau = \frac{AI + OC + MR + T}{C}$$

where AI represents the annualized value of the required investments by the model operator, OC is the annual operating and maintenance costs, MR is the minimum guaranteed returns, T are the taxes that the operator must pay, and C represents the total annual projected water consumption for the next 5 years in the concession area. One of the investment costs considered in the calculation of the AI is the market value of the necessary water rights. Thus, water tariffs should reflect the scarcity value of water (Molinos-Senante and Donoso 2016).

1.3 Methodology

I adopt the double-bootstrap DEA model with a truncated bootstrapped regression proposed by Simar and Wilson (2007) to estimate the efficiency scores and their determinants in the Chilean water industry, since it enables bias-corrected efficiency score estimation and identification of the determinants of efficiency for the water industry.

Efficiency estimation

DEA method has been widely applied to evaluate the efficiency of water utilities (See 2015). DEA is a non-parametric technique based on linear programming that allows for the construction of the efficient production frontier based on the inputs and outputs of the decision making units (DMUs) (Charnes et al. 1978). The relative efficiency for each unit is calculated by comparing its inputs and outputs in relation to the rest of the units. Further details on DEA methodology are provided by Cooper et al. (2007) and Hwang et al. (2016). DEA models can take either an input or output orientation. In the framework of water industry, previous studies (Molinos-Senante and Sala-Garrido 2016; Guerrini et al. 2015, 2011) have adopted input orientation since the aim of the WaSCs is to provide water and sewerage services minimizing the use of inputs.

Given $j = 1, 2, ..., N$ units (WaSCs in my case study), each one using a vector of M inputs $x_j = (x_{1j}, x_{2j}, ..., x_{Mj})$ to produce a vector of S outputs $y_j = (y_{1j}, y_{2j}, ..., y_{Sj})$, the input-oriented DEA model is denoted as follows

$$\begin{array}{llcll} \min & \theta_j & & & \\ \text{s.t.} & \sum_{j=1}^{N} \lambda_j x_{ij} & \leq & \theta x_{i0} & 1 \leq i \leq M \\ & \sum_{j=1}^{N} \lambda_j y_{rj} & \geq & y_{r0} & 1 \leq r \leq S \\ & \lambda_j & \geq & 0 & 1 \leq j \leq N \end{array} \tag{1.1}$$

θ_j indicates the efficiency of the unit evaluated being efficient when $\theta_j = 1$ and inefficient whenever $\theta_j > 1$, M is the number of inputs used; S is the number of outputs generated, N is the number of units analyzed, and λ_j is a set of intensity variables which represent the weighting of each analyzed WaSC j in the composition of the efficient frontier.

Double-bootstrap DEA approach

From the DEA literature, two main approaches are the most used to account for the effects of explanatory and environmental variables on efficiency scores. The first one is Tobit regression analysis in which the efficiency scores are regressed against a set of explanatory variables taking into account the censured nature of the dependent variable distribution (Guerrini et al. 2015). However, this procedure suffers important shortcomings (Badin et al. 2014). Simar and Wilson (2007) proved that if the variables used in specifying the original efficiency model are correlated with the explanatory variables used in the regression analysis, then the second-stage estimates are inconsistent and biased. Conventional inference methods used in the two-stage DEA procedure are based on efficiency estimates that are serially correlated. As a result related statistical inference might not be reliable (Santos et al., 2017).

The second approach is to apply non-parametric statistical tests to verify whether there are significant differences between the efficiency scores of units grouped according to certain factors that appear to be related to efficiency. Nevertheless, this approach does not allow to isolate the influence of the explanatory variables on the efficiency scores and, thus, causality cannot be determined.

To overcome these limitations, Simar and Wilson (2007) proposed a double-bootstrap procedure that provides a confidence interval for the efficiency estimates and yields consistent inferences for factors explaining efficiency (Boamah et al. 2017). The bootstrapping generates new data that are drawn from the original set. This new data are then used to reestimate the DEA model (Eq.1). The distinction between the true and the estimated frontier allows for statistical inferences in DEA (Ananda 2014).

As in many previous studies (e.g., Da Cruz and Marques 2014; Zhang et al. 2016), the double-bootstrap procedure applied in this chapter is referred to as Algorithm 2 of Simar and Wilson (2007) which can be summarized in the following steps:

1. Estimate DEA input-efficiency scores θ_j for all WasCs in the sample by using Eq. (1).

2. Carry out a truncated maximum likelihood estimation to regress θ against a set of explanatory variables z_j, $\theta_j = z_j\beta + \epsilon_j$, and provide an estimate $\hat{\beta}$ of the

coefficient vector β and estimate $\hat{\sigma_\epsilon}$ of σ_ϵ , the standard deviation of the residual errors ϵ_j.

3. For each WaSC $j(j = 1, ..., N)$ repeat the following four steps (3.1-3.4) B_1 times to obtain a set of B_1 bootstrap estimates $\hat{\theta}_{jb}$ for $b = 1, ..., B_1$.

   3.1 Generate the residual error ϵ_j from the normal distribution $N(0, \hat{\sigma}_\epsilon^2)$.

   3.2 Compute $\theta_j^* = z_j \hat{\beta} + \epsilon_j$.

   3.3 Generate a pseudo data set (x_j^*, y_j^*) where $x_j^* = x_j$ and $y_j^* = y_j \left(\dfrac{\theta_j}{\theta_j^*}\right)$.

   3.4 Using the pseudo data set (x_j^*, y_j^*) and Eq.(1) estimate pseudo efficiency estimates $\hat{\theta}_j^*$.

4. Calculate the bias-corrected estimator $\hat{\theta}_j$ for each WaSC $j(j = 1, ..., N)$ using the bootstrap estimator of the bias $\hat{b}_j$ where $\hat{\theta}_j = \theta_j - \hat{b}_j$ and $\hat{b}_j = \left(\dfrac{1}{B_1} \sum_{b=1}^{B_1} \hat{\theta}_{jb}^*\right) - \theta_j$.

5. Use truncated maximum likelihood estimation to regress $\hat{\theta}_j$ on the explanatory variables z_j and provide an estimate $\hat{\beta}^*$ for β and an estimate $\hat{\sigma}^*$ for σ_ϵ.

6. Repeat the following three steps (6.1-6.3) B_2 times to obtain a set of B_2 pairs of bootstrap estimates $\left(\hat{\beta}_j^{**}, \hat{\sigma}_j^{**}\right)$ for $b = 1, ..., B_2$.

   6.1 Generate the residual error ϵ_j from the normal distribution $N(0, \hat{\sigma}^{*2})$.

   6.2 Compute $\hat{\theta}_j^{**} = z_j \hat{\beta}^* + \epsilon_j$.

6.3 Use truncated maximum likelihood estimation to regress ${\hat{\theta}_j}^{**}$ on the explanatory variables z_j and provide an estimate $\hat{\beta}^{**}$ for β and an estimate $\hat{\sigma}^{**}$ for σ_ϵ.

7. Construct the estimated $(1-\alpha)\%$ confidence interval of the n-th element β_n of the vector β, that is:
$[Lower_{\alpha_n}, Upper_{\alpha_n}] = \left[\hat{\beta}^*_n + \hat{a}_\alpha, \hat{\beta}^*_n - \hat{b}_\alpha\right]$ with $Prob\left(-\hat{b}_\alpha \leq \hat{\beta}^{**}_n - \hat{\beta}^*_n \leq \hat{a}_\alpha\right) \approx 1-\alpha$

Sample description

The empirical application carried out in this study focused on the Chilean water and sewerage industry which finished its privatization in 2004. As a result, in 2014, 95.7% of customers were served by private WaSCs and the remaining 4.3public concessionaries, municipalities, and cooperatives (SISS 2014). While the total number of regulated Chilean WaSCs in 2014 was 53, this study evaluates the efficiency of the 23 main WaSCs which provide water and sewerage services to approximately 98% of the total number of urban customers (SISS 2014). The source of the data was the “Management Report for Water and Sewerage Companies in Chile”, published by the national water regulator (Superintendencia de Servicios Sanitarios, SISS) on its webpage for the year 2014. The 23 WaSCs evaluated provide water supply and wastewater treatment services.

The selection of inputs and outputs is essential in DEA studies. In a literature review recently conducted by See (2015), it was evidenced that the input and output variables included in the efficiency assessment of water utilities vary notably in empirical studies. Regarding inputs, the most widely used variables include operating costs (Byrnes et al. 2010), network length (De Witte and Marques 2010c), number of employees, total capital expenditure, etc. Considering previous studies on this topic, this study employed three inputs: operating costs, labor, and network length. Operating costs involve the water and sewerage industry's total expenditure except labor costs which were proxy by the full-time employees. Selecting a variable that represents capital expenditure is a difficult task by valuation disparities (Ananda 2014). The total network length was used as a proxy to capital stock. However, from a theoretical point of view, there is opposition to include fixed capital as it is a sunk cost (Byrnes et al. 2010). Hence, following previous studies (De Witte and Marques 2010c; Ananda 2014; See 2015), the network length, expressed in kilometers, was selected as a proxy to capital costs. In my case study, this variable is the sum of the water delivery network and sewerage network.

From the literature review, the most widely used output variables are the volume of water delivered (Guerrini et al. 2013) and the number of properties connected (Ananda 2014). These variables focus mainly on the water supply service while the water companies evaluated in this study also provide sewerage services. Hence, one or more outputs related to this service should be introduced in the model. Moreover, recent studies (e.g., HernÃ¡ndez-Sancho et al. 2012; Maziotis et al. 2015) have evidenced that the quality of the service affects the performance assessment of water companies. WaSCs incur in considerable expenditures to improve water quality (Ananda 2014), and therefore, quality issues cannot be omitted in the efficiency assessment of WaSCs (Cherchi et al. 2015). Following Saal et al. (2007), two quality-adjusted outputs were used in this study, one focused on water supply and the other is sewerage services. The two outputs considered in the assessment were as follows: (i) distributed water (expressed in thousands of cubic meters) adjusted by its quality (y_1) and (ii) the number of customers with access to wastewater treatment services adjusted by the quality of the treated water (y_2). Information about the quality of the drinking water and water treated is provided by the SISS. Thus, the regulator develops for each WaSC two quality indicators (drinking water and wastewater) that range between 0 and 1. A value of one means that the WaSC has fulfilled all legal requirements regarding quality issues. The two quality-adjusted outputs are defined as follows:

$$y_1 = VDW * Q_1 \tag{1.2}$$

$$y_2 = CWW * Q_2 \tag{1.3}$$

where y_1 is the quality-adjusted drinking water output; VDW is the volume of drinking water delivered; Q_1 is the quality indicator of the drinking water; y_2 is the quality-adjusted wastewater treatment output; CWW is the number of customers with access to wastewater treatment services; and Q_2 is the quality indicator of the treated wastewater.

A wide number of variables are employed in the literature as potential determinants of efficiency of water utilities (See 2015). In general, ownership, customer density, peak factor, and water losses have been considered as environmental variables that explain the efficiency of water companies (Carvalho and Marques 2011; Marques et al. 2014). In this study, the potential explanatory variables were selected taking into account the features of the Chilean water and sewerage industry, the available data and the extant literature (Berg and Marques 2011; See 2015). Five variables, namely customer density, non-revenue water, ownership, water source, and peak factor, were included in the second stage of the double-bootstrap DEA model, as determinants of WaSCs efficiency scores. Table 1.1 provides a snapshot of the statistical data used to compute efficiency scores of Chilean WaSCs and of their potential explanatory variables. The variable ownership and source of water are not quantitative and therefore are introduced in the regression analysis as dummies.

Operating costs are highly variable, presenting an average value of 42,819 US$ per year with a coefficient of variation (CV) of 1.40%. As expected, average operating costs per client are less variable, presenting a CV of 0.51%. Figure 1.2 and 1.3 show that average per client operating costs and average number of employees per client decreases as the number of clients increases, respectively.

Table 1.1: Sample Description

			Average	SD
Inputs	Operating costs (US$/year)		42,819	59,973
	Labor (N° workers)		590	765
	Network length (km)		3,138	4,941
Outputs	Water distributed (103 m^3/year)		47,979	93,080
	Customers with wastewater treatment service		688,434	1,314,390
	Indicator of drinking water quality		0.954	0.073
	Indicator of wastewater treatment quality		0.99	0.014
Continuous variables	Non-revenue water (%)		29.7	11.9
	Peak factor		1.2	0.2
	Customer density (Customers/km)		57.3	14.46
Categorical variables			Number	% of total
	Ownership	Private operator	12	52.2
		Concession	10	43.5
		Municipal operator	1	4.3
	Water Source	Surface water	3	13.1
		Ground water	9	39.1
		Mixed sources	11	47.8

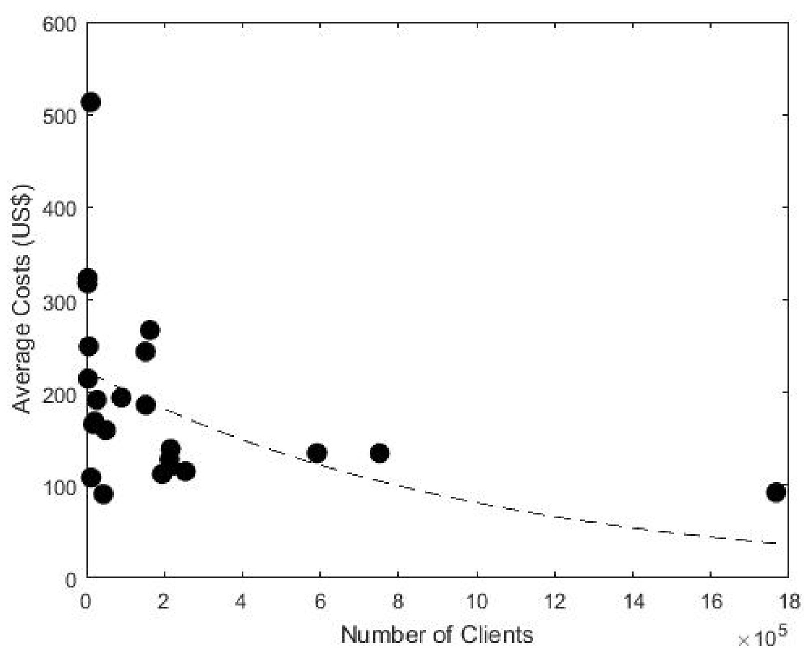

Figure 1.2: Average per client operating costs vs. number of clients.

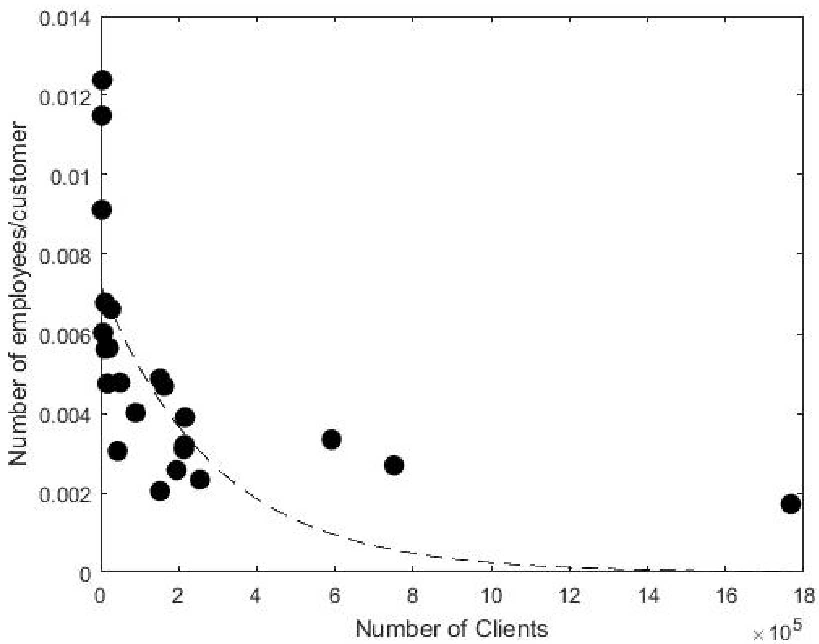

Figure 1.3: Average number of employees vs. number of clients.

Drinking water quality is measured by the compliance with all water quality standards and treated wastewater; the quality indicator is based on compliance with emission standards, which is directly related with the quality of effluent discharge. On average, WaSCs present high drinking water and treated wastewater quality, reaching 95.4% and 99%, respectively. Only two WaSCs present low compliance with drinking water quality, with indicators below 90%. On the other hand, all WaSCs present indicators of wastewater treatment quality above 95%. In spite that the Chilean water regulator establishes that the maximum percentage of leakage in water supply of the efficient WaSC is 20%, most of the water companies in Chile exhibit larger percentages. In 2014 the average percentage of non-revenue water for the Chilean water industry was 29.7% and about 74% of this percentage corresponded

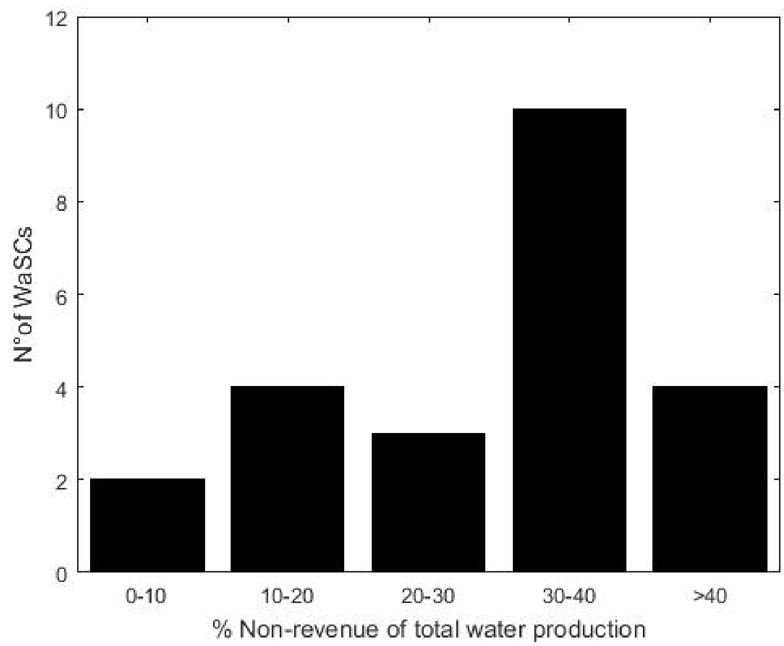

Figure 1.4: Distribution of WaSCs with respect to non-revenue water.

to water losses. As can be seen in Figure 1.4, the majority of Chilean WaSCs (61%) present a percentage of non-revenue water above 30%, while only 26% present a leakage percentage below the regulator?s target of 20%.

The peak factor is defined as the ratio between the maximum daily consumption in the year and the average consumption per day. Table 1.1 shows that the average value is 1.2. Regarding the ownership of WaSCs in Chile, it should be noted that all companies evaluated are private, except one which is municipal. The privatization process carried out in Chile followed two approaches: fully privatized, where utilities were sold either by sales or of shares or transfers of assets to the private sector, and concessionary, where the public sector maintained ownership of the infrastructure and a private contractor undertakes operational and maintenance activities for a fixed term (Molinos-Senante and Sala-Garrido 2016). Our WaSCs sample consists of 52.2% and 43.5% of fully privatized and concessionary WaSCs, respectively. Our sample also considers the only municipal water operator of Chile. Finally, the majority of WaSCs obtain water from both surface water and groundwater (47.8%), followed by those whose only water source is groundwater (39.1%). There are only a few WaSCs whose sole water source is surface water. These are all located in the south of Chile where surface water flows are on average greater than 20,000 m3/s.

