Natural hp-BEM for the electric field integral equation with singular solutions

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Natural *hp*-BEM for the electric field integral equation with singular solutions *

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Abstract

We apply the *hp*-version of the boundary element method (BEM) for the numerical solution of the electric field integral equation (EFIE) on a Lipschitz polyhedral surface Γ. The underlying meshes are supposed to be quasi-uniform triangulations of Γ, and the approximations are based on either Raviart-Thomas or Brezzi-Douglas-Marini families of surface elements. Non-smoothness of Γ leads to singularities in the solution of the EFIE, severely affecting convergence rates of the BEM. However, the singular behaviour of the solution can be explicitly specified using a finite set of power functions (vertex-, edge-, and vertex-edge singularities). In this paper we use this fact to perform an a priori error analysis of the *hp*-BEM on quasi-uniform meshes. We prove precise error estimates in terms of the polynomial degree p, the mesh size h, and the singularity exponents.

Key words: *hp*-version with quasi-uniform meshes, boundary element method, electric field integral equation, singularities, a priori error estimate
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1 Introduction

In this paper we study numerical approximations of the electric field integral equation (EFIE) on a surface $\Gamma = \partial\Omega$, where $\Omega \subset \mathbb{R}^3$ is a Lipschitz polyhedron. The EFIE is a boundary integral formulation of a boundary value problem for the time-harmonic Maxwell equations in the domain exterior to Ω. It models the scattering of electromagnetic waves at a perfectly conducting body.

For the numerical solution of the EFIE we use the Galerkin boundary element method (BEM). It employs $\mathbf{H}(\mathrm{div}_\Gamma)$-conforming families of surface elements, namely Raviart-Thomas (RT) and Brezzi-Douglas-Marini (BDM) elements, to discretise the variational formulation of the EFIE (called Rumsey's principle). This approach is referred to as the natural BEM for the EFIE. As in finite element methods, the convergence may be achieved either by keeping

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polynomial degrees fixed and refining the mesh (h-version), or by fixing the mesh and increasing polynomial degrees (p-version), or by simultaneous h-refinement and p-enrichment (hp-version).

The Galerkin BEM is widely used in the engineering practice for the simulation of electromagnetic scattering. Moreover, it had been used long before a rigorous theoretical analysis of the method became available. Error analyses of different boundary element schemes for the eddy current problem on smooth obstacles are given by MacCamy and Stephan in [24, 25]. Despite these early results the BEM-analysis for non-smooth obstacles started much later. In fact, the convergence and a priori error analysis of the h-BEM for the EFIE on piecewise smooth (open or closed) surfaces has been developed within the last decade (see [13, 22, 16, 10, 15]), and the corresponding results for high-order methods (p- and hp-BEM) are very recent (see [6, 4, 7]). In particular, [6] and [7] establish quasi-optimal convergence of the natural hp-BEM on meshes of shape-regular elements. An essential ingredient for the proofs in these papers are the projection-based interpolation operators developed by Demkowicz and co-authors [19, 20, 18]. These operators also facilitate a priori error analysis, where one needs hp-approximation theory in specific trace spaces. In [4] we prove an a priori error estimate of the hp-BEM on quasi-uniform meshes of affine elements under the assumption that the regularity of the solution to the EFIE is given in Sobolev spaces on Γ. The latter result states that the method converges with the same rate $r + \frac{1}{2}$ in both h and p^{-1} (here, r denotes the Sobolev regularity order, and p is assumed to be large enough). However, it is evident from the numerical results reported in [23] that the p-BEM converges faster than the h-BEM for the EFIE. This is similar to what was observed and proved for the p-BEM and the h-BEM applied to elliptic problems in three-dimensions (cf. [27, 21, 3, 5]).

With this paper we fill a gap in the theory of the BEM for the EFIE on polyhedral surfaces by proving a precise error estimate for the hp-BEM on quasi-uniform meshes. Similarly to the elliptic case we make use of explicit expressions for singularities in electromagnetics fields (these expressions are available from [17, 6]). The established error estimate shows that the convergence rates in h and p depend on the strongest singularity exponent, and that the p-BEM converges twice as fast as the h-BEM on quasi-uniform meshes. This extends the results of [6], where the analysis was restricted to the p-BEM on a plane open surface.

It is now well known that appropriate decompositions of vector fields (on both the continuous and discrete level) are critical for the analysis of electromagnetic problems and their approximations. In the case of the EFIE the main idea is to isolate the kernel of the div_Γ-operator such that the complementary field possesses an enhanced smoothness (cf. [10, 9, 7]). The corresponding decompositions of singular vector fields greatly facilitate our analysis in this paper as well. In particular, the complementary vector fields are singular vector functions belonging to $\mathbf{H}^{1/2}(\Gamma)$. Then, in the case of the BDM-based BEM, these vector functions can be approximated component-wise by continuous piecewise polynomials, and the desired hp-error estimates are derived by using the corresponding results in [3] for scalar singularities (belonging to $H^{1/2}(\Gamma)$). However, this simple approach does not work for the RT-based BEM because the dimension of the underlying RT-space on the reference element is smaller than the dimension of the BDM-counterpart. That is why, the results of [3] cannot be applied directly, and additional technical arguments are needed (see the proof of Lemma 4.2 in Section 5.2).

The rest of the paper is organised as follows. In the next section we formulate the EFIE in its variational form and recall the typical structure of the solution to this model problem. In Section 3 we introduce the *hp*-version of the BEM for the EFIE and formulate the main result of the paper (Theorem 3.1) stating convergence rates of the method. This result follows from the general approximation theorem (Theorem 4.1) established in Section 4. The proof of Theorem 4.1 relies, in particular, on two technical lemmas which are proved in Section 5.

Throughout the paper, C denotes a generic positive constant independent of h and p.

2 Formulation of the problem

Let Γ be a Lipschitz polyhedral surface in $\mathbb{R}^3$. Throughout the paper we will use exactly the same notation as in [4] for all involved differential and boundary integral operators as well as for Sobolev spaces of scalar functions and tangential vector fields on Γ (all essential definitions are given in [4, Section 3.1]). In particular, we use boldface symbols for vector fields, and the spaces (or sets) of vector fields are denoted in boldface as well.

For a given wave number $\kappa > 0$, we denote by Ψ_κ (resp., $\boldsymbol{\Psi}_\kappa$) the scalar (resp., the vectorial) single layer boundary integral operator on Γ for the Helmholtz operator $-\Delta - \kappa^2$ (see [15, Section 5]). The variational formulation for the EFIE will be posed in the following Hilbert space of tangential vector fields on Γ:

$$\mathbf{X} = \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) := \{\mathbf{u} \in \mathbf{H}_\parallel^{-1/2}(\Gamma);\ \mathrm{div}_\Gamma\, \mathbf{u} \in H^{-1/2}(\Gamma)\},$$

which is the trace space of $\mathbf{H}(\mathbf{curl}, \Omega)$, where $\Omega \subset \mathbb{R}^3$ is a Lipschitz polyhedron such that $\Gamma = \partial\Omega$ (we refer to [11, 12, 14] for the definition and properties of this and other trace spaces on Γ).

Let $\mathbf{X}'$ be the dual space of $\mathbf{X}$ (with duality pairing extending the $\mathbf{L}^2(\Gamma)$-inner product for tangential vector fields). Then, for a given source functional $\mathbf{f} \in \mathbf{X}'$ the variational formulation of the EFIE reads as: *find a complex tangential field* $\mathbf{u} \in \mathbf{X}$ *such that*

$$a(\mathbf{u}, \mathbf{v}) := \langle \Psi_\kappa \mathrm{div}_\Gamma\, \mathbf{u}, \mathrm{div}_\Gamma\, \mathbf{v}\rangle - \kappa^2 \langle \boldsymbol{\Psi}_\kappa \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{f}, \mathbf{v}\rangle \quad \forall \mathbf{v} \in \mathbf{X}. \tag{2.1}$$

Here, the brackets $\langle\cdot,\cdot\rangle$ denote dualities associated with $H^{1/2}(\Gamma)$ and $\mathbf{H}_\parallel^{1/2}(\Gamma)$. To ensure the uniqueness of the solution to (2.1) we always assume that κ^2 is not an electrical eigenvalue of the interior problem.

