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Three-mode truncation of a model equation for systems with a long-wavelength oscillatory instability

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We study a three-mode truncation of the equation $u_t+uu_x+\delta u_{xxx}+u_{xx}+u_{xxxx}=0$. This simple model allows us to understand analytically the role played by dispersion in the appearance of well-defined traveling pulses. In the absence of dispersion, that is, for $\delta=0$, the solutions obtained from the truncated model are in quantitative agreement with the known results for the Kuramoto-Sivashinsky equation for small horizontal periodicity.

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I. INTRODUCTION

The nonlinear equation $u_t+uu_x+\delta u_{xxx}+u_{xx}+u_{xxxx}=0$ arises in problems that have a long-wavelength oscillatory instability, such as fluid flow along an inclined plane for large surface tension [1]. It corresponds to a special case of the equation that describes the Eckhaus instability of traveling waves [2], surface waves in a convecting fluid [3], and flow along an inclined plane for moderate surface tension [4]. In the absence of dispersion, that is, for $\delta=0$, this is the well-studied Kuramoto-Sivashinsky (KS) equation, which has become the main example of phase turbulence [5]. Our aim is to provide some understanding of the time evolution of the solution of this equation in periodic intervals, particularly of the role played by dispersion in the appearance of well-defined traveling pulses. We do so considering a three-mode truncated system that is adequate for a small periodicity interval L. The solutions of the KS equation for small L have been described in detail in [6]. Different types of final states are possible, steady states and traveling waves among them. When dispersion is included, the solution is significantly changed. Several numerical studies have shown that for periodic boundary conditions and for sufficiently high dispersion, most initial conditions evolve into a final state consisting of a row of equally spaced, sharply defined pulses of the same height that travel as a whole [7]. The number of pulses that appear depends on the box size and on initial conditions. The sharpness and similarity of the pulses have been used by different authors, who have constructed approximate solutions to this equation as a superposition of single traveling pulses. The dynamics contained in the full equation is then reflected by the interaction between them [8,9]. Other analytical approaches have dealt with perturbations around the known solutions of the Korteweg–de Vries equation [10]. In the two approaches just mentioned, the final state is assumed.

Numerical studies have shown that for a given periodicity interval L, the final state consists of few modes, the linearly unstable ones having much larger amplitudes than the linearly stable ones [10]. Therefore, we expect that the main features of the final state can be explained, keeping just a few stable modes when studying a truncated model. Similar criteria for truncation have been used in different problems [11]. In this article we study analytically the two- and three-mode truncated system, which is adequate for small box size. The results obtained from this truncated system for the Kuramoto-Sivashinsky equation are in good qualitative and quantitative agreement with the results for the small periodicity interval obtained from the integration of the partial differential equation, which gives us some confidence in the results obtained when dispersion is included. Although in this case the long-time solution exhibits only one pulse, the results obtained for large δ are also valid for multiple-pulse solutions generated as periodic repetitions of a single pulse.

II. MATHEMATICAL FORMULATION

Our starting point is the equation

$$u_t+uu_x+\delta u_{xxx}+u_{xx}+u_{xxxx}=0\ , \tag{1}$$

subject to periodic boundary conditions

$$u^{(n)}(0,t)=u^{(n)}(L,t),\quad n=0,1,2,3\ ,$$

where $u^{(n)}$ denotes the nth derivative of u with respect to x, with initial condition

$$u(x,0)=u_0(x)\ .$$

We recall that for $L\le 2\pi$ all initial conditions evolve into $u(x,t)=0$ [9]. This is clearly seen by multiplying (1) by u and integrating between 0 and L. This yields

$$\frac{1}{2}\frac{\partial}{\partial t}\int_0^L u^2dx=\int_0^L u_x^2dx-\int_0^L u_{xx}^2dx\ . \tag{2}$$

For periodic solutions of zero average in the interval $[0,L]$ we know that

$$\int_0^L u_x^2dx\le\frac{L^2}{4\pi^2}\int_0^L u_{xx}^2dx\ ;$$

therefore,

$$\frac{1}{2}\frac{\partial}{\partial t}\int_0^L u^2 dx \le \left[\frac{L^2}{4\pi^2}-1\right]\int_0^L u_{xx}^2 dx \ , \tag{3}$$

so that for $L \le 2\pi$ all initial conditions die away.

