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OPTIMAL DECISION POLICY FOR REAL OPTIONS UNDER GENERAL MARKOVIAN DYNAMICS

FELIPE IGNACIO SAINZ GONZÁLEZ

Thesis submitted to the Office of Research and Graduate Studies in partial fulfillment of the requirements for the Degree of Master of Science in Engineering

Advisor:
GONZALO CORTÁZAR SANZ

Santiago de Chile, August 2018

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Gratefully to my mother, father, and siblings

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ABSTRACT

In this thesis, we propose a novel simulation approach to solve for optimal decision policies in real option problems under general Markovian dynamics. Our algorithm is implemented for the classical commodity mine of Brennan & Schwartz [*The Journal of Business*, 58(2), 135-157, 1985] under a wide variety of underlying dynamics such as stochastic variance, jumps, and regime-dependent parameters. In our numerical analysis, the method provides an accurate approximation of the critical prices when the underlying price follows a standard geometric Brownian motion. Moreover, the optimal policies produced by our algorithm are more profitable than those delivered by the widely-used Least-Squares Monte Carlo Method when the commodity follows more general dynamics. Finally, the algorithm allows to easily obtain the critical prices under regime-dependent dynamics, which are not accessible for backward methods based on forward simulation schemes.

Keywords: Real Options, Optimal Switching Problem, Numerical Methods in Finance

RESUMEN

En esta tesis se propone un novedoso enfoque de simulación para resolver las políticas de decisión en problemas de opciones reales bajo dinámicas Markovianas generales. El algoritmo es implementado para la clásica mina de Brennan & Schwartz [*The Journal of Business*, 58(2), 135-157, 1985] bajo una amplia variedad de dinámicas, tales como varianza estocástica, saltos, y parámetros dependientes del régimen de operación. En nuestro análisis numérico, el método entrega una estimación precisa de los precios críticos cuando el precio del *commodity* sigue un movimiento Browniano geométrico estándar. Además, las políticas óptimas producidas por el algoritmo son más rentables que aquellas entregadas por el ampliamente utilizado *Least-Squares Monte Carlo Method* cuando el *commodity* sigue dinámicas más generales. Finalmente, el algoritmo permite obtener los precios críticos bajo dinámicas dependientes del régimen de operación, las cuales no son abordables por métodos *backward* basados en esquemas de simulación *forward.*

Palabras clave: Opciones Reales, *Optimal Switching Problem*, Métodos numéricos en finanzas

1 ARTICLE BACKGROUND

1.1 Introduction

The term real option was introduced by Myers (1977) to refer to investment opportunities or “options” to be realized in future scenarios depending on the unveiled information. The tools that have been developed to address such options are grouped into what is known as Real Option Valuation (ROV).

Despite some weaknesses still hinder it from being the usual method for practitioners (see e.g. Lambrecht, 2017), ROV has been widely studied to capture the value of flexibilities in several contexts. Applications include the option to expand/contract production capacity, to switch from one operation regime to another, to postpone an investment, among others. All these options may be recognized in investment such as Research & Development, natural resources exploitation, production and manufacturing scheduling, etc. For a summary of the foundation research in the field (from the 80’s and 90’s) we refer to Dixit & Pindyck (1994), and Trigeorgis (1996). Additionally, for a contemporary review of real options research, we refer to the work of Trigeorgis & Tsekrekos (2018), who catalog papers published from 2004 to 2015 in internationally-renowned Operations Research journals.

Among the vast range of applications, a classical real option is related to production activities which allow transitions among different operating regimes, depending on a stochastic process such as an input/output price, supply/demand, or any other economic indicator. This general framework may be formulated as an optimal switching problem and includes activities such as the exploitation of an exhaustible natural resource (Brennan & Schwartz, 1985), electricity generation and tolling agreements (Deng & Xia, 2005), and natural gas storage (Carmona & Ludkovski, 2010). In spite of impressive advances in the field of valuation of such investment projects under complex multifactor processes, little work has been done to understand the decision policies required to value and optimally operate over time. Defined in terms of switching boundaries, such policies are commonly delivered as a byproduct, leading to inaccuracy, suboptimality, and difficulties when it is operationalized in the real world.

The remainder of this chapter is organized as follows. **Section 1.2** formulates the hypothesis and **Section 1.3** states the pursued objectives. **Section 1.4** addresses a literature review and outlines some alternatives to deal with the optimal switching problem. Main results and conclusions of this thesis are presented in **Section 1.5**, while **Section 1.6** provides perspectives for future research. The following chapter contains the main article of this thesis.

1.2 Hypothesis

The hypothesis of this thesis is that decision policies for the optimal switching problem may be addressed directly by means of a simulation-based algorithm. Given the flexibility provided by Monte Carlo simulation, one may expect to deal with different sources of uncertainty in the underlying stochastic process (e.g. stochastic variance, jumps, and regime-dependent parameters).

1.3 Main Objectives

The main objective of this thesis is to develop a simulation-based algorithm to directly address the decision policy for the optimal switching problem. The algorithm must be accurate, robust and easy-to-compute in order to efficiently deal with the problem and also the drawbacks arising under the currently used methods.

In order to accomplish this goal, the following three specific objectives are pursued. First, characterize the optimal switching problem in order to better understanding the switching boundaries. A first approach is given by American options. This is an optimal stopping problem that may be generalized as an optimal switching one, and the underlying exercise policy has been widely analyzed. Second, formulate a simulation-based algorithm from the knowledge about switching boundaries. Finally, implement and evaluate the performance of the algorithm. To achieve this, the classical commodity mine of Brennan & Schwartz (1985) is studied, under different specifications for the commodity price dynamic (e.g. stochastic variance, jumps, and regime-dependent parameters). Moreover, the algorithm will be compared with those methods widely used in literature.

1.4 Literature Review

The solution of the optimal switching problem is given by a system of variational inequalities, which are derived from the Dynamic Programming Principle. Chapter 5 of Pham (2009) provides a comprehensive characterization of the problem and its solution.

Closed-form solutions for the optimal switching problem can only be found in special cases, which generally considers a one-dimensional geometric Brownian motion (GBM) to model the underlying process and infinite decision horizons. For instance, Brennan & Schwartz (1985) propose an exhaustible mine with three regimes (open, closed, or abandoned), which is valued under a riskless strategy. A closed-form solution is provided when both the resource inventory and the concession time are infinite. A simpler model with just two regimes was then extensively studied, which is known as the starting and stopping problem. Dixit (1989) considers a firm under an operating and an idle state, which is valued through the same approach used by Brennan & Schwartz (1985). The problem is then addressed by Brekke & Øksendal (1991; 1994), who add resource depletion at a rate proportional to the remaining reserves, and Duckworth & Zervos (2001), who provide a mathematical framework to value the investment under a more general setup (e.g. general payoff functions). A general specification for the two-regime case was also studied by Ly Vath & Pham (2007), who propose a viscosity solution approach. Finally, recent works have been focused on formulating a Backward Stochastic Differential Equation (BSDE) to solve the optimal switching problem under two or more regimes (Hamadène & Jeanblanc, 2007; Djehiche & Hamadène, 2009; Djehiche, Hamadène, & Popier, 2009).

Analytical solutions (decision policies included) are quite restricted for real-world applications. For instance, the underlying asset could be driven by complex multifactor processes (Gibson & Schwartz, 1990; Sørensen, 2002; Cortazar & Schwartz, 2003; Casassus & Collin-Dufresne, 2005; Cortazar & Naranjo, 2006). Moreover, the longer the decision horizon the more critical the dimensionality of the process since a single risk factor would not allow to properly model future uncertainty. Finally, these closed-form solutions do not answer important questions such as how to manage the investment as time decays. Even though the problem may be approximated by numerical methods, such as finite difference schemes, the curse of dimensionality arise as the problem becomes complex (e.g. underlying dynamics with stochastic variance, jumps, etc.).

To overcome the above limitations, the current literature is focused on probabilistic methods, which allow addressing complex specifications for both the production activity and the underlying stochastic process, like the Quantization Method (QM) of Bally & Pagès (2003), and the Least-Squares Monte Carlo Method (LSM) proposed by Longstaff & Schwartz (2001). On the one hand, QM is a generalized (and sophisticated) version of decision trees. According to Glasserman (2003), the quantization procedure seems to be computationally demanding. Therefore, neither the maximized value nor the decision policy would be easy-to-compute. To the best of our knowledge, Gassiat et al. (2012) are among the few who have addressed the method for the optimal switching problem. Despite formulating the method under a general setup, they just compare numerical results against explicit formulae under a one-dimensional GBM.

On the other hand, LSM has been the most widely applied to the optimal switching problem (Deng & Xia, 2005; Cortazar et al., 2008; Carmona & Ludkovski, 2008; Carmona & Ludkovski, 2010; Aïd et al., 2014). They propose a backward dynamical programming algorithm using Monte Carlo simulation and least-squares regressions in order to compute the maximized value. Even though accurate values are delivered, both the simulation of the stochastic process and the use of regressions could generate undesirable noise in the estimation of the switching boundaries. Moreover, the precision of these boundaries seems to be subordinated to the choice of appropriate regression functions (Andersen & Broadie, 2004; Carmona & Ludkovski, 2008). Although literature recommends functions that resemble the expected shape of the dependent variable, we do not know a priori what these functions are, and one must heuristically propose them.

Considering the above, decision policies by themselves are an interesting field of research. A first approach to deal with them is given by the widely-studied early exercise boundary for American options (Kim, 1990; Kallast & Kivinukk, 2003; Chiarella & Ziogas, 2005; Chockalingam & Muthuraman, 2011; Kim, Jang, & Kim, 2013). A promising method to solve that exercise boundary was proposed by Cortazar et al. (2015) and then extended by Cortazar et al. (2016). In the first paper, the authors derive an analytical fixed-point iteration to compute the exercise boundary under a one-dimensional GBM and the Heston model (stochastic variance). The second paper extends the methodology to general Markovian dynamics by means of a simulation-based scheme.

1.5 Main Conclusions

This thesis proposes the Simulated-Fixed Point Iteration Method for Real Options (SFPI-RO), a novel simulation-based method to directly address the optimal decision policy for real options modeled as an optimal switching problem. Starting from an initial guess of the decision policy (a set of switching boundaries), the method iterates until optimality is reached through the Newton-Kantorovich Method. The algorithm replicates the spirit of

Cortazar et al. (2016), taking advantage of the flexibility provided by Monte Carlo simulation to deal with different sources of uncertainty (e.g. stochastic variance, jumps, and regime-dependent parameters).

SFPI-RO is implemented for the classical copper mine of Brennan & Schwartz (1985), where the decision policy is defined in terms of four critical prices triggering transitions among three possible regimes: open, closed, and abandoned. The method provides an accurate approximation of the switching boundaries when the underlying price follows a standard GBM. Moreover, the optimal policies produced by the algorithm are more profitable than those delivered by the widely-used least-squares Monte Carlo method when the commodity follows underlying dynamics with stochastic variance and jumps. Finally, the algorithm allows to obtain the critical prices under regime-dependent dynamics, which are not accessible for backward methods based on forward simulation schemes.

1.6 Further Research

The next step is to generalize the applicability of SFPI-RO. The formulation provided in this thesis assumes that the switching boundaries may be expressed in terms of one critical state variable, which satisfies a functional form with respect to the remaining variables. Under the model of Brennan & Schwartz (1985), this assumption is reasonable, and the critical variable is given by the commodity price. However, it is not known a priori under which conditions it is still valid. Therefore, further research should be focused on better understanding the switching boundaries in order to extend the methodology proposed in this thesis.

In the short term, feasible extensions consider the introduction of new features and operational constraints to the model of Brennan & Schwartz (1985), as well as new dynamics for the commodity price process. The flexibility provided by Monte Carlo simulation allows that new features are easily addressed. For example, one may consider delays between the decision making and the realization of it, in the same way as in Carmona & Ludkovski (2008). Moreover, we could analyze other production activities such as electricity generation (Deng & Xia, 2005; Carmona & Ludkovski, 2008), natural gas dome storage and hydroelectric pumped storage (Carmona & Ludkovski, 2010), among others.

2 OPTIMAL DECISION POLICY FOR REAL OPTIONS UNDER GENERAL MARKOVIAN DYNAMICS

2.1 Introduction

A classical real option[1] is related to production activities which allow transitions among different operating regimes, depending on a stochastic process such as an input/output price, supply/demand, or any other economic indicator. In spite of impressive advances in the field of valuation of such investment projects under complex multifactor processes (Cortazar et al., 2008; Carmona & Ludkovski, 2008; Gassiat et al., 2012), little work has been done to understand the optimal decision policies required to value and optimally operate over time. Defined in terms of switching boundaries, such policies are commonly delivered as a byproduct, leading to inaccuracy, suboptimality, and difficulties when it is operationalized in the real world. Instead, we formulate a novel method to directly deal with the required optimal decision policies for a wide variety of problems and underlying stochastic dynamics.

The valuation problem may be formulated as an optimal switching one and closed-form solutions can only be found in special cases, which generally considers a one-dimensional geometric Brownian motion (GBM) to model the underlying process, and infinite decision horizons (see e.g. Brennan & Schwartz, 1985; Dixit, 1989; Brekke & Øksendal, 1991, 1994; Duckworth & Zervos, 2001; Ly Vath & Pham, 2007; Zervos et al., 2018). However, these cases are quite restricted for real-world applications. For instance, the underlying

[1] Real options refer to investment opportunities to be realized in future scenarios depending on the unveiled information. They include the option to expand/contract production capacity, to switch from one operation regime to another, to postpone an investment, among others. For a summary of the foundation research in the field (from the 80's and 90's) we refer to Dixit & Pindyck (1994), and Trigeorgis (1996). Additionally, for a contemporary review we refer to Trigeorgis & Tsekrekos (2018), who catalog papers published from 2004 to 2015 in internationally-renowned Operations Research journals.

asset could be driven by complex multifactor processes (Gibson & Schwartz, 1990; Sørensen, 2002; Cortazar & Schwartz, 2003; Casassus & Collin-Dufresne, 2005; Cortazar & Naranjo, 2006). Moreover, the longer the decision horizon the more critical the dimensionality of the process since a single risk factor would not allow to properly model future uncertainty. Finally, these closed-form solutions do not answer important questions such as how to manage the investment as time decays. Even though the problem may be approximated by numerical methods, such as finite difference schemes, the curse of dimensionality arise as the problem complexity rises (e.g. complex multifactor processes).

To overcome the above limitations, the current literature is focused on probabilistic methods, which allow addressing complex specifications for both the production activity and the underlying stochastic process, like the Quantization Method of Bally & Pagès (2003), and the Least-Squares Monte Carlo Method proposed by Longstaff & Schwartz (2001). On the one hand, Quantization Method is a generalized (and sophisticated) version of decision trees. According to Glasserman (2003), the quantization procedure seems to be computationally demanding. Therefore, neither the maximized value nor the decision policy would be easy-to-compute. To the best of our knowledge, Gassiat et al. (2012) are among the few who have addressed the method for the optimal switching problem. Despite formulating the method under a general setup, they just compare numerical results against explicit formulae under a one-dimensional GBM.

On the other hand, the Least-Squares Monte Carlo Method (LSM) has been the most widely applied to the optimal switching problem (Deng & Xia, 2005; Cortazar et al., 2008; Carmona & Ludkovski, 2008; Carmona & Ludkovski, 2010; Aïd et al., 2014). They propose a backward dynamical programming algorithm using Monte Carlo simulation and least-squares regressions in order to compute the maximized value. Even though accurate values are delivered, both the simulation of the stochastic process and the use of regressions could generate undesirable noise in the estimation of the switching boundaries. Moreover, the precision of these boundaries seems to be subordinated to the choice of appropriate regression functions (Andersen & Broadie, 2004; Carmona & Ludkovski, 2008). Although literature recommends functions that resemble the expected shape of the dependent variable, we do not know a priori what these functions are, and one must heuristically propose them.

Considering the above, in this paper we propose a novel simulation-based method that computes the optimal decision policy for a wide variety of problems and underlying dynamics. Starting from an initial guess of the switching boundaries, we iterate until optimality is reached through the Newton-Kantorovich Method, as done by Cortazar et al. (2015) and Cortazar et al. (2016) for the optimal exercise boundary of American options. Our algorithm is implemented for the classical commodity mine of Brennan & Schwartz (1985) under a wide variety of underlying dynamics such as stochastic variance, jumps, and regime-dependent parameters. In our numerical analysis, the method provides an accurate approximation of the switching boundaries when the underlying price follows a standard GBM. Moreover, the optimal policies produced by our algorithm are more profitable than those delivered by the widely-used Least-Squares Monte Carlo Method when the commodity follows underlying dynamics with stochastic variance and jumps. Finally, the algorithm allows to obtain the critical prices under regime-dependent dynamics, which are not accessible for backward methods based on forward simulation schemes.