1.4 Results and discussion

Efficiency assessment

The efficiency of each WaSC was computed by applying the double-bootstrap DEA model. According to the initial DEA model (Eq. 1), 8 out of 23 observations (35%) are efficient, i.e., those whose efficiency score equals one. These WaSCs formed the best practice frontier since they cannot reduce the use of inputs keeping the production of outputs when they are compared with the other assessed WaSCs. As is shown in Table 1.2, the mean efficiency stands at 1.248 with a standard deviation of 0.265.This finding indicates that an average WaSC that performed as efficiently as its benchmark can decrease its inputs by 20% $(1 - 1/1.248)$ while keeping its output constant.

The efficiency estimations of WaSCs level are gathered in Table 1.2.The column named “Bias” provides the bias estimate of the initial efficiency scores obtained with the bootstrap using 2000 iterations. It is illustrated that the bias ranges from a minimum value of -0.004 (WaSC7) to a maximum value of -0.905 (WaSC5). The sign of the bias is negative for all WaSCs which is consistent with previous studies (Ananda 2014; See 2015) whom also obtained negative bias for all water utilities assessed. A bootstrap procedure under the DEA framework allows us to obtain more precise efficiency scores with a limited sample size. Ignoring the need to rescale residuals with small sample sizes, as in this study, leads to strictly negative efficiency score bias in finite samples (Simar and Wilson 2007). The fourth column of Table 1.2 shows the bias-corrected efficiency scores for each WaSC. The average efficiency for the 23 WaSCs evaluated after correcting for the bias stands at 1.631 which means that the potential for input saving among WaSCs is about 39%.

While the difference between the mean biased efficiency (second column of Table 1.2) and the mean bias-corrected efficiency is not large, the ranking of WaSCs based on its performance changes notably. For example, WaSC6 was efficient considering the biased efficiency assessment and therefore, it was in the first position of the ranking. However, when the classification of WaSCs is based on the bias-corrected efficiency score, the WaSC6 occupies the 15th position. From a statistical point of view,the non-parametric test of Mann-Whitney reveals statistically significant differences between the biased and bias-corrected efficiency scores (p-value$\leq$ 0.01) with a 1% of significance. The last three columns of Table 1.2 shows the variance estimates and the lower and upper bounds of the confidence intervals at the 95%.

They illustrate the large variability in the bias estimates among the evaluated WaSCs.

From policy perspective and especially in the context of regulated water industries, the results evidence the importance of estimating bias-corrected efficiency scores. Otherwise, the comparison of the performance of water companies involves biased rankings. This issue is relevant in countries or regions in which tariffs are set based on benchmarking processes. The estimation of biased-corrected efficiency scores provides water regulators with a more complete,reliable and robust in formation to support their decision-making process of setting water tariffs or of introducing incentives to the best performed WaSCs.

Table 1.2: Biased efficiency scores and bias-corrected efficiency scores for WaSCs.

Water and sewerage company	Biased efficiency score	Bias	Bias-corrected efficiency	SD	Lower bound	Upper bound
WaSC1	1.000	-0.671	1.671	0.062	1.106	1.984
WaSC2	1.000	-0.667	1.667	0.052	1.102	1.956
WaSC3	1.158	-0.606	1.764	0.060	1.241	2.172
WaSC4	1.466	-0.414	1.880	0.894	1.809	1.945
WaSC5	1.077	-0.905	1.982	1.249	1.674	2.196
WaSC6	1.000	-0.521	1.521	0.029	1.108	1.783
WaSC7	1.342	-0.004	1.347	0.613	1.046	1.765
WaSC8	1.174	-0.561	1.735	0.874	1.678	1.942
WaSC9	1.883	-0.078	1.961	0.089	1.047	2.195
WaSC10	1.000	-0.455	1.455	0.020	1.096	1.695
WaSC11	1.000	-0.592	1.592	0.037	1.094	1.858
WaSC12	1.624	-0.071	1.694	0.052	1.617	1.723
WaSC13	1.000	-0.530	1.530	0.025	1.084	1.736
WaSC14	1.000	-0.471	1.471	0.028	1.093	1.721
WaSC15	1.000	-0.672	1.672	0.062	1.083	1.968
WaSC16	1.500	-0.055	1.555	0.778	1.543	1.684
WaSC17	1.063	-0.086	1.148	0.280	1.035	1.284
WaSC18	1.333	-0.475	1.807	0.039	1.440	1.921
WaSC19	1.536	-0.072	1.608	0.736	1.506	1.861
WaSC20	1.500	-0.045	1.545	0.856	1.190	1.948
WaSC21	1.282	-0.295	1.578	0.255	1.358	1.955
WaSC22	1.144	-0.474	1.618	0.032	1.241	1.938
WaSC23	1.625	-0.090	1.715	0.564	1.630	1.980
Average	1.248	-0.383	1.631	0.334	1.297	1.879
SD	0.265	0.265	0.188	0.387	0.255	0.198

Exploring the determinants of efficiency

The main implication of considering the uncertainty of the data using bootstrapping is that it is feasible to identify explanatory variables of WaSCs' efficiency. In other words, the second-stage analysis using a regression approach allows for the identification of the environmental factors that significantly influence the efficiency of water companies. Efficiency scores are $\theta_j \geq 1$, being a WaSC efficient when $\theta_j = 1$ and inefficient whenever $\theta_j \geq 1$. Hence, the dependent variable of the regression analysis indicates the inefficiency of the WaSCs. Hence, a positive sign of the estimated regression parameter means higher inefficiency, i.e., lower efficiency, while a negative sign of the estimated parameter means larger efficiency.

The number of potential environmental factors considered was conditioned by the number of WaSCs, in order to ensure enough degrees of freedom for the estimation. Hence, 5 environmental variables were considered, namely: (i) percentage of non-revenue water; (ii) peak factor; (iii) source of water; (iv) customer density and; (v) ownership of water companies. The bias-corrected coefficients of the regressed variables, their standard error and p-values, which indicate the significance of the estimated parameters, are presented in Table 1.3.It illustrates that the percentage of non-revenue water negatively influences WaSCs' efficiency. Hence, WaSCs with large values of this variable correspond to high values of inefficiency since higher percentage of non-revenue water involves higher operating costs influencing the efficiency of water companies. Moreover, the p- value indicates that the relationship between non-revenue water and efficiency is statistically significant. This is explained

Table 1.3: Results of bootstrap truncated regression.

Variable	Bias-corrected coefficients	Standard error	p-value
Intercept	-157.708	358.881	0.558
Non-revenue water	8.923	3.356	0.007**
Peak factor	124.05	214.362	0.473
Groundwater	Reference Variable		
Surface water	-69.462	90.588	0.38
Mixed water	-66.275	88.341	0.394
Customer density	-3.849	2.433	0.99*
Privatized WaSCs	Reference Variable		
Concessionary WaSCs	-10.69	51.588	0.819

* Significant at 10%
** Significant at 10, 5 and 1% level

by the fact that as nonrevenue water increases, the WaSC must increase water extraction and distribute a greater amount of water, so as to produce the same output level; thus, WaSCs with higher non-revenue present higher costs (Hastak et al. 2016). However, previous studies were inconclusive about this issue. For example, Corton and Berg (2009) found that for water companies located in Central America, efficiency and volume of water billed, which is the opposite to non-revenue water, were correlated variables. By contrast, Ananda (2014) for Australian water utilities and Marques et al. (2014) for Japanese water companies evidenced that leakage has no influence on efficiency. In both case studies, the water loss levels were quite low (notably lower than in Chile), and therefore, this variable was irrelevant in terms of efficiency.

The average peak factor of the Chilean WaSCs analyzed in 2014 was 1.2. My results show that parameter associated to peak factor is not statistically significant; thus, this variable does not influence WaSCs' efficiency in Chile. The lack of significance can be explained by two main factors. The first one is the lack of variability of peak factor levels. Table 1.1 indicates that the average peak factor is 1.2 with a standard deviation of 0.2; thus, its coefficient of variation is very low, reaching 14similar peak factors and, thus, I cannot explain inefficiency scores based on this variable. The second explanation might be that these WaSCs supply seasonal and touristic areas, which normally consume more water due to large outdoor water use. Hence, the higher operating costs related to large peak factor are offset by the greater water consumption. In the context of Portuguese water companies, Carvalho and Marques (2011) evidenced that there is a negative influence for peak factors up to 1.2 and for peak factors higher than 1.4. By contrast, when the peak factor is close to 1.4, it has a positive influence on the efficiency.

The sources of water for the Chilean WaSCs are as follows: (i) only surface water, (ii) only ground water, and (iii) mixed surface and ground water. The variable source of water was integrated in the regression analysis as three dummy variables-one for each type of water source. Given that the three dummy variables are perfectly collinear (their sum is always 1), the dummy variable associated to groundwater source was dropped from the estimation. The parameter estimates for the other sources of water are interpreted as intercept shifters. Results show that water source does not have a significant impact on efficiency. Studying water companies located in Southeast Asian and Japan, See (2015) and Marques et al. (2014) also evidenced that the source of water was not significantly related with efficiency.

For the assessed Chilean WaSCs, the average customer density in 2014 was

57.27 number of customers per kilometer of network. As well as water utilities located in several countries (Abbott et al., 2012; Guerrini et al. 2013; Ananda 2014), Chilean WaSCs present economies of density. Thus, the negative sign of the coefficient for customer density indicates that this variable has a positive influence on efficiency. It should be noted that the relationship between efficiency and customer density is significant at 10% level.

WaSC ownership was introduced in the analysis as two artificial dummy variables. Thus, the parameter estimate associated to Bconcessionary WaSCs' must be interpreted with respect to the reference category. The estimated parameter is not statistically significant; hence, ownership type of WaSCs does not influence their efficiency. This may be because the assessment only includes private operators whose only difference is that the concession term for concessionary operators is 30 years, renewable, while fully privatized is for perpetuity. Hence, a concessionary firm that complies with the regulation faces a concession term that tends to perpetuity. This implies that under Chilean regulation, the incentives to be more efficient do not depend on ownership type.

1.5 Conclusions

Benchmarking the efficiency of water companies has acquired a fundamental role in many regulated water industries in the tariff setting procedures so as to increase the competitiveness of water companies and to improve the quality of service to customers. In this context, conventional DEA models have been widely applied to assess the efficiency of water companies. In spite of the positive features of DEA method, one important limitation is that statistical inferences cannot be drawn from conventional DEA models. This drawback is especially relevant in the framework of water companies where controlling the impacts of environmental variables for conducting performance benchmarking is essential. To overcome this limitation, in this chapter, the double-bootstrap DEA method developed by Simar and Wilson (2007) was applied. This approach allowed us to estimate bias-corrected efficiency scores and also to identify determinants of efficiency.

The results for the case study provide three primary conclusions. First, the efficiency ranking of WaSCs based on the conventional DEA model and on the double-bootstrap procedure changes notably. Therefore, the estimation of bias uncorrected efficiency scores generates biased rankings of WaSCs. This issue is essential in countries where benchmarking procedures are used to set water tariffs. The estimation of biascorrected efficiency scores is essential to support the decisionmaking process. Secondly, the percentage of non-revenue water and the density of customers significantly influence the efficiency of WaSCs. On one hand, customer density is an external variable for the WaSCs so they cannot act to improve this factor. However, the regulator should consider this factor while benchmarking WaSCs since customers' density of areas served by each water company impacts the performance of the companies. On the other hand, the percentage of non-revenue water is an environmental factor that WaSCs have some capacity to act upon, especially with respect to leakage reduction. Hence, taking into account that the second-stage analysis presents evidence that non-revenue water negatively influences efficiency, the regulator must introduce public policies to encourage reductions in non-revenue water such as awards or sanctions. Finally, for the Chilean water and sewerage industry, it was illustrated that the peak factor, the source of water, and the ownership of the WaSCs do not significantly influence their efficiency.

Chapter 2

Finite Sample Properties of Estimators of a Commodity Storage Model: A Monte Carlo Study

2.1 Introduction

In a very influential paper, Michaelides and Ng (2000) compares the small sample performance of estimators for models with dynamic structure. They focus on the commodity storage model which they recognize provides a sufficiently demanding context for such comparison. The role of storage in such models is key to characterize price and consumption distributions. However, the particular choice of parameter values of Michaelides and Ng (2000) imply too little role for storage if compared with most of those presented in calibration or econometric estimations of the model. Figure 2.1 shows kernel densities of occurrence of stockouts implied in the simulated price samples to explain how change the levels of storage using different parameterizations from the literature.

They compare the pseudo-maximum likelihood (PML) of Deaton and Laroque (1995, 1996) with those of three simulation-based estimators: the simulated method of moments estimator (SMM) of Duffie and Singleton (1993) the indirect inference estimator (IND) of Gourieroux et al. (1993) and the efficient method of moments estimator (EMM) of Gallant and Tauchen (1996). They conclude that the simulation-based estimators have smaller bias, but are less efficient than PML.

This chapter has three contributions. First, I revisit the Monte Carlo comparisons of Michaelides and Ng (2000), but for a set of storage models including a wider family of parameter values including those more recently proposed as relevant for major commodities. My results strongly differ from those of Michaelides and Ng (2000).

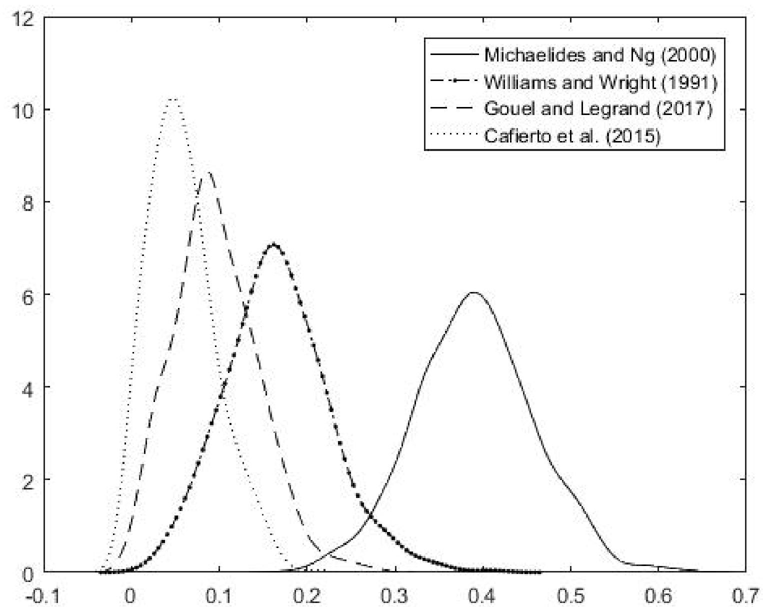

Figure 2.1: Kernel densities of occurrence of stockouts.

Second, I compare the small sample performance of the conditional maximum likelihood estimator (CML) of Cafiero et al. (2015), the unconditional maximum likelihood estimator (UML) of Gouel and Legrand (2017), with those estimators studied by Michaelides and Ng (2000). Third, I provide a proof of consistency for the maximum likelihood estimators of Cafiero et al. (2015) and of Gouel and Legrand (2017).

Although I do not claim that the PML procedure implemented by Deaton and Laroque (1995, 1996) and Cafiero et al. (2011) is asymptotically biased, I do not find evidence in favor of the claim of consistency by Cafiero et al. (2011, p. 48). The Monte Carlo experiments show that biases do not disappear but tend to stabilize as the sample size is significantly increased (up to a sample size of 10,000). This was also observed by Michaelides and Ng (2000, p. 244) for their more modest sample sizes of up to 200.

Michaelides and Ng (2000, p.263) claim that for the storage model "the objective function for classical estimation becomes intractable." Although analytical expressions for the estimators of the storage model are not available, I prove consistency for classical maximum likelihood estimators. I show that such estimators are far more efficient and exhibit substantial less bias in small samples than each of the estimators they consider.

All of the estimators studied in this paper are comparable. They require the same informational structure, in the sense that they all need to assume a particular distribution for the shocks, a particular specification for the consumption demand, and storage cost. Although my results refer to estimation of the commodity storage model, my results provide useful lessons for the estimation of dynamic models that share the non-linearity implied by the non-negativity constraint on stocks, for example in models of consumption with liquidity constraints in the tradition of Deaton (1991).

The rest of the chapter is organized as follows. Section 2.2 describe the speculative storage model and present the endogenous grid method to solve it. Section 2.3 details the Monte Carlo experiments and review the econometrics methods. Section 2.4 present the results of Monte Carlo experiments. Section 2.5 proof the consistency of the Conditional Maximum Likelihood estimator in the context of storage model and section 2.6 concludes.

2.2 The Model

The speculative storage model

I use the framework of a commodity storage model with non-negativity constraints on the amount stored, in the tradition of Gustafson (1958), Scheinkman and Schechtman (1983), Williams and Wright (1991) and Deaton and Laroque (1992). The standard speculative storage model consider two type of agents, consumers and inventory holders. They are competitive and both have rational expectations. I assume risk neutrality, access to a perfect capital market where the rate of interest r is fixed and there is no storage cost.[1] The exogenous supply shocks w_t are i.i.d. on a compact support $[\underline{w}, \overline{w}]$. The availability at period t is $z_t \equiv w_t + (1-d)x_{t-1}$, where x_{t-1} is the storage at time $t-1$, and d is the physical deterioration rate of stocks. The demand for commodities c_t has a linear inverse demand $F(c) = a + bc_t$ with $b < 0$ and $\left(\frac{1-d}{1+r}\right) EF(w_t) > 0$.

Considering the above elements, a stationary rational expectations equilibrium (SREE) is a function $f : Z \to \mathbb{R}$ which describes price as a function of the current availability, and satisfies for all $z_t \in Z$,

$$p_t = f(z_t) = \max\left\{\left(\frac{1-d}{1+r}\right) E_t f\big(w_{t+1} + (1-d)\big[z_t - F^{-1}\big(f(z_t)\big)\big]\big),\ F(z_t)\right\}. \tag{2.1}$$

Since the w_t's are i.i.d., f is the solution to the following functional equation:

$$f(z) = \max\left\{\left(\frac{1-d}{1+r}\right) Ef\big(w + (1-d)\big[z - F^{-1}\big(f(z)\big)\big]\big), F(z)\right\}. \tag{2.2}$$

[1] Cafiero et. al (2011b, 2014) and Gouel and Legrand (2016) consider a marginal cost k of storing x_t units of discretionary stocks.

Existence and uniqueness of the SREE, $f(z)$, are given by the following Theorem:

Theorem 1. *There is a unique stationary rational expectations equilibrium f in the class of continuous non-negative, non-increasing functions. Furthermore, if $p^* \equiv \left(\frac{1-d}{1+r}\right) Ef(w)$, then:*

$$f(z) = F(z), \quad \text{for} \quad z \leq F^{-1}(p^*),$$

$$f(z) > F(z), \quad \text{for} \quad z > F^{-1}(p^*).$$

f is strictly decreasing whenever it is strictly positive. The equilibrium level of inventories, $x(z)$, is strictly increasing for $z > F^{-1}(p^*)$.[2]

Proof. Deaton and Laroque (1992), Theorem 1.

[2]Cafiero et. al (2011b, 2014) extend the theorem for a model with positive marginal cost, possibly unbounded realized production and free disposal.

Numerical method for compute the SREE

There are many numerical methods to solve the equilibrium price function.[3] I estimate the SRRE function f with a linear spline over a grid of 1,000 equally spaced points. To take expectations with respect to the normal shock w, I substitute the integral adopting the same approximation presented in Deaton and Laroque (1995,1996) and used in Michaelides and Ng (2000), with nodes w_s and equiprobly weights π_s.[4]. This procedure allow us write (2.2) as:

$$f(z) = \max\left\{\left(\frac{1-d}{1+r}\right)\sum_{s=1}^{10} f\big(w_s + (1-d)\big[z - F^{-1}\big(f(z)\big)\big]\big)\pi_s, F(z)\right\}. \tag{2.3}$$

Theorem 1 of Deaton and Laroque (1992) studies the mapping operator T which for some $m \in \mathbb{N}$ associates with a function f_m the function f_{m+1}, defined by:

$$f_{m+1}(z) = \max\left\{\left(\frac{1-d}{1+r}\right)\sum_{s=1}^{10} f_m\big(w_s + (1-d)\big[z - F^{-1}\big(f_{m+1}(z)\big)\big]\big)\pi_s, F(z)\right\}. \tag{2.4}$$

and proof that such operator defines a contracting mapping, and given a choice of some f_0 the sequence converges to the SREE f.

