PONTIFICIA UNIVERSIDAD CATOLICA DE CHILE
SCHOOL OF ENGINEERING

A NEW ITERATIVE METHOD FOR PRICING AMERICAN OPTION

LEONARDO ALEXANDER MEDINA VERGARA

Thesis submitted to the Office of Research and Graduate Studies
in partial fulfillment of the requirements for the degree of
Master of Science in Engineering

Advisor:
GONZALO CORTÁZAR

Santiago de Chile, August 2013

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Dedicated to the memory of my mother, Kenny.

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ABSTRACT

In this thesis it is proposed a novel, simple, fast and accurate iterative algorithm for pricing American options based on a direct solution of the early exercise representation. The performance of the proposed iterative method is compared with other existing numerical schemes that have been studied and presented in the literature. A thorough numerical analysis of existing methodologies is conducted and find that the proposed iterative procedure perform significantly better. The proposed method is also found to be stable, robust, and converges monotonically. Additionally, it is showed that the efficiency of the method can be improved further through the use of Richardson extrapolation. The method is well suited for parallel implementations that take advantage of the multi-core processors and graphic processing units.

Keywords: American Options, Iterative Methods, Numerical Methods, Parallel Computing

RESUMEN

En esta tesis es propuesto un novedoso, simple, rápido y preciso método iterativo para valorizar opciones Americanas, el cual se basa en una solución directa del precio crítico. El rendimiento del método iterativo propuesto es comparado con otras aproximaciones numéricas ampliamente estudiadas en la literatura. Un análisis numérico exhaustivo de las principales metodologías se lleva a cabo. Las puebas arrojan que el método propuesto presenta un desempeño significativamente mejor. Además es estable, robusto y converge monótonamente. Del mismo modo, se muestra que la eficiencia del método se puede mejorar aún más mediante el uso de la extrapolación de Richardson. El método es muy adecuado para ser programado en forma vectorizada y paralela, de tal forma de poder sacar ventaja no solo a los procesadores multi-core sino que también a los procesadores de tarjetas gráficas.

Palabras Claves: Opcines Americanas, Métodos Iterativos, Métodos Numéricos, Cálculo Paralelo

1. ARTICLE BACKGROUND

1.1. Introduction

Options have been object of study for decades. Many efforts have been made looking for analytical solutions for all style-options. On the seminal articles of Merton (1973) and Black & Scholes (1973), an analytical formulae for European-style options was successfully derived. American options have also been well studied since then. Unfortunately, no analytical solution have been found for American-style options. American options differ from its European counterpart in that American-style options allow early exercise any time before maturity. This particular feature leads to theoretical difficulties to value the option and find the closed-form formulae since an optimal exercise boundary has to be determined as part of the solution. Following this idea, Kim (1990) derived an integral equation to determine the early exercise boundary for American options on dividend paying assets. In his article, he also showed that the American option price is equal to the corresponding European price plus an early exercise premium, which depends on the optimal exercise boundary. This finding was also derived and proved, using different approaches, by Jacka (1991) and Carr, Jarrow, & Myneni (1992). Despite of their contribution, approximation techniques have to be applied in order to price the option. Thus, having no analytical solution to the American option pricing problem, have encouraged many researchers to come up with alternatives numerical methods to price American-style options concerning not only about the accuracy of pricing but also the computing time.

In this thesis a novel functional iterative method for pricing American options is introduced. The method is based on a fast and accurate solution of the early exercise representation derived by Kim (1990) for American options. Furthermore, exhaustive studies of the convergence and performance of the proposed method are conducted. This is also tested against the main numerical approximation methods widely used, comparing their speed and accuracy. The proposed method is found stable, with fast convergence and having the best speed-accuracy trade-off.

The rest of this chapter is structured as follows: Section 1.2 states the main objectives pursued in this thesis; Section 1.3 presents a literature review of the main theoretical frameworks and methods for American option pricing; and Section 1.4 presents the future research. Following this, Chapter 2 contains the main article of this thesis. Within this, Section 2.1 presents a literature review in conjunction with the novelty of the proposed method; Section 2.2 introduces the theoretical framework and recall some standard results of American option pricing theory. Section 2.3 presents the proposed methodology to price the American options. Section 2.4 describes in detail the numerical implementation of the method to price the American options. In Section 2.5 a battery of numerical tests is performed showing the speed-accuracy trade-off of the proposed procedure compared to others. Section 2.6 finally concludes.

1.2. Main Objectives

The goal of this thesis is to present a novel functional iterative method for pricing American options based on the integral equations of Kim (1990). This method, instead of calculating the points of the critical prices in a sequential way as the traditional method does, computes the early exercise boundary in a parallel procedure which has a flavour of a Newton-Raphson iteration. Additionally, this paper has two main objectives in order to demonstrate the power of speed and accuracy that the proposed method has:

The first objective is to carry out numerical tests in order to illustrate the convergence property and stability of this numerical procedure. Up to now, many methods calculate properly the American option price. In some cases stability and convergence of these methods are not demonstrated and lead to miss pricing when some parameters are set. In order to avoid this difficulty extensive numerical tests are proposed.

The second objective is to compare the performance of the proposed method with others widely used in order to establish the speed-accuracy domain. For this purpose, several methods were programmed and compare. Within this objective, the functional iterative method will be implemented using different approaches such as approximate integrals by using either the trapezoidal rule or the Gauss-Kronrod rule, and use Richardson extrapolation.

1.3. Literature Review

1.3.1. The Free Boundary Problem

The American put is a derivative instrument that gives the right to the holder to sell an established underlying asset at a specified strike price K within a specified period of time T. Unlike the European put, the American counterpart can be exercised any time until maturity. This important feature of American options makes more challenging the valuation process.

Merton (1973) has shown that the price $W(S, T-t; B(\cdot))$ of a contingent claim, satisfies the following partial differential equation (PDE):

$$\frac{1}{2}\sigma^2 S^2 W_{SS} + (r-\delta)SW_S - rW + W_t = 0, \tag{1.1}$$

where r represents the risk-less rate, q the dividend rate, σ the volatility and S the asset price. The partial differential equation (1.1) applies to any claim whose payoff depends on the asset price S. The early exercise boundary is represented by $B(\cdot)$, where critical asset prices $B(t)$ in $B(\cdot)$ are defined for $t \in [0, T]$, and $B(t_1) \leqslant B(t_2)$ for $t_1 \leqslant t_2$ where $t_i \in [0, T]$. $B(\cdot)$ defines two regions. The region above $B(t)$, also known as *continuation region*, defines the asset prices S that make it optimal to keep the American option alive. The region bellow $B(t)$ including the boundary is known as *stopping region*, and it is where it is optimal to exercise the American put. In order to determine the value of the live American Put $W(S, t; B(\cdot))$, it is necessary to define the terminal and boundary conditions:

$$\lim_{t \to T} W\left(S, T-t; B(\cdot)\right) = \max\left[0, (K-S)\right], \tag{1.2}$$

$$\lim_{S \to B(t)} W\left(S, T-t; B(\cdot)\right) = K - B(t), \tag{1.3}$$

$$\lim_{S \to \infty} W\left(S, T-t; B(\cdot)\right) = 0, \tag{1.4}$$

$$\lim_{S \to B(t)} W_S = -1. \tag{1.5}$$

Equation (1.2) refers to the value at maturity of an American Put that has not been exercised. Equation (1.3) specifies the value of an American put when the underlying asset is crossing the early exercise boundary and the put is exercised. It also implies that the American put price is continuous across the exercise boundary. The boundary condition (1.4) is known as *value matching condition* and reflects the fact that if the underlying asset increases without limit, it will never fall back within a finite time period, implying that the American put will not be exercised and therefore it will have no value. The boundary condition (1.5) is known as *high contact condition* and ensures the optimality of the exercise boundary. Equations (1.4) and (1.5) are jointly referred as the *smooth fit conditions*. These conditions ensures that the premature exercise of the American put on the early exercise boundary, will be optimal and self-financing. The partial differential equation (1.1), subject to the conditions (1.2) to (1.5), is known as *the free boundary problem*. Having the definition of $W(S, T-t; B(\cdot))$, the price function of the American put can be separated as a live option price and exercise price as:

$$P(S_t, T-t) = \begin{cases} W(S_t, T-t; B(\cdot)) & \text{if} \quad S_t > B(t) \\ K - S_t & \text{if} \quad S_t \leqslant B(t) \end{cases} \tag{1.6}$$

The reason that $P(S_t, T-t)$ is a piecewise function is because as long as the option is kept alive, $W(S, T-t; B(\cdot))$ holds, and $P(S_t, T-t)$ satisfies the PDE. Once the asset price S_t cross bellow the early exercise boundary, the option is exercised and the payoff is $K - S_t$, and the PDE is not longer satisfied by $P(S_t, T-t)$. Hence, this separation makes it easier to define $P(S_t, T-t)$ by focusing on the value of live American puts.

1.3.2. The integral equation of the early exercise boundary

In order to solve the free boundary problem and determine the live American put value $W(S, T-t; B(\cdot))$, Kim (1990) used the risk-neutral valuation of Cox & Ross (1976) and assumed that the stock price follows a geometric Brownian motion given by:

$$dS = (r - q)Sdt + \sigma SdZ \tag{1.7}$$

where the term dZ denotes increments on a standard Wiener process. He came up with a continuous-time extension of the discrete-time framework proposed by Geske & Johnson (1984), allowing early exercise at any point in continuous time as follows:

$$\begin{aligned} W(S,T;B(\cdot)) = p(S,T) + \int_0^T &[rKe^{-rt}N(-d_2(S,t;B(t))) \\ &- \delta Se^{-qt}N(-d_1(S,t;B(t)))]dt, \end{aligned} \tag{1.8}$$

where $p(S,t)$ represent the Black and Scholes/Merton European put pricing formula given by:

$$p(S,T) = Ke^{-rT}N(-d_2(S,T;K)) - Se^{-qT}N(-d_1(S,T;K)) \tag{1.9}$$

and where $N(\cdot)$ represent the standard cumulative normal distribution function, and

$$d_1(x,t;y) = \frac{\ln(x/y) + (r - q + \sigma^2/2)t}{\sigma\sqrt{t}},$$

$$d_2(x,t;y) = d_1(x,t;y) - \sigma\sqrt{t}.$$

Equation (1.8) expresses the value of the live American put as the sum of an European put price and the early exercise premium. The early exercise premium can be seen as the additional value the option provides due to the possibility of exercises prior to maturity. Thereupon, in order to value the live option it is necessary to calculate the early exercise boundary $B(\cdot)$. This is determined by the following integral equation:

$$\begin{aligned} K - B(t) = p(B(t), T - t) & \\ + \int_t^T &[rKe^{-r(\xi-t)}N(-d_2(B(t), \xi - t; B(\xi - t))) \\ &- qB(\xi - t)e^{-q(\xi-t)}N(-d_1(B(t), \xi - t; B(\xi - t)))]d\xi. \end{aligned} \tag{1.10}$$

This equation reflects the fact that the value of an American put at time of exercise, i.e, when asset price S_t touches the optimal boundary $B(t)$ at time $t \in (0, T)$, is equal to the payoff due to immediate exercise. This issue is shown in condition (1.3). As Kim (1990) shows, the optimal exercise value at expiration is given by:

$$\begin{aligned} &\lim_{t \to T} B(t) = K && \text{if} \quad q \leqslant r \\ &\lim_{t \to T} B(t) = (r/q)K && \text{if} \quad q > r \end{aligned}$$

which is analogous to:

$$B(T) = K \min\left(1, \frac{r}{q}\right). \tag{1.11}$$

Having defined the valuation formulas, the price of the American put option can be computed in two steps. First, Equation (1.10) must be solved backwards from expiration time $(t = T)$. As it is mentioned, the value for the optimal exercise at maturity is defined in (1.11). This is used as a starting point in the procedure of determining $B(\cdot)$. Once the early exercise boundary $B(\cdot)$ is found, Equation (1.8) must be used in order to price the American put option. It is possible to notice that numerical tools, such as numerical integration, have to be applied since equations that Kim (1990) provide are not in a closed-form. This leads to difficulties in the computing time, since the early exercise boundary has to be determined recursively and requires extensive calculation.

1.3.3. General methods

Having in mind the definition of the free boundary problem and the solution of this by the integral equations of Kim (1990), it is possible to describe the main numerical procedures that attempts to solve the American option pricing problem by using this framework. It is possible to split the main method into three groups. One group is based on trying to approximate the path of the underlying asset in order to price the options. Other find a solution to the partial differential equation (PDE) subject to the boundary condition, also known as, free boundary problem, by numerically solving the problem or deriving an explicit solution. Finally, the remaining numerical schemes make an approximation based on the Kim (1990) integral equation and might use Richardson extrapolation in order to improve the accuracy and speed of the method. In this section, the main approximation schemes are briefly described as follows.