We now expand the solution for u in the Fourier series

$$u(x,t)=\sum_{n=-\infty}^{\infty} a_n(t)e^{ik_n x} \ ,$$

where $k_n = 2n\pi/L$ and the coefficients satisfy

$$a_{-n}(t)=\bar{a}_n \ .$$

Here $\bar{a}$ denotes the complex conjugate of a, and, since we have chosen solutions with zero average, $a_0=0$. Replacing the series expansion in the equation, we obtain the following system for the time evolution of the Fourier amplitudes:

$$\dot{a}_n+(k_n^4-k_n^2-i\delta k_n^3)a_n+\frac{ik_n}{2}\sum_{t=0}^{\infty}(a_t a_{n-t}+\bar{a}_t a_{n+t})=0 \ . \tag{4}$$

Including only the first three modes, we obtain the system

$$\dot{a}_1+(\mu_1-i\delta k^3)a_1+ik(\bar{a}_1 a_2+\bar{a}_2 a_3)=0 \ , \tag{5}$$

$$\dot{a}_2+(\mu_2-8i\delta k^3)a_2+ik(a_1^2+2\bar{a}_1 a_3)=0 \ , \tag{6}$$

$$\dot{a}_3+(\mu_3-27i\delta k^3)a_3+3ika_1 a_2=0 \ , \tag{7}$$

where $k=2\pi/L$ and $\mu_n=k_n^4-k_n^2$. The condition $L>2\pi$ implies that at least the first mode is linearly unstable, that is, $\mu_1<0$. It is convenient to introduce the variables

$$Z=a_1\bar{a}_1 \ ,$$

$$X+iY=\frac{\bar{a}_1 a_2}{a_1} \ ,$$

$$U+iV=\frac{\bar{a}_1^2 a_3}{a_1} \ .$$

In these variables, the system of six (three complex) coupled equations is simplified to five real coupled equations and an additional decoupled equation for the phase. We obtain

$$\dot{Z}=-2\mu_1 Z+2k(ZY+VX-UY) \ , \tag{8}$$

$$\dot{X}=-\mu_2 X-6k^3\delta Y-2k\left[XY-V+\frac{1}{Z}(Y^2V+XYU)\right] \ , \tag{9}$$

$$\dot{Y}=-\mu_2 Y+6\delta k^3 X-kZ+2k\left[X^2-U+\frac{1}{Z}(X^2U+XYV)\right] \ , \tag{10}$$

$$\dot{U}=-(\mu_1+\mu_3)U-24\delta k^3 V+3kZY-k(3VX-UY)-\frac{k}{Z}(YU^2+3YV^2+2UVX) \ , \tag{11}$$

$$\dot{V}=-(\mu_1+\mu_3)V+24\delta k^3 U-3kZX+k(3XU+YV)+\frac{k}{Z}(3U^2X+V^2X+2UVY) \ . \tag{12}$$

The missing equation yields an equation for the phase θ_1 of a_1 in terms of the variables Z, X, Y, U, and V:

$$\dot{\theta}_1=\delta k^3-kX-\frac{k}{Z}(XU+YV) \ . \tag{13}$$

One can readily check the system defined by (8)–(12) is contracting if $\mu_1+\mu_2+\mu_3>0$; in fact, one has

$$\frac{\partial \ln\dot{Z}}{\partial \ln Z}+\frac{\partial \dot{X}}{\partial X}+\frac{\partial \dot{Y}}{\partial Y}+\frac{\partial \dot{U}}{\partial U}+\frac{\partial \dot{V}}{\partial V}=-2(\mu_1+\mu_2+\mu_3) \ .$$

