

PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE ESCUELA DE INGENIERÍA

# MODELING OF FLASH FLOODS IN THE ANDEAN FOOTHILLS: REDUCING THE UNCERTAINTY ASSOCIATED WITH THE SEDIMENT CONCENTRATIONS

# MARÍA TERESA CONTRERAS VARGAS

Thesis submitted to the Office of Research and Graduate Studies in partial fulfillment of the requirements for the degree of Master of Science in Engineering

Advisor: CRISTIÁN ESCAURIAZA

Santiago de Chile, May, 2016

C MMXVI, María Teresa Contreras Vargas



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Gratefully to my parents and sibling

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#### ABSTRACT

Rain-induced flash floods are common events in regions near the Andes mountain range. Rapid urban development in this region, combined to the changing climate and ENSO effects have resulted in an alarming proximity of flood-prone streams to densely populated areas in the Andean foothills, increasing the risk for cities and infrastructure. Simulations of rapid floods in these watersheds are particularly challenging, due to the complex morphology, the insufficient hydrometeorological data, and the uncertainty posed by the variability of sediment concentration. High concentrations produced by hillslope erosion and rilling by the overland flow in areas with steep slopes and low vegetational covering, can change significantly the dynamics of the flow as the flood propagates in the channel. In this investigation, we develop a two-dimensional finite-volume numerical model of the non-linear shallow water equations, coupled with the mass conservation of sediment, considering the density effects and the changes on the rheology of the flow. We carry out simulations to evaluate the effects of the sediment concentration on the floods in the Quebrada de Ramón watershed, an Andean basin in central Chile. We simulate a confluence and a total length of the channel of 10.4 km, with the same water hydrographs and different combinations of sediment concentrations in the tributaries. Our results show that the most important effects on the flood propagation are observed in the range of concentrations from 0% to 20%. By comparing simulations with clear-water and a concentration of 60%, we find that the maximum flow depth at different locations along the channel increases by 38%, and the total 2D flooded area is 75% larger in the latter case. Simulations also show that variables such as the arrival time of the peak flow, and the shape of the hydrograph are not significantly affected by the sediment concentration, and depend mostly on the steep channel morphology. Through this work we provide a framework for future studies aimed at improving hazard assessment, urban planning, and early warning systems in urban areas near mountain streams with limited data, and affected by rapid flood events.

**Keywords**: Floods; shallow water equations; numerical model; Andean watershed; sediment concentration

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#### RESUMEN

Crecidas rápidas inducidas por precipitación son eventos comunes en regiones cercanas a la Cordillera de los Andes. El rápido desarrollo urbano en esta región, combinado al cambio climático y los efectos de los eventos ENSO han resultado en una alarmante proximidad de los cauces propensos a inundaciones, a áreas densamente pobladas en la precordillera Andina, incrementando el riesgo de ciudades e infraestructura. Simulaciones de crecidas rápidas en éstas cuencas son particularmente desafiantes, debido a la compleja morfología, la insuficiencia de datos hidrometeorológicos, y la incertidumbre generada por la variabilidad en la concentración de sedimentos. Altas concentraciones producidas por la erosión de las laderas y el escurrimiento de flujos superficiales en áreas con pendientes empinadas y baja cobertura vegetal, pueden cambiar la dinámica del flujo a medida que la inundación se propaga en el canal. En ésta investigación, nosotros desarrollamos un modelo bidimensional en volúmenes finitos de las ecuaciones no lineares de flujos someros, acopladas con la conservación de la masa de sedimentos, considerando los efectos de la densidad y los cambios en la reología del flujo. Nosotros llevamos a cabo simulaciones para evaluar los efectos de la concentración de sedimentos en inundaciones en la Quebrada de Ramón, una cuenca Andina en Chile central. Nosotros simulamos una confluencia y un largo total del canal de 10.4 km, con el mismo hidrograma de agua clara y diferentes concentraciones de sedimentos en los tributarios. Nuestros resultados muestran que los efectos más importantes en la propagación de la inundación son observados en el rango de concentraciones desde 0% a 20%. Comparando simulaciones con agua clara y una concentración de 60%, nosotros encontramos que la máxima profundidad del flujo en diferentes locaciones a lo largo del canal incrementa en un 38%, y el área total inundada es un 75% más grande en el último caso. Simulaciones también muestran que variables tales como el tiempo de arribo del caudal máximo, y la forma del hidrograma no son significativamente afectados por la concentración de sedimentos, y depende mayoritariamente de la empinada morfología del canal. Mediante este trabajo nosotros proveemos un

marco de referencia para futuros estudios dirigidos a mejorar la evaluación del peligro, la planificación urbana, y los sistemas de alerta temprana en áreas urbanas cercanas a ríos de montaña con datos limitados, y afectados por eventos de crecidas rápidas.

Palabras Claves: Inundaciones; ecuaciones de aguas someras; modelo numérico; cuenca Andina; concentración de sedimento

#### 1. INTRODUCTION

Flash floods with high sediment concentrations are common natural events in mountain rivers, which generate hazards in cities and other smaller human communities located near river channels (European Environmental Agency (EEA), 2005; Wilby et al., 2008). In spite of the continued efforts to provide structural and non-structural measures to control flood hazards in general, economical losses have increased in recent decades (Slater et al., 2015), and flood risks and vulnerability associated with various economic, political, and social processes are also expected to increase in the future due to climate change and urban growth (Blaikie et al., 2004; Pelling, 2003; Bankoff et al., 2004).

The spatial and temporal distribution of precipitation, the morphology of the drainage basin, soil properties, and vegetation characteristics, naturally influence the magnitude and frequency of floods and sediment transport. Anthropogenic factors also affect the volume and peak discharges of floods in mountain rivers. Climate models predict a larger frequency of intense precipitation events and cyclonic weather systems that will increase the vulnerability in many mountainous regions in the future (Sanders, 2007; Arnell & Gosling, 2014; Boers et al., 2014). An amplification of the flood hazards is also expected due to the continued expansion of cities located in floodplains (Hirabayashi et al., 2013; Jongman et al., 2012), accelerated urbanization processes (Schubert et al., 2008), lack of urban planning (Rugiero & Wyndham, 2013), and changes in land-use and cover (Kundzewicz et al., 2014).

The effectiveness to assess flood hazards and to design strategies aimed at reducing potential damages caused by flooding are closely related to the understanding of the dynamics of the flow in real conditions. Physical models and experiments have provided relevant insights on the flow physics of flash floods in extreme conditions (e.g. Testa et al., 2007). Field-based and experimental research over complex topography, however, require large facilities with advanced instrumentation to provide high-resolution measurements that are also limited by the spatio-temporal scales at which rapid floods occur. Numerical models, on the other hand, have also become fundamental tools to advance our understanding on the dynamics of floods, evaluating complex scenarios and predicting water depths and flow velocities in arbitrary geometries (Siviglia & Crosato, 2016). Simulations yield detailed information on the flood dynamics, which is sometimes experimentally inaccessible or cannot be directly measured in the field. They can also complement measurements, becoming effective tools for urban planning and for designing early warning systems in flood events (Mignot et al., 2006; Schubert & Sanders, 2012).

In most hydrodynamic models to simulate flood propagation, the nonlinear shallow water equations (NSWE) or Saint Venant equations are employed to describe the dynamics of the flow in homogeneous and incompressible fluids. They are obtained by vertically averaging the three-dimensional Navier-Stokes equations, assuming a hydrostatic pressure distribution, resulting in a set of horizontal two-dimensional (2D) hyperbolic conservation laws that describe the evolution of the water depth and depth-averaged velocities in space and time. In flows where discontinuities or rapid wet-dry interfaces develop, numerical models employ Godunov-type of formulations, solving a Riemann problem at the interfaces of the elements of the discretization (Anastasiou & Chan, 1997; Toro, 2001).

The development of efficient and accurate numerical models to simulate flash floods, however, is far from trivial, since multiple factors control the dynamics of the flow. Especially in mountainous regions, where rivers are characterized by three important features that complicate their representation: (1) Complex bathymetries and steep slopes produce rapid changes on velocities and water depths, formation of bores, and wet-dry interfaces; (2) Large sediment concentrations affect directly the flow hydrodynamics by introducing additional stresses that alter the momentum balance of the instantaneous flow; and (3) Lack of accurate field data due to the difficulties on measuring hydrometeorological variables in high-altitude environments, with difficult access, and during episodes of severe weather.

The Andes mountains in South America incorporate all these characteristics, and they have been the scenario of many recent events with catastrophic consequences, leaving a

significant human toll and economic losses (Wilcox et al., 2016). The region is characterized by rapid floods with high concentrations of sediment, generally produced by hillslope erosion and rilling by the overland flow in areas with steep slopes and low vegetational covering. Additional factors, such as the storms caused by the South-American monsoon (Zhou & Lau, 1998), and El Niño-Southern Oscillation (ENSO) can generate anomalous heavy rainfall (Holton et al., 1989; Díaz & Markgraf, 1992), producing great volume of liquid precipitation and significant erosion and sediment transport in the flow.

High sediment concentrations during floods cause additional stresses produced by the increase of the density and viscosity of the water-sediment mixture. In cases with homogeneous fluids and no significant sediment transport, numerical models only consider a flow resistance term due to the bed shear stresses, represented by coefficients derived for uniform flows, such as Manning, Chezy, and the Darcy-Weisbach friction factor. In hyperconcentrated flows, on the other hand, models need to account for the internal stresses that emerge from the particle-flow and particle-particle interactions in the sediment-laden flow. These stresses transform the rheological behavior of the mixture, represented by additional terms of momentum transfer in the governing equations. A wide variety of rheological models have been proposed depending on the sediment properties and concentration (see for instance Bingham, 1922; Bagnold, 1954; O'Brien & Julien, 1985, among others). These models are based on empirical equations that have been estimated from laboratory studies (Parsons et al., 2001), or back-calibrated from past events (Naef et al., 2006).

Understanding and quantifying the dynamics of hyperconcentrated floods in mountainous regions is thus critical to designing flood hazard mitigation strategies and control measures. The main objective of this investigation is to gain fundamental insights on the effects of high sediment concentrations on the propagation of floods in an Andean watershed. We develop a 2D finite-volume numerical model of the NSWE, building on the work of Guerra et al. (2014), which incorporates the effects of the sediment load on the dynamics of the flow over natural terrains and complex geometries. We carry out simulations of flows with different sediment concentrations in the two main tributaries of the *Quebrada de Ramón* watershed, located at the foothills of the Andes mountain range, to the east of Santiago, Chile, where part of the city occupies the lower section of the river basin. From the simulations we evaluate the effects of the sediment load on the evolution of the flow depth and velocity, and we link the response of the river channel to the variations of sediment concentration. The analysis provides quantitative information of the hyperconcentrated flood propagation, including the changes on the total flooded area and momentum at cross-sections of the flow, reducing the uncertainty associated with flooding in these mountain rivers.

The thesis is organized as follows: In section 2 we provide a brief description of the characteristics of the Andes mountain range and the area of study, the *Quebrada de Ramón* watershed in central Chile, where the confluence of two main channels is the most important morphological feature of the rapid floods that reach the human settlements and city infrastructure. The governing equations of the coupled flow and sediment transport model, and the numerical methods employed in this investigation are explained in section 3. In section 4 we validate the model by comparing the numerical results with analytical solutions and experiments of shallow-mudflows where spatial and temporal gradients of sediment concentration are important. Section 5 contains the analysis of the simulations in the *Quebrada de Ramón* watershed for a large flood, considering different concentrations in each of the tributaries. We study the consequences of the hyperconcentration on the dynamics of the flow, the total momentum in the cross-sections of the river, and local water depths and velocities. Finally, in section 6 we summarize the findings of this investigation and outline topics for future research.

## 2. THE QUEBRADA DE RAMÓN WATERSHED

The Andean foothills are especially susceptible to flash floods with high sediment loads. The complex morphology of the basins, the proximity to urban areas, and the lack of monitoring are some of the factors that increase the flood hazards in the region. In addition, storms that are influenced by ENSO cause significant rainfall with warm temperatures, elevating the 0°C isotherm. These warm conditions generate a large total contributing area of the watershed that receives mostly liquid precipitation, increasing the runoff volume, the peak discharges, and creating high flow velocities and sediment production. Similar conditions are generated by rising local temperatures associated with urban heat island effects due to the environmental degradation of the Andean foothills, and the decrease of vegetation productivity, biomass, and soil moisture (Romero & Ordenes, 2004).

A series of floods produced by these factors have been recorded throughout history in the Andes. Some of the most destructive events have been generated by landslides triggered by rainfall, which have dammed rivers and later collapsed in catastrophic events, such as the *Mayunmarca* in the Mantaro river in Perú in 1974, and the Paute river in Ecuador in 1993 (Voight, 1978; Harden, 2001). High-intensity precipitation during warm storms has also produced rapid hyperconcentrated floods with many fatalities and economic losses in Medellín, Colombia, in 1987 (Aguilar et al., 2008), and in the Atacama desert in northern Chile (Sepúlveda et al., 2006; Wilcox et al., 2016).

To understand the effects of the sediment concentration on the flood hydrodynamics in the Andes, in this investigation we select the *Quebrada de Ramón* watershed as the case of study. This basin has a total area of 38.5 km<sup>2</sup> and it is located in central Chile, to the east of the city of Santiago as shown in Figure 2.1, with elevations that range from 800 to 3,400 m asl (Sepúlveda et al., 2006). The Quebrada de Ramón stream drains the north section of the watershed, flowing in N-S direction in the headwaters, with a mean slope around  $20-30^{\circ}$ . In the middle zone, between 1,450 and 1,650 m asl, the flow is oriented in E-W direction, with a mean slope of nearly  $10^{\circ}$ . The confluence of the Quebrada de

Ramón with the *Quillayes* stream, which is oriented S-N with a mean slope of 20°, is the main morphological feature of the channel located at an elevation of about 1,200 m asl. Downstream of the confluence the mean slope of the stream decreases to  $5-10^{\circ}$  (Lara, 2007).



Figure 2.1. Satellite image of the Andes in central Chile, next to the city of Santiago. The *Quebrada de Ramón* watershed is highlighted in red (Google Earth, 2016).

This watershed has many interesting features due to the proximity of the city and growing urbanization. In the lower section, the natural channel has been covered by the city, and the flow is now diverted into a concrete channel with a design discharge of 20 m<sup>3</sup>/s. The size of the basin and the range of elevations generate high velocities and hyperconcentrated flows during flood events, and the hydrological response of the catchment is very sensitive to the location of the 0°C isotherm elevation, which is usually located

between 1,500 and 2,500 m asl. Small variations of the freezing level position originate large differences in the contributing area.

The most recent and catastrophic flood in this watershed occurred on May 3, 1993. A flood was produced by a warm storm associated to the ENSO phenomenon (Garreaud & Rutllant, 2009), which generated heavy rainfall over the Andean foothills, reaching a total of 30 mm in 16 h, which corresponds to a return period of 25 years of the precipitation, with a maximum intensity of 9.8 mm/h. The temperature increment raised the 0°C isotherm from its average altitude to 4,000 m asl, which has a return period of 10 years (Garreaud & Rutllant, 2009). As a result, a large debris flow reached the city in a few minutes, leaving 26 people dead, 8 missing, and damaging residential areas with a total cost that reached US\$5 million (ONEMI, 1995).

In the following sections we describe and validate the numerical model that couples the flow hydrodynamics with the sediment transport, to simulate a 50-year flood whose peak flow exceeds considerably the capacity of the channelization in the city. The model is tested with different sediment concentrations to evaluate the consequences of high sediment loads on the inundation process and flood propagation.

#### 3. NUMERICAL MODEL FOR HYPERCONCENTRATED FLOWS

#### 3.1. Governing Equations

Rapid floods over the complex topography of mountainous regions are commonly affected by high sediment concentrations, which change the rheology of the flow. By assuming that the mixture preserves the Newtonian constitutive relation between stress and rate of strain, the NSWE equations can be modified to account for the heterogeneous distribution of sediment in space (Loose et al., 2005; Michoski et al., 2013).

The NSWE model implemented in this investigation has the following assumptions: (i) hydrostatic pressure distribution; (ii) negligible vertical velocities; (iii) vertically-averaged horizontal velocities; (iv) horizontal heterogeneous fluid density; (v) homogeneous density in the vertical direction; and (vi) fixed bed. The momentum sources and sinks consider the gravity term, and the bed resistance and rheology of the mixture, including the yield stress, Mohr-Coulomb, viscous stresses, and turbulent and dispersive stresses, as discussed in section 3.2.