Among the earliest numerical methods that value the American contract indirectly through the underlying asset is the Binomial tree of Cox, Ross, & Rubinstein (1979). This lattice method discretizes the time space of the asset price and then discounts, using risk neutral valuation, the cash flows $\max(K - S_t, 0)$ backwards from maturity until the beginning of the contract. This method is still widely used because of its simplicity and ease to adapt to any kind of options. The method is convergent since the pricing accuracy can be improved by setting a higher number of time-steps. Following the same idea, other researchers try to improve the lattice approach in order to gain accuracy. This is the case of Trinomial method of Boyle (1988), that adds an extra branch to the tree in every step. This means that instead of having two discrete jumps as binomial, it has three. This improvement allows the asset price on the Trinomial method to cover more time space than in the Binomial. The rest of the procedure is similar to the Binomial, i.e, the cash flow generated by the asset price path is discounted backwards from maturity. Another lattice method is the Binomial Black and Scholes of Broadie & Detemple (1996). This numerical procedure is similar to the Binomial method, with the difference that in the penultimate discount payoff is approximated as a European option using Black & Scholes (1973) formula. Finally, Longstaff & Schwartz (2001) present a new approach for approximating the value of an American option by simulation. First, they simulate the path of the asset price with equation (1.7). Then, they use a simple least squares to estimate the function of the expected conditional payoff to the option holder from continuation. The intuition behind this method is that at any time the holder of an American option compares the payoff from the immediate exercise with the expected payoff from continuation, and exercises the option if the immediate payoff is higher. Thus, the key insight of this approach is that the conditional expected payoff can be estimated easily from the cross-section information in the simulated paths by using a least squares regression.

Other type of methods are the so called Quasi-analytical approximation methods. One of the first was the Quadratic approximation of Barone-Adesi & Whaley (1987) based on the MacMillan (1986) approach. The Quadratic approximation methods solve the partial differential equation (1.1) governing the price of the American option by introducing a specific approximation in the process. Afterwards, Ju & Zhong (1999) refine the derivation of the Quadratic Approximation by making a similar approximation to the process, but steps later. Another quasi-analytical method is the so called LUBA of Broadie & Detemple (1996) based on a lower and an upper bound price. The lower bound price is based on a capped option, while the upper bound price is based on the integral equations of Kim (1990). The price is finally obtained as a weighted average between the lower and upper-bound prices, where the weights are estimated as a function of model parameters.

The last group of methods use the integral equations of Kim (1990) described previously in Section 1.3.2., in order to price the American option through the early exercise boundary. The accelerated recursive method of Huang, Subrahmanyam, & Yu (1996) approximate the American option as a Bermudan option. The randomization method of Carr (1998) randomizes the time to maturity of the American option and defines a feasible distribution in order to find a simple solution to the American option pricing problem. Ibáñez (2003) refines the recursive method of Huang et al. (1996) by making the bermudan option monotonically convergent to the true American option while the number of exercise times increases. All of the above methods of this group, use Richardson extrapolation for improving the accuracy of pricing. On the other hand, Kallast & Kivinukk (2003) use the trapezoidal rule in order to approximate the integral part of the integral equation (1.10) of Kim (1990). Recently, Kim, Jang, & Kim (2013) came up with an iterative method that also solves the integral equation of Kim (1990) in order to calculate the early exercise boundary. The difference is that they use Little, Pant, & Hou (2000) in order to simplify integrals. The remaining integrals are solved by using the adaptive Gauss-Kronrod rule.

The mentioned numerical approximations will be compared with the proposed procedure on the next Chapter in terms of calculation time and accuracy.

1.4. Future Research

The method proposed in this thesis could be tested in the future against other numerical procedure using GPU codes. Since the proposed method is well suited for calculation in a GPU, a first interesting line of investigation is to determine how much the speed is increased compared with the other methods.

A second line of research could be to extend the model to a more complex factor model, such as adding stochastic volatility to the process. A similar numerical research can be conducted by comparing the accuracy of pricing with similar approaches. It would be necessary to find a feasible benchmark and then program similar methods such as LSM of Longstaff & Schwartz (2001) with stochastic volatility and compare them with the modified functional iterative method.

2. A NEW ITERATIVE METHOD FOR PRICING AMERICAN OPTION

2.1. Introduction

The difficulty of valuing American options stems from the fact that the optimal exercise rule is unknown ex-ante, and must be computed simultaneously with the price of the option. Kim (1990), Jacka (1991), and Carr et al. (1992) derived an integral equation that provides an explicit solution to the optimal early exercise boundary for American options. In this paper we provide a simple iterative method that solves this equation and computes the early exercise boundary explicitly.

A robust method to solve the early exercise boundary equation of Kim (1990) was already studied by Kallast & Kivinukk (2003). The authors show that solving such equation yields a fast and stable method to compute the early exercise boundary. Their approach works **sequentially** in that for a given solution of the boundary up to maturity T, they compute the boundary up to maturity $T + \Delta T$.

In contrast, we propose a method that operates in **parallel**. We iterate over a series of approximating functions $B_\tau^{(n)}$, where $\tau \in [0, T]$ represents the time-to-maturity, in order to compute the early exercise boundary $S_c(\tau)$. In each iteration a new approximation of the **whole** early exercise boundary $B_\tau^{(n+1)}$ is obtained as the result of applying an operator Υ to the previous approximation $B^{(n)}$, i.e. $B^{(n+1)} = \Upsilon(B^{(n)})$. The operator is derived from the equation that determines the early exercise boundary. For each maturity τ, the new value of the early exercise boundary at that point is given by $B_\tau^{(n+1)} = \Phi_\tau(B^{(n)})$, where Φ_τ is a functional that defines the operator Υ for each $\tau \in [0, T]$. Hence we can think of our method as a functional iteration of the early exercise boundary.

Iterative methods are interesting alternatives to traditional numerical methods that price American options in that they are well-suited for parallel implementations. In recent years, parallel algorithms have become attractive with the advent of multi-core processors and graphic processing units (GPUs). Using these novel features of modern hardware can improve the efficiency of algorithms significantly.

In a recent paper, Kim et al. (2013) propose an iterative procedure based on an equation developed by Little et al. (2000) to solve for the early exercise boundary. Even though the method of Kim et al. (2013) and ours are both functional iterative procedures, we show in the paper that our implementation is more efficient in terms of speed, accuracy, and stability. This is consistent with the previous findings of Kallast & Kivinukk (2003) who show that solving the equation of Kim (1990) directly yields stable results.

There are numerous studies in the literature that propose methods to price American options.[1] Brennan & Schwartz (1977) were the first to solve numerically a partial differential equation (PDE) to price American options. Another popular method that discretizes the time space and the asset price is the binomial method of Cox et al. (1979). Both methods are still widely used because of their simplicity.

Following the same ideas, other researchers have tried to improve the lattice approach in order to increase the accuracy and/or reduce the computation time, such as the trinomial method of Boyle (1988) and the improved binomial method presented in Broadie & Detemple (1996). Longstaff & Schwartz (2001) develop a novel method to value options by simulation that determines the conditional expected payoff by least-squares. Although all of these methods are flexible and easily adapted to many kinds of options, their main drawback is that they are time consuming.

A different approach to improve the speed at the expense of precision are the so-called quasi-analytical approximation methods. One of the first of such methods is the quadratic approximation of Barone-Adesi & Whaley (1987). The idea of the method is to solve an approximate version of the PDE governing the price of the American option that yields a closed-form solution. Ju & Zhong (1999) refine the derivation of the quadratic approximation of Barone-Adesi & Whaley (1987) by making a similar approximation to the PDE. Even though Ju & Zhong (1999) appears to be more accurate than Barone-Adesi & Whaley (1987), a main drawback of both methods is that the approximation works well for very

[1]Barone-Adesi (2005) presents a comprehensive survey of existing methods to price American options.

short and very long maturity options, but presents difficulties when applied to medium-term maturity options. Additionally, these methods do not converge to the true price so the estimation cannot be made arbitrarily small. Broadie & Detemple (1996) develop a method along these lines based on a lower and an upper bound. The lower-bound price is computed from a closed-form solution of a capped option, while the upper-bound price is based on the integral equations of Kim (1990). The price is finally obtained as a weighted average between the lower and upper-bound prices, where the weights are estimated as a function of model parameters.

In the literature there are also methods that use Richardson extrapolation in order to improve the accuracy of the computations. For example, Geske & Johnson (1984) find an exact representation of an American put and introduce the Richardson extrapolation to the pricing problem. Huang et al. (1996) approximate the American option as a Bermudan option. Carr (1998) randomizes the time to maturity of the American option and introduces a feasible distribution in order to find a simple solution. Ju (1998) price the option approximating the early exercise boundary as a multipiece exponential function. Ibáñez (2003) refines the recursive method of Huang et al. (1996) by making the Bermudan option monotonically convergent to the true American option as the number of exercise times increases.

Finally, other methods use quadrature formulas in order to price the option. Sullivan (2000) approximates the early exercise premium by using Gaussian quadrature. Kallast & Kivinukk (2003) use the trapezoidal rule in order to approximate the integral part of the equation of Kim (1990).

In our paper we take a different approach, and solve the early exercise equation directly. As we show in the paper, this approach is equivalent to applying a Newton iteration to parallel perturbations of the early exercise boundary. The resulting algorithm requires an initial guess of the early exercise boundary, and a numerical rule to evaluate the integral part of the equation of Kim (1990).

We explore three different alternatives of implementing the method. First, we estimate the integrals appearing in the early exercise boundary equation by use of the trapezoidal rule as in Kallast & Kivinukk (2003). We start the iterations using two different priors: the flat guess of Kim et al. (2013), and the initial guess of Barone-Adesi & Whaley (1987). Second, we follow Kim et al. (2013) and estimate the integrals appearing in the boundary equation by interpolating a few discretized points, and using a more advanced quadrature procedure such as Gauss-Kronrod.

We perform a thorough empirical analysis, and compare the different implementations of our method with the quadratic approximation of Barone-Adesi & Whaley (1987), the least-square Monte-Carlo approach of Longstaff & Schwartz (2001), the refined quadratic approximation of Ju & Zhong (1999), the six-point recursive integration method of Huang et al. (1996), the six-point randomization method of Carr (1998), the three-point modified recursive integration method of Ibáñez (2003), the lower and upper bounds approximation method of Broadie & Detemple (1996), the binomial tree method of Cox et al. (1979), the trinomial tree method of Boyle (1988), the binomial tree using the Black & Scholes formula at the last time-step of Broadie & Detemple (1996), the recursive solution method of Kallast & Kivinukk (2003), and the iterative method of Kim et al. (2013).

The numerical results show that the implementations of our method that use the trapezoidal rule achieve the best performance among all the aforementioned algorithms. In particular, the best performance is obtained by using the smart initial guess of Barone-Adesi & Whaley (1987). Our method is fast because the iterations are performed in parallel. Even though the results that we report in the paper were obtained using a multi-core CPU, the execution speed of our method is even faster using a standard GPU.

In addition, our method is also stable and accurate, which allows to achieve an arbitrary precision. We find that our functional iteration converges monotonically to the true American option price as we increase the number of time-steps in the approximation, regardless of the method employed to estimate the integrals. Finally, we find that the performance of our method can be improved further by the use of Richardson extrapolation.

The rest of the article is organized as follows. In Section 2 we introduce the theoretical framework and recall some standard results of American option pricing theory. Section 3 presents our methodology to price the American options. Section 4 describes in detail the numerical implementation of our method. In Section 5 we perform a battery of numerical tests showing the speed-accuracy trade-off of our procedure compared to other methods. Section 6 finally concludes.

2.2. Theoretical Background

In this section we recall some standard results about American option pricing when the underlying asset follows a geometric Brownian motion. We consider a continuous-time economy in which a complete probability space $(\Omega, F, \mathcal{P})$ and a filtration $\mathbb{F} = \{F_t; t \geq 0\}$ satisfying the usual conditions are defined (see, e.g., Protter, 2005). The risk-free rate r is constant. A risky asset S pays a constant dividend yield q and follows a geometric Brownian motion process with constant volatility under the pricing measure $\mathcal{Q}$, equivalent to $\mathcal{P}$:

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma dW_t.$$

We consider an American put option with maturity T and exercise price K. Denoting by $x^+ = \max(0, x)$, the price P_0 of the American put is given by (see e.g. Schroder, 1999):

$$P_0 = \sup_{\tau \in [0,T]} \mathrm{E}^{\mathcal{Q}} \left(e^{-r\tau} (K - S_\tau)^+ \right),$$

where the supremum is taken over all stopping times $\tau \in [0, T]$. The price p_0 of an equivalent European put option satisfies:

$$p_0 = \mathrm{E}^{\mathcal{Q}} \left(e^{-rT} (K - S_T)^+ \right).$$

The difference between these two prices is called the early exercise premium e_0, i.e.

$$P_0 = p_0 + e_0. \tag{2.1}$$

When the underlying asset follows a diffusion (not necessarily a geometric Brownian motion), the early exercise premium takes the following form (see e.g. Rutkowski, 1994):

$$e_0 = \mathrm{E}^{\mathcal{Q}} \left(\int_0^T e^{-ru} (rK - qS_u) \mathbf{1}_{\{S_u \le S_c(T-u)\}} du \right), \tag{2.2}$$

as long as $r > 0$. In the expression $S_c(T-u)$ denotes the critical spot price that triggers early exercise when the time-to-maturity is $T-u$. For American put options, early exercise is optimal whenever, at time u, the spot price S_u is lower than or equal to $S_c(T-u)$.

In the setup described above, it is well known that we can derive a closed-form expression for the early exercise premium (see e.g. Kim, 1990, Jacka, 1991, Carr et al., 1992). Let:

$$f(x, y, \tau) = rKe^{-r\tau} N(-d_2(x, y, \tau)) - qxe^{-q\tau} N(-d_1(x, y, \tau)), \tag{2.3}$$

$$p(x, y, \tau) = ye^{-r\tau} N(-d_2(x, y, \tau)) - xe^{-q\tau} N(-d_1(x, y, \tau)), \tag{2.4}$$

where:

$$\begin{aligned} d_1(x, y, \tau) &= \frac{\log(x/y) + (r - q + 0.5\sigma^2)\tau}{\sigma\sqrt{\tau}}, \\ d_2(x, y, \tau) &= d_1(x, y, \tau) - \sigma\sqrt{\tau}, \end{aligned}$$

and $N(x) = \mathcal{Q}(X \le x)$, where X is a standard normally distributed random variable under $\mathcal{Q}$. The price the American put is then given by:

$$P_0 = p(S_0, K, T) + \int_0^T f(S_0, S_c(T-u), u) du, \tag{2.5}$$

where $S_0 = S(0)$ denotes the current price of the underlying, the function f is defined as in (2.3), and $p(S_0, K, T)$ is the corresponding Black-Scholes price of a European put option defined in (2.4).