Let us recall the typical structure of the solution $\mathbf{u}$ to problem (2.1), provided that the source functional $\mathbf{f}$ is sufficiently smooth (we note that this regularity assumption is satisfied for the electromagnetic scattering with plane incident wave). We use the results of [6, Appendix A]. These results were derived from the regularity theory in [17] for the boundary value problems for Maxwell's equations in 3D by making use of trace arguments and some technical calculations in the spirit of [29] and [28].

Let $V = \{v\}$ and $E = \{e\}$ denote the sets of vertices and edges of Γ, respectively. For $v \in V$, let $E(v)$ denote the set of edges with v as an end point. Then the solution $\mathbf{u}$ of (2.1) can be written as

$$\mathbf{u} = \mathbf{u}_{\mathrm{reg}} + \mathbf{u}_{\mathrm{sing}}, \tag{2.2}$$

where

$$\mathbf{u}_{\text{reg}} \in \mathbf{X}^k := \{\mathbf{u} \in \mathbf{H}^k_-(\Gamma);\ \operatorname{div}_\Gamma \mathbf{u} \in H^k_-(\Gamma)\} \quad \text{with} \quad k > 0 \tag{2.3}$$

(the spaces $\mathbf{H}^k_-(\Gamma)$ and $H^k_-(\Gamma)$ are defined in a piecewise fashion by localisation to each face of Γ, and the space $\mathbf{X}^k$ is equipped with its graph norm $\|\cdot\|_{\mathbf{X}^k}$, see [4]),

$$\mathbf{u}_{\text{sing}} = \sum_{e\in E} \mathbf{u}^e + \sum_{v\in V} \mathbf{u}^v + \sum_{v\in V} \sum_{e\in E(v)} \mathbf{u}^{ev}, \tag{2.4}$$

and $\mathbf{u}^e$, $\mathbf{u}^v$, and $\mathbf{u}^{ev}$ are the edge, vertex, and edge-vertex singularities, respectively.

In order to write explicit expressions for the above singularities, let us fix a vertex $v \in V$ and an edge $e \in E(v)$. Then, on each face $\Gamma^{ev} \subset \Gamma$ such that $e \subset \partial\Gamma^{ev}$ we will use local polar and Cartesian coordinate systems (r_v, θ_v) and (x_{e1}, x_{e2}), both with the origin at v, such that $e = \{(x_{e1}, x_{e2});\ x_{e2} = 0,\ x_{e1} > 0\}$ and for a sufficiently small neighbourhood B_τ of v there holds $\Gamma^{ev} \cap B_\tau \subset \{(r_v, \theta_v);\ 0 < \theta_v < \omega_v\}$. Here, ω_v denotes the interior angle (on Γ^{ev}) between the edges meeting at v. For simplicity of notation we write out here only the leading singularities in $\mathbf{u}^e$, $\mathbf{u}^v$, and $\mathbf{u}^{ev}$ on the face Γ^{ev}, thus omitting the corresponding terms of higher regularity (see [6, Appendix A] for complete expansions).

For the edge singularities $\mathbf{u}^e$ one has

$$\mathbf{u}^e = \mathbf{curl}_{\Gamma^{ev}} \Big(x_{e2}^{\gamma_1^e} |\log x_{e2}|^{s_1^e}\, b_1^e(x_{e1})\, \chi_1^e(x_{e1})\, \chi_2^e(x_{e2}) \Big) +\, x_{e2}^{\gamma_2^e} |\log x_{e2}|^{s_2^e}\, \mathbf{b}_2^e(x_{e1})\, \chi_1^e(x_{e1})\, \chi_2^e(x_{e2}), \tag{2.5}$$

where $\mathbf{curl}_{\Gamma^{ev}} = (\partial/\partial x_{e2}, -\partial/\partial x_{e1})$ is the tangential vector curl operator $\mathbf{curl}_\Gamma$ restricted to the face Γ^{ev} (cf. [11, 12]), $\gamma_1^e,\ \gamma_2^e > \frac{1}{2}$, and $s_1^e,\ s_2^e \geq 0$ are integers. Here, $\chi_1^e,\ \chi_2^e$ are C^∞ cut-off functions with $\chi_1^e = 1$ in a certain distance to the end points of e and $\chi_1^e = 0$ in a neighbourhood of these vertices. Moreover, $\chi_2^e = 1$ for $0 \leq x_{e2} \leq \delta_e$ and $\chi_2^e = 0$ for $x_{e2} \geq 2\delta_e$ with some $\delta_e \in (0, \frac{1}{2})$. The functions $b_1^e\chi_1^e \in H^{m_1}(e)$ and $\mathbf{b}_2^e\chi_1^e \in \mathbf{H}^{m_2}(e)$ for m_1 and m_2 as large as required.

The vertex singularities $\mathbf{u}^v$ have the form

$$\mathbf{u}^v = \mathbf{curl}_{\Gamma^{ev}} \Big(r_v^{\lambda_1^v} |\log r_v|^{q_1^v}\, \chi^v(r_v)\, \chi_1^v(\theta_v) \Big) +\, r_v^{\lambda_2^v} |\log r_v|^{q_2^v}\, \chi^v(r_v)\, \boldsymbol{\chi}_2^v(\theta_v), \tag{2.6}$$

where $\lambda_1^v,\ \lambda_2^v > -\frac{1}{2}$ are real numbers, $q_1^v,\ q_2^v \geq 0$ are integers, χ^v is a C^∞ cut-off function with $\chi^v = 1$ for $0 \leq r_v \leq \tau_v$ and $\chi^v = 0$ for $r_v \geq 2\tau_v$ with some $\tau_v \in (0, \frac{1}{2})$. The functions χ_1^v, $\boldsymbol{\chi}_2^v$ are such that $\chi_1^v \in H^{t_1}(0, \omega_v)$, $\boldsymbol{\chi}_2^v \in \mathbf{H}^{t_2}(0, \omega_v)$ for t_1, t_2 as large as required.

For the combined edge-vertex singularity $\mathbf{u}^{ev}$ one has

$$\mathbf{u}^{ev} = \mathbf{u}_1^{ev} + \mathbf{u}_2^{ev},$$

where

$$\mathbf{u}_1^{ev} \quad = \quad \mathbf{curl}_{\Gamma^{ev}} \Big(x_{e1}^{\lambda_1^v - \gamma_1^e} x_{e2}^{\gamma_1^e} |\log x_{e1}|^{\beta_1} |\log x_{e2}|^{\beta_2}\, \chi^v(r_v) \chi^{ev}(\theta_v) \Big)$$

$$+ x_{e1}^{\lambda_2^v-\gamma_2^e} x_{e2}^{\gamma_2^e} |\log x_{e1}|^{\beta_3} |\log x_{e2}|^{\beta_4} \, \chi^v(r_v)\chi^{ev}(\theta_v) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.7}$$

and

$$\mathbf{u}_2^{ev} = \mathbf{curl}_{\Gamma^{ev}} \Big(x_{e2}^{\gamma_1^e} \, |\log x_{e2}|^{s_1^e} \, b_3^e(x_{e1}, x_{e2}) \, \chi_2^e(x_{e2}) \Big) + \, x_{e2}^{\gamma_2^e} \, |\log x_{e2}|^{s_2^e} \, \mathbf{b}_4^e(x_{e1}, x_{e2}) \, \chi_2^e(x_{e2}). \tag{2.8}$$

Here, λ_i^v, γ_i^e, s_i^e $(i = 1, 2)$, χ^v, and χ_2^e are as above, $\beta_k \geq 0$ $(k = 1 \ldots, 4)$ are integers, $\beta_1 + \beta_2 = s_1^e + q_1^v$, $\beta_3 + \beta_4 = s_2^e + q_2^v$ with q_1^v, q_2^v being as in (2.6), χ^{ev} is a C^∞ cut-off function with $\chi^{ev} = 1$ for $0 \leq \theta_v \leq \beta_v$ and $\chi^{ev} = 0$ for $\frac{3}{2}\beta_v \leq \theta_v \leq \omega_v$ for some $\beta_v \in (0, \min\{\omega_v/2, \pi/8\}]$. The functions b_3^e and $\mathbf{b}_4^e$, when extended by zero onto $\mathbb{R}^{2+} := \{(x_{e1}, x_{e2});\ x_{e2} > 0\}$, lie in $H^{m_1}(\mathbb{R}^{2+})$ and $\mathbf{H}^{m_2}(\mathbb{R}^{2+})$, respectively, with m_1, m_2 as large as required. Finally, the supports of $\mathbf{u}_1^{ev}$ and $\mathbf{u}_2^{ev}$ are subsets of the sector $\bar{S}_{ev} = \{(r_v, \theta_v);\ 0 \leq r_v \leq 2\tau_v,\ 0 \leq \theta_v \leq \frac{3}{2}\beta_v\}$.