The solution for $u(x,t)$ in terms of these variables is written as

$$u(x,t)=2\sqrt{Z}\,\cos s+2X\cos 2s-2Y\sin 2s+\frac{2U}{\sqrt{Z}}\cos 3s-\frac{2V}{\sqrt{Z}}\sin 3s \ , \tag{14}$$

where $s=kx+\theta_1$. We see that the fixed points of the system for the new variables correspond to traveling waves $u(x-ct)$, with $c=-\dot{\theta}_1/k$. An analogous set of variables can be defined for a truncation of any order due to the particular form of the nonlinearity of the original equation. The problem that we address now is the time evolution of the reduced system and the nature of the solutions of u in the two opposite cases $\delta=0$ and $\delta \gg 1$. Before analyzing the three-mode system we study the large δ behavior of the two-mode system, which is simpler analytically. The argument that we follow is nonetheless analogous to the one needed when more modes are included.

III. TWO-MODE SYSTEM

We consider a small box so that only the first Fourier component is linearly unstable. In a first approximation we shall keep only one linearly stable mode, the least stable, that is, a_2. We obtain then the two-mode system setting $U=V=0$. This system is contracting if $\mu_1+\mu_2>0$. There exist two fixed points, the origin (0,0,0), which is unstable for $L>2\pi$ (or $k<1$), and an additional point (X_0,Y_0,Z_0) given by

$$X_0=-\frac{6\delta k^2\mu_1}{2\mu_1+\mu_2}, \quad Y_0=\frac{\mu_1}{k} \ ,$$

$$Z_0=-\frac{\mu_1\mu_2}{k^2}-\frac{36\delta^2 k^4\mu_1\mu_2}{(2\mu_1+\mu_2)^2} \ .$$

This system also arises in a different context in the opposite case $\mu_1>0$, $\mu_2<0$, which implies a completely different behavior of the solutions [12,13]. For large δ the fixed point is stable if $2\mu_1+\mu_2>0$; that is, if $L<2\sqrt{3}\pi$, or, in terms of k, if $k^2>\frac{1}{3}$. We shall assume that L lies in the range in which this fixed point is stable, so we consider only k^2 in the interval $(\frac{1}{3},1)$ and reconstruct $u(x,t)$ in the final state. All the initial conditions tested in the numerical integrations of this system led to the fixed point. For large δ we obtain

$$u(x,t)=2\delta\sqrt{z_0}F(s)\ , \tag{15}$$

where

$$F(s)=\cos s+\sigma\cos 2s\ ,$$

$\sigma=\sqrt{-\mu_1/\mu_2}$, and $z_0=-36k^4\mu_1\mu_2/(2\mu_1+\mu_2)^2$. The behavior of $u(x,t)$ for large t and δ is entirely described by $F(s)$. The function $F(s)$ defined on the interval $[0,2\pi]$ with periodic boundary conditions has exactly one maximum if $4\sigma\le 1$ and two maxima if $4\sigma>1$. Therefore, $F(s)$ has exactly one bump for $\frac{5}{8}\le k^2<1$ and two bumps for $\frac{1}{3}<k^2<\frac{5}{8}$. In the latter case the central bump at $s=\pi$ is much smaller than the bump at $s=0$. The ratio between the height of these two bumps [measured from the minimum value attained by $F(s)$] is given by $[2\sigma+(1/8\sigma)+1]/[2\sigma+(1/8\sigma)-1]$, which is a decreasing function of σ. For $k^2=\frac{5}{8}$ this ratio is obviously infinity, and as k^2 decreases from $\frac{5}{8}$ to $\frac{1}{3}$ this ratio decreases monotonically to 4.38 at $k^2=\frac{1}{3}$. Even in this, the worst case, $u(x,t)$ exhibits a sharp bump. The speed obtained from (13) is now

$$c=-(\delta k^2-X_0)=-\delta k^2\left[1-\frac{6\mu_1}{2\mu_1+\mu_2}\right].$$

The solution for $u(x,t)$ when $\delta=0$ is given instead by

$$u(x,t)=\frac{2}{k}\sqrt{-\mu_1\mu_2}\left[\cos s+\left[\frac{-\mu_1}{\mu_2}\right]^{1/2}\sin 2s\right],\quad c=0\ .$$

In Fig. 1 we show the solution $u(x,t)$ for $\delta=0$ and for $\delta=3$ after the system has reached a fixed point. For $\delta=0$ the solution is steady; for $\delta=3$ the pulse travels to the left with speed $c=1.169$. The role of dispersion in the two-mode system is the appearance of a traveling well-defined pulse whose amplitude increases with δ.

