



PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE
SCHOOL OF ENGINEERING

EPISTEMIC UNCERTAINTY OF LINEAR BUILDING MODELS AND INELASTIC CONTINUUM CONCRETE MODELS

**MATÍAS FERNANDO NICOLÁS CHACÓN DE LA
CRUZ**

Thesis submitted to the Office of Graduate Studies in partial fulfillment of
the requirements for the Degree of Doctor in Engineering Sciences

Advisor:

JUAN CARLOS DE LA LLERA

MATÍAS HUBE

Santiago of Chile, october 2018

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*To God and my family for all, and
to my grandfather Alberto who
passed on to me the passion for
mathematics and engineering*

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ABSTRACT

Reinforced concrete (RC) building models consider several modeling assumptions that influence the accuracy of the predicted seismic response. Moreover, their nonlinear response is strongly dependent on the concrete model adopted. This study evaluates the epistemic uncertainty inherent to modeling of RC both at the linear building response level and at the non-linear material response level.

The first part of this thesis quantifies the epistemic uncertainty inherent in modeling assumptions by evaluating the seismic response of six instrumented RC free-plan buildings. Four linear modeling assumptions were studied: **(1)** the type of finite element used; **(2)** the in-plane and out-of-plane stiffness of the diaphragms; **(3)** the type of soil-structure-interaction model considered; and **(4)** the decision where to apply fixity to the base. The response's uncertainty was first evaluated comparing predicted and measured periods using ambient vibrations. Additionally, global seismic response parameters such as story shears, torques, and drifts were compared between a *reference* model and a set of *variant* models. In general, uncertainties identified in the core forces were larger than in the story forces, and also larger at the basements than in the upper levels.

The second part of this thesis evaluates the uncertainty associated with the inelastic constitutive concrete models for three-dimensional (3D) and plane-stress formulations. Five concrete models were considered: **(i)** the Hyperbolic Drucker-Prager (DPH) plastic model; **(ii)** the Lubliner-Lee-Fenves (LLF) plastic-damage model; **(iii)** the Wu-Li-Faría (WLF) model; **(iv)** the Faría-Oliver-Cervera (FOC) model; and **(v)** the total strain rotating (ROT) smeared-crack model. New analytical expressions for the numerical integration of the updated stress, and the consistent tangent operator were derived for all models. Results were validated with simple numerical experimental tests subjected to several stress states. Unilateral effects, strain-rate effects, mesh size, and strain-localization phenomenon were evaluated using these models. Furthermore, validated finite element recommendations

were proposed to improve the convergence of the models studied, most notably the use of a smoothed consistent tangent operator and the incorporation of a viscous-regularization technique. Finally, it is concluded that the most important source of epistemic uncertainty of the material models is observed in the dissipated energy and the linearized stiffness of the last unloading-loading cycle in most simulated tests.

Keywords: epistemic uncertainty, free-plan buildings, reinforced concrete, seismic response, finite elements, diaphragm stiffness, soil-structure interaction, basements, instrumentation, plastic-damage models, smeared-crack models, stress updated integration, return-mapping algorithms, projected return-mapping, consistent tangent operator, strain-softening, strain-localization, viscous-regularization

RESUMEN

Los modelos de edificios de hormigón armado (HA) consideran distintos supuestos de modelización que influyen en la precisión para la predicción de la respuesta sísmica. Además, su respuesta no lineal depende fuertemente del modelo de hormigón adoptado. Este estudio evalúa la incertidumbre epistémica inherente a los supuestos de modelización de HA tanto a nivel de respuesta lineal del edificio como a nivel de respuesta no lineal del material.

La primera parte de esta tesis cuantifica la incertidumbre epistémica inherente en los supuestos de modelización mediante la evaluación de la respuesta sísmica de seis edificios instrumentados de HA del tipo planta libre. Se estudiaron cuatro supuestos de modelización lineal: **(i)** el tipo de elemento finito utilizado; **(ii)** la rigidez en el plano y fuera del plano de los diafragmas; **(iii)** el tipo de modelo de interacción suelo-estructura considerado; y **(iv)** la decisión sobre dónde aplicar el empotramiento a la base. La incertidumbre de la respuesta se evaluó primero comparando los períodos simulados con los medidos usando vibraciones ambientales. Además, se compararon parámetros globales de respuesta sísmica, como las fuerzas por piso, torques y desplazamientos entrepiso, entre un modelo de *referencia* y un conjunto de modelos *variantes*. En general, las incertidumbres asociadas en las fuerzas de corte del núcleo de muros fueron mayores que las fuerzas por piso y también mayores en los subterráneos que en los niveles superiores.

