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Phase fluctuations in a laser with atomic memory effects

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We calculate phase fluctuations of the phase operator defined by Pegg and Barnett [Europhys. Lett. **6**, 483 (1988); J. Mod. Opt. **36**, 7 (1989); Phys. Rev. A **39**, 1665 (1989)] in a laser with atomic memory effects. First, we compute the phase fluctuations of a laser, using the Scully-Lamb theory. In a single curve of $\langle(\Delta\phi)^2\rangle$ versus time, we obtain the expected behavior: namely, for $Dt \sim 1$ the Schawlow-Townes linewidth and for $t \gg D^{-1}$ the phase fluctuations corresponding to a random phase. As we include the atomic memory effects, we obtain a noise reduction for times of the order of the atomic decay time.

I. INTRODUCTION

In quantum optics the phase fluctuations of electromagnetic fields play an essential role, since in physical systems such as lasers they are associated with a loss of coherence of the field. Up to now, semiclassical and phenomenological approaches have been employed to determine phase fluctuations. This fact is closely related to the absence of a well-behaved quantum phase operator.

In a series of recent papers, Pegg and Barnett[1-3] defined a phase operator, $\hat{\phi}_\theta$, constructed with the use of sharply defined phase states. Their definition is

$$\hat{\phi}_\theta = \sum_{m=0}^{s} \theta_m |\theta_m\rangle\langle\theta_m| \ , \tag{1.1}$$

and the phase states[4] are defined as

$$|\theta_m\rangle = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} e^{i\theta_m n} |n\rangle \ , \tag{1.2}$$

where $|n\rangle$ are the occupation number states. The $|\theta_m\rangle$ form an orthonormal basis of a $(s+1)$-dimensional Hilbert space. Their eigenvalues are

$$\theta_m = \theta_0 + \frac{2\pi}{s+1} m \ , \quad m = 0,1,\ldots,s \ , \tag{1.3}$$

where θ_0 is arbitrary and defines a particular basis set. After the calculation of the physical quantities, the $s \rightarrow \infty$ limit is taken.

In Sec. II we compute the fluctuations of the phase operator, using the Scully-Lamb[5,6] theory, as a function of an adimensional time, for an extensive range of the parameters. Finally, in Sec. III we include the atomic memory effects, where one can use long-lived atoms as an active medium. Recent results show that the atomic memory effects can lead to a reduction of spontaneous emission noise for short measurement times.[7] Also in this section we find the density matrix and calculate the phase fluctuations, and obtain indeed a reduction in the phase noise for measurement times of the order of the atomic lifetime.

II. FLUCTUATION OF $\hat{\phi}_\theta$ IN THE SCULLY-LAMB LASER THEORY

In order to compute of the fluctuations of a physical observable, we need to know the density operator. In the Scully-Lamb[5,6] laser theory, the density matrix is known, when the diagonal elements reach their stationary values. We consider, as an initial condition, that the laser is in a coherent state[8] (in the limit $s \rightarrow \infty$):

$$|\alpha\rangle = e^{-r^2/2} \sum_{n=0}^{s} \frac{r^n}{\sqrt{n!}} e^{i\zeta_0 n} |n\rangle \ , \tag{2.1}$$

where r and ζ_0 are, respectively, the amplitude and phase of the coherent state. Thus the density matrix at time t is given by

$$\rho(t) = e^{-r^2} \sum_{n,l=0}^{s} \frac{(r)^n}{\sqrt{n!}} \frac{(r)^l}{\sqrt{l!}} e^{i\zeta_0(n-l)} e^{-1/2DT(n-l)^2} |n\rangle\langle l| \ , \tag{2.2}$$

where D is the decay constant of the off-diagonal matrix elements. In the classical limit, the decay constant is associated to the phase diffusion.

In order to determine the variance of $\hat{\phi}_\theta$, we calculate the averages by tracing over the phase states, so

$$\begin{aligned} \langle(\Delta\hat{\phi}_\theta)^2\rangle &= \langle\hat{\phi}_\theta^2\rangle - \langle\phi_\theta\rangle^2 \\ &= \sum_m \theta_m^2 \langle\theta_m|\rho|\theta_m\rangle - \left[\sum_m \theta_m \langle\theta_m|\rho|\theta_m\rangle\right]^2 \ , \end{aligned} \tag{2.3}$$

where we have used

$$\hat{\phi}_\theta |\theta_{m'}\rangle = \theta_{m'} |\theta_{m'}\rangle \ . \tag{2.4}$$