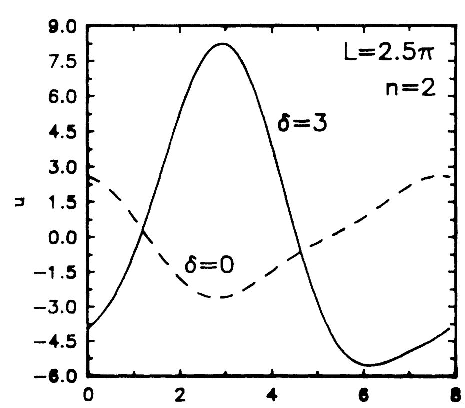

FIG. 1. Wave form $u(x,t)$ obtained from a numerical integration of the two-mode system, shown once the fixed point is reached. In all figures the dotted line shows the final state for $\delta=0$. For $\delta=0$ the solution is steady; for $\delta=3$ it travels to the left with speed 1.169.

IV. THREE-MODE SYSTEM

A. The fixed points

We have studied the three-mode system (8)–(12) numerically and analytically for large δ. Let us consider the range of L in which the origin is unstable and the map contracting, namely $2\pi<L<2\sqrt{7}\pi$, or equivalently, $\frac{1}{7}<k^2<1$. Here too, all the initial conditions tested led to fixed points of the system until a value of L, at which the fixed points lost stability through a Hopf bifurcation.

The fixed points (X_0,Y_0,Z_0) of the system are found in a simpler way from the complex version (5)–(7), making use of the fact that at the fixed point

$$\dot a_n=in\dot\theta_1 a_n,\quad n=1,2,3\ .$$

Replacing this form for a_n in (5)–(7) we obtain a quadratic equation with complex coefficients for Z_0:

$$\alpha Z_0^2+\beta Z_0+\gamma=0\ , \tag{16}$$

where

$$\alpha=k^4(f+2\bar f)-4hk^4|f|^2\ ,$$
$$\beta=k^2\bar g-2k^2h(f\bar g+g\bar f)\ ,$$
$$\gamma=-h|g|^2\ ,$$

with

$$f=-3(\mu_3-27i\delta k^3+3i\dot\theta_1)^{-1}\ ,$$
$$g=-2i\dot\theta_1-\mu_2+8i\delta k^3\ ,$$
$$h=\mu_1-i\delta k^3+i\dot\theta_1\ .$$

Equation (16) is a coupled system for Z_0 and $\dot\theta_1$, which can be easily solved numerically. Once Z_0 and $\dot\theta_1$ have been found, the additional components of the fixed point are obtained from

$$X_0+iY_0=\frac{ikZ_0}{2k^2Z_0f+g}\ ,\quad U_0+iV_0=ikfZ_0(X_0+iY_0)\ . \tag{17}$$

First we notice from Eqs. (8)–(12) that when $\delta=0$, if Z_0,X_0,Y_0,U_0,V_0 is a fixed point with speed c_0, then $Z_0,-X_0,Y_0,U_0,-V_0$ is also a fixed point, but with speed $-c_0$. If a fixed point for $\delta=0$ has vanishing speed $\dot\theta_1=0$, then, since α, β, and γ are real in this case, we obtain $X_0=V_0=0$.