La segunda parte de esta tesis evaluó la incertidumbre asociada con los modelos constitutivos inelásticos de hormigón para las formulaciones tri-dimensional (3D) y de tensiones planas. Se consideraron cinco modelos de hormigón: **(i)** el modelo plástico hiperbólico de Drucker-Prager (DPH); **(ii)** el modelo de plasticidad y daño de Lubliner-Lee-Fenves (LLF); **(iii)** el modelo de Wu-Li-Farías (WLF); **(iv)** el modelo de Farías-Oliver-Cervera (FOC); y **(v)** el modelo de grieta difusa *total strain rotating* (ROT). Se derivaron nuevas

expresiones analíticas para la integración numérica de la tensión actualizada y el operador tangente consistente para todos los modelos. Los resultados se validaron con pruebas experimentales numéricas simples sometidas a diferentes estados de tensiones. Los efectos unilateral, la velocidad de deformación, el tamaño de la malla, y el fenómeno *strain-localization* se evaluaron usando estos modelos. Además, se propusieron recomendaciones de elementos finitos validadas para mejorar la convergencia de los modelos estudiados, en particular el uso de un operador tangente consistente suavizado y la incorporación de una técnica de regularización viscosa. Finalmente, se concluyó que la fuente más importante de incertidumbre epistémica de los modelos de materiales se observa en la energía disipada y la rigidez linealizada del último ciclo de descarga-carga en la mayoría de las pruebas simuladas.

Palabras claves: incertidumbre epistémica; edificios de plan libre, hormigón armado, respuesta sísmica, elementos finitos, rigidez del diafragma, interacción suelo-estructura, subterráneos, instrumentación, modelos de plasticidad y daño; modelos *smeared-crack*, integración de la tensión actualizada, algoritmos de retorno; algoritmos de retorno proyectados; operador tangente consistente, strain-softening, strain-localization, regularización viscosa.

INTRODUCTION

0.1. Overview of this thesis

This research begins with the study of the epistemic uncertainty inherent in linear models of the so-called free-plan reinforced concrete (RC) buildings. These buildings are commonly used in Chile for office space, and characterized by three main structural components: a central shear wall core, a perimeter frame, and a post-tensioned slab that connects the two seismic and vertical-load carrying elements. The motivation of this part of the thesis is two-fold. First, to study the epistemic uncertainty present in computing the linear dynamic response of these structures under different modeling assumptions, following the same procedures as engineers currently use in design practice. And second, to try to unravel the most significant components controlling the seismic response of these structures, which at the time of this work had yet not been exposed to strong ground shaking. However, due to the good seismic performance of free-plan buildings in the 2010, Maule Chile earthquake, the main focus of this research shifted toward evaluating the epistemic uncertainty of residential shear wall buildings. More than 40 of these latter buildings presented an unexpected and rather brittle failure of some of the shear walls primarily in the first basement and lower stories. Consequently, in the attempt to model typical damaged shear wall structures using non-linear finite element (FE) models, aimed to better understand their seismic response, we encountered in different software serious numerical convergence problems of these models. These difficulties forced us to study and possibly improve the convergence characteristics of some of these models. Thus, the second part of this thesis is completely dedicated in Chapters 2 and 3 to study in detail the three-dimensional (3D) and plane-stress available FE models used for concrete, and propose consistent formulations for all of them. Reinforcing steel models were also used and improved. This task ended up being very complex and interesting from a research perspective, so the study of epistemic uncertainty and convergence of complete buildings, moved

one step down into the epistemic uncertainty of the finite elements used in constructing the FE non-linear models required for these structures. This is the sequence and rationale of the material presented in this thesis work, and is reflected in the two different examples of epistemic uncertainty considered in Chapters 1 and 2. An obvious next step of this work is to construct the complete 3D models of typically damaged shear wall structures in 2010, and evaluate the epistemic uncertainty at the aggregated structural model level. This evaluation enables the designer to predict and validate the seismic response using as benchmark the damaged structures, but also to propose for the thousands of similar residential structures, the implementation of a consistent retrofit strategy similar to that used for the damaged buildings strengthened after the 2010, Chile earthquake.