For any given T, the early exercise price $S_c(T)$ satisfies the following integral equation for the American put option:

$$K - S_c(T) = p(S_c(T), K, T) + \int_0^T f(S_c(T), S_c(T-u), u)du. \tag{2.6}$$

We also have that $\lim_{T \to 0^+} S_c(T) = K \min(1, r/q)$ (see e.g. Kim, 1990).

2.3. A New Method

In this section we present a new methodology to price American options. We iterate over a series of approximating functions of the early exercise frontier in order to compute the true early exercise boundary. In each iteration a new approximation of the **whole** early exercise boundary is obtained as the result of applying an operator to the previous approximation. The operator is derived from the equation that determines the early exercise boundary. The resulting method is parallelizable, fast, accurate, and stable.

We first describe the method for the case of American put options. We then show that our method is equivalent to a Newton scheme applied to parallel perturbations of the early exercise boundary. We finally show how the method can be applied to price American call options.

2.3.1. American Put Options

One approach to price an American put option is to solve Equation (2.6) numerically to obtain the function $S_c(T)$. Kallast & Kivinukk (2003) solve (2.6) recursively by finding $S_c(T)$ one time-step at a time. More precisely, they approximate the continuous time interval $[0, T]$ with the discretized set $[0, \Delta T, \ldots, N_T \Delta T]$. Then, given an accurate approximation of $S_c(\tau)$ for all $\tau \in [0, T]$, they compute an approximation of $S_c(T + \Delta T)$. This method can then be used to compute an approximation of $S_c(T)$ for any arbitrary T. Furthermore, they show that solving equation (2.6) directly is robust and efficient.

In what follows we provide a different approach to obtain a solution to (2.6). Our method differs from Kallast & Kivinukk (2003) in that we iterate over a series of approximations $S_c^{(k)}(T)$ that converge to $S_c(T)$ as $k \to \infty$ for all $T > 0$. In other words, our method works in parallel whereas Kallast & Kivinukk (2003) solve (2.6) sequentially.

We start by re-writing equation(2.6) as:

$$S_c(T)U_T(S_c) - KV_T(S_c) = 0 \tag{2.7}$$

where

$$U_T(\phi) = 1 - e^{-qT}N(-d_1(\phi(T), K, T)) - q\int_0^T e^{-qu}N(-d_1(\phi(T), \phi(T-u), u))du,$$
$$V_T(\phi) = 1 - e^{-rT}N(-d_2(\phi(T), K, T)) - r\int_0^T e^{-ru}N(-d_2(\phi(T), \phi(T-u), u))du,$$

for a suitable function ϕ. It is interesting to note that the functionals U_T and V_T are bounded between 0 and 1. Indeed, we can re-write U_T as:

$$\begin{aligned} U_T(\phi) = 1 &- e^{-qT}N(-d_1(\phi(T), K, T)) \\ &- (1 - e^{-qT})\int_0^T \left[\frac{qe^{-qu}}{1 - e^{-qT}}\right] N(-d_1(\phi(T), \phi(T-u), u))du. \end{aligned}$$

We first note that

$$\varphi(u) = \frac{qe^{-qu}}{\int_0^T qe^{-qu}du} = \frac{qe^{-qu}}{1 - e^{-qT}}$$

defines a density over $[0, T]$, and that $0 < N(-d_1(\phi(T), \phi(T-u), u)) < 1$, implying that $0 < \int_0^T \varphi(u)N(-d_1(\phi(T), \phi(T-u), u))du < 1$. Also, $0 < N(-d_1(\phi(T), K, T)) < 1$, which implies that a convex combination between $\int_0^T \varphi(u)N(-d_1(\phi(T), \phi(T-u), u))du$ and $N(-d_1(\phi(T), K, T))$ should also be greater than 0 and less than 1. Thus, we can conclude that $0 < U_T(\phi) < 1$ whenever $T > 0$. A similar argument shows that $0 < V_T(\phi) < 1$ for $T > 0$.

Therefore, we can solve for $S_c(T)$ in (2.7) since $U_T(S_c)$ is well-defined:

$$S_c(T) = K\frac{V_T(S_c)}{U_T(S_c)}. \tag{2.8}$$

In the paper we use (2.8) as the basis of our iterative method. Starting from an initial guess $B_\tau^{(0)}$ of the early exercise boundary for all $\tau \in [0, T]$, we obtain a new approximation $B_\tau^{(1)}$ recursively as follows:

$$B_\tau^{(1)} = K\frac{V_\tau(B^{(0)})}{U_\tau(B^{(0)})}.$$

Hence, given an approximation of the early exercise frontier $B_\tau^{(n)}$ for all $\tau \in [0, T]$ after n iterations, we can find a new approximation $B_\tau^{(n+1)}$ using (2.8):

$$B_\tau^{(n+1)} = K\frac{V_\tau(B^{(n)})}{U_\tau(B^{(n)})}. \tag{2.9}$$

The new approximation for a given maturity τ_1 can be computed independently from the new approximation corresponding to maturity τ_2.

Kim et al. (2013) propose an alternative method to compute the early exercise boundary. Their iteration is based on the equation of Little et al. (2000):

$$S_c(T) = K\frac{\tilde{V}_T(S_c)}{\tilde{U}_T(S_c)} \tag{2.10}$$

where

$$\begin{aligned}\tilde{U}_T(\phi) &= e^{-qT}N(d_1(\phi(T), K, T) + \frac{1}{\sigma\sqrt{2\pi T}}e^{-(qT+\frac{1}{2}d_1(\phi(T),K,T)^2)} \\ &\quad + q\int_0^T e^{-ru}N(d_1(\phi(T), \phi(T-u), u) + \frac{1}{\sigma\sqrt{2\pi u}}e^{-(ru+\frac{1}{2}d_1(\phi(T),\phi(T-u),u)^2)}du,\end{aligned}$$

and

$$\tilde{V}_T(\phi) = \frac{1}{\sigma\sqrt{2\pi T}}e^{-(rT+\frac{1}{2}d_2(\phi(T),K,T)^2)} + r\int_0^T \frac{1}{\sigma\sqrt{2\pi u}}e^{-(ru+\frac{1}{2}d_2(\phi(T),\phi(T-u),u)^2)}du.$$

Even though the method of Kim et al. (2013) appears similar to ours, empirically their approach is much slower. In particular, we find the integrals in $\tilde{U}_T$ and $\tilde{V}_T$ to be harder to estimate than their counterparts in U_T and V_T.

2.3.2. Our Method As a Newton Iteration

In this section we show that our iterative method is equivalent to applying a multivariate Newton iteration to equation (2.7). We define for each $\tau \in (0, T]$ the functional

$$F_\tau(\phi) = \phi(\tau)U_\tau(\phi) - KV_\tau(\phi), \tag{2.11}$$

for a suitable function ϕ. Note that (2.6) is equivalent to $F_\tau(S_c) = 0$ for all $\tau \in (0, T]$. We also define for each $\tau \in (0, T]$ the functional

$$F'_\tau(\phi) = \lim_{\epsilon \to 0} \frac{F_\tau((1+\epsilon)\phi) - F_\tau(\phi)}{\epsilon} = \frac{d}{d\epsilon} F_\tau((1+\epsilon)\phi)\bigg|_{\epsilon=0},$$

which can be interpreted as the derivative of $F_\tau(\phi)$ with respect to a proportional perturbation of ϕ, and is akin to a Gteaux derivative.[2] The above derivative can be computed explicitly as:

$$F'_\tau(\phi) = \phi(\tau)U_\tau(\phi). \tag{2.12}$$

Note that we showed previously that $U_\tau(\phi) > 0$, which also implies that $F'_\tau(\phi) > 0$.

For a given function $h(x)$ of a real variable x, the Newton method provides a way to solve iteratively the equation $h(x) = 0$. If we have an approximation x_n of a root to the above equation, the Newton iteration x_{n+1} solves:

$$h(x_{n+1}) = h(x_n) + h'(x_n)(x_{n+1} - x_n) = 0.$$

In the same way, for a given approximation $B^{(n)}$ of the early exercise frontier, we could fix τ and solve for ϵ_{n+1} in the following equation:

$$F_\tau(B^{(n)}(1 + \epsilon_{n+1})) = F_\tau(B^{(n)}) + F'_\tau(B^{(n)})\epsilon_{n+1} = 0,$$

which gives $\epsilon_{n+1} = -F_\tau(B^{(n)})/F'_\tau(B^{(n)})$.

[2] If we let $\lambda(\tau) = \log(\phi(\tau))$ for all $\tau \in (0, T]$, and define $G_\tau(\lambda) = F_\tau(\phi)$, then we have that

$$F'_\tau(\phi) = \lim_{\epsilon \to 0} \frac{G_\tau(\lambda + \epsilon) - G_\tau(\lambda)}{\epsilon} = \frac{d}{d\epsilon} G_\tau(\lambda + \epsilon)\bigg|_{\epsilon=0}.$$

Hence, we can interpret the function $F'_\tau(\phi)$ as the Gteaux derivative of $G_\tau(\lambda)$ in the direction of a constant function equal to 1 for all $\tau \in (0, T]$, i.e. with respect to a parallel perturbation of λ.

The previous equation defines an iteration that computes ϵ in $F_\tau(B^{(n)}(1+\epsilon)) = 0$, for a fixed $\tau \in (0, T]$. This is not the same equation that we are trying to solve, since we want to have $F_\tau(S_c) = 0$ for all $\tau \in (0, T]$. However, by taking

$$B_\tau^{(n+1)} = B_\tau^{(n)} \left(1 - \frac{F_\tau(B^{(n)})}{F'_\tau(B^{(n)})}\right), \tag{2.13}$$

we obtain an update to $B_\tau^{(n)}$ that is the same as the one given by (2.9). Hence, our functional method is equivalent to a Newton iteration that solves for ϵ in $F_\tau(B^{(n)}(1+\epsilon)) = 0$, and then uses the updated translation of $B^{(n)}$ evaluated at τ as the new update for $B_\tau^{(n+1)}$.

Since $F'_\tau(\phi) > 0$, equation (2.13) also shows that $B_\tau^{(n+1)} < B_\tau^{(n)}$ whenever $F_\tau(\phi) > 0$, and that $B_\tau^{(n+1)} > B_\tau^{(n)}$ if $F_\tau(\phi) < 0$. Hence, the fixed-point iteration in (2.9) adjusts $B_\tau^{(n)}$ downwards or upwards depending on the sign of $F_\tau(\phi)$.

2.3.3. American Call Options

The case of American call options can be treated similarly. In order to compute the price C_0 of an American call option with the same characteristics we first define:

$$g(x, y, \tau) = -rKe^{-r\tau} N(d_2(x, y, \tau)) + qxe^{-q\tau} N(d_1(x, y, \tau)), \tag{2.14}$$

$$c(x, y, \tau) = -ye^{-r\tau} N(d_2(x, y, \tau)) + xe^{-q\tau} N(d_1(x, y, \tau)). \tag{2.15}$$

The price of the American call option is then given by:

$$C_0 = c(S_0, K, T) + \int_0^T g(S_0, S_c(T-u), T-u)du, \tag{2.16}$$

where the function g is defined as in (2.14), and $c(S_0, K, T)$ is the corresponding Black-Scholes price of a European call option defined in (2.15). By letting $S_0 = S_c(T)$ in (2.16) we can obtain an equation similar to (2.8) for the early exercise boundary of the American call:

$$S_c(T) = K \frac{\hat{V}_T(S_c)}{\hat{U}_T(S_c)}, \tag{2.17}$$

where

$$\hat{U}_T(\phi) = e^{-qT}N(d_1(\phi(T), K, T)) + q\int_0^T e^{-qu}N(d_1(\phi(T), \phi(T-u), u))du - 1,$$
$$\hat{V}_T(\phi) = e^{-rT}N(d_2(\phi(T), K, T)) + r\int_0^T e^{-ru}N(d_2(\phi(T), \phi(T-u), u))du - 1.$$

Empirically, we find that the method works similarly well for American call and put options. It should be noted that the value of the American call can also be obtained from the value of an American put by using the parity result of McDonald & Schroder (1998):

$$C(S_0, K, r, q, \sigma, T) = P(K, S_0, q, r, \sigma, T).$$

2.4. Numerical Implementation

We calculate the price of an American option in three steps. First, we start with an initial guess of the early exercise boundary. Second, the initial guess is updated using equation (2.8). We repeat this procedure until the maximum difference among two estimated curves during two sequential iterations is below a specified tolerance. Finally, the price of an American option is obtained by using the estimated early exercise boundary along with equation (2.5).

In order to implement our numerical method we need to estimate the integrals appearing in the numerator and denominator of equation (2.8), as well as the integral appearing in equation (2.5). We proceed by using two different methods: the trapezoidal rule, as implemented in Kallast & Kivinukk (2003), and a numerical quadrature method similar to the one used by Kim et al. (2013).

2.4.1. Trapezoidal Rule

We first divide the time interval $[0, T]$ into N_T subintervals of length $\Delta t = T/N_T$. We keep the number of points fixed through all iterations. Therefore, by varying the number of initial points it is possible to increase the accuracy. However, the trapezoidal method is highly parallelizable so its computational cost does not increase much with the grid size. Our implementation of the trapezoidal rule takes advantage of this feature.