If we denote the dimensional variables of the flow with a hat (<sup>^</sup>), the set of equations can be written as follows,

$$\frac{\partial \hat{\rho}\hat{h}}{\partial \hat{t}} + \frac{\partial \hat{\rho}\hat{h}\hat{u}}{\partial \hat{x}} + \frac{\partial \hat{\rho}\hat{h}\hat{v}}{\partial \hat{y}} = 0$$
(3.1)

$$\frac{\partial \hat{\rho} \hat{h} \hat{u}}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}} \left( \hat{\rho} \hat{u}^2 \hat{h} + \frac{1}{2} \hat{\rho} g \hat{h}^2 \right) + \frac{\partial \hat{\rho} \hat{u} \hat{v} \hat{h}}{\partial \hat{y}} = -\hat{\rho} g \hat{h} \frac{\partial \hat{z}}{\partial \hat{x}} - \hat{\tau}_{\hat{x}}$$
(3.2)

$$\frac{\partial \hat{\rho} \hat{h} \hat{v}}{\partial \hat{t}} + \frac{\partial \hat{\rho} \hat{u} \hat{v} \hat{h}}{\partial \hat{x}} + \frac{\partial}{\partial \hat{y}} \left( \hat{\rho} \hat{v}^2 \hat{h} + \frac{1}{2} \hat{\rho} g \hat{h}^2 \right) = -\hat{\rho} g \hat{h} \frac{\partial \hat{z}}{\partial \hat{y}} - \hat{\tau}_{\hat{y}}$$
(3.3)

$$\frac{\partial C\hat{h}}{\partial \hat{t}} + \frac{\partial C\hat{h}\hat{u}}{\partial \hat{x}} + \frac{\partial C\hat{h}\hat{v}}{\partial \hat{y}} = 0$$
(3.4)

where  $\hat{h}$  is the flow depth, and  $\hat{u}$  and  $\hat{v}$  are the depth-averaged velocities in the cartesian coordinate directions  $\hat{x}$  and  $\hat{y}$ , respectively. The bed elevation is denoted as  $\hat{z}$ , g is the acceleration of gravity,  $\hat{t}$  represents the time,  $\hat{\rho}$  is the density of the water-sediment mixture, C is the volumetric concentration of sediment, and  $\hat{\tau}_x$  and  $\hat{\tau}_y$  are the total stresses.

To solve the system of equations with the conservative variables  $\hat{h}$ ,  $\hat{h}\hat{u}$ ,  $\hat{h}\hat{v}$  and  $\hat{h}C$ , the density of the mixture  $\hat{\rho}$  is replaced by  $\hat{\rho} = C\rho_s + (1 - C)\rho_w$ , where  $\rho_w$  is the water density and  $\rho_s$  the sediment density. Therefore, the conservative form of the equations can be written as:

$$\hat{Q}_{,\hat{t}} + \hat{F}(\hat{Q})_{,\hat{x}} + \hat{G}(\hat{Q})_{,\hat{y}} = \hat{S}_B(\hat{Q}) + \hat{S}_S(\hat{Q}) + \hat{S}_C(\hat{Q})$$
(3.5)

where  $\hat{Q}$  is the vector that contains the conservative hydrodynamic variables, and  $\hat{F}$  and  $\hat{G}$  are the flux vectors in each coordinate direction. The term  $\hat{S}_B$  represents to the bed slope, and  $\hat{S}_S$  correspond to the bed and internal stresses of the flow. The components of this equation 3.5, are expressed as follows:

$$\hat{Q} = \begin{pmatrix} \hat{h} \\ \hat{h}\hat{u} \\ \hat{h}\hat{v} \\ \hat{h}\hat{v} \\ \hat{C}\hat{h} \end{pmatrix}, \hat{F}(\hat{Q}) = \begin{pmatrix} \hat{h}\hat{u} \\ \hat{u}^{2}\hat{h} + \frac{1}{2}g\hat{h}^{2} \\ \hat{u}\hat{v}\hat{h} \\ \hat{C}\hat{h}\hat{u} \end{pmatrix}, \hat{G}(\hat{Q}) = \begin{pmatrix} \hat{h}\hat{v} \\ \hat{u}\hat{v}\hat{h} \\ \hat{v}^{2}\hat{h} + \frac{1}{2}g\hat{h}^{2} \\ \hat{C}\hat{h}\hat{v} \end{pmatrix},$$

$$\hat{S}_{B}(\hat{Q}) = \begin{pmatrix} 0 \\ -g\hat{h}\frac{\partial\hat{z}}{\partial\hat{x}} \\ -g\hat{h}\frac{\partial\hat{z}}{\partial\hat{y}} \\ 0 \end{pmatrix}, \hat{S}_{S}(\hat{Q}) = \begin{pmatrix} 0 \\ -\hat{S}_{\hat{x}} \\ -\hat{S}_{\hat{y}} \\ 0 \end{pmatrix}, \hat{S}_{C}(\hat{Q}) = \begin{pmatrix} 0 \\ -\frac{1}{2}g\left(\frac{\rho_{s}-\rho_{w}}{\hat{\rho}}\right)\hat{h}^{2}\frac{\partial C}{\partial\hat{x}} \\ -\frac{1}{2}g\left(\frac{\rho_{s}-\rho_{w}}{\hat{\rho}}\right)\hat{h}^{2}\frac{\partial C}{\partial\hat{y}} \\ 0 \end{pmatrix}$$
(3.6)

In the vector of source terms  $\hat{S}_C$  of the coupled model presented in equation 3.5, there is a term that does not appear in the conventional NSWE, which incorporates the effects of the spatial gradients of sediment concentration. This term is significant in rapid flows with large concentration gradients, such as dam-break with sediment-laden debris flows (Cao et al., 2004), and in cases with important interactions of clear water and hyperconcentrated flows (i.e. lahars).

Here we follow the same procedure outlined in Guerra et al. (2014), expressing the equations in non-dimensional form using a velocity scale  $\mathcal{U}$ , a scale for the water depth  $\mathcal{H}$ , and a horizontal length scale of the flow  $\mathcal{L}$ , which characterize the flow. In this case,

two non-dimensional parameters appear in the equations, the relative density between the sediment and water  $s = \rho_s / \rho_w$ , and the Froude number  $Fr = U / \sqrt{gH}$ . Therefore the set of dimensionless equations can be written as follows,

$$Q_{,t} + F(Q)_{,x} + G(Q)_{,y} = S_B(Q) + S_S(Q) + S_C(Q)$$
(3.7)

where the vectors correspond to the following expressions:

$$Q = \begin{pmatrix} h \\ hu \\ hv \\ Ch \end{pmatrix}, F(Q) = \begin{pmatrix} hu \\ u^2h + \frac{1}{2Fr^2}h^2 \\ uvh \\ uCh \end{pmatrix}, G(Q) = \begin{pmatrix} hv \\ uvh \\ v^2h + \frac{1}{2Fr^2}h^2 \\ vCh \end{pmatrix},$$

$$S_B(Q) = \begin{pmatrix} 0 \\ \frac{-h}{Fr^2}\frac{\partial z}{\partial x} \\ \frac{-h}{Fr^2}\frac{\partial z}{\partial y} \\ 0 \end{pmatrix} S_S(Q) = \begin{pmatrix} 0 \\ -S_x \\ -S_y \\ 0 \end{pmatrix} S_C(Q) = \begin{pmatrix} 0 \\ \frac{-h^2}{2Fr^2}\left(\frac{s-1}{C(s-1)+1}\right)\frac{\partial C}{\partial x} \\ \frac{-h^2}{2Fr^2}\left(\frac{s-1}{C(s-1)+1}\right)\frac{\partial C}{\partial y} \\ 0 \end{pmatrix}$$
(3.8)

To adapt the computational domain to the complex arbitrary topography in mountainous watersheds, we use a boundary fitted curvilinear coordinate system, denoted by the coordinates  $(\xi, \eta)$ . Through this transformation we can have a better resolution in zones of interest and an accurate representation of the boundaries. We perform a partial transformation of the equations, maintaining the cartesian components in the vector Q, such that the system of equations is written as follows,

$$\frac{\partial Q}{\partial t} + J\frac{\partial F}{\partial \xi} + J\frac{\partial G}{\partial \eta} = S_b(Q) + S_S(Q) + S_C(Q)$$
(3.9)

where the Jacobian of the coordinate transformation J is expressed in terms of the metrics  $\xi_x$ ,  $\xi_y$ ,  $\eta_x$  and  $\eta_y$ , such that  $J = \xi_x \eta_y - \xi_y \eta_x$  (see Lackey & Sotiropoulos, 2005; Guerra et

al., 2014, for details). The fluxes F and G are expressed as follows,

$$F = \frac{1}{J} \begin{pmatrix} hU^{1} \\ uhU^{1} + \frac{1}{2Fr^{2}}h^{2}\xi_{x} \\ vhU^{1} + \frac{1}{2Fr^{2}}h^{2}\xi_{y} \\ ChU^{1} \end{pmatrix}, G = \frac{1}{J} \begin{pmatrix} hU^{2} \\ uhU^{2} + \frac{1}{2Fr^{2}}h^{2}\eta_{x} \\ vhU^{2} + \frac{1}{2Fr^{2}}h^{2}\eta_{y} \\ ChU^{2} \end{pmatrix}$$
(3.10)

and source vectors associated to the bed slope, the shear stress and the spacial gradient of the sediment concentration, respectively,

$$S_{b}(Q) = \begin{pmatrix} 0 \\ \frac{-h(z_{\xi}\xi_{x} + z_{\eta}\eta_{x})}{Fr^{2}} \\ \frac{-h(z_{\xi}\xi_{y} + z_{\eta}\eta_{y})}{Fr^{2}} \\ 0 \end{pmatrix} S_{S}(Q) = \begin{pmatrix} 0 \\ -S_{x} \\ -S_{y} \\ 0 \end{pmatrix}$$

$$S_{C}(Q) = \begin{pmatrix} 0 \\ \frac{-h^{2}(C_{\xi}\xi_{x} + C_{\eta}\eta_{x})}{2Fr^{2}} \left(\frac{s-1}{C(s-1)+1}\right) \\ \frac{-h^{2}(C_{\xi}\xi_{y} + C_{\eta}\eta_{y})}{2Fr^{2}} \left(\frac{s-1}{C(s-1)+1}\right) \\ 0 \end{pmatrix}$$
(3.11)

where  $U^1$  y  $U^2$  represent the contravariant velocity components defined as  $U^1 = u\xi_x + v\xi_y$ and  $U^2 = u\eta_x + v\eta_y$ , respectively.

#### **3.2. Rheological Model**

The classification of gravity-driven flows with higher concentrations usually depend on the rheology of the mixture, sediment size distribution, and sediment composition. Depending on these characteristics, the flows can vary from nearly dry landslides to water flow, with intermediate conditions such as debris flows, mudflows, and mud floods (see Julien & León, 2000; Naef et al., 2006, for details). The rheological behavior that determines the magnitude of the momentum losses is incorporated in additional source terms of the hydrodynamic model previously presented in equation 3.11. As summarized by Ancey (2007), gravity-driven flows can consider rheological models such as Bagnold, Bingham, Voellmy, or Coulomb, depending on the assumptions of the effects of the particles on the dynamics of the flow. In our numerical model we implement the quadratic shear stress model developed by O'Brien and Julien (1985) (see also O'Brien et al., 1993), which represents the total stress  $\hat{\tau}_i$  in each coordinate direction *i*, as follows,

$$\hat{\tau}_i = \hat{\tau}_{yield} + \hat{\mu}_m \frac{\partial \hat{u}_i}{\partial \hat{z}} + \hat{\zeta} \left(\frac{\partial \hat{u}_i}{\partial \hat{z}}\right)^2 \tag{3.12}$$

where  $\hat{\tau}_{yield}$  represents the sum of the Mohr-Coulomb and yield stresses, the second term is the viscous shear-stress that depends on the dynamic viscosity of the mixture  $\hat{\mu}_m$  and the vertical velocity gradient expressed as a function of the cartesian velocity components  $\hat{u}_i$ . The last term corresponds to the sum of the turbulent and dispersive stresses, which depend quadratically of the velocity gradient and the inertial shear stress coefficient  $\hat{\zeta}$ , defined by the following equation,

$$\hat{\zeta} = \hat{\rho} \, \hat{l}_m^2 + c_{Bd} \, \rho_s \, \lambda^2 \, d_s^2 \tag{3.13}$$

where  $\hat{l}_m = 0.4 \,\hat{h}$  is the Prandtl mixing-length (see Julien & León, 2000, for details),  $c_{Bd}$  is an empirical proportionality constant equal to 0.01 according to Bagnold (1954),  $d_s$  is the median sediment diameter, and  $\lambda$  is Bagnold's linear concentration, which corresponds to the ratio between the grain diameter and the mean free dispersion distance. The magnitude of  $\lambda$  is related to the volumetric concentration of the mixture and the maximum volumetric static concentration  $C^*$ , as defined by Bagnold (1954),

$$\lambda^{-1} = \left[ \left( \frac{C^*}{C} \right)^{\frac{1}{3}} - 1 \right] \tag{3.14}$$

In the present numerical model we modify the quadratic model of O'Brien and Julien (1985) to represent better the stresses for a wide range of sediment concentrations, expressing clearly the contribution of each physical mechanism as the combination of relations that account for the stresses, which have been obtained from experiments or physicallybased formulas. To determine the values of the source terms defined as  $S_i = \tau_i / \rho$  in equation 3.11, the stresses are non-dimensionalized, depth-integrated, and added in the source term vector for each coordinate direction, such that the total stresses are expressed as:

$$S_i = S_{yield} + S_{v_i} + S_{td_i} \tag{3.15}$$

where  $S_{yield}$  represents the sum of the yield and Mohr-Coulomb stress,  $S_{v_i}$  the viscous stress and  $S_{td_i}$  the sum of the dispersive and turbulent stresses. Each of these terms are computed separately from empirical formulas.

The yield and Mohr-Coulomb stresses  $S_{yield}$  is calculated from the following expression:

$$S_{yield} = \frac{\mathcal{L}}{\mathcal{H}} \left[ \frac{\tau_{yield}}{\rho} \right]$$
(3.16)

in which the yield shear stress and the density of the mixture are non-dimensionalized with the scale of the inertia  $\rho_w U^2$ , and the water density  $\rho_w$ , respectively. The yield stress is isotropic and calculated using the following empirical relation given in SI units:

$$\hat{\tau}_{yield} = a \ 10^{bC} \tag{3.17}$$

where for typical soils, the experimental coefficients a and b are equal to 0.005 and 7.5, respectively (Julien, 2010).

