FACULTAD DE MATEMÁTICAS

ON THE GEOGRAPHY OF SURFACES OF GENERAL TYPE WITH FIXED FUNDAMENTAL GROUP

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Contents

To my family and Anita, you are very important to me.

Abstract

In this thesis, we study the geography of complex surfaces of general type with respect to the topological fundamental group. The understanding of this general problem can be coarsely divided into geography of simply-connected surfaces and geography of non-simply-connected surfaces.

The geography of simply-connected surfaces was intensively studied in the eighties and nineties by Persson, Chen, and Xiao among others. Due to their works, we know that the set of Chern slopes c_1^2/c_2 of simply-connected surfaces of general type is dense in the interval $[\frac{1}{5}, 2]$. The last result which closes the density problem for this type of surfaces happened in 2015. Roulleau and Urzúa showed the density of the Chern slopes in the interval $[1, 3]$. This completes the study since accumulation points of c_1^2/c_2 belong to the interval $[\frac{1}{5}, 3]$ by the Noether's inequality and the Bogomolov-Miyaoka-Yau inequality for complex surfaces.

The geography of non-simply-connected surfaces is well understood only for small Chern slopes. Indeed, because of works of Mendes, Pardini, Reid, and Xiao, we know that for $c_1^2/c_2 \in [\frac{1}{5}, \frac{1}{3}]$ the fundamental group is either finite with at most nine elements, or the fundamental (algebraic) group is commensurable with the fundamental (algebraic) group of a curve. Furthermore, a well-known conjecture of Reid states that for minimal surfaces of general type with $c_1^2/c_2 < \frac{1}{2}$ the topological fundamental group is either finite or it is commensurable with the fundamental group of a curve. Due to Severi-Pardini's inequality and a theorem of Xiao, Reid's conjecture is true, at least in the algebraic sense for irregular surfaces or surfaces having an irregular étale cover. Keum showed with an example in his doctoral thesis that Reid's conjecture cannot be extended over $\frac{1}{2}$.

For higher slopes essentially there are no general results. In this thesis, we prove that for any topological fundamental group G of a given non-singular complex projective surface, the Chern slopes $c_1^2(S)/c_2(S)$ of minimal non-singular projective surfaces of general type S with $\pi_1(S) \simeq G$ are dense in the interval $[1, 3]$. It remains open the question for non-simply-connected surfaces in the interval $[\frac{1}{2}, 1]$.

1 Introduction

1.1 Surfaces and Moduli

Our principal objects of study are smooth projective surfaces over the complex numbers. Our focus is on the classification of surfaces. For instance, in the case of curves, we know that they are parametrized by their genus (number of holes). If the genus of a curve is zero, then the curve is the projective line; if the genus is one, the curve is a complex torus; and if the genus is at least two, the curve is a connected sum of tori. For surfaces S (smooth projective) the most common biregular invariants are:

- The first Chern number $c_1^2(S) := c_1^2(\Omega_S^*) = K_S^2$, where Ω_S^* is the dual of the sheaf of differentials on S,
- The second Chern number $c_2(S) := c_2(\Omega_S^*)$,
- The geometric genus $p_g(S) := h^2(S, \mathcal{O}_S)$,
- The irregularity $q(S) := h^1(S, \mathcal{O}_S)$,
- The plurigenera $P_n(S) := h^0(S, \mathcal{O}_S(nK_S))$,
- The holomorphic Euler characteristic $\chi(\mathcal{O}_S) := 1 - q(S) + p_g(S)$.

The numbers c_1^2, c_2 and χ are related (and so q, p_g) by the Noether's formula $12\chi = c_1^2 + c_2$, [No1877]. The use of the plurigenera allows us to divide the surfaces into four big birational classes. This classification is known as Enriques classification:

(1) P_n vanish always;

(2) $P_n \in \{0, 1\}$, but, $P_n = 1$ for some n;

(3) P_n grows linearly in n;

(4) P_n grows quadratically in n.

The surfaces in the class (1) are birational to $\mathbb{P}^1 \times C$, where C is a curve. If $C \cong \mathbb{P}^1$ the surfaces are called rational, otherwise they are called ruled.

The surfaces in the class (2) are divided into four subclasses, depending on the values of q and p_g. If $p_g = q = 0$ they are called Enriques surfaces, if $p_g = 0$ and $q = 1$ these surfaces are called Bi-elliptic, if $p_g = 1$ and $q = 0$ the surfaces are called $K3$, and finally, if $p_g = 1$ and $q = 2$ we have Abelian surfaces.

The surfaces in class (3) are called elliptic surfaces, because, for any surface S in that class there exists a fibration $f\colon S \to C$ with generic fibers of genus one.

In class (4), the surfaces are known as surfaces of general type. These surfaces are analogous of the curves of genus at least two. There is no uniform classification for them.

In this thesis, we are interested on minimal surfaces S of general type, i.e., surfaces of general type which does not contain curves $E \simeq \mathbb{P}^1$ such that $E^2 = -1$. Since out of the class (1) there is a unique minimal model in a birational class, it is enough to classify minimal surfaces. To classify minimal surfaces one uses moduli spaces.

For curves, the space of parameters $\mathcal{M}_g$ classifying curves, which is a coarse moduli space, is well known for each genus g. Indeed, if the the genus is zero $\mathcal{M}_0$ is a point, if the genus is one $\mathcal{M}_1$ is isomorphic to the affine line $\mathbb{A}^1$, and if the genus is at least two $\mathcal{M}_g$ is a not empty quasi-projective irreducible variety of dimension $3g-3$, see [DM69].

In the case of surfaces of general type, by Bombieri [Bom73], we know that for every $n \geq 5$, the complete linear system $|nK_S|$ induces a birational morphism $\psi_{|nK_S|}\colon S \to S' \subseteq \mathbb{P}^{P_n(S)-1}$, given by $x \mapsto [s_0(x) : \cdots : s_{P_n-1}(x)]$ where $\{s_i\}$ forms a basis for the space of sections $H^0(S, \mathcal{O}_S(nK_S))$. The image surface S' has degree $n^2c_1^2(S)$. By Kawamata-Viehweg vanishing (e.g. [Kaw82, Vie82]) and Riemann-Roch Theorem, we have that $P_n(S) = \frac{n(n-1)}{2}c_1^2(S) + \chi(S)$. Furthermore, if we take another basis for $H^0(S, \mathcal{O}_S(K_S))$ the image S'' of S by the map $\psi_{|nK_S|}$, using this new basis, is projectively equivalent to S'. Therefore, the possible space of parameters for surfaces is given as a quotient of a closed subscheme of the Hilbert scheme with Hilbert polynomial $P_n = \frac{n(n-1)}{2}c_1^2 + \chi$ by the linear projective group $PGL_{P_n}(\mathbb{C})$, which parametrizes all surfaces (up to isomorphism) of degree $n^2c_1^2$ in $\mathbb{P}^{P_n-1}$. Thus, the space of parameters depends on the invariants c_1^2 and χ, or equivalently on c_1^2 and c_2. Gieseker [G77], showed that the space of parameters for minimal surfaces of general type with c_1^2 and c_2 fixed, denoted by $\mathcal{M}_{c_1^2,c_2}$, is a quasi-projective variety.

There are two general problems about $\mathcal{M}_{c_1^2,c_2}$. First, give a complete answer to the question: for which values of c_1^2 and c_2 the space $\mathcal{M}_{c_1^2,c_2}$ is not empty? This is called **the problem of geography**. Second, describe the quasi-projective variety $\mathcal{M}_{c_1^2,c_2}$ for fixed c_1^2 and c_2. This is known as the **moduli problem**.

For instance, for $\mathcal{M}_{1,11}$ the geographical problem is solved by means of the classical Godeaux surface, which is the quotient of $\{x_0^4+x_1^4+x_2^4+x_3^4=0\} \subseteq \mathbb{P}^3$ by the free automorphism of order 5, $\sigma[x_0 : x_1 : x_2 : x_3] = [x_0, \zeta x_1, \zeta^2 x_2, \zeta^3 x_3]$ where ζ is a primitive 5th roof of the unity. For the same space $\mathcal{M}_{1,11}$, M. Reid conjectured that it has five irreducible components, each one parametrizing surfaces with fundamental group $\pi_1 \in \{1, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/5\mathbb{Z}\}$.

1.2 Geography of surfaces of general type

We are looking for the pairs of integers (c_2, c_1^2) for which the moduli space $\mathcal{M}_{c_1^2,c_2}$ is not empty. Firstly, we must have in mind the inequalities which bound the possibilities for the pairs (c_2, c_1^2), together with the Noether relation $c_1^2 + c_2 = 0 \bmod 12$. Such inequalities are given in Theorem 1.1.

Theorem 1.1. *Let S be a minimal surface of general type. Then,*

CI: $c_1^2, c_2 \geq 1$,

NI: $\frac{1}{5}(c_2 - 36) \leq c_1^2$,

BMYI: $c_1^2 \leq 3c_2$.

Here we have labeled by CI the Castelnuovo inequality, NI the Noether inequality and by BMYI the Bogomolov-Miyaoka-Yau inequality.

NI was proved by Max Noether in [No1877]. The line $c_2 - 36 = 5c_1^2$ is called the Noether line. Thanks to the work of Horikawa, see [Ho76], we know that every surface with Chern numbers in the Noether line are simply-connected. Bogomolov in [Bog77] shows the inequality $c_1^2 \leq 4c_2$, whereas a year latter Miyaoka and Yau in [Miy77, Yau78] proved independently BMYI. The line $c_1^2 = 3c_2$ is called the Bogomolov-Miyaoka-Yau line. Due to Miyaoka and Yau, we have that every surface with Chern numbers in the Bogomolov-Miyaoka-Yau line are free compact quotients of the unitary ball $\mathbb{B}$ of $\mathbb{C}^2$ by an infinite discrete group. In particular, those surfaces are not simply-connected.

If we draw the region bounded by the CI, NI and BMYI, inequalities together with the relation $c_1^2 + c_2 = 0 \bmod 12$, we obtain a picture as in Figure 1.

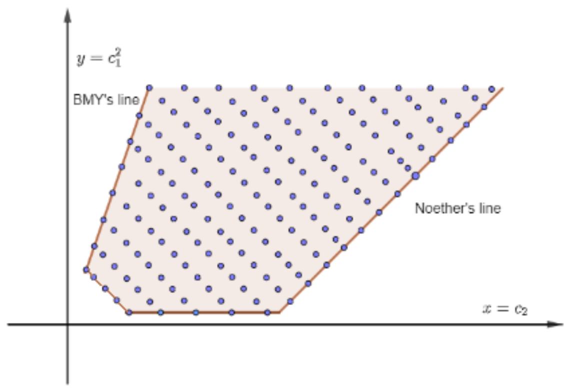


Figure 1: Region of Geography.

Persson in [P81] coined the term "geography". The problem of geography is almost resolved due to the works of Persson and Chen. They proved the following theorems. See for example [BHPV04, Chap. VII, Sect. 8]

Theorem 1.2. *[P81, Theo.2.] Let x, y be positive integers satisfying*

$$\frac{1}{5}(x - 36) \leq y \leq 2x \text{ and } 2y \neq x - k,$$

where $k = 2$, or k is odd and $1 \leq k \leq 15$ or $k = 19$. Then there exists a minimal surface

S of general type such that $c_1^2(S) = y$ and $c_2(S) = x$. Moreover, every of such surfaces is a genus two fibration, and simply-connected $\pi_1(S) = \{1\}$.

Theorem 1.3. *[Ch87, Theo. 1.] there exists a simply-connected surface of general type with $x = c_1^2(S)$ and $y = c_2(S)$. Let x, y be positive integers satisfying*

$$\left(\frac{352}{716}\right) x + C_1 x^{\frac{2}{3}} \leq y \leq \left(\frac{18644}{6904}\right) x - C_2 x^{\frac{2}{3}}$$

and $x > C$, where C_1, C_2, C are constants (C a large constant). Then there exists a simply-connected surface of general type with $x = c_1^2(S)$ and $y = c_2(S)$.

Due to the proved difficulty of studying the problem of simply-connected geography pair by pair, we will turn into the study Chern slopes c_1^2/c_2. By the inequalities of Theorem 1.1 the Chern slopes c_1^2/c_2 can be wrapped asymptotically in $[1/5, 3]$.

Persson and Chen constructions can be used to prove that every rational point in $[\frac{1}{5}, 2]$ and $[\frac{352}{716}, \frac{18644}{6904}] \subseteq [\frac{1}{2}, 2.7005]$ can be approximated by Chern slopes numbers $\frac{c_1^2}{c_2}$ of simply-connected surfaces of general type. A natural question here is: Are there any restrictions for the Chern slopes of minimal surfaces of general type with no condiction on the fundamental group? This question can be easily answered, and it was replayed by Sommese [Som84].

Theorem 1.4. *[Som84, Sect. 2.] Every rational point in $[\frac{1}{5}, 3]$ occurs as the slope of some minimal surface of general type.*

1.3 Geography of surfaces of general type with fixed fundamental group

1.3.1 Simply-connected surfaces

By 1977, Bogomolov conjectured that simply-connected surfaces of general type have Chern slopes less than or equal to 2, i.e., $c_1^2/c_2 \leq 2$. It was known as the Watershed conjecture of Bogomolov. Such assertion is false. For example, Moishezon-Teicher constructed counterexamples in [MT87]. Similarly, by the work of Chen, there are simply-connected surfaces of general type with a Chern slope near 2.7005. Later, for spin surfaces (simply-connected surfaces with canonical class divisible by 2), Persson, Peters and Xiao [PPX96] obtained the following density results.

Theorem 1.5.
- *For any rational number $r \in \mathbb{Q} \cap [\frac{1}{5}, 2)$, there exists a minimal spin surfaces S such that $c_1^2(S) = rc_2(S)$.*
- *The Chern slopes $c_1^2(S)/c_2(S)$ of minimal spin surfaces S are dense in the interval $[2, 2.\overline{703}]$*

Urzuá, in [U10], found a sequence of simply-connected surfaces with Chern slope $c_1^2/c_2 = \frac{71}{26} \approx 2.730796$, which was the record at that moment. Finally, Roulleau and Urzúa in [RU15] via cyclic coverings found very special families of simply-connected surfaces X_p, such that $\{c_1^2(X_p)/c_2(X_p)\}_p$ is a dense set in $[1, 3]$. More precisely:

Theorem 1.6. *[RU15, Theo. 5.6, Theo. 6.3]*

- *For any $r \in [1.375, 3]$, there are minimal Spin surfaces of general type S with $c_1^2(S)/c_2(S)$ arbitrarily close to r.*
- *For any $r \in [1, 3]$, there are minimal simply-connected surfaces of general type S with $c_1^2(S)/c_2(S)$ arbitrarily close to r.*

In summary, the geography in terms of Chern slopes is done for simply-connected surfaces of general type.

Theorem 1.7 (Persson, ..., Roulleau-Urzúa)**.** *For any $r \in [\frac{1}{5}, 3]$, there are minimal simply-connected surfaces of general type S with $c_1^2(S)/c_2(S)$ arbitrarily close to r.*

1.3.2 Geography of non-simply-connected surfaces

By the Lefschetz hyperplane theorem, we know that the fundamental group of any nonsingular projective variety is the fundamental group of some nonsingular projective surface. Groups that are fundamental groups of varieties are abundant. Serre proved, for example, that any finite group is realizable [S58]. See the survey [A95] for more on that topic, and see the book [ABCKT96] for Kähler manifolds.

A natural question is: Are there any restriction for the Chern slope c_1^2/c_2 when we fix a non-trivial group $G \cong \pi_1$? In more generality, this question has been studied for 4-manifolds (cf. [KL09]) with a particular focus on symplectic 4-manifolds (see e.g. [G95], [BK06, BK07, Park07]). For example, Park showed in [Park07] that the set of Chern slopes c_1^2/c_2 of minimal symplectic 4-manifolds S with $\pi_1(S) \cong G$ is dense in the interval $[0, 3]$, for any presented group G.

Due to the works of Mendes-Lopes and Pardini [MP07, MP06], we deduce that if S is a surface of general type with $c_1^2(S) < \frac{1}{3}c_2(S)$ and $\pi_1(S)$ finite, then the order of $\pi_1(S)$ is at most 9. Moreover, if S is a surface of general type having no irregular étale cover, with $3c_1^2(S) \leq c_2(S) - 8$ and $\pi_1(S)$ is finite, then $|\pi_1(S)| \leq 5$. The equality is attained if S is a Godeaux surface.

At this point is important to mention Reid's conjecture.

Conjecture 1.8 (Miles Reid's Conjecture). Let S be a minimal surfaces of general type such that $\frac{c_1^2(S)}{c_2(s)} < \frac{1}{2}$, then $\pi_1(S)$ is either finite or is commensurable with the fundamental group of a curve, i.e., there is an étale cover S' of S and a fibration $f : S' \to C$ such that

$$1 \to K \to \pi_1(S') \to \pi_1(C) \to 1$$

with $|K| < \infty$.

Reid's conjecture is sharp. Indeed, Keum on his doctoral thesis [K88] constructed a surface S of general type such that $c_1^2(S) = 4, c_2(S) = 8, p_g(S) = q = 0$ and $\pi_1(S) \simeq \mathbb{Z}^4 \rtimes (\mathbb{Z}/2\mathbb{Z})^2$.

As in the case of simply-connected surfaces the first results are for low slopes and at least conjecturally, we have information for surfaces with $\frac{1}{5} \leq c_1^2/c_2 \leq \frac{1}{2}$. But, what about high slopes? In this thesis, we solved question for any realizable group and slopes in the interval $[1, 3]$.

Theorem 1.9. *Let G be the (topological) fundamental group of a non-singular complex projective surface. Then the Chern slopes $c_1^2(S)/c_2(S)$ of minimal non-singular projective surfaces of general type S with $\pi_1(S) \cong G$ are dense in the interval $[1, 3]$.*

The ideas and general tools to prove the Theorem 1.9 will be described in the next section.

The one dimensional geography for surfaces of general type with fixed fundamental group can be represented as in Figure 2.

As evidence, not all of the interval $[\frac{1}{5}, 3]$ has total freedom for π_1. To know what is the largest sub-interval with that property is a problem that remains open for future research.

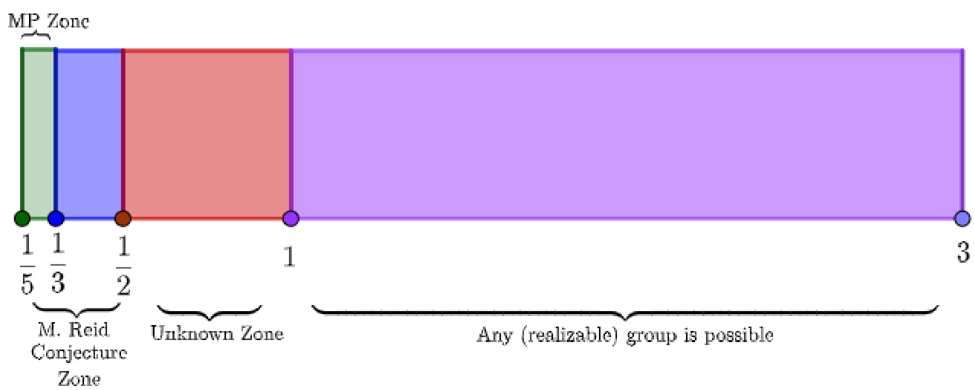


Figure 2: Geography of c_1^2/c_2 and fundamental groups.

1.4 Idea and elements to prove Theorem 1.9

We will construct surfaces S_p of general type with $\lim_{p\to\infty} c_1^2(S_p)/c_2(S_p) = \lim_{p\to\infty} c_1^2(X_p)/c_2(X_p)$, and $\pi_1(S_p) \cong G$ for each prime number p, where the surfaces X_p are the built in [RU15].

Let us explain a key example firstly. Let us choose some surface Y of general type such that $\pi_1(Y) \cong G$ and K_Y being nef, i.e., $K_Y \cdot D \geq 0$ for each divisor D on Y. Next, since X_p is simply-connected, we take the product $X_p \times Y$ to get a variety of dimension 4 with the property $\pi_1 \cong G$. By the Lefschetz hyperplane theorem and Bertini Theorem, for an effective divisor D on $X_p \times Y$, there exist a smooth 3-fold $M \subset X_p \times Y$ such that $\pi_1(M) \cong G$.

Iterating the last argument, we can find a surface $S \subset M$ inside $X_p \times Y$ with $\pi_1(S) \cong G$. Then, the first important question is, how can we have some control for the expressions $c_1^2(S)$ and $c_2(S)$? The answer depends on the election of divisor D to construct the 3-fold M and, consequently, the surface S. It is based on Catanesse's trick.