The solution of Eq. (16) shows that when $\delta=0$ for $2\pi<L<3.6\pi$ there is a single fixed point that has $\dot\theta_1=0$, which is the case described above. At $L\approx 3.6\pi$ it loses stability and a new fixed point appears; in fact, two fixed points appear, which are related by the symmetry just described. One of them leads to right-traveling waves, the other to left-traveling waves. This new fixed point corresponds to the rotating waves described in [6]. Then, at $L\approx 4.01\pi$, it too loses stability through a Hopf bifurcation, the value of which is slightly below the value 4.09π reported in [6] as the first point where pulsing waves were observed. When $\delta\neq 0$, for low values of δ we find one or

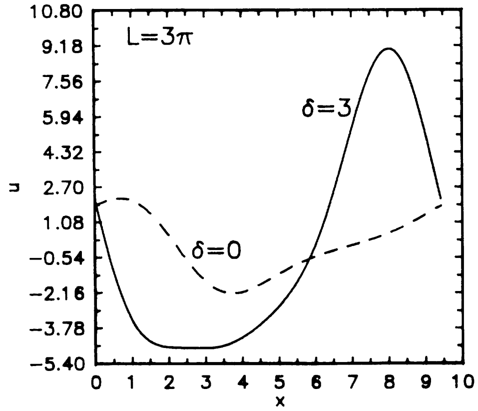

FIG. 2. Solution for $u(x,t)$ obtained from the integration of the three-mode system, after reaching the fixed point. For $\delta=0$ the solution is steady; for $\delta=3$ it is traveling to the left with speed 0.199.

three fixed points as occurs with $\delta=0$. For larger δ only one fixed point exists (fixed points with $Z<0$ are discarded). In Figs. 2 and 3 we show the solution $u(x,t)$ for $\delta=0$ and 3 after the system has reached a fixed point. At $L=3\pi$, the solution of the KS equation is steady ($c=0$), the solution with $\delta=3$ traveling to the left with speed 0.199. At $L=3.8\pi$ we obtain traveling waves to the left or the right, depending on initial conditions, for the KS equation. For $\delta=3$ the wave travels to the right. The speed for $\delta=0$ is $c=\pm 0.369$, and $\delta=3$ it is $c=1.516$. In Fig. 4 we show $|u|$ vs $(L/\pi)^2$ for the complete branch of fixed points between $L=2\pi$ and 4π, where we can observe the fast growth of $|u|$ with δ and the difference in the structure of fixed points as a function of δ. The fixed points for $\delta\neq 0$ lose stability through a Hopf bifurcation as L is increased. For $\delta=1$ the loss of stability of the unique fixed point occurs at $L=4.54\pi$. In Fig. 5 the speed is shown for the solution branches shown in Fig. 4. We have not investigated beyond this, since at $L=4\pi$ a

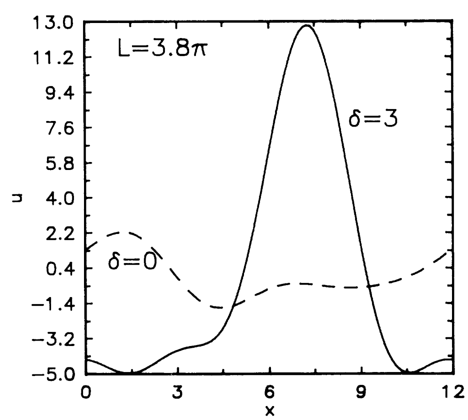

FIG. 3. Solution for $u(x,t)$ for a different value of L obtained from the integration of the three-mode system, after reaching the fixed point. Now for $\delta=0$ the solution may travel to the right or the left with speed 0.369. For $\delta=3$ it travels to the right with speed 1.516.

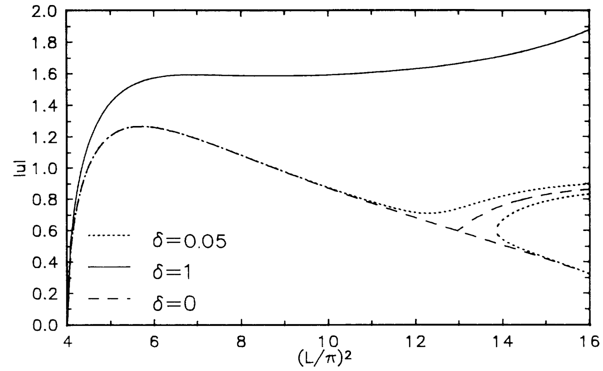

FIG. 4. The norm $|u|$ as a function of $(L/\pi)^2$ for the complete branch of fixed points between $L=2\pi$ and $L=4\pi$, for three different values of δ.

new two-pulse solution bifurcates from $u=0$, a solution that is not included in our system.