The viscous term  $S_{v_i}$  is computed from the bed stress in each cartesian direction  $\tau_x = \rho C_f u \sqrt{u^2 + v^2}$  and  $\tau_y = \rho C_f v \sqrt{u^2 + v^2}$ , using the laminar friction coefficient defined as  $C_f = k/Re$ , where k is the viscous resistance parameter equal to 64 in open-channel flows (Sturm, 2001). The Reynolds number is defined as  $Re = \frac{\rho \sqrt{u^2 + v^2}h}{\mu_m}$ , where the dynamic viscosity of the mixture is non-dimensionalized as  $\mu_m = \frac{\hat{\mu}_m}{\rho_w \mathcal{UH}}$ . Thus, the expression used to represent the viscous losses is written as:

$$S_{v_i} = \frac{\mathcal{L}}{\mathcal{H}} \left[ \frac{k \ \mu_m \ u_i}{8\rho h} \right]$$
(3.18)

To estimate  $\hat{\mu}_m$  we use the formula proposed by Eyring (1964) and Thomas (1965). This relation is a function of the volumetric sediment concentration in the mixture and the dynamic viscosity of water  $\mu_w$  in SI Units:

$$\frac{\hat{\mu}_m}{\mu_w} = 1 + 2.5C + 10.05C^2 + 0.00273 \exp(16.6C)$$
(3.19)

To compute the last term in equation 3.15,  $S_{td}$ , a Manning or Chézy coefficient is used to represent the friction factor  $C_f$  in the bed stress formula, resulting in the following expression for each cartesian coordinate direction:

$$S_{td_i} = \begin{cases} \text{Manning:} \quad \frac{\mathcal{L}}{\mathcal{H}} \begin{bmatrix} \frac{n_{td}^2 u_i \sqrt{u^2 + v^2}}{Fr^2 h^{1/3}} \end{bmatrix} \\ \text{Chézy:} \quad \frac{\mathcal{L}}{\mathcal{H}} \begin{bmatrix} \frac{u_i \sqrt{u^2 + v^2}}{Fr^2 C_{z_{td}}^2} \end{bmatrix} \end{cases}$$
(3.20)

Since  $S_{td}$  represents the sum of friction, turbulence and dispersive stresses, we use either a modified Manning  $n_{td}$  or Chézy  $Cz_{td}$  coefficients. To estimate their value we add two Darcy-Weisbach friction factors, denoted as  $f_t$  and  $f_d$ , representing the turbulent and dispersive effects respectively. To compute  $f_t$ , we use Colebrook's equation:

$$\frac{1}{\sqrt{f_t}} = -2\log\left(\frac{\hat{k}_s}{3.7\mathcal{H}h} + \frac{2.51}{Re\sqrt{f_t}}\right)$$
(3.21)

The value of  $f_t$  is calculated as a function of the depth of the mixture h, the Reynolds number, and the bed specific roughness  $\hat{k}_s$ , which is estimated as follows (Bathurst, 1978),

$$\hat{k}_s = 6.8d_s \text{ [SI Units]} \tag{3.22}$$

To account for the dispersive effects,  $f_d$  is calculated using the relation proposed by Takahashi (2007):

$$\sqrt{\frac{8}{f_d}} = \frac{2\mathcal{H}h}{5d_s} \left\{ \frac{1}{0.02} \left[ C + (1-C) \frac{\rho_w}{\rho_s} \right] \right\}^{\frac{1}{2}} \lambda^{-1}$$
(3.23)

where  $\rho_s$  and  $\rho_w$  are the sediment and water densities, respectively, and  $\lambda$  is Bagnold's linear concentration defined previously in equation 3.14. In general, numerical simulations show that the turbulent friction coefficient  $f_t$  is significantly smaller than the dispersive

factor  $f_d$  (D'Aniello et al., 2015). Dispersive effects, however, become important for low values of relative roughness ( $\frac{h}{d_s} < 50$ ), as discussed in detail by Julien and Paris (2010). To obtain the terms  $S_{td_i}$ , we use the following relation proposed by Julien (2010), to transform the combined Darcy-Weisbach friction coefficient  $f_{td} = f_t + f_d$  in an equivalent Manning o Chézy coefficient:

$$\sqrt{\frac{8}{f_{td}}} = C_{z_{td}}Fr = \frac{h^{1/6}Fr}{n_{td}}$$
(3.24)

#### 3.3. Numerical Method

The numerical solution of the system of equations 3.9 is based on the method developed by Guerra et al. (2014) to solve the NSWE, which has shown great efficiency and precision to simulate extreme flows and rapid flooding over natural terrains and complex geometries. This is a finite-volume formulation that is implemented in two steps: First, in the so-called hyperbolic step, the Riemann problem is solved at each element of the discretization without considering momentum sinks. The flow is reconstructed hydrostatically from the bed slope source-term, adding the effects of the spatial concentration gradients. In the second step we incorporate the shear stress source terms by means of a semi-implicit scheme, correcting the predicted values of the hydrodynamic variables obtained in the previous step.

The initial hyperbolic step consists of solving numerically the following equation:

$$\frac{\partial Q}{\partial t} + J\frac{\partial F}{\partial \xi} + J\frac{\partial G}{\partial \eta} = S_B(Q) + S_C(Q)$$
(3.25)

in which a semi-discrete finite-volume formulation in generalized curvilinear coordinates can be written as follows,

$$\frac{\partial Q_{i,j}}{\partial t} + \frac{J_{i,j}}{\Delta \xi} \left( F_{i+\frac{1}{2},j} - F_{i-\frac{1}{2},j} \right) + \frac{J_{i,j}}{\Delta \eta} \left( G_{i,j+\frac{1}{2}} - G_{i,j-\frac{1}{2}} \right) = S_{B_{i,j}} + S_{C_{i,j}}$$
(3.26)

where  $Q_{i,j}$  and  $J_{i,j}$  represent the vector of hydrodynamic variables and the Jacobian of the coordinate transformation at the center of the discrete elements of the grid (i, j). The

vectors  $F_{i\pm\frac{1}{2},j}$  and  $G_{i,j\pm\frac{1}{2}}$  are the numerical fluxes through each of the cell interfaces. The terms  $\Delta\xi$  and  $\Delta\eta$  correspond to the size of the discretization, and  $S_{B_{i,j}}$  and  $S_{C_{i,j}}$  are the discrete source terms of the bed slope and concentration gradients, respectively.

To compute the numerical fluxes we implement the VFRoe-ncv method (Gallouët et al., 2003; Marche, 2006) to solve equation 3.25, through a non-conservative change of variables, linearizing the Riemann problem (Guerra et al., 2014). The vector of hydrodynamic variables  $Q_{i,j}$  is extrapolated to the boundaries of the each cell to ensure the non-negativity of the intermediate states and flow depths, preserving the dry zones of the terrain. The Monotonic Upwind Scheme for Conservation Laws method (MUSCL), developed by Van Leer (1979), is used to perform the extrapolation with second order accuracy. Finally, the methology developed by Masella et al. (1999) is used to avoid unphysical solutions due to the lack of dissipation to capture shock waves.

The bed-slope source term  $S_{B_{i,j}}$  is computed following the well-balanced methodology developed by Audusse et al. (2004) and adapted to generalized curvilinear coordinates by Guerra et al. (2014). This method reconstructs hydrostatically the free surface by performing a balance between the topographic variations of the domain and the hydrostatic pressure. The hydrodynamic variables and bed elevations are extrapolated to the boundaries of the cells using the MUSCL method, preserving locally and globally the dry zones and stationary steady-states. To ensure the non-negativity of the flow depth and to avoid spurious oscillations, the *minmod* limiter is implemented during the hydrostatic reconstruction of the fluid depth; such that realistic values of the spatial gradients of depth are reached in the shock waves (LeVeque, 2002; Bohorquez & Fernandez-Feria, 2008).

The concentration gradient term  $S_{C_{i,j}}$  is discretized using the following scheme:

$$S_{C_{i,j}} = \frac{-h_{i,j}^2}{2Fr^2} \left(\frac{s-1}{C_{i,j}(s-1)+1}\right) \left(\frac{C_{i+\frac{1}{2},j} - C_{i-\frac{1}{2},j}}{\Delta\xi} \xi_x + \frac{C_{i,j+\frac{1}{2}} - C_{i,j-\frac{1}{2}}}{\Delta\eta} \eta_x\right) \quad (3.27)$$

where,  $h_{i,j}$  and  $C_{i,j}$  are the centered-cell flow depth and sediment concentration, respectively,  $C_{i\pm\frac{1}{2},j}$  and  $C_{i,j\pm\frac{1}{2}}$  are the sediment concentrations at the interfaces of the each cell, obtained from a first order upwind scheme.

In the second step of the numerical solution, we incorporate the momentum source terms in vector  $S_S(Q)$ , solving the following system of equations,

$$\frac{\partial hu}{\partial t} = -S_x; \ \frac{\partial hv}{\partial t} = -S_y \tag{3.28}$$

We use a splitting semi-implicit method (Liang & Marche, 2009), employing a second order Taylor expansion. The limiters developed by Burguete et al. (2007) are implemented to avoid numerical instabilities at the wet/dry interfaces, where the flow depths are shallower. These limiters are designed to prevent unphysical effects, such as reversed flows due to high shear stresses.

Finally, the temporal integration of equation 3.25 is carried out by using a fourth-order Runge-Kutta numerical scheme. The condition for numerical stability of the model is based on the Courant-Friedrichs-Lewy criterion *CFL*.

The boundary conditions are handled by creating two rows of "ghost-cells" outside of the computational domain (Sanders, 2002). We implement three types of boundaries: (1) Open or transmissive boundary at the outlets; (2) Closed reflective boundary for the solid walls; and (3) Inflow boundary to introduce a hydrograph or a controlled discharge toward the computational domain.

#### 4. VALIDATION

A comprehensive validation study of the numerical model was performed by Guerra et al. (2014), considering clear-water flow, and demonstrating the accuracy of our method by qualitative and quantitative comparisons against analytical solutions containing sharp gradients of velocity and water depths, and experimental data of rapid flood propagation (the reader is referred to Guerra et al., 2014, for details). Here we validate the capabilities of the model related to the variable density and the rheology of the flow.

#### 4.1. Quiescent equilibrium in a tank

This benchmark test is developed to demonstrate the capacity of the model to preserve the hydrostatic state with density differences. An analytical solution is obtained from the procedure developed by Leighton et al. (2009), in which the original system of equations 3.1 to 3.4 is simplified by considering steady flow  $(\frac{\partial(\cdot)}{\partial \hat{t}} = 0)$  and a stationary initial state  $(\hat{u} = \hat{v} = 0)$  in inviscid flow with zero stresses  $(\hat{\tau}_{\hat{x}} = \hat{\tau}_{\hat{y}} = 0)$ . Therefore, the equations are reduced to:

$$\frac{\partial}{\partial \hat{x}} \left( \frac{1}{2} \hat{\rho} g \hat{h}^2 \right) = -\hat{\rho} g \hat{h} \frac{\partial \hat{z}}{\partial \hat{x}}$$
(4.1)

which can be written as follows:

$$\frac{\hat{h}}{\hat{\rho}}\frac{\partial\hat{\rho}}{\partial\hat{x}} + 2\frac{\partial\hat{h}}{\partial\hat{x}} = -2\frac{\partial\hat{z}}{\partial\hat{x}}$$
(4.2)

Hence, for a rectangular tank of length L and width A, with a constant initial flow depth  $\hat{h}(\hat{x}) = h_0$ , and a bed described by a cosine function:

$$\hat{z}(\hat{x}) = A \left[ 1 - \cos\left(\frac{2\pi\hat{x}}{L}\right) \right]$$
(4.3)

the analytical solution of equation 4.2 becomes:

$$\hat{\rho}(\hat{x}) = \rho_0 \exp\left[\frac{2A}{h_0} \cos\left(\frac{2\pi\hat{x}}{L}\right)\right]$$
(4.4)

where  $\rho_0$  is the initial reference value of the fluid density.

The dimensions and initial conditions of this test are presented in the Figure 4.1, where the reference density  $\rho_0$  was set to 1,000 kg/m<sup>3</sup>. The 1D computational domain was discretized in a grid of 1,001 cells on the longitudinal direction, with a reflective solid wall boundary condition at each wall of the tank. The total simulated time was 100 s and the CFL number was set to 0.2.



Figure 4.1. Dimensions and initial conditions of the rectangular tank used to the quiescent equilibrium test (Leighton et al., 2009).

Results show that there is an excellent agreement between the analytical and numerical solution for the free surface, as shown in Fig. 4.2(a), and the density profile in Fig. 4.2(b). The maximum error of water depth is equal to  $10^{-7}$  m, and the model is capable of maintaining the steady-state of the flow.

#### 4.2. Density dam-break with two initial discontinuities

To test the model in unsteady conditions, we simulate a density-driven dam-break to evaluate the evolution of the hydrodynamic variables in space and time. The numerical



Figure 4.2. Quiescent equilibrium test. Comparison between theoretical and numerical profiles of hydrodynamic variables. (a) Free surface; and (b) Fluid density.

experiment is based in the work developed by Leighton et al. (2009), which consists of a horizontal rectangular tank of 100 m long, with two fluids of different densities  $\rho_1$  and  $\rho_2$ , as shown in Figure 4.3. The acceleration of gravity is considered equal to 1 m/s<sup>2</sup>, and the shear stresses are neglected.

Two different simulations are performed for  $\rho_2 = 0.1 \text{ kg/m}^3$  and  $\rho_2 = 10 \text{ kg/m}^3$ , while  $\rho_1$  is kept constant and equal to 1 kg/m<sup>3</sup>. Both simulations are implemented on a regular grid with a resolution of 0.005 m during 30 s, using a CFL = 0.2 and reflective boundary conditions at solid walls.

In Figure 4.4 we show the instantaneous flow depth and velocity profiles at 2 and 30 s, from the start of the first simulation ( $\rho_2 = 0.1 \text{ kg/m}^3$ ). A good agreement is found with



Figure 4.3. Initial state of the density-driven dam-break (Leighton et al., 2009).

respect to the solution provided by Leighton et al. (2009), as we capture the propagation of the free surface and velocity magnitudes that are generated by the initial imbalance of the hydrostatic pressure at the interface of the fluids. The amplitudes of the main shock are slightly smaller due to the second-order accuracy of the numerical model. Similar results are obtained for  $\rho_2 = 10 \text{ kg/m}^3$  as shown in Figures 4.5 and 4.6.

Note that when  $\rho_2/\rho_1 < 1$ , the flow velocities are directed toward the middle of the tank, where the fluid is less dense, which increases the flow depth in that zone. Conversely, when  $\rho_2/\rho_1 > 1$ , the fluids move to reach hydrostatic equilibrium, balancing the pressure in the entire domain, which produces higher depths at the sidewalls.

#### 4.3. Large-scale experimental dam-break

To test the rheological model, we simulate the large-scale dam-break experiment with high sediment concentration performed by Iverson et al. (2010). We compare the numerical results with the measurements of flow depth and the arrival time of the wave front. It is important to note that the simulation of this experiment is a very challenging computational test for the numerical model. The slope of the channel, the sediment concentration, and the flow phenomena as the wave advances generates a complex dynamic that is difficult to measure and reproduce with a high resolution numerical model.


Figure 4.4. Density driven dam break. Case  $\rho_2 = 0.1$  kg/m<sup>3</sup>: Comparison of the flow depth (on the left) and velocity profiles (on the right) at (a)  $\hat{t} = 2$  s; and (b)  $\hat{t} = 30$  s from the beginning of the simulation.



Figure 4.5. Density driven dam break. Case  $\rho_2 = 10 \text{ kg/m}^3$ : Comparison of the flow depth at  $\hat{t} = 1 \text{ s}$ , 4 s, 12 s, and 30 s from the beginning of the simulation.

The experiment consists of the sudden release of a large volume of a sediment-water mixture on a 95 m long rectangular channel, with a cross section of 2 m wide by 1.2 m deep. The channel is very steep, with an inclination of 31° on the first 75 m downstream



Figure 4.6. Density driven dam break. Case  $\rho_2 = 10 \text{ kg/m}^3$ : Comparison of velocity profiles at  $\hat{t} = 1 \text{ s}$ , 4 s, 12 s, and 30 s from the beginning of the simulation.

from the gate, and 2.5° on the downstream section. The total volumen released in the dam-break experiment is 6 m<sup>3</sup>, with an initial depth of 2 m and a volumetric sediment concentration of 64.7%. The bed roughness changes along the channel, with a representative roughness height of 1 mm on the first 6 m of the channel measured from the gate, and a roughness of 15 mm in the rest of the channel, downstream. The sediment density considered in this case is  $\rho_s = 2,700 \text{ kg/m}^3$ .

The unsteady inflow condition is the debris flow at a distance of 2 m downstream from the gate, which is shown in Figure 4.7. This was obtained from the simulation of the dam break delayed 1 s, to consider the delay on the opening of the gate, as reported by Iverson et al. (2010). We simulate a total time of 25 s, using a 2D spatial discretization with a uniform resolution of 0.1 m, and a CFL number equal to 0.1.

In Figure 4.8 we compare the flow thickness between the simulation and the experiment at two locations, corresponding to 32 and 66 m downstream of the gate. The experimental data was collected by Iverson et al. (2010) at a frequency of 100 Hz. In our simulation, the grid is fine enough to resolve the well-known roll waves that appear at high Froude numbers in steep channels (Bohorquez & Fernandez-Feria, 2006). This phenomenon has also been recently observed in the simulations of the same experiment by Bohorquez (2011). In this case we capture roll waves with an amplitude close to 0.5 m, as shown by the red line in Figure 4.8.



Figure 4.7. Unsteady inflow boundary condition corresponds to the crosssection flow measured at a location of 2 m downstream of the gate (Iverson et al., 2010).