We notice from Eq. (2.3) that $\langle\theta_m|\rho|\theta_m\rangle$ is the probability distribution of the eigenvalues θ_m. It is straightforward to calculate $\langle\theta_m|\rho|\theta_m\rangle$ from Eq. (2.2) as

$$\langle \theta_m | \rho | \theta_m \rangle = \frac{1}{s+1} \left[1 + 2e^{-r^2} \sum_{n=1}^{s} \sum_{l=0}^{n-1} \frac{r^n}{\sqrt{n!}} \frac{r^l}{\sqrt{l!}} \cos[(\zeta_0 - \theta_m)(n-l)] e^{-1/2Dt(n-l)^2} \right] . \tag{2.5}$$

If we take the continuous limit in Eq. (2.3), one can define a probability density distribution:

$$\begin{aligned} P^{(\zeta_0)}(\theta) &= \frac{s+1}{2\pi} \langle \theta | \rho | \theta \rangle \\ &= \frac{1}{2\pi} \left[1 + 2e^{-r^2} \sum_{n=1}^{\infty} \sum_{l=0}^{n-1} \frac{r^n}{\sqrt{n!}} \frac{r^l}{\sqrt{l!}} \cos[(\zeta_0 - \theta)(n-l)] e^{-1/2Dt(n-l)^2} \right] . \end{aligned} \tag{2.6}$$

It is clear that Eq. (2.6) is normalized over the entire range $[\theta_0, \theta_0 + 2\pi]$, which corresponds to the complete range of eigenvalues given in Eq. (1.3), when $s \to \infty$. Moreover, this function is invariant under $\zeta_0 \to \zeta_0 + 2\pi k$, with k integer, as we expect physically. In general, we can evaluate this probability distribution numerically. However, for $r^2 \gg 1$ we can perform the following approximation:

$$P(k) = \frac{r^{2k}}{k!} e^{-r^2} \simeq \frac{1}{(2\pi r^2)^{1/2}} e^{-(r^2 - k)^2 / 2r^2} . \tag{2.7}$$

We now substitute the square root of Eq. (2.7) into Eq. (2.6), integrate in n and l, and obtain

$$P^{(\zeta 0)}(\theta) = \frac{1}{2\pi} \frac{1}{(2\pi r^2)^{1/2}} \int_0^\infty dn \int_0^\infty dl \exp \left[\frac{-1}{4r^2} [(r^2 - n)^2 + (r^2 - l)^2] + i(\zeta_0 - \theta)(n-l) - \tfrac{1}{2} Dt (n-l)^2 \right] , \tag{2.8}$$

$$P^{(\zeta_0)}(\theta) = \frac{1}{[\pi 2(1/4r^2 + Dt)]^{1/2}} \times \exp \left[\frac{-(\zeta_0 - \theta)^2}{2(1/4r^2 + Dt)} \right] .$$

This probability is normalized for $-\infty \leq \theta \leq \infty$. We notice that Eq. (2.8) is a Gaussian with a variance that increases in time. We would like to define a probability distribution in the $[\theta_0, \theta_0 + 2\pi]$ range. This can be achieved by defining

$$\begin{aligned} P(\theta, \zeta_0) &= \sum_{k=-\infty}^{\infty} P^{(\zeta_0 + 2\pi k)}(\theta) \\ &= \frac{1}{(\pi a^2)^{1/2}} \sum_{k=-\infty}^{\infty} \exp \left[\frac{-(\zeta_0 + 2k\pi - \theta)^2}{a^2} \right] , \end{aligned} \tag{2.9}$$

where

$$a^2 = 2 \left[\frac{1}{4r^2} + Dt \right] . \tag{2.10}$$

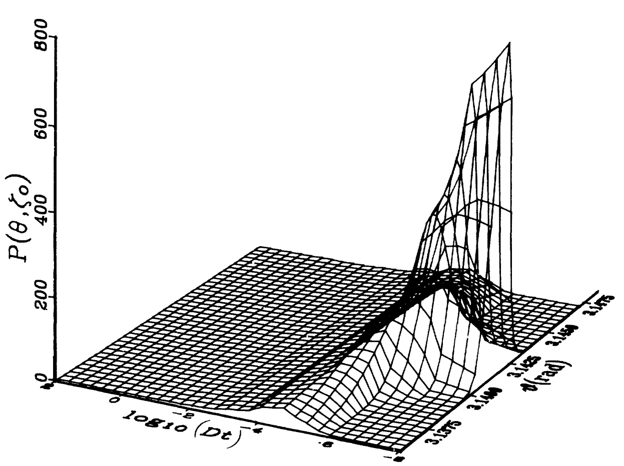

FIG. 1. Probability $P(\theta,\zeta_0)$ vs phase (Dt), as per Eq. (2.9) with $r^2 = 10^6$. For (Dt) small, a strong peak is present at the center of the interval. As (Dt) increases, this peak is reduced, and finally when $Dt \gg 1$, the distribution becomes flat and equal to $1/2\pi$.