Remark 2.1 (i) *The exponents γ_i^e $(i = 1, 2)$ of the edge and vertex-edge singularities in* (2.5), (2.7), (2.8) *satisfy $\gamma_i^e > \frac{1}{2}$. However, for our approximation analysis below it suffices to require that $\gamma_i^e > 0$ $(i = 1, 2)$. Note that $\gamma_i^e > 0$ and $\lambda_i^v > -\frac{1}{2}$ $(i = 1, 2)$ are the minimum requirements to guarantee $\mathbf{u} \in \mathbf{X}$.*

(ii) *As mentioned above, the terms of higher regularity (i.e., with greater singularity exponents) are omitted in* (2.5)–(2.8)*. These terms are necessary to obtain the regular part $\mathbf{u}_{\mathrm{reg}} \in \mathbf{X}^k$ of decomposition* (2.2) *as smooth as required. This can be done by considering sufficiently many (omitted) singularity terms of each type.*

Remark 2.2 (i) *By* (2.5)–(2.8) *we conclude that any singular vector field $\mathbf{u}^s$ in* (2.4) *$(s = e,\ v$, or $ev)$ can be written as*

$$\mathbf{u}^s = \mathbf{curl}_\Gamma \, w^s + \mathbf{v}^s = \mathbf{curl}_\Gamma \, w^s + (v_1^s,\ v_2^s) \tag{2.9}$$

with corresponding (scalar) singular functions w^s, v_1^s, v_2^s being defined on the whole surface Γ. Note that these scalar functions are $H^{1/2}$-regular on Γ, i.e.,

$$w^s \in H^{1/2}(\Gamma), \qquad \mathbf{v}^s = (v_1^s,\ v_2^s) \in \mathbf{H}^{1/2}(\Gamma), \qquad s = e,\ v,\ ev.$$

(ii) *It is important to observe that the functions w^s, v_1^s, v_2^s $(s = e,\ v$, or $ev)$ in* (2.9) *are typical scalar singularities inherent to solutions of the boundary integral equations with hypersingular operator for the Laplacian on Γ and with possibly singular right-hand side. Continuous piecewise polynomial approximations of these scalar singularities in fractional-order Sobolev spaces were analysed in* [3, 1] *and will be used to prove the main result of the present paper.*

3 The *hp*-version of the BEM and the main result

For the approximate solution of (2.1) we apply the *hp*-version of the BEM on quasi-uniform triangulations of Γ. Our BEM is based on Galerkin discretisations with an appropriate family of

$\mathbf{H}(\mathrm{div}_\Gamma, \Gamma)$-conforming surface elements (here, $\mathbf{H}(\mathrm{div}_\Gamma, \Gamma) := \{\mathbf{u} \in \mathbf{L}^2(\Gamma);\ \mathrm{div}_\Gamma \mathbf{u} \in L^2(\Gamma)\}$). In what follows, $h > 0$ and $p \ge 1$ will always specify the mesh parameter and a polynomial degree, respectively. For any $\Omega \subset \mathbb{R}^n$ we will denote $\rho_\Omega = \sup\{\mathrm{diam}(B);\ B \text{ is a ball in } \Omega\}$.

Let $\mathcal{T} = \{\Delta_h\}$ be a family of meshes $\Delta_h = \{\Gamma_j;\ j = 1, \ldots, J\}$. Each mesh is a partition of Γ into triangular elements Γ_j such that $\bar{\Gamma} = \cup_{j=1}^{J} \bar{\Gamma}_j$, and the intersection of any two elements $\bar{\Gamma}_j$, $\bar{\Gamma}_k$ $(j \ne k)$ is either a common vertex, an entire side, or empty.

In the following we always identify a face of the polyhedron Γ with a subdomain of $\mathbb{R}^2$. We denote $h_j = \mathrm{diam}(\Gamma_j)$ for any $\Gamma_j \in \Delta_h$. Furthermore, any element Γ_j is the image of the reference triangle $K = \{(\xi_1, \xi_2);\ 0 < \xi_1 < 1,\ 0 < \xi_2 < 1 - \xi_1\}$ under an affine mapping T_j, more precisely

$$\bar{\Gamma}_j = T_j(\bar{K}), \quad \mathbf{x} = T_j(\boldsymbol{\xi}) = B_j\, \boldsymbol{\xi} + \mathbf{b}_j,$$

where $B_j \in \mathbb{R}^{2\times 2}$, $\mathbf{b}_j \in \mathbb{R}^2$, $\mathbf{x} = (x_1, x_2) \in \bar{\Gamma}_j$, and $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \bar{K}$. Then, the Jacobian matrix of T_j is $B_j \in \mathbb{R}^{2\times 2}$, and its determinant $J_j := \det(B_j)$ satisfies the relation $|J_j| \simeq h_j^2$.

We consider a family $\mathcal{T}$ of quasi-uniform shape-regular meshes Δ_h on Γ in the sense that there exist positive constants σ_1, σ_2 independent of $h = \max\limits_j h_j$ such that for any $\Gamma_j \in \Delta_h$ and arbitrary $\Delta_h \in \mathcal{T}$ there holds

$$h_j \le \sigma_1\, \rho_{\Gamma_j}, \qquad h \le \sigma_2\, h_j. \tag{3.1}$$

Whereas the mapping T_j introduced above is used to associate scalar functions defined on the real element Γ_j and on the reference triangle K, the Piola transformation is used to transform vector-valued functions between K and Γ_j:

$$\mathbf{v} = \mathcal{M}_j(\hat{\mathbf{v}}) = \tfrac{1}{J_j} B_j \hat{\mathbf{v}} \circ T_j^{-1}, \quad \hat{\mathbf{v}} = \mathcal{M}_j^{-1}(\mathbf{v}) = J_j B_j^{-1} \mathbf{v} \circ T_j. \tag{3.2}$$

Let us introduce the needed polynomial sets. By $\mathcal{P}_p(K)$ we denote the set of polynomials of total degree $\le p$ on the reference triangle K. We will use two families of $\mathbf{H}(\mathrm{div}_\Gamma, \Gamma)$-conforming surface elements: the Raviart-Thomas (RT) and Brezzi-Douglas-Marini (BDM) elements. The corresponding spaces of degree $p \ge 1$ on the reference triangle K will be denoted as follows (see, e.g., [8, 26]):

$$\begin{aligned} \boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K) &= (\mathcal{P}_{p-1}(K))^2 \oplus \boldsymbol{\xi} \mathcal{P}_{p-1}(K); \\ \boldsymbol{\mathcal{P}}_p^{\mathrm{BDM}}(K) &= (\mathcal{P}_p(K))^2. \end{aligned}$$

We will use the unified notation $\boldsymbol{\mathcal{P}}_p(K)$ which refers to either the RT- or BDM-space on K for $p \ge 1$. Accordingly, all results in this paper are formulated in a unified way and are valid for both the RT- and BDM-based boundary element spaces defined on the triangulation of Γ. Note, however, that in some cases we will need to provide arguments separately for each type of these boundary elements.