The remainder of this paper is organized as follows. **Section 2.2** outlines our method from the mathematical formulation of the optimal switching problem. **Section 2.3** formulates the method to solve the switching boundaries under the commodity mine proposed by Brennan & Schwartz (1985). Then, numerical results are presented in **Section 2.4**. First, the algorithm is validated when the commodity follows a one-dimensional GBM. We compare our results with those delivered by a finite difference approach and the Least-Squares Monte Carlo Method. Then, the method is extended to general Markovian dynamics, which includes stochastic variance, jumps, and regime-dependent parameters.

Finally, **Section 2.5** proposes extensions and further research lines, and **Section 2.6** provides concluding remarks.

2.2 Optimal Switching Problem

In this section, we formulate the optimal switching problem using the notation of Pham (2009). We then formulate our method using the notion of continuation and switching regions.

2.2.1 Mathematical Formulation

Consider a system operating in a set of possible regimes $\mathbb{I}_m = \{1, \dots, m\}$ over a time interval $[0, T]$. The system is driven by a diffusion process $\mathbf{X} \in \mathbb{R}^n$, which is solution to

$$d\mathbf{X}_u = b(\mathbf{X}_u)\, du + \sigma(\mathbf{X}_u)\, d\mathbf{W}_u, \qquad u \in [t, T],\ \ \mathbf{X}_t = \mathbf{x} \in \mathbb{R}^n \tag{2.1}$$

where $\mathbf{W} \in \mathbb{R}^n$ is a n-dimensional Wiener process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, P)$, b and σ are $\mathbb{R}^n$ and $\mathbb{R}^{n \times n}$ valued–functions, respectively. One may assume b and σ are also functions of the regime at time u.

A payoff rate function $\psi_i : \mathbb{R}^n \to \mathbb{R}$ is defined for each regime i. Furthermore, the transition from regime i to regime j involves a fixed cost k_{ij}, which satisfies the following triangular condition

$$k_{ij} < k_{ik} + k_{kj}, \qquad k \neq i, j, \tag{2.2}$$

implying that the one-step transition from i to j is cheaper than any 2-step alternative. Furthermore, we assume that $k_{ii} = 0$ to avoid arbitrage opportunities by switching back and forth.

Let $\alpha = (\tau_n, \iota_n)_{n\geq 1}$ be an operation strategy where $\{\tau_n\}_{n\geq 1}$ is an increasing sequence of switching times, and $\iota_n \in \mathbb{I}_m$ is the current regime from time τ_n until time τ_{n+1}. Accordingly, the process indicating the regime value over time, given an initial regime $\iota_0 = i$ at time t, is

$$I_u = \sum_{n\geq 0} \iota_n \mathbb{1}_{[\tau_n,\tau_{n+1})}(u)\,, \qquad u \in [t,T] \tag{2.3}$$

where $\tau_0 = t$, and $\mathbb{1}_A(x)$ is the indicator function on A.

Thereby, the expected discounted profit at time t is expressed as

$$H(t,\mathbf{x},i;\alpha) = \mathbb{E}\left[\int_t^T \psi_{I_u}(\mathbf{X}_u)\, e^{-\rho(u-t)}\, du - \sum_{t\leq\tau_n\leq T} k_{\iota_{n-1},\iota_n}\, e^{-\rho(\tau_n-t)}\right] \tag{2.4}$$

where ρ is a positive discount rate and the expectation is conditioned on the initial values $\mathbf{x}$ and i at time t. The goal of a decision maker is to find the operation strategy α that maximize the discounted expected profit in (2.4). Then, the optimal value function is defined as

$$v(t,\mathbf{x},i) = \sup_{\alpha\in\mathcal{A}} H(t,\mathbf{x},i;\alpha)\,, \qquad t \in [0,T],\;\; \mathbf{x} \in \mathbb{R}^n,\;\; i \in \mathbb{I}_m \tag{2.5}$$

where $\mathcal{A}$ denote the set of all possible switching strategies. From the Dynamic Programming Principle, one may derive the well-known system of variational inequalities in (2.6) for which $v_i = v(t, \mathbf{x}, i)$ is a solution, where $\mathcal{L}$ is the generator of the diffusion process $\mathbf{X}$.

$$\min\left[\rho v_i - \frac{\partial v_i}{\partial t} - \mathcal{L}v_i - \psi_i, \quad v_i - \max_{i \neq j}\left(v_j - k_{ij}\right)\right] = 0, \qquad t \in [0, T], \ \ \mathbf{x} \in \mathbb{R}^n, \ \ i \in \mathbb{I}_m \tag{2.6}$$

2.2.2 Switching Boundaries and Optimality

From (2.6), for any regime $i \in \mathbb{I}_m$ we note that the space $[0, T] \times \mathbb{R}^n$ splits into a continuation ($\mathcal{C}_i$) and a switching ($\mathcal{S}_i$) region. We define such regions as sets $(t, \mathbf{x}) \subset [0, T] \times \mathbb{R}^n$ where the optimal decision is to stay in regime i and to switch from regime i to any other, respectively. Let $\mathcal{S}_{ij} \subset \mathcal{S}_i$ be the subset where the optimal transition is to regime j. We have the following definitions

$$\mathcal{C}_i = \left\{(t, \mathbf{x}) \in [0, T] \times \mathbb{R}^n : \rho v_i - \frac{\partial v_i}{\partial t} - \mathcal{L}v_i - \psi_i = 0\right\}, \tag{2.7}$$

$$\mathcal{S}_i = \left\{(t, \mathbf{x}) \in [0, T] \times \mathbb{R}^n : v_i = \max_{j \neq i}\left(v_j - k_{ij}\right)\right\}, \tag{2.8}$$

$$\mathcal{S}_{ij} = \left\{(t, \mathbf{x}) \in [0, T] \times \mathbb{R}^n : v_i = v_j - k_{ij}\right\}. \tag{2.9}$$

Denote by $\partial\mathcal{S}_{ij}$ the boundary of the switching region $\mathcal{S}_{ij}$. Under continuity of the solution to (2.6), the optimality condition for $\partial\mathcal{S}_{ij}$ is given by the following equality

$$v(t, \mathbf{x}, i) = v(t, \mathbf{x}, j) - k_{ij}, \qquad (t, \mathbf{x}) \in \partial\mathcal{S}_{ij}, \ \ i \neq j \in \mathbb{I}_m \tag{2.10}$$

where $v(t, \mathbf{x}, i)$ is the expected cash flow when the transition to regime j at time t is avoided. Since $(t, \mathbf{x})$ is on $\partial\mathcal{S}_{ij}$, note that the value function $v(t, \mathbf{x}, i)$ is given by the operation strategy over the adjacent region to $\mathcal{S}_{ij}$.

Let us outline a novel approach to address the optimal switching problem from the above notion of switching boundaries. Consider a set of switching regions (not necessarily optimal), one may derive the operation strategy α according to the following sequence

$$\begin{gathered} \tau_0 = t, \quad \iota_0 = i \\ \tau_1 = \inf\{u \geq \tau_0 \colon \mathbf{X}_u \in \mathcal{S}_i\}, \quad \iota_1 = j : \mathbf{X}_{\tau_1} \in \mathcal{S}_{ij} \subset \mathcal{S}_i \\ \vdots \\ \tau_n = \inf\{u \geq \tau_{n-1} \colon \mathbf{X}_u \in \mathcal{S}_{\iota_{n-1}}\}, \quad \iota_n = j : \mathbf{X}_{\tau_n} \in \mathcal{S}_{\iota_{n-1}, j} \subset \mathcal{S}_{\iota_{n-1}} \end{gathered} \tag{2.11}$$

and compute the expected discounted profit $H(t, \mathbf{x}, i; \alpha)$ at each point over the switching boundary $\partial\mathcal{S}_{ij}$. If equation (2.10) is hold, then the switching boundary is optimal. If not, we propose an iterative method to update it until equality in (2.10) is achieved.

Let $(S, \mathbf{Z})$ be a partition of the diffusion process $\mathbf{X} \in \mathbb{R}^n$ such that $S \in \mathbb{R}$ and $\mathbf{Z} \in \mathbb{R}^{n-1}$. The process starts from $\mathbf{x} = (s, \mathbf{z})$ at time $t \in [0, T]$. Consider the following operator

$$\Phi(t, s, \mathbf{z}, i, j; \alpha) = H(t, s, \mathbf{z}, i; \alpha) - H(t, s, \mathbf{z}, j; \alpha) + k_{ij} \tag{2.12}$$

Fixing t and $\mathbf{z}$, the equation $\Phi(t, s, \mathbf{z}, i, j; \alpha) = 0$ may be solved for s and then characterize the optimal switching boundary as

$$\partial\mathcal{S}_{ij} = \{(t, s_{ij}^*(t, \mathbf{z}), \mathbf{z}) \in [0, T] \times \mathbb{R} \times \mathbb{R}^{n-1}\} \tag{2.13}$$

where $s_{ij}^{*}(t, \mathbf{z})$ is the solution to (2.12). A key assumption in this first formulation is that $s_{ij}^{*}(t, \mathbf{z})$ is indeed a function of $(t, \mathbf{z})$, which holds in the model we study later.

Based on Cortazar et al. (2016), the equation $\Phi(t, s, \mathbf{z}, i, j; \alpha) = 0$ may be solved using the Newton-Kantorovich method. Starting from an initial guess $s_{ij}^{(0)}$ at some state $(t, \mathbf{z})$, a new approximation is computed according to

$$s_{ij}^{(k+1)} = s_{ij}^{(k)} - \left(\left. \frac{\partial \Phi(t, s, \mathbf{z}, i, j; \alpha)}{\partial s} \right|_{s = s_{ij}^{(k)}} \right)^{-1} \cdot \Phi\left(t, s_{ij}^{(k)}, \mathbf{z}, i, j; \alpha\right) \quad (2.14)$$

Our method shall be referred as the Simulated-Fixed Point Iteration Method for Real Options (SFPI-RO). Note that the implementation of the algorithm depends on: (1) the hypothesis of s_{ij}^{*} as a function of $(t, \mathbf{z})$, and (2) an expression to easily compute $\partial \Phi / \partial s$.

2.3 SFPI-RO Method and the Model of Brennan and Schwartz (1985)

In this section, we formulate the SFPI-RO Method to address the optimal decision policy for the commodity mine of Brennan & Schwartz (1985).

Let us consider a mine of an exhaustible natural resource, which operates under a concession expiring at time T. The possible regimes for the operation are closed, open, and abandoned (permanent closure), which are labeled as $i = \{0, 1, 2\}$, respectively. Note that the mine value depends on the commodity price S, the reserves level, Q, and time, t. In addition, one may consider dependency on $\mathbf{Y} \in D \subseteq \mathbb{R}^{n-2}$, a vector of $(n-2)$ latent state variables for the commodity price process. Thus, the process $\mathbf{X} = (S, \mathbf{Y}, Q) \in \mathbb{R}^{n}$ drives the system.

From Brennan & Schwartz (1985), the mine value $H(t, \mathbf{x}, i; \alpha) = H_i(t, \mathbf{x})$ is given by

$$
\begin{aligned}
H_i(t,\mathbf{x}) = \mathbb{E}\Bigg(& \int_t^T (q(S_u - a) - tax(S_u))\, e^{-R_u}\, \mathbb{1}_{\{I_u=1\}} du \\
& - \int_t^T f\, e^{-R_u}\, \mathbb{1}_{\{I_u=0\}} du - \sum_{t \le \tau_n \le T} k_{\iota_{n-1},\iota_n}\, e^{-R_{\tau_n}} \Bigg)
\end{aligned} \tag{2.15}
$$

$$tax(s) = t_1 q s + \max\{t_2 q(s(1-t_1) - a), 0\}$$

where q is the extraction rate, a is the deflated cost rate of production when open, and f is the deflated cost rate of maintaining the mine when closed. While the mine is open, the tax structure includes a royalty and an income tax at rates t_1 and t_2, respectively. The authors also consider a property tax rate λ_i on the market value of the mine, depending on the operating status ($i = 0, 1$). As point out by them, λ_i may represent the arrival rate of a Poisson process governing the possible expropriation of the mine. Hence, we define a discount factor $R_u = \int_t^u \rho_{I_v} dv$, where $\rho_i = r + \lambda_i$ is a regime-dependent rate and r is the real risk-free rate.

To simplify notation, we reformulate $H_i(t, \mathbf{x})$ as in (2.16), where $A_i(t, \mathbf{x})$ is an expected cash-flow linearly related to the commodity price process while $B_i(t, \mathbf{x})$ is not.

$$
\begin{aligned}
H_i(t,\mathbf{x}) &= \mathbb{E}\left(\int_t^T m_u S_u du \right) - \mathbb{E}\left(\int_t^T n_u du + \sum_{t \le \tau_n \le T} k_{\iota_{n-1},\iota_n} e^{-R_{\tau_n}} \right) \\
&= A_i(t,\mathbf{x}) - B_i(t,\mathbf{x})
\end{aligned} \tag{2.16}
$$

The decision policy is then defined in terms of four critical prices: $S_{02}(t, \mathbf{Y}, Q)$, $S_{01}(t, \mathbf{Y}, Q)$, $S_{12}(t, \mathbf{Y}, Q)$, and $S_{10}(t, \mathbf{Y}, Q)$, each one related to a possible regime transition.

These critical prices satisfy equations (2.17) – (2.20)[2] and trigger regime transitions according to **Figure 2.1** for a given state $(t, \mathbf{Y}, Q)$.

$$H_0(t, S_{02}, \mathbf{Y}, Q) = -k_{02} \tag{2.17}$$

$$H_0(t, S_{01}, \mathbf{Y}, Q) = H_1(t, S_{01}, \mathbf{Y}, Q) - k_{01} \tag{2.18}$$

$$H_1(t, S_{12}, \mathbf{Y}, Q) = -k_{12} \tag{2.19}$$

$$H_1(t, S_{10}, \mathbf{Y}, Q) = \max\{H_0(t, S_{10}, \mathbf{Y}, Q) - k_{10}, -k_{12}\} \tag{2.20}$$

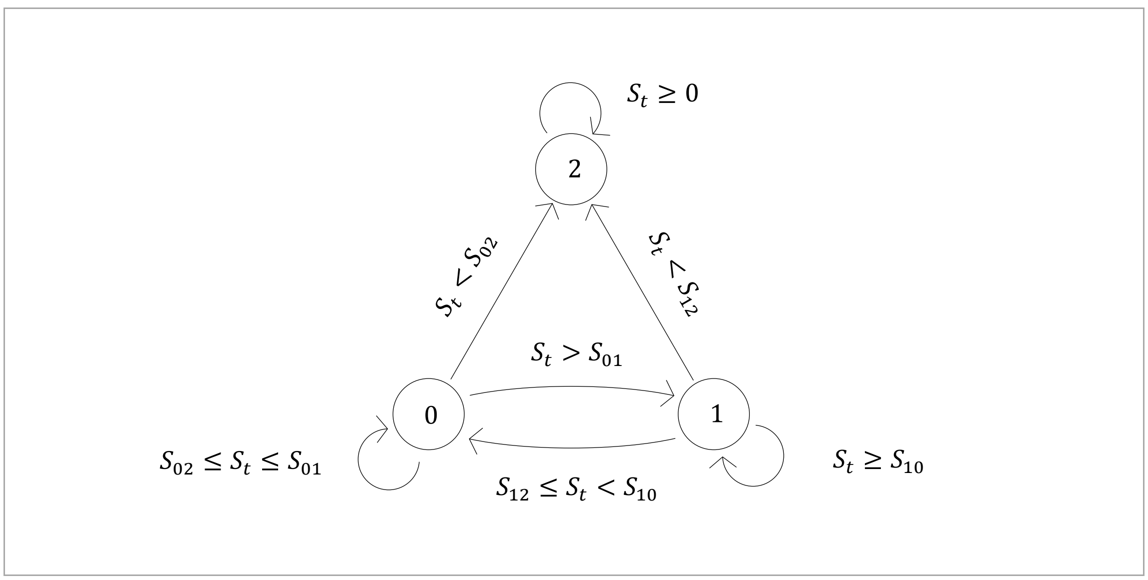

Figure 2.1 Switching conditions for the model of Brennan and Schwartz (1985)

[2] Note that the optimal decision is to abandon instead of close at S_{10} when $H_0(t, S_{10}, \mathbf{Y}, Q) - k_{10} < -k_{12}$. Therefore, the closure region is empty, and we conveniently set $S_{10} = S_{12}$, even though S_{10} is undefined.

In order to implement SFPI-RO, the following proposition gives an expression for the mine delta (a proof is provided in **Appendix A**).