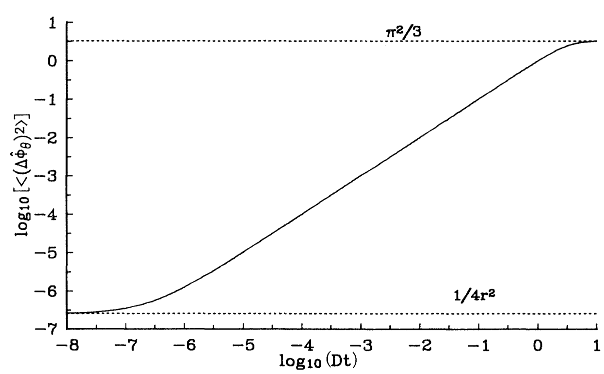

FIG. 2. Phase fluctuations of a laser vs time (Dt). In the lower end $(Dt \ll 1)$ we obtain the shot-noise limit $1/4r^2$ and in the upper end $(Dt \gg 1)$ we obtain the value of the phase fluctuations corresponding to a random phase $(\pi^2/3)$. The region between the two ends is linear and the slope corresponds to the Schawlow-Townes linewidth. The number of photons is $r^2 = 10^6$.

It is simple to prove that this probability is normalized in the range $[\theta_0, \theta_0+2\pi]$.

With Eq. (2.9) we can determine the variance of $\hat{\phi}_0$ for all times. In Fig. 1 we show[9] $P(\theta,\zeta_0)$ for $r^2=10^6$. For Dt small, there is a strong peak at the center of the interval. As Dt increases, this peak is less and less pronounced, and for $Dt \gg 1$ the distribution is already flat, with $P(\theta,\zeta_0)\sim 1/2\pi$.

The density matrix, Eq. (2.2), has two temporal limits. For $t=0$, we have a coherent state, namely

$$P(\theta,\zeta_0)\simeq P^{(\zeta_0)}(\theta)$$

and[2]

$$\langle(\Delta\hat{\phi}_\theta)^2\rangle(t=0)=\frac{1}{4r^2}\ . \tag{2.11}$$

The other limit is $t\to\infty$, ρ is diagonal in the $|n\rangle$ representation, which corresponds to a system with a random phase and phase variance equal to $\pi^2/3$. A way of seeing this is as follows:

$$\begin{aligned} P(\theta,\zeta_0)&=\lim_{l,t\to\infty}\frac{1}{(\pi a^2)^{1/2}}\left[\sum_{k=0}^{l}\exp\left[\frac{-(\zeta_0+2k\pi-\theta)^2}{a^2}\right]+\sum_{k=1}^{l}\exp\left[\frac{-(\zeta_0-2k\pi-\theta)^2}{a^2}\right]\right]\\ &\simeq\lim_{l,t\to\infty}\frac{1}{2\pi}\frac{1}{\sqrt{\pi}}\left[\int_{(\zeta_0-\theta)/a}^{(\zeta_0-\theta+2\pi l)/a}dx\,e^{-x^2}-\int_{(\zeta_0-\theta-2\pi l)/a}^{(\zeta_0-\theta-2\pi)/a}dx\,e^{-x^2}\right]\\ &=\frac{1}{2\pi}\ , \end{aligned} \tag{2.12}$$

which is the result one should expect if one has a uniform distribution (random phase) over a 2π interval.