B. Large δ behavior

From Eq. (16) we find that for large δ the fixed point for Z_0 and $\dot{\theta}_1$ is of the form

$$Z_0=\delta^2 z,\quad \dot{\theta}_1=-\delta A\ ,$$

A is a root of the quartic equation

$$4(\mu_1 A_4 A_9+\mu_2 A_1 A_9+\mu_3 A_1 A_4)^2 + A_9^2(2\mu_3 A_4+\mu_2 A_9)(2\mu_1 A_4+\mu_2 A_1)=0\ ,$$

and

$$z=\frac{A_9^2}{k^2}\frac{(2\mu_1 A_4+\mu_2 A_1)}{(\mu_1 A_9-\mu_3 A_1)}\ ,$$

where we have defined $A_n=A-nk^3$. The quartic equation for A has two complex roots, which we discard, and two real roots. One of the real roots yields $z<0$ and is discarded also. We are left then with a single solution, as mentioned above. Then from Eqs. (17) we see that X_0 is of order δ, Y_0 is of order 1, U_0 is of order δ^2, and V_0 is of order δ. Therefore, in the expression for $u(x,t)$ [Eq. (14)]

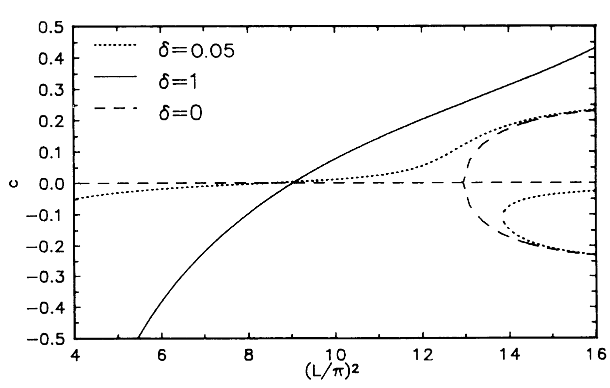

FIG. 5. The frequency of the branches of fixed points shown in Fig. 4.

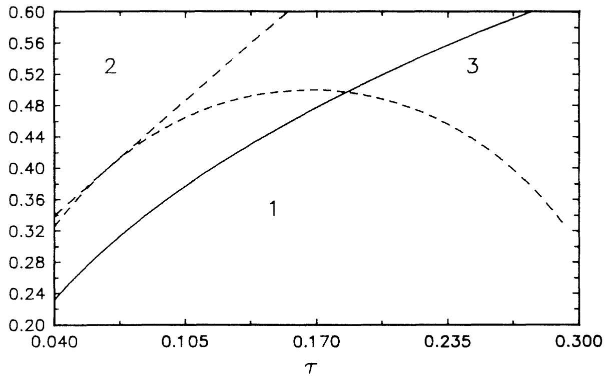

FIG. 6. The solid line indicates the values of σ and τ for Eq. (18) as L increases. The dashed lines are the boundaries between regions corresponding to different numbers of humps.

for large t we may neglect $\sin(s)$ and $\sin(2s)$. For large δ we then have

$$u(x,t) \approx 2\delta\sqrt{z}\,[\cos(s)+\sigma\cos(2s)+\tau\cos(3s)]\ , \qquad c=-\delta[k^2-\sigma(1+\tau)\sqrt{z_0}]\ , \tag{18}$$