Using transformations (3.2), we set

$$\mathbf{X}_{hp} := \{\mathbf{v} \in \mathbf{H}(\mathrm{div}_\Gamma, \Gamma);\ \mathcal{M}_j^{-1}(\mathbf{v}|_{\Gamma_j}) \in \boldsymbol{\mathcal{P}}_p(K),\ j = 1, \ldots, J\}. \tag{3.3}$$

Note that only one type of surface elements (i.e., either the RT- or BDM-elements) is used in (3.3) for all triangles Γ_j. We will denote by $N = N(h,p)$ the dimension of the discrete space $\mathbf{X}_{hp}$. One has $N \simeq h^{-2}$ for fixed p and $N \simeq p^2$ for fixed h.

The hp-version of the Galerkin BEM for the EFIE reads as: *Find* $\mathbf{u}_{hp} \in \mathbf{X}_{hp}$ *such that*

$$a(\mathbf{u}_{hp}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{X}_{hp}. \tag{3.4}$$

Due to the infinite-dimensional kernel of the div_Γ-operator, the bilinear form $a(\cdot,\cdot)$ in (2.1) is not $\mathbf{X}$-coercive, and, hence, the unique solvability of (3.4) cannot be proved by standard arguments. However, the refined analysis in [7] shows that the unique BEM-solution $\mathbf{u}_{hp} \in \mathbf{X}_{hp}$ does exist, and it converges quasi-optimally to the exact solution $\mathbf{u} \in \mathbf{X}$ of the EFIE as $N(h,p) \to \infty$. This result is formulated in the following proposition.

Proposition 3.1 [7, Theorem 1.2] *There exists* $N_0 \geq 1$ *such that for any* $\mathbf{f} \in \mathbf{X}'$ *and for arbitrary mesh-degree combination satisfying* $N(h,p) \geq N_0$ *the discrete problem* (3.4) *is uniquely solvable and the hp-version of the Galerkin BEM converges quasi-optimally, i.e.,*

$$\|\mathbf{u} - \mathbf{u}_{hp}\|_{\mathbf{X}} \leq C \inf\{\|\mathbf{u} - \mathbf{v}\|_{\mathbf{X}};\ \mathbf{v} \in \mathbf{X}_{hp}\}. \tag{3.5}$$

Here, $\mathbf{u} \in \mathbf{X}$ *is the solution of* (2.1)*,* $\mathbf{u}_{hp} \in \mathbf{X}_{hp}$ *is the solution of* (3.4)*,* $\|\cdot\|_{\mathbf{X}}$ *denotes the norm in* $\mathbf{X}$*, and* $C > 0$ *is a constant independent of* h *and* p*.*

The next theorem is the main result of this paper. It states convergence rates of the hp-BEM with quasi-uniform meshes for the EFIE. These convergence rates (in both the mesh parameter h and the polynomial degree p) are given explicitly in terms of the singularity exponents of the vector fields in (2.5)–(2.8).

Theorem 3.1 *Let* $\mathbf{u} \in \mathbf{X}$ *and* $\mathbf{u}_{hp} \in \mathbf{X}_{hp}$ *be the solutions of* (2.1) *and* (3.4)*, respectively. We assume that the source functional* $\mathbf{f}$ *in* (2.1) *is sufficiently smooth such that representation* (2.2)–(2.8) *holds for the solution* $\mathbf{u}$ *of* (2.1)*. Let* $v_0 \in V$*,* $e_0 \in E(v_0)$ *be a vertex-edge pair such that*

$$\min\{\lambda_1^{v_0} + 1/2, \lambda_2^{v_0} + 1/2, \gamma_1^{e_0}, \gamma_2^{e_0}\} = \min_{v \in V, e \in E(v)} \min\{\lambda_1^{v} + 1/2, \lambda_2^{v} + 1/2, \gamma_1^{e}, \gamma_2^{e}\}$$

with λ_i^v *and* γ_i^e *(*$i = 1,2$*) being as in* (2.5)–(2.8)*. Then for any* $h > 0$ *and for every* $p \geq \min\{\lambda_1^{v_0}, \lambda_2^{v_0}, \gamma_1^{e_0} - \frac{1}{2}, \gamma_2^{e_0} - \frac{1}{2}\}$ *there holds*

$$\|\mathbf{u} - \mathbf{u}_{hp}\|_{\mathbf{X}} \leq C \left(\frac{h}{p^2}\right)^{\min\{\lambda_1^{v_0}+1/2, \lambda_2^{v_0}+1/2, \gamma_1^{e_0}, \gamma_2^{e_0}\}} \left(1 + \log\frac{p}{h}\right)^{\beta+\nu}, \tag{3.6}$$

where

$$\beta = \begin{cases} \max\{q_1^{v_0} + s_1^{e_0} + \frac{1}{2},\ q_2^{v_0} + s_2^{e_0} + \frac{1}{2}\} & \textit{if}\ \ \lambda_i^{v_0} = \gamma_i^{e_0} - \frac{1}{2}\ \textit{for}\ i = 1,2, \\ \max\{q_1^{v_0} + s_1^{e_0} + \frac{1}{2},\ q_2^{v_0} + s_2^{e_0}\} & \textit{if}\ \ \lambda_1^{v_0} = \gamma_1^{e_0} - \frac{1}{2},\ \ \lambda_2^{v_0} \neq \gamma_2^{e_0} - \frac{1}{2}, \\ \max\{q_1^{v_0} + s_1^{e_0},\ q_2^{v_0} + s_2^{e_0} + \frac{1}{2}\} & \textit{if}\ \ \lambda_1^{v_0} \neq \gamma_1^{e_0} - \frac{1}{2},\ \ \lambda_2^{v_0} = \gamma_2^{e_0} - \frac{1}{2}, \\ \max\{q_1^{v_0} + s_1^{e_0},\ q_2^{v_0} + s_2^{e_0}\} & \textit{otherwise} \end{cases} \tag{3.7}$$

with the numbers $s_i^{e_0}$ *and* $q_i^{v_0}$ *(*$i=1,2$*) given in* (2.5) *and* (2.6)*, respectively, and*

$$\nu = \begin{cases} \frac{1}{2} & \textit{if } p = \min\{\lambda_1^{v_0}, \lambda_2^{v_0}, \gamma_1^{e_0} - 1/2, \gamma_2^{e_0} - 1/2\}, \\ 0 & \textit{otherwise.} \end{cases} \tag{3.8}$$

If $1 \le p < \min\{\lambda_1^{v_0}, \lambda_2^{v_0}, \gamma_1^{e_0} - \frac{1}{2}, \gamma_2^{e_0} - \frac{1}{2}\}$*, then for any* $h > 0$ *there holds*

$$\|\mathbf{u} - \mathbf{u}_{hp}\|_{\mathbf{X}} \le C\, h^{p+1/2}. \tag{3.9}$$

Proof. Considering enough singularity terms in representation (2.4) the function $\mathbf{u}_{\mathrm{reg}}$ in (2.3) is as regular as needed. Then, due to the quasi-optimal convergence (3.5) of the hp-BEM with quasi-uniform meshes, the assertion follows immediately from the general approximation result given in Theorem 4.1 below. □

Remark 3.1 *We have only considered meshes of triangular elements on* Γ*. If the meshes contain also shape-regular parallelogram elements (i.e., affine images of the reference square), then a priori error estimates of Theorem* 3.1 *remain valid only in the case of the RT-based BEM. This is because all the auxiliary results needed for the proof are valid in this case. This, however, is not true for the BDM-based BEM. In particular, the arguments in the proofs of Proposition* 3.1 *above and Proposition* 4.1 *below rely essentially on the fact that the involved polynomial spaces form the exact* $\mathbf{curl} - \mathrm{div}$ *sequence (on the reference element), a property the BDM-spaces fail to satisfy on the reference square (see, e.g.,* [18]*).*

Remark 3.2 *We have assumed that* Γ *is a polyhedral (closed) surface. However, all arguments in our proofs carry over only with minor modifications to the case of a piecewise plane orientable open surface* Γ*. Note that in this case there are no restrictions needed on the wave number* κ *to ensure the uniqueness of the solution to the EFIE; the strongest edge singularities in* (2.5) *have the exponents* $\gamma_i^e = \frac{1}{2}$ *(*$i = 1,2$*); the energy space for the EFIE is* $\tilde{\mathbf{H}}_0^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ *and the boundary element space* $\mathbf{X}_{hp}$ *consists of* $\mathbf{H}(\mathrm{div}_\Gamma, \Gamma)$*-conforming polynomial vector fields with normal components vanishing on* $\partial\Gamma$ *(see* [6, 4] *for details).*

4 General approximation result

In this section we prove the following general hp-approximation result for the vector field $\mathbf{u}$ given by formulas (2.2)–(2.8) with the singularity exponents γ_i^e and λ_i^v ($i = 1,2$) satisfying the minimum requirements to guarantee $\mathbf{u} \in \mathbf{X}$.