We assume that at $t=0$ we had the laser in the coherent state. This of course assumes that, in the first place, we are well above threshold, so that we have Poissonian statistics. In the second place, t is the phase measurement time, so that t very small implies that during this measurement we do not allow the phase to diffuse or even the atom to decay. Now, ζ_0 is the phase of the coherent state (initial) and $\theta_0\to\theta_0+2\pi$ is the interval. Since all ζ_0 have equal probability, on the average, ζ_0 will be in the center of the interval, that is $\zeta_0=\pi+\theta_0$. Alternatively, one would require physically that $\langle(\Delta\hat{\phi}_\theta)^2\rangle$ is θ_0 independent, which implies a symmetrical distribution in the $[\theta_0,\theta_0+2\pi]$ interval. This is only satisfied if the above condition is verified. Using this and Eq. (2.12), after a straightforward calculation (Appendix A) we get, for the phase fluctuations,

$$\begin{aligned} \langle(\Delta\hat{\phi}_\theta)^2\rangle=&\left[\frac{1}{4r^2}+Dt\right]+4\pi^2\sum_{k=1}^{\infty}k^2\left[\Phi\left[\frac{-\pi+2\pi(k+1)}{a}\right]-\Phi\left[\frac{-\pi+2\pi k}{a}\right]\right]\\ &-4(\pi a^2)^{1/2}\sum_{k=1}^{\infty}k\left[\exp\left[\frac{-(-\pi+2\pi k)^2}{a^2}\right]-\exp\left[\frac{-[-\pi+2\pi(k+1)]^2}{a^2}\right]\right]\ , \end{aligned} \tag{2.13}$$

where $\Phi(\chi)$ is the error function and a^2 is given by (2.10).

In Fig. 2 we plot Eq. (2.13) in a logarithmic scale. We see that in an extensive range of the adimensional parameter Dt, the fluctuations are linear with Dt. This dependence corresponds to the Schawlow-Townes linewidth.[10] Moreover, notice that when $Dt\gg 1$, Eq. (2.13) converges rapidly to $\pi^2/3$.

III. ATOMIC MEMORY IN A LASER

The atomic memory effects are found when the measurement times are of the order of the atomic decay time. Experimentally, this can be achieved with atoms with long lifetimes in the active medium. Recent results show that these effects yield to a reduction of spontaneous emission noise. The phase fluctuations of a laser, considering atomic memory effects, is given by[7,11]

$$\langle\phi^2(t)\rangle=Dt+\frac{1}{(\Gamma/D)}(e^{-\Gamma t}-1)\ , \tag{3.1}$$

where Γ is the atomic decay constant. For some typical lasers, the ratio Γ/D can take values[9] between 10^7 and 10^{10}.

We will demonstrate that in order to include atomic memory effects, we must perform the following change:

$$Dt\to Dt+\frac{D}{\Gamma}(e^{-\Gamma t}-1)\ , \tag{3.2}$$

in Eq. (2.2).

In Appendix B, from Ref. 11, we have derived the master equation for the field density operator, up to order g^2. If we drop the index f in Eq. (B9), we have

$$\begin{aligned} \dot{\rho}(t)=&-ig\sum_j f(t,t_j)[C_j(a\rho-\rho a)+B_j(a^\dagger\rho-\rho a^\dagger)]\\ &-g^2\int_0^t dt'\sum_j f(t,t_j)\Theta(t'-t_j)[A_j(aa^\dagger\rho+\rho aa^\dagger-2a^\dagger\rho a)+D_j(a^\dagger a\rho+\rho a^\dagger a-2a\rho a^\dagger)]+L\rho\ , \end{aligned} \tag{3.3}$$

with

$$f(t,t_j)=\Theta(t,t_j)e^{-\Gamma(t-t_j)} . \tag{3.4}$$

Here, t_j is the injection time of the jth atom into the laser cavity. $\Theta(t)$ is the step function and $L\rho$ accounts for the damping of the field through cavity losses.[12] The coefficients A_j, B_j, C_j, and D_j are given by Eqs. (B7) and (B8).

In order to find the time behavior of the off-diagonal elements of ρ, we take the $N,N+p$ matrix elements of Eq. (3.3). The result is

$$\begin{aligned}\dot{\rho}_{N,N+P}=&-ig\sum_j f(t,t_j)C_j(\sqrt{N+1}\rho_{N+1,N+P}-\sqrt{N+P}\,\rho_{N,N+P-1})\\&+-ig\sum_j f(t,t_j)B_j(\sqrt{N}\,\rho_{N-1,N+P}-\sqrt{N+P+1}\rho_{N,N+P+1})\\&-g^2\int_0^t dt'\sum_j f(t,t_j)\Theta(t'-t_j)A_j[(2N+P+2)\rho_{N,N+P}-2\sqrt{N(N+P)}\rho_{N-1,N+P-1}]\\&-g^2\int_0^t dt'\sum_j f(t,t_j)\Theta(t'-t_j)D_j[(2N+P)\rho_{N,N+P}-2\sqrt{(N+1)(N+P+1)}\rho_{N+1,N+P+1}]\\&-\tfrac{1}{2}\gamma(2N+P)\rho_{N,N+P}+\gamma\sqrt{(N+1)(N+P+1)}\rho_{N+1,N+P+1} ,\end{aligned} \tag{3.5}$$

where γ is the cavity decay constant and R is the pump rate.