In this paper, I use the endogenous grid method proposed by Carroll(2016) to solve the equation (2.4). Gouel (2013a) shows that this method applied in the storage model allows rapid solution compares with Deaton and Laraque (1992). Notice that the function f_m has a kink in its domain at $z^*_{m+1} = F^{-1}\left(\left(\frac{1-d}{1+r}\right)\sum_{s=1}^{10} f_m(w_s)\pi_s\right)$.

For a given iterate f_m defines the storage function:

$$x_{m+1}(z) = z - F^{-1}(f_{m+1}(z)) \tag{2.5}$$

[3] See Gouel (2013)

[4] $S = 10$ nodes, $w_s = (\pm 1.755 \pm 1.045 \pm 0.677 \pm 0.386 \pm 0.126)$ with probabilities $\pi_s = 0.1$ each one.

Then the equation (2.4) can be written as:

$$f_{m+1}(z) = \max\left\{\left(\frac{1-d}{1+r}\right)\sum_{s=1}^{10} f_m\big(w_s + (1-d)\big[x_{m+1}(z)\big]\big)\pi_s, F(z)\right\}. \tag{2.6}$$

The storage function (2.5) implies that, $f_{m+1} = F(z - x_{m+1}(z))$, replacing this expression in (6), I can applying F^{-1} and solving for all $z > z^*_{m+1}$. Hence, the last equation can be written as:

$$z = F^{-1}\left(\left(\frac{1-d}{1+r}\right)\sum_{s=1}^{10} f_m\big(w_s + (1-d)\big[x_{m+1}(z)\big]\big)\pi_s\right) + x_{m+1}(z) \tag{2.7}$$

The first step of the algorithm is define a monotone grid of points for the state variable $\vec{Z}_0$, and a monotone grid for the storage function, $\vec{x}$ which starts at zero. Given F, r, d, f_m and considering the approximation of w, I compute z^*_{m+1} and define the grid $\vec{Z}_{m+1} = \{z \in \vec{Z}_0 : z \leq z^*_{m+1}\}$. Replace the storage grid $\vec{x}$ in equation (7) and compute a new grid for $z > z^*_{m+1}$, $\vec{z}_{m+1}$ defined as:

$$\vec{z}_{m+1} = F^{-1}\left(\left(\frac{1-d}{1+r}\right)\sum_{s=1}^{10} f_m\big(w_s + (1-d)\vec{x}\big)\pi_s\right) + \vec{x} \tag{2.8}$$

construct f_{m+1} interpolating the domain points $\{\vec{Z}_{m+1} \cup z^*_{m+1} \cup \vec{z}_{m+1}\}$ with their prices co-domain $\{F(\vec{Z}_{m+1}) \cup F(z^*_{m+1}) \cup F(\vec{z}_{m+1} - \vec{x})\}$. Finally update f_m by f_{m+1} and repeat the steps until $\|f_{m+1}(\vec{Z}_0) - f_m(\vec{Z}_0)\|_\infty < \epsilon$, where $\|\cdot\|_\infty$ is the supremum norm, and $\epsilon > 0$ is an error of bound to approximate f which is fixed arbitrary. The advantage of this procedure is that the convergence is monotone and I do not need to solve a non-linear equation.

2.3 Montecarlo Analysis

Generation of Prices

I design a Montecarlo experiment based on Michaelides and Ng (2000). To generate sequences of ”observed” prices under the storage model, I compute the equilibrium price function setting the parameters $\theta_0 = (a, b, d)$ and I simulate $\{w_t\}_{t=1}^T$ normal variates fixing the seed and parameters μ and σ. Finally, I construct the sample $\{p_t\}$ of size T. To generate the heuristic models, I use the following parameterizations:

[1] $a = 0.6, b = -0.3$, $d = 0.10, \mu = 0$ and $\sigma = 1$ (Michaelides and Ng 2000)

[2] $a = 1, b = -1, \mu = 0$ and $\sigma = 1$ (Gouel and Legrand 2017)

[3] $a = 1, b = -2, \mu = 0$ and $\sigma = 1$ (Cafiero et. al 2015)

[4] $a = 600, b = -5, \mu = 100$ and $\sigma = 10$ (Williams and Wright 1991)

Econometric Procedures

In this section, I present the econometrics methods in the context of the speculative storage model under rational expectations. Deaton and Laroque (1995) presented pioneer estimator of Pseudo Maximum Likelihood (PML). Michaelides and Ng(2000) implemented three simulation estimators: The Indirect Inference Estimator (IND) of Gourieroux et al. (1993), the Simulated Method of Moments Estimator (SMM) of Duffie and Singleton (1993), and the Efficient Method of Moments Estimator (EMM) by Gallant and Tauchen (1996). Cafiero et. al (2014) proposed the Conditional Maximum Likelihood Estimator (CML) with stock-outs based only in prices. Finally, Gouel (2017) extend the CML to its unconditional counterpart (UML). With these econometrics tools, I estimate a set of parameters $\hat{\theta} = \{a, b, d\}$ setting μ and σ.[5], and I analyze the performance of each estimator for different sample sizes.

The Simulation Estimators

This kind of estimators require generate ”simulated” prices. First, I simulate normal variates $\{\tilde{w}_t\}_{t=1}^{N}$ with the same mean μ and standard deviation σ as in 3.1, but using a different seed. I define $N = TH$, where H are paths, each of length T. Start at the initial guess $\tilde{\theta}_0$, in each iteration solve the equilibrium price function and simulate prices $\{\tilde{p}_t\}$ of size N. For all estimators, $\hat{\theta}$ is determined as: $\mathrm{Argmin}_\theta\, \xi'\Omega\xi$. Bellow, I present the three estimators which differ in the choice of ξ and the weighting matrix Ω.

- SMM

$$\xi = \left(\frac{1}{T} \sum_{t=1}^{T} m(p_t) - \frac{1}{TH} \sum_{t=1}^{TH} m(\tilde{p}_t) \right)$$
$$\Omega = I_0^{-1} = \lim_{T \to \infty} Var \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} m(p_t) \right)^{-1}$$

[5]Deaton and Laroque (1996), Proposition 1, prove that the mean and variance of harvest cannot be separately identified from the demand parameters

- IND

$$\xi = \hat{\beta}_T(p_{[T]}) - \tilde{\beta}_{TH}(\tilde{p}_{[TH]}, \theta)$$

$$\Omega = J_0 I_0^{-1} J_0$$

$$I_0^{-1} = \lim_{T\to\infty} Var\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{\partial m(p_t,\beta)}{\partial\beta}\right)^{-1}$$

$$J_0 = -\frac{1}{T}\sum_{t=1}^{T}\frac{\partial^2 m(p_t,\beta)}{\partial\beta\partial\beta'}$$

- EMM

$$\xi = E_p\left[\frac{\partial m(p_{[\infty]},\beta)}{\partial\beta'}\right]_{\beta=\hat{\beta}} \approx \frac{1}{N}\sum_{n=1}^{N}\frac{\partial m(\tilde{p}_n,\hat{\beta})}{\partial\beta'}$$

$$\Omega = I_0^{-1} = Var\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{\partial m(p_t,\beta)}{\partial\beta}\right)^{-1}$$

I use the moments conditions m_1 and m_2 for SMM. The main difference is that m_1 do not explicitly recognize the non-linear nature of the price process while m_2 incorporate more relevant information by taking account the skewness and kurtosis. In case of IND, I consider two auxiliary equations: M_1 and M_2. In this case, M_2 recognize the non-linearity in the conditional mean. Finally for EMM, I use four specifications: M_1, M_2, M_3 and M_4 to estimate the simulated model: The third auxiliary model M_3, allows for a time-conditional heteroskedasticity variance , but does not capture the two regime nature of the prices process as M_4.

$$m_1 : m(p_t) = [p_t, (p_t - \bar{p})^2, (p_t - \bar{p})(p_{t-j} - \bar{p})]', j = 1, 2, 3$$

$$m_2 : m(p_t) = [p_t, (p_t - \bar{p})^i, (p_t - \bar{p})(p_{t-1} - \bar{p})]', i = 2, 3, 4$$

$$M1 : m(p_t, \beta) = -(T-3)\log(2\pi) - \frac{T-3}{2}\log(\sigma^2) - \sum_{t=4}^{T} \frac{(p_t - \beta_0 - \beta_1 p_{t-1} - \beta_2 p_{t-2} - \beta_3 p_{t-3})^2}{2\sigma^2}$$

$$M2 : m(p_t, \beta) = -(T-1)\log(2\pi) - \frac{T-1}{2}\log(\sigma^2) - \sum_{t=2}^{T} \frac{(p_t - \beta_0 - \beta_1 p_{t-1} - \beta_2 p_{t-1}^2 - \beta_3 p_{t-1}^2)^2}{2\sigma^2}$$

$$M3 : m(p_t, \beta) = -(T-1)\log(2\pi) + \sum_{\tau=2}^{T} \left(-0.5h_t - 0.5\frac{e_t^2}{exp(h_t)} \right)$$

$$e_t = p_t - \beta_0 - \beta_1 p_{t-1} - \beta_2 p_{t-1}^2$$

$$\log \sigma_t^2 = h_t = \alpha_0 + \alpha_1 p_{t-1}^2$$

$$M4 : m(p_t, \beta) = -(T-1)\log(2\pi) + \sum_{t=2}^{T} \left(-0.5h_t - 0.5\frac{e_t^2}{exp(h_t)} \right)$$

$$p_t = \alpha_1 p^* - \frac{\alpha_1 p^* - \alpha_1 p_{t-1}}{\triangle_t} + e_t$$

$$\log \sigma_t^2 = h_t = \alpha_0 + \frac{\beta_0 - \alpha_0 + \beta_1 p_{t-1}^2}{\triangle_t}$$

$$\triangle_t = 1 + \exp\left(\gamma(p_{t-1} - p^*)\right), \text{with } \gamma \text{ fixed at } 10/\text{standard deviation of prices.}$$

For SMM and EMM the optimal weighting matrix is given by $\Omega = I_0^{-1}$. The long-run covariance matrix is estimated using the Newey-West approximation (Newey and West 1987), a heteroskedasticity and autocorrelation consistent procedure, with appropriate weights. That is: $I_0^{-1} = \Gamma_0 + \sum_{j=1}^{J} \lambda_j(\Gamma_j + \Gamma_j')$ where Γ_j is the j-th order estimated autocovariance matrix $\Gamma_j = \frac{1}{T}\sum_{t=j+1}^{T} m_t m_{t-j}'$, J is referred to as the bandwitdth and the λ_j are weights. The specification for the weighting function is the Parzen's window:

$$\lambda_j = \begin{cases} 1 - 6x^2 + 6|x|^3, & \text{if } 0 \leq |x| \leq 0.5 \\ 2(1-|x|)^3, & \text{if } 0.5 \leq |x| \leq 1 \end{cases}$$

where $x = 1 - j/(J+1)$. Consider the last specification, the optimal weighting for IND is given by:

$$\Omega = \left[\frac{1}{T}\sum_{t=1}^{T}\frac{\partial^2 m(p_t,\beta)}{\partial\beta\partial\beta'}\right]' \left[\Gamma_0 + \sum_{j=1}^{J}\lambda_j(\Gamma_j + \Gamma_j')\right]\left[\frac{1}{T}\sum_{t=1}^{T}\frac{\partial^2 m(p_t,\beta)}{\partial\beta\partial\beta'}\right]$$

PML

The log-pseudo likelihood function is formed as follows:

$$\log L = 0.5\left(-(T-1)\log(2\pi) - \sum_{t=1}^{T-1}\log s(p_t) - \sum_{t=1}^{T-1}\frac{\left(p_{t+1} - m(p_t)\right)^2}{s(p_t)}\right)$$

we calculate the first two moments of p_{t+1} conditional on p_t using the approximate SREE price function f in (3):

$$m(p_t) = \sum_{s=1}^{S} f\left(w_{t+1}^s + (1-d)(f^{-1}(p_t|\theta) - F^{-1}(p_t|a,b))\right)(\pi_{t+1}^s)$$

$$s(p_t) = \sum_{s=1}^{S} f\left(w_{t+1}^n + (1-d)(f^{-1}(p_t|\theta) - F^{-1}(p_t|a,b))\right)^2(\pi_{t+1}^s) - m^2(p_t|\theta)$$

CML

Given the SREE function f in (3), for positive prices the model implicitly defines a mapping from harvests w_{t+1} to prices p_{t+1}, conditional on the previous price p_t: The likelihood function is:

$$L(\theta|p_1,...,p_T) = \prod_{t=1}^{T} l_\theta(p_{t+1}|p_t) = \prod_{t=1}^{T}\frac{\phi(w_{t+1})}{\Phi(\overline{w}) - \Phi(\underline{w})}|J_{t+1}|$$

where $J_{t+1} = \dfrac{df^{-1}}{dp_{t+1}}(p_{t+1})$ is the Jacobian of the mapping $p_{t+1} \mapsto w_{t+1}$. The conditional log-likelihood function is:

$$\log L(\theta|p_1,...,p_T) = -\frac{T}{2}\log(2\pi) - T\log[\Phi(\overline{w}) - \Phi(\underline{w})] + \sum_{t=1}^{T}\log|J_{t+1}| - \frac{1}{2}\sum_{t=1}^{T}[f^{-1}(p_{t+1}|\theta) - (1-d)\{f^{-1}(p_t|\theta) + F^{-1}(p_t|a,b)\}]^2$$

where:

$$J(p_t) = \begin{cases} \frac{dF^{-1}}{dp_t}(p_t) & ,if \quad p_t \geq p^* \\ \frac{df^{-1}}{dp_t}(p_t) & ,if \quad p_t < p^* \end{cases}$$

UML

The purpose is to extend the CML using the information from the first price by accounting for the marginal density $l_\theta(p_1)$.

$$L(\theta|p_1,...,p_T) = l_\theta(p_1)\prod_{t=1}^{T} l_\theta(p_{t+1}|p_t)$$

where $l_\theta(p_1) \approx \frac{1}{M}\sum_{m=1}^{M} l_\theta(p_1|p_0^m)$ is the Montercalo integration by simulating the model on the invariant distribution. Follow Gouel and Legrand (2007) I draw a 10,000 trajectories from the steady state for 100 periods, generating a sample of 1,000,000 prices. The shocks that generate the price simulations are drawn at the beginning of the estimation procedure and remain fixed throughout. The unconditional log-likelihood function is:

$$\log L(\theta|p_1,...,p_T) = l_\theta(p_1) - \frac{T}{2}\log(2\pi) - T\log[\Phi(\overline{w}) - \Phi(\underline{w})] + \sum_{t=1}^{T}\log|J_{t+1}| - \frac{1}{2}\sum_{t=1}^{T}[f^{-1}(p_{t+1}|\theta) - (1-d)\{f^{-1}(p_t|\theta) + F^{-1}(p_t|a,b)\}]^2$$

2.4 Results

To study the finite sample performance of the different econometric estimators, I conduct four Monte Carlo experiments using the parameterizations presented in the previous chapter for different sample sizes: T = 100, 200, 500, 1000, 5000, 10000. The size of simulated data to estimate Simulation Methods varies for each auxiliary model and ranges from: N = 500, 1000, 2000, 2500, 10000, 20000. The number of simulations for each experiment is 500. Results for econometrics methods of each experiment are reported in Appendix A, where the Mean and the Root Mean Square Error (RMSE) of the parameters distribution are relevant to assess their performance.

Tables 1 to 9 present the Monte Carlo experiment results; whose parameterization (3.1[1]) implies low average storage and frequent stockouts. Results of the Simulation Methods estimators are similar to those obtained by Michaelides and Ng (2000) for samples T = 100, 200 that conclude that PML is more efficient in terms of RMSE for all estimated parameters. Yet Table 9 shows that CML and UML estimators yield to precise and more efficient estimates of the parameters of the model, same conclusion presented by Cafiero et al. (2015). By increasing the sample size, Simulation estimators tend to significantly reduce the bias and the RMSE, converging to the true parameters of the heuristic model. According to Michaelides and Ng (2000, p.251),“increasing the length of the observed data has a stronger influence on the estimates than increasing the length of the simulated series”. My results prove such statement for all Simulation Models as well as for CML and UML. However, biases for PML estimates do not disappear but tend to stabilize as sample size increase close to the heuristic model values. For both CML and UML, the RMSE for each estimated parameter is substantially lower than the corresponding value obtained by the other methods. They do not have a significant difference and their estimates tend to converge when the sample size increases to T = 10000, with a RMSE of 0.0014 for parameter a, 0.0018 for b and 0.001 for d.

Next, I assessed a Monte Carlo simulation model where demand functions are steeper thus storage plays a greater role and is more frequent. Tables 10 to 18 present the results of the second parameterization (3.1[2]). Results show that small samples bias increases for the Simulation estimators. In particular SMM, IND (M1), EMM (M1), and EMM (M2) tend to underestimate parameters a and b. I observe that IND (M2) underestimate a and over-estimates b while EMM (M3) and EMM (M4), over-estimates a and underestimate b. As the sample size increases, the bias is reduced and for a sample size T = 10000 the average RMSE for all estimates converge to 0.01 for a and 0.03 for b. For small sample, the performance of PML estimator underestimates a and over-estimates b. In general terms, both parameter estimates are more efficient than the Simulation estimators. Yet, as sample size increases parameter a tend to stabilize at 0.94. I observe that although bias might be small compared to CML and UML methods for a sample T = 10000, estimators have substantially better precision where the bias and the RSME converge to almost zero.

Tables 19 to 27 show the Monte Carlo results whose parameterization (3.1[3]) increases the slope of the previous case, implying greater storage. I note that small sample performance for all Simulation estimators underestimate a and b. The bias in b in terms of magnitude is greater than that presented in the previous simulation. By increasing the slope, the RMSE present higher values that decrease as the sample size increases. When I compare to the PML method, the small samples estimators are more efficient than the Simulation estimators for both parameters. However, as sample size increases, the estimators tend to underestimate a and b, whose mean rises to 0.9085 for a and -1.9073 for b. Current estimation presents a greater bias and RMSE compared to both CML and UML which are more efficient than all previous methods studied.

The result of the last Monte Carlo (3.1[4]) are present in tables 28-36. They are similar to those obtained in the previous experiments: when sample size increases Simulation estimators tend to converge to the true values of the heuristic model. PML keeps a bias as sample size increases. The estimates of CML and UML perform better in terms of efficiency for all models.

The Monte Carlo experiments suggest that CML and UML tend to converge to the true values of each heuristic model, which is related with the consistency property of these estimators. In the next section I prove that the CML estimator in the context of storage model is consistent.