Proposition 1.10. *[Cat00, Sect. 1.] Consider the $4-$fold $X \times Y$, where X, Y are minimal surfaces of general type. Take Γ, B very ample divisors on X and Y respectively. Let S be the complete intersection surface on $X \times Y$ defined by the linear system $|\Gamma \boxtimes B|$. Then, due to Lefschetz hyperplane theorem $\pi_1(S) \cong \pi_1(X) \times \pi_1(Y)$. Also, we have*

$$c_1^2(S) = c_1^2(X)B^2 + c_1^2(Y)\Gamma^2 + 8c(\Gamma, B) - 4\Gamma^2 B^2,$$

and

$$c_2(S) = c_2(X)B^2 + c_2(Y)\Gamma^2 + 4c(\Gamma, B) + 4\Gamma^2 B^2,$$

where

$$c(\Gamma, B) = \frac{7}{2}\Gamma^2 B^2 + \frac{3}{2}(\Gamma \cdot K_X)B^2 + \frac{3}{2}(B \cdot K_Y)\Gamma^2 + \frac{1}{2}(\Gamma \cdot K_X)(B \cdot K_Y).$$

We note that both numbers $c_1^2(S)$ and $c_2(S)$ are depending of the Chern number of X and Y. In our case, if Γ_p is a very ample divisor on X_p, and called S_p the resulting surface of applying Theorem 5.36 on each surface X_p, the limit of $c_1^2(S_p)/c_2(S_p)$ depends only on $c_1^2(X_p), c_2(X_p), \Gamma_p^2$ and $\Gamma_p \cdot K_{X_p}$.

By the work of Roulleau and Urzúa mentioned above, we know that $c_1^2(X_p)$ and $c_2(X_p)$ have polynomial expressions of degree 4 in p, see [RU15, Pag. 302]. This way, the naive idea is to find a very ample divisor Γ_p in each X_p, such that both formulas Γ_p^2 and $\Gamma_p \cdot K_{X_p}$ are polynomials of degree at most 3 in p. It is a difficult task because of the condition of very ampleness, so we must change the requirement for the divisors Γ_p. However, there are additional difficulties: does the Catanesse's trick still work for another kind of divisor? What type of divisor can be used?

As we noted, the answers have to be interlaced in some way, because the type of divisor used must be good enough so that it allows us to use Bertini and Lefschetz theorems, and have control on the numbers Γ_p^2 and $\Gamma_p \cdot K_{S_p}$.

The lef line bundles are the appropriate replacement of very ample divisors to solve the questions above. These type of line bundles were introduced by De Cataldo and Migliorini in [CM02]. The name lef is the contraction of *Lefschetz effettivamente funziona*. De Cataldo and Migliorini in several papers have justified the name. For instance they proved that the Hard Lefschetz theorem holds for a line bundle L if and only if L is lef, see [CM02, Prop. 2.2.7] . Here we consider a sophisticated version of the Lefschetz hyperplane theorem for homotopy groups due to Goreski and MacPherson, [GM88], see Theorem 4.16, which allows to use lef divisors instated of ample divisors. Under this framework, we improve Catanesse's trick by the use of the lef line bundles. Also we find a special one of them L_p in each X_p, such that L_p^2 and $L_p \cdot K_{X_p}$ are polynomials of order 3 in p. Therefore, we construct a collection of surfaces S_p (by a generalized Catanesse's trick) such that $\pi_1(S_p) \cong \pi_i(Y) \cong G$ (by the generalized Lefschetz theorem for lef divisors), and populating densely the interval $[1, 3]$ (as RU surfaces).

We believe that both questions below have a positive answer. Our belief are based on the proof of Theorem 1.9.

Problem A. Given a realizable group G, i.e., there exists a non-singular complex projective variety Y such that $\pi_1(Y) \cong G$. Are the Chern slopes $c_1^2(S)/c_2(S)$ dense in $[1, 3]$ with K_S ample?

Let us recall that a variety X is said to be (Brody) hyperbolic if any holomorphic morphism $\mathbb{C} \to X$ is constant, i.e., they have no entire curves. In particular, they have no rational nor elliptic curves.

Problem B. Let Y be a Brody hyperbolic nonsingular projective surface. Are then Chern slopes of Brody hyperbolic nonsingular projective surfaces S with $\pi_1(S)$ isomorphic to $\pi_1(Y)$ dense in $[1, 3]$?

Notes on terminology

Let X be a smooth variety. We denote by $Pic(X)$ the free abelian group generated by all the codimension one subvarieties of X modulo linear equivalence.

Let X be a variety and let $\mathcal{L}$ be an invertible sheaf. We denote $\mathcal{L}^n$ the $n-$fold product $\mathcal{L}^{\otimes n}$.

Let X and Y be two varieties. Let $\mathcal{F}$ be a sheaf on X. Let $f\colon X \to Y$ be a morphism, we denote by $R^i f_*(\mathcal{F})$ for $i \geq 1$, the $i-$th higher direct image of $\mathcal{F}$ via f.

We denote as usual by $\Sigma_{n,m}\colon \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$ the Segree embedding.

Let X be a connected topological space. We denote by $\pi_1(X)$ the topological fundamental group of X and by $\pi_i(X)$ for $i \geq 2$ the higher homotopy groups of X. Similarly, we denote by $\pi_1^{alg}(X)$ the étale fundamental group of X.

Given a variety X and a divisor D on it, we denote by $H^i(X, \mathcal{O}(D))$ the $i-$th cohomology group of D and $h^i(X, D) = \dim_{\mathbb{C}} H^i(X, \mathcal{O}(D))$.

Given a smooth variety X, we denote by K_X its canonical divisor.

Let x be real number. We denote by $\lceil x \rceil$ smallest integer value that is bigger than or equal to x, and $\lfloor x \rfloor$ greatest integer less than or equal to x.

2 Semi-small morphisms, lef line bundles

2.1 Semi-small morphisms

The following definition can be found in several places, e.g. [GM88, p.151], [Mig95, Def. 4.1] or [CM02, Def.2.1.1].

Definition 2.1. Let X be a smooth variety, and let Y be a normal variety. For a proper surjective morphism $f\colon X \to Y$, we define

$$Y_f^k = \{y \in Y \,|\, \dim f^{-1}(y) = k\}.$$

We say that f is *semi-small* if $\dim(Y_f^k) + 2k \leq \dim X$ for every $k \geq 0$. (Note that $\dim(\emptyset) = -\infty$.) If no confusion can arise, the subscript f will be suppressed.

Proposition 2.2. *Let $f\colon X \to Y$ be a semi-small morphism, then* $\dim(X) = \dim(Y)$.

Proof. By generic flatness, see [Gro65, Theo. 6.9.2, and Coro. 6.1.2], there is a Zariski dense open subset $U \subseteq Y$ such that for any $y \in U$ we have:

$$\dim f^{-1}(y) = \dim(X) - \dim(Y) =: k_{gen}.$$

In particular, since f is semi-small we have that $\dim(Y^{k_{gen}}) + 2k_{gen} = \dim(Y) + 2\dim(X) - \dim(Y) \leq \dim(X)$, then $\dim(X) \leq \dim(Y)$. But, since f is proper and surjective we have that $\dim(Y) \leq \dim(X)$. Therefore, $\dim(X) = \dim(Y)$.

□

Remark 2.3.

- Let $n = \dim X = \dim Y$. If f is a semi-small morphism, then $Y^k = \emptyset$ if $k > \lceil \frac{n}{2} \rceil$, and for any $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$ the inequality $\dim(Y^k) \leq n - 2k$ holds. Therefore, f is a generically finite map due to $\dim(Y^0) = n$.

- If $f\colon C_1 \to C_2$ is a proper morphism between curves, then f is a finite if and only if f is semi-small.

- A morphism $f\colon T_1 \to T_2$ between three-folds is semi-small if and only if f does not contract subvarieties S of T_1 with $1 \leq \operatorname{codim}_{T_1}(S) \leq \lfloor \frac{n}{2} \rfloor$ to a point.
- More generally, if $f\colon X \to Y$ is semi-small with $\dim(X) \geq 3$, then f does not contract subvarieties S of X with $1 \leq \operatorname{codim}_X(S) \leq \lceil \frac{n}{2} \rceil$. In particular, the blow-up in a point of a smooth projective variety of dimension at least three is not semi-small.

Proposition 2.4. *Let X be a smooth variety, and let Y be a normal variety. For a proper surjective morphism $f\colon X \to Y$. Then, f is semi-small if and only if* $\dim(X \times_Y X) \leq \dim(X)$.

Proof. Suppose that f is a semi-small morphism, then for every $y \in Y^k$,

$$\dim(Y^k) + 2k = \dim(Y^k) + \dim(f \times f)^{-1}(y, y) \leq \dim(X),$$

therefore, $\dim(f \times f)^{-1}(Y_k \times Y_k) \leq \dim(X)$. On the other hand, $X \times_Y X = \bigsqcup_k (f \times f)^{-1}(Y_k \times Y_k)$, thus $\dim(X \times_Y X) = \max_k((f \times f)^{-1}(Y_k \times Y_k)) \leq \dim(X)$.

Conversely, suppose that $\dim(X \times_Y X) \leq \dim(X)$. Then, $(f \times f)^{-1}(Y_k \times Y_k) = 2k + \dim(Y^k) \leq \dim(X \times_Y X) \leq \dim(X)$, so f is semi-small. $\square$

Lemma 2.5. *Let X be a smooth surface, and let Y be a normal surface. If $f\colon X \to Y$ is a proper surjective morphism, then f is semi-small.*

Proof. It is clear that $\dim(Y^1) = 0$ and $\dim(Y^0) = 2$, since f is surjective. Then the inequality $\dim(Y^k) + 2k \leq \dim(X)$ holds for any $k \geq 0$. $\square$

Proposition 2.6. *Let $f\colon X \to Y$ and $g\colon Z \to W$ be two semi-small morphisms. Then the product morphism $f \times g\colon X \times Z \to Y \times W$ is a semi-small morphism.*

Proof. Let $n = \dim(X)$ and $m = \dim(Z)$. Since f and g are semi-small, then we have that $\dim(Y^k) \leq n - 2k$ for any $k \geq 0$, $\dim(Z^l) \leq m - 2l$ for any $l \geq 0$, and $\dim(Y^0) = n, \dim(W^0) = m$. We also have $(Y \times W)^q = \bigcup_{i+j=q} Y^i \times W^j$, and so

$$\dim(Y \times W)^q \leq \max_{i+j=q} \dim(Y^i \times W^j) \leq n + m - 2i - 2j = n + m - 2q.$$

Hence $f \times g$ is semi-small. $\square$

Proposition 2.7. *Let X, Y, Z be smooth projective varieties of the same dimension. Assume that $f\colon X \to Y$ is semi-small, and that $g\colon Y \to Z$ is finite morphism. Then, $h = g \circ f\colon X \to Z$ is semi-small.*

Proof. Since g is a finite, we have $Z_h^k = g(Y_f^k)$ for each $k \geq 0$, and so $\dim(Z_h^k) = \dim(Y_f^k)$. Thus $\dim(Z_h^k) + 2k \leq \dim(X)$, and so h is semi-small. □

2.2 Lef line bundles

Definition 2.8. ([CM02, Def. 2.1.3]) Let X be a smooth projective variety, and let M be a line bundle on X. We say that M is *lef* if there exists $n > 0$ such that $|nM|$ is generated by global sections, and the morphism $\psi_{|nM|}$ associated to $|nM|$ is semi-small onto its image. The *exponent* of M is the smallest n so that M is lef. We denote it by $\exp(M)$.

Proposition 2.9. *Let X be smooth variety. If L is an ample divisor on X, then L is lef. If moreover L is very ample, then* $\exp(L) = 1$.

Proof. Since L is ample there is an integer $n > 0$ such that nL is a very ample divisor on X. It follows that nL is generated by global sections and the induced morphism $\psi_{|nL|} : X \to \mathbb{P}(|nL|)$ is an immersion, then $\psi_{|nL|}$ is an isomorphism onto its image. Therefore, L is a lef divisor on X. □

Proposition 2.10. *Let $f\colon X \to Y$ be semi-small between smooth projective varieties, and let L be very ample on Y. Then $f^*(L)$ is lef with* $\exp(f^*(L)) = 1$.

Proof. Let $\psi_{|L|}\colon Y \to \mathbb{P}(|L|)$ be the map defined by the linear system $|L|$. It is clear that, $\psi_{|L|}^*(\mathcal{O}(1)) \cong L$, thus $f^*(\psi_{|L|}^*(\mathcal{O}(1))) \cong f^*(L)$ is globally generated by $\{f^*(s_i)\}_{i=0}^n$, where $\{s_i\}_{i=0}^n$ is a basis of $H^0(Y, L)$. Due to the fact that f is semi-small, $\psi_{|L|}$ is an immersion and using Proposition 2.7, we have $\psi_{|L|} \circ f$ is semi-small. Since $\psi_{|f^*(L)|} = \psi_{|L|} \circ f$, we obtain $f^*(L)$ is semi-small with $\exp(f^*(L)) = 1$. □

A useful Bertini type theorem for lef line bundles is the following. (See [CM02, Prop. 2.1.7] or [Mig95, Lemma 4.3].)

Theorem 2.11. *Let X be a nonsingular projective variety of dimension at least 2. Let M be a lef line bundle on X. Assume that M is globally generated and with $\exp(M) = e$. Then any generic member $Y \in |M|$ is a nonsingular projective variety, and the restriction $M|_Y$ is lef on Y with $\exp(M|_Y) \leq e$.*

Corollary 2.12. *Let $f\colon X \to Y$ be a semi-small morphism between smooth projective varieties and let L be an ample line bundle on Y. If m is the smallest positive integer such that mL is very ample. Then, $M = f^*(L)$ is a lef line bundle on X with $\exp(M) \leq m$.*

Proposition 2.13. *Let D be a lef divisor on a variety X with $\exp(D) = m$. Then there exists an ample divisor L on a variety Y and a projective semi-small morphism $\psi\colon X \to Y$ such that $\psi^*(L) \cong mD$.*

Proof. Consider $\psi\colon = \psi_{|mD|}\colon X \to \psi(X) \subseteq \mathbb{P}(|mD|)$ and denote $Y\colon = \psi(X)$. The morphism ψ is projective and semi-small on Y. Let $L\colon = \iota^*(\mathcal{O}(1))$ where $\iota\colon Y \to \mathbb{P}(|mD|)$ is the closed immersion. Therefore, $\psi^*(L) \cong mD$. $\square$

Definition 2.14. Let N be a divisor on a variety X, we say that N *nef* if $D.N \geq 0$ for any divisor D on X.

Let X, Y be two varieties and let $f\colon X \to Y$ be a proper morphism. Given any curve C on X, the 1-cycle $f_*(C)$ is defined as follows: if C is contracted by f, put $f_*(C) = 0$; if $f(C)$ is a curve, put $f_*(C) = \deg(f|_C)C$, where $f|_C : C \to f(C)$ is the restriction of f to C. Note that if $f : S \to S'$ is a morphism between surfaces, for any divisor D' on S' and any curve C of S, we have that $f_*(C) \cdot D' = \deg(f|_C) f(C) \cdot D'$. More generally;

Proposition 2.15 (Projection formula)**.** *Let X, Y be two varieties, and let $f\colon X \to Y$ be a surjective proper morphism. Let $D_1, \ldots, D_r$ be divisors on X with $r \geq \dim(Y)$. Then,*

$$f_*(D_1) \cdots f_*(D_r) = \deg(f)(D_1 \cdots D_r)$$

Proof. See [Deb01, Prop. 1.10]. $\square$

Corollary 2.16. *Let L be a lef divisor on a variety X, then L is nef.*

Proof. It follows by Proposition 2.15. $\square$

3 Algebraic Surfaces

For us a surface S is a smooth complex projective variety of dimension 2.

3.1 Basic facts on Surfaces

Let S be a surface. We denote the Picard group of S by $\mathrm{Pic}(S)$. This group is the free abelian group of line bundles on S module isomorphism, or equivalently the free abelian group of divisors on S modulo linearly equivalence.

There exists a $\mathbb{Z}-$bilinear and symmetric map $(\bullet, \bullet)\colon \mathrm{Pic}(S) \times \mathrm{Pic}(S) \to \mathbb{Z}$. It is the intersection map and it is defined as follows:

Theorem 3.1. *Given D and D' two divisors on S, the pairing*

$$(D, D') = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-D)) - \chi(\mathcal{O}_S(-D')) + \chi(\mathcal{O}_S(-D) \otimes \mathcal{O}_S(-D')),$$

where χ is the Euler characteristic, is symmetric and $\mathbb{Z}-$bilinear.

Proof. See e.g. [Bea96, Theo. I.4], [BHPV04, Chap. II, Sect. 10, pags. 83-84], or [Har77, Chap. V, Theo 1.1]. □

Notation 3.2. Given D, D' two divisors on a surface S, we denote (D, D') the pairing of the above theorem just by $D \cdot D'$. By the sake of simplicity we will write $\chi(D)$ instead of $\chi(\mathcal{O}_S(D))$. Thus,

$$D \cdot D' = \chi(\mathcal{O}_S) - \chi(-D) - \chi(-D') + \chi(-D - D').$$

The following relation is important in the theory of surfaces.

Theorem 3.3 (Noether's formula)**.** *Let S be a surface, then*

$$12\chi(\mathcal{O}_S) = c_1^2(S) + c_2(S),$$

where $c_1^2(S) = K_S^2$ is the self-intersection of the canonical class of S and $c_2(S) = e(S)$ denote the topological Euler characteristic for S.

Proof. See for example [Har77, Appen. A, Exam. 4.1.2]. The first proof of this classical theorem was given by Noether on [No1886]. Nowadays it is seen as a consequence of the sophisticated Hirzebruch-Riemann-Roch theorem.

Note that by Hirzebruch-Riemman-Roch Theorem we have that

$$c_2(S) = \sum_{\substack{j=0\\k+l=j}}^{2} h^k(S, \Omega^l)$$

and by Hodge decomposition

$$\sum_{\substack{j=0\\k+l=j}}^{2} (-1)^{k+l} h^k(S, \Omega^l) = \sum_{i=0}^{4} (-1)^i b_i(S) = e(S),$$

where $b_i(S) = \dim H^i(S, \mathbb{R})$.

$\square$

In order to understand surfaces or varieties, in general, it is useful to have a lower bound for the dimension of vector spaces of sections for a given line bundle, and the next theorem is a tool to obtain such bounds.

Theorem 3.4 (Riemann-Roch for Surfaces)**.** *Let S be a surface and let D be a divisor on S, then*

$$\chi(D) = \chi(\mathcal{O}_S) + \frac{1}{2}(D^2 - D \cdot K_S).$$

Corollary 3.5. *Let D be a divisor on a surface S, then*

$$h^0(S, D) + h^0(S, K_S - D) \geq \chi(\mathcal{O}_S) + \frac{1}{2}(D^2 - D \cdot K_S).$$

Proof. It follows from Serre duality, and the equality

$$h^0(S, D) + h^0(S, K_S - D) = \chi(\mathcal{O}_S) + \frac{1}{2}(D^2 - D \cdot K_S) + h^1(S, D).$$

$\square$

Lemma 3.6 (Riemann-Roch for Curves)**.** *Let C be a smooth projective curve, and let D be a divisor on C, then*

$$h^0(C, D) - h^0(C, K_C - D) = \deg(D) + 1 - g(C),$$

where $g(C) = h^1(C, \mathcal{O}_C)$ is the geometric genus of C.

Proof. See [Har77, Chap. IV, sect. 1, Theorem 1.3.] $\square$

Now we recall an important general formula to compute the canonical divisor of a given non-singular hypersurface on a smooth variety X.

Proposition 3.7 (Adjunction formula)**.** *Let X be a smooth variety and let Y be a smooth divisor of X. Then $K_Y \sim K_X \otimes \mathcal{O}_X(Y) \otimes \mathcal{O}_Y$.*

Example 3.8. Let $X := X_{d_1,\ldots,d_r}$ be a complete intersection surface on $\mathbb{P}^{r+2}$. Then $K_X \sim \mathcal{O}_{\mathbb{P}^{r+2}}(-(r+3)) \otimes \mathcal{O}_{\mathbb{P}^{r+2}}(\sum_i d_i) \otimes \mathcal{O}_X \sim \mathcal{O}_X(\sum_i d_i - (r+3))$.

Given any irreducible smooth curve C on a surface S we can estimate its genus in terms of K_S^2 and $K_S \cdot C$, the formula to do that is known as the genus formula.

Proposition 3.9 (The Genus Formula)**.** *Let C be an irreducible smooth curve on a surface S. Denote by $g(C) := g$ the geometric genus of C. Then,*

$$2g - 2 = C^2 + C \cdot K_S.$$

Example 3.10.