Theorem 4.1 *Let the vector field* $\mathbf{u}$ *be given by* (2.2)–(2.8) *on* Γ *with* $\gamma_1^e, \gamma_2^e > 0$ *and* $\lambda_1^v, \lambda_2^v > -\frac{1}{2}$ *for each edge* e *and every vertex* v*. Also, let* $v_0 \in V$*,* $e_0 \in E(v_0)$ *be such that*

$$\min\{\lambda_1^{v_0} + 1/2, \lambda_2^{v_0} + 1/2, \gamma_1^{e_0}, \gamma_2^{e_0}\} = \min_{v \in V, e \in E(v)} \min\{\lambda_1^v + 1/2, \lambda_2^v + 1/2, \gamma_1^e, \gamma_2^e\}$$

with λ_i^v *and* γ_i^e $(i = 1, 2)$ *being as in* (2.5)–(2.8). *Then for any* $h > 0$ *and for every* $p \geq \min\{\lambda_1^{v_0}, \lambda_2^{v_0}, \gamma_1^{e_0} - \frac{1}{2}, \gamma_2^{e_0} - \frac{1}{2}\}$, *there exists* $\mathbf{u}^{hp} \in \mathbf{X}_{hp}$ *such that*

$$\|\mathbf{u} - \mathbf{u}^{hp}\|_{\mathbf{X}} \leq C \max\Big\{h^{\min\{k,p\}+1/2}\, p^{-(k+1/2)}, \left(\frac{h}{p^2}\right)^{\min\{\lambda_1^{v_0}+1/2, \lambda_2^{v_0}+1/2, \gamma_1^{e_0}, \gamma_2^{e_0}\}} \Big(1 + \log\frac{p}{h}\Big)^{\beta+\nu}\Big\}, \tag{4.1}$$

where β *and* ν *are defined by* (3.7) *and* (3.8), *respectively.*

If $1 \leq p < \min\{\lambda_1^{v_0}, \lambda_2^{v_0}, \gamma_1^{e_0} - \frac{1}{2}, \gamma_2^{e_0} - \frac{1}{2}\}$, *then for any* $h > 0$ *there exists* $\mathbf{u}^{hp} \in \mathbf{X}_{hp}$ *such that*

$$\|\mathbf{u} - \mathbf{u}^{hp}\|_{\mathbf{X}} \leq C\, h^{\min\{k,p\}+1/2}. \tag{4.2}$$

In order to prove this theorem one needs to find discrete vector fields belonging to $\mathbf{X}_{hp}$ and approximating the smooth and singular parts of $\mathbf{u}$ such that the approximation errors satisfy the upper bounds in (4.1) and (4.2).

We start with formulating the following *hp*-approximation result for regular vector fields on Γ. This result will be used, in particular, to approximate the vector field $\mathbf{u}_{\mathrm{reg}} \in \mathbf{X}^k$ in (2.2).

Proposition 4.1 *Let* $P_{hp} : \mathbf{X} \to \mathbf{X}_{hp}$ *be the orthogonal projection with respect to the norm in* $\mathbf{X}$. *If* $\mathbf{u} \in \mathbf{X}^k$ *with* $k > 0$, *then*

$$\|\mathbf{u} - P_{hp}\mathbf{u}\|_{\mathbf{X}} \leq C\, h^{\min\{k,p\}+1/2}\, p^{-(k+1/2)}\, \|\mathbf{u}\|_{\mathbf{X}^k} \tag{4.3}$$

with a positive constant C *independent of* h, p, *and* $\mathbf{u}$.

The proof is given in [4, Theorem 4.1] for the case of RT-spaces, and it carries over without essential modifications to the case of BDM-spaces on *triangular* elements (cf. Remark 3.1).

Now, we will study approximations of the singular part $\mathbf{u}_{\mathrm{sing}}$ in representation (2.2). By (2.4)–(2.8) and due to the arguments in Remark 2.2 (i), we conclude that $\mathbf{u}_{\mathrm{sing}}$ can be written as

$$\mathbf{u}_{\mathrm{sing}} = \mathbf{curl}_\Gamma\, w + \mathbf{v} = \mathbf{curl}_\Gamma\, w + (v_1,\, v_2), \tag{4.4}$$

where $w \in H^{1/2}(\Gamma)$ and $\mathbf{v} \in \mathbf{H}^{1/2}(\Gamma)$.

Let us define the following discrete space (of continuous piecewise polynomials) over the mesh Δ_h:

$$S_{hp}(\Gamma) := \{v \in C^0(\Gamma);\ v|_{\Gamma_j} \circ T_j \in \mathcal{P}_p(K),\ j = 1, \ldots, J\}.$$

We will also need the following functions of h and p:

$$f_j(h,p) := h^{\alpha_j} p^{-2\alpha_j} (1 + \log(p/h))^{\tilde{\beta}_j + \nu_j}, \tag{4.5}$$

where $j = 1, 2$,

$$\alpha_j := \min\{\lambda_j^{v_0} + 1/2, \gamma_j^{e_0}\}, \tag{4.6}$$

$$\tilde{\beta}_j := \begin{cases} q_j^{v_0} + s_j^{e_0} + \frac{1}{2} & \text{if } \lambda_j^{v_0} = \gamma_j^{e_0} - \frac{1}{2}, \\ q_j^{v_0} + s_j^{e_0} & \text{otherwise,} \end{cases} \tag{4.7}$$

$$\nu_j := \begin{cases} \frac{1}{2} & \text{if } p = \alpha_j - \frac{1}{2}, \\ 0 & \text{otherwise,} \end{cases} \tag{4.8}$$

and the numbers $\gamma_j^{e_0}$, $\lambda_j^{v_0}$, $s_j^{e_0}$, $q_j^{v_0}$ $(j = 1, 2)$ are given in (2.5)–(2.8) for the vertex-edge pair (v_0, e_0) introduced in the formulation of Theorem 4.1.

In the next two lemmas we formulate approximation results for the vector fields $\mathbf{curl}_\Gamma\, w$ and $\mathbf{v}$ on the right-hand side of (4.4). The proofs are given in Section 5 below.

Lemma 4.1 *Let $w \in H^{1/2}(\Gamma)$ be the scalar singular function in representation (4.4). Then for any $h > 0$ and $p \geq 1$, there exists $w^{hp} \in S_{hp}(\Gamma)$ such that $\mathbf{curl}_\Gamma\, w^{hp} \in \mathbf{X}_{hp}$ and there holds*

$$\|\mathbf{curl}_\Gamma\, w - \mathbf{curl}_\Gamma\, w^{hp}\|_{\mathbf{X}} \leq \begin{cases} C\, f_1(h,p) & \text{if } \; p \geq \alpha_1 - \frac{1}{2}, \\ C\, h^{p+1/2} & \text{if } \; 1 \leq p < \alpha_1 - \frac{1}{2}, \end{cases} \tag{4.9}$$

where $f_1(h,p)$ and α_1 are defined by (4.5) and (4.6), respectively.