2.5 Consistency of Maximum Likelihood Estimator

Assumptions:

1. Linear inverse consumption demand $F(c_t) = a + bc_t,$ where $a > 0,$ and $b < 0$ are real constants. The deterioration rate of stocks is $d,\ 0 \leq d \leq 1.$

2. The interest rate $r,\ r > 0,$ is fixed.

3. The shocks are given for an i.i.d. sequence of random variables $\{w_t\}_{t \in \mathbb{N}}$, with a Normal $N(0, 1)$ truncated distribution. The support of w_t is $[\underline{w}, \overline{w}],$ where $-\infty < \underline{w} < \overline{w} < +\infty.$

4. The SREE is a function which describes price as a function of the current availability, and satisfies for all $z_t \in Z,$

$$p_t = f(z_t) = \max\left\{\left(\frac{1-d}{1+r}\right) E_t f\big(w_{t+1} + (1-d)\big[z_t - F^{-1}\big(f(z_t)\big)\big]\big),\ F(z_t)\right\}.$$

Conclusions, Results:

1. The model implicitly defines a mapping from the harvests w_{t+1} to prices $p_{t+1},$ conditional on the previous price p_t.

$$p_{t+1} = f[w_{t+1} + (1-d)\{f^{-1}(p_t) - F^{-1}(p_t)\}]$$

For a vector parameter $\theta = (a, b, d),$ the conditional density $l_\theta(p_{t+1}|p_t)$ is equal to:

$$l_\theta(p_{t+1}|p_t) \ = \frac{\phi(w_{t+1}^{(\theta)})}{\Phi(\overline{w}) - \Phi(\underline{w})} \left|\frac{df_\theta^{-1}}{dp_{t+1}}(p_{t+1})\right|,$$

where Φ and ϕ are the accumulated and density functions corresponding to the normal $N(0,1)$, and $w_{t+1}^{(\theta)} \equiv f_\theta^{-1}(p_{t+1}) - (1-d)\{f_\theta^{-1}(p_t) - F_\theta^{-1}(p_t)\}$.

For a sample of prices, the likelihood function is:

$$L(\theta|p_1,...,p_T) = \prod_{t=1}^{T} l_\theta(p_{t+1}|p_t) = \prod_{t=1}^{T} \frac{\phi(w_{t+1}^{(\theta)})}{\Phi(\overline{w}) - \Phi(\underline{w})} |J_{t+1}^{(\theta)}|$$

where $J_{t+1}^{(\theta)} = \dfrac{df_\theta^{-1}}{dp_{t+1}}(p_{t+1})$ is the Jacobian of the mapping $p_{t+1} \mapsto w_{t+1}^{(\theta)}$.

2. The log-likelihood function is:

$\log L(\theta|p_1,...,p_T) = -\dfrac{T}{2}\text{log}(2\pi) - T\log[\Phi(\overline{w}) - \Phi(\underline{w})] + \sum_{t=1}^{T} \log|J_{t+1}^{(\theta)}| - \dfrac{1}{2}\sum_{t=1}^{T}[f_\theta^{-1}(p_{t+1}) - (1-d)\{f_\theta^{-1}(p_t) - F_\theta^{-1}(p_t)\}]^2$

where:

$$J_{t+1}^{(\theta)} = \begin{cases} \dfrac{dF_\theta^{-1}}{dp_{t+1}}(p_{t+1}) & ,\text{if} \quad p_{t+1} > p^* \\ \dfrac{df_\theta^{-1}}{dp_{t+1}}(p_{t+1}) & ,\text{if} \quad p_{t+1} < p^* \end{cases}$$

3. Let $\{p_t\}_{t\in\mathbb{N}}$ be the storage price process. Let $l_\theta(p_{t+1}|p_t)$ be the conditional density.

Claim: $l_\theta(p_{t+1}|p_t)$ is continuous in θ, for all (p_t, p_{t+1}).

Proof of Claim:

$$l_\theta(p_{t+1}|p_t) \;=\; \frac{e^{-\frac{\left[f_\theta^{-1}(p_{t+1})-(1-d)\{f_\theta^{-1}(p_t)-F_\theta^{-1}(p_t)\}\right]^2}{2}}}{\sqrt{2\pi}\,(\Phi(\overline{w}) - \Phi(\underline{w}))} \left|\frac{df_\theta^{-1}}{dp_{t+1}}(p_{t+1})\right|$$

For any given p_{t+1}, $f_\theta^{-1}(p_{t+1})$ depends continuously on θ. Indeed, if $g = g(z,\theta)$ is continuous, then $\left\{\dfrac{1-d}{1+r}\right\} Eg\left(w_{t+1} + (1-d)(z - F_\theta^{-1}(q)), \theta\right)$ is continuous (see Lemma 1 in Deaton and Laroque, 1992). The Jacobian of the price function, $\dfrac{df_\theta^{-1}}{dp_{t+1}}(p_{t+1}),$ exists almost everywhere. At $p^*,$ we consider a small smooth perturbation of f_θ (see for example Bertsekas, 1975). Q.E.D.

4. **Theorem 1:**

   (i) The random vector process $\{(w_{t+1}, p_t)\}_{t\in\mathbb{N}}$ is ergodic, that is, it has a unique invariant vector distribution $\pi_\infty,$ which is a global attractor.

   (ii) The vector process $\{(p_{t+1}, p_t)\}_{t\in\mathbb{N}}$ satisfies the Strong Law of Large Numbers. That is, for any given Borel measurable function $\varphi : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ with finite limit expectation, $\dfrac{1}{T}\sum_{t=1}^{T}\varphi(p_{t+1}, p_t) \to E_{\pi_\infty}\Psi(w_{t+1}, p_t),$ where $\Psi(w_{t+1}, p_t) \equiv \varphi(f[w_{t+1} + (1-d)\{f^{-1}(p_t) - F^{-1}(p_t)\}], p_t).$

   Proof of Theorem 1:

   (i) Since w_{t+1} is independent of $p_t,$ the distribution of the random vector (w_{t+1}, p_t) can be expressed as :

   $$\text{Prob}\,\{(w_{t+1}, p_t) \in]-\infty, \alpha]\times]-\infty, \beta]\} \;=\; \text{Prob}[w_{t+1} \le \alpha] \;\cdot\; \text{Prob}[p_t \le \beta].$$

   The first factor, Prob$[w_{t+1} \le \alpha]$ does not depend on $t,$ and the second factor Prob$[p_t \le \beta]$ converges to a unique invariant distribution, by the ergodicity of prices $\{p_t\}_{t\in\mathbb{N}}.$

   (ii) Since $\Psi(w_{t+1}, p_t) \equiv \varphi(f[w_{t+1} + (1-d)\{f^{-1}(p_t) - F^{-1}(p_t)\}], p_t)$ is measurable with finite expectation with respect to $\pi_\infty,$ by Breiman (1960) the result follows (see Theorem 17.1.7 in Meyn and Tweedie, 1993). Q.E.D.

5. Let $\{p_t\}_{t\in\mathbb{N}}$ be the price process of the commodity storage model.

Claim: There exists a finite constant K such that $U_\theta(\{p_{t+1}, p_t\}) \equiv \log l_\theta(p_{t+1}|p_t) - \log l_{\theta_0}(p_{t+1}|p_t)$ satisfies $|U_\theta(\{p_{t+1}, p_t\})| \leq K$ for all θ and almost surely in $\{p_t, p_{t+1}\}$. By Theorem 1 (see 4.), this fact implies that the sequence $\{U_\theta(\{p_{t+1}, p_t\})\}_{t\in\mathbb{N}}$ satisfies the law of large numbers.

Proof of Claim: $U_\theta(\{p_{t+1}, p_t\}) =$

$$= \log\left(\exp\frac{-(w_{t+1}^{(\theta)}))^2}{2}|J_{t+1}^{(\theta)}|\right) - \log\left(\exp\frac{-(w_{t+1}^{(\theta_0)})^2}{2}|J_{t+1}^{(\theta_0)}|\right)$$

$$= \frac{-(w_{t+1}^{(\theta)})^2}{2} + \log|J_{t+1}^{(\theta)}| + \frac{(w_{t+1}^{(\theta_0)})^2}{2} - \log|J_{t+1}^{(\theta_0)}|$$

$$= \frac{1}{2}\left[(w_{t+1}^{(\theta_0)})^2 - (w_{t+1}^{(\theta)})^2\right] + \log\left[\frac{|J_{t+1}^{(\theta)}|}{|J_{t+1}^{(\theta_0)}|}\right]$$

Therefore:

$$|U_\theta(\{p_{t+1}, p_t\})| \;\leq\; \max\{\overline{w}^2, \underline{w}^2\} \;+\; \left|\log\left[\frac{|J_{t+1}^{(\theta)}|}{|J_{t+1}^{(\theta_0)}|}\right]\right|.$$

The absolute value of Jacobian, $|J_{t+1}^{(\theta)}|,$ is bounded from below by the positive number $-1/b,$ and it is bounded from above by a finite bound obtained from the deterministic model with $w_t \equiv \underline{w}$ (see Theorem 1.1 (i) in Schechtman and Escudero, 1977). The analytical expression for such deterministic model is presented for example in Bobenrieth, Bobenrieth and Wright (2012, pp. 4-5). Taking a bounded and closed parameter space (that is a compact parameter space), $\Theta \equiv [\underline{a}, \overline{a}] \times [\underline{b}, \overline{b}] \times [0, 1] \ni \theta_0 = (a_0, b_0, d_0),$ where $0 < \underline{a} < \overline{a} < \infty,$ $-\infty < \underline{b} < \overline{b} < 0,$ we get a finite upper bound for $|J_{t+1}^{(\theta)}|,$ for all $\theta \in \Theta.$ Hence, $|U_\theta(\{p_{t+1}, p_t\})| \;\leq\; K < +\infty, \quad \forall\, \theta \in \Theta,$ where K is a real constant. Q.E.D.

6. Claim: For all $\theta \in \Theta$ and sufficiently small $\xi > 0$, $\sup\limits_{||\theta' - \theta|| < \xi} l_{\theta'}(p_{t+1}|p_t)$ is Borel-measurable in $\{p_t, p_{t+1}\}$.

   Proof of Claim: $l_{\theta'}(p_{t+1}|p_t)$ is Borel-measurable in $\{p_t, p_{t+1}\}$, and it is continuous in θ, this condition is satisfied because $\sup\limits_{||\theta' - \theta|| < \xi} l_{\theta'}(p_{t+1}|p_t) = \sup\limits_{\theta'_n \in D} l_{\theta'_n}(p_{t+1}|p_t)$ for any denumerable set D, dense in the ball $B(\theta, \xi) \equiv \{\theta' \ : \ ||\theta' - \theta|| < \xi\}$ Q.E.D.

7. Claim:(Identifiability)

   $\theta \neq \theta_0 \Rightarrow l_\theta(p_{t+1}|p_t) \neq l_{\theta_0}(p_{t+1}|p_t)$ with positive probability in (p_t, p_{t+1}), for t large enough.

   Proof of Claim: I consider two cases:

   First case: Let $\theta = (a, b, d)$ and $\theta_0 = (a_0, b_0, d_0)$ with $d \neq d_0$. By the continuity of f_θ and F_θ in θ, and the compactness of Θ, there exists a set A of prices, a set of positive probability, such that:

$$x_\theta(p_t) \equiv (f_\theta^{-1}(p_t) - F_\theta^{-1}(p_t)) \ > \ 0, \quad \forall \, p_t \in A, \quad \text{and} \quad \forall \, \theta \in \Theta.$$

   For prices $p_t \in A$, the Euler condition is satisfied with equality, and therefore: $\left\{\frac{1+r}{1-d}\right\} p_t = E_t(p_{t+1}^{(\theta)}) \neq \left\{\frac{1+r}{1-d_0}\right\} p_t = E_t(p_{t+1}^{(\theta_0)})$, $\forall \, p_t \in A$ implying that: $l_\theta(p_{t+1}|p_t) \neq l_{\theta_0}(p_{t+1}|p_t)$, with positive probability.

   Second case: Let $\theta = (a, b, d)$ and $\theta_0 = (a_0, b_0, d_0)$ with $d = d_0$, and therefore $(a, b) \neq (a_0, b_0)$.

By the continuity of f_θ and F_θ in $\theta,$ and the compactness of $\Theta,$ there is a set B of prices, a set of positive probability, such that:

$$x_\theta(p) \equiv (f_\theta^{-1}(p) - F_\theta^{-1}(p)) \ = \ 0, \quad \forall\, p \in B, \quad \text{and} \quad \forall\, \theta \in \Theta.$$

Consider the joint probability:

$$\text{Prob}[p_{t+1} \in B,\ p_t \in B] \ = \ \text{Prob}[p_{t+1} \in B \mid p_t \in B] \cdot \text{Prob}[p_t \in B]$$

$$= \ \text{Prob}[F_{\theta_0}(w_{t+1}) \ \in B] \cdot \text{Prob}[p_t \in B] \ = \ \text{Prob}[w_{t+1} \ \in F_{\theta_0}^{-1}(B)] \cdot \mu_{t,\theta_0}(B)$$

Therefore, for t large enough, the joint probability that $p_t,$ and p_{t+1} are in the stock-out region of prices $B,$ is bounded by below by a strictly positive constant, for all $\theta \in \Theta.$ However, note that the conditional densities for prices $p_t,$ and p_{t+1} in $B,$ satisfies:

$$l_\theta(p_{t+1}|p_t) = l_{\theta_0}(p_{t+1}|p_t) \implies$$

$$\frac{1}{b} \exp \frac{-\left(\frac{a}{b} - \frac{p_{t+1}}{b}\right)^2}{2} = \frac{1}{b_0} \exp \frac{-\left(\frac{a_0}{b_0} - \frac{p_{t+1}}{b_0}\right)^2}{2} \implies$$

$$\left(\frac{1}{b^2} - \frac{1}{b_0^2}\right) p_{t+1}^2 - 2\left(\frac{a}{b^2} - \frac{a_0}{b_0^2}\right) p_{t+1} + \left[\left(\frac{a^2}{b^2} - \frac{a_0^2}{b_0^2}\right) - 2\log\left(\frac{b_0}{b}\right)\right] = 0$$

The quadratic equation indicates that p_{t+1} takes at most two values. The continuity of the distribution of the shocks, and continuity of the price function, implies that this is a zero probability event. Q.E.D.

8. Theorem 2: Consider the model satisfying Assumptions 1,2,3. Then the ML estimator $\hat{\theta}_T \equiv \text{argmax}_{\theta\in\Theta} \sum_{t=1}^{T} \log\left[l_\theta(p_{t+1}|p_t)\right]$ is strongly consistent, that is $\lim_{T\to\infty}\hat{\theta}_T = \theta_0,$ almost surely.

   Proof of Theorem 2: The idea of this proof is based on Wald (1949). Notwithstanding that there are proofs of consistency of maximum likelihood for generic cases, to the best of myr knowledge there is no proof for the case of the storage model considered here.

   $U_\theta(\{p_{t+1},p_t\}) \equiv \log\left[\frac{l_\theta(p_{t+1}|p_t)}{l_{\theta_0}(p_{t+1}|p_t)}\right]$ It is clear that $\hat{\theta}_T = \text{argmax}_{\theta\in\Theta} \sum_{t=1}^{T} U_\theta(\{p_{t+1},p_t\}).$

   Since:

   - $l_\theta(p_{t+1}|p_t)$ is continuous in $\theta,$
   - The vector process $\{(p_{t+1},p_t)\}_{t\in\mathbb{N}}$ satisfies the Strong Law of Large Numbers (Theorem 1).
   - There exists a finite constant $K \in \mathbb{R},$ such that $|U_\theta(\{p_{t+1},p_t\})| \leq K < +\infty, \quad \forall\, \theta \in \Theta,$ almost surely in $\{p_{t+1},p_t\},$ and
   - The parameter vector θ is identified at θ_0(identifiability),

   Then, for any given $\delta > 0,$ and for the complement of a $\delta-$neighborhood of $\theta_0,$ $S \equiv \{\theta \in \Theta \,:\, ||\theta-\theta_0|| \geq \delta\},$ we have that, with probability one:

   Claim:

$$\text{limsup}_{T\to\infty}\left\{\sup_{\theta\in S}\frac{1}{T}\sum_{t=1}^{T} U_\theta(\{p_{t+1},p_t\})\right\} \leq \sup_{\theta\in S} E_{\pi_\infty,\theta_0}\left\{U_\theta(\{p_{t+1},p_t\})\right\}$$

Proof of Claim: Indeed, if we define for each $\xi > 0$:

$$\varphi(\{p_{t+1}, p_t\}, \theta, \xi) \equiv \sup_{||\theta' - \theta|| < \xi} U_{\theta'}(\{p_{t+1}, p_t\}),$$

then φ is measurable in $\{p_{t+1}, p_t\}$(by the measurability of $l_\theta(p_{t+1}|p_t)$ in $\{p_{t+1}, p_t\}$ and by the continuity of $l_\theta(p_{t+1}|p_t)$ in θ.) Furthermore, $|\varphi(\{p_{t+1}, p_t\}, \theta, \xi)| \leq K < +\infty, \quad \forall\, \theta \in \Theta,\ \forall\, \xi > 0,$ almost surely in $\{p_{t+1}, p_t\},$ and, by the continuity in θ :

$$\varphi(\{p_{t+1}, p_t\}, \theta, \xi) \ \downarrow\ U_\theta(\{p_{t+1}, p_t\}), \qquad \text{as } \xi \ \downarrow\ 0.$$

Hence, by the Dominated Convergence Theorem:

$$\text{lim}_{\xi \downarrow 0} E_{\pi_\infty, \theta_0}\left[\varphi(\{p_{t+1}, p_t\}, \theta, \xi)\right] \ = \ E_{\pi_\infty, \theta_0}\left[U_\theta(\{p_{t+1}, p_t\})\right] \qquad (*)$$

Let $\varepsilon > 0$ be any small positive number, fixed. By $(*)$ for each $\theta \in S,$ there is $\xi_\theta > 0,$ such that:

$$E_{\pi_\infty, \theta_0}\Big[\varphi(\{p_{t+1}, p_t\}, \theta, \xi_\theta)\Big] \ < \ E_{\pi_\infty, \theta_0}\Big[U_\theta(\{p_{t+1}, p_t\})\Big] \ + \varepsilon.$$

The balls $\{B(\theta, \xi_\theta) \ : \ \theta \in S\}$ cover the compact $S,$ and therefore there is a finite subcover, that is : $S \subseteq \bigcup_{j=1}^{m} B(\theta_j, \xi_{\theta_j}).$ For each $T \in \mathbb{N},$ by definition of $\varphi,$

$$\sup_{\theta \in S} \frac{1}{T} \sum_{t=1}^{T} U_\theta(\{p_{t+1}, p_t\}) \ \leq \ \sup_{1 \leq j \leq m} \frac{1}{T} \sum_{t=1}^{T} \varphi(\{p_{t+1}, p_t\}, \theta_j, \xi_{\theta_j})$$

Furthermore, by the strong law of large numbers, for each $j \in \{1, \cdots, m\},$ with probability one:

$$\text{lim}_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \varphi(\{p_{t+1}, p_t\}, \theta_j, \xi_{\theta_j}) \ = \ E_{\pi_\infty, \theta_0}\Big[\varphi(\{p_{t+1}, p_t\}, \theta_j, \xi_{\theta_j})\Big]$$

Hence, with probability one:

$$\text{lim}_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\varphi(\{p_{t+1},p_t\},\theta_j,\xi_{\theta_j}) \;\leq\; E_{\pi_\infty,\theta_0}\Big[U_{\theta_j}(\{p_{t+1},p_t\})\Big]+\varepsilon, \quad \text{for } j=1,\cdots,m.$$

Therefore, with probability one:

$$\text{limsup}_{T\to\infty}\left\{\sup_{1\leq j\leq m}\frac{1}{T}\sum_{t=1}^{T}\varphi(\{p_{t+1},p_t\},\theta_j,\xi_{\theta_j})\right\} \;\leq\; \sup_{1\leq j\leq m} E_{\pi_\infty,\theta_0}\Big[U_{\theta_j}(\{p_{t+1},p_t\})\Big]+\varepsilon$$

Noting that $\varepsilon>0$ is arbitrary, we conclude, with probability one,

$$\text{limsup}_{T\to\infty}\left\{\sup_{\theta\in S}\frac{1}{T}\sum_{t=1}^{T}U_\theta(\{p_{t+1},p_t\})\right\} \;\leq\; \sup_{\theta\in S}\; E_{\pi_\infty,\theta_0}\left\{U_\theta(\{p_{t+1},p_t\})\right\},$$

showing the Claim in this way.