To compare the numerical results directly with data provided by the experiments, we apply the same moving-average filter used to smooth the experimental data (black line in Figure 4.8). The model reproduces with good agreement the arrival time of the wave front in both gauges, with delays smaller than 0.2 s. The maximum flow depth computed at the location of 32 m and 66 m downstream from the gate is over- and underestimated by just 1.78 cm and 1.6 cm respectively.

The simulated and observed wave-front position in time are very similar (Figure 4.9) with values of the mean square error and the coefficient of determination of the fit being 1.98 m and  $R^2 = 99.28\%$ , respectively. The sudden discontinuity in the computed front velocity at 7.6 s is due to a roll wave advancing through the front, which briefly slows down the flow. Due to the resolution of the experimental results, we cannot compare directly this phenomenon captured in the simulation.



Figure 4.8. Comparison of the flow thickness measured in the experiment of Iverson et al. (2010), and computed with our model. Flow thickness at two locations: (a) 32 m; and (b) 66 m downslope from the gate.

Overall, the validation study shows that the numerical model in these extreme cases is very robust and it is able to reproduce many of the phenomena of interest that appear in hyperconcentrated flash floods.



Figure 4.9. Comparison of the position of the flow front as a function of time: (a) The position of the flow front over the time (b) Comparison of the experimental and simulated flow front position. The dashed line is the perfect fit with slope equal to 1.

# 5. FLOOD PROPAGATION IN THE QUEBRADA DE RAMÓN

To study the effects of sediment concentration on flash floods in mountain rivers, we selected the Quebrada de Ramón watershed in the Andes of central Chile to evaluate different scenarios regarding the sediment load. We not only characterize locally the depths and velocities of the flow as the flood propagates, but we also evaluate the extension of flooded area and the momentum of the flow in the lower section of the watershed, within the city of Santiago.

As described in section 2, there are two major tributaries draining the north and south sections of the watershed. We define the computational domain shown in Figure 5.1, which comprises a total distance of 10.4 km along the main channel. The highest part of the computational domain is located at an elevation of 2,212 m asl, with the Quebrada de Ramón stream (QR) and the Quillayes stream (Qui) approaching from the north and south respectively. The upstream boundary of the domain is located 3 km upstream of the confluence (Figure 5.2c). The channel downstream the confluence continues to the flood zone, with a single main river channel shown in Figure 5.2b, and ends at an elevation of 652 m asl, where the stream has been channelized in the city, as shown in Figure 5.2a. A curvilinear boundary-fitted grid is used to perform the simulations, consisting of a total of  $10,070 \times 218$  grid nodes. The grid resolution varies progressively in the flow direction from 0.5 m upstream and near the confluence, to 2 m of resolution within the flooding zone. On the cross-stream direction, the mean resolution of the grid close to the channels is approximately 1 m. To construct the grid, we use a 1m resolution LIDAR of the area around the channels. The LIDAR data is coupled to a 30 m resolution digital elevation model (DEM) from satellite images for the rest of the watershed.

The bed roughness is represented by a mean sediment grain diameter  $d_s$ . Field measurements are used to interpolate the values of  $d_s$  in the entire computational domain using the nearest neighbor method. The mean sediment grain size distribution is shown in Figure 5.3, along with the 7 points where we report the dynamics of the flow depth, which include



Figure 5.1. Satellite image of the Quebrada de Ramón watershed and the computational domain. The area enclosed by the black line is defined from the LIDAR topography, and incorporates the section of the city around the river channel.



Figure 5.2. Three photos along the Quebrada de Ramón stream from downstream to upstream: a) The channelized section, b) The floodplain, and c) The confluence of the Quebrada de Ramón and the Quillayes stream.

locations at the highest elevation of the domain (denoted as QR-U and Qui-U), upstream of the confluence (QR-D and Qui-D), downstream of the confluence (QR-C), the flooding zone (QR-F), and at the outlet (QR-E).



Figure 5.3. Distribution of the mean sediment size in millimeters. White circles denote the measurement sites where we report the propagation of the flood.

The hydrographs of the events studied in this investigation for the two main rivers that correspond to the tributaries of the confluence are shown in Figure 5.4. The return period of these hydrographs have been is estimated in 50 years, and they were obtained from a continuous semi-distributed hydrological model built in HEC-HMS, for the 1971 - 2010 period (Ríos, 2016). This case is selected since the peak flow at the outlet is expected to exceed significantly the capacity of the channelized section in the city. The river beds downstream are considered dry at the beginning of the simulation, to avoid the any additional uncertainty associated to the sediment concentration of the initial flow.

We perform the simulations for a total physical time of 1 day, using a simulation time step defined by the stability criterion with a CFL = 0.9. The inflow boundary



Figure 5.4. 50-year return period hydrographs in both streams obtained from a hydrological model of the catchment (Ríos, 2016). This conditions in the Quebrada de Ramón and at the Quillayes stream can potentially produce a large-scale flood in the watershed.

condition in Figure 5.4 is used at the eastward boundaries, and open boundary conditions are considered at all the other boundaries of the computational domain.

### 5.1. Simulations of hyperconcentrated flows

The actual sediment concentration during flash floods in Andean watersheds is unknown in most cases. In addition to the lack of information in these rivers, landslides due to erosion produced by soil saturation in the steep slopes of the mountains are common. These conditions can increase considerably the sediment supply to the streams during flood events. To reduce the uncertainty regarding the effects of the sediment concentration, we study the dynamics of a flood for different scenarios by carrying out a series of simulations to compare and understand the flood hazards and effects of hyperconcentration on the two main streams of the Quebrada de Ramón watershed.

We simulate four different scenarios considering different concentrations of 0%, 20%, 40% and 60% equal in both streams. Two another cases are simulated, with concentrations of 20% in the Quebrada de Ramón stream and 60% in the Quillayes stream, and viceversa. To evaluate the impacts of these concentrations on the flood dynamics, in the following sections we analyze the flow hydrodynamics including: (1) the position and velocity of the flood wave front; (2) the peak flow and arrival time, (3) the flooded areas; (4) the effect of the sediment concentration on the depth and flow velocity; and (5) the momentum of the flow in the urban zone.

### 5.1.1. Position and mean velocity of the wave front

To quantify the propagation of the flood along the channels and the arrival time of the flood to the city, we compute the mean velocity of the wave front by tracking its position in time. Table 5.1 shows the mean velocity in the section upstream of the confluence, for the Quebrada de Ramón and Quillayes stream. As it can be anticipated, the velocity of the wave front decreases with the concentration, as interparticle collisions and internal stresses reduce the momentum of the flow, increasing the flow resistance. Note that the flood propagation velocity is very sensitive to variations of concentration in more dilute conditions. Overall, the velocity for a concentration of 20% is 60% slower than that of clear water. On the other hand, when the concentration increases from 40% to 60%, the velocity is reduced by only 10%. The mean wave front propagation velocity seems to be decreasing exponentially with the concentration in this case.

Figure 5.5 shows the location of the wave front vs time for both streams upstream of the confluence. The inverse slopes of these curves represent the instantaneous velocity of the front for each sediment concentration we have simulated.

Volumetric	Mean velocity of the wave front [km/h]	
concentration [-]	Quebrada de Ramón stream	Quillayes stream
0%	2.88	4.99
20%	1.88	3.10
40%	1.64	1.27
60%	1.54	1.12

Table 5.1. Mean velocity of the front for different sediment concentrations

The numerical results show that the sediment concentration produces a significant change on the evolution of the flood, as it is the only factor that we modify in these simulations. The local variations in these velocities are produced by the gradual change of bed roughness and the slope of the river channels, which is approximately constant in large portions of the reaches. The Quillayes stream exhibits higher propagation velocities, which are also consistent with the steeper slopes and finer sediment diameters on the bed.

Figure 5.5 shows that the flood propagation has deceleration stages of the front, seen as steps in the location of the front in time. Two clear steps are observed in the Quebrada de Ramón stream. The first is located at 800 m from the inflow boundary of the computational domain, which is produced by a local widening of the channel. The second deceleration, at 4,500 m, is generated by a narrowing of the river that accumulates a large volume and reduces the velocity of the flow, increasing the depth upstream of this section due to backwater effects. In the Quillayes stream, we observe three deceleration stages at 400, 800, and 2,800 m, which are also caused by local widening of the channel. These detailed dynamic features of the flood are modulated by the sediment concentration, as the hyperconcentrated cases show a more uniform propagation of the front.

Downstream of the confluence, the flood exhibits a similar dynamics. Figure 5.6 shows the distance traveled by the flood from the confluence to the outlet of the watershed. The origin in this case is defined at the junction of the the tributaries, and the time starts running when the flood from the Quillayes stream reaches the confluence.



Figure 5.5. Location of the wave front in time for both streams, considering four different sediment concentrations, to characterize the advance of the flood in the river: (a) Quebrada de Ramón; and (b) The Quillayes stream.



Figure 5.6. Location of the wave front in time for the Quebrada de Ramón downstream of the confluence considering four sediment concentrations in both streams.

The velocities in the four cases show that the dynamics of the flow changes along the channel, as a function of the concentration. The clear-water flow is faster at the beginning but decelerates along the channel, while hyperconcentrated flows accelerate for all the sediment concentrations. The flow in this second section of the river channel is affected by the different arrival times of the flood to the confluence from each of the tributaries. The time lapse between the arrivals of the flood from each tributary to the confluence is larger for flows with lower concentrations. This difference is equal to 3 hours in clear-water flow, but only 1 hour for a concentration of 60%, which changes the hydrodynamics of the wave-front when it arrives to the lower section of the channel, in the urban area.

### 5.1.2. Peak flow and its arrival time

By comparing the hydrographs computed using different sediment concentrations in the four points monitored upstream of the confluence, we observe that the most important difference is the magnitude of the peak flow for different concentrations. The relative difference between the peak discharge simulated with clear-water flow compared to a sediment concentration of 60% is 44% at QR-U, and 67% at QR-D. To remove the effects of the additional volume that are produced by the sediment concentration in each stream, in Figures 5.7 and 5.8 we show the normalized hydrographs, which are obtained by dividing the discharge by the total volume of the mixture that we obtain at each gauged point defined in Figure 5.3. We can observe that the difference in the peak flows for different concentrations is only produced by the bulking effect of the sediments.

In both streams, the time to the peak of the hydrograph, however, is not significantly affected by the different concentrations. This seems to be related to the shape of the inflow hydrograph, and to the location of the gauged points. The time to reach the peak discharge is around of 7 h from the start of the simulation at QR-U, and 10 min later the maximum discharge reaches QR-D. At the Quillayes stream, the peak flow reaches Qui-U after 8 h from the start of the simulation, and Qui-D 13 min later.

When we analyze the hydrographs downstream of the confluence we observe similar results, as shown in Figure 5.8. Due to the progressive reduction of the bed roughness, the peak flows increase in sections closer to the outlet of the watershed. The maximum peak flow is reached at the station QR-F, since the city park located at the north side of the main channel, and between QR-F and QR-E, is flooded and attenuates the peak flow near the outlet.

### 5.1.3. Total flooded area

The total area in the watershed that is inundated for different sediment concentrations is depicted in Figure 5.9. No significant differences are noticed for most of the length of



Figure 5.7. Normalized hydrographs at the gauged points upstream of the confluence. In the right panel, we show the upper and lower zones in the Quebrada de Ramón stream, and in the left panel, the points at a similar elevation in the Quillayes stream.

the river channel. Both streams have steep slopes in confined canyons, and the maximum increments of flow depth, reaching up to 3 m, do not alter significantly the horizontal extension of the 2D area affected by the flood.

Major differences, however, appear in regions with milder slopes, around the confluence and in the city, near the outlet of the watershed. At the confluence, simulations with higher concentrations of 40% and 60% overflow the natural channels, due to the fast arrival of a large volume of the mixture to this region. In the city, near the outlet of the domain, all the flows inundate the urban park located at the north bank of the main channel,



Figure 5.8. Normalized hydrographs at the gauged points downstream of the confluence. In the left panel we depict the point closest to the confluence; in the middle the flooding zone; and in the right panel the outlet of the computational domain.

downstream of a large extension of an urbanized area. In this section, the most important increment of the total flooded area occurred when we increase the concentration from clear water to 20%, where the total flooded area increases by 36%. For larger concentrations the affected area grows gradually compared to the clear-water flooding case, as the fluid is more concentrated. Increments of the total area of 46% and 75% are observed for concentrations 40% and 60%, respectively.



Figure 5.9. Contours of the flooded area for different sediment concentrations.

### 5.1.4. Maximum flow depth and mean velocity

Figure 5.10 shows the maximum depth registered at each gauged point along the channels for the different sediment concentrations we simulate. The numerical results show that the depth increases with concentration, and the largest differences are obtained between the clear-water case and 20% concentratiom, in 6 of the measurement points we analyze. In QR-C for instance, the first increment of the sediment concentration, from clear-water to 20%, produces a maximum depth that is 24.1% larger, whereas increasing the concentration from 20% to 40%, and then to 60%, the flow depth increases in only 7.5% and 5.1%.



Figure 5.10. Maximum depth computed in every gauged point depending of the sediment concentration.

By comparing the flow depths in the simulations, we note that the deepest flow is always located downstream of the confluence (QR-C). In this location, a difference of 0.80 m is measured between the clear-water and the flow with the maximum concentration of 60%. In the urban areas (points QR-F and QR-E), depths larger than 2 m are observed. Here, it is important to have a precise solution for different sediment concentrations, since there is a difference of 0.5 m between clear-water and the maximum concentration, which can have significant impacts on the design of flood control measures.

Additionally, in Figure 5.11 we show the mean velocity at each measurement point for the range of sediment concentrations. In this case, we cannot observe a clear trend of velocity changes as a function of the concentration. For this flow variable, it seems that the local topographic conditions affect considerably the averages of the flow hydrodynamics.

In Figure 5.12 we relate the magnitude of the hydrodynamic variables, velocity vs depth, computed at each time step at QR-U on the left panel, and Qui-U on the right panel. These plots are similar to a stage-velocity relation that links the flow depth and the total velocity at the same instant in time. The plots are constructed using data every 30 s, for a total time of 24 hours.

Even though a direct relation is not observed between mean velocity and sediment concentration in points QR-U and Qui-U (Figure 5.11), the depth-velocity plots in Figure 5.12 confirm the relation of large depths and lower velocities as the concentration increases. This is closely related to the changes on the flow resistance. Larger concentrations increase the yield stress, the fluid viscosity, and the dispersive effects, producing additional momentum losses, which reduce the flow velocity. The stage-velocity relation is different for clear-water flow as compared to the sediment laden cases. For hyperconcentrated flows, the relationship between the depth and velocity is fitted to a quadratic regression that always increases. In clear water the relation is linear in shallow flows under 0.05 m deep, where the Froude number is larger than 1, reaching 1.43 and 1.53 in the QR-U and Qui-U respectively. Then, a transition zone with depths between 0.05 m



Figure 5.11. Mean velocity computed in each gauged point depending on the sediment concentration.

and 0.2 m, and almost critical Froude number is observed. Above 0.2 m depth, the velocity increases quadratically as seen in the flows with sediments. In this zone, the flow is dominated by gravity with subcritical Froude numbers of around 0.25.

### 5.1.5. Flow momentum in the urban area

To evaluate the potential damage to the infrastructure generated by floods, we can compute the flow momentum at each cross section of the flooded area. In this case we compare the maximum force produced by the flow in the urban area of the watershed, considering flows with different sediment concentrations coming from the Quebrada de Ramón and the Quillayes stream. Figure 5.13 shows contours of maximum cross-section momentum



Figure 5.12. Stage-velocity relation between the computed flow depths and velocities for different sediment concentrations. The data at the point QR-U is shown on the left, and Qui-U on the right.

along the river. The top figure shows the momentum for a sediment concentration of 20% in the Quebrada de Ramón and 60% in the Quillayes stream. The opposite case, 60% in the Quebrada de Ramón and 20% in the Quillayes stream, is depicted in the lower image.

The approaching flow has an approximate force of 700 kN in both simulations. For these two cases, the areas with highest momentum correspond to: (1) The confined zone on the right of the image; and (2) At the outlet of the basin in the urban area. However, the force is on average 14.5% higher in the second case, which could be related to the higher flow density of the flow that is obtained downstream of the confluence.