We first describe the method when applied to American put options. We denote by $B_i^{(k)}$ the estimated early exercise boundary after k iterations for time-to-maturity $\tau = i\Delta t$, where $i = 0, \ldots, N_T$. We also denote by $B^{(k)} = \left(B_0^{(k)}, B_1^{(k)}, \ldots, B_{N_T}^{(k)}\right)'$ a $N \times 1$ column vector containing estimates of the early exercise frontier after k iterations.

In our empirical tests, we initialize the method for American put options either by setting:

$$B_i^{(0)} = K \min(1, r/q), \quad \forall i = 0, \ldots, N_T \tag{2.18}$$

as in Kim et al. (2013), or by using the initial guess of Barone-Adesi & Whaley (1987):

$$B_i^{(0)} = B_\infty + (K - B_\infty)\left(1 - e^{-\left[\left((r-q)i\Delta t + 2\sigma\sqrt{i\Delta t}\right)K/(K-B_\infty)\right]}\right), \quad \forall i = 0, \ldots, N_T \tag{2.19}$$

where B_∞ represents the critical price of a perpetual American put option.[3] Also, since we know that $S_c(0+) = K \min(1, r/q)$, we fix $B_0^{(k)} = K \min(1, r/q)$ for all $k \geq 0$.

Given a set of estimates $B^{(k)}$ obtained after k iterations, we find a new set of estimates $B^{(k+1)}$ by using equation (2.8):

$$B_i^{(k+1)} = K \frac{V_i(B^{(k)})}{U_i(B^{(k)})}, \quad \forall i = 0, \ldots, N_T, \tag{2.20}$$

[3]The critical price of the perpetual American put option is equal to $B_\infty = \dfrac{K}{1 - 1/q_\infty^1}$, where $q_\infty^1 = -\frac{1}{2}(N - 1) - \frac{1}{2}\sqrt{(N-1)^2 + 4M}$, $M = \dfrac{2r}{\sigma^2}$ and $N = \dfrac{2(r-q)}{\sigma^2}$.

where

$$
\begin{aligned}
U_i(B^{(k)}) &= 1 - e^{-qi\Delta t} N(-d_1(B_i^{(k)}, K, i\Delta t)) - q\Delta t \sum_{j=1}^{i-1} e^{-qj\Delta t} N(-d_1(B_i^{(k)}, B_{i-j}^{(k)}, j\Delta t)) \\
&\quad - \frac{q\Delta t}{2} \left[N(-d_1(B_i^{(k)}, B_i^{(k)}, 0)) + e^{-qi\Delta t} N(-d_1(B_i^{(k)}, B_0^{(k)}, i\Delta t)) \right], \\
V_i(B^{(k)}) &= 1 - e^{-ri\Delta t} N(-d_2(B_i^{(k)}, K, i\Delta t)) - r\Delta t \sum_{j=1}^{i-1} e^{-rj\Delta t} N(-d_2(B_i^{(k)}, B_{i-j}^{(k)}, j\Delta t)) \\
&\quad - \frac{r\Delta t}{2} \left[N(-d_2(B_i^{(k)}, B_i^{(k)}, 0)) + e^{-ri\Delta t} N(-d_2(B_i^{(k)}, B_0^{(k)}, i\Delta t)) \right],
\end{aligned}
$$

and $N(-d_1(B_i^{(k)}, B_i^{(k)}, 0)) = N(-d_2(B_i^{(k)}, B_i^{(k)}, 0)) = 0.5$. We continue the process until:

$$
\max_{i=1,\ldots,N_T} \left| \frac{B_i^{(k+1)} - B_i^{(k)}}{K} \right| \leq \varepsilon,
$$

where ε is a relative tolerance threshold. The method usually converges very fast, taking for example between 5 to 6 iterations when $\varepsilon = 10^{-3}$.

Figure A.1 presents an example of how the method works. In this example we use 20 time-intervals to approximate the early exercise boundary. The strike price is $K = 100$, the time-to-maturity is $T = 1$, the risk-free rate is $r = 0.04$, the dividend rate is $q = 0.08$, and the constant volatility is $\sigma = 0.2$. In this example the method converges in 5 iterations.

Once an estimate of the early exercise boundary $B = (B_0, B_1, \ldots, B_{N_T})'$ is determined as in the previous example, we compute the premium of an American put option with spot S_0, strike K and maturity T using equation (2.5). We follow Kallast & Kivinukk (2003) and use Simpson's rule to approximate the integral:

$$
\begin{aligned}
P_0 = p(S_0, K, T) + &\frac{\Delta t}{3} \left[f(S_0, B_{N_T}, 0) + 4f(S_0, B_{N_T-1}, \Delta t) + 2f(S_0, B_{N_T-2}, 2\Delta t) + \cdots \right. \\
&\left. \cdots + 2f(S_0, B_2, (N_T - 2)\Delta t) + 4f(S_0, B_1, (N_T - 1)\Delta t) + f(S_0, B_0, N_T\Delta t) \right]. \quad (2.21)
\end{aligned}
$$

We only use the equation when the option is alive, i.e. when $S_0 > K$. Also, note that we assume that N_T is even.

For pricing American call options we follow a similar approach. We denote by $G_i^{(k)}$ the estimated early exercise boundary after k iterations for each time-to-maturity $i\Delta t$, where $i = 0, \ldots, N_T$, and by $G^{(k)} = \left(G_0^{(k)}, G_1^{(k)}, \ldots, G_{N_T}^{(k)}\right)'$ a $N \times 1$ column vector containing estimates of the early exercise frontier after k iterations. We initialize the method either by setting:

$$G_i^{(0)} = K \max(1, r/q), \quad \forall i = 0, \ldots, N_T, \tag{2.22}$$

or by using the initial guess of Barone-Adesi & Whaley (1987):

$$G_i^{(0)} = K + (G_\infty - K)\left(1 - e^{-\left[\left((r-q)i\Delta t + 2\sigma\sqrt{i\Delta t}\right)K/(G_\infty - K)\right]}\right), \quad \forall i = 0, \ldots, N_T, \tag{2.23}$$

where G_∞ represents the critical price of a perpetual American call option.[4] Also, since we know that $S_c(0+) = K \max(1, r/q)$, we fix $G_0^{(k)} = K \max(1, r/q)$ for all $k \geq 0$.

Given a set of estimates $G^{(k)}$ obtained after k iterations, we find a new set of estimates $G^{(k+1)}$ by using equation (2.17):

$$G_i^{(k+1)} = K \frac{\hat{V}_i(G^{(k)})}{\hat{U}_i(G^{(k)})}, \quad \forall i = 0, \ldots, N_T, \tag{2.24}$$

where

$$\begin{aligned}
\hat{U}_i(G^{(k)}) &= e^{-qi\Delta t} N(d_1(B_i^{(k)}, K, i\Delta t)) + q\Delta t \sum_{j=1}^{i-1} e^{-qj\Delta t} N(d_1(B_i^{(k)}, B_{i-j}^{(k)}, j\Delta t)) \\
&\quad + \frac{q\Delta t}{2}\left[N(d_1(B_i^{(k)}, B_i^{(k)}, 0)) + e^{-qi\Delta t} N(d_1(B_i^{(k)}, B_0^{(k)}, i\Delta t))\right] - 1, \\
\hat{V}_i(B^{(k)}) &= e^{-ri\Delta t} N(d_2(B_i^{(k)}, K, i\Delta t)) + r\Delta t \sum_{j=1}^{i-1} e^{-rj\Delta t} N(d_2(B_i^{(k)}, B_{i-j}^{(k)}, j\Delta t)) \\
&\quad + \frac{r\Delta t}{2}\left[N(d_2(B_i^{(k)}, B_i^{(k)}, 0)) + e^{-ri\Delta t} N(d_2(B_i^{(k)}, B_0^{(k)}, i\Delta t))\right] - 1,
\end{aligned}$$

[4]The critical price of the perpetual American call option is equal to $G_\infty = \dfrac{K}{1 - 1/q_\infty^2}$, where $q_\infty^2 = -\frac{1}{2}(N - 1) + \frac{1}{2}\sqrt{(N-1)^2 + 4M}$, $M = \dfrac{2r}{\sigma^2}$ and $N = \dfrac{2(r-q)}{\sigma^2}$.

and $N(d_1(B_i^{(k)}, B_i^{(k)}, 0)) = N(d_2(B_i^{(k)}, B_i^{(k)}, 0)) = 0.5$. For an estimate of the early exercise boundary G, the premium of an American call is computed as:

$$C_0 = c(S_0, K, T) + \frac{\Delta t}{3}\left[g(S_0, G_{N_T}, 0) + 4g(S_0, G_{N_T-1}, \Delta t) + 2g(S_0, G_{N_T-2}, 2\Delta t) + \cdots\right.$$
$$\left.\cdots + 2g(S_0, G_2, (N_T-2)\Delta t) + 4g(S_0, G_1, (N_T-1)\Delta t) + g(S_0, G_0, N_T\Delta t)\right]. \quad (2.25)$$

This equation is used only when the option is alive, i.e. when $S_0 < K$. We also assume that N_T is even.

2.4.2. Numerical Quadrature

We also implement our method by estimating the integrals using a Gauss-Kronrod adaptive procedure. In this section we only describe the method when applied to American put options. The method is adapted in a similar manner as it was explained for the trapezoidal rule.

We work with a discrete set of estimated points from the early exercise curve denoted by $B_i^{(k)}$. In this implementation we follow Kim et al. (2013) and initialize the method only with the flat prior given in equation (2.18). Given a set of estimates $B^{(k)}$ obtained after k iterations, we build an estimated curve $B_\tau^{(k)}$ for $\tau \in [0, T]$ by using a spline that goes through the points $B_i^{(k)}$ for all $i = 0, \ldots, N_T$.

We find a new set of estimates $B^{(k+1)}$ for each discrete point $B_i^{(k+1)}$ by using equation (2.8) where the integrals in U_T and V_T are estimated using the adaptive Gauss-Kronrod procedure. We continue the process until:

$$\max_{i=1,\ldots,N_T} \left| \frac{B_i^{(k+1)} - B_i^{(k)}}{K} \right| \leq \varepsilon,$$

where ε is a relative tolerance threshold.

Once an estimate of the early exercise boundary B is determined, we build a curve $B(\tau)$ for $\tau \in [0, T]$ using splines, and compute the premium of an American put option with spot S_0, strike K and maturity T using equation (2.5) and the adaptive Gauss-Kronrod procedure.

2.5. Numerical Results

In this section we present numerical tests that compare the pricing accuracy and the speed of our method with other numerical techniques that have been studied in the literature. We present several examples in which our method converges to the "true" American option price. Furthermore, we provide numerical tests for several pricing methods under different scenarios in terms of underlying asset prices, maturities and volatilities.

In the following, we denote our functional iteration of Kim's equation by FIK. We implement the method in three different ways:

- Using the trapezoidal rule with a flat initial guess [FIK-F]
- Using the trapezoidal rule with the initial guess of Barone-Adesi & Whaley (1987) [FIK-BAW]
- Using the Gauss-Kronrod quadrature with splines and starting from a flat initial guess [FIK-GK]

We compare our results with the following methods that have been implemented in the literature:

- Quadratic approximation of Barone-Adesi & Whaley (1987) [BAW]
- Least-square Monte-Carlo approach of Longstaff & Schwartz (2001) [LS]
- Refined quadratic approximation of Ju & Zhong (1999) [JZ]
- Six-point recursive integration method of Huang et al. (1996) [HSY]
- Six-point randomization method of Carr (1998) [CARR]
- Three-point modified recursive integration method of Ibáñez (2003) [IBN]
- Lower and upper bounds approximation method of Broadie & Detemple (1996) [LUBA]
- Binomial tree method of Cox et al. (1979) [BIN]
- Trinomial tree method of Boyle (1988) [TRI]

- Binomial tree using Black & Scholes formula at the last time-step of Broadie & Detemple (1996) [BIN-BS]
- Recursive solution method of Kallast & Kivinukk (2003) [KK]
- Iterative method of Kim et al. (2013) [KJK]

In order to assess the accuracy of all our computations, the true value is computed with a binomial tree model with 15 000 time steps in which the Black & Scholes formula is used at the penultimate node. Broadie & Detemple (1996) show that the traditional binomial method can have an oscillatory convergence whereas the binomial tree with Black & Scholes at the end converges faster to the true price.[5] We use the root mean squared error (RMSE) and the relative RMSE[6] as the main measures of errors.

All codes were programmed in Matlab and all tests were performed using the same hardware. For every method we recalculate the early exercise frontier when pricing each option. We do not take advantage of the fact that for some methods (such as ours) once the early exercise frontier is known, it is possible to price options with different spot prices without having to recompute it. We do this to make all methods comparable.

2.5.1. Pricing Accuracy of Our Method

In this section we analyze the pricing accuracy of our methodology. Table A.1 compares the pricing performance of our functional iterative method FIK-F with the true American option price. In the table we fix the following parameters: strike price $K = 100$, time-to-maturity $T = 3$, and volatility $\sigma = 0.2$. We generate 12 different examples by using different spot prices (80, 100, and 120), interest rates (0.04 and 0.08) and dividend yields (0.04 and 0.12), as shown in the table. Column (1) reports the "true" value of the American put option calculated using a binomial tree with 15 000 steps in which the Black & Scholes formula is used at the last time-step (Broadie & Detemple, 1996). Columns (2)$-$(8) report

[5]Kallast & Kivinukk (2003) use the binomial method with 10 000 steps as their benchmark. Broadie & Detemple (1996) use the convergent binomial method proposed by Amin & Khanna (1994) with 15 000 steps as their benchmark. We tried both but found that the binomial tree with Black & Scholes at the end possess the best convergence among all three of them.