- Let C be a smooth curve of degree d on $\mathbb{P}^2$, then $2g(C) - 2 = d^2 - 3d$, so $g(C) = \frac{(d-1)(d-2)}{2}$.
- Let C be a curve on $\mathbb{P}^1 \times \mathbb{P}^1$ of type (a, b) on $Pic(\mathbb{P}^1 \times \mathbb{P}^1)$, then $g(C) = ab - a - b + 1$.

3.2 Birational Transformations

Definition 3.11. Two surfaces S, S' are birational if there exist a rational map $\epsilon \colon S \dashrightarrow S'$ which has an inverse or equivalently if there are X, X' closed subsets such that $\epsilon \colon S \setminus X \to S' \setminus X'$ is an isomorphism.

Definition 3.12. Given a surface S and a point p in S, we can construct a new surface $\hat{S}$ and a morphism $\epsilon \colon \hat{S} \to S$, such that $E := \epsilon^{-1}(p) \cong \mathbb{P}^1$ and $\epsilon : \hat{S} \setminus E \to S \setminus \{p\}$ is an isomorphism. The pair $(\hat{S}, \epsilon)$ is called the blow-up of S at the point p, and the surface $\hat{S}$ is denoted by $Bl_p(S)$, and the divisor E is called the exceptional divisor of ϵ. See for example [Bea96, Chap. II, Sect. 1].

Proposition 3.13 (Properties of $Bl_p(S)$)**.** *Let S be a surface and let $p \in S$. Then*

1. $Pic(\hat{S}) \cong f^*(\mathrm{Pic}(S)) \oplus \mathbb{Z}E$,
2. $f^*(D) \cdot f^*(D') = D \cdot D', f^*(D) \cdot E = 0$, *where* $D, D' \in \mathrm{Pic}(S)$,
3. $E^2 = -1$,
4. *Given C an irreducible curve on S, then $f^*(C) = \hat{C} + mE$, where $m = \mathrm{mult}_p(C)$, and $\hat{C}$ is the strict transform of C,*
5. $K_{\hat{S}} = f^*(K_S) + E$ *and* $K_{\hat{S}}^2 = K_S^2 - 1$.

Proof. Statements 1,2,3 and 5 can be found in [Bea96, Lemma II.3.] and statement 4 in [Bea96, Lemma II.2] or statements 1 and 2 can be found in [Har77, Chap. V, Prop. 3.2], statement 3 in [Har77, Chap. V, Prop. 3.1] and statement 5 in [Har77, Chap. V, Prop. 3.3]. □

Theorem 3.14. *Let $f \colon S' \to S$ be a birational map of surfaces, then f is the composition of a finite number of blow-ups and blow-downs.*

Proof. See [Bea96, Theo.II. 11] □

Definition 3.15. We say that a surface S is minimal if there are no (-1)-curves, i.e. there is no curves $E \simeq \mathbb{P}^1$ such that $E^2 = -1$.

Proposition 3.16. *For any surface S' there exists a birational map $f \colon S' \to S$, where S is minimal surface.*

Proof. See [Bea96, Prop. II.16] □

Theorem 3.17 (Castelnouvo's Contractibility Criterion)**.** *Let S be a surface and let E be a $(-1)-$curve on S. Then E is an exceptional divisor.*

Proof. See [Bea96, Theo. II. 17] □

Definition 3.18. Let D be a divisor on X, we denote by $|D|$ its complete linear system

$$|D| := \{L \sim D, L \geq 0\}.$$

The base locus of D is by definition $\mathrm{B_S}(|D|) = \bigcap_{L\in|D|} L$. We say that D is base point free if its base locus is empty, i.e, $\mathrm{B_S}(|D|) = \emptyset$.

For any effective divisor D on S, we can define a map $\psi_{|D|}\colon S \dashrightarrow \mathbb{P}(H^0(S,D)^*)$ as follows. Let $s_0,\dots,s_d$ be a basis for $H^0(S,D)$, then define $\psi_{|D|}(x) := [s_0(x),\dots,s_d(x)]$. It is clear that $\psi_{|D|}$ is not well-defined at points $x \in X$ such that $s_0(x) = \cdots = s_d(x) = 0$. On the other hand, if $L \in |D|$, then L is the zero locus of a section $s = \sum a_i s_i \in H^0(S,D)$, and conversely any section $s \in H^0(S,D)$ define a divisor $L = \{s = 0\} \in |D|$. Thus, the map $\psi_{|D|}$ is a morphisms (well-defined map) if and only if D is base point free. We can identify $\mathbb{P}(|D|)$ with $\mathbb{P}(H^0(S,D)^*)$.

Let S be a surfaces and D a divisor on S such that $h^0(S,D) \geq 2$. A divisor T on S is said to be fixed by the linear system $|D|$ if T belong to all $D' \in |D|$. The maximal divisor F fixed by the linear system $|D|$ is called the fixed part of $|D|$. There is a bijection between $|D|$ and $|D-F|$. The divisor $M := D - F$ is called the moving part of $|D|$. If $x \in \mathrm{B_S}(|D|) \setminus F$, then $x \in M$. Therefore, we have that $M^2 \geq |B_S(|M|)|$. The image of $\psi_{|D|}\colon = Y$, i.e., $\psi_{|D|}(S)$, is contained in $\mathbb{P}(|D|) \cong P(|M|)$, such dimension can be one or two under our hypothesis. Let $\overline{S} \xrightarrow{\epsilon} S$ be the birational map such that $\hat{\psi} := \psi_{|D|} \circ \epsilon\colon \overline{S} \to \mathbb{P}(|D|)$ is defined everywhere. Let $Z := \hat{\psi}(\overline{S})$ and suppose that $\dim(Z) = 1$. Then by the Stein factorization $\hat{\psi}$ can be factorized as $\hat{\psi} = \sigma \circ \tau$, with $\tau\colon \overline{S} \to W$ is fibre-connected morphism where W is a smooth curve, and σ is a finite morphism. In the last case we say that $|D|$ is composed with a pencil, and the pencil is rational if $g(W) = 0$, otherwise the pencil is irrational, the last case only happens if D is base point free, see [Zar12, pag. 25-26].

Proposition 3.19. *As above, Suppose that $D \in \mathrm{Pic}(S)$ is such that $h^0(D) \geq 2$ and is composed with a pencil. Let T be a fiber of τ. Then:*

1. *If $g(W) = 0$, then $D \sim F + aT$, where $h^0(D) = h^0(aT) = h^0(\mathbb{P}^1, \mathcal{O}(a)) = a + 1$*
2. *If $g(W) \geq 1$, then $D \sim F + \tau^*(E_a)$, where E_a is divisor of degree a on W.*

3.3 Kodaira Dimension

Given any variety X there is an important notion of dimension which help us in the process of classification. Such dimension is called Kodaira dimension, and it is defined as follows: (See for example [Bea79, Chap. VII], [BHPV04, Chap. I, Sect. 7], or [Laz17, Vol. I, Chap. 2, Sect. 1.A])

Definition 3.20 (Kodaira dimension)**.** Let X be any smooth variety and denoted by K_X its canonical divisor, the Kodaira dimension of X is by definition

$$\mathrm{kod}(X) := \begin{cases} -\infty & , \text{ if } H^0(X, nK_X) = 0, \forall n \in \mathbb{N} \\ \min\limits_{k \in \mathbb{N}} \left\{ \dfrac{h^0(X, nK_X)}{n^k} \text{ is bounded} \right\} & , \text{ if } \exists n >> 0, H^0(X, nK_X) \neq 0. \end{cases}$$

Or equivalently

$$\mathrm{kod}(X) := \begin{cases} -\infty & , \text{ if } H^0(X, nK_X) = 0, \forall n \in \mathbb{N} \\ \max\limits_{k \in \mathbb{N}} \left\{ \dim(\psi_{|nK_X|}(X)) \right\} & , \text{ if } \exists n >> 0, H^0(X, nK_X) \neq 0. \end{cases}$$

Where $\psi_{|nK_X|} \colon X \dashrightarrow \mathbb{P}(|nK_X|)$ is the map defined by the linear system $|nK_X|$.

$$\mathrm{kod}(X) \in \{-\infty, 0, 1, 2, \ldots, \dim(X)\}.$$

Definition 3.21. Let X be a variety. If $\mathrm{kod}(X) = \dim(X)$, then we say that X is of general type.

Proposition 3.22. *Let X, Y be smooth varieties, then*

$$\operatorname{kod}(X \times Y) = \operatorname{kod}(X) + \operatorname{kod}(Y).$$

Proof. See for example [BHPV04, Chap. I, Theo. 7.3]. □

Classification of Curves

Let C be a curve. Then $\operatorname{kod}(C) \in \{-\infty, 0, 1\}$. We have the following table describing the topological type of curves according to their Kodaira dimension.

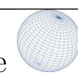
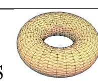
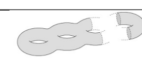
$\operatorname{kod}(C)$	$g(C)$	Curve C
$-\infty$	0	$\mathbb{P}^1$ Riemann Sphere 
0	1	Torus 
1	≥ 2	General Type 

Table 1: Classification of curves

3.4 Enriques classification

The process of classifying surfaces is much more complicated than the classification of curves. The particular parameter to classify is the Kodaira dimension, but as we will see, the difficulties grow as the Kodaira dimension of the surfaces grows.

3.4.1 $\operatorname{kod}(S) = -\infty$

Definition 3.23. A surface S is called rational if is birational to $\mathbb{P}^2$.

For any $n > 0$, denote by $\mathbb{F}_n := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ the $n-$th Hirzebruch surface, and denote $\mathbb{F}_0 := \mathbb{P}^1 \times \mathbb{P}^1$.

Proposition 3.24 (Properties of $\mathbb{F}_n$)**.**

1. *Let f be the class of a fiber, $\mathbb{F}_n \to \mathbb{P}^1$, and let h be the class of $\mathcal{O}_{\mathbb{F}_n}(1)$ on $Pic(\mathbb{F}_n)$. Then, $Pic(\mathbb{F}_n) = \mathbb{Z}f \oplus \mathbb{Z}h$, where $f^2 = 0, h^2 = n, f \cdot h = 1$.*

2. *For any $n > 0$, there exists an unique smooth curve $C \subset \mathbb{F}_n$, such that $C^2 = -n$, and $C \sim h - nf \in Pic(\mathbb{F}_n)$.*

3. *$\mathbb{F}_n \cong \mathbb{F}_m$ if and only if $m = n$.*

4. *$K_{\mathbb{F}_n} \sim -2h + (n-2)f$ and $K^2_{\mathbb{F}_n} = 8$.*

Proof. See [Bea96, Prop. IV.10]. □

Another construction of the Hirzebruch surfaces $\mathbb{F}_n$'s, via blow-ups and blow-downs.

We start with $\mathbb{P}^2$. Set $\mathbb{F}_1 := Bl_p(\mathbb{P}^2)$ the blow-up of $\mathbb{P}^2$ at some point p. Let E be the corresponding exceptional divisor (red line), and let F be a fiber (blue line) passing through p' (image of p via the blow-up).

Then we do a blow-up of $\mathbb{F}_1$ at p'. Set $S := Bl_p(\mathbb{F}_1) \xrightarrow{f} \mathbb{F}_1$. Let D the exceptional divisor of f (black line on S). Then, the divisor $E' = f^*(E) + D$ (red line) has self-intersection -2, and $F' = f^*(F) + D$ (blue line on S) has self-intersection -1. So, we can do a blow-down on S to contract the $(-1)-$curve F'. Thus, we get a surface birrational to $\mathbb{P}^2$ with only one curve with negative self-intersection, a $(-2)-$curve, it is $\mathbb{F}_2$. This process can be iterated to obtain all Hirzebruch surfaces $\mathbb{F}_n$. (See Image 3.4.1).

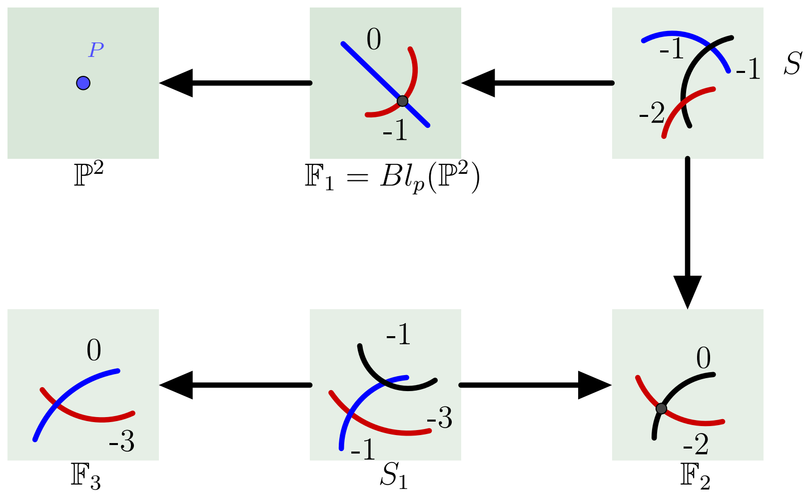


Figure: Construction of the Hirzebruch Surfaces F_n

Proposition 3.25. *Let S be a ruled surface over C, then $q(S) = g(C), p_g(C) = 0$ and $K_S^2 = 8(1 - g(C))$.*

See [Bea96, Prop. III.21] or [Har77, Cap. V, Cor. 2.11].

Proposition 3.26. *Any minimal rational surface is isomorphic either to $\mathbb{P}^2$ or or $\mathbb{F}_n$ for $n \neq 0$.*

Proof. See [Bea96,] or [BHPV04, Chap. V, Prop. 4.3] $\square$

Corollary 3.27. *Let S be any rational surfaces, then, $q(S) = p_g(S) = 0$.*

Definition 3.28. A surfaces S is called ruled (non–rational) if S is birational to $\mathbb{P}^1 \times C$, where C is a curve with $g(C) \geq 1$.

3.4.2 $\text{kod}(S) = 0$

For proof of the followings results, you can consult [Bea79, Chap. VIII]

If S is a surface such that $\text{kod}(S) = 0$, then its plurigenus satisfies $P_n(S) = 0$ or 1 for all n, and there exist some n_0 such that $P_{n_0}(S) = 1$. In particular, $p_g(S) = 0, 1$.

Proposition 3.29. *Let S be a minimal surface such that* $\text{kod}(S) = 0$. *Then,* $K_S^2 = 0$ *and* $\chi(S) \geq 0$.

Corollary 3.30. *Let S be a minimal surfaces with* $\text{kod}(S)$. *Then* $q(S) \in \{0, 1, 2\}$

The possible values for $q(S)$ are bounded by the expressions $\chi(\mathcal{O}_S) = 1 - q(S) + p_g(S) \geq 0$, and since $p_g(S) = p_1(S)$ is either 0 or 1. Hence $q(S) \in \{0, 1, 2\}$. Furthermore, note that if $p_g(S) = 0$ and $q(S) = 2$, then $\chi(\mathcal{O}_S) = 1 - q(S) + p_g(S) = -1$, which is absurd by the theorem 3.29. The alternative $p_g(S) = q(S) = 1$ is neither allowed, as we state in the following proposition.

Proposition 3.31. *Let S be a minimal surface with* $\text{kod}(S) = 0$. *Then it is impossible that* $p_g = q = 1$.

Finally, for minimal surfaces with $\text{kod}(S) = 0$ we present the behavior (definition) in each possible case.

Definition 3.32. Let S be a minimal surfaces such that $\text{kod}(S) = 0$, then S belong to one of the following types:

1. $p_g = q = 0$, we say that S is a "Enriques" surface.
2. $p_g = 0, q = 1$, we say that S is a "Bielliptic" surface.
3. $p_g = 1, q = 0$, we say that S is a "K3" surface.
4. $p_g = 1, q = 2$, we say that S is an "Abelian" surface.

Example 3.33.

(Enriques) Let $X = \{x_0^4 + x_1^4 - x_2^4 - x_3^4 = 0\} \subseteq \mathbb{P}^3$. Thus, X is a $K3$ surface, since $\pi_1(X) = 0$ due to Lefschetz hyperplane theorem, so $q(X) = 0$ and $K_X \sim \mathcal{O}_X$ by adjunction formula. Consider $\sigma[x_0 : x_1 : x_2 : x_4] = [x_0 : ix_1, -x_2 : -ix_3]$, where $i^2 = -1$. Clearly $\sigma^4 = id_X$, and $\sigma(X) \subset X$. Moreover, σ has no fixed points on X. Consider $S := X/\sigma$, then S is a Enriques surface, since $\chi(\mathcal{O}_X) = 2$ (is a $K3$) we have that $\chi(\mathcal{O}_S) = 1 - q(S) + p_g(S) = 1$ (see, [Bea96, Lemma VI.3]), thus $q(S) = p_g(S)$. Finally, since S is a quotient of X by a finite group, the surface X is an universal cover of S, thus $\pi_1(S)$ is finite, so $q(S) = rank\left(\frac{\pi_1(S)}{[\pi_1(S), \pi_1(S)]}\right) = 0$.

(Bielliptic) A Bielliptic surface S is a surface $S \cong (E \times F)/G$, where E and F are elliptic curves and G is a finite group of translations of E acting on F such that $F/G \cong \mathbb{P}^1$.

Theorem 3.34 (Bagnera-de Franchis)**.** *Every Bielliptic surface belong to one of the followings types:*

(a) $(E \times F)/G$, *where* $G \cong \mathbb{Z}/2\mathbb{Z}$, *acting on* F *by* $x \mapsto -x$.

(b) $(E \times F)/G$, *where* $G \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, *acting on* F *by* $x \mapsto -x, x \mapsto x + \epsilon, \epsilon \in F_2$, *where* $F_2 = \{$ *points of order 2 on* $F\}$.

(c) $(E \times F_i)/G$, *where* $G \cong \mathbb{Z}/4\mathbb{Z}$, *acting on* F_i *by* $x \mapsto ix, i^2 = -1$, *where* $F_i = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i)$.

(d) $(E \times F_i)/G$, *where* $G \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, *acting on* F_i *by* $x \mapsto ix, x \mapsto x + \frac{1+i}{2}$.

(e) $(E \times F_\rho)/G$, *where* $G \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, *acting on* F *by* $x \mapsto \rho x, \rho^3 = 1$, *where* $F_\rho = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\rho)$.

(f) $(E \times F_\rho)/G$, *where* $G \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, *acting on* F_{rho} *by* $x \mapsto \rho x, x \mapsto \frac{1-\rho}{3}$.

(g) $(E \times F_\rho)/G$, *where* $G \cong \mathbb{Z}/6\mathbb{Z}$, *acting on* F *by* $x \mapsto -\rho x$.

($K3$) (a) Consider the complete intersection surfaces $S_{d_1,\dots,d_r} := \{F_{d_1} = \cdots = F_{d_r} = 0\} \subseteq \mathbb{P}^{r+2}$, then $S_4, S_{2,3}$ and $S_{2,2,2}$ are $K3$ surfaces,

(b) let C be a sextic on $\mathbb{P}^2$ and consider S as the double cover of $\mathbb{P}^2$ with branch locus C. Then, S is a $K3$ surface.

(Abelian) (a) Let C_1, C_2 be two elliptic curves, then $A = C_1 \times C_2$ is an abelian surface.

(b) Let S be a surface with $q(S) = 2$, then its Albanese variety, $Alb(S)$ is an abelian surface (see [Bea96, Chap. V.]). Recall, that for any variety X its Albanese variety is defined by

$$\mathrm{Alb}(X) := H^0(X, \Omega^1_x)^*/H,$$

where H is the image of the morphism,

$$\iota \colon H_1(X, \mathbb{Z}) \to H^0(X, \Omega^1_X)^*, \iota(\gamma)(\omega) = \int_\gamma \omega \in \mathbb{C}.$$

3.4.3 $\mathrm{kod}(S) = 1$

Definition 3.35. A minimal surface S is called elliptic if there exists a curve C and a fibration $f \colon S \to C$ such that the generic fiber is an elliptic curve.

Theorem 3.36. *Every minimal surface with* $\mathrm{kod}(S) = 1$ *is an elliptic fibration.*

Proof. See [Bea96, Theo. IX.2] □

Example 3.37.

1. Let C_1 be an elliptic curve and let C_2 be a curve of genus ≥ 2. Then, $S = C_1 \times C_2$ has $\mathrm{kod}(S) = 1$.

2. Consider the family of elliptic curves $F_t := \{y^2 = x^3 + A(t)x + B(t)\}$ where $t \in \mathbb{C}$ and $A(t), B(t) \in \mathbb{C}[t]$. The curve F_t is non-singular if $4A^3(t) + 27B^2(t) \neq 0$. Compatifying we obtain a fibration $\phi : S \to \mathbb{P}^1$ with generic fiber F a smooth curve of genus one. The canonical class of S is $K_S \sim (\chi(S) - 2)F$. If we consider $A_{4n}(t)$ and $B_{6n}(t)$ be polynomials of degree $4n$ and $6n$, respectively. The polynomial $4A^3_{4n}(t) + 27B^2_{6n}(t)$ has degree $12n$. Thus, the fibration ϕ has $12n$ singular fibers. So, $\chi(\mathcal{O}_S) = \frac{12n}{12} = n$. Therefore, for $n \geq 3$, the surface S has Kodaira dimension 1. If $n = 0, 1$, S is a rational surface, and if $n = 2$ the surfaces S is a K3.