Lemma 4.2 *Let $\mathbf{v} \in \mathbf{H}^{1/2}(\Gamma)$ be the singular vector field in representation (4.4). Then for any $h > 0$ and $p \geq 1$, there exists $\mathbf{v}^{hp} \in \mathbf{X}_{hp}$ satisfying*

$$\|\mathbf{v} - \mathbf{v}^{hp}\|_{\mathbf{X}} \leq \begin{cases} C\, f_2(h,p) & \text{if } \; p \geq \alpha_2 - \frac{1}{2}, \\ C\, h^{p+1/2} & \text{if } \; 1 \leq p < \alpha_2 - \frac{1}{2}, \end{cases} \tag{4.10}$$

where $f_2(h,p)$ and α_2 are defined by (4.5) and (4.6), respectively.

Now we are able to prove Theorem 4.1.

Proof of Theorem 4.1. For the regular vector field $\mathbf{u}_{\mathrm{reg}} \in \mathbf{X}^k$ in (2.2) we use the orthogonal projection $P_{hp} : \mathbf{X} \to \mathbf{X}_{hp}$ with respect to the norm in $\mathbf{X}$ to define $\mathbf{u}_{\mathrm{reg}}^{hp} := P_{hp}\mathbf{u}_{\mathrm{reg}} \in \mathbf{X}_{hp}$. Then we have by Proposition 4.1

$$\|\mathbf{u}_{\mathrm{reg}} - \mathbf{u}_{\mathrm{reg}}^{hp}\|_{\mathbf{X}} \leq C\, h^{\min\{k,p\}+1/2}\, p^{-(k+1/2)}. \tag{4.11}$$

Since the singular part of decomposition (2.2) can be written as in (4.4), we use approximations $w^{hp} \in S_{hp}(\Gamma)$ and $\mathbf{v}^{hp} \in \mathbf{X}_{hp}$ from the above two lemmas to define $\mathbf{u}_{\mathrm{sing}}^{hp} := \mathbf{curl}_\Gamma\, w^{hp} + \mathbf{v}^{hp} \in \mathbf{X}_{hp}$. Then, applying the triangle inequality, we obtain by (4.9) and (4.10)

$$\|\mathbf{u}_{\mathrm{sing}} - \mathbf{u}_{\mathrm{sing}}^{hp}\|_{\mathbf{X}} \leq \begin{cases} C\, \max\{f_1(h,p),\; f_2(h,p)\} & \text{if } \; p \geq \min\{\alpha_1, \alpha_2\} - \frac{1}{2}, \\ C\, h^{p+1/2} & \text{if } \; 1 \leq p < \min\{\alpha_1, \alpha_2\} - \frac{1}{2}. \end{cases} \tag{4.12}$$

Now, we set $\mathbf{u}^{hp} := \mathbf{u}_{\mathrm{reg}}^{hp} + \mathbf{u}_{\mathrm{sing}}^{hp} \in \mathbf{X}_{hp}$. Combining estimates (4.11) and (4.12), applying the triangle inequality, and using expressions (4.5) and (4.6) for the functions $f_j(h,p)$ and the parameters α_j, respectively, we prove the desired estimates in (4.1) and (4.2). $\square$

5 Proofs of technical lemmas

In this section we prove Lemmas 4.1 and 4.2.

5.1 Proof of Lemma 4.1

Recalling Remark 2.2 (ii), we use the results of [3, 1] to find the desired piecewise polynomial $w^{hp} \in S_{hp}(\Gamma)$ such that the norm $\|w - w^{hp}\|_{H^{1/2}(\Gamma)}$ is bounded as in (4.9) (see Theorems 5.1, 5.2 in [3] and Theorem 4.1 in [1]). Then, recalling the fact that the the operator $\mathbf{curl}_\Gamma : H^{1/2}(\Gamma) \to \mathbf{H}_{\|}^{-1/2}(\Gamma)$ is continuous (see [12]), we derive the estimate in (4.9):

$$\begin{aligned} \|\mathbf{curl}_\Gamma\, w - \mathbf{curl}_\Gamma\, w^{hp}\|_{\mathbf{X}} &= \|\mathbf{curl}_\Gamma(w - w^{hp})\|_{\mathbf{H}_{\|}^{-1/2}(\Gamma)} \\ &\le C\|w - w^{hp}\|_{H^{1/2}(\Gamma)} \le \begin{cases} C\, f_1(h,p) & \text{if } \ p \ge \alpha_1 - \frac{1}{2}, \\ C\, h^{p+1/2} & \text{if } \ 1 \le p < \alpha_1 - \frac{1}{2}. \end{cases} \end{aligned}$$

It remains to prove that $\mathbf{curl}_\Gamma\, w^{hp} \in \mathbf{X}_{hp}$. In fact, it is easy to check (see [6, p. 615]) that

$$\mathcal{M}_j^{-1}(\mathbf{curl}_\Gamma\, w^{hp}|_{\Gamma_j}) = \mathbf{curl}\,(w^{hp}|_{\Gamma_j} \circ T_j), \qquad \text{where } \mathbf{curl} = \left(\tfrac{\partial}{\partial \xi_2}, -\tfrac{\partial}{\partial \xi_1}\right).$$

Hence, recalling that $w^{hp}|_{\Gamma_j} \circ T_j \in \mathcal{P}_p(K)$, we conclude that $\mathcal{M}_j^{-1}(\mathbf{curl}_\Gamma\, w^{hp}|_{\Gamma_j}) \in (\mathcal{P}_{p-1}(K))^2 \subset \boldsymbol{\mathcal{P}}_p(K)$ (this is because $(\mathcal{P}_{p-1}(K))^2$ is a subset of both $\boldsymbol{\mathcal{P}}_p^{\rm BDM}(K)$ and $\boldsymbol{\mathcal{P}}_p^{\rm RT}(K)$). Moreover, $\mathbf{curl}_\Gamma\, w^{hp} \in \mathbf{H}(\mathrm{div}_\Gamma, \Gamma)$, because $\mathrm{div}_\Gamma(\mathbf{curl}_\Gamma\, w^{hp}) \equiv 0$ on Γ. Therefore, $\mathbf{curl}_\Gamma\, w^{hp} \in \mathbf{X}_{hp}$, and the proof is finished.

5.2 Proof of Lemma 4.2

Let $\mathbf{v} = (v_1, v_2)$ be the second term in decomposition (4.4) of the singular vector field $\mathbf{u}_{\rm sing}$. Again, using the results in [3, 1], we find continuous piecewise polynomial approximations to the scalar components v_1, v_2 of $\mathbf{v}$: for any $h > 0$ and every $p \ge 1$, there exist v_1^{hp}, $v_2^{hp} \in S_{hp}(\Gamma)$ such that for $i = 1, 2$ there holds

$$\|v_i - v_i^{hp}\|_{H^{1/2}(\Gamma)} \le \begin{cases} C\, f_2(h,p) & \text{if } \ p \ge \alpha_2 - \frac{1}{2}, \\ C\, h^{p+1/2} & \text{if } \ 1 \le p < \alpha_2 - \frac{1}{2} \end{cases} \tag{5.1}$$

with positive constants $C > 0$ independent of h and p.