Now, the identifiability of θ_0 implies that for each $\theta\in S$:

$$E_{\pi_\infty,\theta_0}\left\{U_\theta(\{p_{t+1},p_t\})\right\} \;=E_{\pi_\infty,\theta_0}\left\{\log\left[\frac{l_\theta(p_{t+1}|p_t)}{l_{\theta_0}(p_{t+1}|p_t)}\right]\right\}<\log E_{\pi_\infty,\theta_0}\left[\frac{l_\theta(p_{t+1}|p_t)}{l_{\theta_0}(p_{t+1}|p_t)}\right],$$

where in the last strict inequality we are using the strict concavity of $\log$ and Claim 7 (identifiability). Furthermore, the last expectation satisfies:

$$E_{\pi_\infty,\theta_0}\left[\frac{l_\theta(p_{t+1}|p_t)}{l_{\theta_0}(p_{t+1}|p_t)}\right]=\int\frac{l_\theta(p_{t+1}|p_t)}{l_{\theta_0}(p_{t+1}|p_t)}l_{\theta_0}(p_{t+1}|p_t)\,d\pi_{\infty,\theta_0}(w_{t+1},p_t)\leq 1,$$

concluding that for each $\theta\in S$: $\quad E_{\pi_\infty,\theta_0}\left\{U_\theta(\{p_{t+1},p_t\})\right\}<0.$

By the continuity in θ of $E_{\pi_\infty,\theta_0}\left\{U_\theta(\{p_{t+1},p_t\})\right\}$, and the compactness of S, we have:

$$\sup_{\theta\in S}\; E_{\pi_\infty,\theta_0}\left\{U_\theta(\{p_{t+1},p_t\})\right\} \;<\; 0.$$

Therefore, with probability one,

$$\text{limsup}_{T\to\infty}\left\{\sup_{\theta\in S}\frac{1}{T}\sum_{t=1}^{T}U_\theta(\{p_{t+1},p_t\})\right\} \ < \ 0,$$

and then, with probability one, there exists $T_1 \in \mathbb{N},$ such that:

$$\sup_{\theta\in S}\frac{1}{T}\sum_{t=1}^{T}U_\theta(\{p_{t+1},p_t\}) \ < \ 0, \quad \forall\, T \geq T_1.$$

Finally, since :

$$\hat{\theta}_T \equiv \text{argmax}_{\theta\in\Theta}\frac{1}{T}\sum_{t=1}^{T}U_\theta(\{p_{t+1},p_t\}) \geq \frac{1}{T}\sum_{t=1}^{T}U_{\theta_0}(\{p_{t+1},p_t\}) = 0,$$

I conclude that $\hat{\theta}_T$ does not belong to $S,$ concluding that:

$$||\hat{\theta}_T - \theta_0|| < \delta, \quad \forall\, T \geq T_1. \quad Q.E.D.$$

2.6 Conclusions

In this chapter I conduct Monte Carlo experiments with different parameterizations to compare finite sample performance of the Simulation and Likelihood estimators. The results suggest that for parameterizations that imply low average storage and frequent stockouts, the PML estimator for small sample presents low bias and is more efficient than Simulations estimators. However, for parameterizations that imply a more significant role of storage, the Simulations estimators present bias that decrease with sample size increase, while the PML estimator biases do not disappear but instead tend to stabilize. I prove theoretically and numerically that Maximum Likelihood estimator is consistent and achieves better finite sample performance than the others.

Bibliography

[1] Abbott M, Cohen B, Wang WC (2012): “The performance of the urban water and wastewater sectors in Australia,” Utilities Policy, 20, 52-63.

[2] Ananda J (2014): “Evaluating the performance of urban water utilities: robust nonparametric approach,” Journal of Water Resources Planning and Management, 140, 04014021.

[3] Badin L, Daraio C, Simar L (2014): “Explaining inefficiency in nonparametric production models: the state of the art,” Annals of Operations Research, 214, 5-30.

[4] Benito-López B, Moreno-Enguix MDR, Solana-Ibañez J (2011): “Determinants of efficiency in the provision of municipal street-cleaning and refuse collection services,” Waste Management, 31, 1099-1108.

[5] Berg S, Marques R (2011): “Quantitative studies of water and sanitation utilities: a benchmarking literature survey,” Water Policy, 13, 591-606.

[6] Bertsekas, D. P. (1975): "Nondifferentiable optimization via approximation," Mathematical Programming Study, 3, 1-25.

[7] Boamah KB, Du J, Bediako IA, Boamah AJ, Abdul-Rasheed AA, Owusu SM (2017): "Carbon dioxide emission and economic growth of China-the role of international trade," Environmental Science and Pollution Research, 24, 13049-13067.

[8] Bobenrieth, E.S.A., Bobenrieth, J.R.A., and B.D. Wright (2012): "Strict Concavity of the Value Function for a Family of Dynamic Accumulation Models," The B.E. Journal of Theoretical Economics, 12, 1-9.

[9] Breiman, L. (1960): "The Strong Law of Large Numbers for a class of Markov Chains," The Annals of Mathematical Statistics, 31, 801-803.

[10] Byrnes J, Crase L, Dollery B, Villano R (2010): "The relative economic efficiency of urban water utilities in regional new South Wales and Victoria," Resource and Energy Economics, 32, 439-455.

[11] Cafiero, C., Bobenrieth, E.S.A., Bobenrieth, J.R.A., and B. D. Wright (2015): "Maximum Likelihood Estimation of the Standard Commodity Storage Model. Evidence from Sugar Prices," American Journal of Agricultural Economics, 97 (1), 122-136.

[12] Carroll, Christopher, (2006): "The method of endogenous gridpoints for solving dynamic stochastic optimization problems", Economics Letters, 91, 312-320.

[13] Carvalho P, Marques RC (2011): “The influence of the operational environment on the efficiency of water utilities,” Journal of Environmental Management, 92, 2698-2707.

[14] Carvalho P, Marques RC (2014): “Computing economies of vertical integration, economies of scope and economies of scale using partial frontier nonparametric methods,” European Journal of Operational Research, 234, 292-307.

[15] Cazals C, Florens J-P, Simar L (2002): “Nonparametric frontier estimation: a robust approach,” Journal of Econometrics, 106, 1-25.

[16] Charnes A, Cooper WW, Rhodes E (1978): “Measuring the efficiency of decision making units,” European Journal of Operational Research, 2, 429-444.

[17] Cherchi C, Badruzzaman M, Oppenheimer J, Bros CM, Jacangelo JG (2015): “Energy and water quality management systems for water utility’s operations: a review,” Journal of Environmental Management, 153, 108-120.

[18] Cooper WW, Seiford LM, Zhu J (2007): “Handbook on data envelopment analysis”. Springer

[19] Corton ML, Berg SV (2009): “Benchmarking central American water utilities,” Utilities Policy, 17, 267-275.

[20] Da Cruz NF, Marques RC (2014): “Revisiting the determinants of local government performance,” Omega (United Kingdom), 44, 91-103.

[21] Daraio C, Simar L (2006): “Advanced robust and nonparametric methods in efficiency analysis: methodology and applications”. Springer, New York

[22] Deaton, A., and G. Laroque (1992): “On the Behaviour of Commodity Prices.” Review of Economic Studies, 59, 1–23.

[23] Deaton, A., and G. Laroque (1995): “Estimating a nonlinear rational expectations commodity price model with unobservable state variables,” Journal of Applied Econometrics, 10, S9–S40.

[24] Deaton, A., and G. Laroque (1996): “Competitive Storage and Commodity Price Dynamics,” Journal of Political Economy, 104(5), 896–923.

[25] De Witte K, Marques RC (2010a): “Influential observations in frontier models, a robust non-oriented approach to the water sector,” Annals of Operations Research, 181, 377-392.

[26] De Witte K, Marques RC (2010b): “Designing performance incentives, an international benchmark study in the water sector,” CEJOR, 18, 189-220.

[27] De Witte K, Marques RC (2010c): “Incorporating heterogeneity in non parametric models: a methodological comparison,” International Journal of Operational Research, 9, 188-204.

Donoso, G. (2017): “Urban water pricing in Chile: Cost recovery, affordability, and water conservation”. Wiley Interdisciplinary Reviews: Water, 4(2), 1â10.

[28] Duffie, D., Singleton, K., 1993: "Simulated moments estimation of Markov models of asset prices," Econometrica, 61 (4), 929-952.

[29] Gallant, R., Tauchen, G., 1996: "Which moments to match," Econometric Theory, 61 (4), 12, 657-681.

[30] Gourieroux, C., Monfort, A., Renault, E., 1993: "Indirect inference," Journal of Applied Econometrics, 8, S85-S118

[31] Gouel, C. (2013): "Comparing Numerical Methods for Solving the Competitive Storage Model," Computational Economics, 41(2), 267-295.

[32] Gouel, C., and N. Legrand (2017): "Estimating the Competitive Storage Model with Trending Commodity Prices," Journal of Applied Econometrics, 32(4), 744-763.

[33] Guerrini A, Romano G, Campedelli B (2011): "Factors affecting the performance of water utility companies," International Journal of Public Sector Management, 24, 543-566.

[34] Guerrini A, Romano G, Campedelli B (2013): "Economies of scale, scope, and density in the Italian water sector: a two-stage data envelopment analysis approach," Water Resources Management, 27, 4559-4578.

[35] Guerrini A, Romano G, Leardini C, Martini M (2015): "The effects of operational and environmental variables on efficiency of Danish water and wastewater utilities," Water (Switzerland), 7, 3263-3282.

[36] Gustafson, R. L. (1958): Carryover Levels for Grains. Washington D.C.: USDA, Technical bulletin.

[37] Hastak S, Labhasetwar P, Kundley P, Gupta R (2016): “Changing from intermittent to continuous water supply and its influence on service level benchmarks: a case study in the demonstration zone of Nagpur. India, Urban Water Journal,. In press

[38] Hwang S-N, Lee H-S, Zhu J (2016): “Handbook of operations analytics using data envelopment analysis”. Springer

[39] HernÃ¡ndez-Sancho F, Molinos-Senante M, Sala-Garrido R, Del SazSalazar S (2012): “Tariffs and efficient performance by water suppliers: an empirical approach,” Water Policy, 14, 854-864.

[40] Marques, R.C. (2011): “Regulation of water and wastewater services”. London: IWA Publishing.

[41] Marques RC, Berg S, Yane S (2014): “Nonparametric benchmarking of Japanese water utilities: institutional and environmental factors affecting efficiency,” Journal of Water Resources Planning and Management, 140, 562-571.

[42] Maziotis A, Saal DS, Thanassoulis E, Molinos-Senante M (2015): “Profit, productivity and price performance changes in the water and sewerage industry: an empirical application for England and Wales,” Clean Technologies and Environmental Policy, 17, 1005-1018.

[43] Michaelides, A., and S. Ng (2000): “Estimating the rational expectations model of speculative storage: A Monte Carlo comparison of three simulation estimators,” Journal of Econometrics, 96, 231-266.

[44] Molinos-Senante M, Mocholi-Arce M, Sala-Garrido R (2016): “Efficiency assessment of water and sewerage companies: a disaggregated approach accounting for service quality,” Water Resources Management, 30, 4311-4328.

[45] Molinos-Senante M, Sala-Garrido R (2017): “Decomposition of productivity growth of water and sewerage companies: an empirical approach for Chile,” Water Resources Management, 31, 4309-321.

[46] Molinos-Senante M, Sala-Garrido R (2016): “Performance of fully private and concessionary water and sewerage companies: a metafrontier approach,” Environmental Science and Pollution Research, 23, 11620-11629.

[47] Molinos-Senante M, Sala-Garrido R, Lafuente M (2015): “The role of environmental variables on the efficiency of water and sewerage companies: a case study of Chile,” Environmental Science and Pollution Research, 22, 10242-10253.

[48] Molinos-Senante, M., Donoso, G. (2016): “Water scarcity and affordability in urban water pricing: A case study of Chile”. Utilities Policy, 43, 106â117.

[49] Meyn, S. P., and R.L. Tweedie (1993): Markov Chains and Stochastic Stability. Springer. London.

[50] Romano G, Guerrini A (2011): "Measuring and comparing the efficiency of water utility companies: a data envelopment analysis approach," Utilities Policy, 19, 202-209.

[51] Saal DS, Parker D, Weyman-Jones T (2007): "Determining the contribution of technical change, efficiency change and scale change to productivity growth in the privatized English and welsh water and sewerage industry: 1985-2000," Journal of Productivity Analysis, 28, 127-139.

[52] Santos P, Amado C, Coelho ST, Leitão JP (2017): "Stochastic data mining tools for pipe blockage failure prediction" Urban Water Journal, 14, 343-353.

[53] Scheinkman, J.A., and V. L. Escudero (1977): "Some Results on "An Income Fluctuation Problem"," Journal of Economic Theory, 16, 151-166.

[54] Scheinkman, J. A., and J. Schechtman (1983): "A simple competitive model with production and storage," Review of Economic Studies, 50, 427?441

See KF (2015): "Exploring and analysing sources of technical efficiency in water supply services: some evidence from southeast Asian public water utilities," Water Resources and Economics, 9, 23-44.

[55] Simar L, Wilson PW (2007): "Estimation and inference in two-stage, semiparametric models of production processes". Journal of Econometrics, 136, 31-64.

[56] SISS (2014): "Report about water and sewerage services in Chile 2014".

[57] SISS. (2015b): “VI Proceso de FijaciÃ³nTarifariaPerÃ-odo2016â2021EmpresaEssbioS.A.Div.ConcesionesAreaTarifas”.MinisteriodeObrasPÃºblicas.Sant

[58] Wald, A. (1949): “Note on The Consistency of the Maximum Likelihood Estimate,” The Annals of Mathematical Statistics, 20, 595-601.

[59] Williams, J. C., and B. D. Wright (1991): Storage and Commodity Markets. Cambridge University Press.

[60] Worthington AC (2014): “A review of frontier approaches to efficiency and productivity measurement in urban water utilities,” Urban Water Journal, 11, 55-73.

[61] Zhang S, Lundgren T, Zhou W (2016): “Energy efficiency in Swedish industry: a firm-level data envelopment analysis,” Energy Economics, 55, 42-51.

Appendix A

Results Monte Carlo Experiments

Table A.1: Results for SMM ($a = 0.6$ $b = -0.3$ $d = 0.10$)

Model	T	$N = TH$	Statistics	a	b	d
m1	100	500	Mean	0.5464	-0.2820	0.1106
			Median	0.5570	-0.2754	0.0921
			Std	0.0522	0.0637	0.0732
			RMSE	0.0748	0.0661	0.0739
m1	100	1000	Mean	0.5490	-0.2792	0.1101
			Median	0.5539	-0.2745	0.0889
			Std	0.0495	0.0610	0.0703
			RMSE	0.0711	0.0644	0.0709
m1	100	2500	Mean	0.5508	-0.2772	0.1110
			Median	0.5564	-0.2730	0.0951
			Std	0.0487	0.0571	0.0658
			RMSE	0.0692	0.0614	0.0666
m1	200	500	Mean	0.5683	-0.2906	0.1069
			Median	0.5728	-0.2856	0.0989
			Std	0.0363	0.0528	0.0552
			RMSE	0.0482	0.0536	0.0556
m1	200	1000	Mean	0.5726	-0.2881	0.1067
			Median	0.5770	-0.2828	0.0992
			Std	0.0329	0.0485	0.0505
			RMSE	0.0427	0.0499	0.0509
m1	200	2500	Mean	0.5753	-0.2862	0.1083
			Median	0.5793	-0.2811	0.1032
			Std	0.0317	0.0449	0.0474
			RMSE	0.0401	0.0470	0.0481
m1	500	1000	Mean	0.5866	-0.2961	0.1023
			Median	0.5884	-0.2927	0.0990
			Std	0.0208	0.0340	0.0348
			RMSE	0.0247	0.0342	0.0348
m1	1000	2000	Mean	0.5938	-0.2977	0.1014
			Median	0.5950	-0.2962	0.1008
			Std	0.0127	0.0223	0.0233
			RMSE	0.0141	0.0224	0.0233
m1	5000	10000	Mean	0.5986	-0.2995	0.1002
			Median	0.5986	-0.2990	0.0994
			Std	0.0049	0.0107	0.0113
			RMSE	0.0051	0.0107	0.0113
m1	10000	20000	Mean	0.5993	-0.3001	0.1000
			Median	0.5993	-0.3003	0.1001
			Std	0.0034	0.0072	0.0077
			RMSE	0.0034	0.0072	0.0077

Table A.2: Continued ($a = 0.6$ $b = -0.3$ $d = 0.10$)

Model	T	$N = TH$	Statistics	a	b	d
m2	100	500	Mean	0.5693	-0.2713	0.1234
			Median	0.5666	-0.2694	0.1185
			Std	0.0564	0.0541	0.0477
			RMSE	0.0641	0.0611	0.0531
m2	100	1000	Mean	0.5678	-0.2699	0.1221
			Median	0.5664	-0.2668	0.1180
			Std	0.0545	0.0540	0.0455
			RMSE	0.0632	0.0617	0.0506
m2	100	2500	Mean	0.5691	-0.2684	0.1231
			Median	0.5658	-0.2648	0.1182
			Std	0.0520	0.0512	0.0451
			RMSE	0.0605	0.0601	0.0506
m2	200	500	Mean	0.5810	-0.2843	0.1128
			Median	0.5839	-0.2806	0.1091
			Std	0.0468	0.0438	0.0356
			RMSE	0.0504	0.0465	0.0378
m2	200	1000	Mean	0.5807	-0.2821	0.1116
			Median	0.5808	-0.2781	0.1091
			Std	0.0433	0.0399	0.0328
			RMSE	0.0473	0.0437	0.0347
m2	200	2500	Mean	0.5824	-0.2815	0.1115
			Median	0.5799	-0.2786	0.1101
			Std	0.0401	0.0368	0.0300
			RMSE	0.0438	0.0411	0.0320
m2	500	1000	Mean	0.5905	-0.2925	0.1050
			Median	0.5911	-0.2897	0.1044
			Std	0.0318	0.0289	0.0229
			RMSE	0.0332	0.0298	0.0234
m2	1000	2000	Mean	0.5956	-0.2957	0.1027
			Median	0.5974	-0.2955	0.1024
			Std	0.0209	0.0197	0.0152
			RMSE	0.0214	0.0202	0.0154
m2	5000	10000	Mean	0.5985	-0.2986	0.1009
			Median	0.5990	-0.2985	0.1010
			Std	0.0092	0.0090	0.0070
			RMSE	0.0093	0.0091	0.0070
m2	10000	20000	Mean	0.5992	-0.2996	0.1004
			Median	0.5997	-0.2996	0.1003
			Std	0.0066	0.0062	0.0049
			RMSE	0.0067	0.0062	0.0049

Table A.3: Results for IND ($a = 0.6$ $b = -0.3$ $d = 0.10$)

Model	T	$N = TH$	Statistics	a	b	d
M1	100	500	Mean	0.5972	-0.2880	0.1240
			Median	0.5976	-0.2788	0.1070
			Std	0.0329	0.0646	0.0785
			RMSE	0.0330	0.0657	0.0820
M1	100	1000	Mean	0.5969	-0.2863	0.1186
			Median	0.5962	-0.2783	0.1011
			Std	0.0312	0.0614	0.0702
			RMSE	0.0313	0.0629	0.0726
M1	100	2500	Mean	0.5974	-0.2838	0.1206
			Median	0.5983	-0.2786	0.1044
			Std	0.0305	0.0586	0.0693
			RMSE	0.0306	0.0607	0.0723
M1	200	500	Mean	0.5991	-0.2961	0.1157
			Median	0.5985	-0.2902	0.1070
			Std	0.0228	0.0534	0.0608
			RMSE	0.0228	0.0535	0.0627
M1	200	1000	Mean	0.5989	-0.2923	0.1130
			Median	0.5990	-0.2868	0.1063
			Std	0.0208	0.0485	0.0525
			RMSE	0.0208	0.0491	0.0540
M1	200	2500	Mean	0.5995	-0.2902	0.1132
			Median	0.5990	-0.2840	0.1080
			Std	0.0200	0.0455	0.0481
			RMSE	0.0200	0.0465	0.0499
M1	500	1000	Mean	0.5996	-0.2983	0.1059
			Median	0.5997	-0.2950	0.1014
			Std	0.0152	0.0342	0.0352
			RMSE	0.0152	0.0342	0.0356
M1	1000	2000	Mean	0.6003	-0.2988	0.1034
			Median	0.6004	-0.2968	0.1019
			Std	0.0100	0.0224	0.0235
			RMSE	0.0100	0.0224	0.0237
M1	5000	10000	Mean	0.6000	-0.2998	0.1006
			Median	0.6002	-0.2993	0.0997
			Std	0.0046	0.0107	0.0113
			RMSE	0.0046	0.0107	0.0113
M1	10000	20000	Mean	0.5999	-0.3003	0.1001
			Median	0.5998	-0.3004	0.1001
			Std	0.0033	0.0072	0.0077
			RMSE	0.0033	0.0072	0.0077