3.4.4 $\mathrm{kod}(S) = 2$

Definition 3.38. A surface S with $\mathrm{kod}(S) = 2$ is called of general type.

Proposition 3.39. *Let S be a minimal surface of general type, then $K_S^2 \geq 1$ and $\chi(\mathcal{O}_S) \geq 1$.*

Proof. Let H be smooth very ample divisor on S. then $HK_S > 0$, since nK_S is effective for $n >> 0$. Consider the short exact sequence

$$0 \to \mathcal{O}_S(nK_S - H) \to \mathcal{O}_S(nK_S) \to \mathcal{O}_H(nK_S) \to 0,$$

then taking long exact sequence of cohomology, $h^0(S, nK_S) \leq h^0(H, nK_S|_H) + h^0(S, nK_S - H)$, but $h^0(S, nK_S)$ grows as n^2 and $h^0(H, nK_S|_H)$ at most as n, thus exists $E \in |nK_S - H|$ for $n >> 0$. So, $E \cdot K_S = (nK_S - H)K_S = nK_S^2 - H \cdot K_S \geq 0$, then $nK_S^2 \geq H \cdot K_S > 0$. For Castelnouvo's inequality $\chi(\mathcal{O}_S) \geq 1$, see [Bea96, Theo. X.4] □

The followings two theorems are profound and essential in order to study the "Geography of Surfaces."

Theorem 3.40 (Noether inequality 1875)**.** *Let S be a minimal surface of general type over any algebraically closed field* k*. Then,*

$$2\chi(\mathcal{O}_S) - 6 \leq c_1^2(S)$$

or equivalently

$$\frac{1}{5}(c_2(S) - 36) \leq c_1^2(S)$$

Theorem 3.41 (Bogomolov-Miyaoka-Yau inequiality 1977)**.** *Let S be a minimal surface of general type, then*

$$c_1^2(S) \leq 9\chi(\mathcal{O}_S)$$

or equivalently

$$c_1^2(S) \leq 3c_2(S).$$

The proofs of the last two inequalities can be found on [BHPV04, Chap. VII, Theo. 3.1 and Theo. 4.1].

Example 3.42.

(Compl. int) Let $S = S_{d_1,\dots,d_r} \subset \mathbb{P}^{r+2}$ be a complete intersection surface. If $r+3 < \sum_i d_i$, then S is of general type. By adjointion formula $K_S = \mathcal{O}_S(\sum_i d_i - (r+3))$, so $K_S^2 = (\sum d_i - (r+3))^2 d_1 \cdots d_r$. Taking the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^{r+2}}(-\sum_i d_i) \to \mathcal{O}_{\mathcal{P}^{r+2}}(\sum_i d_i - (r+3)) \to \mathcal{O}_S(\sum_i d_i - (r+3)) \to 0,$$

and long exact sequence of cohomology we get

$$H^0(\mathbb{P}^{r+2}, \mathcal{O}_{\mathbb{P}^{r+2}}(\sum_i d_i - (r+3))) \cong H^0(S, \mathcal{O}_S(\sum_i d_i - (r+3))),$$

so $h^0(S, \mathcal{O}_S(n(\sum_i d_i - (r+3)))) = \binom{n\sum_i d_i + (1-n)(r+3)-1}{n\sum_i d_i - n(r+3)} \sim n^2$. Therefore, $\mathrm{kod}(S) = 2$. In particular, for hypersurfaces $S = S_d$ on $\mathbb{P}^3$, we obtain $K_S^2 = (d-4)d^2$ and $\chi(\mathcal{O}_S) = 1 + p_g(S) = 1 + \binom{d-1}{d-4} = \frac{(d-3)(d-2)(d-1)+6}{6}$. Hence,

$$K_S^2 \leq 6\chi(\mathcal{O}_S) \text{ or equivalently } c_1^2(S) \leq c_2(S).$$

More generally;

Proposition 3.43. *If S is a complete intersection surfaces of general type. Then,*

$$c_1^2(S) \leq 2c_2(S).$$

Proof. See [P87, pag. 207]. □

(Godeaux) Let $S = \{x_0^5 + x_1^5 + x_2^5 + x_3^5\}$ be the Quintic surface on $\mathbb{P}^3$. The group $G := \mathbb{Z}/5\mathbb{Z}$ acts on S, as follow. Let ζ be a primitive 5-th root of the unity. We define the automorphism σ by $\sigma[x_0, x_1, x_2, x_3] = [x_0, x_1, \zeta^2 x_2, \zeta^3 x_3]$, easily we can see that σ is an automorphism with no fixed points. Define the smooth surface $S' := S/G$, the surface S' is known as the Godeaux surface. This surface has the smallest possible values for K_S^2 and $\chi(S)$. Notice that by Riemann-Hurwitz $5e(S') = e(S)$ and $5\chi(S') = \chi(S)$. Now, since $K_S \sim \mathcal{O}_S(1)$ then $K_S^2 = 5$ and by Lefschetz hyperplane theorem $\pi_1(S) = \{1\}$, and

$p_g = h^0(\mathbb{P}^3, \mathcal{O}(1)) = 4$ so $\chi(S) = 1 + 4 = 5$. Then $\chi(S') = 1$, so $12 = 1 + e(S')$ by Noether's formula. Therefore, $K_{S'}^2 = 1$ and $e(S') = 11$, then S' is of general type.

A classical question of the surface theory was: there exists a surface S of general type with rational invariants?, i.e., $q(S) = p_g(S) = 0$. It was answered in a positive way by Godeaux in [Godeaux31], presenting the example explained above. Nowadays, any smooth minimal projective surface S satisfying $c_1^2(S) = 1, q(S) = p_g(S) = 0$, and $\pi_1(S) \cong \mathbb{Z}/5\mathbb{Z}$ is called a Godeaux surface.

(Product of curves) Let C_1 and C_2 be two smooth projective curves of genus g_1 and g_2 respectively. If $g_1, g_2 \geq 2$, then $S = C_1 \times C_2$ is a surface of general type, because $\mathrm{kod}(S) = \mathrm{kod}(C_1) + \mathrm{kod}(C_2) = 1 + 1 = 2$. We can compute all the invariant of S. Note that $\Omega_S^1 = p^*(\Omega_{C_1}) \oplus q^*(\Omega_{C_2})$, where p, q are the projections to C_1 and C_2 respectively. Thus, $K_S = p^*(K_{C_1}) \otimes q^*(K_{C_2})$, hence, $K_S^2 = 2deg(K_{C_1})deg(K_{C_2}) = 8(g_1 - 1)(g_2 - 1)$. Similarly, we compute $q(S) = g_1 + g_2$ and $p_g(S) = g_1 g_2$. Then, $\chi(S) = 1 - g_1 - g_2 + g_1 g_2 = (g_1 - 1)(g_2 - 1)$. Therefore, $c_2(S) = 4(g_1 - 1)(g_2 - 1)$ and $c_1^2(S)/c_2(S) = 2$. Moreover, it implies:

Proposition 3.44. *If $\mathcal{M}_{c_1^2, c_2}$ denote the coarse moduli space of minimal surfaces S of general type with $c_1^2(S) = c_1^2$ and $c_2(S) = c_2$. Then, $\mathcal{M}_{8ab, 4ab} \neq \emptyset$ for all $a, b \geq 1$.*

The next result will be important in Chapter 5, Section 5.2, in order to study the fundamental groups of surfaces of general type with small c_1^2.

Proposition 3.45. *Let S be a minimal surface of general type, let M the moving part of K_S. Then, $K_S^2 \geq M^2$ and if $\psi_{|K_S|}(S) = S' \subset \mathbb{P}^{p_g - 1}$ is a surface, we have that $M^2 \geq (\deg(\psi_{|K_S|})) \deg(S')$.*

Proof. We write $K_S^2 = K(F + M) = K_S \cdot F + K_S \cdot M = K_S \cdot F + (F + M)M = K_S \cdot F + F \cdot M + M^2$, then since K_S is nef we have $K_S^2 \geq M^2$. Now let $\psi_{|K_S|} \colon S \dashrightarrow S' \subset \mathbb{P}^{p_g - 1}$, where S' is a surface. Let $\epsilon \colon \hat{S} \to S$ be the blow-up such that $\psi_{|K_S|} \circ \epsilon$ is well-defined. Then, $\epsilon^* M^2 = \deg((\psi_{|K_S|})) \deg(S')$ and by Proposition 3.13 we get that $M^2 \geq \deg((\psi_{|K_S|})) \deg(S')$. $\square$

Proposition 3.46. *Let S be an irreducible surface contained in $\mathbb{P}^n$ but no in any hyperplane. Then $\deg(S) \geq n-1$. If the surface S is not ruled, then $\deg(S) \geq 2n-2$.*

Proof. Let $\hat{S} \xrightarrow{\beta} S$ be a desingularization of S, let H be a hyperlane on the projective space $\mathbb{P}^n$, and denote by $\iota \colon S \to \mathbb{P}^n$ the inclusion. We denote by $|\hat{H}| = |(\beta \circ \iota)^* H|$. A generic curve $C \in |\hat{H}|$ is smooth. Considering the short exact sequence

$$0 \to \mathcal{O}_{\hat{S}} \to \mathcal{O}_{\hat{S}}(\hat{H}) \to \mathcal{O}_C(\hat{H}|_C) \to 0,$$

and by taking long exact sequence of cohomology we have that $n \leq h^0(C, \hat{H}|_C)$. We have two cases to analyze, case 1) $K_{\hat{S}} \cdot H \geq 0$, case 2) $K_{\hat{S}} \cdot H < 0$. On case 1), by the genus formula 3.9 we have that $2g(C) - 2 = C^2 + \hat{H} \cdot K_{\hat{S}} \geq \hat{H}^2 > 0$. Then by Clifford's lemma, see [Har77, Chap. IV, Theo. 5.4], we obtain $h^0(C) \leq \hat{H}^2/2 + 1$, so $\deg(S) = H^2 \leq 2n-2$. For the case 2), again by the genus formula we get the equality $2g(C) - 2 = \hat{H}^2 - \hat{H} \cdot K_{\hat{S}} < \hat{H}^2$, therefore by Riemann-Roch Theorem 3.4, we have $h^0(C, \hat{H}|_C) = \hat{H}^2 + 1 - g(C)$, thus $n - 1 \leq \deg(S)$. □

We summarize the information about classification of surfaces in the following table.

kod(C)	$p_g(S)$	$q(S)$	Minimal Surfaces
$-\infty$	0	0	$\mathbb{P}^2$ or $\mathbb{F}_n$ for $n \neq 1$
	0	≥ 0	$\mathbb{P}_C(\mathcal{E})$ where $\mathcal{E}$ is a rank 2 bundle on C
0	0	0	Enriques Surfaces
	0	1	Bielliptic Surfaces
	1	0	$K3$ Surfaces
	1	2	Abelian Surfaces
1	≥ 0	≥ 0	Elliptic fibration
2	≥ 0	≥ 0	?

Table 2: Enriques' Classification

4 Lefschetz type theorems

This chapter is more about algebraic topology than about algebraic geometry. However, the principal results (Lefschetz's Theorems) are standard tools in algebraic geometry to understand the topology of algebraic varieties. Lefschetz proved the classical Lefschetz Theorem 4.1 in [Lef24]. Lefschetz's tool was to use a hyperplane section pencil (Lefschetz pencil). Latter, Andreotti and Frankel in [AF59] proved the Lefschetz Theorem using differential topology. More precisely, they used Morse theory. This method has been developed further, and it has allowed obtaining generalizations of the Theorems 4.1 and 4.10. For instance, Goresky and MacPherson in [GM88] present six generalizations [GM88, Part. II, Chap. 1, Theorem 1.1, Theorem 1.1*, Theorem 1.2, Theorem 1.2*, Theorem 1.3, Theorem 1.3*], all of them based on a generalized Morse Theory.

The first part of the next exposition is based on the book Morse theory wrote by Milnor [MSW69]. The second part is based on the book "Stratified Morse Theory" wrote by Goresky and MacPherson [GM88]. We will focus specially in the proof of the Theorem "Relative Lefschetz Theorem with Large Fibres", [GM88, Part. II, Chap. 5, sect. 5.1]. It will be key to obtain our Theorem 5.37.

4.1 Classical Lefschetz Theorems

This section is based in the results presented by Milnor in [MSW69, Part I].

Theorem 4.1 (Lesfchetz Hyperplane Theorem)**.** *Let X be a smooth projective complex variety of dimension n and let D be an ample effective divisor on X. Then, the restriction homomorphism*

$$r_i : H^i(X,\mathbb{Z}) \to H^i(D,\mathbb{Z})$$

is an isomorphism if $i \leq n-2$ and is surjective if $i = n-1$.

We follow the proof presented by Andreotti and Frankel in [AF59]. The idea is to use an appropriate Morse function f from a manifold M to the real numbers $\mathbb{R}$, to determine the homotopy type of M in terms of the critical points of f and their index.

Definition 4.2. Let $f\colon M \to \mathbb{R}$ be a C^∞ function. A *critical point* of f is a point $p \in M$, such that $df_p = 0$, and $q = f(p)$ is called a *critical value* of f. A critical point p of f is *non-degenerate* if its Hessian matrix $Hess_p(f) = \left(\frac{\partial f(p)}{\partial x_i \partial x_j}\right)$ is non-degenerate, where $x_1, x_2, \ldots, x_n$ is a local system coordinate around p.

Example 4.3. 1. The function $f(x, y) = x^3 - 3xy^2$, has a degenerate critical point in $p = (0, 0)$. It is easy to see, since $df = (3x^2 - 3y^2, -6xy)$ and $Hess(f) = \begin{pmatrix} 6x & -6y \\ -6y & -6x \end{pmatrix}$.

2. The function $f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^2$ has only one critical point $p = (0, 0, \ldots, 0)$, and this point is non-degenerate, because $df = 2(x_1, \ldots, x_n)$ and $Hess(f) = 2I_{n\times n}$

3. The function $f(x, y) = sin(\frac{1}{x})y$ is such that every point of the form $(\frac{1}{(n+1)\pi}, 0)$ for $n \neq -1$ is a critical point. However, none of these points are degenerate, because f has differential equal to $df = \left(-\frac{ycos(\frac{1}{x})}{x^2}, sin(\frac{1}{x})\right)$ with Hessian matrix

$$Hess(f) = \begin{pmatrix} y\dfrac{(2xcos(\frac{1}{x}) - sin(\frac{1}{x}))}{x^4} & -\dfrac{cos(\frac{1}{x})}{x^2} \\ -\dfrac{cos(\frac{1}{x})}{x^2} & 0 \end{pmatrix}.$$

The determinant of $Hess(f)$ is equal to $\dfrac{cos^2(\frac{1}{x})}{x^4} \neq 0$ for $x = \dfrac{1}{(n+1)\pi}$,$(n \neq -1)$.

Definition 4.4. A function C^∞ function $f : M \to \mathbb{R}$ is said to be a *Morse function* if only has finitely many number of critical points, all the critical values are different and every critical point is non-degenerate.

The unique Morse function on the above Examples 4.3 is the second one; the other two have either a degenerate critical point or infinitely many critical points.

Definition 4.5. Let $f\colon M \to \mathbb{R}$ be a Morse function, and let p be one critical point of f. The index of f at p is the number λ_p of negative eigenvalues of $Hess_p(f)$.

Now, we state a theorem that summarizes how we can decompose a manifold M in terms of cells, which are parametrized by the index of the critical points of a Morse function $f\colon M \to \mathbb{R}$.

Theorem 4.6. *Let $f\colon M \to \mathbb{R}$ be a Morse function. For any $a \in \mathbb{R}$ we define $M_a := f^{-1}((-\infty, a])$. Suppose that the set M_a is compact. Then:*

1. *Let $a, b \in Im(f)$ such that $a < b$, $f^{-1}([a, b])$ is compact and with no critical points between them. Then, M_a is a deformation retract of M_b.*

2. *Let p be a critical point of f with index $\lambda_p = \lambda$, and $c = f(p)$. Then, for $0 < \epsilon << 1$, the set $M_{c+\epsilon}$ has the homotopy type of $M_{c-\epsilon}$ with a cell of dimension λ ($\lambda - cell$) attached.*

3. *M has the homotopy type of a CW-complex, with one cell of dimension λ_p for each critical point p of f.*

Proof. See [MSW69, Chap. 3]. ☐

Example 4.7. Let T be a torus, and we consider the height function f, i.e., if we suppose $T \subseteq \mathbb{R}^3$ the function $f\colon T \to \mathbb{R}$ is defined by $f(x, y, z) = z$. The points p_0, p_1, p_2, p_3 are the critical points of f. Now, we denote by $v_i = f(p_i)$ the critical value associated to the critical point p_i, see Figure 3.

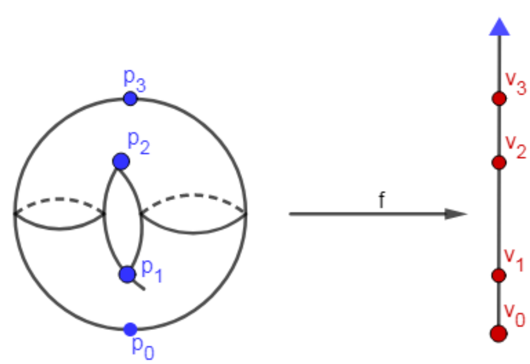


Figure 3: Morse theory for the standard torus.

Taking appropriate coordinates, we can write f in a small neighborhood p_0 as $f(x,y,z) = x^2+y^2+1$, f in a small neighborhood p_1 as $f(x,y,z) = x^2-y^2+1$, f in a small neighborhood p_2 as $f(x,y,z) = -x^2+y^2+1$, and f in a small neighborhood p_3 as $f(x,y,z) = -x^2-y^2+1$. Then, the index of each critical point are $\lambda_{p_0} = 0$, $\lambda_{p_1} = \lambda_{p_2} = 1$, and $\lambda_{p_3} = 2$. Therefore, the torus T has the type of homotopy of a CW-complex formed by one cell of dimension 0, two of dimension 1 and one of dimension 2, respectively. See Figure 4.

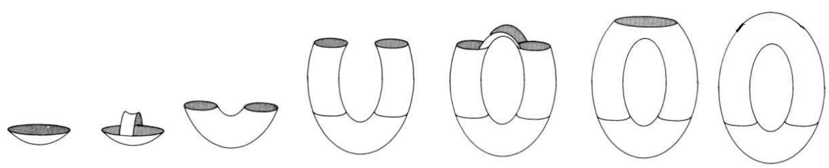


Figure 4: Decomposition in Cells

Theorem 4.8 (Andreotti-Frankel)**.** *Let $Y \subset \mathbb{C}^N$ be a complex smooth affine submanifold of real dimension $2n$. Then, Y has the same homotopy type as a CW-complex with dimension at most n. Therefore,*

$$H^i(Y,\mathbb{Z}) = H_i(Y,\mathbb{Z}) = 0$$

for any $i > n$.

Proof. See [MSW69, Theo. 7.1]. ☐

By Theorem 4.6 part 3, it suffices to find a proper Morse function such that every critical point has an index $\lambda \leq n$. The adequate function is a square distance function (see Example 4.3,2). These classes of functions are the most important examples of Morse functions.

Lemma 4.9. *1. Let $M \subset \mathbb{R}^r$ be a closed submanifold. Then, for almost all $c \in \mathbb{R}^r$, the squared distance function*

$$\phi_c(x) = ||x-c||^2 = \sum_i (x_i - c_i)^2$$

is a Morse function (restricted to M).

2. Let $Y \subset \mathbb{C}^N$ be a closed complex submanifold of dimension n. Let $c \in \mathbb{R}^{2N}$, then for any critical point p of the squared distance function ϕ_c, we have that

$$index_p(\phi_c) = \lambda_p \leq n.$$

Proof. For (1) see [MSW69, Chap. 6, Theo. 6.6], and for (2) see [MSW69, Chap. 7, Proof of Theo. 7.2, pag. 40-41]. □

Proof of Theorem 4.8. Choose a point $c \in \mathbb{R}^{2N}$ such that the squared distance function ϕ_c is a Morse function. It can be done by Lemma 4.9 part 1, and by part 2, we have that the index of every critical point p has index $\leq n$.

Therefore, by Theorem 4.3 part 3, Y has the homotopy type of a CW-complex of dimension $\leq n$.