Let $\mathbf{v}^{hp} = (v_1^{hp}, v_2^{hp})$. Then $\mathbf{v}^{hp} \in \mathbf{H}(\mathrm{div}_\Gamma, \Gamma)$. Furthermore, in the case of BDM-elements we observe that for any Γ_j there holds

$$\mathcal{M}_j^{-1}(\mathbf{v}^{hp}|_{\Gamma_j}) = J_j B_j^{-1}(\mathbf{v}^{hp}|_{\Gamma_j}) \circ T_j \in (\mathcal{P}_p(K))^2 = \boldsymbol{\mathcal{P}}_p^{\rm BDM}(K).$$

Therefore, $\mathbf{v}^{hp} \in \mathbf{X}_{hp}$ in this case. Moreover, since $\mathbf{v} \in \mathbf{H}^{1/2}(\Gamma)$ and $v_i^{hp} \in S_{hp}(\Gamma)$, we use estimate (5.1) and the continuity of the operator $\mathrm{div}_\Gamma : \mathbf{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ to obtain

$$\begin{aligned}
\|\mathbf{v} - \mathbf{v}^{hp}\|_{\mathbf{X}} &= \|\mathbf{v} - \mathbf{v}^{hp}\|_{\mathbf{H}_\|^{-1/2}(\Gamma)} + \|\mathrm{div}_\Gamma(\mathbf{v} - \mathbf{v}^{hp})\|_{H^{-1/2}(\Gamma)} \\
&\le C\|\mathbf{v} - \mathbf{v}^{hp}\|_{\mathbf{H}^{1/2}(\Gamma)} \le C \sum_{i=1}^{2} \|v_i - v_i^{hp}\|_{H^{1/2}(\Gamma)} \\
&\le \begin{cases} C\, f_2(h,p) & \text{if } \; p \ge \alpha_2 - \frac{1}{2}, \\ C\, h^{p+1/2} & \text{if } \; 1 \le p < \alpha_2 - \frac{1}{2}. \end{cases}
\end{aligned} \tag{5.2}$$

Unfortunately, in the case of RT-elements this component-wise approximation of $\mathbf{v}$ does not work since the dimension of the RT-space on the reference triangle is smaller than the dimension of the BDM-space, and, in general, $\mathcal{M}_j^{-1}(\mathbf{v}^{hp}|_{\Gamma_j}) \notin \boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K)$, so that $\mathbf{v}^{hp} = (v_1^{hp}, v_2^{hp}) \notin \mathbf{X}_{hp}$. In this case we follow the procedure described in [3] (for the scalar case). More precisely, we use appropriate h-scaled cut-off functions and represent the vector field $\mathbf{v}$ as the sum of a singular vector field $\boldsymbol{\varphi}$ with small support in the vicinity of the edges and a sufficiently smooth vector field $\boldsymbol{\psi}$. In the rest of this subsection we will demonstrate how this procedure works in the vector case. We will give a concise step-by-step outline of the procedure referring frequently to [3] for particular error estimates and other technical details.

Step 1: decomposition of v. Let $h_0 = (\sigma_1\sigma_2)^{-1}h$ with σ_1, σ_2 from (3.1). Using the scaled cut-off functions $\chi_2^e(x_{e2}/h_0)$ and $\chi^v(r_v/h_0)$ with χ_2^e and χ^v from (2.5) and (2.6), respectively, one can decompose $\mathbf{v}$ as follows (cf. [3, eqs. (5.4), (5.21), (6.4)])

$$\mathbf{v} = \boldsymbol{\varphi} + \boldsymbol{\psi}, \tag{5.3}$$

where $\mathrm{supp}\,\boldsymbol{\varphi} \subset \cup_{v\in V} \cup_{e\in E(v)} (\bar{A}_e \cup \bar{A}_v)$, $\boldsymbol{\varphi} \in \mathbf{H}^{1/2}(\Gamma)$, $\boldsymbol{\psi} \in \mathbf{X}^m$ (see (2.3) for the notation), and $\boldsymbol{\psi}$ vanishes in small (h-dependent) neighbourhoods of each vertex and each edge of Γ. Here, A_e is the union of elements at one edge e, i.e., $\bar{A}_e := \cup\{\bar{\Gamma}_j;\ \bar{\Gamma}_j \cap e \ne \emptyset\}$ (note that the endpoints of e are not included in e), A_v is the union of elements at a vertex v, i.e., $\bar{A}_v := \cup\{\bar{\Gamma}_j;\ v \in \bar{\Gamma}_j\}$, m is sufficiently large and depends on the parameters t_2 and m_2 specified for the singularities $\mathbf{u}^v$ and $\mathbf{u}_2^{ev}$, respectively.

Step 2: approximation of $\boldsymbol{\varphi}$ for $p = 1$. If $p = 1$, then one can approximate $\boldsymbol{\varphi}$ by zero. One has $\boldsymbol{\varphi}_{hp} \equiv \mathbf{0} \in \mathbf{X}_{hp}$, and, recalling the continuity of the operator div_Γ, we derive

$$\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_{hp}\|_{\mathbf{X}} = \|\boldsymbol{\varphi}\|_{\mathbf{H}_\|^{-1/2}(\Gamma)} + \|\mathrm{div}_\Gamma\, \boldsymbol{\varphi}\|_{H^{-1/2}(\Gamma)} \le C\|\boldsymbol{\varphi}\|_{\mathbf{H}^{1/2}(\Gamma)}. \tag{5.4}$$

We will obtain h-estimates for the norms of $\boldsymbol{\varphi}$ in $\mathbf{H}^s(\Gamma)$ with $s = 0$ and $s = \frac{1}{2} + \varepsilon$ (for sufficiently small $\varepsilon > 0$) by using the fact that $\boldsymbol{\varphi}$ has a small support. First, we apply Lemma 3.1 of [2] and Lemma 3.5 of [3] to localise the norm $\|\boldsymbol{\varphi}\|_{\mathbf{H}^{1/2+\varepsilon}(\Gamma)}$ to the faces $\Gamma^f \subset \Gamma$ and to the elements $\Gamma_j \subset \Gamma^f$, respectively. Then, we use scaling on each element $\Gamma_j \subset \mathrm{supp}\,\boldsymbol{\varphi}$ (cf. [3, Lemma 3.1, eqs. (5.6), (5.11), (5.23), (6.7)]). As a result we have

$$\|\boldsymbol{\varphi}\|^2_{\mathbf{H}^s(\Gamma)} \;\le\; C \sum_{f:\,\Gamma^f \subset \Gamma} \|\boldsymbol{\varphi}\|^2_{\mathbf{H}^s(\Gamma^f)}$$

$$
\begin{aligned}
&\le C \sum_{f:\,\Gamma^f\subset\Gamma}\ \sum_{j:\,\Gamma_j\subset\Gamma^f} \Big(h^{-2s}\|\boldsymbol{\varphi}\|^2_{\mathbf{L}^2(\Gamma_j)} + |\boldsymbol{\varphi}|^2_{\mathbf{H}^s(\Gamma_j)} \Big) \\
&\le C h^{2(\alpha_2+1/2-s)}(1+\log(1/h))^{2\tilde\beta_2} \qquad \text{for } s\in\{0, 1/2+\varepsilon\}.
\end{aligned}
$$

Here, α_2 and $\tilde\beta_2$ are defined by (4.6) and (4.7), respectively. Hence, using the interpolation between $\mathbf{H}^0(\Gamma)$ and $\mathbf{H}^{1/2+\varepsilon}(\Gamma)$, we obtain by (5.4)

$$
\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_{hp}\|_{\mathbf{X}} \le C h^{\alpha_2}(1+\log(1/h))^{\tilde\beta_2}. \tag{5.5}
$$

Step 3: approximation of $\boldsymbol{\varphi}$ for $p\ge 2$. We approximate each component φ_i of $\boldsymbol{\varphi}$ by a piecewise polynomial $\varphi_i^{hp}\in S_{h,p-1}(\Gamma)$ $(i=1,2)$. Here we can use the results of [3, 1] for each type of singularity: there exist $\varphi_i^{hp}\in S_{h,p-1}(\Gamma)$ such that for $i=1,2$ there holds (cf. [3, eqs. (5.12), (5.13), (5.24), (6.9)])

$$
\|\varphi_i-\varphi_i^{hp}\|_{H^{1/2}(\Gamma)} \le C\,h^{\alpha_2}\,(p-1)^{-2\alpha_2}\,(1+\log\tfrac{p-1}{h})^{\tilde\beta_2} \le C\,h^{\alpha_2}\,p^{-2\alpha_2}\,(1+\log\tfrac{p}{h})^{\tilde\beta_2}, \tag{5.6}
$$

where α_2 and $\tilde\beta_2$ are the same as in (5.5).