Table A.4: Continued ($a = 0.6$ $b = -0.3$ $d = 0.10$)

Model	T	$N = TH$	Statistics	a	b	d
M2	100	500	Mean	0.5820	-0.3067	0.0975
			Median	0.5893	-0.2934	0.0979
			Std	0.0583	0.0876	0.0435
			RMSE	0.0610	0.0877	0.0436
M2	100	1000	Mean	0.5785	-0.3073	0.0976
			Median	0.5925	-0.2900	0.0948
			Std	0.0818	0.1319	0.0450
			RMSE	0.0845	0.1320	0.0450
M2	100	2500	Mean	0.5807	-0.2977	0.0986
			Median	0.5912	-0.2851	0.0971
			Std	0.0609	0.0871	0.0432
			RMSE	0.0638	0.0870	0.0432
M2	200	500	Mean	0.5888	-0.3114	0.0963
			Median	0.5958	-0.2969	0.0944
			Std	0.0562	0.0764	0.0333
			RMSE	0.0572	0.0771	0.0334
M2	200	1000	Mean	0.5912	-0.3030	0.0966
			Median	0.5965	-0.2949	0.0954
			Std	0.0395	0.0609	0.0316
			RMSE	0.0404	0.0609	0.0317
M2	200	2500	Mean	0.5922	-0.2982	0.0983
			Median	0.5968	-0.2928	0.0978
			Std	0.0368	0.0525	0.0319
			RMSE	0.0376	0.0525	0.0319
M2	500	1000	Mean	0.5960	-0.3058	0.0973
			Median	0.5975	-0.3023	0.0967
			Std	0.0226	0.0371	0.0209
			RMSE	0.0229	0.0375	0.0210
M2	1000	2000	Mean	0.5986	-0.3034	0.0990
			Median	0.5992	-0.3005	0.0986
			Std	0.0132	0.0226	0.0134
			RMSE	0.0133	0.0228	0.0134
M2	5000	10000	Mean	0.6000	-0.3000	0.1002
			Median	0.5998	-0.2994	0.1003
			Std	0.0053	0.0081	0.0059
			RMSE	0.0053	0.0081	0.0059
M2	10000	20000	Mean	0.5999	-0.3003	0.0998
			Median	0.5998	-0.3006	0.1000
			Std	0.0038	0.0052	0.0043
			RMSE	0.0038	0.0052	0.0043

Table A.5: Results for GT ($a = 0.6$ $b = -0.3$ $d = 0.10$)

Model	T	$N = TH$	Statistics	a	b	d
M1	100	500	Mean	0.5934	-0.2778	0.1128
			Median	0.5925	-0.2729	0.0925
			Std	0.0331	0.0659	0.0798
			RMSE	0.0337	0.0695	0.0807
M1	100	1000	Mean	0.5936	-0.2761	0.1102
			Median	0.5939	-0.2715	0.0878
			Std	0.0315	0.0637	0.0739
			RMSE	0.0321	0.0679	0.0745
M1	100	2500	Mean	0.5946	-0.2759	0.1092
			Median	0.5951	-0.2717	0.0927
			Std	0.0306	0.0593	0.0708
			RMSE	0.0310	0.0640	0.0713
M1	200	500	Mean	0.5960	-0.2900	0.1063
			Median	0.5959	-0.2844	0.0969
			Std	0.0231	0.0527	0.0579
			RMSE	0.0235	0.0536	0.0582
M1	200	1000	Mean	0.5964	-0.2888	0.1037
			Median	0.5959	-0.2837	0.0994
			Std	0.0212	0.0490	0.0495
			RMSE	0.0214	0.0502	0.0496
M1	200	2500	Mean	0.5974	-0.2870	0.1048
			Median	0.5971	-0.2819	0.0987
			Std	0.0203	0.0451	0.0469
			RMSE	0.0205	0.0469	0.0471
M1	500	1000	Mean	0.5980	-0.2964	0.1012
			Median	0.5983	-0.2942	0.0983
			Std	0.0154	0.0350	0.0351
			RMSE	0.0155	0.0351	0.0351
M1	1000	2000	Mean	0.5994	-0.2980	0.1007
			Median	0.5996	-0.2962	0.0996
			Std	0.0101	0.0225	0.0233
			RMSE	0.0101	0.0225	0.0233
M1	5000	10000	Mean	0.5998	-0.2996	0.1001
			Median	0.5999	-0.2988	0.0991
			Std	0.0046	0.0107	0.0113
			RMSE	0.0046	0.0107	0.0113
M1	10000	20000	Mean	0.5999	-0.3002	0.0999
			Median	0.5998	-0.3003	0.0999
			Std	0.0033	0.0072	0.0077
			RMSE	0.0033	0.0072	0.0077

Table A.6: Continued ($a = 0.6$ $b = -0.3$ $d = 0.10$)

Model	T	$N = TH$	Statistics	a	b	d
M2	100	500	Mean	0.5927	-0.2526	0.1065
			Median	0.5940	-0.2477	0.1006
			Std	0.0417	0.0484	0.0428
			RMSE	0.0423	0.0677	0.0433
M2	100	1000	Mean	0.5909	-0.2493	0.1071
			Median	0.5928	-0.2466	0.1020
			Std	0.0428	0.0453	0.0464
			RMSE	0.0437	0.0679	0.0469
M2	100	2500	Mean	0.5911	-0.2487	0.1067
			Median	0.5937	-0.2456	0.1016
			Std	0.0414	0.0450	0.0445
			RMSE	0.0423	0.0682	0.0450
M2	200	500	Mean	0.5970	-0.2711	0.1038
			Median	0.5969	-0.2691	0.1015
			Std	0.0303	0.0388	0.0287
			RMSE	0.0304	0.0484	0.0289
M2	200	1000	Mean	0.5976	-0.2685	0.1037
			Median	0.5964	-0.2665	0.1009
			Std	0.0256	0.0351	0.0273
			RMSE	0.0256	0.0472	0.0275
M2	200	2500	Mean	0.5981	-0.2679	0.1044
			Median	0.5982	-0.2651	0.1020
			Std	0.0265	0.0333	0.0265
			RMSE	0.0266	0.0463	0.0268
M2	500	1000	Mean	0.5982	-0.2830	0.1016
			Median	0.5984	-0.2822	0.1010
			Std	0.0181	0.0274	0.0184
			RMSE	0.0182	0.0323	0.0184
M2	1000	2000	Mean	0.5995	-0.2900	0.1013
			Median	0.5997	-0.2894	0.1005
			Std	0.0116	0.0190	0.0121
			RMSE	0.0116	0.0214	0.0122
M2	5000	10000	Mean	0.5999	-0.2970	0.1005
			Median	0.5998	-0.2965	0.1008
			Std	0.0052	0.0084	0.0059
			RMSE	0.0052	0.0089	0.0059
M2	10000	20000	Mean	0.5999	-0.2988	0.1000
			Median	0.5998	-0.2992	0.1001
			Std	0.0038	0.0056	0.0042
			RMSE	0.0038	0.0058	0.0042

Table A.7: Continued ($a = 0.6$ $b = -0.3$ $d = 0.10$)

Model	T	$N = TH$	Statistics	a	b	d
M3	100	500	Mean	0.6016	-0.2723	0.1167
			Median	0.6027	-0.2631	0.1148
			Std	0.0333	0.0657	0.0503
			RMSE	0.0333	0.0712	0.0529
M3	100	1000	Mean	0.6021	-0.2710	0.1167
			Median	0.6010	-0.2629	0.1126
			Std	0.0314	0.0604	0.0490
			RMSE	0.0314	0.0669	0.0517
M3	100	2500	Mean	0.6027	-0.2698	0.1171
			Median	0.6020	-0.2623	0.1149
			Std	0.0307	0.0586	0.0462
			RMSE	0.0308	0.0659	0.0492
M3	200	500	Mean	0.6017	-0.2801	0.1123
			Median	0.6018	-0.2751	0.1111
			Std	0.0234	0.0469	0.0336
			RMSE	0.0235	0.0509	0.0358
M3	200	1000	Mean	0.6024	-0.2800	0.1119
			Median	0.6024	-0.2741	0.1107
			Std	0.0211	0.0440	0.0321
			RMSE	0.0212	0.0483	0.0342
M3	200	2500	Mean	0.6033	-0.2797	0.1125
			Median	0.6035	-0.2756	0.1124
			Std	0.0201	0.0407	0.0293
			RMSE	0.0203	0.0454	0.0318
M3	500	1000	Mean	0.6027	-0.2887	0.1090
			Median	0.6020	-0.2876	0.1089
			Std	0.0151	0.0282	0.0211
			RMSE	0.0153	0.0303	0.0229
M3	1000	2000	Mean	0.6035	-0.2916	0.1084
			Median	0.6034	-0.2915	0.1086
			Std	0.0099	0.0179	0.0138
			RMSE	0.0105	0.0198	0.0161
M3	5000	10000	Mean	0.6026	-0.2957	0.1050
			Median	0.6026	-0.2957	0.1049
			Std	0.0045	0.0078	0.0063
			RMSE	0.0052	0.0089	0.0081
M3	10000	20000	Mean	0.6022	-0.2970	0.1038
			Median	0.6023	-0.2973	0.1038
			Std	0.0033	0.0053	0.0046
			RMSE	0.0039	0.0061	0.0059

Table A.8: Continued ($a = 0.6$ $b = -0.3$ $d = 0.10$)

Model	T	$N = TH$	Statistics	a	b	d
M4	100	500	Mean	0.6090	-0.2816	0.0971
			Median	0.6086	-0.2752	0.0950
			Std	0.0389	0.0638	0.0387
			RMSE	0.0399	0.0664	0.0388
M4	100	1000	Mean	0.6083	-0.2869	0.0907
			Median	0.6081	-0.2824	0.0906
			Std	0.0376	0.0649	0.0370
			RMSE	0.0385	0.0661	0.0381
M4	100	2500	Mean	0.6067	-0.2804	0.0910
			Median	0.6060	-0.2779	0.0895
			Std	0.0373	0.0620	0.0366
			RMSE	0.0379	0.0650	0.0377
M4	200	500	Mean	0.6037	-0.2957	0.0946
			Median	0.6024	-0.2922	0.0939
			Std	0.0266	0.0497	0.0289
			RMSE	0.0268	0.0499	0.0294
M4	200	1000	Mean	0.6037	-0.2943	0.0929
			Median	0.6022	-0.2926	0.0921
			Std	0.0235	0.0456	0.0276
			RMSE	0.0238	0.0459	0.0285
M4	200	2500	Mean	0.6039	-0.2938	0.0930
			Median	0.6053	-0.2921	0.0922
			Std	0.0233	0.0454	0.0267
			RMSE	0.0236	0.0458	0.0276
M4	500	1000	Mean	0.6004	-0.2980	0.0970
			Median	0.6004	-0.2949	0.0975
			Std	0.0171	0.0321	0.0202
			RMSE	0.0171	0.0321	0.0204
M4	1000	2000	Mean	0.6009	-0.2990	0.0983
			Median	0.6013	-0.2984	0.0983
			Std	0.0110	0.0205	0.0132
			RMSE	0.0110	0.0206	0.0133
M4	5000	10000	Mean	0.6000	-0.2993	0.0998
			Median	0.6001	-0.2992	0.1000
			Std	0.0049	0.0088	0.0061
			RMSE	0.0049	0.0088	0.0061
M4	10000	20000	Mean	0.6000	-0.3000	0.0998
			Median	0.6000	-0.2997	0.0997
			Std	0.0035	0.0060	0.0043
			RMSE	0.0035	0.0059	0.0043

Table A.9: Pseudo and Maximum Likelihood Estimators ($a = 0.6$ $b = -0.3$ $d = 0.10$)

Size	Statistics	PML			CML			UML		
		a	b	d	a	b	d	a	b	d
100	Mean	0.6005	-0.3050	0.1080	0.6023	-0.2956	0.1031	0.6034	-0.2984	0.1012
	Median	0.5997	-0.2980	0.1040	0.6000	-0.2963	0.1021	0.6013	-0.2997	0.1006
	Std	0.0291	0.0506	0.0360	0.0163	0.0205	0.0106	0.0147	0.0186	0.0097
	Bias	0.0009	0.0167	0.0790	0.0038	-0.0146	0.0306	0.0056	-0.0053	0.0117
	RMSE	0.0291	0.0508	0.0370	0.0164	0.0209	0.0111	0.0151	0.0187	0.0098
200	Mean	0.5984	-0.3060	0.1050	0.6014	-0.2977	0.1019	0.6018	-0.2995	0.1009
	Median	0.5978	-0.3010	0.1030	0.6008	-0.2985	0.1014	0.6007	-0.3000	0.1007
	Std	0.0203	0.0354	0.0240	0.0099	0.0139	0.0071	0.0093	0.0129	0.0067
	Bias	-0.0030	0.0185	0.0520	0.0023	-0.0078	0.0191	0.0030	-0.0015	0.0095
	RMSE	0.0204	0.0358	0.0250	0.0100	0.0140	0.0073	0.0095	0.0128	0.0068
500	Mean	0.5963	-0.3080	0.1040	0.6012	-0.2991	0.1011	0.6013	-0.3000	0.1008
	Median	0.5954	-0.3060	0.1040	0.6010	-0.2992	0.1011	0.6007	-0.3000	0.1005
	Std	0.0135	0.0237	0.0160	0.0055	0.0102	0.0044	0.0052	0.0093	0.0042
	Bias	-0.0060	0.0251	0.0410	0.0020	-0.0029	0.0111	0.0022	-0.0001	0.0079
	RMSE	0.0140	0.0250	0.0160	0.0056	0.0102	0.0045	0.0054	0.0093	0.0043
1000	Mean	0.5958	-0.3080	0.1030	0.6009	-0.2996	0.1009	0.6010	-0.2997	0.1006
	Median	0.5958	-0.3080	0.1030	0.6010	-0.2999	0.1008	0.6008	-0.3000	0.1004
	Std	0.0092	0.0161	0.0110	0.0041	0.0063	0.0033	0.0033	0.0059	0.0028
	Bias	-0.0070	0.0275	0.0330	0.0016	-0.0012	0.0085	0.0017	-0.0009	0.0058
	RMSE	0.0101	0.0181	0.0110	0.0042	0.0063	0.0034	0.0035	0.0059	0.0029
5000	Mean	0.5948	-0.3090	0.1030	0.6005	-0.3000	0.1005	0.6005	-0.3001	0.1004
	Median	0.5949	-0.3080	0.1030	0.6005	-0.3001	0.1005	0.6004	-0.3001	0.1004
	Std	0.0041	0.0073	0.0050	0.0016	0.0025	0.0015	0.0016	0.0023	0.0015
	Bias	-0.0090	0.0283	0.0340	0.0009	0.0000	0.0050	0.0008	0.0003	0.0042
	RMSE	0.0067	0.0112	0.0060	0.0017	0.0025	0.0016	0.0017	0.0023	0.0016
10000	Mean	0.5948	-0.3090	0.1030	0.6002	-0.3000	0.1002	0.6003	-0.3000	0.1002
	Median	0.5949	-0.3090	0.1030	0.6002	-0.3000	0.1002	0.6001	-0.3000	0.1002
	Std	0.0030	0.0049	0.0040	0.0014	0.0018	0.0013	0.0014	0.0018	0.0012
	Bias	-0.0090	0.0300	0.0320	0.0004	0.0000	0.0023	0.0005	0.0000	0.0024
	RMSE	0.0060	0.0103	0.0050	0.0014	0.0018	0.0013	0.0014	0.0018	0.0012

Table A.10: Results for SMM ($a = 1$ $b = -1$)

Model	T	$N = TH$	Statistics	a	b
m1	100	500	Mean	0.8568	-0.7635
			Median	0.8750	-0.7529
			Std	0.1379	0.2673
			RMSE	0.1987	0.3566
m1	100	1000	Mean	0.8678	-0.7549
			Median	0.8829	-0.7386
			Std	0.1292	0.2512
			RMSE	0.1847	0.3508
m1	200	500	Mean	0.9113	-0.8578
			Median	0.9211	-0.8445
			Std	0.1068	0.2422
			RMSE	0.1388	0.2807
m1	200	1000	Mean	0.9213	-0.8486
			Median	0.9311	-0.8319
			Std	0.1002	0.2199
			RMSE	0.1274	0.2668
m1	500	1000	Mean	0.9566	-0.9253
			Median	0.9611	-0.9159
			Std	0.0668	0.1526
			RMSE	0.0796	0.1698
m1	1000	2000	Mean	0.9806	-0.9609
			Median	0.9826	-0.9511
			Std	0.0421	0.1072
			RMSE	0.0464	0.1141
m1	5000	10000	Mean	0.9954	-0.9915
			Median	0.9961	-0.9887
			Std	0.0180	0.0498
			RMSE	0.0186	0.0505
m1	10000	20000	Mean	0.9976	-0.9960
			Median	0.9978	-0.9963
			Std	0.0126	0.0343
			RMSE	0.0128	0.0345

Table A.11: Continued ($a = 1$ $b = -1$)

Model	T	$N = TH$	Statistics	a	b
m2	100	500	Mean	0.8661	-0.8199
			Median	0.8743	-0.8132
			Std	0.1502	0.2622
			RMSE	0.2011	0.3179
m2	100	1000	Mean	0.8700	-0.8155
			Median	0.8740	-0.8085
			Std	0.1551	0.2364
			RMSE	0.2022	0.2997
m2	200	500	Mean	0.9172	-0.8845
			Median	0.9228	-0.8854
			Std	0.1131	0.2165
			RMSE	0.1401	0.2452
m2	200	1000	Mean	0.9216	-0.8768
			Median	0.9310	-0.8755
			Std	0.1040	0.1896
			RMSE	0.1301	0.2260
m2	500	1000	Mean	0.9558	-0.9298
			Median	0.9608	-0.9325
			Std	0.0705	0.1454
			RMSE	0.0832	0.1614
m2	1000	2000	Mean	0.9797	-0.9584
			Median	0.9844	-0.9530
			Std	0.0453	0.0965
			RMSE	0.0496	0.1050
m2	5000	10000	Mean	0.9936	-0.9869
			Median	0.9941	-0.9848
			Std	0.0208	0.0498
			RMSE	0.0217	0.0515
m2	10000	20000	Mean	0.9968	-0.9944
			Median	0.9975	-0.9933
			Std	0.0142	0.0336
			RMSE	0.0145	0.0341

Table A.12: Results for IND ($a = 1$ $b = -1$)