[6]We define the relative RMSE as the root mean squared percentage errors.

American put option prices calculated using our FIK-F method where the number of time-steps vary from 20 to 140 in increments of 20, while columns (9)$-$(11) report results using 200, 300 and 400 time-steps respectively.

The table shows that the method using a simple trapezoidal rule to estimate the integral requires a small number of time-steps to attain high precision, even when the maturity of the option is large as in the example. Using 20 time-steps we obtain a relative RMSE less than 10^{-4}. With 60 time-steps we achieve a relative RMSE less than 10^{-5}. The last two columns show relative RMSEs less than 10^{-6}. We can also see that errors decrease monotonically as the number of time-steps increase.

2.5.2. Comparison with Other Methods

In this section we compare the performance of our proposed implementations with other methods that have been studied in the literature. We first compare all methods described at the beginning of this section across twelve combinations of the spot price (80, 100, and 120), time-to-maturity (0.5 and 3), and volatility (0.2 and 0.5). The following parameters are fixed throughout the evaluations: $K = 100$, $q = 0.04$, and $r = 0.04$.

Tables A.2 and A.3 report the results where the methods are sorted from left to right by decreasing relative RMSE (RRMSE). We find that BAW has the worst RRMSE. JZ attains a better accuracy than BAW since it is a refined version of the later, reaching close to a tenth of the RRMSE in BAW (Ju & Zhong, 1999). LUBA has a RRMSE close to but better than BIN with 1000 time-steps, which is consistent with the accuracy reported in Broadie & Detemple (1996). CARR has around a third of the accuracy of LUBA, and around double of the accuracy of HSY, which is consistent with results reported in Ju (1998). We also find that IBN gains an extra decimal in accuracy over HSY, despite the fact that IBN uses half the points of HSY. This is consistent with the findings reported in Ibáñez (2003) who introduces the use of Richardson extrapolation applied to American option methods. Finally, our approach with only 60 time-steps achieves better accuracy than BAW, LS, JZ, HSY, CARR, BIN 500, TRI 500, BIN 1000, BIN BS 500, LUBA, BIN BS 1000, BIN 2500, IBN and TRI 2500. Also, our method FIK-F with 400 time-steps reaches the same accuracy as KK with the same number of time-steps, which is expected since both methods use the same equation to solve for the early exercise boundary.

We also perform a comprehensive analysis in which we compare the pricing performance of each method using the following different combinations of parameters: spot price $S = 75, 80, ..., 120, 125$ (11 values); maturity $T = 1/12, 3/12, 6/12, 9/12, 1, 2, 3$; volatility $\sigma = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6$; risk-free rate $r = 0.02, 0.04, 0.06, 0.08, 0.1$ and dividend yield $q = 0.0, 0.04, 0.08, 0.12$. This generates a set of 9 240 different combinations. Without loss of generality, the strike price is fixed at $K = 100$. In the analysis we only focus on American put options since the put-call symmetry identity of McDonald & Schroder (1998) suggests that similar results should be obtained for American call options. In unreported results, we verify that this is indeed the case.

We exclude from our tests options with prices of less than 50 cents and we do not include samples in which a certain method computes a negative early exercise premia.[7] After applying these filters, we obtain a total of 7 865 sample points over which we test the accuracy and speed of all methods.

Table A.4 reports summary statistics of the empirical performance of each method. As a measure of accuracy we report the root mean squared error (RMSE). We also report the average time in seconds for each method to price the 7 865 options. To compare the speed-accuracy trade-off across different methods, we define a new measure of efficiency:

$$\text{Efficiency} = -\log(\text{RMSE} \times \text{Time}).$$

According to this definition, a method performs better the higher its Efficiency.

When compared using our measure of efficiency, the best method is FIK-F 60, followed by FIK-BAW 400, FIK-F 400, LUBA[8], and JZ (in that order). It is interesting to note that

[7] Binomial and trinomial methods give in some cases negative early exercise premia. This is probably due to the oscillatory convergence of these kind of methods. On the other hand, our method never computes negative early exercise premia under the same set of parameters.

[8] We use the same parameters λ_1 and λ_2 as calibrated in Broadie & Detemple (1996). Even though these parameters were calibrated using a dividend rate $q = [0, ..., 0.1]$, we test LUBA under same constraints of q and obtain similar RMSEs compared with our testing set that has in addition q = 0.12.

FIK-GK 24 and KJK 24 score much lower, even though they are also iterative methods. Also, KK 400 also scores lower, even though the method solves the same equation as FIK-F or FIK-BAW. Finally, it is surprising to note that the highly popular LS method is the second worst in terms of RMSE, and the worst in terms of speed.

Table A.4 also reports information on the dispersion of the pricing accuracy among the methods. Columns (4) to (6) report the percentage of options for which the absolute error (AE) is lower than the corresponding threshold. It is interesting to note that FIK-BAW 400 achieves the highest percentage of options priced with an absolute error less than 10^{-5}. It is followed by FIK-F 400, KK 400, FIK-GK 24, and KJK 24 (in that order).

We continue our analysis by plotting the speed-accuracy trade-off as shown in Figures A.2 to A.8. In the figures we measure accuracy as the root mean squared error (RMSE), and speed as the number of options priced per second. In all figures the axis are in $\log_{10}$-scale. Hence, the performance of a given method increases as we move towards the north-east.

Consistent with Table A.4, Figure A.2 shows that FIK-F and FIK-BAW outperform all other methods. We can also observe that for lower precision LUBA and BIN-BS are efficient. Also, note that FIK-BAW, FIK-F, KK, FIK-GK, BIN-BS, TRI and BIN converge monotonically in accuracy at the expense of a lower speed.

As a robustness check, we analyze if the results are affected by time-to-maturity (T), moneyness (S/K), and volatility (σ). Figures A.3 and A.4 split the results by short ($T < 1$) and long maturity ($T \geq 1$) options, respectively. Figures A.5 and A.6 split the results by moneyness, where we consider at-the-money ($0.9 < S/K < 1.1$), and in or out-the-money ($S/K \leq 0.9$ and $S/K \geq 1.1$) options, respectively. Finally, Figures A.7 and A.8 split the results by low ($\sigma \leq 0.3$) and high ($\sigma > 0.3$) volatility. In the figures we exclude FIK-BAW since its performance is very similar to FIK-F as shown in Figure A.2.

Overall, all figures confirm the convergence of our methods in terms of pricing errors. As a general result, the more time-steps we use in the computation of the early exercise frontier, the higher the accuracy we obtain in pricing the options. Furthermore, our approach seems to dominate in terms of speed-accuracy trade-off all other methods including

KK, even though the later is based on solving the same equation. Our method is fast because the iterations are performed in parallel. Hence, the speed could be increased even further without sacrificing the accuracy by the use of modern hardware such as graphic processing units (GPUs).[9]

Finally, it is also worth noticing that methods such as BIN, TRI and BIN-BS, among others, need to recalculate all steps if one wishes to price an option with a different spot price. On the other hand, in the case of our method we could compute the early exercise frontier up to a time-to-maturity T, say, for fixed r, q, and σ. We could then use the same early exercise boundary to price options with different spot and strike prices, increasing the speed even more. We choose not to exploit this natural ability of our method in order to make the benchmark comparable across different methods.

2.5.3. Richardson Extrapolation

Several authors in the literature have proposed the use of Richardson extrapolation to improve the pricing accuracy (see e.g. Geske & Johnson, 1984, Bunch & Johnson, 1992, Ibáñez, 2003). Let P_i denote the option price obtained using N_i time-steps. The 2-point Richardson extrapolation (Bunch & Johnson, 1992) is equal to $P = 2P_2 - P_1$, whereas the 3-point Richardson extrapolation (Geske & Johnson, 1984) is obtained as $P = P_3 + 7/2P_2 - 1/2P_1$.

Richardson extrapolation works on methods on which convergence can be improved by using more time-steps. Therefore, the extrapolation cannot be used on methods such as LUBA, BAW and JZ, for example. We decide to check if Richardson extrapolation would improve the efficiency of our method and compare the results to two other methods that improve their efficiency as the number of time-steps increases, namely BIN-BS and KK.

Table A.5 reports the results. We try five different combinations of 2 and 3-point Richardson extrapolation for BIN-BS, FIK-F and KK. Overall, FIK-F improves its efficiency considerably and in all five cases achieves a higher efficiency than BIN-BS and KK,

[9] In unreported results, we increased the speed of our method by 10 to 100 times by using a standard NVIDIA GTX-570 GPU with 480 cores.

with FIK-F 200/150/100 achieving the highest efficiency. The reason why we are able to improve the efficiency by using Richardson extrapolation is that the RMSE decreases linearly in that region with the number of time-steps (see Ibáñez, 2003), as observed in Figure A.2.

For BIN-BS, the 2-point Richardson extrapolation performs better than the 3-point Richardson Extrapolation in terms of efficiency, which is consistent with the results reported by Broadie & Detemple (1996). The 2-point Richardson extrapolation of BIN-BS shows a significant improvement over the regular BIN-BS. On the other hand, KK does not benefit much from the use of Richardson extrapolation.

2.6. Concluding Remarks

We introduce a novel, simple, fast and accurate iterative method to price an American option and solve for its early exercise boundary. The approach is equivalent to applying a Newton iteration to parallel perturbations of the early exercise boundary. Our method shows improved performance in terms of speed-accuracy efficiency over existing numerical methodologies. Moreover, we find that the method is stable, converges monotonically, and is well-suited for vectorized and parallel implementations.

Numerical results show that using the trapezoidal rule in our method achieves the best performance among all existing methods analyzed in the paper. The performance can be improved even further by using a smart guess for the initial early exercise boundary as in Barone-Adesi & Whaley (1987). We find that our functional iteration converges monotonically to the true American option price as we increase the number of time-steps in the approximation, regardless of the method employed to estimate the integrals. Finally, we find that the performance of our method can also be improved further by the use of Richardson extrapolation.

Therefore, our analysis shows that a direct solution of the early exercise representation of Kim (1990) seems to be robust and efficient. This point was already raised by Kallast & Kivinukk (2003) who solved the same equation sequentially. Our results confirm that the method of Kallast & Kivinukk (2003) and the one we propose in this paper share the same convergence property. On the other hand, by solving the early exercise boundary equation as a functional iteration as in Kim et al. (2013) we can accelerate the solution process through the parallelization of the iterations.

In summary, the analysis shows that our proposed method is superior to all commonly used algorithms to price American options, and seems promising for solving early exercise boundaries of even more complicated models.

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APPENDIX A. FIGURES AND TABLES

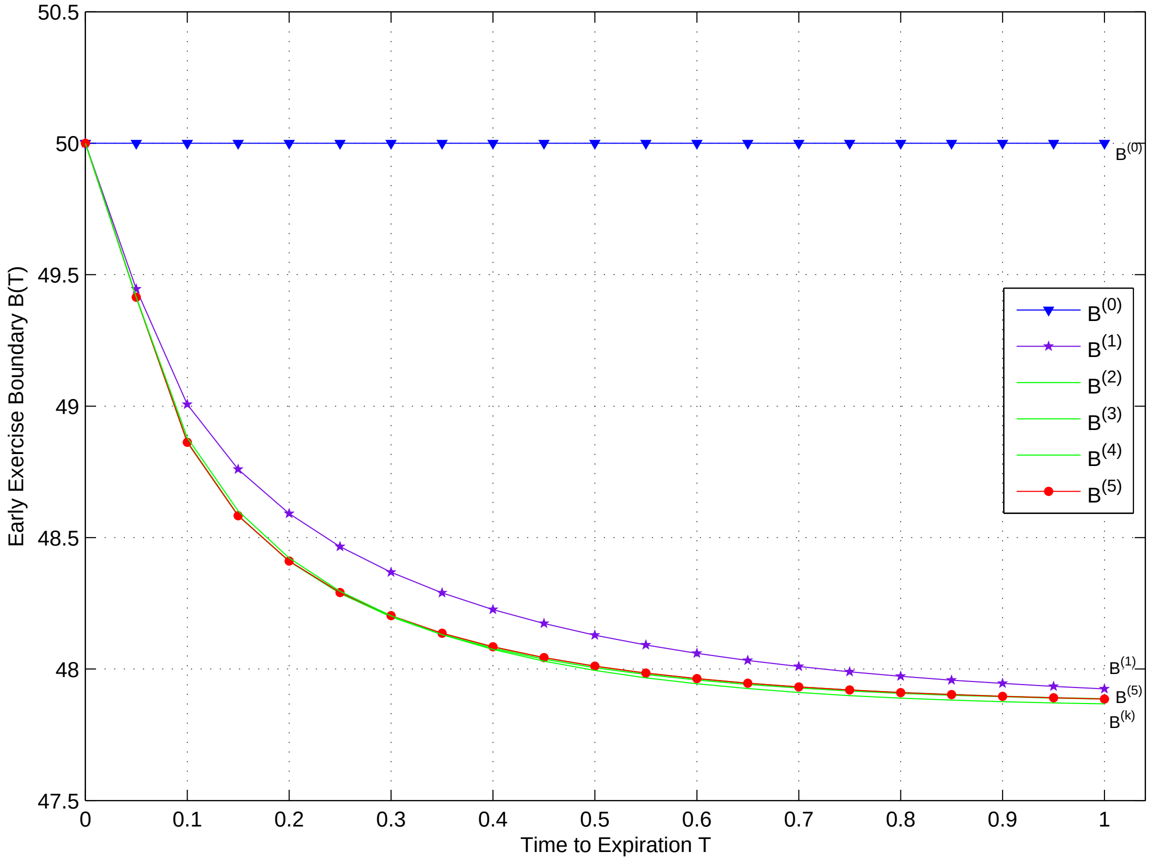

Figure A.1 – Iterations of the early exercise curve $B^{(k)}$ using our functional iterative method.