Proposition 1. If S exhibits constant returns to scale, then the partial derivative of the mine value with respect to $S_t = s$ is given by

$$\Delta = \frac{\partial H_i(t, \mathbf{x})}{\partial s} = \frac{\mathbb{E}\left(\int_t^T m_u S_u du\right)}{s} = \frac{A_i(t, \mathbf{x})}{s} \tag{2.21}$$

From (2.14) and **Proposition 1**, critical prices defined in equations (2.17) – (2.20) may be computed with the following iteration method at a given state $(t, \mathbf{Y}, Q)$, where we denote $A_i\left(t, S_c^{(k)}, \mathbf{Y}, Q\right) = A_i\left(S_c^{(k)}\right)$ and $B_i\left(t, S_c^{(k)}, \mathbf{Y}, Q\right) = B_i\left(S_c^{(k)}\right)$:

$$S_{02}^{(k+1)} = S_{02}^{(k)} \cdot \frac{B_0\left(S_{02}^{(k)}\right) - k_{02}}{A_0\left(S_{02}^{(k)}\right)}, \tag{2.22}$$

$$S_{01}^{(k+1)} = S_{01}^{(k)} \cdot \frac{B_0\left(S_{01}^{(k)}\right) - B_1\left(S_{01}^{(k)}\right) - k_{01}}{A_0\left(S_{01}^{(k)}\right) - A_1\left(S_{01}^{(k)}\right)}, \tag{2.23}$$

$$S_{12}^{(k+1)} = S_{12}^{(k)} \cdot \frac{B_1\left(S_{12}^{(k)}\right) - k_{12}}{A_1\left(S_{12}^{(k)}\right)}, \tag{2.24}$$

$$S_{10}^{(k+1)} = \begin{cases} S_{12}^{(k+1)} & \text{if } H_0\left(S_{10}^{(k)}\right) - k_{10} < -k_{12} \\ S_{10}^{(k)} \cdot \dfrac{B_1\left(S_{10}^{(k)}\right) - B_0\left(S_{10}^{(k)}\right) - k_{10}}{A_1\left(S_{10}^{(k)}\right) - A_0\left(S_{10}^{(k)}\right)} & \text{otherwise.} \end{cases} \tag{2.25}$$

To operationalize the method, we address a discrete version of the problem, where the number of regime transitions is bounded. Let x be the number of switching opportunities per year, implying that transitions are allowed every $\Delta t = 1/x$ years and thus the reserves level is restricted to multiples of $\Delta q = q\Delta t$. Let $\mathcal{P}$ be a discretization of the space $[0, T] \times D \times [0, Q^*]$, where Q^* is the initial state of reserves. We use a time-grid given by $\mathcal{T} = \{0 = t_0, t_1, \ldots, t_{Nt} = T\}$, where t_k is an increasing sequence of multiples of Δt, and the reserves grid is defined as $\mathcal{Q} = \left\{0 = Q_0, Q_1, \ldots, Q_{Nq} = Q^*\right\}$, where Q_k is an increasing sequence of multiples of Δq. Since states attainable by **Y** are stochastic, one may bound the process within a feasible range and then discretize such interval.

The set of critical prices is defined for each $p = (\mathcal{t}, \boldsymbol{\mathcal{y}}, \mathcal{q}) \in \mathcal{P}$ and it is obtained by linear interpolation otherwise. As an initial guess, we derive an expression of the critical prices at time $T - \Delta t$ as in **Appendix B**. We do so since the optimal decision is to abandon under any scenario at time T, for both the open and the closed mine, and therefore the critical prices go to infinity. The same behavior arises when the mine is exhausted ($Q = 0$).

Starting from the nodes along $Q = \Delta q$ and $t = T - \Delta t$, the resulting set of initial prices $\left\{S_{02}^{(0)}, S_{01}^{(0)}, S_{12}^{(0)}, S_{10}^{(0)}\right\}$ is refined simultaneously through equations (2.22) – (2.25). To estimate $A_i\left(S_c^{(k)}\right)$ and $B_i\left(S_c^{(k)}\right)$ at each $p = (\mathcal{t}, \mathcal{y}, \mathcal{q}) \in \mathcal{P}$, a set of M trajectories of S and **Y** are simulated with an initial value $(S_c^{(k)}, \boldsymbol{\mathcal{y}})$, starting at time $t = \mathcal{t}$, and initial reserves level $Q = \mathcal{q}$. Simulation stops once: (1) drop below abandonment price, (2) concession expires ($t = T$), or (3) the mine is exhausted ($Q = 0$).

Finally, for each node $p \in \mathcal{P}$ the algorithm iterates until convergence is achieved. For example, one may consider the stopping criterion in (2.26), where $\epsilon > 0$.

$$\max_{c \in \{02,01,12,10\}} \left| \frac{S_c^{(k+1)} - S_c^{(k)}}{S_c^{(k)}} \right| \leq \epsilon \tag{2.26}$$

2.4 Numerical Results

SFPI-RO is implemented for the mine of Brennan & Schwartz (1985). First, we validate the method under a one-dimensional GBM. As a reference, we numerically address (2.6) by means of a Finite Difference Method (FDM). Then, we introduce a general stochastic variance model with jumps in price and variance to exhibit the performance of our method under different sources of uncertainty. We compare our results with those delivered by the Least-Squares Monte Carlo Method (LSM). Finally, we address a regime-dependent model to highlight the versatility of our algorithm when methods like LSM are not applicable. The source code was developed in MATLAB and executed on an Intel laptop i5-8250U 6GB with 3.40 GHz.

2.4.1 SFPI-RO under Geometric Brownian Model

Consider a mine with finite reserves, $Q^* = 150$ million pounds, finite concession time, $T = 30$ years, and $x = 4$ switching opportunities per year. The deflated spot price under the risk-free measure follows a one-dimensional GBM given by

$$\frac{dS_t}{S_t} = (r - d)dt + \sigma dW_t \tag{2.27}$$

where $r = 2\%$ is the real risk-free rate, $d = 1\%$ is the convenience yield, and $\sigma = \sqrt{0.08}$ is the price volatility. The remaining parameters are obtained from Table 1 of Brennan & Schwartz (1985).

In what follows we provide a series of results in order to show the performance of the algorithm.

2.4.1.1 Critical Prices delivered by SFPI-RO

Figure 2.2 and **Figure 2.3** show the critical prices delivered by SFPI-RO (black lines) and the switching regions obtained with FDM[3] as a colored background. Moreover, we display the critical prices delivered by LSM[4] (blue lines). We show cross-sections of the decision policy across the time-to-maturity axis ($\tau = T - t$) in **Figure 2.2** and across the reserves axis in **Figure 2.3**. SFPI-RO considers M = 250,000 paths for each critical price, and a convergence threshold ϵ = 1%. Moreover, we define a mesh $\mathcal{P}$ with 26 time-nodes and 33 reserves-nodes ($\#\mathcal{P}$ = 858 nodes). **Appendix E** provides results under different specification of $\mathcal{P}$ (changing the number of nodes in both the time and reserves axis).

Note that SFPI-RO provides smooth critical prices which closely approximate the reference delivered by FDM. From the management perspective, non-noisy estimations are an important feature in order to avoid arbitrary correction procedures before operationalizing the decision policy. In fact, any correction attempt could add new sources of error, leading to suboptimal decisions. Therefore, our method can help to easily operationalize an optimal operation strategy. On the other hand, critical prices delivered by LSM are slightly suboptimal and even noisy for some cross-sections. This is to be expected since LSM was developed to accurately address the optimal value and not the decision policy.

---

[3] See **Appendix C** for further details on FDM. When implementing the method, we consider 64 time-steps per year to discretize the time-axis. Furthermore, we use 25,000 points to discretize the price-axis between 0 and S_{max}, which is defined as the 99th percentile of S_T when $S_0 = a$.

[4] See **Appendix D** for further details on LSM. We consider 400,000 prices paths with 64 time-steps per year when implementing the method. Moreover, as regression functions we use polynomials of order 1 to 3, and three European options on the commodity price, strike price at multiples of the production cost rate a (0.50 a, a, and 1.50 a), and time to maturity $1/x$ (time between consecutive transition opportunities).

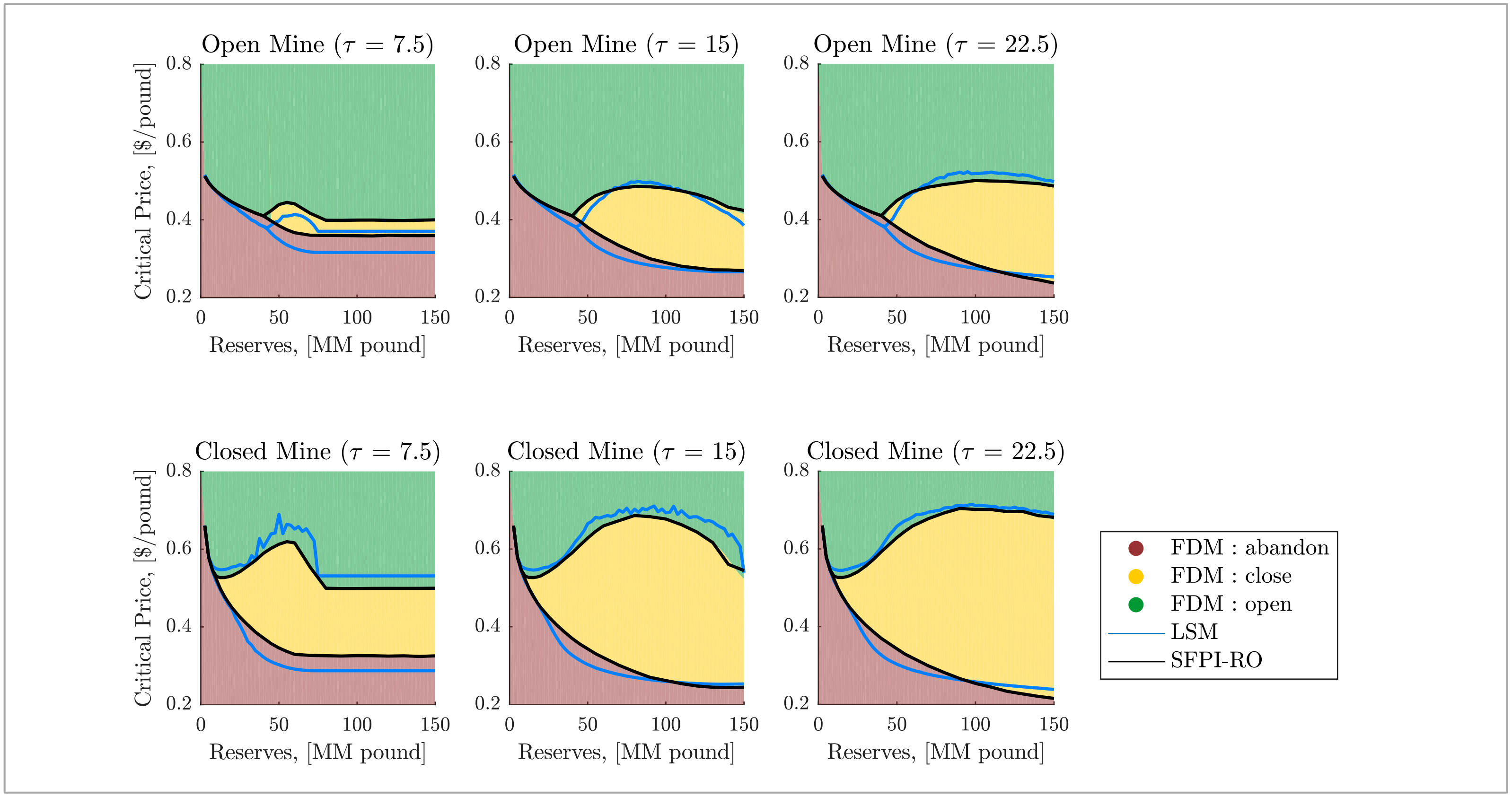

Figure 2.2 Cross-sections across the τ-axis [year] of critical prices under a one-dimensional GBM

We compare the critical prices delivered by SFPI-RO (black lines) and LSM (blue lines) with the decision policy delivered by FDM (colored background) when the mine is initially open (top row) and closed (bottom row). As a reference, the cost rate of production is a = 0.50 \$/pound.

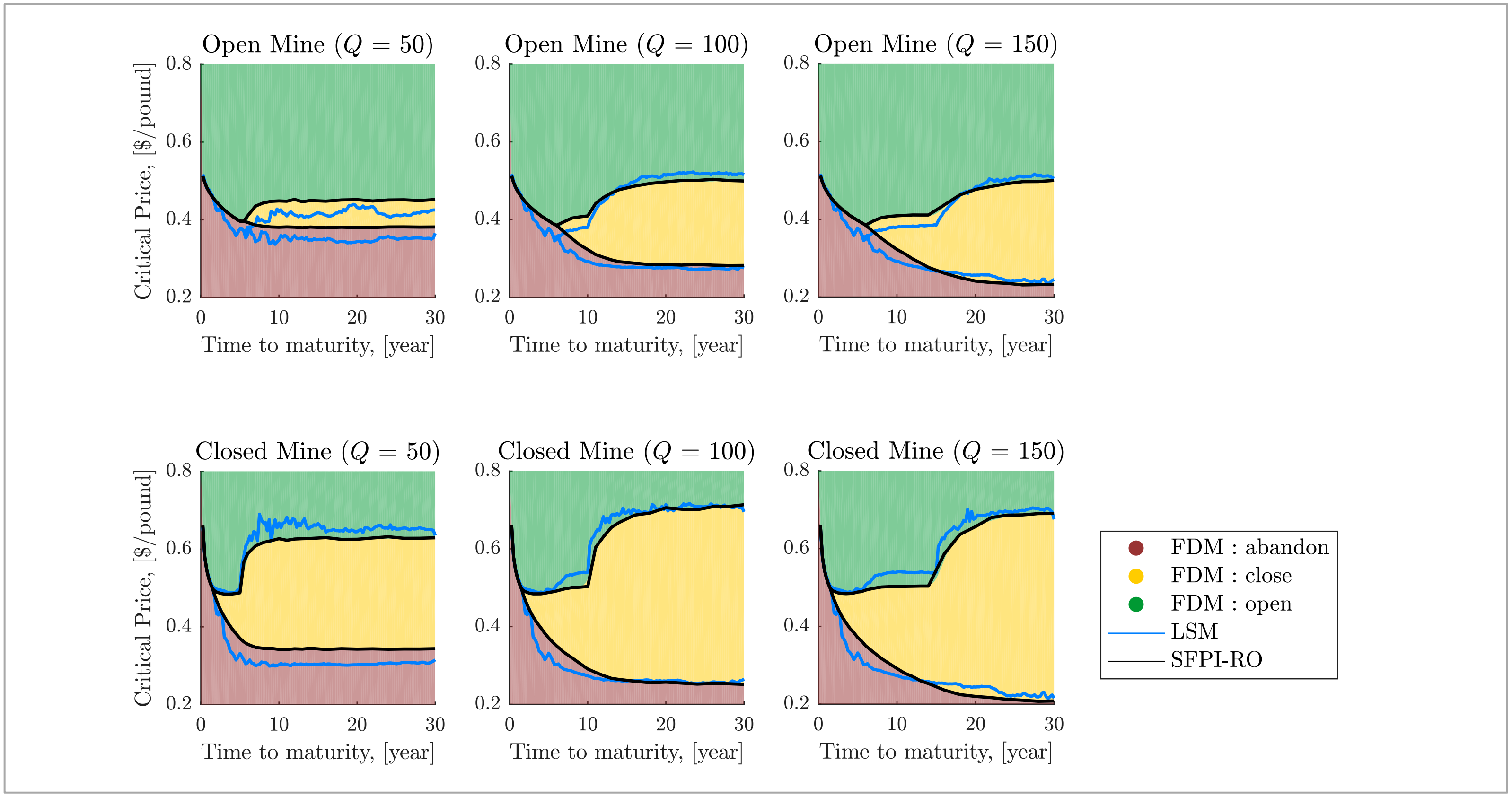

Figure 2.3 Cross-sections across the Q-axis [MM pound] of critical prices under a one-dimensional GBM

We compare the critical prices delivered by SFPI-RO (black lines) and LSM (blue lines) with the decision policy delivered by FDM (colored background) when the mine is initially open (top row) and closed (bottom row). As a reference, the cost rate of production is $a = 0.50$ \$/pound.

Regarding the estimation accuracy, **Table 2.1** displays the overall RMSE and the computational runtime when varying the number of nodes in the domain discretization, $\#\mathcal{P}$, and the number of simulated paths, M. The first one is measured by calculating the RMSE of the four critical prices delivered by SFPI-RO, with respect to its FDM counterpart. Since SFPI-RO mesh is incomplete, the critical prices are interpolated for the missing points.