Since the density in these simulations is different in both streams upstream the confluence, the density of the fluid in the main channel varies in time and space, both along and across the flow. The mean concentration in the main channel, downstream the confluence, is around 30% and 44% considering a sediment concentration of 20% in the Quebrada de Ramón and 60% in the Quillayes stream and viceversa, respectively. These values are consistent with the theoretical concentrations computed from the fully mixed conditions.



Figure 5.13. Maximum momentum of the flow in the urban area. The picture above shows results for a concentration of 20% coming from the Quebrada de Ramón and 60% from the Quillayes stream. The image below corresponds to the opposite case, 60% from the Quebrada de Ramón and 20% from the Quillayes stream.

### 6. CONCLUSIONS AND FUTURE RESEARCH

The primary emphasis of this work is to examine the effects of the sediment concentration on the flood dynamics in an Andean watershed. To simulate different scenarios, we developed a finite-volume numerical model that solves the hydrodynamics of hyperconcentrated fluids in complex natural topographies. The model is based on the work of Guerra et al. (2014), and it is employed to solve the non-linear shallow water equations coupled to a transport equation for the sediment in generalized curvilinear coordinates. To consider the effects of the sediment concentration, we implement a new version of the quadratic rheological model (O'Brien et al., 1993) to calculate the stresses produced by high concentrations, separating the turbulent and dispersive effects of the sediment concentration.

The numerical method considers the semi-implicit splitting scheme developed by Liang and Marche (2009) for the treatment of the source terms in the governing equations. The model is well-balanced to incorporate the source term associated to gravity and slope (Audusse et al., 2004), and the VFRoe-ncv methodology (Gallouët et al., 2003; Marche, 2006) is used to solve the numerical fluxes. We test and validate the model by comparing the results against analytical solutions (Leighton et al., 2009) and observations from a hyperconcentrated dam-break flow on a steep slope experiment (Iverson et al., 2010).

To investigate the effects of the sediment concentration in floods that occur in mountain rivers, we perform simulations in the Quebrada de Ramón watershed, an Andean catchment located in central Chile. We analyze the changes on hydrodynamic variables such as peak discharge, arrival time of the flood wave, cross-section momentum, flow depth, mean velocity, and total flooded area. Most the these results are compared and analyzed in seven points along the channel.

The most important effects on the flood propagation are observed for the increments of sediment concentration just above the clear-water flow, in the range of concentrations from 0% to 20%. Even though the channel slope is the most important morphological feature

that controls the dynamics of the flow, local factors such as channel widening can change significantly the propagation of the flood wave. High sediment concentrations modulate these morphodynamic effects, producing larger flow depths and slower velocities overall.

Major momentum loss produced by the increment of the total shear stress as the fluid becomes hyperconcentrated, cause larger flow depths as the concentration increases. The differences in depth between no-sediment conditions and a concentration of 60%, is around 38% in one of the monitored point indentified as QR-C, where the largest value of flow depth was computed, and 22.5% in the point QR-E, which is located in the urban area. A clear relationship between velocity and sediment-load was not observed.

Some of the hydrodynamic variables analyzed were more sensitive to changes in sediment concentration. We observed significant effects on the total flooded area and momentum of the flow as the flood arrives to the urban area. While the extension of the 2D flooded area in the entire basin remains more or less constant for different concentrations, the largest difference is observed in the city, where the slopes are milder. The simulations show a difference of 76% in the total 2D flooded area when we compare the clear-water conditions with the 60% concentration. Regarding the cross-section momentum as the flood advances in the urban zone, we show that the maximum momentum of the flow increases 14% on average for a 20% concentration in the Quebrada de Ramón, and 60% concentration in the Quillayes stream.

We also observe that the arrival time of the peak discharge at different locations of the basin, and the shape of the hydrograph, are not modified significantly with the magnitude of the sediment concentration, but they are associated to the local morphological conditions.

Future developments of the model will incorporate the erosion and sedimentation of the channel by coupling the flow to a sediment transport and a morphodynamic model represented by the Exner equation. This analysis will require to address questions regarding the applicability of empirically derived bedload transport formulas to unsteady nonequilibrium conditions, which can add more uncertainty to the flood simulation in realistic conditions. To improve the flow description in some cases, we will consider the effects of the non-hydrostatic pressure generated in shallow-water flow over rugged or very steep terrain (Denlinger & O'Connell, 2008). The efficiency and accuracy of the model open new possibilities to study additional phenomena, in which the influence of density and internal stresses are relevant to the flow hydrodynamics. Some examples could include the study of lahars and their interaction with natural water bodies, saline intrusion processes in estuaries, and contamination processes in natural environments.

In future research we will also use the simulations to develop real-time automated systems of flood hazard prediction. Taking a probabilistic framework based on the numerical results, and coupling the hazard to the assessment of vulnerability, we will develop surrogate models for rapid evaluation of the flood risk.

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APPENDIX

# A. NON-LINEAR SHALLOW EQUATIONS FOR FLUIDS WITH VARIABLE DEN-SITY

# A.1. The original version of the non-linear equations for shallow flows with variable density

Since the model solves the non-shallow water equations coupled to the sediment transport equation; here, the mass and momentum conservation law are presented in cartesian coordinates.

### A.1.1. Continuity equation

Firstly, the continuity law is applied to the mixture of water and sediment, which looks as:

$$\frac{\partial \hat{\rho}\hat{h}}{\partial \hat{t}} + \frac{\partial \hat{\rho}\hat{h}\hat{u}}{\partial \hat{x}} + \frac{\partial \hat{\rho}\hat{h}\hat{v}}{\partial \hat{y}} = 0$$
(A.1)

where  $\hat{h}$ ,  $\hat{u}$ ,  $\hat{v}$  and  $\hat{\rho}$  represent the depth, velocity in the  $\hat{x}$  and  $\hat{y}$  direction, and density of the flow respectively.

### A.1.2. Momentum equations

In the  $\hat{x}$  direction, the momentum law applied to the mixture is:

$$\frac{\partial \hat{\rho} \hat{h} \hat{u}}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}} \left( \hat{\rho} \hat{u}^2 \hat{h} + \frac{1}{2} g \hat{\rho} \hat{h}^2 \right) + \frac{\partial \hat{\rho} \hat{u} \hat{v} \hat{h}}{\partial \hat{y}} = -\hat{\rho} g \hat{h} \frac{\partial \hat{z}}{\partial \hat{x}} - \hat{\tau}_{\hat{x}}$$
(A.2)

And, respectively in the  $\hat{y}$  direction:

$$\frac{\partial \hat{\rho} h \hat{v}}{\partial \hat{t}} + \frac{\partial \hat{\rho} \hat{u} \hat{v} h}{\partial \hat{x}} + \frac{\partial}{\partial \hat{y}} \left( \hat{\rho} \hat{v}^2 \hat{h} + \frac{1}{2} g \hat{\rho} \hat{h}^2 \right) = -\hat{\rho} g \hat{h} \frac{\partial \hat{z}}{\partial \hat{y}} - \hat{\tau}_{\hat{y}}$$
(A.3)

Here, the terms  $\hat{\tau}_{\hat{x}}$  and  $\hat{\tau}_{\hat{y}}$  groups the stresses associated to the friction and hyperconcentration of the sediment, i.e. the high concentration of sediment in a fluid change the hydrodynamic of the fluid since additional shear stresses occurred. In this model, the stresses considered are the yield, Mohr-Coulomb, viscous, turbulent and dispersive shear stresses, by means the rheological model that is described below.

### A.1.3. Advection-Diffusion Equation

The mass conservation law applied to the solid phase is written as:

$$\frac{\partial C\hat{h}}{\partial \hat{t}} + \frac{\partial \hat{u}C\hat{h}}{\partial \hat{x}} + \frac{\partial \hat{v}C\hat{h}}{\partial \hat{y}} = \hat{h}\frac{\partial}{\partial \hat{x}}\left(\hat{D}_{\hat{x}}\frac{\partial C}{\partial \hat{x}}\right) + \hat{h}\frac{\partial}{\partial \hat{y}}\left(\hat{D}_{\hat{y}}\frac{\partial C}{\partial \hat{y}}\right)$$
(A.4)

where C represents the volumetric sediment concentration in the mixture, and  $\hat{D}_{\hat{x}}$  and  $\hat{D}_{\hat{y}}$  are the molecular diffusion coefficient in the  $\hat{x}$  and  $\hat{y}$  direction respectively.

### A.1.4. Matrix form

The set of equations can be written in the matrix form as:

$$\hat{Q}_{,\hat{t}} + \hat{F}(\hat{Q})_{,\hat{x}} + \hat{G}(\hat{Q})_{,\hat{y}} = \hat{S}(\hat{Q})$$
(A.5)

where  $\hat{Q}$  contains the hydrodynamic variables, which is temporally derivatived, and  $\hat{F}(\hat{Q})$ and  $\hat{G}(\hat{Q})$  are the mass and momentum fluxes partially derivatived respect to the  $\hat{x}$  and  $\hat{y}$ coordinate respectively. To continuation, each vector is shown in detail:

$$\begin{split} \hat{Q} &= \begin{pmatrix} \hat{\rho}\hat{h} \\ \hat{\rho}\hat{h}\hat{u} \\ \hat{\rho}\hat{h}\hat{v} \\ \hat{\rho}\hat{h}\hat{v} \\ \hat{C}\hat{h} \end{pmatrix}, \hat{F}(\hat{Q}) &= \begin{pmatrix} \hat{\rho}\hat{h}\hat{u} \\ \hat{\rho}\hat{u}^{2}\hat{h} + \frac{1}{2}g\hat{\rho}\hat{h}^{2} \\ \hat{\rho}\hat{u}\hat{v}\hat{h} \\ \hat{u}\hat{C}\hat{h} \end{pmatrix}, \hat{G}(\hat{Q}) &= \begin{pmatrix} \hat{\rho}\hat{h}\hat{v} \\ \hat{\rho}\hat{u}\hat{v}\hat{h} \\ \hat{\rho}\hat{v}^{2}\hat{h} + \frac{1}{2}g\hat{\rho}\hat{h}^{2} \\ \hat{v}\hat{C}\hat{h} \end{pmatrix}, \\ \hat{S}(\hat{Q}) &= \begin{pmatrix} 0 \\ -\hat{\rho}g\hat{h}\frac{\partial\hat{z}}{\partial\hat{x}} - \hat{\tau}_{\hat{x}} \\ -\hat{\rho}g\hat{h}\frac{\partial\hat{z}}{\partial\hat{y}} - \hat{\tau}_{\hat{y}} \\ \hat{h}\frac{\partial}{\partial\hat{x}}\left(\hat{D}_{\hat{x}}\frac{\partial\hat{C}}{\partial\hat{x}}\right) + \hat{h}\frac{\partial}{\partial\hat{y}}\left(\hat{D}_{\hat{y}}\frac{\partial\hat{C}}{\partial\hat{y}}\right) \end{split}$$

### A.2. Modified non-linear equations for shallow flows with variable density

In order to express every equation in terms of C, the mixture density  $\hat{\rho}$  has been replaced by the expression  $\hat{\rho} = \rho_s + (1-C)\rho_w$ . Where,  $\rho_s$  and  $\rho_w$  represent the sediment and clear water density, respectively. Thus, the set of equations has been modified following the next procedure.

### A.2.1. Continuity equation

The expression for mixture density has been replaced in equation A.1:

$$\frac{\partial(\rho_s + (1-C)\rho_w)\hat{h}}{\partial\hat{t}} + \frac{\partial(\rho_s + (1-C)\rho_w)\hat{h}\hat{u}}{\partial\hat{x}} + \frac{\partial(\rho_s + (1-C)\rho_w)\hat{h}\hat{v}}{\partial\hat{y}} = 0 \quad (A.6)$$

Regrouping terms:

$$\rho_w \left[ \frac{\partial \hat{h}}{\partial \hat{t}} + \frac{\partial \hat{u}\hat{h}}{\partial \hat{x}} + \frac{\partial \hat{v}\hat{h}}{\partial \hat{y}} \right] = (\rho_w - \rho_s) \left[ \frac{\partial \hat{h}C}{\partial \hat{t}} + \frac{\partial \hat{u}\hat{h}C}{\partial \hat{x}} + \frac{\partial \hat{v}\hat{h}C}{\partial \hat{y}} \right]$$
(A.7)

Simplifing, the mixture mass conservation law looks as:

$$\left[\frac{\partial \hat{h}}{\partial \hat{t}} + \frac{\partial \hat{u}\hat{h}}{\partial \hat{x}} + \frac{\partial \hat{v}\hat{h}}{\partial \hat{y}}\right] = \hat{h}\left(\frac{\rho_w - \rho_s}{\rho_w}\right) \left[\frac{\partial}{\partial \hat{x}}\left(\hat{D}_{\hat{x}}\frac{\partial C}{\partial \hat{x}}\right) + \frac{\partial}{\partial \hat{y}}\left(\hat{D}_{\hat{y}}\frac{\partial C}{\partial \hat{y}}\right)\right]$$
(A.8)

### A.2.2. Momentum equations

In the same way, the fluid density is replaced by the volumetric concentration in both momentum equations A.2 and A.3:

$$(\rho_s - \rho_w) \left[ \frac{\partial \hat{u}\hat{h}C}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}} \left( \hat{u}^2 \hat{h}C + \frac{1}{2}g\hat{h}^2C \right) + \frac{\partial \hat{u}\hat{v}\hat{h}C}{\partial \hat{y}} \right] + \\\rho_w \left[ \frac{\partial \hat{u}\hat{h}}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}} \left( \hat{u}^2 \hat{h} + \frac{1}{2}g\hat{h}^2 \right) + \frac{\partial \hat{u}\hat{v}\hat{h}}{\partial \hat{y}} \right] = -\hat{\rho}g\hat{h}\frac{\partial \hat{z}}{\partial \hat{x}} - \hat{S}_{\hat{x}}$$
(A.9)

$$\hat{\rho} \left[ \frac{\partial \hat{u}\hat{h}}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}} \left( \hat{u}^2 \hat{h} + \frac{1}{2} g \hat{h}^2 \right) + \frac{\partial \hat{u} \hat{v} \hat{h}}{\partial \hat{y}} \right] + \frac{1}{2} g \left( \rho_s - \rho_w \right) \hat{h}^2 \frac{\partial C}{\partial \hat{x}} = -\hat{\rho} g \hat{h} \frac{\partial \hat{z}}{\partial \hat{x}} - \hat{S}_{\hat{x}} + \left( \rho_w - \rho_s \right) \hat{h} \hat{u} \left[ \frac{\partial C}{\partial \hat{t}} + \hat{u} \frac{\partial C}{\partial \hat{x}} + \hat{v} \frac{\partial C}{\partial \hat{y}} \right]$$
(A.10)

Thus, the modified momentum law in the  $\hat{x}$  and  $\hat{y}$  directions can be written respectively as:

$$\frac{\partial \hat{u}\hat{h}}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}} \left( \hat{u}^2 \hat{h} + \frac{1}{2}g\hat{h}^2 \right) + \frac{\partial \hat{u}\hat{v}\hat{h}}{\partial \hat{y}} = -g\hat{h}\frac{\partial \hat{z}}{\partial \hat{x}} - \hat{S}_{\hat{x}} - \frac{1}{2}g\left(\frac{\rho_s - \rho_w}{\hat{\rho}}\right)\hat{h}^2\frac{\partial C}{\partial \hat{x}} - \left(\frac{\rho_s - \rho_w}{\rho_w}\right)\hat{h}\hat{u}\left[\frac{\partial}{\partial \hat{x}}\left(\hat{D}_{\hat{x}}\frac{\partial C}{\partial \hat{x}}\right) + \hat{h}\frac{\partial}{\partial \hat{y}}\left(\hat{D}_{\hat{y}}\frac{\partial C}{\partial \hat{y}}\right)\right] \quad (A.11)$$

$$\frac{\partial \hat{v}\hat{h}}{\partial \hat{t}} + \frac{\partial \hat{u}\hat{v}\hat{h}}{\partial \hat{x}} + \frac{\partial}{\partial \hat{y}}\left(\hat{v}^{2}\hat{h} + \frac{1}{2}g\hat{h}^{2}\right) = -g\hat{h}\frac{\partial \hat{z}}{\partial \hat{y}} - \hat{S}_{\hat{y}} - \frac{1}{2}g\left(\frac{\rho_{s} - \rho_{w}}{\hat{\rho}}\right)\hat{h}^{2}\frac{\partial C}{\partial \hat{y}} - \left(\frac{\rho_{s} - \rho_{w}}{\rho_{w}}\right)\hat{h}\hat{v}\left[\frac{\partial}{\partial \hat{x}}\left(\hat{D}_{\hat{x}}\frac{\partial C}{\partial \hat{x}}\right) + \hat{h}\frac{\partial}{\partial \hat{y}}\left(\hat{D}_{\hat{y}}\frac{\partial C}{\partial \hat{y}}\right)\right] \quad (A.12)$$