We now expand A_j,D_j and B_j,C_j to zero and first order in g, respectively. Moreover, we will perform the following changes: $\epsilon\rightarrow\sqrt{N}$ and $\epsilon^*\rightarrow\sqrt{N+1}$. This is equivalent to taking ϵ as a classical electromagnetic field. The result is

$$\begin{aligned}\frac{1}{R}\frac{d}{dt}\rho_{N,N+P}(t)=&\frac{g^2}{\Gamma^2}e^{-\Gamma t}[(N+1)\rho_{N+1,N+P}-\sqrt{(N+1)(N+P+1)}\rho_{N,N+P-1}]\\&+\frac{g^2}{\Gamma^2}e^{-\Gamma t}[N\rho_{N-1,N+P}-\sqrt{N(N+P+1)}\rho_{N,N+P+1}]\\&+2\frac{g^2}{\Gamma^2}(1-e^{-\Gamma t})\sqrt{N(N+P)}\rho_{N-1,N+P+1}+\frac{\gamma}{R}\sqrt{(N+1)(N+P+1)}\rho_{N+1,N+P+1}\\&-\left[\frac{g^2}{\Gamma^2}(1-e^{-\Gamma t})(2N+P+2)+\frac{1}{2}\frac{\gamma}{R}(2N+P)\right]\rho_{N,N+P} .\end{aligned} \tag{3.6}$$

To obtain Eq. (3.6) the $\sum_j$ has been replaced by $R\int dt_j$, assuming a regular injection.[13]

In Appendix C we show that the solution of Eq. (3.6) is given by

$$\rho_{N,N+P}(t)=\rho_{N,N+P}(t=0)\exp\left[-\frac{D(t)}{2}\right] , \tag{3.7}$$

where

$$D(t)=p^2\left[Dt+\frac{1}{\Gamma/D}(e^{-\Gamma t}-1)\right] . \tag{3.8}$$

Thus, we proved Eq. (3.2).

Using Eq. (2.13) with the above-mentioned substitution, we readily get

$$\begin{aligned}\langle(\Delta\phi)^2\rangle=&\frac{a'^2}{2}+4\pi^2\sum_{k=1}^{\infty}k^2\left[\Phi\left[\frac{-\pi+2\pi(k+1)}{(a'^2)^{1/2}}\right]-\Phi\left[\frac{-\pi+2\pi k}{(a'^2)^{1/2}}\right]\right]\\&-4(\pi a'^2)^{1/2}\sum_{k=1}^{\infty}k\left[\exp\left[\frac{-(-\pi+2\pi k)^2}{a'^2}\right]-\exp\left[\frac{-[-\pi+2\pi(k+1)]^2}{a'^2}\right]\right]\end{aligned} \tag{3.9}$$

with

$$a'^2=2\left[\frac{1}{4r^2}+Dt+\frac{1}{\Gamma/D}(e^{-\Gamma t}-1)\right] . \tag{3.10}$$

In Fig. 3 we plot Eq. (3.9) for Γ/D between 10^4 and 10^7. In this figure we notice a noise reduction for $t\sim\Gamma^{-1}$. This reduction is more noticeable when Γ/D is small. This can be achieved in a micromaser rather than a laser.[14]