Setting $\boldsymbol{\varphi}^{hp}=(\varphi_1^{hp},\varphi_2^{hp})$ we observe that $\boldsymbol{\varphi}^{hp}\in\mathbf{H}(\mathrm{div}_\Gamma,\Gamma)$, and for any element Γ_j one has

$$
\mathcal{M}_j^{-1}(\boldsymbol{\varphi}^{hp}|_{\Gamma_j}) = J_j B_j^{-1}(\boldsymbol{\varphi}^{hp}|_{\Gamma_j})\circ T_j \in (\mathcal{P}_{p-1}(K))^2 \subset \boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K).
$$

Hence, $\boldsymbol{\varphi}^{hp}\in\mathbf{X}_{hp}$. Moreover, since $\boldsymbol{\varphi}\in\mathbf{H}^{1/2}(\Gamma)$ and $\varphi_i^{hp}\in S_{h,p-1}(\Gamma)$ for $i=1,2$, we estimate by analogy with (5.4) and using (5.6):

$$
\begin{aligned}
\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_{hp}\|_{\mathbf{X}} &\le C\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_{hp}\|_{\mathbf{H}^{1/2}(\Gamma)} \\
&\le C\sum_{i=1}^{2}\|\varphi_i-\varphi_i^{hp}\|_{H^{1/2}(\Gamma)} \le C\,h^{\alpha_2}\,p^{-2\alpha_2}\,(1+\log\tfrac{p}{h})^{\tilde\beta_2}.
\end{aligned} \tag{5.7}
$$

Comparing (5.5) and (5.7) we observe that one can use estimate (5.7) also in the case $p=1$.

Step 4: approximation of $\boldsymbol{\psi}$. Recalling that $\boldsymbol{\psi}\in\mathbf{X}^m$ we apply Proposition 4.1: there exists $\boldsymbol{\psi}^{hp}\in\mathbf{X}_{hp}$ such that

$$
\|\boldsymbol{\psi}-\boldsymbol{\psi}^{hp}\|_{\mathbf{X}} \le C\,h^{\min\{k,p\}+1/2}\,p^{-(k+1/2)}\,\|\boldsymbol{\psi}\|_{\mathbf{X}^k}, \qquad 0\le k\le m. \tag{5.8}
$$

We also recall that $\boldsymbol{\psi}$ vanishes in the h-neighbourhood of all edges and vertices of Γ. Therefore, having explicit expressions of the singular vector field $\mathbf{v}$ (and thus, of $\boldsymbol{\psi}$) on each face $\Gamma^f\subset\Gamma$, one can derive upper bounds for the norms $\|\boldsymbol{\psi}\|_{\mathbf{H}^k(\Gamma^f)}$ and $\|\mathrm{div}_{\Gamma^f}\boldsymbol{\psi}\|_{H^k(\Gamma^f)}$ in terms of the mesh size h, the parameter k, and the singularity exponents. For the norm of $\boldsymbol{\psi}$ this can be done component-wise using the same calculations as in [3] (see, e.g., inequalities (5.15) and (6.10) therein):

$$
\|\boldsymbol{\psi}\|_{\mathbf{H}^k(\Gamma^f)} \le C\,h^{\alpha_2+1/2-k}\,(\log(1/h))^{\tilde\beta_2+\tilde\nu_2}, \qquad \alpha_2+\tfrac12\le k\le m, \tag{5.9}
$$

where α_2 and $\tilde{\beta}_2$ are defined by (4.6) and (4.7), respectively, $\tilde{\nu}_2 = \frac{1}{2}$ if $k = \alpha_2 + \frac{1}{2}$ and $\tilde{\nu}_2 = 0$ otherwise.

To estimate the norm $\|\mathrm{div}_{\Gamma^f}\boldsymbol{\psi}\|_{H^k(\Gamma^f)}$ we observe that the operator div_{Γ^f} reduces all singularity exponents by one (while preserving the structure of the corresponding singularity). Then, using similar calculations as indicated above, we obtain

$$\|\mathrm{div}_{\Gamma^f}\,\boldsymbol{\psi}\|_{H^k(\Gamma^f)} \le C\,h^{\alpha_2-1/2-k}\,(\log(1/h))^{\tilde{\beta}_2+\bar{\nu}_2}, \qquad \alpha_2 - \tfrac{1}{2} \le k \le m, \tag{5.10}$$

where α_2, $\tilde{\beta}_2$ are the same as in (5.9), whereas $\bar{\nu}_2 = \frac{1}{2}$ if $k = \alpha_2 - \frac{1}{2}$ and $\bar{\nu}_2 = 0$ otherwise.

By (5.8)-(5.10) we conclude that

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}^{hp}\|_{\mathbf{X}} \le C\,h^{\min\{k,p\}+\alpha_2-k}\,p^{-(k+1/2)}\,(\log(1/h))^{\tilde{\beta}_2+\bar{\nu}_2}, \qquad \alpha_2 - \tfrac{1}{2} \le k \le m \tag{5.11}$$

with the same α_2, $\tilde{\beta}_2$, and $\bar{\nu}_2$ as in (5.10).

Let $p > 2\alpha_2 - \frac{1}{2}$. Since m is large enough, we can select an integer k satisfying

$$2\alpha_2 - \tfrac{1}{2} < k \le \min\{m, p\}.$$

Then $\min\{k,p\} = k$, and $p^{-(k+1/2)} \le p^{-2\alpha_2}$.

If $\alpha_2 - \frac{1}{2} < p \le 2\alpha_2 - \frac{1}{2}$ (i.e., p is bounded), we choose an integer $k \in (\alpha_2 - \frac{1}{2}, p]$, and if $p = \alpha_2 - \frac{1}{2}$, then we take $k = p = \alpha_2 - \frac{1}{2}$. In both these cases $\min\{k,p\} = k$, and $p^{-(k+1/2)} \le C\,p^{-2\alpha_2}$.

Thus, for any $p \ge \alpha_2 - \frac{1}{2}$, selecting k as indicated above we find by (5.11)

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}^{hp}\|_{\mathbf{X}} \le C\,h^{\alpha_2}\,p^{-2\alpha_2}\,(\log(1/h))^{\tilde{\beta}_2+\nu_2} \tag{5.12}$$

with α_2, $\tilde{\beta}_2$, and ν_2 being defined by (4.6), (4.7), and (4.8), respectively.

Step 5: approximation of $\mathbf{v} = \boldsymbol{\varphi} + \boldsymbol{\psi}$. Let us define $\mathbf{v}^{hp} := \boldsymbol{\varphi}^{hp} + \boldsymbol{\psi}^{hp} \in \mathbf{X}_{hp}$, where $\boldsymbol{\varphi}^{hp}$ and $\boldsymbol{\psi}^{hp}$ are approximations constructed above (see Steps 2–4). Then combining estimates (5.7) and (5.12) and using the triangle inequality we prove (4.10) in the case $p \ge \alpha_2 - \frac{1}{2}$.

It remains to consider the case $1 \le p < \alpha_2 - \frac{1}{2}$. In this case one does not need decomposition (5.3). Observe that for each face Γ^f one has $\mathrm{div}_{\Gamma^f}\,\mathbf{v} \in H^k(\Gamma^f)$ with $1 \le k < \alpha - \frac{1}{2}$. Therefore, $\mathbf{v} \in \mathbf{X}^k$ with $1 \le k < \alpha - \frac{1}{2}$, and applying Proposition 4.1 we find $\mathbf{v}^{hp} \in \mathbf{X}_{hp}$ satisfying

$$\|\mathbf{v} - \mathbf{v}^{hp}\|_{\mathbf{X}} \le C\,h^{\min\{k,p\}+1/2}\,\|\mathbf{v}\|_{\mathbf{X}^k}.$$

Hence, selecting $k \in [p, \alpha - \frac{1}{2})$ we arrive at the desired upper bound in (4.10), and the proof is finished.

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