□

Proof of the Theorem 4.1. Since the divisor D is ample, then mD is very ample for some $m >> 0$. Thus, there exists an embedding $X \to \mathbb{P}^N$, such that $X \cap H = mD$, so $X \setminus D = X \setminus mD = X \setminus (X \cap H)$ is an affine subset of $\mathbb{C}^N$. Then by Theorem 4.8 we have that $H_i(X, D) = 0$ for any $i < n$. On the other hand by Lefschetz duality, see for example [Hat05, Theo. 3.43] , $H_i(X, D) = H^{2n-i}(X, D)$. Then taking long exact sequence of relative cohomology we obtain the result. □

Theorem 4.10 (Lesfchetz Hyperplane Theorem for Homotopy groups)**.** *Let X be a smooth projective complex variety of dimension n and let D be an ample effective divisor on X. Let $\iota\colon D \to X$ be the immersion morphism. Then, the induced homomorphism*

$$\iota_*\colon \pi_i(D) \to \pi_i(X)$$

is an isomorphism if $i \leq n-2$ and is surjective if $i = n-1$.

We want to give a proof in all detail of the Theorem 4.10, to make a step-by-step comparison with the work done by Goresky-MacPherson in demonstrating its generalizations. In particular for the "Relative Lefschetz Theorem with Large Fibers" Theorem 4.16.

Definition 4.11. Let X be CW-complex and A subcomplex of X, we say that the pair (X, A) is a CW-pair.

Lemma 4.12. *Let (Y, A) a CW-pair, if $Y \setminus A$ only has cell of dimension at least k. Then, $\pi_i(Y) \cong \pi_i(A)$ for any $i < k$.*

Proof. See [Hat05, Cor. 4.12]. □

Proof of Theorem 4.10.

Step 1. Embed $X \setminus D$ into an affine space.

As in the proof of the Theorem 4.1, we can show that $X \setminus D \subseteq \mathbb{C}^N \cong \mathbb{R}^{2N}$, for some $N >> 0$.

Step 2. Define an adequate function and compute the index λ of its critical points.

Define the function

$$f(x) = \begin{cases} \dfrac{1}{\phi_c(x)}, & x \in X \setminus D \\ 0, & x \in D \end{cases}$$

Where $c \in \mathbb{R}^{2N} \setminus X$. Then, away of D, the critical points of $f(X)$ and $\phi_c(X)$ are the same since $df_p = (\phi_c(p))^{-2}(-\frac{\partial \phi_c(p)}{\partial x_i})$ and $Hess_p(f) = (\phi_c(p))^{-2}(-\frac{\partial^2 \phi_c(p)}{\partial x_j \partial x_i})$, so the eigenvalue of the Hessian matrix of f at a critical p is the eigenvalue of the Hessian matrix of ϕ_c at p with reversed sing. Therefore, the index of $\lambda_p^f \geq 2n - \lambda_p^{\phi_c} \geq 2n - n$, by Lemma 4.9.

Step 3. Show that X has the homotopy type of X_ϵ attached with cells of dimension $\geq n$.

We consider X as a CW-complex where D is a subcomplex. Let U be a neighborhood of D such that D is a deformation retract of U. Let $\epsilon << 1$, such that the set

$X_\epsilon := f^{-1}[0,\epsilon] \subseteq U$. Then, by Theorem 4.6 X has the type of homotopy of X_ϵ attached cell of dimension $\geq n$. Hence,

$$\pi_i(X) \cong \pi_i(X_\epsilon \cup \text{ Cells of dimension } \geq n)$$

Step 4. Show That the pair (X, X_ϵ) is n-connected.

By Lemma 4.12, putting $Y = X$ and $A = X_\epsilon$, we get that $\pi_i(X) \cong \pi_i(X_\epsilon)$ for $i < n$, since $X \setminus X_\epsilon$ only has cell of dimension $\geq n$.

Step 5. Show that $\pi_i(X, D) \cong \pi_i(X_\epsilon, D)$ for all $i \leq n-1$.

For that, we take both long exact sequences of homotopy for (X, D) and (X_ϵ, D) and compare step by step.

$$\begin{tikzcd}
\dots \arrow[r] & \pi_i(D) \arrow[r] \arrow[d] & \pi_i(X_\epsilon) \arrow[r] \arrow[d, "\cong"] & \pi_i(X_\epsilon, D) \arrow[r] \arrow[d] & \pi_{i-1}(D) \arrow[r] \arrow[d] & \dots \arrow[r] & \pi_0(X_\epsilon) \arrow[d] \\
\dots \arrow[r] & \pi_i(D) \arrow[r] & \pi_i(X) \arrow[r] & \pi_i(X, D) \arrow[r] & \pi_{i-1}(D) \arrow[r] & \dots \arrow[r] & \pi_0(X)
\end{tikzcd}$$

Therefore, we have that $\pi_i(X, D) \cong \pi_i(X_\epsilon, D)$ for all $i \leq n-1$.

Step 6. Show that $\pi_i(X, D) \cong \{1\}$ for $i \leq n-1$.

By the nested inclusions $D \subset X_\epsilon \subset U \subset X$ and since D is a deformation retract of U, we obtain the commutative diagram

$$\begin{tikzcd}
\pi_i(X_\epsilon, D) \arrow[r] \arrow[d, "\cong"] & \{1\} \cong \pi_i(U, D) \arrow[dl] \\
\pi_i(X, D).
\end{tikzcd}$$

Therefore, we get the desired isomorphism $\pi_i(X, D) \cong \{1\}$ for $i \leq n-1$ or equivalently $\pi_i(D) \cong \pi_i(X)$ for $i \leq n-2$ and $\iota_* \colon \pi_{n-1}(D) \to \pi_{n-1}(X)$ is a surjection.

$\square$

Remark 4.13 (Some applications).

1. Ample divisors are connected. Let D be an ample divisor on a smooth complex projective variety X with dimension at least 2. Then, by Theorem 4.1 we have $H^0(X,\mathbb{Z}) \cong H^0(D,\mathbb{Z})$, so $H^0(D,\mathbb{Z}) \cong \mathbb{Z}$.

2. Complete intersection surfaces on $\mathbb{P}^N$ are regular, i.e, $q = 0$. Let S be a complete intersection surfaces on $\mathbb{P}^N$, then S can be write as $S = \{H_1 = H_2 = \cdots = H_{N-2} = 0\}$, where each H_i is a smooth hypersurface. Thus, by Theorem 4.10 we know that $\pi_1(\{H_i = 0\}) \cong \pi_1(\mathbb{P}^N) \cong \{1\}$.Then, by induction we get that

$$\pi_1(S) \cong \pi_1(\{H_1 = H_2 = \cdots = H_{N-1} = 0\}) \cong \pi_1(\mathbb{P}^N) \cong \{1\}.$$

   Therefore, $q(S) = 0$.

We finish this subsection presenting a Weak Lefschetz Theorem for lef line bundle.

Theorem 4.14. *Let X be a smooth projective variety and let L be a lef line bundle on X. If $E \in \Gamma(X,L)$ is a non-singular divisor of X. Then, the morphism $\iota^*\colon H^r(X) \to H^r(E)$ induced by the inlcusion $\iota : E \to X$, is an isomorphis if $r < \dim(X) - 1$ and it is injective if $r = \dim(X) - 1$.*

Proof. See [CM02, Prop. 2.1.5] □

4.2 Relative Lefschetz Theorem with Large Fibres

Definition 4.15. Let X be a topological space and $(I, \leq)$ a partial ordered set. We say that X is $(I, \leq)$-stratified if there is a collection $\{S_i\}_{i \in I}$ of disjoint locally closed subsets $S_i \subset X$ such that:

(i) $X = \bigcup_{i \in I} S_i$,

(ii) $S_i \cap \overline{S_j} \neq \emptyset$ if and only if $S_i \subset \overline{S_j}$ if and only if $i \leq j$.

If X and Y are two $(I, \leq)$-stratified spaces a map $f : X = \bigcup_{i \in I} R_i \to Y = \bigcup_{i \in I} S_i$ is called stratified if f is continuous and $f(R_i) \subset S_i$ for all $i \in I$.

The last definition can be found in [GM88, Part. I, Chap. 1, Sect. 1].

Theorem 4.16 (Relative Lefschetz Theorem with Large Fibres)**.** *Let X be an smooth projective variety of dimension n. Let $f : X \to \mathbb{P}^N$ a proper morphism, and let H be a linear subspace of $\mathbb{P}^N$ with codimension c. Define $\phi(k) := \dim((\mathbb{P}^N \setminus H)^k_f)$.*

Then the induced homorphism

$$\pi_i(f^{-1}(H)) \to \pi_i(X)$$

is an isomorphism if $i < \hat{n}$, and it is surjective if $i = \hat{n}$, where

$$\hat{n} = n - 1 - \sup_k \{2k - n + \phi(k) + \inf(\phi(k), c - 1)\}.$$

Corollary 4.17. *If H is a hyperplane in Theorem 4.16 and $f \colon X \to \mathbb{P}^N$ is semi-small into its image, then $\pi_i(f^{-1}(H)) \cong \pi_i(X)$ if $i < n - 1$.*

Proof. The case $f(X) \subset H$ is trivial. In the computation of $\hat{n}$ we can ignore the values $\phi(k) = -\infty$. Then we compute $\hat{n} = n - 1$ since $\dim(f(X)) = n$, the codimension of H is $c = 1$, and we have the inequality $\phi(k) \leq dim((f(X))^k_f)$. The last inequality is because $(\mathbb{P}^N \setminus H)^k_f = (f(X) \setminus (f(X) \cap H))^f_k \subset f(X)^k_f$. $\square$

The proof the Theorem 4.16 that we will outline here is given by Goresky and MacPherson in [GM88, Part II. Chap. 5, Sect. 5.1].

Before to start the outline, we will present some necessary definitions and lemmas.

Lemma 4.18. *Let $Z \subset \mathbb{P}^N$ be a stratified space, and let H be a linear subspace of $\mathbb{P}^N$ of codimension c. Let $G \subset \mathbb{P}^N$ be a complementary space of H, i.e., a linear subspace such that $G \cap H = \emptyset$ and $\dim(G) = c - 1$. For instance, if $H = \{x_0 = \cdots = x_{c-1} = 0\}$, then $G = \{x_c = \cdots = x_N = 0\}$. Let $h \colon \mathbb{P}^N \to \mathbb{R}$, given by*

$$h([x_0 : \cdots : x_N]) := \frac{\sum_{i=0}^{c-1} x_i \overline{x_i}}{\sum_{i=0}^{N} x_i \overline{x_i}}.$$

Then, for every critical point p of $h|_Z$ the index $\lambda := \lambda_p^{h|_Z}$ satisfies:

$$\dim(Z) - \lambda \leq c - 1.$$

Proof. See [GM88, Part. II, Sect. 4.4.]. □

Definition 4.19. Let $f\colon X \to Z$ be a *semi-small stratified* map and $h\colon Z \to \mathbb{R}$ be a proper Morse function. If $p \in Z$ is a non-degenerate critical point $p \in S_\alpha$ for some stratum $S_\alpha \subset Z$, we define the convexity defect of h at the point p by $\Gamma_h(p) := \dim(S_\alpha) - \lambda_p$.

Lemma 4.20. *Consider $f\colon X \to Z$ and $h\colon Z \to \mathbb{R}$ as in the definition above. Suppose that the interval $[a, b]$ contains no critical values of h except $v = h(p) \in (a, b)$. Denote $X_{\leq t} := X \cap f^{-1}(h^{-1}(-\infty, t])$. Then, the pair $(X_{\leq a}, X_{\leq b})$ is $(n - \Gamma_h(p))$-connected.*

Proof. By [GM88, Part II, Sect. 4.2, Prop. 4.3] the pair $(X_{\leq a}, X_{\leq b})$ is $(n - (\Delta(p) + \Gamma_h(p)))-$connected, where $\Delta(p)$ is the normal defect of h at p, see [GM88, Part II, Sect. 4.2]. But, by [GM88, Part II, Sect. 4.5, Prop. 4.5.1$_m$] and the semi-smallness condition of f, we have that $\Delta(p) = 0$. □

The following is a sketch of a proof of Theorem 4.16.

Proof of Theorem 4.16. Since we are interested in apply the theorem for semi-small morphisms, so we will assume that the map $f\colon X \to \mathbb{P}^N$ is semi-small into its image. In this case, the integer $\hat{n} = n - 1 - \inf_k(\phi(k), c - 1)$.

Let us denote $Z := f(X) \subset \mathbb{P}^N$.

Step 1. Stratify Z and X such that π is an stratified morphism.

For each non-negative integer define $Z^k := \{z \in Z | \dim(f^{-1}(z)) = k\}$. Then, X can be stratified by $X^k := f^{-1}(Z^k)$, so f is an stratified morphism.

Step 2. Define an appropriate function in order to use stratified Morse theory.

Let $G \subset \mathbb{P}^N$ be a complementary space to H. define $\overline{h} : \mathbb{P}^N \to \mathbb{R}$ by

$$\overline{h}([x_0 : \cdots : x_n]) \colon = \frac{\sum_{i=0}^{c-1} x_i \overline{x_i}}{\sum_{i=0}^{n} x_i \overline{x_i}},$$

as in Lemma 4.18. Then, the function $\overline{f}$ satisfies that:

1. $0 \leq \overline{h}(x) \leq 1$,
2. $\overline{h}^{-1}(0) = H$ and $, \overline{h}^{-1}(1) = G$,
3. $\overline{h}$ is analytic.

Step 3. Show that we can replace H for $H_\epsilon := \overline{h}^{-1}[0, \epsilon]$.

The morphism $\overline{h} \circ f \colon X \to \mathbb{R}$ has finitely many critical points. Thus, we can choose $0 < \epsilon << 1$ such that the interval $(0, \epsilon]$ contains no critical values of $\overline{h} \circ f$. Therefore, by [GM88, Part. II, Sect.5.A, Prop. 5.A.1.], the inclusion

$$\iota \colon X \cap f^{-1}(H) \hookrightarrow X \cap f^{-1}(H_\epsilon)$$

yields isomorphisms on all the homotopy groups π_i.

Step 4. Approximate $\overline{h}$ by a Morse function h.

By [GM88, Part I, Sect. 2.2] or [Pig80], we can approximate $\overline{h}$ by a C^∞ function h, such that $h|_{H_\epsilon} = \overline{h}|_{H_\epsilon}$ and,

1. $0 \leq h(x) \leq 1, h^{-1}(0) = H$ and, $h^{-1}(1) = G$,
2. $h^{-1}([0, \epsilon]) = H_\epsilon$,
3. the function h is a stratified Morse function for the strata Z_k of Z with different critical values on $Z \cap h^{-1}(\epsilon/2, 1)$ and,

By Lemma 4.18, for each critical point $p \in Z$ of h we have that

$$\Gamma_h(p) = \dim(Z_k) - \lambda_p^{h|_{Z_k}} \leq \min(\dim(Z_k), c-1),$$

where $p \in Z_k$.

We have built the function Morse function $h \circ f \colon X \to \mathbb{R}$. As in steps 3,4 and 5 of the proof of Theorem 4.10, we will use Morse theory applied to $h \circ f$ to estimate the connectivity of the pair $(X, X \cap f^{-1}(H_\epsilon))$.

Step 5. Estimate the connectivity of the pair $(X_{1-\theta}, X \cap f^{-1}(H_\epsilon))$, for $0 < \theta << 1$, where $X_{1-\theta} := X \cap f^{-1}(h^{-1}[0, 1-\theta])$.

First, we can write $\hat{n} = \inf_k(n(Z_k))$, where

$$n(Z_k) = n - 1 - (2\dim(\pi^{-1}(z)) - (n - \dim(Z_k)) + \min(\dim(Z_k), c-1))),$$

it is clear because $\phi(x) = \dim(Z_k)$ and $2\dim(f^{-1}(z)) + \dim(Z_k) \leq n$ if $z \in Z_k$. Let $v_1, \ldots, v_k$ be the critical values of $h \circ f$ in the interval $[\epsilon, 1-\theta]$ with $f(h(p_i)) = v_i$, and let $\epsilon = \epsilon_0 < \epsilon_1 < \cdots < \epsilon_{k-1} < \epsilon_k = 1-\theta$, such that $v_i \in (\epsilon_{i-1}, \epsilon_i)$. By the Lemma 2, the pair $(X_{\epsilon_i}, X_{\epsilon_{i+1}})$ is $\hat{m_i} = n - 1 - \Gamma(p_i)$ connected, for all $0 \leq i \leq k-1$, where $p_i \in Z_k$. But by Step 4, we have the inequality $\Gamma(p_i) \leq \min(d(Z_k), c-1)$. Hence, taken infimum over all the index i, we have $\hat{m_i} \geq \hat{n}$. So, $\pi_i(X_{1-\theta}, X \cap f^{-1}(H_\epsilon)) = \{1\}$ for all $i \leq \hat{n}$.

Step 6. Go from $X_{1-\theta}$ to X.

The idea is the following.

First show that the pair $(X \cap f^{-1}h^{-1}[1-\theta, 1], X \cap f^{-1}h^{-1}(1-\theta))$ is such that $X_{[1-\theta,1]} := X \cap f^{-1}h^{-1}[1-\theta, 1]$ is homeomorphic to the disk bundle over a vector bundle of $X \cap f^{-1}h^{-1}(1) := X_1$ and $X_{1-\theta} := X \cap f^{-1}h^{-1}(1-\theta)$ is homeomorphic to the boundary sphere bundle over X_1 with fibre $\mathbb{C}^{N-c+1}$.

Then, prove that the long exact sequence of homotopy for the pair $(X_{[1-\theta,1]}, X_{1-\theta})$

$$\longrightarrow \pi_i(X_{1-\theta}) \longrightarrow \pi_i(X_{[1-\theta,1]}) \longrightarrow \pi_i(X_{[1-\theta,1]}, X_{1-\theta}) \longrightarrow \pi_{i-1}(X_{1-\theta}) \longrightarrow$$

coincides with long exact sequence of homotopy for the boundary sphere bundle,

$$X_{1-\theta} \to h^{-1}(1),$$

$$\longrightarrow \pi_i(\mathbb{S}^{2N-2c+1}) \longrightarrow \pi_i(X_{1-\theta}) \longrightarrow \pi_i(f^{-1}(1)) \longrightarrow \pi_{i-1}(\mathbb{S}^{2N-2c+1}) \longrightarrow$$

So we get that

$$\pi_i(X_{[1-\theta,1]}, X_{1-\theta}) \cong \pi_i(\mathbb{S}^{2N-2c+1}),$$

but $\pi_i(\mathbb{S}^{2N-2c+1}) \cong \{1\}$ if $i < 2N-2c+1$, see [Hat05, Coro. 4.9]. We only have to show that $\hat{n} \leq 2N-2c+1$. But, since $\dim(X) = n$ and f is semi-small then $X^0 = f^{-1}(Z^0)$ has dimension n. Therefore, $\hat{n} \leq n-(c-1) = n-c+1 \leq N-c+1 \leq 2N-2c+1$.

□

5 Geography of Surfaces of General type

5.1 Geography of simply-connected Surfaces of General type

5.1.1 Persson and Chen's Results

Persson in [P81] showed the following theorems which imply the density of Chern slopes of minimal simply-connected surfaces of general type in the interval $[\frac{1}{5}, 2]$.

Theorem 5.1. *[P81, Prop. 3.23] Given x, y positive integers satisfying the inequality*

$$\frac{1}{5}(x - 36) \leq y \leq \frac{1}{2}(x - 8),$$

then there exists a minimal simply-connected surface of general type S, such that $c_1^2(S) = y$ and $c_2(S) = x$.

Theorem 5.2. *[P81, Theo. 3] Let x, y be positive integers such that the inequality*

$$\frac{1}{2}(x - 8) \leq y \leq 2x - \frac{9}{4}(x + y)^{\frac{2}{3}}$$

holds, then there exists a minimal simply-connected surface of general type S with $c_1^2(S) = y$ and $c_2(S) = x$.

Corollary 5.3. *The set Chern slopes c_1^2/c_2 of minimal simply-connected surfaces of general type is dense in the interval $[\frac{1}{5}, 2]$.*

Bogomolov conjectured that any simply-connected surface of general type satisfy that $c_1^2/c_2 < 2$. However, Xiao in [X85a] found various examples of such surfaces with $c_1^2/c_2 \geq 2$. Following the ideas of Xiao, Chen in [Ch87] show the following theorem.

Theorem 5.4. *[Ch87, Theo. 1] Let x, y be positive integers satisfying the inequality*

$$\frac{352}{89}x + 140.2x^{\frac{2}{3}} < y < \frac{18644}{2129}x - 365.7x^{\frac{2}{3}},$$

for $x > C$, where C is a large constant. Then, there exists a simply-connected minimal surfaces of general type S with $c_1^2(S) = y$ and $\chi(\mathcal{O}_S) = x$.

Using Noether's formula $12\chi = c_1^2 + c_2$ is easy to deduce the density of Chern slopes of simply-connected surfaces of general type in the set $(\frac{352}{716}, \frac{18644}{6904}) \subseteq [0.49, 2.7005]$.