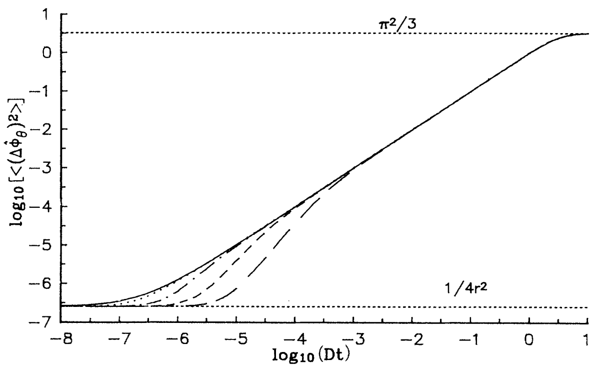

FIG. 3. Phase fluctuations vs time (Dt) with atomic memory effects. The various curves correspond to the following: ——, $\Gamma/D=\infty$; $\cdot\cdot\cdot\cdot$, $\Gamma/D=10^7$; $-\cdot-\cdot-$, $\Gamma/D=10^6$; $---$ $\Gamma/D=10^5$; — —, $\Gamma/D=10^4$. We notice a reduction of noise in the $t\approx\Gamma^{-1}$ region. The stronger noise reduction corresponds to the smaller (Γ/D). For all the curves we took $r^2=10^6$.

IV. SUMMARY

To summarize, we found, using the Pegg and Barnett formalism, the full-time behavior of the phase fluctuations in a laser. For very short times (of the order of the atomic decay times) we found a phase noise quieting. The present model is based on the Scully-Lamb laser theory and we assumed a large number of photons, with steady-state Poissonian photon statistics. The laser near the threshold or with a small number of photons requires a different model, and it is planned to be the subject of a future publication.

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APPENDIX A

Here we determine the phase fluctuations in the laser with the Scully-Lamb theory. Using Eqs. (2.2) and (2.6) we get

$$\langle\hat{\phi}_\theta\rangle=\frac{1}{(\pi a^2)^{1/2}}\times\sum_{k=-\infty}^{\infty}\int_{\theta_0}^{\theta_0+2\pi}d\theta\,\theta\exp\left[\frac{-(\zeta_0+2\pi k-\theta)^2}{a^2}\right], \tag{A1}$$

where a^2 is given by 2.10. Performing a change of variables, we find

$$\begin{aligned}\langle\hat{\phi}_\theta\rangle&=\frac{1}{(\pi a^2)^{1/2}}\sum_{k=-\infty}^{\infty}\int_{-\pi+2\pi k}^{-\pi+2\pi(k-1)}d\theta(\theta+\zeta_0-2\pi k)e^{-\theta^2/a^2}\\&=\zeta_0+\frac{2\pi}{(\pi a^2)^{1/2}}\sum_{k=1}^{\infty}k\left[\int_{-\pi+2\pi k}^{-\pi+2\pi(k-1)}d\theta e^{-\theta^2/a^2}-\int_{-\pi-2\pi k}^{-\pi-2\pi(k-1)}d\theta e^{-\theta^2/a^2}\right].\end{aligned} \tag{A2}$$

To obtain Eq. (A2), we used the condition $\zeta_0=\theta_0+\pi$ as discussed in Sec. II. From parity considerations, the second term in Eq. (A2) vanishes. Thus, we obtain

$$\langle\hat{\phi}_\theta\rangle=\zeta_0\,. \tag{A3}$$

We now calculate the fluctuations of $\hat{\phi}_\theta$:

$$\begin{aligned}\langle(\Delta\hat{\phi}_\theta)^2\rangle&=\langle(\hat{\phi}_\theta-\langle\hat{\phi}_\theta\rangle)^2\rangle\\&=\frac{1}{(\pi a^2)^{1/2}}\sum_{k=-\infty}^{\infty}\int_{-\pi+2\pi k}^{-\pi+2\pi(k-1)}d\theta(\theta-2\pi k)^2\exp\left[\frac{-\theta^2}{a^2}\right].\end{aligned} \tag{A4}$$

It is easily to show that

$$\begin{aligned}\langle(\Delta\hat{\phi}_\theta)^2\rangle=&\left[\frac{1}{4r^2}+Dt\right]+4\pi^2\sum_{k=1}^{\infty}k^2\left[\Phi\left[\frac{-\pi+2\pi(k+1)}{a}\right]-\Phi\left[\frac{-\pi+2\pi k}{a}\right]\right]\\&-4(\pi a^2)^{1/2}\sum_{k=1}^{\infty}k\left[\exp\left[\frac{-(-\pi+2\pi k)^2}{a^2}\right]-\exp\left[\frac{-[-\pi+2\pi(k+1)]^2}{a^2}\right]\right],\end{aligned} \tag{A5}$$

where $\Phi(x)$ is the error function.

APPENDIX B

Here, for completeness, we derive a master equation for the field density operator $\rho^f(t)$ in a laser with atomic memory effects.