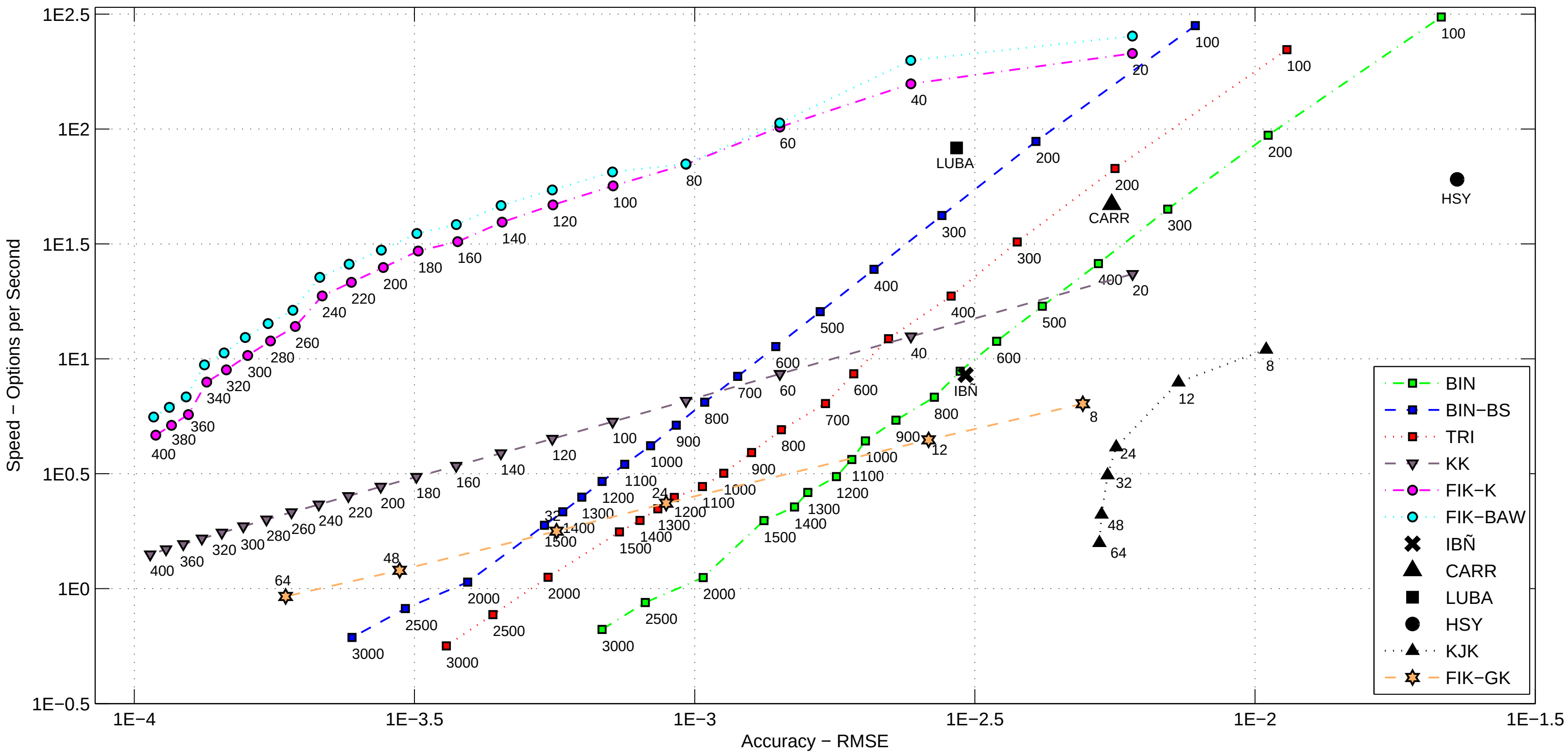

Figure A.2 – Speed-accuracy trade-off for a testing set corresponding to 7 865 options.

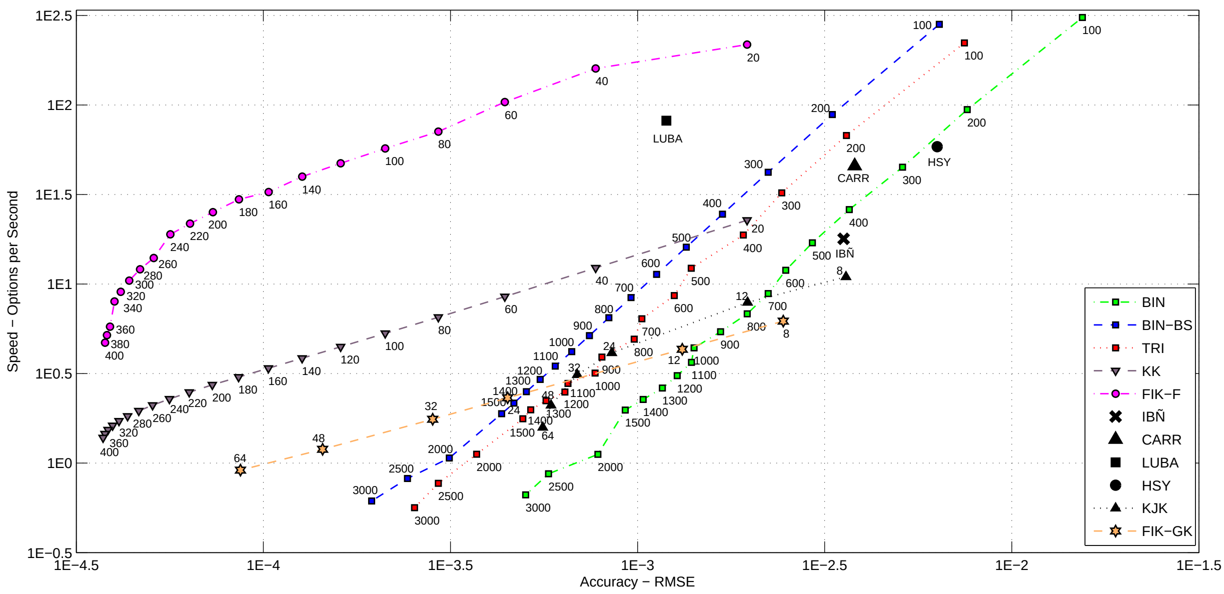

Figure A.3 – Speed-accuracy trade-off for short maturity options ($T < 1$).

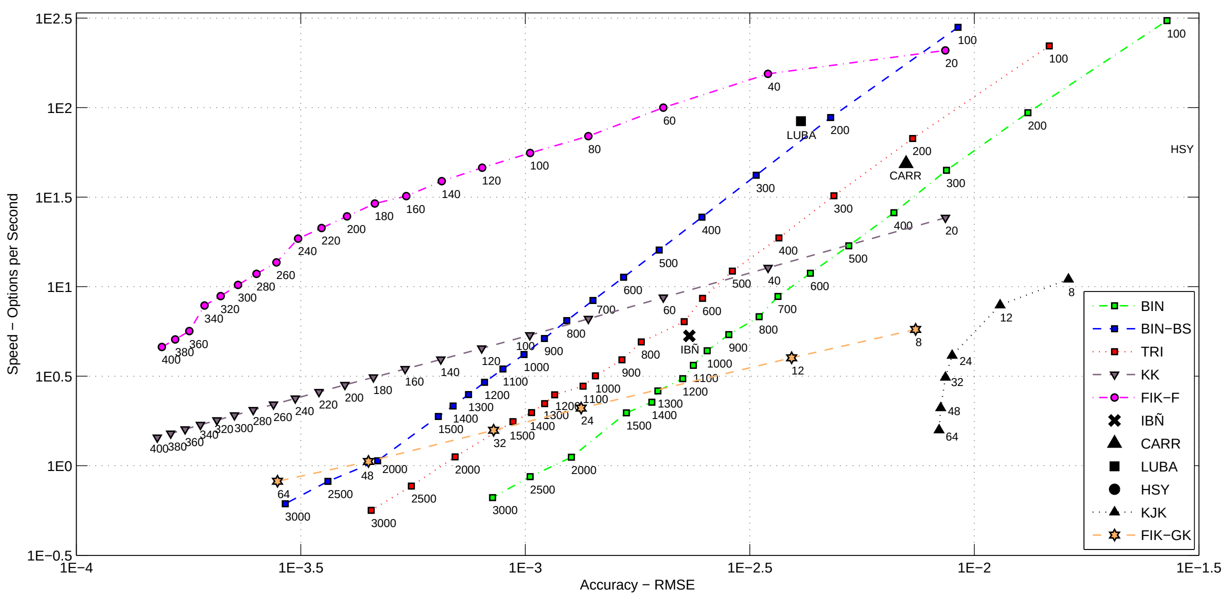

Figure A.4 – Speed-accuracy trade-off for long maturity options ($T \geq 1$).

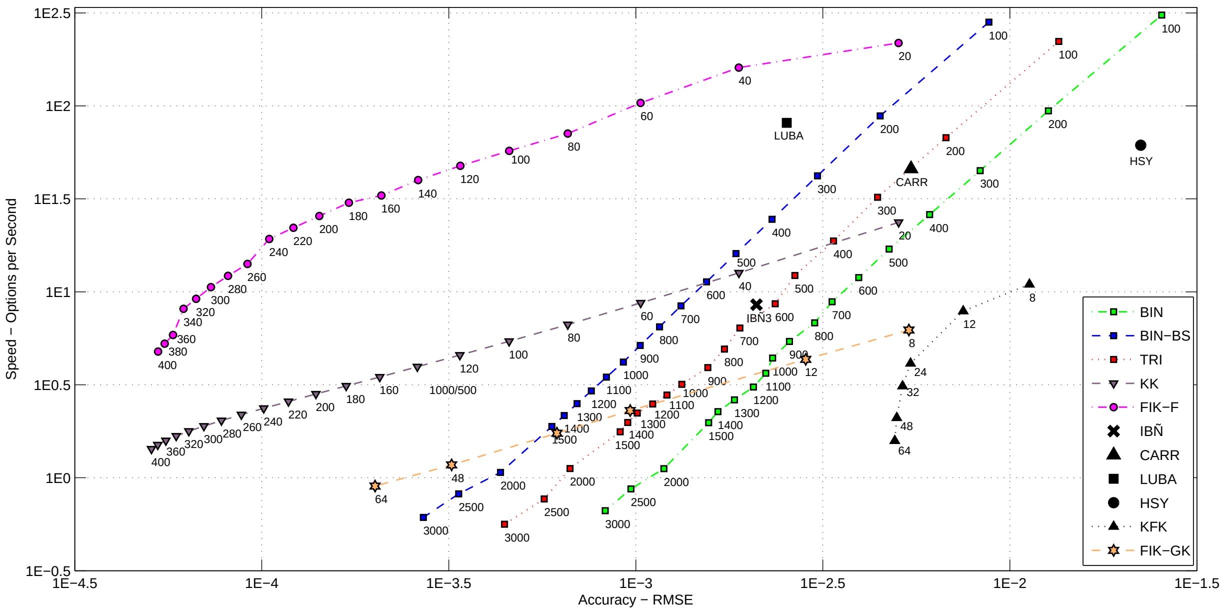

Figure A.5 – Speed-accuracy trade-off for at-the-money-options ($0.9 < S/K < 1.1$).

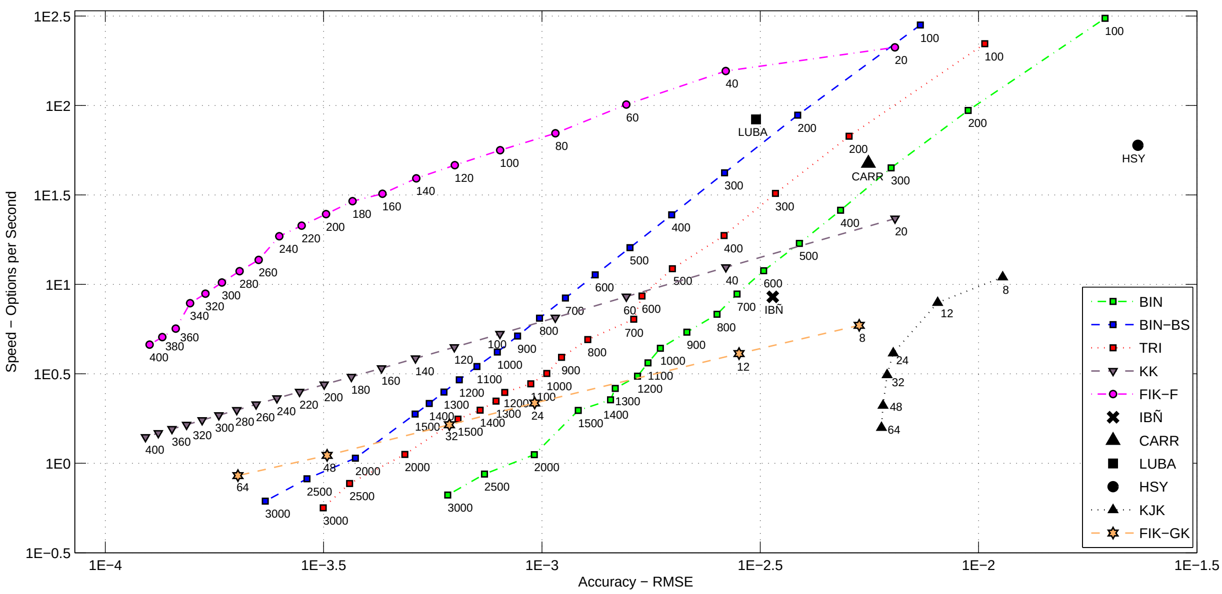

Figure A.6 – Speed-accuracy trade-off for in- and out-of-the-money options ($S/K \leq 0.9$ or $S/K \geq 1.1$).