The idea behind the last theorems is to use double covers of rational surfaces allowing some special type of singularities in the branch locus.

Latter, Persson, Peters and Xiao [PPX96] obtained the following density results.

Definition 5.5. A surface S is called a spin surface if it is simply-connected and its canonical class is divisible is divisible by 2.

Theorem 5.6.

- *For any rational number $r \in \mathbb{Q} \cap [\frac{1}{5}, 2)$, there exists a minimal spin surfaces of general type S such that $c_1^2(S) = rc_2(S)$.*
- *The Chern slopes $c_1^2(S)/c_2(S)$ of minimal spin surfaces of general type S are dense in the interval* $[2, 2.\overline{703}]$

Urzuá, in [U10] using cyclic coverings of $\mathbb{P}^2$, found a sequence of simply-connected surfaces with Chern slope $c_1^2/c_2 = \frac{71}{26} \approx 2.730796$, which was the record at that moment.

Finally, Roulleau and Urzúa in [RU15] complete the work showing the density in the interval $[1, 3]$. In the next section, we will explain the proof of such result.

5.1.2 Roulleau-Urzúa's surfaces

In this section, we recall some surfaces of general type X_p from [RU15, Section 6] which are key in the main result of this thesis.

Definition 5.7. An arrangement of curves is a collection of curves $\{C_1, \ldots, C_r\}$ on a non-singular surface S. An arrangement $\mathcal{C}$ on S is said to be simple crossing if any two curves $C_i, C_j \in \mathcal{C}$ intersect transversely.

A $k-$point of an arrangement $\mathcal{C}$ is a point locally of the form $(0, 0) \in \{(x - \xi_1 y) \ldots (x - \xi_k y) = 0\} \subset \mathbb{C}^2_{x,y}$ where $\xi_i \neq \xi_j$ for $i \neq j$, we denote by t_k the number of $k-$points on $\mathcal{C}$. If $\mathcal{C}$ only have 2-point it is called simple normal crossing.

Let $p \geq 5$ be a prime number, and let $\alpha > 0, \beta > 0$ be integers. Let $n = 3\alpha p$. Let H be the blow-up at the twelve 3-points of the dual Hesse arrangement of 9 lines

$$(x^3 - y^3)(y^3 - z^3)(x^3 - z^3) = 0$$

in $\mathbb{P}^2$.

As defined in [RU15, Sect. 3 and 5] we will consider the diagram of varieties and morphism, where $i \in \{0, 1, \infty, \zeta\}$.

$$\begin{array}{ccccccc} Y_n & \xrightarrow{\sigma_n} & Z_n & \xrightarrow{\varphi_n} & H & \xrightarrow{\tau} & \mathbb{P}^2 \\ & & & & \downarrow{\scriptstyle \pi_i'} & & \\ & & & & \mathbb{P}^1 & & \end{array}$$

The three singular fibers of π_i' are denoted by $F_{i,1}, F_{i,2}, F_{i,3}$. Each $F_{i,j}$ consists of four $\mathbb{P}^1$'s: one central curve $N_{i,j}$ with multiplicity 3, and three reduced curves transversal to $N_{i,j}$ at one each point. We write $N_i = \sum_{j=1}^{3} N_{i,j}$. Let M the nine $\mathbb{P}^1$'s from the dual Hesse arrangement, and N be the twelve exceptional $\mathbb{P}^1$'s from its twelve 3-points. We have $N = \sum_i N_i$, and

$$F_{i,1} + F_{i,2} + F_{i,3} = M + 3N_i$$

We consider the very special arrangement of $\frac{4n^2-12}{3}$ elliptic curves $\mathcal{H}_n' = \mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_\infty + \mathcal{E}_\zeta$ in H. Let $\mathcal{E}_i'$ be $\beta^2 p^2$ general fibers of π_i' (defined also in [RU15, Section 3]), and let $\mathcal{A}_{2d} = L_1 + \ldots + L_{2d}$ be the strict transform of an arrangement of $2d$ general lines in $\mathbb{P}^2$, where $3 \leq 2d \leq p$. We define $a_0 = a_1 = b_i = 1$ for $1 \leq i \leq d$, and $a_\infty = a_\zeta = b_i = p - 1$ for $d + 1 \leq i \leq 2d$. Then

$$\mathcal{O}_H\Big(\sum_{i=0,1,\zeta,\infty} 3a_i \mathcal{E}_i + \sum_{i=0,1,\zeta,\infty} 3a_i \mathcal{E}_i' + \sum_{i=0,1,\zeta,\infty} a_i (F_{i,1} + F_{i,2} + F_{i,3}) + \sum_{i=1}^{2d} 3b_i L_i \Big)$$

is isomorphic to

$\mathcal{L}_0^p$ where

$$\mathcal{L}_0 := \mathcal{O}_H\Big(3p(3\alpha^2 + \beta^2)\big(\sum_{i=0,1,\zeta,\infty} a_i F_i\big) + 3dL\Big),$$

and all symbols have been defined in [RU15, Section 5]. For each i, we denote the strict transform of $\mathcal{E}_i$, $\mathcal{E}'_i$, L_j, $F_{i,j}$ in Z_n by the same symbol, where $\varphi_n \colon Z_n \to H$ is the blow-up of H at all the $\frac{(n^2-3)(n^2-9)}{3}$ 4-points in $\mathcal{H}'_n$. Then

$$\mathcal{O}_{Z_n}\Big(\sum_{i=0,1,\zeta,\infty} 3a_i\mathcal{E}_i + \sum_{i=0,1,\zeta,\infty} 3a_i\mathcal{E}'_i + \sum_{i=0,1,\zeta,\infty} a_i(F_{i,1}+F_{i,2}+F_{i,3}) + \sum_{i=1}^{2d} 3b_iL_i\Big)$$

is $\mathcal{L}_1^p$ where $\mathcal{L}_1 := \varphi_n^*(\mathcal{L}_0) \otimes \mathcal{O}_{Z_n}(-6E)$, and E is the exceptional divisor of φ_n. Again, we denote the strict transform of $\mathcal{E}_i$, $\mathcal{E}'_i$, L_j, $F_{i,j}$, M, N_i, N in Y_n by the same symbol, where $\sigma_n \colon Y_n \to Z_n$ is the blow-up at all the $4(n^2-3)$ 3-points in $\mathcal{H}'_n$. Then we have

$$\mathcal{O}_{Y_n}\Big(\sum_{i=0,1,\zeta,\infty} 3a_i\mathcal{E}_i + \sum_{i=0,1,\zeta,\infty} 3a_i\mathcal{E}'_i + \sum_{i=0,1,\zeta,\infty} 3a_iN_i + \sum_{i=1}^{2d} 3b_iL_i\Big) \simeq \mathcal{L}^p$$

where $\mathcal{L} := \sigma_n^*(\mathcal{L}_1) \otimes \mathcal{O}_{Y_n}(-2M-6G)$.

With this data, we construct a p-th root cover of Y_n branch along

$$A := \sum_{i=0,1,\zeta,\infty} \mathcal{E}_i + \sum_{i=0,1,\zeta,\infty} \mathcal{E}'_i + \sum_{i=0,1,\zeta,\infty} N_i + \sum_{i=1}^{2d} L_i.$$

Let $f \colon X_p \to Y_n$ be the corresponding morphism for the p-th root cover, as in [RU15, Section 5]. The nonsingular projective surface X_p is simply-connected [RU15, Prop.6.1], and minimal [RU15, Prop.6.2].

Let us write

$$A = \sum_j \nu_jA_j = \sum_{i=0,1,\zeta,\infty} 3a_i\mathcal{E}_i + \sum_{i=0,1,\zeta,\infty} 3a_i\mathcal{E}'_i + \sum_{i=0,1,\zeta,\infty} 3a_iN_i + \sum_{i=1}^{2d} 3b_iL_i$$

where A_j are the irreducible curves in A. Hence ν_j is equal to either $3a_i$ or $3b_k$ for some i,k. The arrangement A has only 2-points, and its number is

$$t_2 = 108\alpha^2\beta^2p^4 + 18\beta^4p^4 + 72d\alpha^2p^2 - 25d + 24d\beta^2p^2 + 2d^2.$$

Definition 5.8. Given the pair $(S,\mathcal{C})$, where S is a smooth surface and $\mathcal{C} = \{C_1,\dots,C_d\}$ is a simple crossing arrangement on S. The log Chern numbers of $(S,\mathcal{C})$ are:

$$\bar{c}_1^2(S,\mathcal{C}) = c_1^2(S) - \sum_{i=1}^{d} C_i^2 + \sum_{k\geq 2}(3k-4)t_k + 4\sum_{i=1}^{d}(g(C_i)-1)$$

and

$$\bar{c}_2(S,\mathcal{C}) = c_2(S) - + \sum_{k\geq 2}(k-1)t_k + 2\sum_{i=1}^{d}(g(C_i)-1).$$

By [RU15, Prop.4.1], the log Chern numbers of A are

$$\bar{c}_1^2 = n^4 + 2t_2 - 10d - 48 \quad \text{and} \quad \bar{c}_2 = \frac{n^4}{3} + t_2 - 4d - 12.$$

As in [RU15, Section 5], the Chern numbers of X_p are

$$c_1^2(X_p) = p\,\bar{c}_1^2 - 2\Big(t_2 + 2\sum_j (g(A_j)-1)\Big) + \frac{1}{p}\sum_j A_j^2 - \sum_{i<j} c(q_{i,j},p)A_i\cdot A_j$$

and

$$c_2(X_p) = p\,\bar{c}_2 - \Big(t_2 + 2\sum_j (g(A_j)-1)\Big) + \sum_{i<j} l(q_{i,j},p)A_i\cdot A_j$$

where $0 < q_{i,j} < p$ with $\nu_i + q_{i,j}\nu_j \equiv 0 \pmod{p}$,

$$c(q_{i,j},p) := 12s(q_{i,j},p) + l(q_{i,j},p),$$

and $s(q_{i,j},p)$ and $l(q_{i,j},p)$ are the numbers that we recall below.

Definition 5.9. Let q,p be coprime integers such that $0 < q < p$.

(1) The associated *Hirzebruch-Jung continued fraction* is

$$\frac{p}{q} = e_1 - \cfrac{1}{e_2 - \cfrac{1}{\ddots - \frac{1}{e_l}}} := [e_1,\ldots,e_l]$$

We denote its *length* as $l(q,p) := l$.

(2) The *Dedekind sum* associated to the pair (q,p) is defined as

$$s(q,p) := \sum_{i=1}^{p-1}\Big(\Big(\frac{i}{p}\Big)\Big)\Big(\Big(\frac{iq}{p}\Big)\Big),$$

where $((x)) := x - \lfloor x\rfloor - \frac{1}{2}$.

For the particular multiplicities $a_0 = a_1 = b_i = 1$ for $1 \le i \le d$ and $a_\infty = a_\zeta = b_i = p-1$ for $d+1 \le i \le 2d$ we chose, we have to consider only the numbers $c(p-1,p) = \frac{2p-2}{p}$ and $c(1,p) = \frac{p^2-2p+2}{p}$, and $l(p-1,p) = p-1$ and $l(1,p) = 1$. Therefore,

$$\sum_{i<j} c(q_{i,j}, 4p) A_i \cdot A_j = \frac{(2p-2)}{p} t_{2,1} + \frac{(p^2-2p+2)}{p} t_{2,2}$$

and

$$\sum_{i<j} l(q_{i,j}, 4p) A_i \cdot A_j = (p-1)t_{2,1} + t_{2,2}$$

where $t_{2,1}$ and $t_{2,2}$ are the number of 2-points corresponding to the singularities $\frac{1}{p}(1, p-1)$ and $\frac{1}{p}(1,1)$ respectively. Hence

$$t_{2,1} = 6\beta^4 p^4 + 36\alpha^2\beta^2 p^4 + 36d\alpha^2 p^2 - 13d + 12d\beta^2 p^2 + d^2$$

and

$$t_{2,2} = 12\beta^4 p^4 + 72\alpha^2\beta^2 p^4 + 36d\alpha^2 p^2 - 12d + 12d\beta^2 p^2 + d^2.$$

By plugging in the formulas for Chern numbers, we obtain that

$$c_1^2(X_p) = (81\alpha^4 + 144\alpha^2\beta^2 + 24\beta^4)p^5 + l.o.t.$$

and

$$c_2(X_p) = (27\alpha^4 + 144\alpha^2\beta^2 + 24\beta^4)p^5 + l.o.t.,$$

where l.o.t. (lower order terms) is a Laurent polynomial in p of degree less than 5. In this way, we obtain that

$$\lim_{p\to\infty} \frac{c_1^2(X_p)}{c_2(X_p)} = \frac{27x^4 + 48x^2 + 8}{9x^4 + 48x^2 + 8} =: \lambda(x)$$

where $x := \alpha/\beta$. We note that $\lambda\big([0, \infty^+]\big) = [1, 3]$. This allows to prove the following theorem (see [RU15, Theorem 6.3]).

Theorem 5.10. *For any number $r \in [1,3]$, there are simply-connected minimal surfaces of general type X with $c_1^2(X)/c_2(X)$ arbitrarily close to r.*

Proposition 5.11. *Let $\Gamma_p := f^*(L)$, where as before L is the pull-back in Y_p of a general line in $\mathbb{P}^2$. Then we have $\Gamma_p^2 = p$ and $\Gamma_p \cdot K_{X_p} = -3p + (p-1)(2d + 36\alpha^2 p^2 - 12 + 12\beta^2 p^2)$.*

Proof. As f is a generically finite morphism of degree p, we have $\Gamma_p^2 = p$. Let us consider L generic, so that $f^*(L)$ is a nonsingular curve. We note that $L \cdot N_i = 0$ for all i, $L \cdot \sum_{i=1}^{2d} L_i = 2d$, $L \cdot \sum_{i=0,1,\zeta,\infty} \mathcal{E}_i = 36\alpha^2 p^2 - 12$, and $L \cdot \sum_{i=0,1,\zeta,\infty} \mathcal{E}'_i = 12\beta^2 p^2$. Therefore, the morphism $f_{\Gamma_p} \colon \Gamma_p \to L = \mathbb{P}^1$ is totally ramified at $2d + 36\alpha^2 p^2 - 12 + 12\beta^2 p^2$ points, and so, by the Riemann-Hurwitz formula and adjunction, we obtain the desired equality for $\Gamma_p \cdot K_{X_p}$. □

We finish this section with a proof that the best lower bound for Chern slopes in this construction is indeed 1. As it was shown above, the values of the b_i's do not contribute in the asymptotic final result. We also point out that it is enough to have either $\sum_{i=0,1,\zeta,\infty} a_i = p$ or $\sum_{i=0,1,\zeta,\infty} a_i = 2p$ by considering $0 < a_i < p$ and multiplying by units modulo p. In fact, we can and do take $a_0 = 1$, $a_1 = a$, $a_\zeta = b$, and $a_\infty = c$ with $1 + a + b + c = mp$ for m either equal to 1 or 2.

Through the formulas obtained above, we have

$$\lim_{x \to 0} \frac{c_1^2(X_p)}{c_2(X_p)} = \frac{12 - \frac{1}{p} C}{6 + \frac{1}{p} J}$$

where $C := c(-a,p) + c(-b,p) + c(-c,p) + c(-ba^{-1},p) + c(-ca^{-1},p) + c(-cb^{-1},p)$, $J := l(-a,p) + l(-b,p) + l(-c,p) + l(-ba^{-1},p) + l(-ca^{-1},p) + l(-cb^{-1},p)$, and all the q's in these expressions are taken modulo p with $0 < q < p$. For example, for generic a, b, c one can prove that C/p and J/p tend to 0 as p approaches infinity, and so the limit of the Chern slopes is 2 (see [U10] for these generic behaviours).

Since $c(q,p) = 12s(q,p) + l(q,p)$, it is enough to show that

$$6S + J \leq 3p + 3 - \frac{6}{p}$$

any p, where $S := s(-a,p) + s(-b,p) + s(-c,p) + s(-ba^{-1},p) + s(-ca^{-1},p) + s(-cb^{-1},p)$. The proof will use the following numerical lemma.

Lemma 5.12. *Let $0 < q < p$ be coprime integers. Let $\frac{p}{q} = [e_1, \ldots, e_l]$. Then $\sum_{i=1}^{l} (e_i - 1) \leq p - 1$.*

Proof. We do induction on p. Say for all coprime pairs (q', p') with $p' < p$ we have that the statement is true. We write $\frac{p}{q} = [e_1, \ldots, e_l]$. Then $e_1 = [p/q] + 1$, and $\frac{q}{r} = [e_2, \ldots, e_l]$ with (r, q) coprime and $q < p$. Hence

$$\sum_{i=1}^{l}(e_i - 1) = [p/q] + \sum_{i=2}^{l}(e_i - 1) \leq [p/q] + q - 1$$

by the induction hypothesis. Therefore, we should prove that $[p/q] + q \leq p$. Let $q \neq 1$ (otherwise we are done). Let $1 \leq r < q$ be the unique integer such that $[p/q]q + r = p$. Then $[p/q] + q \leq p$ is equivalent to $\frac{q-r}{q-1} + q \leq p$. But $\frac{q-r}{q-1} \leq 1$ if $r \geq 1$, and $q + 1 \leq p$. $\square$

Proposition 5.13. *We have* $6S + J \leq 3p + 3 - \frac{6}{p}$.

Proof. Let $0 < q < p$ integers where p is a prime number. Then (see e.g. [U10, Example 3.5])

$$12s(q,p) = \frac{q + q^{-1}}{p} + \sum_{i=1}^{l}(e_i - 3)$$

where $\frac{p}{q} = [e_1, \ldots, e_l]$ and q^{-1} is the integer between 0 and p such that $qq^{-1} \equiv 1$ modulo p. Hence $6s(q,p) + l = \frac{q+q^{-1}}{2p} + \frac{1}{2}\sum_{i=1}^{l}(e_i - 1)$. We note that always $\frac{q+q^{-1}}{2p} \leq \frac{p-1}{p}$. We now run this equality for each of the terms in S and in L, and use Lemma 5.12 to conclude that

$$6S + J \leq 3p - 3 + 6\frac{(p-1)}{p} = 3p + 3 - \frac{6}{p}.$$

$\square$

All in all, joining the works of Persson, Roulleau, and Urzúa, we obtain the full density of Chern slopes for simply-connected surfaces of general type in the interval $[1, 3]$.

Theorem 5.14 (Persson-Roulleau-Urzúa)**.** *For any* $r \in [\frac{1}{5}, 3]$*, there are minimal simply-connected surfaces of general type* S *with* $c_1^2(S)/c_2(S)$ *arbitrarily close to* r*.*

5.2 Geography of surfaces with fixed non-trivial fundamental group

5.2.1 Lower Chern Slopes

In this section, we present a survey of different works of Beauville, Ciliberto, Mendes-Lopes, Pardini and Xiao, which allows us to deduce information about the algebraic fundamental group of surfaces of general type with low Chern Slopes c_1^2/c_2.

Definition/ Theorem 5.15 (Grothendieck). The algebraic fundamental group $\pi_1^{alg}(X)$ of a complex projective variety X is the pro-finite completion of the (topological) fundamental group $\pi_1(X)$, i.e., $\pi_1^{alg}(X) := \hat{\pi}_1(X) = \varprojlim \pi_1(X)/N$, where the limit runs over all normal subgroups $N \trianglelefteq \pi_1(X)$ with finite index.

Example 5.16.

1. If $\pi_1(X)$ is finite, then both algebraic and topological fundamental group coincide.
2. If C is a curve of genus one, then $\pi_1(C) \cong \mathbb{Z} \times \mathbb{Z}$ and $\pi_1^{alg}(C) \cong \Pi_p \mathbb{Z}/p\mathbb{Z} \times \Pi_p \mathbb{Z}/p\mathbb{Z}$.

Definition 5.17. A group G is called residually finite if the natural homomorphism $G \to \hat{G} = \varprojlim G/N$ is injective.

In general, fundamental groups of surfaces are not residually finite. See for example [Tol93] and [CK92].

For surfaces of general type with $p_g = 0$ and $c_1^2 \leq 7$ there exists a almost complete list that yields the information available in the literature comparing the algebraic and topological fundamental groups of such surfaces. See [BCR11, Table 1]

For us a *fibration* is a surjective morphism $f\colon S \to C$ with connected fibers from a smooth surface to a smooth curve. The fibration is said to be *relatively minimal* if S has no (-1)-curves contained in fibers of f.

Definition 5.18. Let $F = \sum_{i=1}^{n} c_i C_i$ be a fiber of f, where C_i are irreducible curves. We say that F is a multiple fiber of index $m = \gcd(c_1, \ldots, c_n)$ if $m > 1$.