The model consists of three level atoms that are injected in a laser cavity, in the upper excited state. The upper atomic levels interact with a mode of the radiation, and these levels decay to the ground state with a rate Γ. The Hamiltonian is given by[11]

$$H=\hbar w a^\dagger a+\sum_j(\varepsilon_a|a\rangle\langle a|+\varepsilon_b|b\rangle\langle b|+\varepsilon|c\rangle\langle c|)_j+\hbar g\sum_j\Theta(t-t_j)V_j\,, \tag{B1}$$

with

$$V_j = a^\dagger \sigma^j + \sigma^{j\dagger} a \ . \tag{B2}$$

Here, σ^j is the atomic polarization, while a and $a^\dagger$ are, respectively, the creation and annihilation operators. $\Theta(t)$ is the step function and g specifies the coupling constant between atoms and field.

The evolution of the physical system ρ is given by Liouville equation. It is convenient to change this equation from the Schrödinger to the interaction picture.[15] Tracing over the atoms, we get

$$\dot{\rho}^f(t) = -i\sum_j \Theta(t-t_j)\mathrm{tr}_{A^j}[V_j, \rho_j^f(t)] + L\rho^f \ , \tag{B3}$$

where the term $L\rho^f$ accounts for the damping of the field.[12] For simplicity, we have assumed resonance between the upper atomic levels and the mode of the field, i.e., $w = (\varepsilon_a - \varepsilon_b)/\hbar$.

To derive an equation motion for $\rho_j^f(t)$, we will assume that the evolution of an atom is independent of all other atoms. In addition, we neglect the influence of the cavity damping in the interaction of the field with a single atom. Thus, an effective equation for ρ_j^f is given by

$$\tilde{\rho}_j^f(t) = \tilde{\rho}_j^f(0) - ig\int_0^t dt'\Theta(t-t_j)[V_j, \tilde{\rho}_j^f(t')] \ , \tag{B4}$$

with

$$\tilde{\rho}_j^f(t) = e^{\Gamma(t-t_j)}\rho_j^f(t) \ . \tag{B5}$$

Equation (B4) is the same as for a single, nondecaying atom. This problem is known as the Jayners-Cummings model.[16] Thus, the initial value of the density operator $\tilde{\rho}_j^f$ in a matrix notation:

$$\tilde{\rho}_f(0) = \begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix} \ , \tag{B6}$$

with

$$A_j = \cos^2(gt_j\sqrt{\epsilon\epsilon^*}) \ , \tag{B7a}$$

$$B_j = \frac{i\epsilon}{2\sqrt{\epsilon\epsilon^*}}\sin(2gt_j\sqrt{\epsilon\epsilon^*}) \ , \tag{B7b}$$

$$C_j = B_j^* \ , \tag{B7c}$$

$$D_j = \sin^2(gt_j\sqrt{\epsilon\epsilon^*} \quad \text{for } t_j < 0 \ , \tag{B7d}$$

and

$$A_j = 1 \ , \tag{B8a}$$

$$B_j = C_j = D_j = 0 \quad \text{for } t_j \geq 0 \ . \tag{B8b}$$

Here, ϵ is a classical electromagnetic field and t_j is the injection time of the jth atom in the cavity.

Now, to obtain the field density operator, we substitute Eqs. (B4)–(B8) into (B3). The result is

$$\begin{aligned} \dot{\rho}^f(t) = {} & -ig\sum_j f(t,t_j)[C_j(a\rho^f - \rho^f a) + B_j(a^\dagger\rho^f - \rho^f a^\dagger)] \\ & -g^2\int_0^t dt'\sum_j f(t,t_j)\Theta(t'-t_j)[A_j(aa^\dagger\rho^f + \rho^f aa^\dagger - 2a^\dagger\rho^f a) + D_j(a^\dagger a\rho^f a^\dagger a -- 2a\rho^f a^\dagger)] + L\rho^f \ . \end{aligned} \tag{B9}$$

Equation (B9) is the master equation for the field density operator ρ^f, when we include atomic memory effects in the laser.