Table 2.1 Overall RMSE and runtime for SFPI-RO under a one-dimensional GBM

M	Overall RMSE [$\times 10^{-5}$] when $\#\mathcal{P} =$				Runtime [min] when $\#\mathcal{P} =$			
	56	143	357	858	56	143	357	858
25,000	30.9450	6.1238	1.5972	1.0993	5.90	17.40	54.99	160.71
50,000	28.8757	5.5252	1.2560	0.8050	7.71	25.85	70.64	185.36
100,000	29.0744	5.3613	1.0768	0.6960	10.04	33.73	86.97	275.66
250,000	29.2625	5.2223	0.9948	0.5796	22.71	53.75	132.39	392.77

We note that the algorithm converges almost monotonically while increasing $\#\mathcal{P}$ and M. For low values of $\#\mathcal{P}$, discretization error is large enough to control the accuracy of the method. In fact, the expected benefit of more price paths (in terms of lower RMSE) becomes significant as $\#\mathcal{P}$ increases. As a reference, the overall RMSE of the reported LSM critical prices is 52.0381×10^{-5}, which is higher than those provided by our method.

In terms of computational runtime, the marginal time required increases quickly while increasing $\#\mathcal{P}$ and M. However, the reader may explore alternatives to optimize the performance of the algorithm if required. In fact, one could parallelize the method in the same way as Cortazar et al. (2016) in order to speed it up. Moreover, if the runtime is still a drawback, it is possible to combine the advantages of both LSM and SFPI-RO. It is well-known that LSM is faster. For instance, our implementation with 400,000 and 800,000 price paths takes around 18.4 and 37.9 minutes, respectively. But the method delivers imprecise and noisy estimates of the switching boundaries as observed in **Figure 2.2** and **Figure 2.3**. Then, our proposal is to obtain a rough estimate of the critical prices with LSM, run our method starting from these boundaries and deliver an accurate output faster.

2.4.1.2 Valuation under Critical Prices delivered by SFPI-RO

In this section, we simulate price paths to estimate the mine value under the decision policy obtained with SFPI-RO. Our purpose is to establish the performance of the critical prices delivered by the method when valuing the mine, under the four grid-specifications defined in the previous section (increasing $\#\mathcal{P}$ and $M = 250{,}000$). We consider 50 replications with 50,000 price paths per replication and initial price. Results are summarized in **Table 2.2** for an initially open mine while we omit results for an initially closed one since we observe the same behavior. Column (1) presents the mine value obtained by simulating price paths under the FDM decision policy. Columns (2) and (3) display the mine value obtained by LSM and the RMSE with respect to FDM results. From column (3) to (6) the simulated values under the SFPI-RO decision policy are exhibited, for each of the four grid specifications. Finally, we present the RMSE for the previous estimates from Column (7) to (10), with respect to the simulated value under the FDM decision policy.

Note that SFPI-RO provides accurate mine values for the four grid specifications. The benefit of refining the mesh $\#\mathcal{P}$ is insignificant at least from $\#\mathcal{P}$ = 357 nodes, which is to be expected from the accuracy reported for the critical prices. Nevertheless, the method provides reasonably good results even when using just $\#\mathcal{P}$ = 56 nodes to compute the critical prices. In fact, valuation error is less than 1% in that case, decreasing as the initial price increases. Therefore, SFPI-RO shows to be robust as an approximation procedure.

Regarding LSM, note that we show the method is indeed a good estimator of the mine value. However, SFPI-RO is even more accurate under the four grid-specifications, although the value difference tends to zero as the initial price increases. The last statement is to be expected since the optimal strategy tends to no-switches (continuous operation) as the initial price increases. Hence, an accurate set of switching boundaries is irrelevant for such cases.

The above results highlight the usefulness of our method. In addition to being convenient for a manager when operationalizing an investment strategy in the real world, SFPI-RO also leads to a robust and accurate valuation tool.

Table 2.2 Values [MM$] for an initially open mine under a one-dimensional GBM

Initial Price [$/pound]	FDM	LSM		SFPI-RO							
				Simulated Value [MM$] when $\#\mathcal{P} =$				RMSE when $\#\mathcal{P} =$			
	Sim. Value [MM$]	Sim. Value [MM$]	RMSE	56 nodes	143 nodes	357 nodes	858 nodes	56 nodes	143 nodes	357 nodes	858 nodes
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
0.3	1.2143	1.1437	0.0798	1.2042	1.2133	1.2141	1.2138	0.0125	0.0045	0.0039	0.0041
0.4	4.0807	4.0341	0.0532	4.0674	4.0787	4.0809	4.0801	0.0159	0.0050	0.0034	0.0032
0.5	7.8445	7.8184	0.0307	7.8286	7.8456	7.8476	7.8446	0.0186	0.0099	0.0100	0.0031
0.6	12.4646	12.4423	0.0314	12.4499	12.4629	12.4640	12.4643	0.0170	0.0041	0.0022	0.0026
0.7	17.4903	17.4714	0.0239	17.4768	17.4887	17.4904	17.4908	0.0167	0.0045	0.0018	0.0026
0.8	22.8291	22.8133	0.0203	22.8186	22.8290	22.8296	22.8293	0.0130	0.0041	0.0020	0.0025
0.9	28.3748	28.3622	0.0252	28.3655	28.3736	28.3747	28.3753	0.0116	0.0033	0.0020	0.0023
1.0	34.0283	34.0292	0.0220	34.0220	34.0283	34.0290	34.0289	0.0085	0.0027	0.0019	0.0019

2.4.2 SFPI-RO under General Dynamics

Consider the 2-dimensional affine jump-diffusion process proposed by Duffie et al. (2000). Denote the price process by $\mathbf{X} = (Y \quad V)^{\mathrm{T}}$, where $Y = \ln(S)$ and V are the log-price and the variance process, respectively. Assuming a constant real risk-free rate r and convenience yield d, the dynamic of **X** under the risk-free measure is given by

$$d\begin{pmatrix} Y_t \\ V_t \end{pmatrix} = \begin{pmatrix} r - d - \overline{\lambda\mu} - \frac{1}{2}V_t \\ \kappa(\theta - V_t) \end{pmatrix} dt + \sqrt{V_t}\begin{pmatrix} 1 & 0 \\ \rho\sigma & \sqrt{1-\rho^2}\sigma \end{pmatrix} d\mathbf{W}_t + d\mathbf{J}_t \qquad (2.28)$$

The variance is modeled as a mean-reverting process where θ is the long-term level, κ is the mean-reverting rate, and σ is the volatility-of-variance. Moreover, ρ is the correlation coefficient, $\mathbf{W}_t \in \mathbb{R}^2$ is an uncorrelated Wiener process, and $\mathbf{J}_t \in \mathbb{R}^2$ is a pure jump process with three components:

(i) **Jump in log-price only**, which size is normally distributed with mean μ_y and variance σ_y^2, and arrives according to a Poisson process at rate λ^y.

(ii) **Jump in variance only**, which size is exponentially distributed with mean μ_v, and arrives according to a Poisson process at rate λ^v.

(iii) **Simultaneous and correlated jumps** in Y and V with arrival intensity λ^c. Jump in variance has an exponentially distributed size with mean μ_{cv}. Conditional on the size of the jump in variance (z_v), the jump size in log-price is normally distributed with mean ($\mu_{cy} + \rho_J z_v$) and variance σ_{cy}^2.

Finally, $\overline{\lambda\mu}$ correspond to the jump compensator term for the log-price process under the risk-free measure, which is given by

$$\overline{\lambda\mu} = \lambda^y \left(\exp\left(\mu_y + \frac{1}{2}\sigma_y^2 \right) - 1 \right) + \lambda^c \left(\frac{\exp\left(\mu_{cy} + \frac{1}{2}\sigma_{cy}^2 \right)}{1 - \rho_J \mu_{cv}} - 1 \right) \tag{2.29}$$

We implement SFPI-RO using this model in order to highlight the ability of the algorithm to address price dynamics with different sources of uncertainty. First, we consider stochastic variance only (SV model) to then add a general jump process (SVGJ model) in order to evaluate the performance of the method as complexity increases. The SV model, initially proposed by Heston (1993), is obtained by letting $\lambda^y = \lambda^v = \lambda^c = 0$, while the SVGJ model is obtained by letting $\lambda^y, \lambda^v, \lambda^c > 0$.

Numerical implementation uses $M = 250{,}000$ paths for each critical price, and a convergence threshold ϵ = 1%. We address the case with finite reserves ($Q^* = 150$ million pounds), infinite concession time, $x = 4$ transition opportunities per year, and the same parameters introduced by Brennan & Schwartz (1985), for both the operation of the mine and the price dynamics (except for those related to the price variance). Moreover, the algorithm uses 11 nodes along the Q-grid, between 0 and Q^*, and 10 nodes uniformly distributed in $[0, 1.50]$ for the variance-axis. Since the variance process is mean-reverting, $V = 1.50$ is large enough to guarantee that the most likely variance levels are appropriately covered.

2.4.2.1 SV Model

Rather than considering constant variance equal to 0.08, one may model stochastic variance with long-term mean $\theta = 0.08$. We address the case $(\kappa, \sigma) = (2.5, 0.40)$, and $\rho = \{-0.07, 0.00, +0.70\}$. **Figure 2.4** shows cross-sections of the critical prices for 4 levels of variance (0.02, 0.08, 0.32, and 1.28), where the three correlation cases are overlaid.

Once again, SFPI-RO provides well-behaved switching boundaries for the three correlation levels in terms of smoothness, convergence, etc. Regarding optimality, one may conceptually validate the results from the price process features. For instance, as the initial variance increases, the continuation region for a closed mine (area bounded by S_{02} and S_{01}) wides considerably. When $V = 1.28$ it is even convenient to keep closed when there is only one reserve unit left to extract (gap between S_{02} and S_{01} when Q goes to zero). One may expect this behavior since a high variance level could trigger a very favorable price in the short-term, which could even pay the cost of keeping closed (maintenance cost) while waiting. Moreover, we also note that the abandoning price for both an open and a closed mine (S_{12} and S_{02}) decreases as ρ increases. It is well-known that the correlation coefficient ρ induces skewness on the price distribution: $\rho > 0$ is related to positive skewness (fat right tail) and $\rho < 0$ to negative skewness (fat left tail). Therefore, it is natural that the optimal abandonment strategy is riskier under positive correlation since greater correlation gives greater weight to high prices.

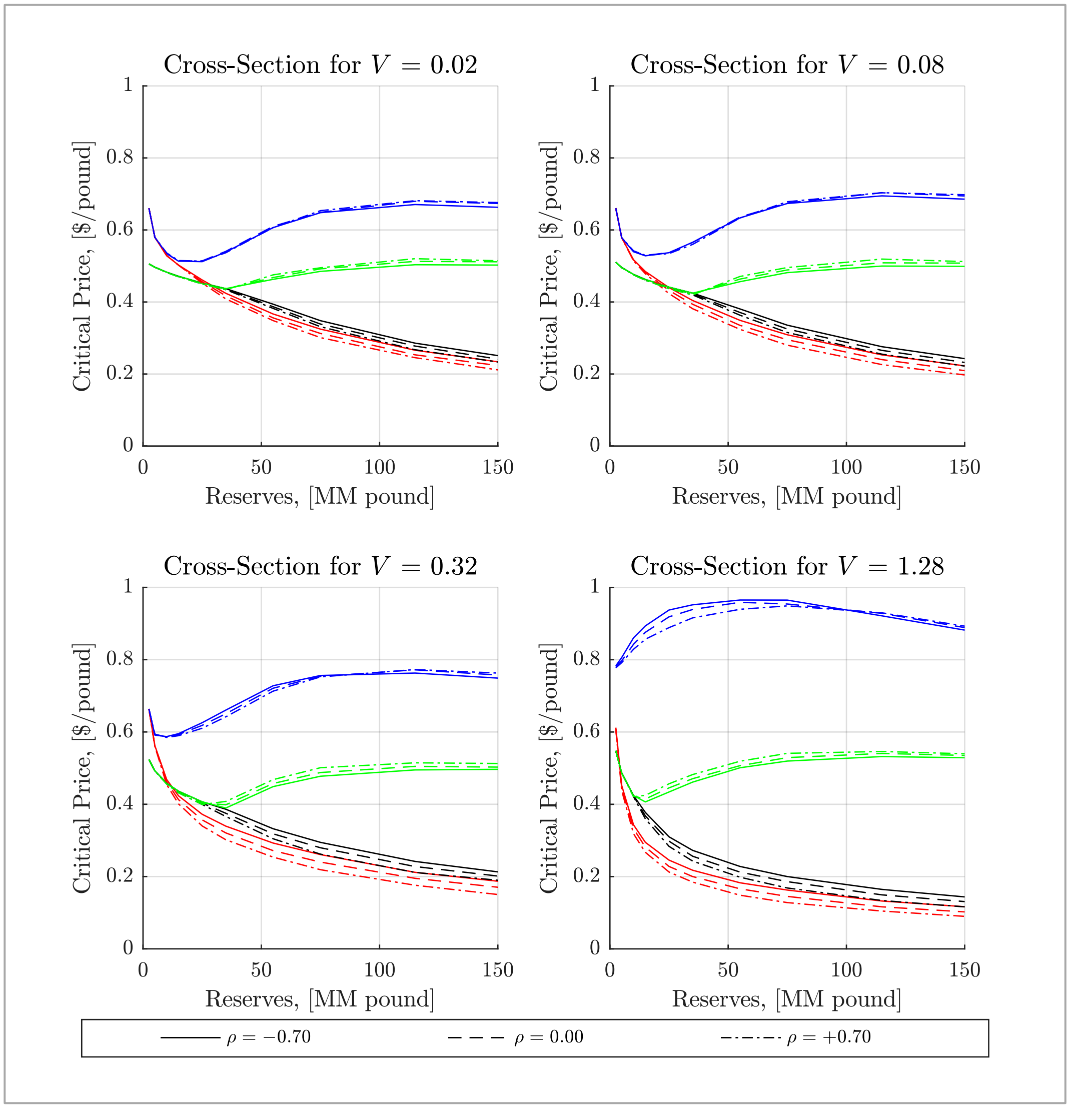

Figure 2.4 Cross-sections across the V-axis of critical prices under the SV model

Color legend for critical prices: S_{02} = red; S_{01} = blue; S_{12} = black; S_{10} = green

Additionally, we compare the values delivered under the SFPI-RO decision policy with results obtained with LSM[5] in order to numerically confirm the optimality of our method. **Table 2.3** shows values for an initially open mine when varying the correlation coefficient. We report the SFPI-RO and LSM values with the corresponding standard deviation (50 replications with the same 50,000 price paths per initial state are considered). Also, we report the difference between both values, indicating the p-value for an upper-tailed t-test according to: (*) for p < 10%, (**) for p < 5%, and (***) for p < 10%.

The results show that SFPI-RO delivers higher values than LSM for almost every case. As expected, when $\rho = \{-0.70, 0.00\}$ we observe the same behavior exhibited under GBM in **Section 2.4.1.2**: value difference decreases as the initial price increases. However, when $\rho = +0.70$ we do not clearly observe such a behavior. Probably, this effect is due to the model itself is more volatile when correlation is positive and therefore one may expect higher and more dispersed errors when LSM is computing linear regressions. From the previous results, one may conclude SFPI-RO provides profitable decision policies, empirically confirming the optimality of our method at least when compared with LSM.

[5] For LSM we use the same setup considered in **Section 2.4.1** but redefining the regressions functions. We use polynomials of S and V up to order 2 (considering cross products) and three European Options under a GBM with constant variance equal to V. Although in this case one may choose options under the SV model, this would suppose a time demanding algorithm since there is no analytical expressions for the option value. Moreover, we set T = 75 years in order to address the infinite decision horizon (the mine has 15 years of production under the parameters of Brennan & Schwartz, 1985).