Model	T	$N = TH$	Statistics	a	b
M1	100	500	Mean	0.9615	-0.7517
			Median	0.9583	-0.7154
			Std	0.1166	0.3699
			RMSE	0.1227	0.4452
M1	100	1000	Mean	0.9637	-0.7355
			Median	0.9560	-0.7049
			Std	0.1061	0.3598
			RMSE	0.1121	0.4463
M1	200	500	Mean	0.9805	-0.8726
			Median	0.9726	-0.8458
			Std	0.0859	0.3091
			RMSE	0.0880	0.3340
M1	200	1000	Mean	0.9811	-0.8646
			Median	0.9791	-0.8482
			Std	0.0791	0.2793
			RMSE	0.0812	0.3101
M1	500	1000	Mean	0.9940	-0.9555
			Median	0.9933	-0.9383
			Std	0.0591	0.1832
			RMSE	0.0594	0.1883
M1	1000	2000	Mean	0.9987	-0.9782
			Median	0.9992	-0.9673
			Std	0.0393	0.1187
			RMSE	0.0393	0.1206
M1	5000	10000	Mean	0.9994	-0.9941
			Median	0.9994	-0.9938
			Std	0.0187	0.0550
			RMSE	0.0187	0.0552
M1	10000	20000	Mean	0.9996	-0.9979
			Median	0.9995	-1.0000
			Std	0.0135	0.0379
			RMSE	0.0135	0.0380

Table A.13: Continued ($a = 1$ $b = -1$)

Model	T	$N = TH$	Statistics	a	b
M2	100	500	Mean	0.9270	-1.0173
			Median	0.9525	-0.9607
			Std	0.1939	0.3966
			RMSE	0.2070	0.3988
M2	100	1000	Mean	0.9043	-1.0182
			Median	0.9432	-0.9553
			Std	0.2059	0.4255
			RMSE	0.2269	0.4254
M2	200	500	Mean	0.9722	-1.0160
			Median	0.9851	-0.9750
			Std	0.1351	0.2866
			RMSE	0.1378	0.2868
M2	200	1000	Mean	0.9582	-0.9974
			Median	0.9725	-0.9462
			Std	0.1229	0.2810
			RMSE	0.1297	0.2807
M2	500	1000	Mean	0.9842	-1.0067
			Median	0.9896	-0.9908
			Std	0.0790	0.1952
			RMSE	0.0805	0.1952
M2	1000	2000	Mean	0.9885	-1.0112
			Median	0.9960	-0.9880
			Std	0.0609	0.1428
			RMSE	0.0619	0.1431
M2	5000	10000	Mean	0.9975	-1.0006
			Median	0.9976	-0.9971
			Std	0.016	0.0566
			RMSE	0.0161	0.0565
M2	10000	20000	Mean	0.9990	-0.9999
			Median	0.9991	-0.9983
			Std	0.0106	0.0372
			RMSE	0.0106	0.0371

Table A.14: Results for GT ($a = 1$ $b = -1$)

Model	T	$N = TH$	Statistics	a	b
M1	100	500	Mean	0.9525	-0.6487
			Median	0.9549	-0.6154
			Std	0.1142	0.3067
			RMSE	0.1236	0.4662
M1	100	1000	Mean	0.9547	-0.6494
			Median	0.9583	-0.6292
			Std	0.1111	0.3013
			RMSE	0.1199	0.4621
M1	200	500	Mean	0.9636	-0.7706
			Median	0.9594	-0.7465
			Std	0.0882	0.2902
			RMSE	0.0954	0.3697
M1	200	1000	Mean	0.9651	-0.7709
			Median	0.9664	-0.7516
			Std	0.0828	0.2685
			RMSE	0.0898	0.3527
M1	500	1000	Mean	0.9775	-0.8715
			Median	0.9773	-0.8731
			Std	0.0603	0.1899
			RMSE	0.0643	0.2291
M1	1000	2000	Mean	0.9890	-0.9303
			Median	0.9898	-0.9251
			Std	0.0412	0.1278
			RMSE	0.0426	0.1455
M1	5000	10000	Mean	0.9967	-0.9813
			Median	0.9971	-0.9821
			Std	0.0192	0.0571
			RMSE	0.0195	0.0601
M1	10000	20000	Mean	0.9982	-0.9911
			Median	0.9983	-0.9937
			Std	0.0136	0.0388
			RMSE	0.0138	0.0397

Table A.15: Continued ($a = 1$ $b = -1$)

Model	T	$N = TH$	Statistics	a	b
M2	100	500	Mean	0.9949	-0.6733
			Median	0.9870	-0.6792
			Std	0.1377	0.2707
			RMSE	0.1377	0.4241
M2	100	1000	Mean	0.9915	-0.6472
			Median	0.9864	-0.6474
			Std	0.1392	0.2557
			RMSE	0.1393	0.4356
M2	200	500	Mean	0.9982	-0.7891
			Median	0.9936	-0.8151
			Std	0.1097	0.2279
			RMSE	0.1096	0.3103
M2	200	1000	Mean	0.9927	-0.7712
			Median	0.9898	-0.7804
			Std	0.1095	0.2211
			RMSE	0.1097	0.3180
M2	500	1000	Mean	1.0041	-0.8662
			Median	0.9991	-0.8748
			Std	0.0714	0.1782
			RMSE	0.0714	0.2227
M2	1000	2000	Mean	0.9970	-0.9209
			Median	0.9945	-0.9260
			Std	0.0445	0.1260
			RMSE	0.0446	0.1486
M2	5000	10000	Mean	0.9968	-0.9789
			Median	0.9962	-0.9789
			Std	0.0166	0.0554
			RMSE	0.0168	0.0592
M2	10000	20000	Mean	0.9985	-0.9889
			Median	0.9988	-0.9886
			Std	0.0110	0.0378
			RMSE	0.0111	0.0394

Table A.16: Continued ($a = 1$ $b = -1$)

Model	T	$N = TH$	Statistics	a	b
M3	100	500	Mean	1.0344	-0.7950
			Median	1.0370	-0.7810
			Std	0.0990	0.3332
			RMSE	0.1047	0.3909
M3	100	1000	Mean	1.0348	-0.7804
			Median	1.0377	-0.7814
			Std	0.0963	0.3171
			RMSE	0.1023	0.3854
M3	200	500	Mean	1.0256	-0.8773
			Median	1.0219	-0.8676
			Std	0.0699	0.2692
			RMSE	0.0744	0.2956
M3	200	1000	Mean	1.0247	-0.8775
			Median	1.0231	-0.8658
			Std	0.0636	0.2624
			RMSE	0.0682	0.2894
M3	500	1000	Mean	1.0138	-0.9472
			Median	1.0098	-0.9392
			Std	0.0491	0.1945
			RMSE	0.0510	0.2013
M3	1000	2000	Mean	1.0088	-0.9790
			Median	1.0081	-0.9674
			Std	0.0352	0.1317
			RMSE	0.0362	0.1333
M3	5000	10000	Mean	1.0018	-0.9943
			Median	1.0011	-0.9908
			Std	0.0154	0.0572
			RMSE	0.0155	0.0574
M3	10000	20000	Mean	1.0008	-0.9961
			Median	1.0004	-0.9944
			Std	0.0112	0.0388
			RMSE	0.0113	0.0389

Table A.17: Continued ($a = 1$ $b = -1$)

Model	T	$N = TH$	Statistics	a	b
M4	100	500	Mean	1.0237	-0.7857
			Median	1.0269	-0.7887
			Std	0.1351	0.3100
			RMSE	0.1371	0.3766
M4	100	1000	Mean	1.0237	-0.7796
			Median	1.0326	-0.7638
			Std	0.1437	0.3142
			RMSE	0.1455	0.3835
M4	200	500	Mean	1.0122	-0.8298
			Median	1.0111	-0.8311
			Std	0.0829	0.2399
			RMSE	0.0837	0.2940
M4	200	1000	Mean	1.0140	-0.8283
			Median	1.0155	-0.8305
			Std	0.0785	0.2466
			RMSE	0.0796	0.3002
M4	500	1000	Mean	0.9996	-0.8793
			Median	0.9978	-0.8827
			Std	0.0562	0.2023
			RMSE	0.0562	0.2354
M4	1000	2000	Mean	0.9985	-0.9295
			Median	0.9975	-0.9410
			Std	0.0380	0.1487
			RMSE	0.0380	0.1644
M4	5000	10000	Mean	0.9990	-0.9787
			Median	1.0003	-0.9796
			Std	0.0177	0.0729
			RMSE	0.0177	0.0759
M4	10000	20000	Mean	0.9992	-0.9892
			Median	0.9994	-0.9892
			Std	0.0123	0.0482
			RMSE	0.0123	0.0493

Table A.18: Pseudo and Maximum Likelihood Estimators ($a = 1$ $b = -1$)

Size	Statistics	PML		CML		UML	
		a	b	a	b	a	b
100	Mean	0.9612	-1.0560	0.9954	-0.9986	1.0073	-0.9994
	Median	0.9610	-1.0039	0.9974	-0.9997	1.0010	-0.9999
	Std	0.1005	0.2995	0.0364	0.0068	0.0315	0.0055
	Bias	-0.0388	0.0560	-0.0046	-0.0014	0.0073	-0.0006
	RMSE	0.1076	0.3044	0.0366	0.0069	0.0323	0.0055
200	Mean	0.9591	-1.0279	0.9986	-0.9992	1.0050	-0.9997
	Median	0.9589	-0.9999	0.9990	-0.9998	1.0009	-0.9999
	Std	0.0690	0.2225	0.0226	0.0055	0.0192	0.0025
	Bias	-0.0409	0.0279	-0.0014	-0.0008	0.0050	-0.0003
	RMSE	0.0801	0.2240	0.0227	0.0056	0.0199	0.0025
500	Mean	0.9517	-1.0115	0.9982	-0.9997	1.0019	-0.9999
	Median	0.9511	-1.0063	0.9990	-0.9999	1.0004	-0.9999
	Std	0.0444	0.1197	0.0131	0.0016	0.0108	0.0005
	Bias	-0.0483	0.0115	-0.0018	-0.0003	0.0019	-0.0001
	RMSE	0.0656	0.1201	0.0132	0.0016	0.0110	0.0005
1000	Mean	0.9517	-1.0098	0.9994	-0.9998	1.0012	-0.9999
	Median	0.9517	-1.0088	0.9994	-0.9999	1.0002	-1.0000
	Std	0.0330	0.0876	0.0079	0.0010	0.0069	0.0002
	Bias	-0.0483	0.0098	-0.0006	-0.0002	0.0012	-0.0001
	RMSE	0.0585	0.0880	0.0079	0.0010	0.0070	0.0002
5000	Mean	0.9493	-1.0104	0.9997	-0.9999	1.0002	-1.0000
	Median	0.9481	-1.0053	0.9998	-1.0000	1.0000	-1.0000
	Std	0.0154	0.0478	0.0022	0.0002	0.0019	0.0003
	Bias	-0.0507	0.0104	-0.0003	-0.0001	0.0002	0.0000
	RMSE	0.0530	0.0489	0.0023	0.0002	0.0019	0.0003
10000	Mean	0.9486	-1.0079	0.9999	-1.0000	1.0001	-1.0000
	Median	0.9485	-1.0042	0.9999	-1.0000	1.0000	-1.0000
	Std	0.0107	0.0333	0.0012	0.0003	0.0011	0.0002
	Bias	-0.0514	0.0079	-0.0001	0.0000	0.0001	0.0000
	RMSE	0.0525	0.0342	0.0012	0.0003	0.0011	0.0002

Table A.19: Results for SMM ($a = 1$ $b = -2$)

Model	T	$N = TH$	Statistics	a	b
m1	100	500	Mean	0.8676	-1.4541
			Median	0.8861	-1.3902
			Std	0.2142	0.6512
			RMSE	0.2516	0.8492
m1	100	1000	Mean	0.8787	-1.4418
			Median	0.8991	-1.3830
			Std	0.2108	0.5868
			RMSE	0.2431	0.8094
m1	200	500	Mean	0.9051	-1.6526
			Median	0.9135	-1.5912
			Std	0.1699	0.6367
			RMSE	0.1945	0.7247
m1	200	1000	Mean	0.9152	-1.6245
			Median	0.9252	-1.5972
			Std	0.1666	0.5403
			RMSE	0.1868	0.6575
m1	500	1000	Mean	0.9494	-1.8017
			Median	0.9550	-1.7878
			Std	0.1180	0.3955
			RMSE	0.1283	0.4421
m1	1000	2000	Mean	0.9802	-1.9007
			Median	0.9828	-1.8983
			Std	0.0733	0.2817
			RMSE	0.0759	0.2984
m1	5000	10000	Mean	0.9953	-1.9773
			Median	0.9968	-1.9720
			Std	0.0326	0.1301
			RMSE	0.0329	0.1319
m1	10000	20000	Mean	0.9974	-1.9886
			Median	0.9979	-1.9851
			Std	0.0230	0.0916
			RMSE	0.0231	0.0922

Table A.20: Continued ($a = 1$ $b = -2$)

Model	T	$N = TH$	Statistics	a	b
m2	100	500	Mean	0.8812	-1.6039
			Median	0.8885	-1.5592
			Std	0.2123	0.5918
			RMSE	0.2431	0.7116
m2	100	1000	Mean	0.8818	-1.6038
			Median	0.8966	-1.5312
			Std	0.3645	0.5979
			RMSE	0.3828	0.7167
m2	200	500	Mean	0.9124	-1.7362
			Median	0.9117	-1.6940
			Std	0.1691	0.5536
			RMSE	0.1903	0.6127
m2	200	1000	Mean	0.9119	-1.6963
			Median	0.9327	-1.6460
			Std	0.3142	0.5188
			RMSE	0.3260	0.6007
m2	500	1000	Mean	0.9514	-1.8276
			Median	0.9540	-1.8058
			Std	0.1122	0.3664
			RMSE	0.1222	0.4046
m2	1000	2000	Mean	0.9773	-1.8920
			Median	0.9850	-1.8911
			Std	0.0741	0.2564
			RMSE	0.0774	0.2780
m2	5000	10000	Mean	0.9922	-1.9635
			Median	0.9926	-1.9528
			Std	0.0352	0.1303
			RMSE	0.0360	0.1352
m2	10000	20000	Mean	0.9960	-1.9821
			Median	0.9959	-1.9824
			Std	0.0243	0.0878
			RMSE	0.0246	0.0895

Table A.21: Results for IND ($a = 1$ $b = -2$)

Model	T	$N = TH$	Statistics	a	b
M1	100	500	Mean	0.9506	-1.3761
			Median	0.9446	-1.2273
			Std	0.2212	0.9000
			RMSE	0.2264	1.0943
M1	100	1000	Mean	0.9474	-1.3657
			Median	0.9510	-1.1837
			Std	0.2421	0.9451
			RMSE	0.2475	1.1374
M1	200	500	Mean	0.9707	-1.6236
			Median	0.9526	-1.5255
			Std	0.1856	0.8636
			RMSE	0.1877	0.9412
M1	200	1000	Mean	0.9710	-1.5674
			Median	0.9616	-1.4758
			Std	0.1719	0.7642
			RMSE	0.1741	0.8775
M1	500	1000	Mean	0.9924	-1.8372
			Median	0.9806	-1.7705
			Std	0.1269	0.5325
			RMSE	0.1270	0.5563
M1	1000	2000	Mean	1.0007	-1.9345
			Median	0.9993	-1.9008
			Std	0.0848	0.3678
			RMSE	0.0847	0.3732
M1	5000	10000	Mean	0.9995	-1.9838
			Median	0.9990	-1.9811
			Std	0.0401	0.1649
			RMSE	0.0401	0.1656
M1	10000	20000	Mean	0.9996	-1.9929
			Median	0.9992	-1.9854
			Std	0.0289	0.1133
			RMSE	0.0289	0.1134

Table A.22: Continued ($a = 1$ $b = -2$)

Model	T	$N = TH$	Statistics	a	b
M2	100	500	Mean	0.8926	-1.9595
			Median	0.9135	-1.8713
			Std	0.3048	0.7597
			RMSE	0.3229	0.7600
M2	100	1000	Mean	0.8518	-1.9463
			Median	0.9168	-1.8267
			Std	0.3175	0.8299
			RMSE	0.3501	0.8308
M2	200	500	Mean	0.9465	-2.0525
			Median	0.9726	-1.9629
			Std	0.2524	0.6911
			RMSE	0.2578	0.6924
M2	200	1000	Mean	0.9202	-1.9898
			Median	0.9582	-1.8720
			Std	0.2441	0.6764
			RMSE	0.2566	0.6758
M2	500	1000	Mean	0.9714	-1.9895
			Median	0.9789	-1.9162
			Std	0.1458	0.4999
			RMSE	0.1484	0.4995
M2	1000	2000	Mean	0.9816	-1.9862
			Median	0.9954	-1.9432
			Std	0.1130	0.3542
			RMSE	0.1144	0.3541
M2	5000	10000	Mean	0.9948	-1.9903
			Median	0.9973	-1.9806
			Std	0.0500	0.1372
			RMSE	0.0502	0.1374
M2	10000	20000	Mean	0.9981	-1.9948
			Median	0.9968	-1.9947
			Std	0.0206	0.0946
			RMSE	0.0207	0.0947

Table A.23: Results for GT ($a = 1$ $b = -2$)

Model	T	$N = TH$	Statistics	a	b
M1	100	500	Mean	0.9397	-1.1761
			Median	0.9447	-1.0367
			Std	0.2193	0.7423
			RMSE	0.2272	1.1085
M1	100	1000	Mean	0.9416	-1.1549
			Median	0.9588	-1.0695
			Std	0.2142	0.6913
			RMSE	0.2218	1.0914
M1	200	500	Mean	0.9423	-1.3801
			Median	0.9339	-1.2783
			Std	0.1625	0.6936
			RMSE	0.1723	0.9298
M1	200	1000	Mean	0.9452	-1.3758
			Median	0.9381	-1.3190
			Std	0.1568	0.6632
			RMSE	0.1660	0.9103
M1	500	1000	Mean	0.9612	-1.6152
			Median	0.9608	-1.5959
			Std	0.1176	0.5088
			RMSE	0.1237	0.6375
M1	1000	2000	Mean	0.9805	-1.7922
			Median	0.9815	-1.7896
			Std	0.0821	0.3659
			RMSE	0.0843	0.4205
M1	5000	10000	Mean	0.9934	-1.9401
			Median	0.9939	-1.9448
			Std	0.0403	0.1702
			RMSE	0.0408	0.1802
M1	10000	20000	Mean	0.9964	-1.9695
			Median	0.9966	-1.9665
			Std	0.0289	0.1148
			RMSE	0.0291	0.1187

Table A.24: Continued ($a = 1$ $b = -2$)

Model	T	$N = TH$	Statistics	a	b
M2	100	500	Mean	1.0120	-1.1614
			Median	1.0027	-1.1664
			Std	0.2411	0.6469
			RMSE	0.2412	1.0587
M2	100	1000	Mean	1.0115	-1.0765
			Median	0.9873	-1.0544
			Std	0.2228	0.6141
			RMSE	0.2229	1.1087
M2	200	500	Mean	1.0254	-1.4391
			Median	1.0055	-1.4751
			Std	0.2060	0.6187
			RMSE	0.2074	0.8346
M2	200	1000	Mean	1.0094	-1.3442
			Median	0.9963	-1.4230
			Std	0.1976	0.5519
			RMSE	0.1976	0.8568
M2	500	1000	Mean	1.0077	-1.6210
			Median	0.9871	-1.6489
			Std	0.1431	0.5043
			RMSE	0.1432	0.6304
M2	1000	2000	Mean	0.9968	-1.7401
			Median	0.9857	-1.7715
			Std	0.1085	0.3594
			RMSE	0.1084	0.4433
M2	5000	10000	Mean	0.9889	-1.9251
			Median	0.9900	-1.9240
			Std	0.0319	0.1420
			RMSE	0.0338	0.1604
M2	10000	20000	Mean	0.9942	-1.9584
			Median	0.9935	-1.9601
			Std	0.0217	0.0943
			RMSE	0.0225	0.1030

Table A.25: Continued ($a = 1\ b = -2$)