APPENDIX C

In this Appendix, we determine the decay law of the off-diagonal matrix elements when we include atomic memory effects in a laser, by using the δ-expansion method.[17]

We define a large parameter N_0 as

$$N_0 = \frac{R}{\gamma} \ , \tag{C1}$$

where R is atomic pump rate and γ is the cavity decay constant. Furthermore, we define

$$\Omega^2 \equiv \frac{2g^2}{\Gamma^2}N_0 \ , \quad \delta \equiv \frac{1}{N_0} \ , \quad n_1 \equiv \frac{N}{N_0} \ ,$$

$$n_2 \equiv \frac{(N+p)}{N_0} \ , \quad \rho_{N,N+p}(t) \equiv h(n_1, n_2\tau) \ ,$$

and

$$\tau \equiv Rt \ . \tag{C2}$$

If we use definitions (C1) and (C2) in Eq. (3.6) we find

$$\begin{aligned} \frac{d}{d\tau}h(n_1,n_2,\tau) = {} & \frac{\theta^2}{2}e^{-\Gamma t}\{(n_1+\delta)h(n_1+\delta,n_2) - [(n_1+\delta)n_2]^{1/2}h(n_1,n_2-\delta)\} \\ & + \frac{\theta^2}{2}e^{-\Gamma t}\{n_1 h(n_1-\delta,n_2) - [n_1(n_2+\delta)]^{1/2}h(n_1,n_2+\delta)\} \\ & + \theta^2(1-e^{-\Gamma t})(n_1 n_2)^{1/2}h(n_1,\delta,n_2-\delta) + [(n_1+\delta)(n_2+\delta)]^{1/2}h(n_1+\delta,n_2+\delta) \\ & - \left[\frac{\theta^2}{2}(1-e^{-\Gamma t})[2n_2 + (2-p)\delta] + \tfrac{1}{2}(2n_2 - p\delta)\right]h(n_1,n_2) \ . \end{aligned} \tag{C3}$$

Expanding Eq. (C3) in δ, we easily obtain a Fokker-Planck equation for $h(n_1,n_2,\tau)$:

$$\frac{d}{d\tau}h(n_1,n_2,\tau)=\frac{-\mu}{2}h(n_1,n_2,\tau)+\delta\left[\frac{\partial}{\partial n_1}\alpha_1+\frac{\partial}{\partial n_2}\alpha_2\right]h(n_1,n_2,\tau)+\frac{\delta^2}{2}\left[\frac{\partial^2}{\partial n_1^2}\beta_1+2\frac{\partial^2}{\partial n_1\partial_2}\beta_2+\frac{\partial^2}{\partial n_2^2}\beta_3\right]h(n_1,n_2,\tau)\ , \tag{C4}$$

with

$$-\frac{\mu}{2}=\frac{\theta^2}{2}e^{-\Gamma t}\{2n_1+\delta-[(n_1+\delta)(n_2+\delta)]^{1/2}-(n_1n_2)^{1/2}\}$$
$$+\theta^2(1-e^{-\Gamma t})[(n_1+\delta)(n_2+\delta)]^{1/2}+(n_1n_2)^{1/2}+\frac{\theta^2}{2}(e^{-\Gamma t}-1)[2n_2+(2-p)\delta]-\tfrac{1}{2}(2n_2-p\delta)\ . \tag{C5}$$

The coefficients α_i,β_j can be obtained from Eq. (C3). However, we are interested in μ. In a straightforward calculation, we get

$$\mu=\frac{p^2\delta^2}{4n_1}[1+\theta^2(1-2e^{-\Gamma t})]\ , \tag{C6}$$

where we have approximated

$$(n_1n_2)^{1/2}\simeq n_1+\tfrac{1}{2}p\delta-\frac{1}{8}\frac{P^2\delta^2}{n_1}\ , \tag{C7}$$

$$[(n_1+\delta)(n_2+\delta)]^{1/2}\simeq n_1+\left[1+\frac{p}{2}\right]\delta-\frac{1}{8}\frac{p^2\delta^2}{n_2}\ . \tag{C8}$$

Using the usual ansatz of Scully *et al.*,[8] one can write

$$h(n_1,n_2,\tau)\simeq\rho_s(n_1)\exp\left[-\tfrac{1}{2}\int_0^\tau\mu d\tau\right]\ , \tag{C9}$$

where $\rho_s(n_1)$ is a stationary function that depends on the photon statistics.

The exponent in (C9) can be written, now in real time, as

$$D(t)=\int_0^\tau\mu d\tau=p^2\left[Dt+\frac{D}{\Gamma}(e^{-\Gamma t}-1)\right] \tag{C10}$$

with

$$D=\frac{2g^2}{\Gamma^2}\frac{R}{N_s}\ . \tag{C11}$$

Here, N_s is the average photon number, D is the usual Schawlow-Townes linewidth. Moreover, to obtain (C10), we have used $2g^2\Gamma^2\sim\gamma$.

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