# A.2.3. Advection-Diffusion Equation

Since, the original version of the advection-diffusion equation is presented with the volumetric concentration in the equation A.13, this does not change.

$$\frac{\partial C\hat{h}}{\partial \hat{t}} + \frac{\partial \hat{u}C\hat{h}}{\partial \hat{x}} + \frac{\partial \hat{v}C\hat{h}}{\partial \hat{y}} = \hat{h}\frac{\partial}{\partial \hat{x}}\left(\hat{D}_{\hat{x}}\frac{\partial C}{\partial \hat{x}}\right) + \hat{h}\frac{\partial}{\partial \hat{y}}\left(\hat{D}_{\hat{y}}\frac{\partial C}{\partial \hat{y}}\right)$$
(A.13)

### A.2.4. Matrix form

$$\hat{Q}_{,\hat{t}} + \hat{F}(\hat{Q})_{,\hat{x}} + \hat{G}(\hat{Q})_{,\hat{y}} = \hat{S}(\hat{Q})$$
(A.14)

$$\begin{split} \hat{Q} &= \begin{pmatrix} \hat{h} \\ \hat{h}\hat{u} \\ \hat{h}\hat{v} \\ \hat{h}\hat{v} \\ C\hat{h} \end{pmatrix}, \hat{F}(\hat{Q}) &= \begin{pmatrix} \hat{h}\hat{u} \\ \hat{u}^{2}\hat{h} + \frac{1}{2}g\hat{h}^{2} \\ \hat{u}\hat{v}\hat{h} \\ \hat{u}\hat{c}\hat{h} \end{pmatrix}, \hat{G}(\hat{Q}) &= \begin{pmatrix} \hat{h}\hat{v} \\ \hat{u}\hat{v}\hat{h} \\ \hat{v}^{2}\hat{h} + \frac{1}{2}g\hat{h}^{2} \\ \hat{v}\hat{c}\hat{h} \end{pmatrix}, \\ \hat{S}(\hat{Q}) &= \begin{pmatrix} \begin{pmatrix} \rho_{w} - \rho_{s} \\ \rho_{w} \end{pmatrix} \hat{h} \left[ \frac{\partial}{\partial\hat{x}} \left( \hat{D}_{\hat{x}} \frac{\partial C}{\partial\hat{x}} \right) + \frac{\partial}{\partial\hat{y}} \left( \hat{D}_{\hat{y}} \frac{\partial C}{\partial\hat{y}} \right) \right] \\ -g\hat{h}\frac{\partial\hat{z}}{\partial\hat{x}} - \hat{S}_{\hat{x}} - \frac{1}{2}g \left( \frac{\rho_{s} - \rho_{w}}{\hat{\rho}} \right) \hat{h}^{2}\frac{\partial C}{\partial\hat{x}} - \left( \frac{\rho_{s} - \rho_{w}}{\rho_{w}} \right) \hat{h}\hat{u} \left[ \frac{\partial}{\partial\hat{x}} \left( \hat{D}_{\hat{x}} \frac{\partial C}{\partial\hat{x}} \right) + \frac{\partial}{\partial\hat{y}} \left( \hat{D}_{\hat{y}} \frac{\partial C}{\partial\hat{y}} \right) \right] \\ -g\hat{h}\frac{\partial\hat{z}}{\partial\hat{y}} - \hat{S}_{\hat{y}} - \frac{1}{2}g \left( \frac{\rho_{s} - \rho_{w}}{\hat{\rho}} \right) \hat{h}^{2}\frac{\partial C}{\partial\hat{y}} - \left( \frac{\rho_{s} - \rho_{w}}{\rho_{w}} \right) \hat{h}\hat{v} \left[ \frac{\partial}{\partial\hat{x}} \left( \hat{D}_{\hat{x}} \frac{\partial C}{\partial\hat{x}} \right) + \frac{\partial}{\partial\hat{y}} \left( \hat{D}_{\hat{y}} \frac{\partial C}{\partial\hat{y}} \right) \right] \\ \hat{h} \left[ \frac{\partial}{\partial\hat{x}} \left( \hat{D}_{\hat{x}} \frac{\partial C}{\partial\hat{x}} \right) + \frac{\partial}{\partial\hat{y}} \left( \hat{D}_{\hat{y}} \frac{\partial C}{\partial\hat{y}} \right) \right] \end{split}$$

Assuming molecular diffusivity is neglected:

$$\hat{\mathbf{Q}}_{,\hat{\mathbf{t}}} + \hat{\mathbf{F}}(\hat{\mathbf{Q}})_{,\hat{\mathbf{x}}} + \hat{\mathbf{G}}(\hat{\mathbf{Q}})_{,\hat{\mathbf{y}}} = \hat{\mathbf{S}}(\hat{\mathbf{Q}})$$
(A.15)

$$\begin{split} \hat{Q} &= \begin{pmatrix} \hat{h} \\ \hat{h}\hat{u} \\ \hat{h}\hat{v} \\ \hat{h}\hat{v} \\ C\hat{h} \end{pmatrix}, \hat{F}(\hat{Q}) &= \begin{pmatrix} \hat{h}\hat{u} \\ \hat{u}^{2}\hat{h} + \frac{1}{2}g\hat{h}^{2} \\ \hat{u}\hat{v}\hat{h} \\ \hat{u}\hat{v}\hat{h} \\ \hat{u}\hat{v}\hat{h} \end{pmatrix}, \hat{G}(\hat{Q}) &= \begin{pmatrix} \hat{h}\hat{v} \\ \hat{u}\hat{v}\hat{h} \\ \hat{v}^{2}\hat{h} + \frac{1}{2}g\hat{h}^{2} \\ \hat{v}\hat{v}\hat{h} \end{pmatrix}, \\ \hat{S}(\hat{Q}) &= \begin{pmatrix} 0 \\ -g\hat{h}\frac{\partial\hat{z}}{\partial\hat{x}} - \hat{S}_{\hat{x}} - \frac{1}{2}g\left(\frac{\rho_{s} - \rho_{w}}{\hat{\rho}}\right)\hat{h}^{2}\frac{\partial C}{\partial\hat{x}} \\ -g\hat{h}\frac{\partial\hat{z}}{\partial\hat{y}} - \hat{S}_{\hat{y}} - \frac{1}{2}g\left(\frac{\rho_{s} - \rho_{w}}{\hat{\rho}}\right)\hat{h}^{2}\frac{\partial C}{\partial\hat{y}} \\ 0 \end{pmatrix} \end{split}$$
### **B. DIMENSIONLESS EQUATIONS**

Firstly, dimensional scales are defined:

Horizontal:  $\mathcal{L}$ 

Vertical:  $\mathcal{H}$ 

Velocity:  $\mathcal{U}$ 

Froud Number:  $Fr = \frac{\mathcal{U}}{\sqrt{g\mathcal{H}}} \longrightarrow g = \frac{\mathcal{U}^2}{Fr^2\mathcal{H}}$ 

Relative density:  $s = \frac{\rho_s}{\rho_w} \longrightarrow \rho_s = s\rho_w$ 

$$\longrightarrow \frac{\hat{\rho}}{\rho_w} = (s+1-C) = \rho$$

Then, the transformed variables are:

$$\begin{split} \hat{x} &\longrightarrow \mathcal{L}x \\ \hat{y} &\longrightarrow \mathcal{L}y \\ \hat{z} &\longrightarrow \mathcal{H}z \\ \hat{h} &\longrightarrow \mathcal{H}h \\ \hat{u} &\longrightarrow \mathcal{U}u \\ \hat{v} &\longrightarrow \mathcal{U}v \\ \hat{t} &\longrightarrow \frac{\mathcal{L}}{\mathcal{U}}t \end{split}$$

## **B.1.** Continuity equation

Starting from the continuity equation for variable density flow presented in A.8:

$$\mathcal{H}\frac{\mathcal{U}}{\mathcal{L}}\frac{\partial h}{\partial t} + \mathcal{H}\mathcal{U}\frac{1}{\mathcal{L}}\frac{\partial hu}{\partial x} + \mathcal{H}\mathcal{U}\frac{1}{\mathcal{L}}\frac{\partial hv}{\partial y} = 0$$
(B.1)

Simplifying:

$$\frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} + \frac{\partial hv}{\partial y} = 0 \tag{B.2}$$

# **B.2.** Momentum equations

From the momentum equation in the x direction A.11:

$$\mathcal{H}\mathcal{U}\frac{\mathcal{U}}{\mathcal{L}}\frac{\partial hu}{\partial t} + \mathcal{U}^{2}\mathcal{H}\frac{1}{\mathcal{L}}\frac{\partial}{\partial x}\left(u^{2}h\right) + \frac{\mathcal{U}^{2}}{Fr^{2}\mathcal{H}}\mathcal{H}^{2}\frac{1}{\mathcal{L}}\frac{\partial}{\partial x}\left(\frac{1}{2}h^{2}\right) + \mathcal{U}^{2}\mathcal{H}\frac{1}{\mathcal{L}}\frac{\partial uvh}{\partial y} = -\frac{\mathcal{U}^{2}}{Fr^{2}\mathcal{H}}\mathcal{H}\frac{\mathcal{H}}{\mathcal{L}}h\frac{\partial z}{\partial x} - S_{x} - \frac{1}{2}\left(\frac{s-1}{C(s-1)+1}\right)\frac{\mathcal{U}^{2}}{Fr^{2}\mathcal{H}}\frac{\mathcal{H}^{2}}{L}h^{2}\frac{\partial C}{\partial x}$$
(B.3)

Simplifying:

$$\frac{\partial hu}{\partial t} + \frac{\partial}{\partial x} \left( u^2 h + \frac{h^2}{2Fr^2} \right) + \frac{\partial uvh}{\partial y} = -\frac{h}{Fr^2} \frac{\partial z}{\partial x} - S_x - \frac{h^2}{2Fr^2} \left( \frac{s-1}{C(s-1)+1} \right) \frac{\partial C}{\partial x}$$
(B.4)

Similarly, for the y direction, from equation A.12:

$$\frac{\partial hv}{\partial t} + \frac{\partial uvh}{\partial x} + \frac{\partial}{\partial y} \left( v^2 h + \frac{h^2}{2Fr^2} \right) = -\frac{h}{Fr^2} \frac{\partial z}{\partial y} - S_y - \frac{h^2}{2Fr^2} \left( \frac{s-1}{C(s-1)+1} \right) \frac{\partial C}{\partial y}$$
(B.5)

# **B.3.** Advection-Diffusion Equation

The advection-diffusion equation is exhibited in A.13. To write in terms of dimensionless variables:

$$\mathcal{H}\frac{\mathcal{U}}{\mathcal{L}}\frac{\partial Ch}{\partial t} + \mathcal{U}\mathcal{H}\frac{1}{\mathcal{L}}\frac{\partial uCh}{\partial x} + \mathcal{U}H\frac{1}{\mathcal{L}}\frac{\partial vCh}{\partial y} = 0$$
(B.6)

Simplifying:

$$\frac{\partial Ch}{\partial t} + \frac{\partial uCh}{\partial x} + \frac{\partial vCh}{\partial y} = 0$$
(B.7)

# **B.4.** Matrix form

$$\mathbf{Q}_{,\mathbf{t}} + \mathbf{F}(\mathbf{Q})_{,\mathbf{x}} + \mathbf{G}(\mathbf{Q})_{,\mathbf{y}} = \mathbf{S}_{\mathbf{B}}(\mathbf{Q}) + \mathbf{S}_{\mathbf{S}}(\mathbf{Q}) + \mathbf{S}_{\mathbf{C}}(\mathbf{Q})$$
(B.8)

$$Q = \begin{pmatrix} h \\ hu \\ hv \\ Ch \end{pmatrix}, F(Q) = \begin{pmatrix} hu \\ u^2h + \frac{1}{2Fr^2}h^2 \\ uvh \\ uCh \end{pmatrix}, G(Q) = \begin{pmatrix} hv \\ uvh \\ v^2h + \frac{1}{2Fr^2}h^2 \\ vCh \end{pmatrix},$$
$$S_B(Q) = \begin{pmatrix} 0 \\ -\frac{1}{Fr^2}h\frac{\partial z}{\partial x} \\ -\frac{1}{Fr^2}h\frac{\partial z}{\partial y} \\ 0 \end{pmatrix}, S_S(Q) = \begin{pmatrix} 0 \\ -S_x \\ -S_y \\ 0 \end{pmatrix}, S_C(Q) = \begin{pmatrix} 0 \\ -\frac{h^2}{2Fr^2}\left(\frac{s-1}{C(s-1)+1}\right)\frac{\partial C}{\partial x} \\ -\frac{h^2}{2Fr^2}\left(\frac{s-1}{C(s-1)+1}\right)\frac{\partial C}{\partial y} \\ 0 \end{pmatrix}$$

#### C. TRANSFORMATION TO CURVILINEAR COORDINATES

In this section, a detailed description of the coordinate transformation from cartesian coordinates to a generalized, boundary fitted, curvilinear coordinate is given. The new coordinate system changes the (x, y) cartesian directions to the curvilinear system  $(\xi, \eta)$ .

Since the model considers a partial transformation of coordinates, the hydrodynamic variables h, u, v and C remain referenced to the cartesian coordinates. In consequence, just the derivates respect to the x and y must to be modified to the new system coordinate.

## C.1. Continuity

Applying the chain rule to modify the partial derivates of the mass fluxes:

$$\frac{\partial h}{\partial t} + \frac{\partial uh}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial uh}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial vh}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial vh}{\partial \eta} \frac{\partial \eta}{\partial y} = 0$$
(C.1)

Besides, to rewrite the equation to a conservative form, the equation is multiplied and divided by the Jacobian J. This is defined as  $J = \xi_x \eta_y + \xi_y \eta_x$ , in which each term represents the metrics of the transformation (i.e. the variation of the  $\xi$  and  $\eta$  respect to x and y respectively).

$$\frac{\partial h}{\partial t} + J\left(\frac{\partial uh}{\partial \xi}\frac{\partial \xi}{\partial x}\frac{1}{J} + \frac{\partial uh}{\partial \eta}\frac{\partial \eta}{\partial x}\frac{1}{J} + \frac{\partial vh}{\partial \xi}\frac{\partial \xi}{\partial y}\frac{1}{J} + \frac{\partial vh}{\partial \eta}\frac{\partial \eta}{\partial y}\frac{1}{J}\right) = 0$$
(C.2)

Regrouping terms:

$$\frac{\partial h}{\partial t} + J \left[ \frac{\partial}{\partial \xi} \left( \frac{h(u\xi_x + v\xi_y)}{J} \right) + \frac{\partial}{\partial \eta} \left( \frac{h(u\eta_x + v\eta_y)}{J} \right) \right] = 0$$
(C.3)

Defining the contravariant velocity components as  $U^1 = u\xi_x + v\xi_y$  and  $U^2 = u\eta_x + v\eta_y$ respectively, the continuity equation can be written as:

$$\frac{\partial h}{\partial t} + J \left[ \frac{\partial}{\partial \xi} \left( \frac{hU^1}{J} \right) + \frac{\partial}{\partial \eta} \left( \frac{hU^2}{J} \right) \right] = 0$$
 (C.4)

#### C.2. Momentum equations

Following the same procedure with the momentum equation in the x direction, the chain rule is applied to the equation B.4:

$$\frac{\partial hu}{\partial t} + \xi_x \frac{\partial}{\partial \xi} \left( u^2 h + \frac{1}{2Fr^2} h^2 \right) + \eta_x \frac{\partial}{\partial \eta} \left( u^2 h + \frac{1}{2Fr^2} h^2 \right) + \xi_y \frac{\partial uvh}{\partial \xi} + \eta_y \frac{\partial uvh}{\partial \eta} = \frac{-h}{Fr^2} \left( z_\xi \xi_x + z_\eta \eta_x \right) - S_x - \frac{h^2 \left( C_\xi \xi_x + C_\eta \eta_x \right)}{2Fr^2} \left( \frac{s-1}{C(s-1)+1} \right)$$
(C.5)

Note that the shear stress term  $S_x$  is not modify since it does not includes spatial derivates.