with

$$\sigma=\frac{X_0}{\sqrt{Z_0}}=kz^{1/2}\frac{b}{e}$$

and

$$\tau=\frac{U_0}{Z_0}=-k^2z\frac{b}{e}\frac{1}{A_9}\ .$$

We have defined here $b=3A_9$ and $e=6k^2z-6A_4A_9$. Next, we analyze the behavior of $u(x,t)$ for large t, which, as in the two-mode truncation, is entirely specified by a single function $F(s)$, which is now given by $F(s)=\cos(s)+\sigma\cos(2s)+\tau\cos(3s)$. A simple analysis shows that $F(s)$ has one bump if $\sigma<[3\tau(1-3\tau)]^{1/2}$ when $\tau>\frac{1}{15}$ or if $\sigma<(1+9\tau)/4$ when $\tau<\frac{1}{15}$. It has two bumps if $\sigma>(1+9\tau)/4$ for any τ and it has three bumps if $\tau>\frac{1}{15}$ and $[3\tau(1-3\tau)]^{1/2}<\sigma<(1+9\tau)/4$. In the present case the values of σ and τ are such that the final state has either one sharp bump or three bumps, of which two are small. In Fig. 6 the σ-τ plane is shown, where the solid line corresponds to the values of σ and τ as L increases. For low values of L ($2\pi<L<11.52$ approximately), the final state lies in the region where only one hump develops (see Fig. 2). As L increases beyond 11.52, two small bumps develop (see Fig. 3). The ratio between the height of the large hump to the height of the small humps is ≈ 19.13 for $L=11.52$, which shows that the main hump is sharply defined.

V. CONCLUSION

We have studied a three-mode truncation of an equation that describes the evolution of systems with a long-wavelength oscillatory instability. This truncation has enabled us to understand the evolution of the system from an arbitrary initial condition to a final state consisting of a solitarylike traveling wave. The introduction of a convenient set of variables that enables us to express explicitly the solution as a Fourier series expansion in the variable $s=kx+\theta_1$, simplifies the problem to the study of the time evolution of the coefficients of the expansion. We find that the system evolves into a fixed point due to the combined action of diffusion and instability. This occurs even in the absence of dispersion, that is, for $\delta=0$, but the final state for $u(x,t)$ in this case does not develop a localized feature. For the KS equation the results obtained from the truncated system are in quantitative agreement with known results. The steady solution that bifurcates from $u=0$ at $L=2\pi$ and the bifurcation from this branch of right- or left-rotating waves $L\approx 3.6\pi$ correspond to fixed points of the system of equations for the amplitudes of the expansion of u in traveling waves. This expansion allows us to obtain an explicit expression for the speed of the traveling wave. As dispersion increases, a localized pulse develops; the amplitude of this pulse, for large δ, increases linearly with δ; since $\bar{u}=0$, as δ increases the pulse becomes sharper. The stability of the fixed point depends both on dispersion and on the box size. The fixed point loses stability through a Hopf bifurcation. Based on the quantitative agreement obtained with the KS equation for $0<L<4\pi$, we expect that the results including dispersion are correspondingly accurate in this range. However, the results obtained for large δ have a wider range of applicability. Multipulse solutions that bifurcate from $u=0$ can be constructed as periodic repetitions of the single pulse. If a_1, a_2, a_3 are the main components of the single pulse, then a_n, a_{2n}, a_{3n} will be the main components of the n pulse (this is confined by the numerical simulations of [10]), so that the asymptotic form of $u(x,t)$ and c for large δ are valid for these types of multipulse solutions after replacing k by nk. There exist other multipulse solutions that do not bifurcate from $u=0$, about which we cannot make any statement. An important point is to determine the stability of the n pulse solutions: A three-mode truncation is adequate only for small L, and all the unstable modes for a given L must be included to obtain reliable information on their stability. Based on the results for the three-mode truncation, we anticipate that in a higher-order truncation several fixed points may exist, all of which give rise to uniformly traveling waves, the number of humps of the final state depending on the precise value of the fixed point. As in the three-mode case, and as is indicated from the numerical results [7], these fixed points are stable or not depending on the value of the box size L; the fact that different initial conditions lead to different final states is possibly due to the fact that several fixed points may be locally stable at the same time, each with its own basin of attraction. One may conjecture that once the fixed points lose stability, the solution will wander in the neighborhood of the fixed points, leading to the creation and destruction of pulses depending on the proximity to fixed points that correspond to different numbers of humps. More precise statements will be made [14] from the analysis of the five-mode truncated system, which presents a wider variety of possibilities.

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