Let $f\colon S \to C$ be a fibration, we denote by $g(C)$ the genus of the curve C and g the genus of a generic fiber F. The relative canonical sheaf of f is $\omega_{S/C} = \omega_S \otimes f^*(\omega_C^{\otimes -1})$, and so the relative canonical class is $K_{S/C} = K_S - f^*(K_C)$.

Theorem 5.19. *Let S be a surface and let $f\colon S \to C$ be a fibration. Then the sheaf $f_*(\omega_{S/C})$ is locally free of rank g and of degree $\chi(\mathcal{O}_S) - (g(C)-1)(g-1)$. If f is not locally trivial, then $deg(f_*(\omega_{S/C})) > 0$.*

Proof. See [X85b, Theo. 1.1]. □

Definition 5.20. The slope of f (non-locally trivial) is $\lambda(f) := K^2_{S/C}/\chi_f$, where $\chi_f := \deg f_*(\omega_{S/C})$. Note that

$$K^2_{S/C} = K^2_S - 8(g(C)-1)(g-1)$$

and

$$\chi_f = \chi(\mathcal{O}_S) - (g(C)-1)(g-1).$$

Let $f\colon S \to C$ be a fibration with a multiple fibers $F_1, \ldots, F_t$, we can find an étale cover $\overline{C} \to C$ such that the induce fibration $\overline{f}\colon \overline{S} \to \overline{C}$ has no multiple fibers, as follows. Let t be the number of multiple fibers of index m. Take $p_1, \ldots, p_k$ points of multiple fibers of index m, with $k \leq t$. We take $p_{k+1}, \ldots, p_{am}$ points with non-multiple fibers and such that am is the smallest multiple of m greater than or equal to k. Let $D = \sum_{i=1}^{am} p_i$ be a divisor on C, hence there exist a line bundle $\mathcal{L}$ such that $\mathcal{L}^m = \mathcal{O}_C(D)$. We consider the corresponding mth-cover $g_m\colon \overline{C}_m \to C$, see [U10, Sect. 2]. Let $\overline{S}_m$ be the normalization of $\hat{S}_m := S \times_C \overline{C}_m$. We have the commutative diagram

$$\begin{tikzcd} \overline{S}_m \arrow[r, "\overline{\alpha}_m"] \arrow[d, "\overline{f}_m"] & S \arrow[d, "f"] \\ \overline{C}_m \arrow[r, "\alpha_m"'] & C \end{tikzcd}$$

where the fibration $\overline{f}_m\colon \overline{S}_m \to \overline{C}_m$ has no multiple fibers of index m. Note that for every multiple fiber of index $m' \neq m$ the fibration $\overline{f}_m$ has m multiple fibers of index m', but since f has finitely many multiple fibers we can iterate the last process a finite times and obtain a diagram

$$\begin{tikzcd} \overline{S} \arrow[r, "\overline{\alpha}"] \arrow[d, "\overline{f}"] & S \arrow[d, "f"] \\ \overline{C} \arrow[r, "\alpha"'] & C \end{tikzcd}$$

where $\alpha\colon \overline{C} \to C$ is a finite degree cover, the fibration $\overline{f}\colon \overline{S} \to \overline{C}$ has no multiple fibers and the morphism $\overline{\alpha}\colon \overline{S} \to S$ in an étale cover. Note that α has a high degree, so by Riemann-Hurwitz's Theorem, the genus of $\overline{C}$, is high too.

Let $f\colon S \to C$ be a fibration with $g(C) \geq 1$ and let F be a generic fiber of f, then the inclusion $F \to S$ yields a sequence of groups

$$\pi_1^{alg}(F) \xrightarrow{\alpha} \pi_1^{alg}(S) \to L \to 1,$$

moreover, we have that $Im(\alpha)$ is a normal subgroup of $\pi_1^{alg}(S)$ and $L = \pi_1^{alg}(S)/Im(\alpha)$ is a quotient of $\pi_1^{alg}(C - \{p_1, \ldots, p_t\})$, where $F_i = f^*(p_i)$ is a multiple fiber. In particular if f has no multiple fibers $L \cong \pi_1^{alg}(C)$. See [Xi87b, Sect. 2]. We know by [X91, Sect. 1], that the last results are also true in the case of π_1, the topological fundamental group.

Theorem 5.21. *[Xi87a, Theo. 1] Let f a fibration as above, if*

$$\lambda(f) < 4 \textit{ or equivalently } K_S^2 < 4\chi(\mathcal{O}_S) + 4(g(C) - 1)(g - 1).$$

Then $Im(\alpha)$ is trivial if f is non-hyperelliptic, and trivial or $\mathbb{Z}/2\mathbb{Z}$ if f is hyperelliptic.

Reid in [Re79, Theo. 1] showed that if S is a minimal surface of general type with $c_1^2(S) < \frac{1}{3}c_2(S)$, then either $\pi_1^{alg}(X)$ is finite or there is an étale Galois cover $Y \to S$, having a fibration $f\colon Y \to C$, where $g(C) \geq 1$, which induces an isomorphism $f_* : \pi_1^{alg}(Y) \xrightarrow{\cong} \pi_1^{alg}(C)$. Reid in [Re79, Conjec. 4], also presented the following conjecture in the algebraic (étale) case weakening the condition $c_1^2(S) < \frac{1}{3}c_2(S)$ for $c_1^2(S) < \frac{1}{2}c_2(S)$. It suggests a possible treatment for the topological fundamental group, Conjecture 5.22. The conjecture is sharp in the sense that exists surfaces with $c_1^2 = \frac{1}{2}c_2$ and π_1 neither finite nor commensurable with the fundamental group of a curve, see Theorem 5.30.

Conjecture 5.22 (Reid's Conjecture). Let S be a minimal surfaces of general type such that $\frac{c_1^2(S)}{c_2(s)} < \frac{1}{2}$, then $\pi_1(S)$ is either finite or is commensurable with the fundamental group of a curve, i.e., there is an étale cover S' of S and a fibration $f : S' \to C$ such that

$$1 \to K \to \pi_1(S') \to \pi_1(C) \to 1$$

with $|K| < \infty$.

Due to the Severi's inequality proved by Pardini in [Par05], Conjecture 5.22 can be proved for irregular minimal surfaces of general type or having an irregular étale cover.

In fact, the Severi inequality asserts that if S is a smooth minimal surfaces of maximal Albanese dimension, i.e., such that its Albanese map $a\colon S \to \mathrm{Alb}(S)$ is generically finite onto its image, then $c_1^2(S) \geq 4\chi(\mathcal{O}_S)$, and hence if S is a minimal irregular surface of general type with $c_1^2(S) < 4\chi(\mathcal{O}_S)$ then the Albanese map $a\colon S \to C$ is a pencil, whose general fiber will be denote by F. Then the inclusion $F \to S$ induce a homomorphism $\alpha : \pi_1^{alg}(F) \to \pi_1^{alg}(S)$. Note that $c_1^2(S) < 4\chi(\mathcal{O}_S) < 4\chi(\mathcal{O}_S) + 4(g(C) - 1)(g - 1)$, implies that $\lambda(a) < 4$, so by Theorem 5.21 the $Im(\alpha)$ is either trivial or $\mathbb{Z}/2\mathbb{Z}$. If a has no multiple fiber we have the exact sequence of groups

$$1 \to Im(\alpha) \to \pi_1^{alg}(S) \to \pi_1^{alg}(C) \to 1,$$

if a has multiple fibres there exists a base change $\overline{C} \to C$ such that we have a commutative diagram

$$\begin{tikzcd}
\overline{S} \arrow[r, "\overline{g}"] \arrow[d, "\overline{a}"] & S \arrow[d, "a"] \\
\overline{C} \arrow[r, "g"'] & C
\end{tikzcd}$$

where $g : \overline{S} \to S$ is an étale Galois cover and $\overline{a}$ has no multiple fibres, so

$$1 \to Im(\alpha) \to \pi_1^{alg}(\overline{S}) \to \pi_1^{alg}(\overline{C}) \to 1,$$

is an exact sequence of groups. If S has an irregular étale cover S', the surface S' holds $c_1^2(S') < 4\chi(\mathcal{O}_{S'})$, then the same argument runs for S', and so we are done.

For minimal surfaces S of general type with no irregular étale covers and such that $c_1^2(S)/c_2(S) < \frac{1}{3}$ or equivalently $c_1^2(S) < 3\chi(\mathcal{O}_S)$, through several works [Bea79, Xi87a, MP07, CMR07], we have a good view of the fundamental group $\pi_1(S)$ when it is finite, indeed, we can deduce that under such hypothesis $\pi_1(S)$ has at most nine elements, see Corollary 5.26.

Theorem 5.23. *[Bea79, Theo. 5.5] Let S be a minimal surface of general type, such that the inequality $c_1^2(S) < 3p_g(S) - 7$ holds. Then, the canonical map $\psi_{|K_S|} : S \dashrightarrow \mathbb{P}^{p_g(S)-1}$ is a degree-2 rational application on a ruled surface S'.*

We need the following lemma,

Lemma 5.24. *Let S be a minimal surface of general type, suppose that the linear system $|K_S|$ is composed with a pencil. Then, the inequality $c_1^2(S) \geq 3p_g(S) - 6$ holds.*

Proof of the theorem 5.23. First, since S is a minimal surface of general type such that $c_1^2(S) < 3p_g(S) - 7$ by Lemma 5.2.1 the canonical map $\psi_{|K_S|} \colon S \dashrightarrow S' \subset \mathbb{P}^{p_g(S)-1}$ has as image a surface S'. Using Proposition 3.45 we get that

$$\deg(\psi_{|K_S|})(p_g(S) - 2) < \deg(\psi_{|K_S|}) \deg(S') < c_1^2(S) < 3p_g(S) - 7,$$

thus $\deg(\psi_{|K_S|}) := d \leq 2$. If we assume that S' is not ruled and $d = 2$, then by Proposition 3.46, we have that $4(p_g(S) - 2) < 3p_g(S) - 7$, which is a contradiction. So, only reminds to prove that $\psi_{|K_S|}$ is not a birational map. In the case that it was true, then there exists a chain of blow-ups $\epsilon : \hat{S} \to S$ such that $\psi_{|K_S|} \circ \epsilon : \hat{S} \to S'$ is defined everywhere. When can write the canonical divisor of $\hat{S}$ as $K_{\hat{S}} \sim Z + \hat{M}$, where Z is the fixed part of $K_{|\hat{S}|}$ and its moving part is $\hat{M}$, so $\psi_{|K_{\hat{S}}|} = \psi_{|\hat{M}|}$. By Bertini's Theorem any generic curve $C \in |\hat{M}|$ is irreducible and smooth. Then, by adjunction formula, see 3.7, we get $K_C \sim (K_{\hat{S}} + \hat{M})|_C = (Z + 2\hat{M})|_C$, therefore $0 \leq \deg(\hat{M}|_C) \leq g(C) - 1$, so the map $\psi_{|\hat{M}|_C|}$ is birational. Then, by [Bea79, Lemma 5.1] we have that $h^0(C, \hat{M}|_C) \leq \frac{1}{3}(\deg(\hat{M}|_C + 4))$, and by genus formula $2g(C) - 2 = C^2 = \hat{M}^2$, so $\deg(\hat{M}^2) \leq \frac{1}{2}\hat{M}^2$, therefore $h^0(C, \hat{M}|_C) \leq \frac{1}{3}(\hat{M}^2 + 4)$.

Now by the (usual) short exact sequence

$$0 \to \mathcal{O}_{\hat{S}} \to \mathcal{O}_{\hat{S}}(\hat{M}) \to \mathcal{O}_C(\hat{M}|_C) \to 0,$$

and taking the long exact sequence of cohomology we obtain $h^0(\hat{S}, \hat{M}) \leq 1 + h^0(C, C_{\hat{M}})$, and since $p_g(S) = h^0(\hat{S}, K_{\hat{S}}) = h^0(\hat{S}, \hat{M})$, then

$$
\begin{aligned}
p_g(S) &\le 1 + \frac{1}{3}(\hat{M}^2 + 4) \\
&= 1 + \frac{1}{3}(\deg(\psi_{|K_S|}(S)) + 4) \\
&\le \frac{1}{3}(K_S^2 + 7),
\end{aligned}
$$

so $3p_g(S) - 7 \le K_S^2$, but it is a contradiction. $\square$

Now, suppose that S is a surfaces of general type with no irregular étale covers, and satisfying that $c_1^2(S) < 3\chi(\mathcal{O}_S)$. If $Y \to S$ is an étale cover with Galois group G, such that $|G| \ge 11$. Thus, $c_1^2(Y) - 3\chi(\mathcal{O}_Y) = |G|(c_1^2(S) - 3\chi(\mathcal{O}_S)) \le -11$, so since $\chi(\mathcal{O}_Y) = 3 + p_g(Y)$, we obtain the inequality $c_1^2(Y) < p_g(Y) - 7$. By Theorem 5.23, the last inequality implies the canonical map $\psi := \psi_{|K_Y|}$ is 2-to-1 to a ruled surface $\hat{Y}$, but, since $q(Y) = 0$ the surface $\hat{Y}$ is rational. Note that any automorphism g of rational surface $\hat{Y}$ always have a fixed point, by the Lefschetz fixed-point theorem. In other way, we obtain an étale morphism between rational surfaces $Y \to Y/(g)$ and it is impossible. We want to show that every non-trivial element of G has order two. In fact, since the $\psi : Y \dashrightarrow \hat{Y}$ is 2-to-1, we can blow-up and obtain a generic 2-to-1 morphisms $\hat{\psi} : \overline{Y} \to \hat{Y}$. Hence, G act in $\hat{Y}$ via $g\hat{y} = g\hat{\psi}(y)$ where $\psi(y) = \hat{y}$. Now if σ is the involution on Y moving two fibres of $\hat{\psi}$, and because g always has a fixed point on $\hat{Y}$ we can take an element $y \in Y$ such that $g\hat{\psi}(x) = \hat{\psi}(x)$. So, due to G act freely on Y we have that

$$
g^2 y = g\sigma(y) = \sigma(gy) = \sigma(\sigma(y)) = y.
$$

Therefore $G \cong (\mathbb{Z}/2\mathbb{Z})^r$.

We thus get the following possibilities, first $G \cong (\mathbb{Z}/2\mathbb{Z})^r$ for $r \ge 4$ and second $|G| \le 10$. If $G \cong (\mathbb{Z}/2\mathbb{Z})^4$, we thus get $c_1^2(Y) < 3(\chi(\mathcal{O}_Y) - 5)$, and since $\chi(\mathcal{O}_S) \ge 2$, we get that $\chi(\mathcal{O}_Y) \ge 32$. Consequently, by the following Xiao's Theorem;

Theorem 5.25. *[Xi87b, Theo. 1] Let $g \ge 2$ be an integer and let Y be a regular minimal hyperelliptic surface of general type such that*

$$
c_1^2(Y) < \frac{4g}{g+1}(\chi(\mathcal{O}_Y) - \frac{9}{8}g - 2).
$$

Then Y has a unique pencil of hyperelliptic curves of genus $\leq g$. Such a pencil has no base point. Moreover, if $\chi(\mathcal{O}_Y) > (2g-1)(g+1)+2$, it is enough to assume $c_1^2(Y) < \frac{4g}{g+1}(\chi(\mathcal{O}_Y) - g - 2)$

We deduce that Y has an unique pencil $|F|$ of hyperelliptic curves of genus $g \leq 3$. By the uniqueness of the pencil $|F|$, the group G act on it. From this we can see G as a subgroup of $Aut(\mathbb{P}(|F|)) \cong Aut(\mathbb{P}^1)$. However, neither $(\mathbb{Z}/2\mathbb{Z})^4$ nor $(\mathbb{Z}/2\mathbb{Z})^3$ are subgroups of $PGL(2,\mathbb{C})$. We conclude that there is a subgroup $H \triangleleft G$ with order $|H| \geq 4$ fixing every element of $|F|$. We may now use the genus formula to conclude that 4 divides $g-1$, but $g \leq 3$. Thus, $g = 1$ but it makes a contradiction because Y is of general type.

If $|G| = 10$, and $c_1^2(S) < 3\chi(\mathcal{O}_S) - 1$. By Theorem 5.23 and the last discussion, we can conclude that the group G is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^r$, but it is a contradiction.

In closing, all the discussion above can be sum up in the following theorem.

Theorem 5.26. *[MP07, Theo. 4.3, Prop. 4.4] Let S be a minimal surface of general type such that $c_1^2(S) < \frac{1}{3}c_2(S)$, and with no irregular étale covers: If $\pi_1^{alg}(S)$ is finite, then $|\pi_1^{alg}(S)| \leq 9$. More over if $|\pi_1(S)| = 9$, then $\chi(\mathcal{O}_S) = 1$ and $c_1^2(S) = 2$, i.e., S is a numerical Campedelli Surface.*

Corollary 5.27. *Let S be a minimal surface of general type with finite fundamental group $\pi_1(S)$. If $c_1^2(S) < \frac{1}{3}c_2(S)$, then $|\pi_1(S)| \leq 9$. More over if $|\pi_1(S)| = 9$, then $\chi(\mathcal{O}_S) = 1$ and $c_1^2(S) = 2$. More over if $|\pi_1(S)| = 9$, then S is a numerical Campedelli Surface.*

Proof. Since π_1^{alg} is the profinite completion of π_1 and π_1 is finite, we have $\pi_1^{alg} = \pi_1$. So by Theorem 5.26, we must prove that S has no irregular étale covers. By the sake of contradiction, suppose that $Y \to S$ is an irregular étale cover of S. The condiction $q(Y) > 0$ implies that $\pi_1(Y)$ has a copy of $\mathbb{Z}$, but, the profinite completion of $\mathbb{Z}$ is $\hat{\mathbb{Z}} = \Pi_p \mathbb{Z}/p\mathbb{Z}$, then $\pi_1^{alg}(Y)$ is infinite. However, $\pi_1^{alg}(Y)$ is normal subgroup of $\pi_1^{alg}(S)$, which make a contradiction. $\square$

Refining the last argument we can deduce that if S satisfies the hypothesis of Theorem 5.26 and $c_1^2(S) \leq 3\chi(\mathcal{O}_S) - 2$, then $|\pi_1^{alg}(S)| \leq 5$.

Theorem 5.28. *[MP06, Theo. 1.1] Let S be a minimal surface of general type such that $c_1^2(S) \leq 3\chi(\mathcal{O}_S) - 2$ not having any irregular étale cover. Then $|\pi_1^{alg}(S)| \leq 5$. Moreover, $|\pi_1^{alg}(S)| = 5$ if and only if S is a numerical Godeaux surface, i.e., $\chi(\mathcal{O}_S) = c_1^2(S) = 1$.*

The last work (until now) in this way, is [CMR07], where they proved by different methods to the last ones the theorem;

Theorem 5.29. *[CMR07, Theo. 1.1] Let S be a minimal surface of general type such that $c_1^2(S) = 3\chi(\mathcal{O}_S) - 1$, and $|\pi_1^{alg}(S)| = 8$. Then, $\chi(\mathcal{O}_S) = 1$. More over if $|\pi_1(S)| = 8$, then S is a numerical Campedelli Surface.*

Finally, we finish this section with Keum's example, see [K88, Chap. III, Theo. 1]. It is a minimal surface of general type such that $c_1^2 = 4$ and $c_2 = 8$, and fundamental group is $\pi_1 = \mathbb{Z}^4 \rtimes (\mathbb{Z}/2\mathbb{Z})^2$, which is not commensurable with the fundamental group of any curve. It shows the sharpness of Conjecture 5.22.

Theorem 5.30 (Keum's example)**.** *There is a surface S of general type such that $c_1^2(S) = 4, c_2(S) = 8, p_g(S) = 0$ and $\pi_1(S) = \mathbb{Z}^4 \rtimes (\mathbb{Z}/2\mathbb{Z})^2$.*

Idea of the proof. We start with two elliptic curves E_1, E_2. We consider the abelian surface $A = E_1 \times E_2$, and an automorphism $\theta : A \to A$, which induces an involution with non fixed points on the Kummer surface $K := Kum(A)$. Then the surface $Y := K/\theta$ is an Enriques surface.

Construct a divisor B on Y, such that $B^2 = -8$, such that there is some divisor D on Y holding that $2D \sim B$, $|K_Y + D| = \emptyset$, and with eight rational disjoint curves no meeting any other component.

Let $\overline{X}$ be double cover of Y with branch locus B. Then, $K_{\overline{X}}^2 = -4, \chi(\mathcal{O}_{\overline{X}}) = 1, p_g(\overline{X}) = 0$ and $\overline{X}$ has only eight $(-1)-$curves. Let S be the blow-down of $\overline{X}$ at these eight exceptional curves. Then, $K_S^2 = 4, c_2(S) = 8$ and $p_g(S) = 0$. Moreover, $\pi_1(S) \cong \mathbb{Z}^4 \rtimes (\mathbb{Z}/2\mathbb{Z})^2$, see [K88, Prop. 3.14].