Multiplying and dividing the equation by the Jacobian:

$$\frac{\partial hu}{\partial t} + J \left[ \frac{\xi_x}{J} \frac{\partial}{\partial \xi} \left( u^2 h + \frac{1}{2Fr^2} h^2 \right) + \frac{\eta_x}{J} \frac{\partial}{\partial \eta} \left( u^2 h + \frac{1}{2Fr^2} h^2 \right) + \frac{\xi_y}{J} \frac{\partial uvh}{\partial \xi} + \frac{\eta_y}{J} \frac{\partial uvh}{\partial \eta} \right] = \frac{-h}{Fr^2} \left( z_\xi \xi_x + z_\eta \eta_x \right) - S_x - \frac{h^2 \left( C_\xi \xi_x + C_\eta \eta_x \right)}{2Fr^2} \left( \frac{s-1}{C(s-1)+1} \right) \quad (C.6)$$

Regrouping terms and considering the conservations relations  $\frac{\partial}{\partial \xi} \left(\frac{\xi_x}{J}\right) + \frac{\partial}{\partial \eta} \left(\frac{\eta_x}{J}\right) = 0$  and  $\frac{\partial}{\partial \xi} \left(\frac{\xi_y}{J}\right) + \frac{\partial}{\partial \eta} \left(\frac{\eta_y}{J}\right) = 0$  valide to generalized curvilinear coordinates, the equation is simplified as:

$$\frac{\partial hu}{\partial t} + J \left[ \frac{\partial}{\partial \xi} \left( \frac{uh(u\xi_x + v\xi_y)}{J} \right) + \frac{\partial}{\partial \eta} \left( \frac{uh(u\eta_x + v\eta_y)}{J} \right) \right] + \frac{1}{2Fr^2} \left[ \frac{\partial}{\partial \xi} \left( \frac{h^2\xi_x}{J} \right) + \frac{\partial}{\partial \eta} \left( \frac{h^2\eta_x}{J} \right) \right] = \frac{-h}{Fr^2} (z_\xi\xi_x + z_\eta\eta_x) - S_x \qquad (C.7) - \frac{h^2 (C_\xi\xi_x + C_\eta\eta_x)}{2Fr^2} \left( \frac{s-1}{C(s-1)+1} \right)$$

Finally, replacing by the contravariant velocity components:

$$\frac{\partial hu}{\partial t} + J \left[ \frac{\partial}{\partial \xi} \left( \frac{uhU^1 + \frac{1}{2Fr^2}h^2\xi_x}{J} \right) + \frac{\partial}{\partial \eta} \left( \frac{uhU^2 + \frac{h^2}{2Fr^2}\eta_x}{J} \right) \right] = \frac{-h}{Fr^2} \left( z_{\xi}\xi_x + z_{\eta}\eta_x \right) - S_x - \frac{h^2 \left( C_{\xi}\xi_x + C_{\eta}\eta_x \right)}{2Fr^2} \left( \frac{s-1}{C(s-1)+1} \right)$$
(C.8)

Following the same procedure, the momentum equation in the direction y is transformed to:

$$\frac{\partial hv}{\partial t} + J \left[ \frac{\partial}{\partial \xi} \left( \frac{vhU^1 + \frac{1}{2Fr^2}h^2\xi_y}{J} \right) + \frac{\partial}{\partial \eta} \left( \frac{vhU^2 + \frac{1}{2Fr^2}h^2\eta_y}{J} \right) \right] = \frac{-h}{Fr^2} \left( z_{\xi}\xi_y + z_{\eta}\eta_y \right) - S_y - \frac{h^2 \left( C_{\xi}\xi_y + C_{\eta}\eta_y \right)}{2Fr^2} \left( \frac{s-1}{C(s-1)+1} \right)$$
(C.9)

## C.3. Advection-Diffusion Equation

To transform the sediment transport equation, the same procedure followed with the continuity equation is adopted. Applying the chain rule to the equation B.7:

$$\frac{\partial Ch}{\partial t} + \frac{\partial uCh}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial uCh}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial vCh}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial vCh}{\partial \eta} \frac{\partial \eta}{\partial y} = 0$$
(C.10)

Multiplying and dividing by the Jacobian J:

$$\frac{\partial Ch}{\partial t} + J\left(\frac{\partial uCh}{\partial \xi}\frac{\partial \xi}{\partial x}\frac{1}{J} + \frac{\partial uCh}{\partial \eta}\frac{\partial \eta}{\partial x}\frac{1}{J} + \frac{\partial vCh}{\partial \xi}\frac{\partial \xi}{\partial y}\frac{1}{J} + \frac{\partial vCh}{\partial \eta}\frac{\partial \eta}{\partial y}\frac{1}{J}\right) = 0 \quad (C.11)$$

Regrouping terms:

$$\frac{\partial Ch}{\partial t} + J \left[ \frac{\partial}{\partial \xi} \left( \frac{Ch(u\xi_x + v\xi_y)}{J} \right) + \frac{\partial}{\partial \eta} \left( \frac{Ch(u\eta_x + v\eta_y)}{J} \right) \right] = 0$$
(C.12)

And replacing by the contravariant velocity components, the advection-diffusion equation is written as:

$$\frac{\partial Ch}{\partial t} + J \left[ \frac{\partial}{\partial \xi} \left( \frac{ChU^1}{J} \right) + \frac{\partial}{\partial \eta} \left( \frac{ChU^2}{J} \right) \right] = 0$$
 (C.13)

## C.4. Matrix form

Thus, the set of equation transformed to curvilinear coordinates in matrix form is written as follows:

$$\mathbf{Q}_{,t} + \mathbf{J}\mathbf{F}_{,\xi} + \mathbf{J}\mathbf{G}_{,\eta} = \mathbf{S}_{\mathbf{B}}(\mathbf{Q}) + \mathbf{S}_{\mathbf{S}}(\mathbf{Q}) + \mathbf{S}_{\mathbf{C}}(\mathbf{Q})$$
(C.14)

$$Q = \begin{pmatrix} h \\ hu \\ hv \\ Ch \end{pmatrix}, F = \frac{1}{J} \begin{pmatrix} hU_1 \\ uhU^1 + \frac{1}{2Fr^2}h^2\xi_x \\ vhU^1 + \frac{1}{2Fr^2}h^2\xi_y \\ ChU^1 \end{pmatrix}, G = \frac{1}{J} \begin{pmatrix} hU_2 \\ uhU^2 + \frac{1}{2Fr^2}h^2\eta_x \\ vhU^2 + \frac{1}{2Fr^2}h^2\eta_y \\ ChU^2 \end{pmatrix},$$
$$S_B = \begin{pmatrix} 0 \\ \frac{-h}{Fr^2}(z_\xi\xi_x + z_\eta\eta_x) \\ \frac{-h}{Fr^2}(z_\xi\xi_y + z_\eta\eta_y) \\ 0 \end{pmatrix}, S_S(Q) = \begin{pmatrix} 0 \\ -S_x \\ -S_y \\ 0 \end{pmatrix},$$
$$S_C(Q) = \begin{pmatrix} 0 \\ -\frac{h^2}{2Fr^2}\left(\frac{s-1}{C(s-1)+1}\right)\left(C_\xi\xi_x + C_\eta\eta_x\right) \\ -\frac{h^2}{2Fr^2}\left(\frac{s-1}{C(s-1)+1}\right)\left(C_\xi\xi_y + C_\eta\eta_y\right) \\ 0 \end{pmatrix}$$

#### D. NON-CONSERVATIVE FORM OF THE EQUATIONS

#### **D.1.** Continuity equation

Replacing the terms  $U^1$  and  $U^2$  in the equation C.4 and expanding the derivatives, it is possible to consider the conservation relations described in the previous section to simplify the following terms:

$$\frac{\partial h}{\partial t} + J \left[ hu \frac{\partial}{\partial \xi} \left( \frac{\xi_x}{J} \right) + \frac{\xi_x}{J} \frac{\partial (hu)}{\partial \xi} + hu \frac{\partial}{\partial \eta} \left( \frac{\eta_x}{J} \right) + \frac{\eta_x}{J} \frac{\partial (hu)}{\partial \eta} \right] + J \left[ hv \frac{\partial}{\partial \xi} \left( \frac{\xi_y}{J} \right) + \frac{\xi_y}{J} \frac{\partial (hv)}{\partial \xi} + hv \frac{\partial}{\partial \eta} \left( \frac{\eta_y}{J} \right) + \frac{\eta_y}{J} \frac{\partial (hv)}{\partial \eta} \right] = 0$$
(D.1)

Since the Jacobians are preserved just outside the derivatives, they can be simplified:

$$\frac{\partial h}{\partial t} + \mathcal{J}\left[\frac{\xi_x}{\mathcal{J}}\frac{\partial(hu)}{\partial\xi} + \frac{\xi_y}{\mathcal{J}}\frac{\partial(hv)}{\partial\xi} + \frac{\eta_x}{\mathcal{J}}\frac{\partial(hu)}{\partial\eta} + \frac{\eta_y}{\mathcal{J}}\frac{\partial(hv)}{\partial\eta}\right] = 0$$
(D.2)

Thus, the non-conservative form of the continuity equation looks as:

$$\frac{\partial h}{\partial t} + \xi_x \frac{\partial (hu)}{\partial \xi} + \xi_y \frac{\partial (hv)}{\partial \xi} + \eta_x \frac{\partial (hu)}{\partial \eta} + \eta_y \frac{\partial (hv)}{\partial \eta} = 0$$
(D.3)

#### **D.2.** Momentum equations

To simplify the description of the procedure carried out to write the momentum equations in a non-conservative form, just the left-hand side of the equation will be explained. The right-hand side remains unchanged.

Considering the conservation relations to curvilinear coordinates, the momentum equation in the x direction (Eq. C.8) is simplified as:

$$\frac{\partial hu}{\partial t} + J \left[ \frac{\partial}{\partial \xi} \left( \frac{uhU^1}{J} \right) + \frac{h^2}{2Fr^2} \frac{\partial}{\partial \xi} \left( \frac{\xi_x}{J} \right)^0 + \frac{h}{Fr^2} \frac{\xi_x}{J} \frac{\partial h}{\partial \xi} \right] + \frac{\partial}{\partial \eta} \left( \frac{uhU^2}{J} \right) + \frac{h^2}{2Fr^2} \frac{\partial}{\partial \eta} \left( \frac{\eta_x}{J} \right)^0 + \frac{h}{Fr^2} \frac{\eta_x}{J} \frac{\partial h}{\partial \eta} = S_B + S_S + S_C$$
(D.4)

Expanding the derivates:

$$\frac{\partial hu}{\partial t} + J \left[ \frac{U^1}{J} \frac{\partial uh}{\partial \xi} + uh \frac{\partial}{\partial \xi} \left( \frac{U^1}{J} \right) + \frac{h}{Fr^2} \frac{\xi_x}{J} \frac{\partial h}{\partial \xi} \right] \\ + \frac{U^2}{J} \frac{\partial uh}{\partial \eta} + uh \frac{\partial}{\partial \eta} \left( \frac{U^2}{J} \right) + \frac{h}{Fr^2} \frac{\eta_x}{J} \frac{\partial h}{\partial \eta} = S_B + S_S + S_C$$
(D.5)

And rearranging terms:

$$\frac{\partial hu}{\partial t} + \frac{h\xi_x h_\xi}{Fr^2} + U^1 \frac{\partial uh}{\partial \xi} + \frac{h\eta_x h_\eta}{Fr^2} + U^2 \frac{\partial uh}{\partial \eta} + J\left[\frac{\partial}{\partial \xi}\left(\frac{uhU^1}{J}\right) - \frac{U^1}{J}\frac{\partial uh}{\partial \xi} + \frac{\partial}{\partial \eta}\left(\frac{uhU^2}{J}\right) - \frac{U^2}{J}\frac{\partial uh}{\partial \eta}\right] = S_B + S_S + S_C$$
(D.6)

Applying the product rule for derivatives to separate the u variable:

$$\frac{\partial hu}{\partial t} + \frac{h\xi_x}{Fr^2}\frac{\partial h}{\partial \xi} + U^1\frac{\partial uh}{\partial \xi} + \frac{h\eta_x}{Fr^2}\frac{\partial h}{\partial \eta} + U^2\frac{\partial uh}{\partial \eta} + J\left[u\frac{\partial}{\partial \xi}\left(\frac{hU^1}{J}\right) - \frac{uU^1}{J}\frac{\partial h}{\partial \xi} + u\frac{\partial}{\partial \eta}\left(\frac{hU^2}{J}\right) - \frac{uU^2}{J}\frac{\partial h}{\partial \eta}\right] = S_B + S_S + S_C$$
(D.7)

And regrouping terms:

$$\frac{\partial hu}{\partial t} + \left(\frac{h\xi_x}{Fr^2} - uU^1\right)\frac{\partial h}{\partial\xi} + U^1\frac{\partial uh}{\partial\xi} + Ju\frac{\partial}{\partial\xi}\left(\frac{hU^1}{J}\right) + \frac{\partial h}{\partial\eta}\left(\frac{h\eta_x}{Fr^2} - uU^2\right) + U^2\frac{\partial uh}{\partial\eta} + Ju\frac{\partial}{\partial\eta}\left(\frac{hU^2}{J}\right) = S_B + S_S + S_C$$
(D.8)

Expanding the contravariant velocity components  $U_1$  and  $U_2$ :

$$\frac{\partial hu}{\partial t} + \left(\frac{h\xi_x}{Fr^2} - uU^1\right)\frac{\partial h}{\partial \xi} + \left(U^1 + u\xi_x\right)\frac{\partial hu}{\partial \xi} + u\xi_y\frac{\partial hv}{\partial \xi} + \left(\frac{h\eta_x}{Fr^2} - uU^2\right)\frac{\partial h}{\partial \eta} + \left(U^2 + u\eta_x\right)\frac{\partial hu}{\partial \eta} + u\eta_y\frac{\partial hv}{\partial \eta} = S_B + S_S + S_C$$
(D.9)

Rearranging, the non-conservative form of the x direction of the momentum equations is written as follow:

$$\frac{\partial hu}{\partial t} + \left(\frac{h\xi_x}{Fr^2} - uU^1\right)\frac{\partial h}{\partial \xi} + \left(U^1 + u\xi_x\right)\frac{\partial hu}{\partial \xi} + u\xi_y\frac{\partial hv}{\partial \xi} + \left(\frac{h\eta_x}{Fr^2} - uU^2\right)\frac{\partial h}{\partial \eta} + \left(U^2 + u\eta_x\right)\frac{\partial hu}{\partial \eta} + u\eta_y\frac{\partial hv}{\partial \eta} =$$
(D.10)  
$$\frac{-h}{Fr^2}\left(z_\xi\xi_x + z_\eta\eta_x\right) - S_x - \frac{h^2}{2Fr^2}\left(\frac{s-1}{C(s-1)+1}\right)\left(C_\xi\xi_x + C_\eta\eta_x\right)$$

And similarly in the y direction:

$$\frac{\partial hv}{\partial t} + \left(\frac{h\xi_y}{Fr^2} - vU^1\right)\frac{\partial h}{\partial \xi} + v\xi_x\frac{\partial hu}{\partial \xi} + \left(U^1 + v\xi_y\right)\frac{\partial hv}{\partial \xi} + \left(\frac{h\eta_y}{Fr^2} - vU^2\right)\frac{\partial h}{\partial \eta} + v\eta_x\frac{\partial hu}{\partial \eta} + \left(U^2 + v\eta_y\right)\frac{\partial hv}{\partial \eta} =$$
(D.11)  
$$\frac{-h}{Fr^2}\left(z_\xi\xi_y + z_\eta\eta_y\right) - S_y - \frac{h^2}{2Fr^2}\left(\frac{s-1}{C(s-1)+1}\right)\left(C_\xi\xi_y + C_\eta\eta_y\right)$$