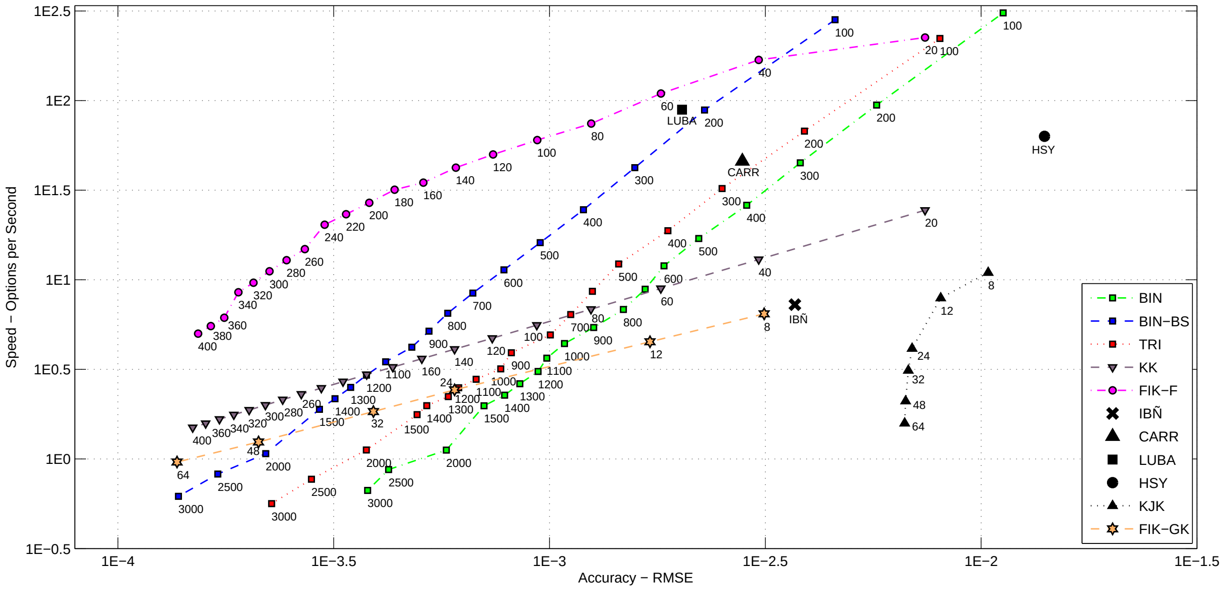

Figure A.7 – Speed-accuracy trade-off for low volatility options ($\sigma \leq 0.3$).

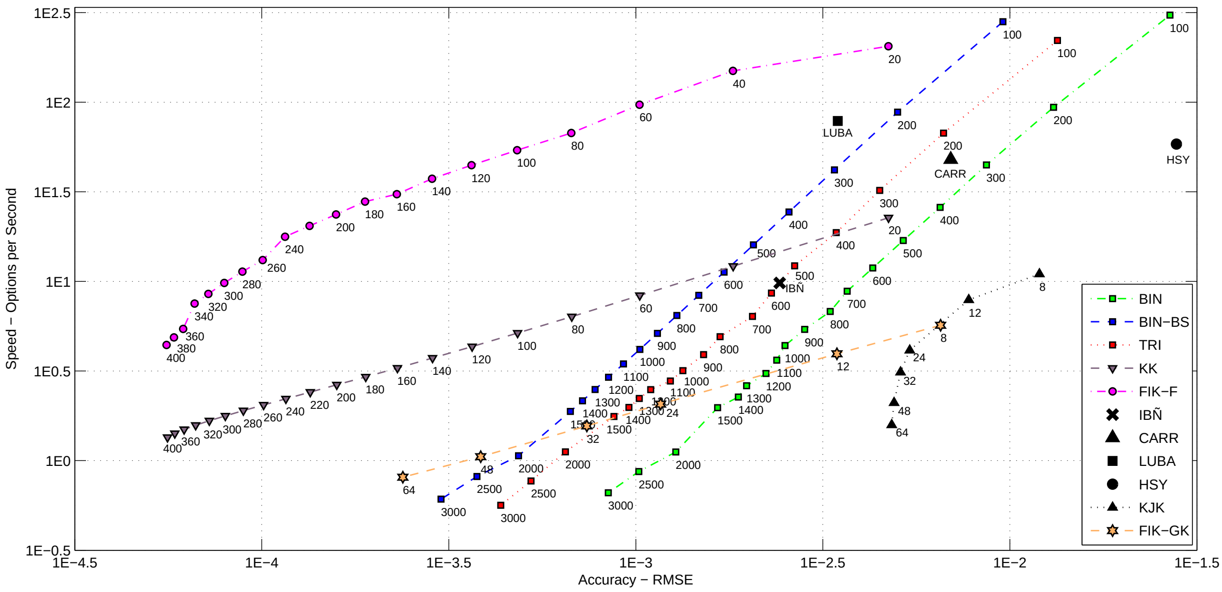

Figure A.8 – Speed-accuracy trade-off for high volatility options ($\sigma > 0.3$).

Table A.1 – Pricing Accuracy of the Functional Iterative Method FIK-F ($K = 100$, $T = 3$, $\sigma = 0.2$)

	Spot	(1) True	(2) $N_T = 20$	(3) $N_T = 40$	(4) $N_T = 60$	(5) $N_T = 80$	(6) $N_T = 100$	(7) $N_T = 120$	(8) $N_T = 140$	(9) $N_T = 200$	(10) $N_T = 300$	(11) $N_T = 400$
$r = 0.04$	80	23.22837	23.23249	23.22990	23.22921	23.22892	23.22876	23.22866	23.22860	23.22850	23.22843	23.22840
$q = 0.04$			*0.00412*	*0.00153*	*0.00084*	*0.00055*	*0.00039*	*0.00029*	*0.00023*	*0.00013*	*0.00006*	*0.00003*
	100	12.60529	12.60836	12.60644	12.60592	12.60568	12.60556	12.60548	12.60543	12.60534	12.60529	12.60526
			0.00308	*0.00116*	*0.00063*	*0.00040*	*0.00027*	*0.00019*	*0.00014*	*0.00006*	*0.00000*	*0.00002*
	120	6.48247	6.48442	6.48323	6.48289	6.48274	6.48266	6.48261	6.48257	6.48252	6.48248	6.48246
			0.00194	*0.00075*	*0.00042*	*0.00027*	*0.00018*	*0.00013*	*0.00010*	*0.00004*	*0.00000*	*0.00002*
$r = 0.04$	80	33.90208	33.90228	33.90216	33.90213	33.90212	33.90211	33.90210	33.90210	33.90210	33.90209	33.90209
$q = 0.12$			*0.00019*	*0.00008*	*0.00005*	*0.00003*	*0.00003*	*0.00002*	*0.00002*	*0.00001*	*0.00001*	*0.00001*
	100	22.83353	22.83359	22.83357	22.83357	22.83356	22.83356	22.83356	22.83356	22.83356	22.83356	22.83356
			0.00006	*0.00004*	*0.00003*	*0.00003*	*0.00003*	*0.00003*	*0.00003*	*0.00002*	*0.00002*	*0.00002*
	120	14.50205	14.50215	14.50215	14.50215	14.50215	14.50215	14.50215	14.50215	14.50215	14.50215	14.50215
			0.00010	*0.00010*	*0.00010*	*0.00010*	*0.00010*	*0.00010*	*0.00010*	*0.00010*	*0.00010*	*0.00010*
$r = 0.08$	80	20.35002	20.34699	20.35250	20.35163	20.35080	20.35031	20.35003	20.34990	20.34982	20.34993	20.35000
$q = 0.04$			*0.00303*	*0.00248*	*0.00161*	*0.00078*	*0.00029*	*0.00001*	*0.00012*	*0.00020*	*0.00009*	*0.00003*
	100	8.94399	8.94509	8.94445	8.94427	8.94419	8.94414	8.94411	8.94409	8.94405	8.94402	8.94401
			0.00110	*0.00045*	*0.00027*	*0.00019*	*0.00014*	*0.00011*	*0.00009*	*0.00006*	*0.00003*	*0.00001*
	120	3.89743	3.90030	3.89855	3.89808	3.89787	3.89775	3.89768	3.89763	3.89755	3.89749	3.89747
			0.00287	*0.00111*	*0.00064*	*0.00044*	*0.00032*	*0.00025*	*0.00020*	*0.00012*	*0.00006*	*0.00003*
$r = 0.08$	80	25.65774	25.66244	25.65946	25.65870	25.65838	25.65821	25.65811	25.65804	25.65793	25.65786	25.65783
$q = 0.12$			*0.00470*	*0.00172*	*0.00096*	*0.00064*	*0.00047*	*0.00036*	*0.00029*	*0.00019*	*0.00011*	*0.00009*
	100	15.49841	15.50063	15.49924	15.49887	15.49871	15.49863	15.49858	15.49854	15.49849	15.49845	15.49844
			0.00222	*0.00082*	*0.00046*	*0.00030*	*0.00021*	*0.00016*	*0.00013*	*0.00008*	*0.00004*	*0.00003*
	120	8.88548	8.88646	8.88587	8.88571	8.88564	8.88560	8.88558	8.88556	8.88554	8.88552	8.88552
			0.00098	*0.00038*	*0.00022*	*0.00015*	*0.00012*	*0.00009*	*0.00008*	*0.00005*	*0.00004*	*0.00003*
	RMSE		2.530E-03	1.142E-03	6.799E-04	3.978E-04	2.502E-04	1.808E-04	1.486E-04	1.039E-04	5.950E-05	4.308E-05

Note – Column (1) reports the true value of an American put option based on a binomial tree with 15 000 time-steps that uses Black & Scholes at the last time-step. Columns (2) to (11) report prices of American put options calculated using the FIK-F method with different number of time-steps (N_T). Absolute errors between the true value and each American option price are displayed in italics. The last row reports the root mean squared error (RMSE).

Table A.2 – Prices of American Put Options ($K = 100$, $q = 0.04$, $r = 0.04$)

	Spot	True	BAW	LS	JZ	HSY	CARR	BIN 500	TRI 500	LUBA	BIN-BS 500	BIN 1000
$T = 0.5$	80	20.14372	20.12517	20.13108	20.13987	20.13870	20.14286	20.14385	20.14333	20.14433	20.14360	20.14344
$\sigma = 0.2$			*0.01854*	*0.01264*	*0.00384*	*0.00502*	*0.00086*	*0.00013*	*0.00039*	*0.00061*	*0.00012*	*0.00027*
	100	5.54634	5.55141	5.52130	5.54328	5.54794	5.54499	5.54378	5.54501	5.54600	5.54735	5.54504
			0.00507	*0.02504*	*0.00306*	*0.00160*	*0.00135*	*0.00256*	*0.00133*	*0.00034*	*0.00101*	*0.00130*
	120	0.70724	0.71067	0.70584	0.70776	0.70738	0.70686	0.70776	0.70747	0.70719	0.70697	0.70747
			0.00342	*0.00140*	*0.00052*	*0.00014*	*0.00039*	*0.00051*	*0.00023*	*0.00005*	*0.00028*	*0.00023*
$T = 0.5$	80	24.67736	24.67381	24.61927	24.65983	24.67603	24.67401	24.67449	24.67513	24.67858	24.67871	24.67528
$\sigma = 0.5$			*0.00355*	*0.05810*	*0.01754*	*0.00133*	*0.00335*	*0.00287*	*0.00223*	*0.00122*	*0.00135*	*0.00208*
	100	13.80581	13.81879	13.73791	13.79824	13.80981	13.80246	13.79944	13.80249	13.80651	13.80824	13.80258
			0.01298	*0.06790*	*0.00757*	*0.00400*	*0.00335*	*0.00637*	*0.00332*	*0.00070*	*0.00243*	*0.00323*
	120	7.28738	7.30229	7.25333	7.28621	7.28808	7.28518	7.28982	7.28716	7.28743	7.28901	7.28720
			0.01491	*0.03405*	*0.00117*	*0.00070*	*0.00220*	*0.00244*	*0.00022*	*0.00005*	*0.00163*	*0.00018*
$T = 3$	80	23.22837	23.31946	23.12495	23.17341	23.24958	23.22514	23.22921	23.22781	23.22219	23.22922	23.22864
$\sigma = 0.2$			*0.09109*	*0.10342*	*0.05496*	*0.02121*	*0.00323*	*0.00084*	*0.00056*	*0.00618*	*0.00085*	*0.00027*
	100	12.60529	12.76321	12.56193	12.58919	12.59927	12.60001	12.60044	12.60236	12.60296	12.60732	12.60282
			0.15792	*0.04335*	*0.01610*	*0.00602*	*0.00528*	*0.00484*	*0.00293*	*0.00233*	*0.00204*	*0.00246*
	120	6.48247	6.62559	6.41858	6.49419	6.49209	6.47798	6.48133	6.48398	6.48009	6.48385	6.48423
			0.14312	*0.06390*	*0.01171*	*0.00961*	*0.00449*	*0.00115*	*0.00151*	*0.00239*	*0.00137*	*0.00176*
$T = 3$	80	37.97483	38.31502	37.98111	37.90752	38.02363	37.96429	37.97287	37.97120	37.97472	37.97786	37.97255
$\sigma = 0.5$			*0.34019*	*0.00628*	*0.06731*	*0.04880*	*0.01054*	*0.00196*	*0.00363*	*0.00011*	*0.00302*	*0.00228*
	100	30.74247	31.13286	30.70746	30.71458	30.72989	30.73039	30.73046	30.73526	30.74201	30.74631	30.73639
			0.39039	*0.03502*	*0.02790*	*0.01258*	*0.01208*	*0.01201*	*0.00721*	*0.00046*	*0.00383*	*0.00609*
	120	25.21333	25.61935	25.20967	25.21672	25.20719	25.20052	25.22383	25.21716	25.21079	25.21707	25.21810
			0.40603	*0.00365*	*0.00339*	*0.00614*	*0.01281*	*0.01050*	*0.00384*	*0.00253*	*0.00375*	*0.00477*
	RMSE		2.016E-01	4.819E-02	2.758E-02	1.633E-02	6.517E-03	5.367E-03	3.016E-03	2.206E-03	2.162E-03	2.751E-03