Table 2.3 Values [MM$] for an initially open mine under the SV model

(Initial Variance: $V_0 = 0.08$)

Initial Price [$/pound]	Correlation coefficient $\rho = -0.70$					
	SFPI-RO		LSM		Difference	
	Value [MM$]	Standard Deviation	Value [MM$]	Standard Deviation	Average [MM$]	
0.3	1.0629	0.0427	0.9951	0.0489	0.0678	(***)
0.4	3.9677	0.0714	3.9232	0.0750	0.0445	(***)
0.5	7.7942	0.0667	7.7622	0.0691	0.0320	(***)
0.6	12.4797	0.0896	12.4575	0.0921	0.0222	(***)
0.7	17.6554	0.0965	17.6371	0.0982	0.0184	(***)
0.8	23.0571	0.1321	23.0453	0.1331	0.0118	(**)
0.9	28.5895	0.1381	28.5763	0.1398	0.0133	(***)
1.0	34.2459	0.1566	34.2435	0.1693	0.0024	

Initial Price [$/pound]	Correlation coefficient $\rho = 0.00$					
	SFPI-RO		LSM		Difference	
	Value [MM$]	Standard Deviation	Value [MM$]	Standard Deviation	Average [MM$]	
0.3	1.2047	0.0473	1.0556	0.0512	0.1490	(***)
0.4	4.0797	0.0864	3.9560	0.0899	0.1236	(***)
0.5	7.8514	0.0998	7.7544	0.1067	0.0970	(***)
0.6	12.4285	0.1018	12.3726	0.1105	0.0559	(***)
0.7	17.4634	0.1202	17.4254	0.1363	0.0380	(***)
0.8	22.8659	0.1459	22.8291	0.1443	0.0369	(***)
0.9	28.3952	0.1650	28.3680	0.1810	0.0272	(***)
1.0	34.0399	0.2149	34.0255	0.2234	0.0144	

Initial Price [$/pound]	Correlation coefficient $\rho = +0.70$					
	SFPI-RO		LSM		Difference	
	Value [MM$]	Standard Deviation	Value [MM$]	Standard Deviation	Average [MM$]	
0.3	1.3336	0.0913	0.9622	0.2739	0.3714	(***)
0.4	4.1661	0.1009	3.7974	0.4444	0.3687	(***)
0.5	7.8700	0.1590	7.4699	0.4199	0.4001	(***)
0.6	12.3323	0.1163	12.0323	0.4537	0.3000	(***)
0.7	17.2874	0.1611	17.0551	0.3417	0.2323	(***)
0.8	22.6086	0.1695	22.4004	0.5625	0.2082	(***)
0.9	28.1188	0.2037	27.8811	0.5671	0.2377	(***)
1.0	33.7497	0.2376	33.5685	0.6946	0.1812	(**)

2.4.2.2 SVGJ Model

Given the flexibility provided by Monte Carlo simulation, the general process in (2.28) may be easily addressed by SFPI-RO. Consider $(\kappa, \sigma, \rho) = (2.50, 0.40, 0.00)$ and jumps according to parameters in **Table 2.4**. Under the previous values, the expected price jump size per year is 12.5%, while the expected variance jump is 0.05 per year.

Table 2.4 Model specification under the SVGJ model

Jump in log-price only		Jump in variance only		Simultaneous and correlated jumps	
λ^y	0.50	λ^v	0.50	λ^c	0.50
μ_y	0.10	μ_v	0.05	μ_{cv}	0.05
σ_y	0.10			μ_{cy}	0.10
				σ_{cy}	0.10
				ρ_J	0.50

Figure 2.5 shows the critical prices delivered by SFPI-RO while **Table 2.5** displays the performance of these boundaries when valuing an initially open mine. For LSM we use the same code considered for stochastic variance without jumps.

Regarding the critical prices, the stability of the algorithm is maintained (e.g. smoothness, convergence, etc.). In terms of mine values, SFPI-RO again delivers higher values than LSM for almost every case. Moreover, compared with the case of stochastic variance without jumps, the results exhibit the same behavior in terms of standard deviation, while LSM exhibit considerably more volatile estimates. As observed for the positive correlation case in the previous section, it seems that as more volatile the model, more difficult is to accurately estimate the critical prices through LSM. In this case, the model dispersion is considerably broadened when adding different types of jumps in both the price and the variance process.

With the above results, the versatility and stability of SFPI-RO when different sources of uncertainty are treated at the same time are highlighted.

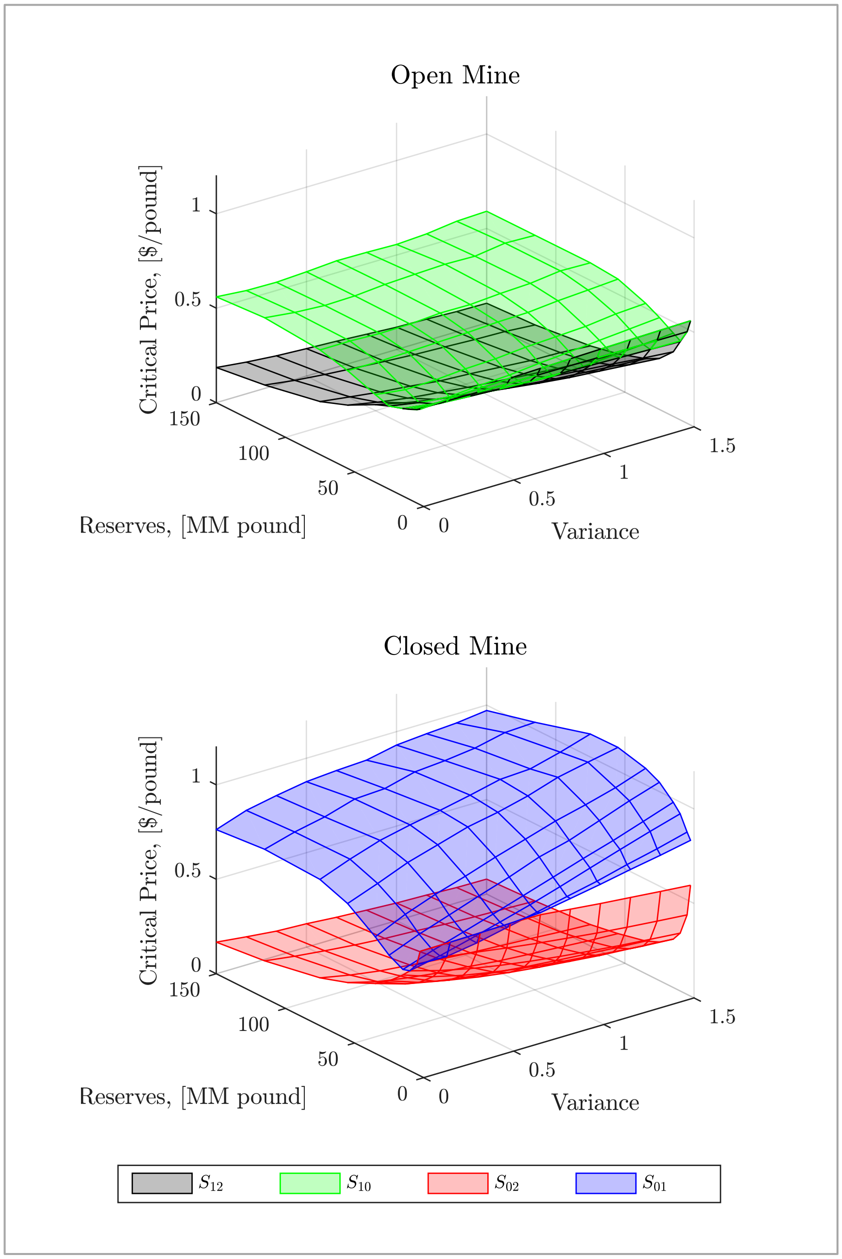

Figure 2.5 Critical prices under the SVGJ model

Table 2.5 Values [MM$] for an initially open mine under the SVGJ model
(Initial Variance: $V_0 = 0.08$)

Initial Price [$/pound]	Stochastic Variance with General Jumps					
	SFPI-RO		LSM		Difference	
	Value [MM$]	Standard Deviation	Value [MM$]	Standard Deviation	Average [MM$]	
0.3	2.6697	0.0915	2.0511	0.6942	0.6185	(***)
0.4	6.0125	0.1314	5.2523	1.3515	0.7602	(***)
0.5	10.0068	0.1569	9.0759	1.5914	0.9309	(***)
0.6	14.5188	0.1581	13.8741	1.5370	0.6446	(***)
0.7	19.5454	0.2105	18.6059	2.3526	0.9395	(***)
0.8	24.7212	0.2274	23.8186	2.7389	0.9026	(**)
0.9	30.0948	0.2159	29.1874	3.1336	0.9074	(**)
1.0	35.6375	0.2718	34.8249	2.9981	0.8126	(**)

2.4.3 SFPI-RO under Regime-Dependent Dynamics

As the last example of the scope of SFPI-RO, consider a firm which is a major player in the commodity market and therefore the price could be affected by the operating status of the firm through changes in supply. Then, we say the diffusion process is controlled by the operation strategy and the price parameters exhibit regime-dependency.

The problem could be addressed by approximating the system of variational inequalities (e.g. finite differences), but dimensionality drawbacks arise under multifactor processes. On the other hand, simulation-based methods like LSM could not solve the optimal switching problem under the above specification. When introducing regime-dependent parameters in the price process, the forward simulation used by LSM is not feasible because a decision policy is not known a priori. Instead, our method provides a simple and robust solution since we simulate under a given set of critical prices (even though they are not optimal). Thereby, our method allows us to widen the set of configurations for which the optimal switching problem may be solved in a straightforward fashion.

As an illustration, one may consider a price process with stochastic variance, such that variance is more volatile when the firm is closed. We use the same values of r, d, and θ. For the variance process, we consider $(\kappa, \sigma, \rho) = (2.50, 0.40, 0.00)$ when the operation is closed, while $(\kappa, \sigma, \rho) = (5.00, 0.10, 0.00)$ when it is open. **Figure 2.6** shows the critical prices delivered by SFPI-RO. Furthermore, **Figure 2.7** shows the evolution of the mine for a given trajectory of the price and variance process. Given the delivered critical prices, we obtain the evolution of the reserves in the mine and the cumulated wealth for that simulated path. We show the operating status over time as a colored background.

From **Figure 2.6**, note that the regions bounded by S_{12} and S_{10} for an open mine and by S_{02} and S_{01} for a closed one are considerably large. Therefore, we conclude the closed regime is particularly valuable for the proposed price dynamic. In fact, **Figure 2.7** shows that we can benefit from closed periods since variance could disperse more under that regime and then push the price to a higher value more quickly.

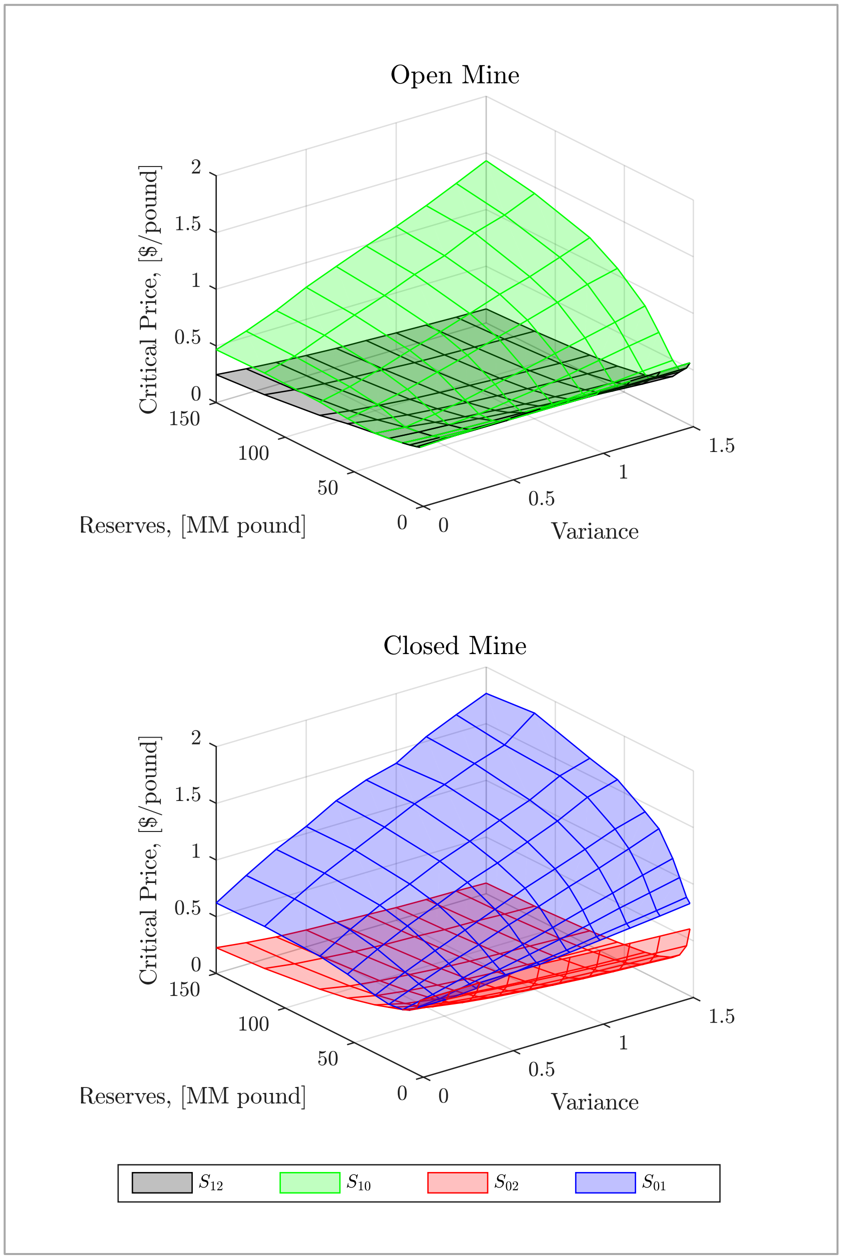

Figure 2.6 Critical prices under the regime-dependent SV model

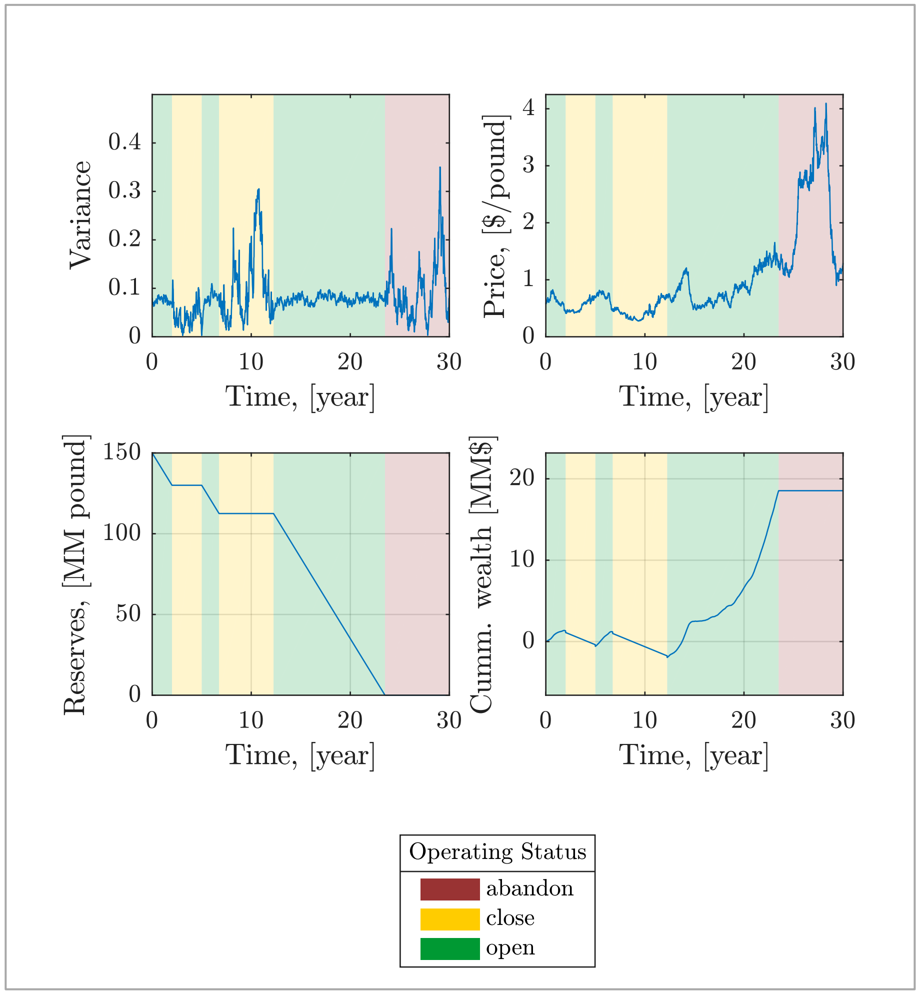

Figure 2.7 Evolution of the mine under the regime-dependent SV model

2.5 Extensions and Further Research

The next step is to generalize the applicability of SFPI-RO. According to **Section 2.2.2**, the algorithm assumes that the switching boundaries may be expressed in terms of one critical state variable, which satisfies a functional form with respect to the remaining variables. Under the model of Brennan & Schwartz (1985), this assumption is reasonable, and the critical variable is given by the commodity price. However, it is not known a priori under which conditions it is still valid. Therefore, further research should be focused on better understanding the switching boundaries in order to extend the methodology proposed by us.