# **D.3.** Advection-Diffusion Equation

Expanding the derivatives in the equation C.13, and considering the conservation relations to curvilinear coordinates:

$$\frac{\partial Ch}{\partial t} + J \left[ uCh \frac{\partial}{\partial \xi} \left( \frac{\xi_x}{J} \right) + \frac{\xi_x}{J} \frac{\partial (uCh)}{\partial \xi} + uCh \frac{\partial}{\partial \eta} \left( \frac{\eta_x}{J} \right) + \frac{\eta_x}{J} \frac{\partial (uCh)}{\partial \eta} \right] + J \left[ vCh \frac{\partial}{\partial \xi} \left( \frac{\xi_y}{J} \right) + \frac{\xi_y}{J} \frac{\partial (vCh)}{\partial \xi} + vCh \frac{\partial}{\partial \eta} \left( \frac{\eta_y}{J} \right) + \frac{\eta_y}{J} \frac{\partial (vCh)}{\partial \eta} \right] = 0$$
(D.12)

Simplifying the Jacobian out side of the derivatives:

$$\frac{\partial Ch}{\partial t} + \xi_x \frac{\partial (uCh)}{\partial \xi} + \xi_y \frac{\partial (vCh)}{\partial \xi} + \eta_x \frac{\partial (uCh)}{\partial \eta} + \eta_y \frac{\partial (vCh)}{\partial \eta} = 0$$
(D.13)

Expanding the derivatives:

$$\frac{\partial Ch}{\partial t} - C\left(u\xi_x + v\xi_y\right)\frac{\partial h}{\partial \xi} + C\xi_x\frac{\partial hu}{\partial \xi} + C\xi_y\frac{\partial hv}{\partial \xi} + \left(u\xi_x + v\xi_y\right)\frac{\partial Ch}{\partial \xi} - C\left(u\eta_x + v\eta_y\right)\frac{\partial h}{\partial \eta} + C\eta_x\frac{\partial hu}{\partial \eta} + C\eta_y\frac{\partial hv}{\partial \eta} + \left(u\eta_x + v\eta_y\right)\frac{\partial Ch}{\partial \eta} = 0$$
(D.14)

And replacing by the expression for  $U_1$  and  $U_2$ , the non-conservative form of the advection-diffusion equation is written as:

$$\frac{\partial Ch}{\partial t} - CU^{1}\frac{\partial h}{\partial \xi} + C\xi_{x}\frac{\partial hu}{\partial \xi} + C\xi_{y}\frac{\partial hv}{\partial \xi} + U^{1}\frac{\partial Ch}{\partial \xi} - CU^{2}\frac{\partial h}{\partial \eta} + C\eta_{x}\frac{\partial hu}{\partial \eta} + C\eta_{y}\frac{\partial hv}{\partial \eta} + U^{2}\frac{\partial Ch}{\partial \eta} = 0$$
(D.15)

# **D.4.** Matrix Form

$$\mathbf{U}_{,\mathbf{t}} + \mathbf{A}^{1}\mathbf{Q}_{,\xi} + \mathbf{A}^{2}\mathbf{Q}_{,\eta} = \mathbf{S}_{\mathbf{B}} + \mathbf{S}_{\mathbf{S}} + \mathbf{S}_{\mathbf{C}}$$
(D.16)

Where,

$$U = \begin{pmatrix} h \\ hu \\ hv \\ Ch \end{pmatrix},$$

$$A^{1} = \begin{pmatrix} 0 & \xi_{x} & \xi_{y} & 0 \\ \frac{h\xi_{x}}{Fr^{2}} - uU^{1} & U^{1} + u\xi_{x} & u\xi_{y} & 0 \\ \frac{h\xi_{y}}{Fr^{2}} - vU^{1} & v\xi_{x} & U^{1} + v\xi_{y} & 0 \\ -CU^{1} & C\xi_{x} & C\xi_{y} & U^{1} \end{pmatrix},$$

$$A^{2} = \begin{pmatrix} 0 & \eta_{x} & \eta_{y} & 0 \\ \frac{h\eta_{x}}{Fr^{2}} - uU^{2} & U^{2} + u\eta_{x} & u\eta_{y} & 0 \\ \frac{h\eta_{y}}{Fr^{2}} - vU^{2} & v\eta_{x} & U^{2} + v\eta_{y} & 0 \\ -CU^{2} & C\eta_{x} & C\eta_{y} & U^{2} \end{pmatrix},$$

$$S_{B} = \begin{pmatrix} 0 \\ \frac{-h}{Fr^{2}} (z_{\xi}\xi_{x} + z_{\eta}\eta_{x}) \\ \frac{-h}{Fr^{2}} (z_{\xi}\xi_{y} + z_{\eta}\eta_{y}) \\ 0 \end{pmatrix}, S_{S} = \begin{pmatrix} 0 \\ -S_{x} \\ -S_{y} \\ 0 \end{pmatrix},$$

$$S_{C} = \begin{pmatrix} 0 \\ -\frac{h^{2}}{2Fr^{2}} \left( \frac{s-1}{C(s-1)+1} \right) (C_{\xi}\xi_{x} + C_{\eta}\eta_{x}) \\ -\frac{h^{2}}{2Fr^{2}} \left( \frac{s-1}{C(s-1)+1} \right) (C_{\xi}\xi_{y} + C_{\eta}\eta_{y}) \\ 0 \end{pmatrix}$$

# **D.5.** Eigenvalues of Jacobian matrices

Considering:

$$U^1 = u\xi_x + v\xi_y \quad U^2 = u\eta_x + v\eta_y \tag{D.17}$$

The eigenvalues are:

$$\lambda_1^1 = U^1 - \sqrt{\frac{h(\xi_x^2 + \xi_y^2)}{Fr^2}}, \lambda_2^1 = U^1, \lambda_3^1 = U^1, \lambda_4^1 = U^1 + \sqrt{\frac{h(\xi_x^2 + \xi_y^2)}{Fr^2}}$$
(D.18)

$$\lambda_1^2 = U^2 - \sqrt{\frac{h(\eta_x^2 + \eta_y^2)}{Fr^2}}, \lambda_2^2 = U^2, \lambda_3^2 = U^2, \lambda_4^2 = U^2 + \sqrt{\frac{h(\eta_x^2 + \eta_y^2)}{Fr^2}}$$
(D.19)

#### **E. SOURCE TERMS TREATMENT**

#### E.1. Concentration's Source

The source term  $S_C$  is represented in  $\xi$  and  $\eta$  coordinates respectively as follows:

$$S_{C_{\xi}} = \frac{h^2}{2Fr^2} \left( \frac{s-1}{C(s-1)+1} \right) \left( C_{\xi} \xi_x + C_{\eta} \eta_x \right)$$
(E.1)

$$S_{C_{\eta}} = \frac{h^2}{2Fr^2} \left( \frac{s-1}{C(s-1)+1} \right) \left( C_{\xi} \xi_y + C_{\eta} \eta_y \right)$$
(E.2)

Discretising each term:

$$S_{C_{\xi(n+1,i)}} = \frac{h_{(n,i)}^2}{2Fr^2} \left( \frac{s-1}{C_{(n,i)}(s-1)+1} \right) \left( \frac{\tilde{C}_{\xi(n,i+\frac{1}{2})} - \tilde{C}_{\xi(n,i-\frac{1}{2})}}{\Delta \xi} \xi_x + \frac{\tilde{C}_{\eta(n,i+\frac{1}{2})} - \tilde{C}_{\eta(n,i-\frac{1}{2})}}{\Delta \eta} \eta_x \right)$$
(E.3)

$$S_{C_{\eta_{(n+1,i)}}} = \frac{h_{(n,i)}^2}{2Fr^2} \left( \frac{s-1}{C_{(n,i)}(s-1)+1} \right) \left( \frac{\tilde{C}_{\xi_{(n,i+\frac{1}{2})}} - \tilde{C}_{\xi_{(n,i-\frac{1}{2})}}}{\Delta \xi} \xi_y + \frac{\tilde{C}_{\eta_{(n,i+\frac{1}{2})}} - \tilde{C}_{\eta_{(n,i-\frac{1}{2})}}}{\Delta \eta} \eta_y \right)$$
(E.4)

Where  $\tilde{C}_{\xi}$  and  $\tilde{C}_{\eta}$  represents the value of C in the position  $i \pm \frac{1}{2}$  in the direction  $\xi$  and  $\eta$  respectively. They are computed using a first order upwind method.

## E.2. Shear Stress

To solve the shear stress source term, we solve the equations E.5 on the x and y direction, by means a splitting semi-implicit method (Liang & Marche, 2009).

$$\frac{\partial \hat{h}\hat{u}}{\partial \hat{t}} = -\hat{S}_{\hat{x}}; \ \frac{\partial \hat{h}\hat{v}}{\partial \hat{t}} = -\hat{S}_{\hat{y}}$$
(E.5)

Here,  $\hat{S}_x$  and  $\hat{S}_y$  are the source term representing the total shear stress on the  $\hat{x}$  and  $\hat{y}$  directions respectively. The rheological model implemented considers total shear stress

considers the linear sum of the three terms: (i) the yield and Mohr-Coulomb  $\hat{S}_{yield}$ , (ii) viscous  $\hat{S}_{v_i}$ , and (iii) dispersive and turbulent stresses  $\hat{S}_{td_i}$ :

$$\hat{S}_i = \hat{S}_{yield} + \hat{S}_{v_i} + \hat{S}_{td_i} \tag{E.6}$$

On the other hand, the generic source term can be related to the shear stress by means:

$$\hat{S}_i = \frac{\hat{\tau}_i}{\hat{\rho}} \tag{E.7}$$

where,  $\hat{\rho}$  represents the mixture density and  $\hat{\tau}_i$  the bed shear stress, which can be calculated as follow,

$$\hat{\tau}_i = \hat{\rho} C_f \, \hat{u}_i \sqrt{\hat{u}^2 + \hat{v}^2} \tag{E.8}$$

Here,  $C_f$  represents the non-dimensional friction coefficient, which will be computed depending on the physic effect that it is representing. In the following sections, the derivation of the expressions to compute each term in the equation E.6 are described.

#### E.2.1. Yield and Mohr-Coulomb term

The model considers that the yield stress is isotropic in both flow directions. Thus, the source term  $\hat{S}_{yield}$  can be expressed as:

$$\hat{S}_{yield} = \frac{\hat{\tau}_{yield}}{\hat{\rho}} \tag{E.9}$$

Besides, considering the yield stress as a function exclusively of the sediment concentration in the fluid (Julien, 2010):

$$\hat{\tau}_{yield} = a \ 10^{bC} \tag{E.10}$$

Here a and b are experimental coefficients equal to 0.005 and 7.5 for typical soils (Julien, 2010). The source term representing the yield and Mohr-Coulomb stresses can be expressed as:

$$\hat{S}_{yield} = \frac{a \ 10^{bC}}{\hat{\rho}} \tag{E.11}$$

#### E.2.2. Viscous term

To compute the viscous effects, we considered the definition of the bed shear stress in the equation E.8, and used the laminar friction coefficient to represent the non-dimensional friction coefficient  $C_f$ :

$$C_f = \frac{k}{Re} \tag{E.12}$$

where k is the viscous resistance parameter equal to 64 in open-channel flows (Sturm, 2001), and Re the Reynold number defined as:

$$Re = \frac{\hat{\rho}\sqrt{\hat{u}^2 + \hat{v}^2}\hat{h}}{\hat{\mu}_m} \tag{E.13}$$

Here,  $\hat{\mu}_m$  is the dynamic viscosity of the mixture, which can be computed depending on the sediment concentration, by means (Eyring, 1964; Thomas, 1965):

$$\frac{\hat{\mu}_m}{\mu_w} = 1 + 2.5C + 10.05C^2 + 0.00273 \exp(16.6C)$$
(E.14)

Thus, replacing the laminar friction coefficient (equation E.12), in the equation E.8, we obtained the expression to the source term representing the viscous stress:

$$\hat{S}_{v_i} = \frac{k \ \hat{\mu}_m \ \hat{u}_i}{8 \hat{\rho} \hat{h}} \tag{E.15}$$

### E.2.3. Dispersive and turbulent term

In this case, the source term  $\hat{S}_{td_i}$  represent the sum of the dispersive and turbulent effects. Thus, we used the expression E.8 to compute the bed shear stress, however, the friction coefficient  $C_f$  is considered as the sum of two Darcy-Weisbach, the coefficient representing the turbulent losses  $f_t$  and the dispersive losses  $f_d$ :

$$C_f = \frac{f_{td}}{8} = \frac{f_t + f_d}{8}$$
(E.16)

To compute  $f_t$  we used Colebrook's equation:

$$\frac{1}{\sqrt{f_t}} = -2\log\left(\frac{\hat{k}_s}{3.7\mathcal{H}h} + \frac{2.51}{Re\sqrt{f_t}}\right) \tag{E.17}$$

in which,  $\hat{k}_s$  is the specific roughness, which is estimated from the mean sediment diameter  $d_s$ , as follow (Bathurst, 1978):

$$\hat{k}_s = 6.8d_s \tag{E.18}$$

On the other hand,  $f_d$  is computed using the following equation (Takahashi, 2007):

$$\sqrt{\frac{8}{f_d}} = \frac{2\mathcal{H}h}{5d_s} \left\{ \frac{1}{0.02} \left[ C + (1-C) \frac{\rho_w}{\rho_s} \right] \right\}^{\frac{1}{2}} \lambda^{-1}$$
(E.19)

Here,  $\lambda$  is Bagnold's linear concentration defined as (Bagnold, 1954):

$$\lambda^{-1} = \left[ \left( \frac{C^*}{C} \right)^{\frac{1}{3}} - 1 \right] \tag{E.20}$$

Finally, we use the following relation to transform the turbulent-dispersive Darcy-Weisbach friction coefficient in an equivaleent Manning o Chézy coefficient (Julien, 2010):

$$\sqrt{\frac{8}{f_{td}}} = C_{z_{td}}Fr = \frac{h^{1/6}Fr}{n_{td}}$$
(E.21)

The source term representing the added effect of the turbulence and dispersion  $\hat{S}_{td_i}$  is computed replacing the friction coefficient  $C_f$  (equation E.16) in the equation E.8. Thus, the expression considering the Manning coefficient looks as:

$$\hat{S}_{td_i} = \frac{\hat{n}_{td}^2 \,\hat{u}_i \,\sqrt{\hat{u}^2 + \hat{v}^2}g}{\hat{h}^{1/3}} \tag{E.22}$$

and the Chézy coefficient:

$$\hat{S}_{td_i} = \frac{\hat{u}_i \sqrt{\hat{u}^2 + \hat{v}^2}g}{C_{z_{td}}^2}$$
(E.23)

#### **E.2.4.** Dimensionless version

To obtain the non-dimensional version of the equation E.5, we consider the dimensional scales  $\mathcal{L}$ ,  $\mathcal{H}$  and  $\mathcal{U}$ , the definition of the Froude Number and the non-dimensional version of the hydrodynamic variables, described in section B.

Besides, we defined the non-dimensional transformation of  $\hat{\tau}_{yield}$  and  $\hat{\mu}_m$  in equations E.24 and E.25:

$$\tau_{yield} = \frac{\hat{\tau}_{yield}}{\rho_w \mathcal{U}^2} \tag{E.24}$$

$$\mu_w = \frac{\hat{\mu}_m}{\rho_w \mathcal{UH}} \tag{E.25}$$

We transform the equation E.5 in its non-dimensional version:

$$\mathcal{H}\mathcal{U}\frac{\mathcal{U}}{\mathcal{L}}\frac{dhu_i}{dt} = \mathcal{U}^2 S_i \tag{E.26}$$

Finally, simplifying and replacing the values of  $S_i$  by the sum of the equations E.11, E.15 and E.22 or E.23, according equation E.6, we obtained the expression to compute the total shear stress in the numerical model.

$$\frac{dhu_i}{dt} = \frac{\mathcal{L}}{\mathcal{H}}S_i = \frac{\mathcal{L}}{\mathcal{H}}\left(\frac{\tau_{yield}}{\rho} + \frac{k\mu_m u_i}{8h\rho} + \frac{n_{td}^2 u_i \sqrt{u^2 + v^2}}{Fr^2 h^{1/3}} \text{ or } \frac{u_i \sqrt{u^2 + v^2}}{Fr^2 C_{z_{td}}^2}\right)$$
(E.27)