Note – Each column in the table reports the price of an American put option using a different method. The true value is based on a binomial tree with 15 000 time-steps that uses Black & Scholes at the last time-step. BAW is the quadratic approximation of Barone-Adesi & Whaley (1987). LS is the least square Monte Carlo approach of Longstaff & Schwartz (2001). JZ is the refined quadratic approximation of Ju & Zhong (1999). HSY is the six-point recursive integration method of Huang et al. (1996). CARR is the six-point randomization method of Carr (1998). BIN is the binomial tree method of Cox et al. (1979). TRI is the trinomial tree method of Boyle (1988). LUBA is the lower and upper bound approximation method of Broadie & Detemple (1996). BIN-BS is the binomial model using Black & Scholes at the last time-step of Broadie & Detemple (1996). Numbers below some methods indicate the number of time-steps. Absolute errors between the true value and each American option price are displayed in italics. The last row reports the root mean squared error (RMSE).

Table A.3 – Prices of American Put Options ($K = 100$, $q = 0.04$, $r = 0.04$)

	Spot	True	BIN-BS 1000	BIN 2500	IBN	FIK-F 60	TRI 2500	BIN-BS 2500	KJK 32	FIK-GK 32	KK 400	FIK-F 400
$T = 0.5$	80	20.14372	20.14366	20.14376	20.14094	20.14384	20.14369	20.14370	20.14355	20.14373	20.14373	20.14373
$\sigma = 0.2$			*0.00005*	*0.00004*	*0.00277*	*0.00012*	*0.00003*	*0.00002*	*0.00016*	*0.00001*	*0.00001*	*0.00001*
	100	5.54634	5.54683	5.54580	5.54661	5.54638	5.54605	5.54652	5.54633	5.54633	5.54631	5.54631
			0.00049	*0.00054*	*0.00027*	*0.00004*	*0.00029*	*0.00018*	*0.00001*	*0.00001*	*0.00003*	*0.00003*
	120	0.70724	0.70711	0.70722	0.70727	0.70727	0.70730	0.70720	0.70727	0.70726	0.70725	0.70725
			0.00013	*0.00003*	*0.00003*	*0.00002*	*0.00006*	*0.00005*	*0.00003*	*0.00002*	*0.00001*	*0.00001*
$T = 0.5$	80	24.67736	24.67802	24.67764	24.67861	24.67757	24.67761	24.67760	24.67722	24.67737	24.67733	24.67733
$\sigma = 0.5$			*0.00066*	*0.00028*	*0.00125*	*0.00021*	*0.00025*	*0.00023*	*0.00014*	*0.00001*	*0.00003*	*0.00003*
	100	13.80581	13.80699	13.80447	13.80647	13.80592	13.80508	13.80623	13.80579	13.80580	13.80574	13.80574
			0.00118	*0.00134*	*0.00066*	*0.00011*	*0.00073*	*0.00042*	*0.00002*	*0.00001*	*0.00007*	*0.00007*
	120	7.28738	7.28818	7.28695	7.28775	7.28745	7.28754	7.28767	7.28742	7.28739	7.28733	7.28733
			0.00080	*0.00043*	*0.00037*	*0.00007*	*0.00016*	*0.00029*	*0.00004*	*0.00001*	*0.00005*	*0.00005*
$T = 3$	80	23.22837	23.22880	23.22817	23.22796	23.22921	23.22830	23.22853	23.22855	23.22848	23.22840	23.22840
$\sigma = 0.2$			*0.00043*	*0.00020*	*0.00041*	*0.00084*	*0.00007*	*0.00016*	*0.00018*	*0.00011*	*0.00002*	*0.00003*
	100	12.60529	12.60629	12.60426	12.60514	12.60592	12.60465	12.60565	12.60548	12.60544	12.60526	12.60526
			0.00100	*0.00103*	*0.00015*	*0.00063*	*0.00064*	*0.00036*	*0.00019*	*0.00015*	*0.00003*	*0.00002*
	120	6.48247	6.48314	6.48242	6.48245	6.48289	6.48284	6.48272	6.48268	6.48264	6.48246	6.48246
			0.00066	*0.00005*	*0.00002*	*0.00042*	*0.00036*	*0.00025*	*0.00020*	*0.00017*	*0.00002*	*0.00002*
$T = 3$	80	37.97483	37.97633	37.97666	37.97435	37.97655	37.97426	37.97536	37.97528	37.97519	37.97484	37.97485
$\sigma = 0.5$			*0.00150*	*0.00183*	*0.00048*	*0.00172*	*0.00057*	*0.00053*	*0.00044*	*0.00036*	*0.00001*	*0.00002*
	100	30.74247	30.74437	30.73995	30.74213	30.74404	30.74093	30.74316	30.74297	30.74290	30.74245	30.74245
			0.00190	*0.00252*	*0.00034*	*0.00157*	*0.00154*	*0.00069*	*0.00049*	*0.00043*	*0.00002*	*0.00002*
	120	25.21333	25.21518	25.21458	25.21311	25.21475	25.21429	25.21401	25.21386	25.21380	25.21330	25.21330
			0.00186	*0.00126*	*0.00022*	*0.00142*	*0.00096*	*0.00068*	*0.00054*	*0.00047*	*0.00003*	*0.00003*
	RMSE		1.066E-03	1.108E-03	9.350E-04	8.558E-04	6.387E-04	3.857E-04	2.714E-04	2.243E-04	3.228E-05	3.225E-05

Note – Each column in the table reports the price of an American put option using a different method. The true value is based on a binomial tree with 15 000 time-steps that uses Black & Scholes at the last time-step. BIN-BS is the binomial model using Black & Scholes at the last time-step of Broadie & Detemple (1996). BIN is the binomial tree method of Cox et al. (1979). IBN is the tree-point modified recursive integration method of Ibáñez (2003). FIK- F is the functional iterative method using a flat guess and trapezoidal rule. TRI is the trinomial tree method of Boyle (1988). KJK is the iterative method of Kim et al. (2013). FIK-GK is the functional iterative method using a flat guess and Gauss-Kronrod adaptive integral. KK is the recursive method of Kallast & Kivinukk (2003). Numbers below some methods indicate the number of time-steps. Absolute errors between the true value and each American option price are displayed in italics. The last row reports the root mean squared error (RMSE).

Table A.4 – Performance Statistics

	(1) RMSE	(2) Time	(3) Efficiency	(4) AE $< 10^{-3}$	(5) AE $< 10^{-4}$	(6) AE $< 10^{-5}$
BAW	2.189E-01	0.001	8.43	16.63%	11.57%	8.30%
BIN 500	4.172E-03	0.059	8.31	33.18%	10.64%	7.31%
BIN 1000	2.017E-03	0.228	7.68	49.28%	13.94%	7.57%
BIN 2500	8.166E-04	1.154	6.97	80.34%	22.75%	9.07%
BIN-BS 500	1.675E-03	0.063	9.16	48.75%	13.48%	7.27%
BIN-BS 1000	8.345E-04	0.239	8.52	75.04%	19.26%	8.05%
BIN-BS 2500	3.047E-04	1.232	7.89	99.87%	33.87%	10.40%
CARR	5.542E-03	0.021	9.06	46.34%	24.11%	8.19%
FIK-BAW 400	1.083E-04	0.176	10.87	99.91%	96.13%	32.21%
FIK-F 60	1.419E-03	0.010	11.16	82.20%	42.10%	13.81%
FIK-F 400	1.092E-04	0.209	10.69	99.92%	96.01%	32.17%
FIK-GK 24	9.636E-04	0.452	7.74	99.06%	69.43%	24.44%
HSY	2.294E-02	0.017	7.85	40.90%	26.19%	10.96%
IBN	3.044E-03	0.117	7.94	81.84%	39.86%	12.68%
JZ	3.665E-02	0.001	10.21	24.31%	15.07%	8.96%
KJK 24	6.101E-03	0.242	6.52	81.72%	50.21%	20.20%
KK 400	1.067E-04	0.711	9.49	99.92%	96.21%	32.02%
LS	7.737E-02	1.780	1.98	2.28%	0.17%	0.01%
LUBA	2.933E-03	0.012	10.25	64.28%	36.39%	12.97%
TRI 500	2.218E-03	0.082	8.61	47.74%	12.22%	7.20%
TRI 1000	1.127E-03	0.315	7.94	70.39%	17.30%	8.01%
TRI 2500	4.364E-04	1.354	7.43	96.07%	30.63%	10.44%

Note – The table reports performance statistics for each numerical method over 7 865 option values. BAW is the quadratic approximation of Barone-Adesi & Whaley (1987). BIN is the binomial tree method of Cox et al. (1979). BIN-BS is the binomial model using Black & Scholes at the last time-step of Broadie & Detemple (1996). CARR is the six-point randomization method of Carr (1998). FIK-BAW is the functional iterative method with the initial guess of Barone-Adesi & Whaley (1987) and the trapezoidal rule. FIK-F is the functional iterative method with a flat initial guess and the trapezoidal rule. FIK-GK is is the functional iterative method with a flat initial guess and the Gauss-Kronrod quadrature. HSY is the six-point recursive integration method of Huang et al. (1996). IBN is the tree-point modified recursive integration method of Ibáñez (2003). JZ is the refined quadratic approximation of Ju & Zhong (1999). KJK is the iterative method of Kim et al. (2013). KK is the recursive method of Kallast & Kivinukk (2003). LS is the least-squares Monte Carlo approach of Longstaff & Schwartz (2001). LUBA is the lower and upper bound approximation method of Broadie & Detemple (1996). TRI the trinomial tree method of Boyle (1988). Numbers next to some methods specify the number of time-steps. Column (1) reports the root mean squared error (RMSE) with respect to the true value that was computed with a binomial tree with 15 000 time-steps that uses Black & Scholes at the last time-step. Column (2) reports the average time in seconds needed to value the 7 865 options. Efficiency in (3) is computed as $-\log(\text{RMSE} \times \text{Time})$. The last three columns (4) to (6) report the percentage of options for which the absolute error (AE) is lower than the corresponding threshold.

Table A.5 – Performance Statistics Using Richardson Extrapolation

	(1) RMSE	(2) Time	(3) Efficiency
BIN-BS $600/400/200$	9.234E-04	0.140	8.95
BIN-BS $400/200$	5.890E-04	0.052	10.39
BIN-BS $600/300$	3.456E-04	0.112	10.16
BIN-BS $800/400$	2.874E-04	0.195	9.79
BIN-BS $1000/500$	2.264E-04	0.301	9.59
FIK-F $100/50$	4.936E-04	0.035	10.97
FIK-F $150/100/50$	1.771E-04	0.073	11.26
FIK-F $200/150/100$	1.257E-04	0.086	11.44
FIK-F $250/200/150$	1.061E-04	0.124	11.24
FIK-F $300/250/200$	8.978E-05	0.182	11.02
KK $100/50$	4.951E-04	0.286	8.86
KK $160/80$	2.861E-04	0.444	8.97
KK $200/100$	2.238E-04	0.547	9.01
KK $300/150$	1.498E-04	0.809	9.02
KK $150/100/50$	1.813E-04	0.560	9.20

Note $-$ The table reports performance statistics for three different methods that are accelerated with the use of Richardson extrapolation. BIN-BS is the binomial model using Black & Scholes at the last time-step of Broadie & Detemple (1996). FIK-F is the functional iterative method with a flat initial guess and the trapezoidal rule. KK is the recursive method of Kallast & Kivinukk (2003). The notation $N_3/N_2/N_1$ stands for the 3-point Richardson extrapolation (see e.g. Geske & Johnson, 1984) where the price of the option is found as $P = P_3 + 7/2P_2 - 1/2P_1$. The notation N_2/N_1 stands for the 2-point Richardson extrapolation (see e.g. Bunch & Johnson, 1992) where the price of the option is found as $P = 2P_2 - P_1$. In the formulas P_1, P_2, and P_3 are the prices obtained using N_1, N_2, and N_3 time-steps, respectively. Column (1) reports the root mean squared error (RMSE) with respect to the true value that was computed with a binomial tree with 15 000 time-steps that uses Black & Scholes at the last time-step. Column (2) reports the average time in seconds needed to value the 7 865 options. Efficiency in (3) is computed as $-\log(\text{RMSE} \times \text{Time})$.