As a first approach, Zervos et al. (2018) propose an interesting study about a similar model to that of Brennan & Schwartz (1985). They provide a comprehensive characterization of the critical prices under an infinite decision horizon and infinite reserves. Their explicit solution to (2.6) is given when the maintenance cost is zero[6], the closing (k_{10}) and opening cost (k_{01}) are positive, and the abandoning costs satisfy $k_{02} = k_{12} = K \in \mathbb{R}$. According to their solution, the critical prices take eight qualitatively different forms and two of them exhibit the behavior in **Figure 2.8** when the system is open.

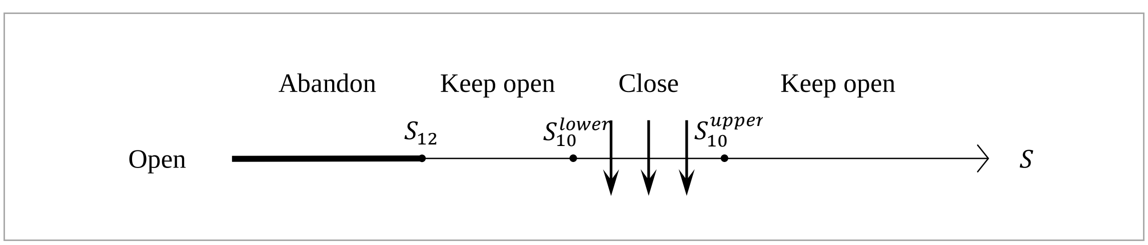

Figure 2.8 A special case for critical prices when the system is open

We refer to Case II.3 and Case III.2 defined by Zervos et al. (2018)

[6] As pointed out by the authors, when the maintenance cost is constant, the problem may be reformulated in terms of a zero-maintenance cost case.

Note that the requirement of functionality does not hold. Instead, under such a case we would expect an upper and a lower critical price for the regime transition $(0 \rightarrow 1)$ at a given state $(t, \mathbf{z})$. Hence, we face an important challenge: how to embrace the cases analyzed by Zervos et al. (where decision horizon, reserves level, and the number switching opportunities per year are infinite), in order to deal with a general formulation with additional state variables and finite reserves level, decision horizon, and number of switching opportunities. The answer will be the start point to formulate the algorithm when S_{10} is no longer a function of the remaining state variables.

Nevertheless, feasible extensions consider the introduction of new features and operational constraints to the mine of Brennan & Schwartz (1985), as well as new specifications for the commodity price process. The flexibility provided by Monte Carlo simulation allows that new features are easily addressed. For example, delays between the decision making and the realization of it could be considered, in the same way as in Carmona & Ludkovski (2008).

Finally, notice also that what has been developed up to now assumes a price process that exhibits constant returns to scale. Under this assumption, we derive an expression for the mine delta, but it would be interesting to explore other models. For example, a classical mean-reverting process (Schwartz, 1997) is given by

$$\frac{dS_t}{S_t} = \alpha(\kappa - \log S_t)dt + \sigma dW_t \tag{2.30}$$

which does not satisfy the aforementioned assumption. Accordingly, formulae (2.22) – (2.25) are no longer valid to run SFPI-RO.

2.6 Concluding Remarks

We propose the Simulated-Fixed Point Iteration Method for Real Options (SFPI-RO), a novel simulation-based method to directly address the optimal decision policy for real options modeled as an optimal switching problem. Starting from an initial guess of the decision policy (a set of switching boundaries), we iterate until optimality is reached through the Newton-Kantorovich Method.

Our algorithm is implemented for the classical copper mine of Brennan & Schwartz (1985), under different Markovian Dynamics for the commodity price process. In our numerical analysis, the method provides an accurate approximation of the switching boundaries when the underlying price follows a standard GBM. Moreover, the optimal policies produced by our algorithm are more profitable than those delivered by the widely-used Least-Squares Monte Carlo method when the commodity follows underlying dynamics with stochastic variance and jumps. Finally, the algorithm allows to obtain the critical prices under regime-dependent dynamics, which are not accessible for backward methods based on forward simulation schemes.

Our results show that SFPI-RO is a useful tool to solve the optimal switching problem under a wide variety of underlying stochastic dynamics. The algorithm is easy to implement and exhibits stability, robustness, and convergence when estimating critical prices under different sources of uncertainty. In addition to being convenient for a manager when operationalizing an investment strategy in the real world, SFPI-RO also leads to a robust and accurate valuation tool.

We conclude this section by noting that our method could also be adapted to handle other real option features, such as operational delays (Carmona & Ludkovski, 2008), and other production activities like electricity generation (Deng & Xia, 2005; Carmona & Ludkovski, 2008) or gas dome storage (Carmona & Ludkovski, 2010).

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APPENDICES

A Proof of Proposition 1

From Brennan & Schwartz (1985), the mine value is given by

$$
\begin{aligned}
H(t,\mathbf{x},i;\alpha) = \mathbb{E}\Bigg(&\int_t^T (q(S_u - a) - tax(S_u))\, e^{-R_u}\, \mathbb{1}_{\{I_u=1\}} du \\
&- \int_t^T f\, e^{-R_u}\, \mathbb{1}_{\{I_u=0\}} du - \sum_{t \le \tau_n \le T} k_{\iota_{n-1},\iota_n}\, e^{-R_{\tau_n}} \Bigg) \qquad \text{(A.1)} \\
tax(s) &= t_1 q s + \max\{t_2 q(s(1-t_1) - a), 0\} \\
R_u &= \int_t^u (r + \lambda_{I_v}) dv
\end{aligned}
$$

Let S be a diffusion process for the commodity price such that exhibits constant returns to scale and the initial state is $S_t = s$. When all cash-flows parameters (i.e. s, a, f, and k_{ij}) and the critical prices are rescaled by a factor β, the mine value is rescaled by the same factor. Therefore, we say H is homogenous of degree 1 and Euler's theorem for homogenous functions yields (A.2) for the mine delta.

$$
\Delta = \frac{\partial H}{\partial s} = \frac{H - aH_a - fH_f - \sum_{i,j} k_{ij} H_{k_{ij}}}{s} \qquad \text{(A.2)}
$$

Note that the above expression does not consider derivatives with respect to critical prices. From the formulation of the optimal switching problem, H depends on critical prices through the regime indicator function I_u. When a critical price S_c changes in dS_c, it would be required a price path within $S_c + dS_c$ at some time to modify I_u and, consequently, the mine value for that path. Since dS_c is close to zero, this is almost impossible even for just one price path. Therefore, one may assume that the mine value stays constant under differential changes in critical prices.

Regarding the remaining derivatives, the following is hold given f and $\{k_{ij}\}$ a set of constant parameters

$$fH_f = -\mathbb{E}\left(\int_t^T f\ e^{-R_u}\ \mathbb{1}_{\{I_u=0\}}du\right) \tag{A.3}$$

$$\sum_{i,j} k_{ij} H_{k_{ij}} = -\mathbb{E}\left(\sum_{t\le\tau_n\le T} k_{\iota_{n-1},\iota_n}\ e^{-R_{\tau_n}}\right) \tag{A.4}$$

Moreover, H_a is expressed as

$$H_a = \frac{\partial}{\partial a}\left\{\mathbb{E}\left(\int_t^T q(S_u - a)\ e^{-R_u}\ \mathbb{1}_{\{I_u=1\}}du\right)\right\} - \frac{\partial}{\partial a}\left\{\mathbb{E}\left(\int_t^T \max\{t_2 q(S_u(1-t_1) - a), 0\}\ e^{-R_u}\ \mathbb{1}_{\{I_u=1\}}du\right)\right\}$$

The first derivative is given by

$$\frac{\partial(\cdot)}{\partial a} = -\mathbb{E}\left(\int_t^T q e^{-R_u}\ \mathbb{1}_{\{I_u=1\}}du\right)$$

On the other hand, let $\varphi_{S_u}(s)$ be the $\mathcal{F}_t$-conditional density function of S_u. Thus, the second derivative is given by

$$\frac{\partial(\cdot)}{\partial a} = \frac{\partial}{\partial a}\left\{\mathbb{E}\left(\int_t^T \max\{t_2 q(S_u(1-t_1)-a),0\}\, e^{-R_u}\, \mathbb{1}_{\{I_u=1\}}du\right)\right\}$$

$$= \frac{\partial}{\partial a}\left\{\int_t^T \mathbb{E}\left[\max\{t_2 q(S_u(1-t_1)-a),0\}\, e^{-R_u}\, \mathbb{1}_{\{I_u=1\}}\right]du\right\}$$

$$= \frac{\partial}{\partial a}\left\{\int_t^T \left(\int_{a/(1-t_1)}^{\infty} t_2 q(s(1-t_1)-a)\, e^{-R_u}\, \mathbb{1}_{\{I_u=1\}}\, \varphi_{S_u}(s)ds\right)du\right\}$$

From de Leibniz integral rule, we have

$$= \int_t^T \left(\frac{\partial}{\partial a}\int_{a/(1-t_1)}^{\infty} t_2 q(s(1-t_1)-a)\, e^{-R_u}\, \mathbb{1}_{\{I_u=1\}}\, \varphi_{S_u}(s)ds\right)du$$

$$= -\int_t^T \left(\int_{a/(1-t_1)}^{\infty} t_2 q\, e^{-R_u}\, \mathbb{1}_{\{I_u=1\}}\, \varphi_{S_u}(s)ds\right)du$$

$$= -\int_t^T \left(\int_{a/(1-t_1)}^{\infty} t_2 q\, e^{-R_u}\, \mathbb{1}_{\{I_u=1\}}\, \varphi_{S_u}(s)ds\right)du$$

$$= -\int_t^T \left(\mathbb{E}\left(t_2 q\, e^{-R_u}\, \mathbb{1}_{\{I_u=1\}}\, \mathbb{1}_{\{S_u \geq a/(1-t_1)\}}\right)\right)du$$

$$= -\mathbb{E}\left(\int_t^T t_2 q\, \mathbb{1}_{\{S_u \geq a/(1-t_1)\}}\, e^{-R_u}\, \mathbb{1}_{\{I_u=1\}}du\right)$$

Finally,

$$H_a = -\mathbb{E}\left(\int_t^T q\left(1 - t_2 \mathbb{1}_{\{S_u \geq a/(1-t_1)\}}\right) e^{-R_u}\, \mathbb{1}_{\{I_u=1\}} du\right) \tag{A.5}$$

Replacing into (A.2), the delta of the mine value is given by

$$\frac{\partial H}{\partial s} = \frac{\mathbb{E}\left(\int_t^T S_u q(1 - t_1)\left(1 - t_2 \mathbb{1}_{\{S_u \geq a/(1-t_1)\}}\right) e^{-R_u} \mathbb{1}_{\{I_u=1\}} du\right)}{s} \tag{A.6}$$

B Initial Guess for Critical Prices under the Model of Brennan and Schwartz (1985)

In this section, we provide an initial guess for the critical prices, i.e., $\left\{S_{02}^{(0)}, S_{01}^{(0)}, S_{12}^{(0)}, S_{10}^{(0)}\right\}$. Let x be the number of switching opportunities per year, implying that $\Delta t = 1/x$ years is the time between consecutive opportunities. We consider the behavior of the mine at time $T - \Delta t$, assuming that the cash-flows of the period $[T - \Delta t, T]$ are paid at the beginning.

1. Initial critical price S_{02}. If $-k_{02} \geq -f\Delta t - k_{02}e^{-\rho_0 \Delta t}$, then the closure option is not optimal and to compute the critical price we compare opening against abandonment. Therefore, the initial critical price $S_{02}^{(0)}$ solves

$$-k_{01} + q\Delta t(S_{02} - a) - tax(S_{02})\Delta t - k_{12}e^{-\rho_1 \Delta t} = -k_{02} \qquad \text{(B.1)}$$

On the other hand, if $-k_{02} < -f\Delta t - k_{02}e^{-\rho_0 \Delta t}$, the abandoning option is worthless at any time[7], and the critical price is defined as $S_{02} = 0$ (even when the critical price is undefined, $S_{02} = 0$ guarantees that the abandoning option will never be exercised).

---

[7] Let τ be the remaining concession time. We note that the closed mine value H_0 is an increasing function with respect to τ. Then

$$-k_{02} < -f\Delta t - k_{02}e^{-\rho_0 \Delta t} \leq H_0(\tau), \qquad \forall \tau \geq \Delta t$$

In other words, if the abandoning option is worthless at $\tau = \Delta t$, it is also worthless at $\tau > \Delta t$. Thus, the assignment $S_{02} = 0$ at $\tau = \Delta t$ should be extended for all $\tau > \Delta t$.

2. Initial critical price S_{01}. As pointed out earlier, the closure option is not optimal when $-k_{02} \geq -f\Delta t - k_{02}e^{-\rho_0 \Delta t}$, and we conveniently set $S_{01} = S_{02}$. On the other hand, if $-k_{02} < -f\Delta t - k_{02}e^{-\rho_0 \Delta t}$, we compare the opening option against the keep closed option. Thus, the critical price $S_{01}^{(0)}$ is given by the solution to

$$-k_{01} + q\Delta t(S_{01} - a) - tax(S_{01})\Delta t - k_{12}e^{-\rho_1 \Delta t} = -f\Delta t - k_{02}e^{-\rho_0 \Delta t} \qquad \text{(B.2)}$$

3. Initial critical price S_{12} and S_{10}. If the mine is open at $t = T - \Delta t$, an approximation of the mine value is given by $q\Delta t(S - a) - tax(S)\Delta t - k_{12}e^{-\rho_1 \Delta t}$ (running cash-flows plus discounted abandoning cost at expiration). On the other hand, if the mine is closed or abandoned, the discounted cash-flows are $-f\Delta t - k_{02}e^{-\rho_0 \Delta t}$ (maintenance cost plus discounted abandoning cost at maturity) and $-k_{12}$ (immediate abandoning cost), respectively.

If $-k_{12} \geq -k_{10} - f\Delta t - k_{02}e^{-\rho_0 \Delta t}$, then the closure option is worthless, $S_{10}^{(0)} = S_{12}^{(0)}$ and $S_{12}^{(0)}$ is given by the solution to

$$q\Delta t(S_{12} - a) - tax(S_{12})\Delta t - k_{12}e^{-\rho_1 \Delta t} = -k_{12} \qquad \text{(B.3)}$$

On the other hand, if $-k_{12} < -k_{10} - f\Delta t - k_{02}e^{-\rho_0 \Delta t}$, the abandonment option is worthless, and we conveniently set $S_{12} = 0$[8]. Moreover, $S_{10}^{(0)}$ solves the following equation

$$q\Delta t(S_{10} - a) - tax(S_{10})\Delta t - k_{12}e^{-\rho_1 \Delta t} = -k_{10} - f\Delta t - k_{02}e^{-\rho_0 \Delta t} \qquad \text{(B.4)}$$

[8] By the same argument given in footnote 7, the abandonment option is worthless for the whole domain of the problem and so also the assignment $S_{12} = 0$.

C Finite Difference Method

From (2.6). the optimal value v_i of the optimal switching problem is given by

$$\min\left[rv_i - \frac{\partial v_i}{\partial t} - \mathcal{L}v_i - \psi_i, \quad v_i - \max_{j\neq i}\left(v_j - k_{ij}\right)\right] = 0$$
$$t \in [0,T], \;\; x \in \mathbb{R}^n, \;\; i \in \mathbb{I}_m \tag{C.1}$$

and it is solvable in a backward fashion by the following scheme

$$\begin{aligned} t = T \;\; &: \qquad v_i = h_i(\,\cdot\,) \\ t < T \;\; &: \qquad rv_i - \frac{\partial v_i}{\partial t} - \mathcal{L}v_i - \psi_i = 0, \\ & \qquad v_i \geq \max_{j\neq i}\left(v_j - k_{ij}\right) \end{aligned} \tag{C.2}$$

where $h_i(\,\cdot\,)$ is a terminal payoff function, which we assume equal to the abandoning cost $-k_{i2}$. Note that we start at the concession expiration and then we move backward in time. At each time, we solve the differential equation for each regime and then compare the continuation against switching to obtain the optimal decision and the corresponding value.