□

Observation 5.31. The semidirect product $\mathbb{Z}^4 \rtimes (\mathbb{Z}/2\mathbb{Z})^2$ is defined as follows. Let ϕ : $(\mathbb{Z}/2\mathbb{Z})^2 \to Aut(\mathbb{Z}^4) \cong GL_4(\mathbb{Z})$ be the homomorphism defined by

$$\phi((0,0)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \phi((1,0)) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\phi((1,0)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \phi((1,1)) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

So, the operation on $\mathbb{Z}^4 \rtimes \mathbb{Z}/2\mathbb{Z}$ is the following one

$$((a,b,c,d),(u,v)) \cdot ((a',b',c',d'),(u',v')) = ((a,b,c,d)\phi_{(u,v)}(a',b',c',d'),(u+u',v+v')),$$

therefore, $x^2 \in \mathbb{Z}^4$ for any $x \in \mathbb{Z}^4 \rtimes \mathbb{Z}/2\mathbb{Z}$.

Proposition 5.32. $\pi_1(S) \cong \mathbb{Z}^4 \rtimes (\mathbb{Z}/2\mathbb{Z})^2$ *is not commensurable with the fundamental group of any smooth projective curve C.*

Proof. Suppose that $G := \pi_1(S)$ is commensurable with the fundamental group of a curve C. It means, that there exists an étale cover S' of S such that

$$1 \to K \to \pi_1(S') \to \pi_1(C) \to 1,$$

where $|K| < \infty$. We divide the proof into three cases, depending on the genus $g(C)$ of C.

$g(C) = 0$. In this case, $K \cong \pi_1(S')$, since $\pi_1(C) \cong \{1\}$. Then, $[\pi_1(S) : \pi_1(S')] = \infty$, which is a contradiction.

$g(C) = 1$. We have the short exact sequence,

$$1 \to K \to \pi_1(S') \xrightarrow{f} \mathbb{Z}^2 \to 1,$$

since $\pi_1(C) \cong \mathbb{Z}^2$. Let a, b be generators of $\mathbb{Z}^2$ and denote by x, y be some elements on $\pi_1(S')$ such that $f(x) = a, f(y) = b$. Note that x, y have infinite order. Denote x, y its images into G, then $x^2, y^2 \in \mathbb{Z}^4$ and none is trivial. Let $z \notin \langle x^2, y^2 \rangle_{\mathbb{Z}}$, consequently $kz \notin \langle x^2, y^2 \rangle_{\mathbb{Z}}$ for any $k \in \mathbb{Z}$. Let $\hat{z}$ be the image of z on $\pi_1(S)$, then, there are infinitely many different lateral classes $k\overline{z}$ mod $\pi_1(S')$, so $[\pi_1(S) : \pi_1(S')] = \infty$, which makes a contradiction.

$g(C) \geq 2$. We have the short exact sequence of groups

$$1 \to K \to \pi_1(S') \xrightarrow{f} \langle a_1, \ldots, a_g, b_1, \ldots, b_g | [a_1, b_1] \cdot \cdots \cdot [a_g, b_g] = 1 \rangle \to 1,$$

where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$, since

$$\pi_1(C) \cong \langle a_1, \ldots, a_g, b_1, \ldots, b_g | [a_1, b_1] \cdot \cdots \cdot [a_g, b_g] = 1 \rangle$$

Let $x_i, y_i \in \pi_1(S')$, such that $f(x_i) = a_i, f(y_i) = b_i$ for $i = 1, \ldots, g$. Denoted by $\hat{x}_i, \hat{y}_i$ the images of x_i, y_i on G, then $\hat{x_i}^2, \hat{y_i}^2 \in \mathbb{Z}^4$. Therefore, $a_i^2 a_j^2 = a_j^2 a_i^2$ for $i, j = 1, \ldots, g$, $b_k^2 b_l^2 = b_l^2 b_k^2$ for $k, l = g+1, \ldots 2g$ and $a_i^2 b_j^2 = b_j^2 a_i^2$ for $i = 1, \ldots, g$ and $j = g+1, \ldots, 2g$. But this produce a contradiction, because there are no other independent relation on $\pi_1(C)$ than $\Pi_{i=1}^g [a_i, b_i]$, see [Bob, Chap. 3, Sect. 3, Example. 2]. In fact, if there is some relation would mean that some composition of curves a_i, b_i can be contracted to a point $p \in C$, on C every pairs of points are equivalents then such relation is a multiple (power) of $\Pi_{i=1}^g [a_i, b_i]$.

□

5.2.2 Higher Chern Slopes

Key Construction and new density theorem

In this section, we generalize the construction used in [Cat00, Section 1] in the context of lef line bundles, which will be used for the main theorem.

Proposition 5.33. *Let X and Y be non-singular projective surfaces. Let $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ be the canonical projections. Let Γ and B be lef line bundles on X and Y respectively. Assume that* $\exp(\Gamma) = \exp(B) = 1$. *Then $p^*(\Gamma) \otimes q^*(B)$ is a lef line bundle on $X \times Y$ of exponent* 1.

Proof. This is elementary, we briefly give an argument. Let $M := p^*(\Gamma) \otimes q^*(B)$. Let $s_0, \ldots, s_l$ be a basis of $H^0(X, \Gamma)$, and let $t_0, \ldots, t_b$ be basis of $H^0(Y, B)$. Since $H^0(X, \Gamma) \otimes H^0(Y, B) \simeq H^0(X \times Y, M)$ (see e.g. [Bea96, Fact III.22, i]), then M is generated by the global sections $s_i t_j$ with $0 \leq i \leq l$ and $0 \leq j \leq b$. The morphism $\psi_{|M|}\colon X \times Y \to \mathbb{P}(|M|)$ is $\Sigma_{l,b} \circ (\psi_{|\Gamma|} \times \psi_{|B|})$, where $\Sigma_{l,b}$ is the Segre embedding. Therefore $\psi_{|M|}$ is semi-small into its image as $\psi_{|\Gamma|} \times \psi_{|B|}$ is semi-small by Proposition 2.6. It follows that M is lef and $\exp(M) = 1$. ☐

Proposition 5.34. *Let X be a non-singular projective variety with* $\dim(X) \geq 3$. *Let M be a lef line bundle on X with* $\exp(M) = 1$. *If $E \in |M|$, then $\pi_1(E) \cong \pi_1(X)$.*

Proof. It follows by Theorem 4.16 and Theorem 2.11. ☐

Corollary 5.35. *Let X be a non-singular projective variety with* $\dim(X) \geq 4$. *Let M be a lef line bundle with* $\exp(M) = 1$. *Then a generic member $E \in |M|$ is nonsingular projective variety, and $M_E := M|E$ is lef. Moreover, if $F \in |M_E|$, then $\pi_1(F) \cong \pi_1(X)$.*

Proof. The first part is just Theorem 2.11. If $F \in M_E$, then by Corollary 5.35 we obtain that $\pi_1(F) \cong \pi_1(E) \cong \pi_1(X)$ ☐

Theorem 5.36. *Let X and Y be a non-singular projective surfaces with nef canonical class, and $K_X^2 > 0$. Let B be a very ample lne bundle on Y and let Γ be a lef line bundle on X with* $\exp(\Gamma) = 1$.

Then there exist a non-singular projective surface $S \subset X \times Y$ with the following properties

1. $\pi_1(S) \simeq \pi_1(X) \times \pi_1(Y)$.

2. *The morphisms $p|_S\colon S \to X$ and $q|_S\colon S \to Y$ have degrees* $\deg(p|_S) = B^2$ *and* $\deg(q|_S) = \Gamma^2$.

3. We have

$$c_1^2(S) = c_1^2(X)B^2 + c_1^2(Y)\Gamma^2 + 8c(\Gamma, B) - 4\Gamma^2 B^2$$

and

$$c_2(S) = c_2(X)B^2 + c_2(Y)\Gamma^2 + 4c(\Gamma, B) + 4\Gamma^2 B^2$$

where

$$c(\Gamma, B) = \frac{7}{2}\Gamma^2 B^2 + \frac{3}{2}(\Gamma \cdot K_X)B^2 + \frac{3}{2}(B \cdot K_Y)\Gamma^2 + \frac{1}{2}(\Gamma \cdot K_X)(B \cdot K_Y).$$

4. K_S is big and nef.

Proof. We first construct a surface $S \subset X \times Y$ which satisfies (1) and (2). Let $M := p^*(\Gamma) \otimes q^*(B)$. Then, by Proposition 5.33, we have that M is lef with $\exp(M) = 1$. We take general sectioins E, E' of M, and define $S := E \cap E'$. Note that since $\exp(M) = 1$, then M is base point free, and due to the ampleness of B the line bundle M has enough sections. So, by Bertini's theorem S is non-empty and non-singular. By Theorem 5.35, we have that E is a non-singular projective 3-fold and M_E is lef with $\exp(M_E) = 1$. Since $S = E'|_E$ is smooth, we have by Theorem 4.14 that $H^0(S, \mathbb{Z}) \cong H^0(E, \mathbb{Z}) \cong \mathbb{Z}$, and thus S is a non-singular projective surface. Finally, by Corollary 5.35 we get $\pi_1(S) \cong \pi_1(X) \times \pi_1(Y)$. We also have that the degree of $p|_S$ is $((p^*(\Gamma) \otimes q^*(B))|_Y)^2 = B^2$. Similarly the morphism $q|_S$ has degree Γ^2.

Now we prove (3). By the adjunction formula applied twice, and since $K_{X\times Y} \sim p^*(K_X) + q^*(K_Y)$, we obtain

$$K_S \sim p|_S^*(K_X + 2\Gamma) + q|_S^*(K_Y + 2B).$$

To compute K_S^2, we first consider two curves C and C' in X, Y respectively, we have

$$p|_S^*(C).q|_S^*(C') = p^*(C).q^*(C')E \cdot E' = (C \times C')M^2 = (M|_{C \times C'})^2 = 2(\Gamma \cdot C)(B \cdot C').$$

This extends to find the intersection $p|_S^*(D).q|_S^*(D')$, for divisors D, D' in X, Y respectively. Thus,

$$\begin{aligned}K_S^2 =&(p|_S^*(K_X+2\Gamma)+q|_S^*(K_Y+2B))^2\\ =&B^2(K_X+2\Gamma)^2+2p|_S^*(K_X+2\Gamma).q|_S^*(K_Y+2B)+\Gamma^2(K_Y+2B)^2\\ =&B^2(K_X+2\Gamma)^2+4(\Gamma\cdot(K_X+2\Gamma)).(B\cdot(K_Y+2B))+\Gamma^2(K_Y+2B)^2\\ =&K_X^2B^2+K_Y^2\Gamma^2+24\Gamma^2B^2+12((\Gamma\cdot K_X)B^2+(B\cdot K_Y)\Gamma^2)+4(\Gamma\cdot K_X)(B\cdot K_Y).\end{aligned}$$

To calculate $\chi(S)$, we use the following exact Koszul complex. Since S is a complete intersection of two sections of M and $X\times Y$ is non-singular, then we have, see e.g. [FL85, Pags. 76-77]

$$0\to\mathcal{O}_{X\times Y}(-2M)\to\mathcal{O}_{X\times Y}^{\oplus 2}(-M)\to\mathcal{O}_{X\times Y}\to\mathcal{O}_S\to 0.$$

Then by the additivity of the Euler characteristic and the Künneth formula, see e.g. [Cut18, Theo. 17.23],

$$H^n(X\times Y, M)=\bigoplus_{i+j=n}H^i(X,\Gamma)\otimes H^j(Y,B),$$

we obtain

$$\begin{aligned}\chi(\mathcal{O}_S)&=\chi(\mathcal{O}_{X\times Y})+\chi(\mathcal{O}_{X\times Y}(-2\Gamma-2B))-2\chi(\mathcal{O}_{X\times Y}(-\Gamma-B))\\ &=\chi(\mathcal{O}_X)\chi(\mathcal{O}_Y)+\chi(\mathcal{O}_X(-2\Gamma))\chi(\mathcal{O}_Y(-2B))\\ &-2\chi(\mathcal{O}_X(-\Gamma))\chi(\mathcal{O}_Y(-B))\\ &=\chi(\mathcal{O}_X)\chi(\mathcal{O}_Y)\\ &+(\chi(\mathcal{O}_X)+\frac{1}{2}(\Gamma^2+2(\Gamma\cdot K_X)))(\chi(\mathcal{O}_Y)+\frac{1}{2}(4B^2+2(B\cdot K_Y)))\\ &-2(\chi(\mathcal{O}_X)+\frac{1}{2}(\Gamma^2+\Gamma\cdot K_X))(\chi(\mathcal{O}_Y)+\frac{1}{2}(B^2+B\cdot K_Y))\\ &=\chi(\mathcal{O}_X)B^2+\chi(\mathcal{O}_Y)\Gamma^2+c(\Gamma,B),\end{aligned}$$

where

$$c(\Gamma,B)=\frac{7}{2}\Gamma^2B^2+\frac{3}{2}(\Gamma\cdot K_X)B^2+\frac{3}{2}(B\cdot K_Y)\Gamma^2+\frac{1}{2}(\Gamma\cdot K_X)(B\cdot K_Y).$$

Finally we show (4). Let C be an irreducible curve on S. Let $a = \deg p|_C$ and $b = \deg q|_C$. Then, by the projection formula for generically finite morphisms, see Proposition 2.15, we have

$$C \cdot K_S = C \cdot p|_S^*(K_X + 2\Gamma) + C \cdot q|_S^*(K_Y + 2B) \tag{1}$$

$$= ap(C) \cdot (K_X + 2\Gamma) + bq(C) \cdot (K_Y + 2B). \tag{2}$$

We note that K_X, K_Y, and Γ are nef, and B is very ample, and so $C \cdot K_S \geq 0$. Using the formula for K_S^2 above and by the same previous reasons, we obtain $K_S^2 > 0$. □

We now present our main result, which puts together all the ingredients elaborated until now.

Theorem 5.37. *Let Y be a non-singular projective surface with K_Y nef, and let $r \in [1, 3]$ be a real number. Then there are minimal nonsingular projective surfaces S with $c_1^2(S)/c_2(S)$ arbitrarily close to r, and $\pi_1(S) \simeq \pi_1(Y)$.*

Proof. Let X_p be the collection of simply-connected surfaces described in Section 5.1.2. Let Γ_p be the line bundle defined in Proposition 5.11. For any p we have that Γ_p is lef by Proposition 2.10. (We note that Γ_p is not ample because of the resolution of singularities involved in the construction of the surfaces X_p.) Let B be a very ample divisor on Y. Note that we satisfy all the hypothesis in Theorem 5.36 with $X = X_p$ and $\Gamma = \Gamma_p$. Therefore, there are surfaces $S_p := S$ such that all the conclusions in Theorem 5.36 hold. In particular, we have $\pi_1(S_p) \simeq \pi_1(Y)$.

The formulas in Theorem 5.36 part (3) are

$$c_1^2(S_p) = c_1^2(X_p)B^2 + c_1^2(Y)\Gamma_p^2 + 8c(\Gamma_p, B) - 4\Gamma_p^2 B^2$$

and

$$c_2(S_p) = c_2(X_p)B^2 + c_2(Y)\Gamma_p^2 + 4c(\Gamma_p, B) + 4\Gamma_p^2 B^2,$$

where $c(\Gamma_p, B)$ is an Theorem 5.36.

By Proposition 5.11 we have that $\Gamma_p^2 = p$ and $\Gamma_p \cdot K_{X_p}$ is a polynomial in p of degree 3. Thus $c(\Gamma_p, B)$ is a polynomial in p of degree 3. By Section 5.1.2, the invariants $c_1^2(X_p)$ and $c_2(X_p)$ are Laurent polynomials in p of degree 5. Therefore, by the formulas above, we have

$$\lim_{p\to\infty} \frac{c_1^2(S_p)}{c_2(S_p)} = \lim_{p\to\infty} \frac{c_1^2(X_p)}{c_2(X_p)} = \frac{27x^4 + 48x^2 + 8}{9x^4 + 48x^2 + 8} =: \lambda(x)$$

where $x := \alpha/\beta$, as in Section 5.1.2. In this way, just as in [RU15, Thereom 6.3], we obtain the desired surfaces $S = S_p$ with $c_1^2(S)/c_2(S)$ arbitrarily close to r. ☐

Corollary 5.38. *Let G be the fundamental group of a non-singular projective surface. Then the Chern slopes $c_1^2(S)/c_2(S)$ of nonsingular projective surfaces S with $\pi_1(S) \simeq G$ are dense in $[1, 3]$.*

Proof. Since π_1 is invariant under birational transformations between non-singular projective surfaces, then it is enough to consider surfaces with no (-1)-curves. If G is the fundamental group of $\mathbb{P}^1 \times C$, where C is a non-singular projective curve, then, for example, we can take as Y a surface in [RU15, Corollary 6.4] to apply Theorem 5.37. Otherwise, we have a non-ruled surface with nef canonical class, and we can directly use Theorem 5.37. ☐

5.3 Further directions

By the Lefschetz hyperplane theorem, the fundamental group of a non-singular complex projective variety is the fundamental of some smooth projective surface. Thus, Corolary 5.38 involves the fundamental group any non-singular projective variety.

One may be tempted to use the result of Persson and Chen (Corollary 5.3, Theorem 5.4) on density of Chern slopes of simply-connected minimal surfaces of general type in $[\frac{1}{5}, 2]$ as an imput in Theorem 5.37, but the strategy does not work. It is not clear in that case how to find a suitable Γ_m which makes things work. On the top of that, this cannot work in full generality since, for example, by Theorem 5.26, we can deduce that: If S is a surface of general type with $c_1^2(S) < \frac{1}{3}c_2(S)$ and $\pi_1(S)$ finite, then the order of $\pi_1(S)$ is at most 9.

In this way, the question of “freedom" of fundamental groups remains open for the interval $[\frac{1}{3}, 1]$.

We want to present two conjectures in relation to geography of Chern slopes for surfaces with ample canonical class, and for Brody hyperbolic surfaces. They could be proved through the theorems in this section if we can show that the projection

$$q|_{S_p} \colon S_p \to Y$$

is a finite morphism (see Theorem 5.36). This depends on the line bundles Γ_p. Catanese proves in [Cat00, Lemma 1.1] that $q|_{S_p}$ is a finite morphism if Γ_p is very ample. We note that in [RU15] it is proved that Chern slopes c_1^2/c_2 of simply-connected minimal surfaces of general type are dense in $[1, 3]$, but canonical class for all the constructed surfaces was not ample, because of the presence of arbitrarily many (-2)-curves.

Conjecture 5.39. Let G be the (topological) fundamental group of a non-singular complex projective surface. Then Chern slopes $c_1^2(S)/c_2(S)$ of minimal non-singular projective surfaces of general type S with $\pi_1(S)$ isomorphic to G and ample canonical class are dense in $[1, 3]$.

Conjecture 5.40. Let Y be a Brody hyperbolic non-singular projective surface. Then Chern slopes of hyperbolic non-singular projective surfaces S with $\pi_1(S)$ isomorphic to $\pi_1(Y)$ are dense in $[1, 3]$.

Note that, if we suppose that the morphism $q|_S$ of Theorem 5.36 is finite. Then by the Formulas in (1) and (2) we obtain that $ap(\Gamma)(K_X + 2A) \geq 0$ and $bq(\Gamma)(K_Y + 2A) > 0$, thus $\Gamma \cdot K_S > 0$. Therefore, by Nakai-Moishezon criterion K_S is ample. It shows Conjecture 5.39.

Conjecture 5.39 evidence the unpredictable behaviour of the minimal surfaces of general type. In particular, it can help us to understand why there is not a characterization for surfaces of general type.

Naturally the next two geographical questions arise.

(Q1): Are there any constraints for the Chern slope of surfaces of general type after fixing the fundamental group?

(Q2): What is the optimal subinterval of $[\frac{1}{5}, 3]$ with the full freedom for π_1?

For the Question Q1, as we showed in this thesis there are restrictions. However, we do not know which is the maximum length where we can find such restrictions. We believe that following the ideas presented by Beauville, Mendes, Pardini, Reid, Xiao, we can extend the length beyond $[\frac{1}{5}, \frac{1}{2}]$.

Question Q2 is related with the last question, in fact we believe that the optimal subinterval with full freedom of $[\frac{1}{5}, 3]$ is $[1, 3]$ which say that the maximum length wanted in Q1 is $[\frac{1}{5}, 1]$. The naive idea to resolved Q2 is find a new family of minimal simply-connected surfaces of general type with Cher slopes dense in $[r, 3]$ with $r < 1$, and such that we can find suitable line bundles in order to apply Theorem 5.37. The problem of find such families of surfaces is challenging and interesting, since we have remarked before the traditional methods follow by Persson and Chen does not work.

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