Model	T	$N = TH$	Statistics	a	b
M3	100	500	Mean	1.1069	-1.4350
			Median	1.1081	-1.3188
			Std	0.1888	0.7913
			RMSE	0.2168	0.9716
M3	100	1000	Mean	1.1091	-1.4404
			Median	1.1092	-1.3641
			Std	0.1806	0.7741
			RMSE	0.2108	0.9545
M3	200	500	Mean	1.0760	-1.6137
			Median	1.0701	-1.4882
			Std	0.1492	0.7473
			RMSE	0.1673	0.8406
M3	200	1000	Mean	1.0755	-1.5588
			Median	1.0750	-1.4753
			Std	0.1396	0.6611
			RMSE	0.1586	0.7942
M3	500	1000	Mean	1.0495	-1.7082
			Median	1.0497	-1.6420
			Std	0.1009	0.5754
			RMSE	0.1123	0.6447
M3	1000	2000	Mean	1.0368	-1.7871
			Median	1.0380	-1.7155
			Std	0.0742	0.5104
			RMSE	0.0828	0.5525
M3	5000	10000	Mean	1.0079	-1.9574
			Median	1.0125	-1.8948
			Std	0.0419	0.3644
			RMSE	0.0426	0.3665
M3	10000	20000	Mean	1.0003	-2.0042
			Median	1.0053	-1.9482
			Std	0.0364	0.3145
			RMSE	0.0363	0.3142

Table A.26: Continued ($a = 1$ $b = -2$)

Model	T	$N = TH$	Statistics	a	b
M4	100	500	Mean	0.9917	-1.5667
			Median	1.0364	-1.5022
			Std	0.4112	0.7781
			RMSE	0.4109	0.8899
M4	100	1000	Mean	0.9484	-1.6230
			Median	1.0483	-1.5068
			Std	0.5935	1.1119
			RMSE	0.5951	1.1730
M4	200	500	Mean	1.0317	-1.6420
			Median	1.0298	-1.6495
			Std	0.1400	0.6688
			RMSE	0.1435	0.7580
M4	200	1000	Mean	1.0049	-1.6214
			Median	1.0415	-1.6062
			Std	0.3635	0.6803
			RMSE	0.3631	0.7779
M4	500	1000	Mean	1.0135	-1.7449
			Median	1.0123	-1.7139
			Std	0.0958	0.5691
			RMSE	0.0967	0.6231
M4	1000	2000	Mean	1.0051	-1.8395
			Median	1.0030	-1.8234
			Std	0.0677	0.4072
			RMSE	0.0678	0.4373
M4	5000	10000	Mean	1.0010	-1.9600
			Median	1.0013	-1.9602
			Std	0.0311	0.2040
			RMSE	0.0311	0.2077
M4	10000	20000	Mean	1.0007	-1.9837
			Median	1.0006	-1.9770
			Std	0.0204	0.1280
			RMSE	0.0204	0.1289

Table A.27: Pseudo and Maximum Likelihood Estimators ($a = 1$ $b = -2$)

Size	Statistics	PML		CML		UML	
		a	b	a	b	a	b
100	Mean	0.9254	-2.0956	0.9959	-1.9547	1.0110	-2.0140
	Median	0.9223	-1.9977	0.9971	-1.9753	1.0011	-2.0060
	Std	0.1474	0.6739	0.0607	0.1259	0.0517	0.0916
	Bias	-0.0746	0.0478	-0.0041	-0.0227	0.0110	0.0070
	RMSE	0.1651	0.6799	0.0608	0.1337	0.0528	0.0925
200	Mean	0.9312	-2.0489	1.0021	-1.9794	1.0077	-2.0141
	Median	0.9272	-2.0037	1.0032	-1.9827	1.0031	-2.0031
	Std	0.1071	0.4889	0.0374	0.0945	0.0353	0.0638
	Bias	-0.0688	0.0244	0.0021	-0.0103	0.0077	0.0071
	RMSE	0.1272	0.4908	0.0374	0.0966	0.0361	0.0653
500	Mean	0.9242	-1.9996	1.0015	-1.9907	1.0033	-2.0095
	Median	0.9235	-1.9897	1.0008	-1.9926	1.0006	-2.0057
	Std	0.0742	0.3328	0.0243	0.0595	0.0227	0.0552
	Bias	-0.0758	0.0002	0.0015	-0.0047	0.0033	0.0047
	RMSE	0.1060	0.3325	0.0243	0.0601	0.0229	0.0560
1000	Mean	0.9187	-1.9564	1.0016	-1.9962	1.0022	-2.0049
	Median	0.9159	-1.9591	1.0005	-1.9984	1.0003	-2.0043
	Std	0.0516	0.2268	0.0159	0.0451	0.0159	0.0410
	Bias	-0.0813	-0.0218	0.0016	-0.0019	0.0022	0.0025
	RMSE	0.0962	0.2307	0.0159	0.0452	0.0160	0.0413
5000	Mean	0.9097	-1.9172	1.0001	-1.9982	1.0002	-2.0018
	Median	0.9093	-1.9069	1.0002	-1.9981	1.0000	-2.0006
	Std	0.0239	0.1125	0.0075	0.0180	0.0071	0.0158
	Bias	-0.0903	-0.0414	0.0001	-0.0009	0.0002	0.0009
	RMSE	0.0934	0.1396	0.0075	0.0181	0.0071	0.0159
10000	Mean	0.9085	-1.9073	1.0004	-1.9989	1.0003	-2.0003
	Median	0.9095	-1.9024	1.0001	-1.9993	1.0000	-2.0003
	Std	0.0175	0.0792	0.0052	0.0112	0.0049	0.0106
	Bias	-0.0915	-0.0463	0.0004	-0.0005	0.0003	0.0002
	RMSE	0.0932	0.1219	0.0052	0.0113	0.0049	0.0105

Table A.28: Results for SMM ($a = 600$ $b = -5$)

Model	T	$N = TH$	Statistics	a	b
m1	100	500	Mean	497.2039	-4.0991
			Median	490.8640	-3.9955
			Std	114.8648	1.0985
			RMSE	154.0554	1.4198
m1	100	1000	Mean	536.5978	-4.4476
			Median	531.2719	-4.3692
			Std	98.8391	0.9475
			RMSE	117.3429	1.0959
m1	200	500	Mean	505.1230	-4.1763
			Median	489.5937	-4.0142
			Std	189.7449	1.9904
			RMSE	211.9634	2.1521
m1	200	1000	Mean	536.4926	-4.4356
			Median	532.7233	-4.3794
			Std	91.7087	0.8783
			RMSE	111.4755	1.0433
m1	500	1000	Mean	568.9188	-4.7273
			Median	567.7719	-4.6833
			Std	61.3887	0.5917
			RMSE	68.7538	0.6509
m1	1000	2000	Mean	583.3728	-4.8521
			Median	580.0225	-4.8164
			Std	41.5158	0.4035
			RMSE	44.6831	0.4294
m1	5000	10000	Mean	596.2600	-4.9670
			Median	595.5766	-4.9605
			Std	19.4570	0.1917
			RMSE	19.7941	0.1943
m1	10000	20000	Mean	598.5690	-4.9879
			Median	599.2193	-4.9941
			Std	13.1525	0.1296
			RMSE	13.2170	0.1300

Table A.29: Continued ($a = 600$ $b = -5$)

Model	T	$N = TH$	Statistics	a	b
m2	100	500	Mean	527.9823	-4.3949
			Median	517.0194	-4.3377
			Std	114.0970	1.0830
			RMSE	134.8180	1.2395
m2	100	1000	Mean	546.7608	-4.5423
			Median	549.7148	-4.5429
			Std	97.1959	0.9266
			RMSE	110.7354	1.0326
m2	200	500	Mean	518.3551	-4.2962
			Median	517.9202	-4.3314
			Std	104.9562	0.9980
			RMSE	132.8807	1.2202
m2	200	1000	Mean	544.7007	-4.5190
			Median	550.4452	-4.5437
			Std	93.1796	0.9149
			RMSE	108.2727	1.0328
m2	500	1000	Mean	570.1166	-4.7383
			Median	572.7126	-4.7441
			Std	62.0150	0.5925
			RMSE	68.7835	0.6472
m2	1000	2000	Mean	582.5474	-4.8440
			Median	581.5468	-4.8236
			Std	39.7741	0.3816
			RMSE	43.3983	0.4119
m2	5000	10000	Mean	595.1586	-4.9568
			Median	594.3108	-4.9478
			Std	19.6214	0.1901
			RMSE	20.1908	0.1948
m2	10000	20000	Mean	598.1520	-4.9841
			Median	598.3952	-4.9899
			Std	13.4525	0.1304
			RMSE	13.5655	0.1313

Table A.30: Results for IND ($a = 600$ $b = -5$)

Model	T	$N = TH$	Statistics	a	b
M1	100	500	Mean	510.3457	-4.1259
			Median	498.7019	-4.0050
			Std	140.9853	1.3797
			RMSE	166.9515	1.6321
M1	100	1000	Mean	553.5491	-4.5484
			Median	546.7013	-4.4958
			Std	99.8893	0.9770
			RMSE	110.0604	1.0754
M1	200	500	Mean	500.9580	-4.0343
			Median	488.2273	-3.9210
			Std	134.3376	1.3144
			RMSE	166.7818	1.6298
M1	200	1000	Mean	542.1322	-4.4358
			Median	533.2960	-4.3263
			Std	94.9570	0.9282
			RMSE	111.1021	1.0852
M1	500	1000	Mean	579.5301	-4.8017
			Median	574.4918	-4.7590
			Std	61.9788	0.6056
			RMSE	65.1952	0.6365
M1	1000	2000	Mean	590.0107	-4.9030
			Median	589.4814	-4.9017
			Std	42.2682	0.4135
			RMSE	43.3685	0.4241
M1	5000	10000	Mean	598.5667	-4.9857
			Median	597.9037	-4.9768
			Std	20.2012	0.1979
			RMSE	20.2165	0.1981
M1	10000	20000	Mean	599.9815	-4.9999
			Median	600.2217	-5.0055
			Std	13.6940	0.1335
			RMSE	13.6713	0.1333

Table A.31: Continued ($a = 600$ $b = -5$)

Model	T	$N = TH$	Statistics	a	b
M2	100	500	Mean Median Std RMSE	604.3078 586.1466 166.4384 166.2679	-5.0683 -4.8816 1.6766 1.6757
M2	100	1000	Mean Median Std RMSE	617.9128 597.8249 132.9859 133.9900	-5.1899 -4.9702 1.3294 1.3409
M2	200	500	Mean Median Std RMSE	588.5175 577.9398 152.2356 152.4541	-4.9138 -4.7904 1.5371 1.5373
M2	200	1000	Mean Median Std RMSE	594.9143 593.5029 103.1122 103.0669	-4.9625 -4.9444 1.0311 1.0301
M2	500	1000	Mean Median Std RMSE	599.7526 598.5612 76.7453 76.6080	-5.0025 -4.9732 0.7721 0.7707
M2	1000	2000	Mean Median Std RMSE	603.9002 599.3215 49.9823 50.0468	-5.0425 -4.9940 0.5023 0.5032
M2	5000	10000	Mean Median Std RMSE	601.5065 601.2103 21.6131 21.6261	-5.0157 -5.0107 0.2157 0.2159
M2	10000	20000	Mean Median Std RMSE	601.2168 601.1404 15.3442 15.3647	-5.0124 -5.0071 0.1538 0.1540

Table A.32: Results for GT ($a = 600$ $b = -5$)

Model	T	$N = TH$	Statistics	a	b
M1	100	500	Mean	464.8138	-3.6770
			Median	459.7403	-3.6320
			Std	135.5756	1.3159
			RMSE	191.3617	1.8651
M1	100	1000	Mean	517.3378	-4.1938
			Median	520.3069	-4.2193
			Std	111.3285	1.0821
			RMSE	138.5724	1.3485
M1	200	500	Mean	465.9827	-3.6871
			Median	455.1236	-3.5545
			Std	132.9884	1.2915
			RMSE	188.7092	1.8408
M1	200	1000	Mean	519.2223	-4.2112
			Median	513.3373	-4.1642
			Std	103.9601	1.0112
			RMSE	131.5717	1.2816
M1	500	1000	Mean	558.0183	-4.5912
			Median	559.1889	-4.5867
			Std	69.3694	0.6747
			RMSE	81.0244	0.7883
M1	1000	2000	Mean	577.6936	-4.7820
			Median	575.9946	-4.7684
			Std	46.0824	0.4487
			RMSE	51.1558	0.4984
M1	5000	10000	Mean	594.5522	-4.9468
			Median	594.6187	-4.9480
			Std	20.4865	0.2003
			RMSE	21.1786	0.2070
M1	10000	20000	Mean	597.8651	-4.9794
			Median	598.0758	-4.9822
			Std	13.8782	0.1354
			RMSE	14.0277	0.1368

Table A.33: Continued ($a = 600$ $b = -5$)

Model	T	$N = TH$	Statistics	a	b
M2	100	500	Mean	478.0481	-3.8095
			Median	474.6868	-3.7535
			Std	119.3750	1.1806
			RMSE	170.5693	1.6759
M2	100	1000	Mean	521.8684	-4.2342
			Median	515.4722	-4.1648
			Std	94.3566	0.9393
			RMSE	122.4333	1.2112
M2	200	500	Mean	473.3456	-3.7700
			Median	472.8052	-3.7536
			Std	123.7129	1.2294
			RMSE	176.9611	1.7382
M2	200	1000	Mean	517.7174	-4.1961
			Median	512.1424	-4.1390
			Std	93.5112	0.9356
			RMSE	124.4878	1.2328
M2	500	1000	Mean	553.7192	-4.5423
			Median	553.0773	-4.5445
			Std	67.0170	0.6699
			RMSE	81.3892	0.8107
M2	1000	2000	Mean	574.0279	-4.7435
			Median	574.8012	-4.7466
			Std	46.0837	0.4609
			RMSE	52.8584	0.5271
M2	5000	10000	Mean	592.9154	-4.9304
			Median	592.5503	-4.9270
			Std	22.0929	0.2207
			RMSE	23.1800	0.2312
M2	10000	20000	Mean	596.5063	-4.9654
			Median	596.9415	-4.9694
			Std	15.5573	0.1560
			RMSE	15.9296	0.1596

Table A.34: Continued ($a = 600$ $b = -5$)

Model	T	$N = TH$	Statistics	a	b
M3	100	500	Mean	503.5479	-4.0287
			Median	502.8393	-4.0233
			Std	131.0933	1.2838
			RMSE	162.6471	1.6088
M3	100	1000	Mean	500.4626	-3.9971
			Median	501.2850	-4.0174
			Std	126.8113	1.2428
			RMSE	161.1105	1.5960
M3	200	500	Mean	548.0457	-4.4741
			Median	543.7376	-4.4166
			Std	110.3548	1.0872
			RMSE	121.8731	1.2067
M3	200	1000	Mean	545.7874	-4.4514
			Median	544.0687	-4.4229
			Std	100.9280	0.9941
			RMSE	114.4775	1.1345
M3	500	1000	Mean	578.5912	-4.7813
			Median	570.4354	-4.7068
			Std	83.9542	0.8260
			RMSE	86.5595	0.8537
M3	1000	2000	Mean	588.6449	-4.8831
			Median	586.0304	-4.8530
			Std	73.0148	0.7192
			RMSE	73.8203	0.7279
M3	5000	10000	Mean	594.0562	-4.9398
			Median	595.7983	-4.9545
			Std	50.0033	0.4918
			RMSE	50.3057	0.4950
M3	10000	20000	Mean	600.9093	-5.0086
			Median	600.4886	-5.0016
			Std	48.8528	0.4804
			RMSE	48.8124	0.4800

Table A.35: Continued ($a = 600$ $b = -5$)

Model	T	$N = TH$	Statistics	a	b
M4	100	500	Mean	528.2140	-4.2649
			Median	525.8295	-4.2499
			Std	136.6993	1.3283
			RMSE	154.2795	1.5170
M4	100	1000	Mean	544.4593	-4.4414
			Median	541.1960	-4.4054
			Std	111.6182	1.0872
			RMSE	124.5731	1.2214
M4	200	500	Mean	515.8175	-4.1421
			Median	514.5777	-4.1406
			Std	129.1316	1.2538
			RMSE	154.0388	1.5182
M4	200	1000	Mean	537.1270	-4.3660
			Median	540.8369	-4.3877
			Std	107.4813	1.0470
			RMSE	124.4272	1.2231
M4	500	1000	Mean	564.5931	-4.6449
			Median	565.9669	-4.6321
			Std	71.6023	0.6983
			RMSE	79.8140	0.7828
M4	1000	2000	Mean	580.1958	-4.8015
			Median	578.1393	-4.7782
			Std	52.8990	0.5176
			RMSE	56.4351	0.5538
M4	5000	10000	Mean	594.5181	-4.9453
			Median	592.4386	-4.9224
			Std	24.2697	0.2374
			RMSE	24.8574	0.2434
M4	10000	20000	Mean	597.4753	-4.9749
			Median	596.9135	-4.9689
			Std	16.4944	0.1612
			RMSE	16.6702	0.1630

Table A.36: Pseudo and Maximum Likelihood Estimators ($a = 600$ $b = -5$)

Size	Statistics	PML		CML		UML	
		a	b	a	b	a	b
100	Mean	629.9104	-5.3158	598.7129	-4.9895	603.0852	-5.0306
	Median	611.8145	-5.1559	600.9977	-5.0109	601.2422	-5.0109
	Std	139.0217	1.3629	14.8260	0.1503	9.5747	0.0978
	Bias	0.0499	0.0632	-0.0021	-0.0021	0.0051	0.0061
	RMSE	142.0669	1.3976	14.8670	0.1505	10.0504	0.1024
200	Mean	620.6766	-5.2256	599.3386	-4.9947	602.0959	-5.0201
	Median	615.2680	-5.1687	600.5140	-5.0067	600.7777	-5.0069
	Std	85.7622	0.8414	10.5729	0.1070	6.5761	0.0665
	Bias	0.0345	0.0451	-0.0011	-0.0011	0.0035	0.0040
	RMSE	88.1361	0.8703	10.5829	0.1070	6.8957	0.0694
500	Mean	618.1547	-5.2037	600.4591	-5.0051	601.6421	-5.0162
	Median	615.0317	-5.1841	601.0898	-5.0113	600.6896	-5.0075
	Std	53.6683	0.5261	7.1347	0.0738	5.7014	0.0586
	Bias	0.0303	0.0407	0.0008	0.0010	0.0027	0.0032
	RMSE	56.6050	0.5636	7.1422	0.0739	5.9277	0.0608
1000	Mean	618.3625	-5.2057	600.5703	-5.0061	600.9427	-5.0091
	Median	614.8426	-5.1700	601.0020	-5.0112	600.5151	-5.0048
	Std	37.4499	0.3680	4.9605	0.0514	3.9702	0.0412
	Bias	0.0306	0.0411	0.0010	0.0012	0.0016	0.0018
	RMSE	41.6758	0.4212	4.9881	0.0517	4.0767	0.0421
5000	Mean	618.4924	-5.2086	600.7797	-5.0083	600.5557	-5.0057
	Median	616.6668	-5.1894	600.8904	-5.0098	600.2301	-5.0023
	Std	17.2490	0.1694	2.0094	0.0210	2.2255	0.0233
	Bias	0.0308	0.0417	0.0013	0.0017	0.0009	0.0011
	RMSE	25.2765	0.2686	2.1535	0.0226	2.2917	0.0239
10000	Mean	619.0792	-5.2141	600.7376	-5.0078	600.3378	-5.0035
	Median	619.2819	-5.2143	600.9583	-5.0101	600.1952	-5.0019
	Std	11.9005	0.1166	1.4693	0.0154	1.6487	0.0173
	Bias	0.0318	0.0428	0.0012	0.0016	0.0006	0.0007
	RMSE	22.4800	0.2437	1.6426	0.0173	1.6813	0.0176