Consider a commodity mine under the model of Brennan & Schwartz (1985). The commodity price process satisfies a one-dimensional geometric Brownian motion, the initial reserves level is Q^*, and the concession time is T. Let $v = v_1(t,S,Q)$ and $w = v_0(t,S,Q)$ the value of an open and a closed mine. Then, the differential equation in (C.1) for v and w becomes

$$(r+\lambda_1)v - v_t - (r-d)Sv_S + qv_Q - \frac{1}{2}\sigma^2 S^2 v_{SS} - q(S-a) + tax(S) = 0 \quad \text{(C.3)}$$

$$(r+\lambda_0)w - w_t - (r-d)Sw_S - \frac{1}{2}\sigma^2 S^2 w_{SS} + f = 0 \quad \text{(C.4)}$$

A finite difference scheme is implemented considering a discretization of t, S, and Q, such that $t_i = i \cdot \Delta t$, $S_j = j \cdot \Delta S$, and $Q_k = k \cdot \Delta Q = k \cdot (q\Delta t)$, where $i = \{0, \dots, N_t = T/\Delta t\}$, $j = \{0, \dots, N_S = S_{\max}/\Delta S\}$, and $k = \{0, \dots, N_Q = Q^*/\Delta Q\}$. In the sequel, we refer only to function v, but the results are the same for w.

At each point over the discretized space, the mine value function is defined as $v_{i,j,k} = v(i \cdot \Delta t, j \cdot \Delta S, k \cdot \Delta Q)$. Moreover, the derivatives of v are approximated as follows

$$v_t = \frac{v_{i+1,j,k} - v_{i,j,k}}{\Delta t} \quad \text{(C.5)}$$

$$v_S = \frac{v_{i,j+1,k} - v_{i,j-1,k}}{2\,\Delta S} \quad \text{(C.6)}$$

$$v_{SS} = \frac{v_{i,j+1,k} - 2v_{i,j,k} + v_{i,j-1,k}}{(\Delta S)^2} \quad \text{(C.7)}$$

$$v_Q = \frac{v_{i,j,k} - v_{i,j,k-1}}{\Delta Q} \quad \text{(C.8)}$$

Replacing into equation (C.3), the PDE is formulated as a system of linear equations[9]

$$
\begin{aligned}
&\beta_j \cdot v_{i,j,k} = \delta_{i,j,k} && j = 0 \\
&\alpha_j \cdot v_{i,j-1,k} + \beta_j \cdot v_{i,j,k} + \gamma_j \cdot v_{i,j+1,k} = \delta_{i,j,k} && j = 1, \dots, N_S - 1 \\
&(\alpha_j - \gamma_j) \cdot v_{i,j-1,k} + (\beta_j + 2\gamma_j) \cdot v_{i,j,k} = \delta_{i,j,k} && j = N_S
\end{aligned}
\qquad \text{(C.9)}
$$

where the coefficients are defined as

$$
\begin{aligned}
\alpha_j &= \frac{j}{2}(r-d)\Delta t - \frac{1}{2}\sigma^2 j^2 \Delta t \\
\beta_j &= 1 + \frac{q\Delta t}{\Delta q} + (r + \lambda_1)\Delta t + \sigma^2 j^2 \Delta t \\
\gamma_j &= -\frac{j}{2}(r-d)\Delta t - \frac{1}{2}\sigma^2 j^2 \Delta t \\
\delta_{i,j,k} &= v_{i+1,j,k} + \left(\frac{q\Delta t}{\Delta q}\right) v_{i,j,k-1} + q(j\Delta S - a)\Delta t - tax(j\Delta S)\Delta t
\end{aligned}
$$

[9] When $S = 0$, equation (C.3) is formulated as

$$(r - \pi + \lambda_1)v - v_t + qv_Q + qa = 0$$

On the other hand, when $S \to \infty$, the optimal strategy is to keep always open until one of the following events happen: (1) reserves are exhausted, or (2) concession time is over. Furthermore, both the price process and the cash-flows rate when open are linear with respect to S. Then, it shall be assumed $v_{SS} = 0$ as $S \to \infty$ and, consequently, equation (C.3) is approximated as

$$(r + \lambda_1)v - v_t - (r - d)Sv_S + qv_Q - q(S - a) + \tau ax = 0$$

Finally, since $j = N_S + 1$ is an unavailable point, we use a backward approximation for the first derivative at $j = N_S$

$$v_S = \frac{v_{i,j,k} - v_{i,j-1,k}}{\Delta S}$$

In the same way, we have the following system of linear equations for the closed mine

$$\begin{aligned}
&\beta_j \cdot w_{i,j,k} = \delta_{i,j,k} && j = 0 \\
&\alpha_j \cdot w_{i,j-1,k} + \beta_j \cdot w_{i,j,k} + \gamma_j \cdot w_{i,j+1,k} = \delta_{i,j,k} && j = 1, \dots, N_S - 1 \\
&\left(\alpha_j - \gamma_j\right) \cdot w_{i,j-1,k} + \left(\beta_j + 2\gamma_{jh}\right) \cdot w_{i,j,k} = \delta_{i,j,k} && j = N_S
\end{aligned} \tag{C.10}$$

where the coefficients are defined as

$$\begin{aligned}
\alpha_j &= \frac{j}{2}(r-d)\Delta t - \frac{1}{2}\sigma^2 j^2 \Delta t \\
\beta_j &= 1 + (r + \lambda_0)\Delta t + \sigma^2 j^2 \Delta t \\
\gamma_j &= -\frac{j}{2}(r-d)\Delta t - \frac{1}{2}\sigma^2 j^2 \Delta t \\
\delta_{i,j,k} &= w_{i+1,j,k} - f
\end{aligned}$$

The above finite difference formulation shall be referred as the standard scheme, and it is solved according to (C.2). Unless stated otherwise, when we refer to results obtained with FDM we consider this scheme, letting $N_S = 25{,}000$ and $N_t = 64 \cdot T$ (64 time-steps per year). Moreover, the S-axis is truncated at a price level $S_{\max}$ defined as

$$S_{\max} = \max\{25 \cdot a,\ 99\text{th percentil of } S(T)\} \tag{C.11}$$

where a is the production cost rate.

As an alternative to this standard version, one may propose the following formulations, depending on how the PDE at $j = N_S$ is addressed:

Alternative 1: Backward scheme. Instead of assuming the behavior of v as $S \to \infty$ (see footnote 9), we use the following backward approximations for v_S and v_{SS} at $j = N_S$

$$v_S = \frac{v_{i,j,k} - v_{i,j-1,k}}{\Delta S} \tag{C.12}$$

$$v_{SS} = \frac{v_{i,j,k} - 2v_{i,j-1,k} + v_{i,j-2,k}}{(\Delta S)^2} \tag{C.13}$$

Then, the equation for v at $j = N_S$ is approximated as

$$\begin{aligned}\left(\frac{1}{2}\alpha_j + \frac{1}{2}\gamma_j\right) \cdot v_{i,j-2,k} + (-2\gamma_i) \cdot v_{i,j-1,k} + \left(\frac{1}{2}\alpha_j + \beta_j + \frac{5}{2}\gamma_j\right) \\ \cdot v_{i,j,k} = \delta_{i,j,k}, \quad j = N_s\end{aligned} \tag{C.14}$$

When closed, the derivatives of the mine value are approximated in the same way.

Alternative 2: Limiting scheme. As pointed out in footnote 9, we expect the optimal operating policy is to keep always open an already open mine when $S \to \infty$. Under such a strategy, the mine value is given by

$$\begin{aligned} v^*(t,S,Q) &= \mathbb{E}\left(\int_t^\theta (q(S_u - a) - (tax)_{lim})\, e^{-\rho_1 u}\, du\right) \\ &= qS(1-t_1)(1-t_2)\left(\frac{1 - \mathrm{e}^{-(d+\lambda_1)\theta}}{d + \lambda_1}\right) \\ &\quad - qa(1-t_2)\left(\frac{1 - \mathrm{e}^{-(r+\lambda_1)\theta}}{r + \lambda_1}\right) \end{aligned} \tag{C.15}$$

where $\theta = \min\{T, t + Q/q\}$ is the instant when one of the following events occur: (1) reserves are exhausted, or (2) concession time is over. Note that the tax structure tends to $(tax)_{lim} = t_1 qs + t_2 q(s(1 - t_1) - a)$ when $S \to \infty$.

Assuming $N_S\, \Delta S$ is large enough, the mine value when open is arbitrary set at the limiting value provided in (C.15).

$$v_{i,N_S,k} = v^*(i\, \Delta t, N_S\, \Delta S, k\, \Delta Q) \tag{C.16}$$

If the mine is initially closed, the optimal decision at $S = N_S\, \Delta S$ is to open immediately and the closed mine value is given by

$$w_{i,N_S,k} = -k_{01} + v^*(i\, \Delta t, N_S\, \Delta S, k\, \Delta Q) \tag{C.17}$$

D Least-Squares Monte Carlo Method

The Dynamic Programming Principle for the optimal switching problem is formulated as

$$\begin{aligned} v_i(t, \mathbf{x}) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}\Bigg[\int_t^\theta \psi_{I_u}(\mathbf{X}_u)\, e^{-\rho(u-t)}\, du - \sum_{t \le \tau_n \le \theta} k_{\iota_{n-1},\iota_n}\, e^{-\rho(\tau_n - t)} \\ + v(\theta, \mathbf{X}_\theta, I_\theta)\, e^{-\rho(\theta - t)}\Bigg] \\ t \in [0,T],\ \ \theta \in [t,T],\ \ x \in \mathbb{R}^n,\ \ i \in \mathbb{I}_m \end{aligned} \tag{D.1}$$

Consider a discretization of the time interval $[0,T]$ such that such that $t_n = n\,\Delta t$, where Δt is the time between consecutive switching opportunities and $n = \{0, \dots, N_t = T/\Delta t\}$. Under the triangular condition for switching costs, the DPP becomes in the following backward formulation (see Section 4 in Gassiat et al., 2012)

$$\begin{aligned} t = T \ &: \qquad v_i(T, \mathbf{x}) = h_i(\,\cdot\,) \\ t < T \ &: \qquad v_i(t_n, x) = \max_{j \in \mathbb{I}_m}\{-k_{ij} + \mathbb{E}[\mathrm{C}_j(t_n, X_{t_n})]\} \end{aligned} \tag{D.2}$$

where $h_i(\,\cdot\,)$ is a terminal payoff function, which we assume equal to the abandoning cost $-k_{i2}$, and

$$\mathrm{C}_j(t_n, \mathbf{x}_{t_n}) = \int_{t_n}^{t_{n+1}} \psi_j(\mathbf{X}_u) e^{-\rho_j(u - t_n)}\, du + v_j(t_{n+1}, \mathbf{X}_{t_{n+1}}) e^{-\rho_j \Delta t} \tag{D.3}$$

is the discounted cash-flow of continuation under regime j, when the initial state is $\mathbf{x}_{t_n}$.

To be precise, one may approximate the profit from t_n to t_{n+1} (stochastic integral term) when simulating the process. But, when Δt is small enough, it may be reduced to a deterministic term given by

$$\int_{t_n}^{t_{n+1}} \psi_j(\mathbf{X}_u) e^{-\rho_j(u-t_n)}\, du \simeq \psi_j(\mathbf{x}_{t_n}) \Delta t \tag{D.4}$$

Consider a commodity mine under the model of Brennan & Schwartz (1985). The commodity price process satisfies a one-dimensional geometric Brownian motion, the initial reserves level is Q^*, and the concession time is T. Let $v = v_1(t, S, Q)$ and $w = v_0(t, S, Q)$ the value of an open and a closed mine. Since the time is discretized, there is a finite number of feasible states of reserves, say $Q = \{0, q\Delta t, \dots, Q^*\}$, where q is the extraction rate.

Moreover, the continuation value C_j becomes

$$\begin{aligned}
&\mathrm{C}_0(t_n, s_{t_n}, Q) = -\int_{t_n}^{t_{n+1}} f\, e^{-\rho_0(u-t_n)}\, du + v_0(t_{n+1}, s_{t_n+1}, Q)\, e^{-\rho_0 \Delta t} \\
&\mathrm{C}_1(t_n, s_{t_n}, Q) = \int_{t_n}^{t_{n+1}} \psi_1(s_u) e^{-\rho_1(u-t_n)}\, du + v_1(t_{n+1}, s_{t_n+1}, Q - q\Delta t) e^{-\rho_1 \Delta t} \\
&\mathrm{C}_2(t_n, s_{t_n}, Q) = 0
\end{aligned} \tag{D.5}$$

where $\psi_1(s) = q(s-a) - tax(s)$.

For a numerical valuation of the mine value, we simulate a set of price paths and then recursively compute v and w from time T to 0 according to (D.2). Moreover, we compute the mine value for every state of reserves at every time step. Since $C_1(t_n, s_{t_n}, Q)$ depends on $v_1(t_{n+1}, s_{t_n+1}, Q - q\Delta t)$, the algorithm goes from Δq to Q.

The conditional expectation in (D.2) may be approximated with a regression of realized values of $\mathrm{C}_j\big(t_n, s_{t_n}, Q\big)$ on the information available up to t_n, i.e., s_{t_n}. Then, under a one-dimensional diffusion process, we have

$$\mathbb{E}\big[\mathrm{C}_j\big(t_n, s_{t_n}, Q\big)\big] \simeq \widehat{\mathbb{E}}\big[\mathrm{C}_j\big(t_n, s_{t_n}, Q\big)\big] = \sum_{k=1}^{M} \alpha_k B_k(s_{t_n}) \qquad \text{(D.6)}$$

where $\{B_k(s)\}$ is a basis of M functions and α_k are the regression coefficients. After that, the optimal decision is made considering the regression expectation $\widehat{\mathbb{E}}\big[\mathrm{C}_j\big(t_n, s_{t_n}, Q\big)\big]$ for each regime j according to (D.2). When the optimal transition is defined, the value $v_j\big(t_n, s_{t_n}, Q\big)$ at each simulated trajectory is computed with the realized value $\mathrm{C}_j\big(t_n, s_{t_n}, Q\big)$. If the expected value is used instead of the realized value, we induce an upward positive bias as pointed out by Longstaff & Schwartz (2001).

The most relevant decision is about regressor functions. This is a subject of extensive literature and it is possibly what determines the estimation accuracy. Discussions about this choice may be revised in Cortazar et al. (2008), Carmona & Ludkovski (2008), and Carmona & Ludkovski (2010). In our implementation, the first three powers of S, and three European options were used[10]. Our choice of regressors is based on the usual practice, adding functions with financial sense in the spirit of those proposed by Andersen & Broadie (2004). European options are used because we intuit these may resemble the shape of the dependent variable in the linear regression.

[10] European option on the commodity, maturity at Δt, and strike price in $\{0.5a, a, 1.5\,a\}$, where a is the production cost.

E Extended Version of SFPI-RO under Geometric Brownian Model

We solve the commodity mine considered by Brennan & Schwartz (1985) with finite reserves, $Q^* = 150$ million pounds, finite concession time, $T = 30$ years, and $x = 4$ switching opportunities per year. Our implementation considers M = 250,000 paths for each critical price, convergence tolerance $\epsilon = 1\%$, and the four grid specifications illustrated in **Figure E.1**. We exhibit the critical prices estimated with SFPI-RO in **Figure E.2** to **Figure E.5.** We show cross-sections of the decision policy across the time-to-maturity axis (top row) and across the reserves axis (bottom row). Note that **Figure E.5** replicates the critical prices shown in **Section 2.4.1.1**, but adding the nodes estimated by the algorithm.

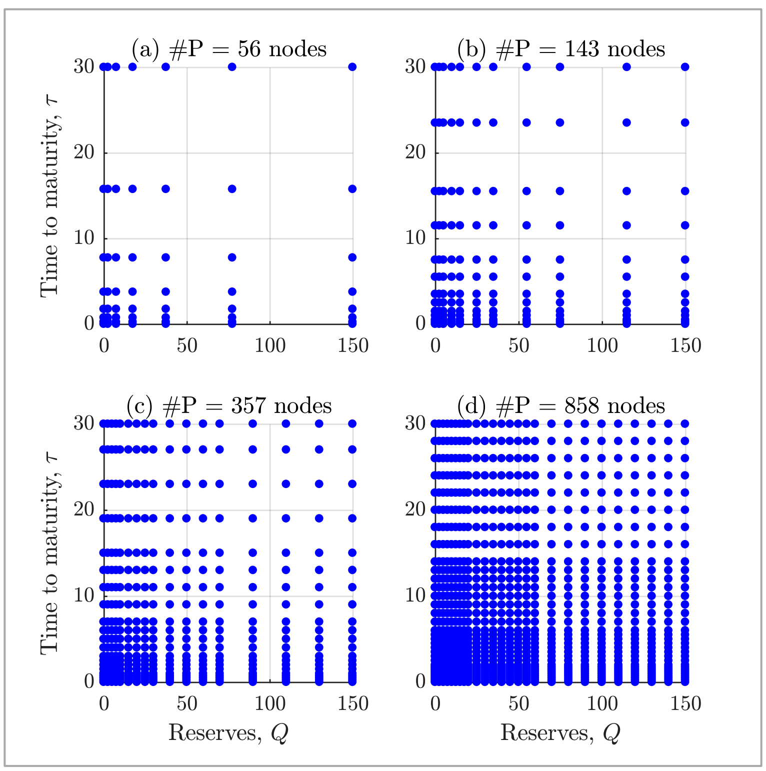

Figure E.1 Four mesh specifications for the reserves and time axis

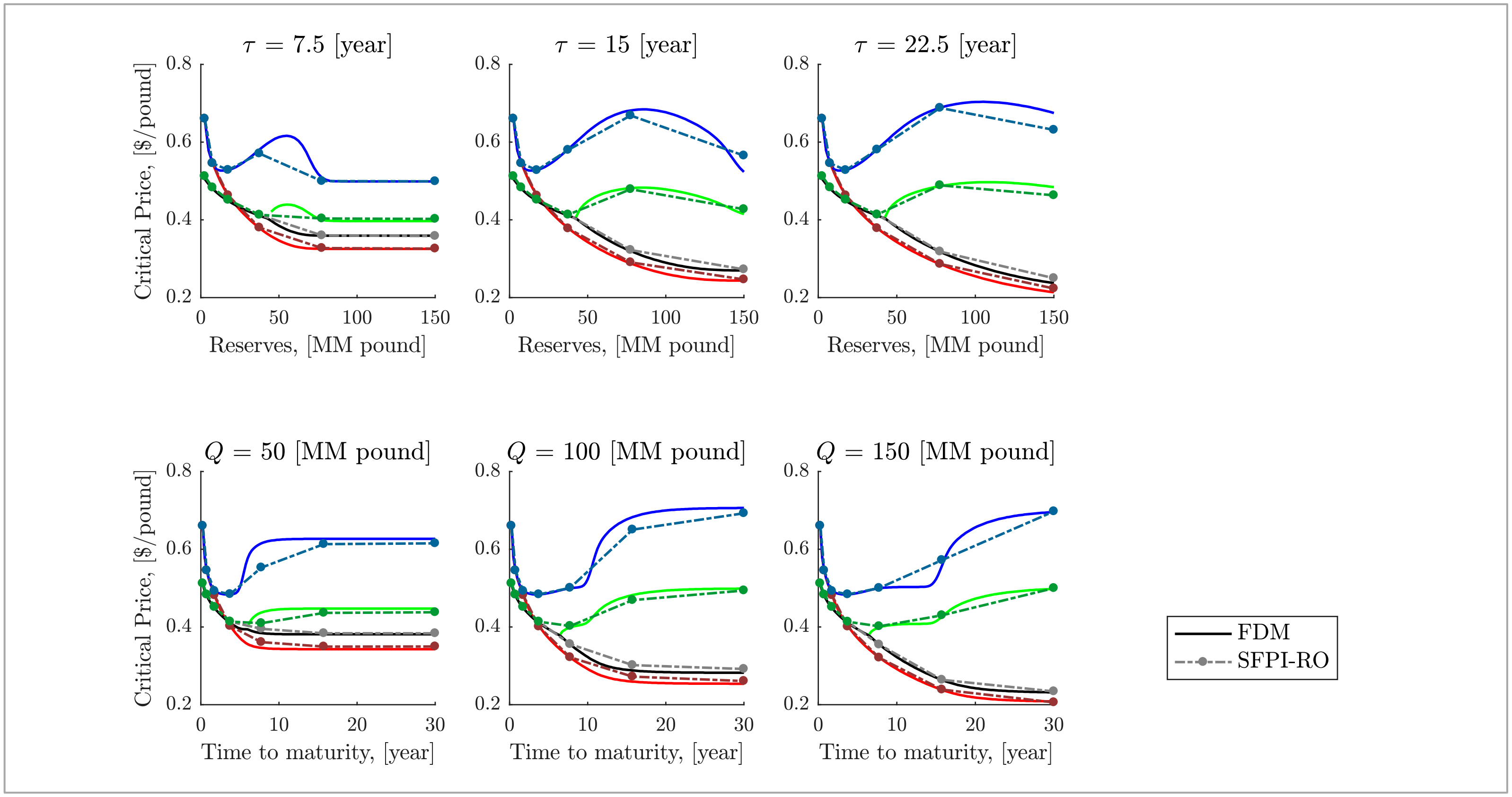

Figure E.2 Cross-sections of critical prices under a one-dimensional GBM (mesh with $\#\mathcal{P}$ = 56 nodes)

Color legend for critical prices: S_{02} = red; S_{01} = blue; S_{12}= black; S_{10} = green

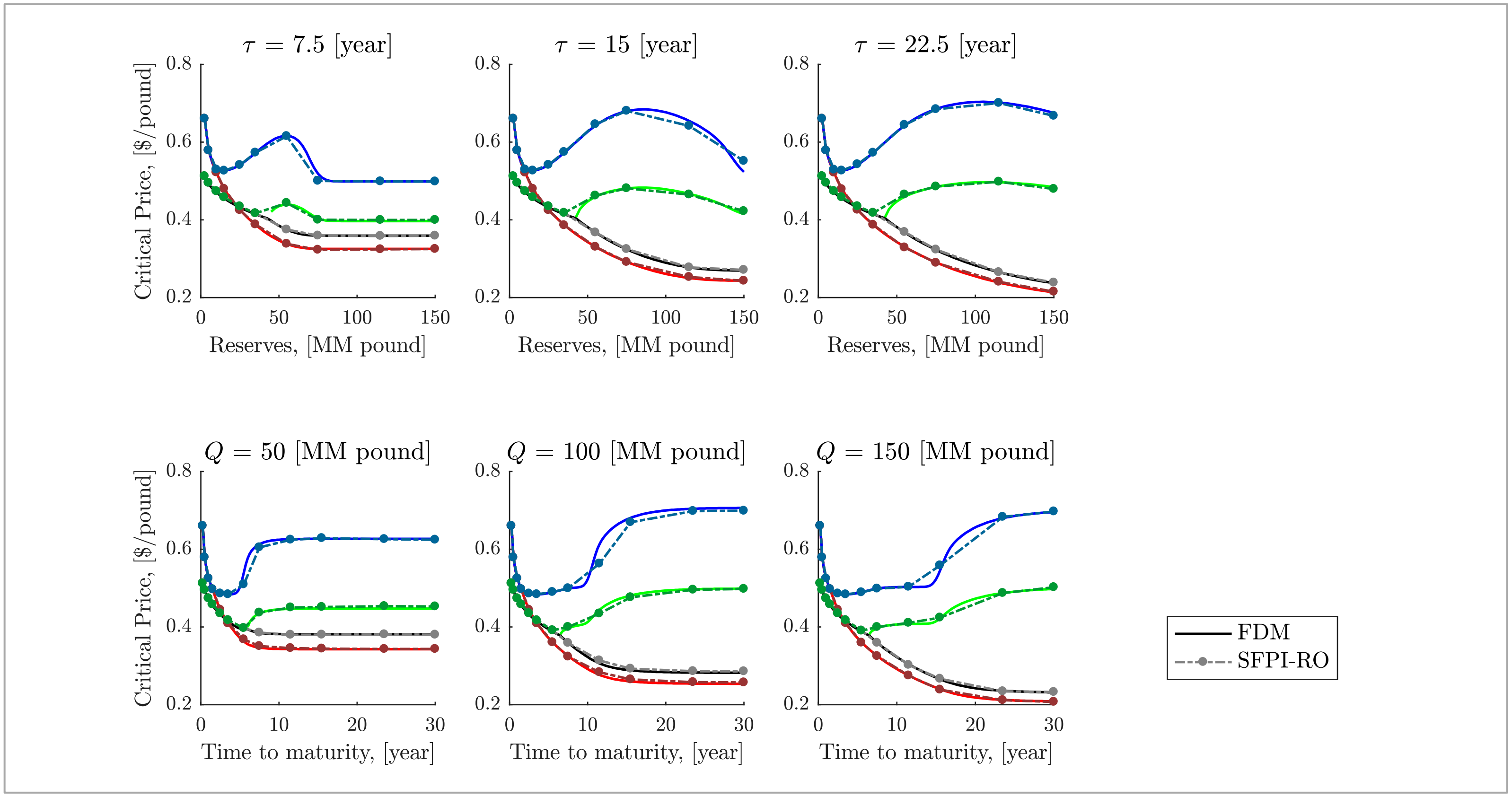

Figure E.3 Cross-sections of critical prices under a one-dimensional GBM (mesh with $\#\mathcal{P}$ = 143 nodes)

Color legend for critical prices: S_{02} = red; S_{01} = blue; S_{12}= black; S_{10} = green

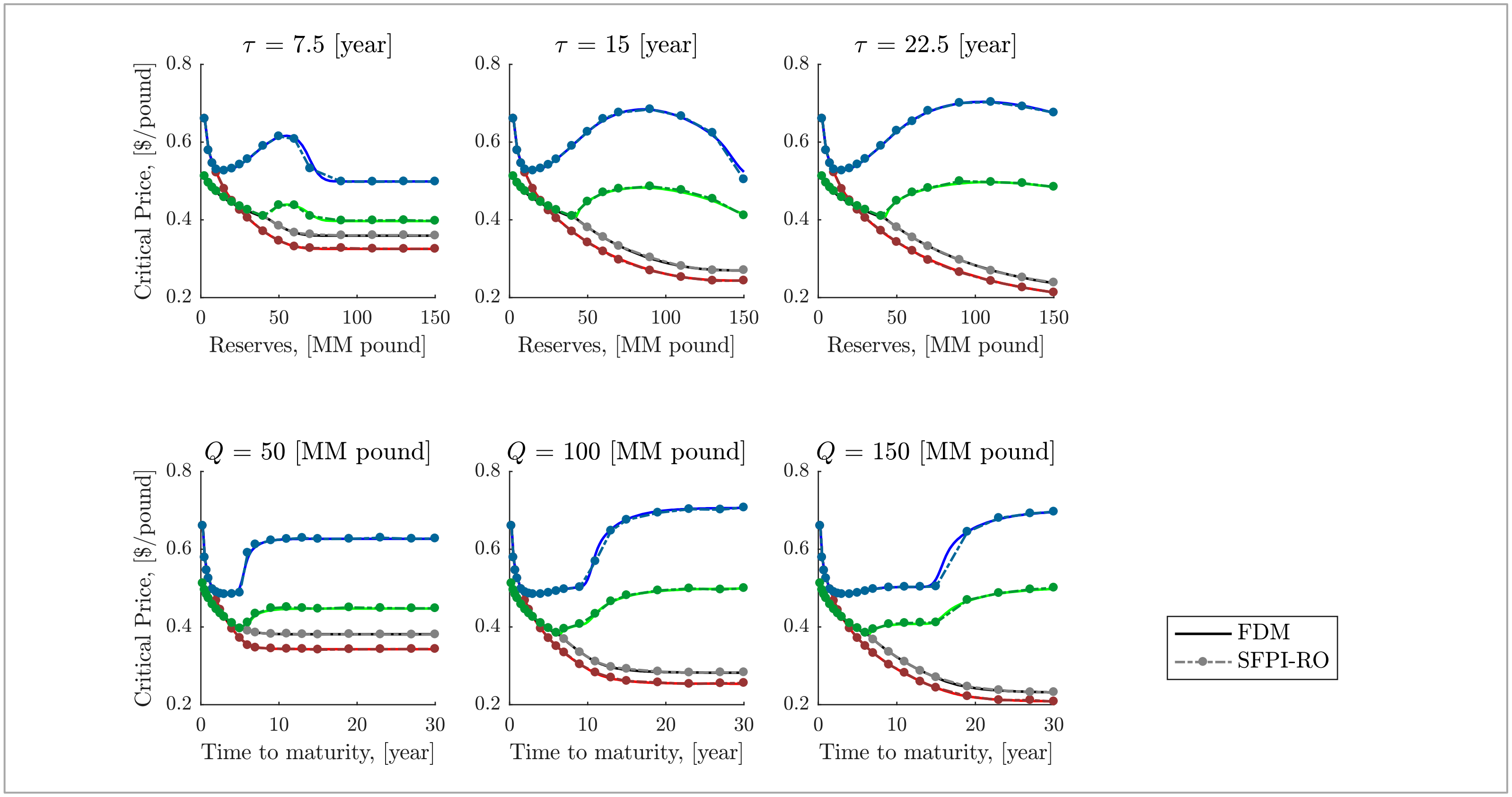

Figure E.4 Cross-sections of critical prices under a one-dimensional GBM (mesh with $\#\mathcal{P}$ = 357 nodes)

Color legend for critical prices: S_{02} = red; S_{01} = blue; S_{12}= black; S_{10} = green

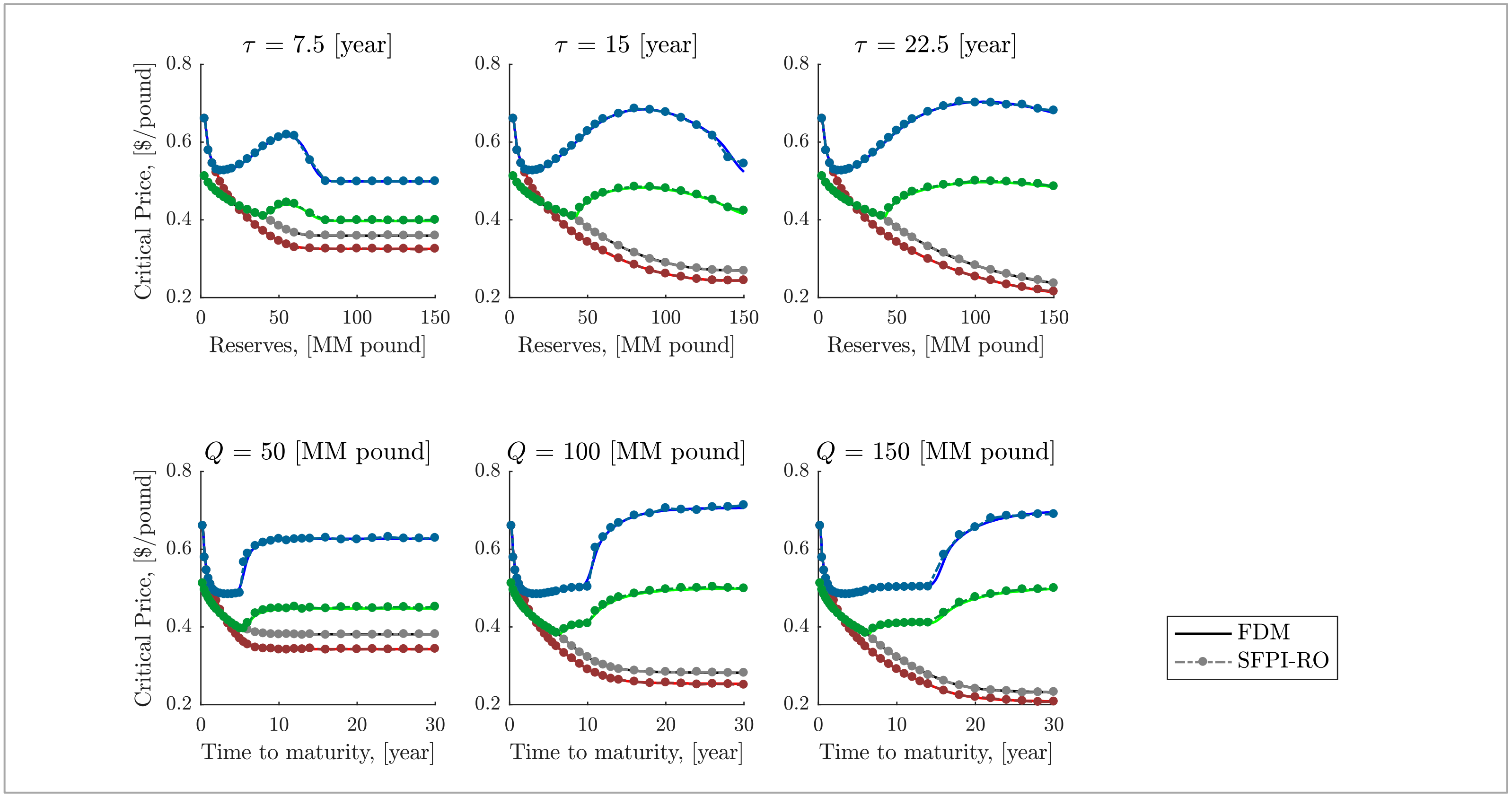

Figure E.5 Cross-sections of critical prices under a one-dimensional GBM (mesh with $\#\mathcal{P}$ = 858 nodes)

Color legend for critical prices: S_{02} = red; S_{01} = blue; S_{12}= black; S_{10} = green