0.2. Description of problem

The predicted seismic response of RC structures is strongly dependent on the modeling assumptions used in their computational simulations. The uncertainty generated by these assumptions is within the realm of epistemic uncertainty and is due to the lack of knowledge associated with each assumption and parameter considered in the model.

Until the 1990's, RC shear-wall buildings were the predominant type of structure used in residential and commercial buildings in countries such as Chile, New Zealand and the US. However, in the past decade, RC free-plan buildings have become more popular for office buildings use. Basically, they consider a lateral force resisting system composed by a shear wall core and a RC moment-resisting perimeter frame, both coupled in bending and shear usually by a post-tensioned floor slab. Before the 2010 Maule, earthquake in Chile, no building of this class had been tested in a real earthquake "experiment", and there existed significant uncertainty and questions about their potential seismic behavior. Fortunately, a good performance of these structures was observed after the earthquake ([Naeim et al., 2011](#); [Lemnitzer, Massone, Skolnik, de la Llera, & Wallace, 2014](#)) probably due to an adequate structural design, a detailed structural review process, and the excellent local soil conditions.

Several models exist to evaluate the seismic response of RC buildings, from simplified models using a single beam (e.g. [Encina & de la Llera, 2013](#); [Sepúlveda, de la Llera, & Jacobsen, 2012](#)) to complex FE models to assess super-tall buildings ([Besjak, McElhatten, & Biswas, 2010](#); [Lu, Lu, Zhang, & Ye, 2011](#)). Recent technical documents provide guidelines to create structural models for tall buildings, e.g., PEER/ATC-72 ([ATC-72, 2010](#)) and LATBSDC ([LATBSDC, 2014](#)), which focus on Performance-Based Seismic Design (PBSD) procedures. However, these guidelines provide limited prediction capability for building simulations and are a source of inherent epistemic uncertainty. Moreover, free-plan buildings are particularly sensitive to model uncertainty given their low structural redundancy. These modeling assumptions include aspects such as the in-plane and out-of-plane diaphragm stiffness considered for each floor slab, the soil-structure interaction effects (SSI) considered, and the fixity level of the structure to the ground. These effects and others can have a large influence in the seismic response. Moreover, some of these modeling assumptions are still today a matter of discussion in design offices.

With the advent of supercomputers, complex inelastic constitutive concrete models are more common today in FE structural software and in engineering design offices. However, these inelastic models may lead to considerably different results due to the use of different input parameters and assumptions, which generates uncertainty in the responses and design. Thus, it is necessary to improve the qualitative and quantitative epistemic uncertainty inherent in these concrete models.

Concrete as a quasi-brittle material exhibits a strongly nonlinear behavior due to cracking in tension and crushing in compression. Cracking generates an asymmetric damage behavior between the tension and compression regimes, and an irreversible strength and stiffness degradation ([Krajcinovic, 1996](#)). Tension is characterized by strain-softening behavior after peak strength due to crack propagation, whereas concrete in compression exhibits nonlinear hardening, non-negligible plastic strains, and volumetric expansion due to dilatancy. Moreover, pressure-sensitive behavior is observed when subject to lateral confinement, followed by material compaction under high confinement loads. Also, in

cyclic loading cases, the cracks can close under load reversals from tension to compression with partial stiffness recovery, phenomenon known as unilateral effect (Mazars et al., 1990; Ramtani, 1990). In addition, it is observed that concrete strength depends on strain rate due to growth delay of internal micro-cracks with strain-rate (Suaris & Shah, 1985).

In the last three decades, several 3D and plane-stress constitutive models have been proposed to simulate the mechanical characteristics of concrete under multi-axial loading conditions. Five main groups of models can be identified: (i) plastic models, based on flow theory of plasticity to describe the irreversible plastic strains and hardening behavior, (Drucker & Prager, 1952; Willam & Warnke, 1975; DiMaggio & Sandler, 1971); (ii) damage models, based on continuum damage mechanics (CDM) (Kachanov, 1958) and defined within the thermodynamics of irreversible processes to predict the stiffness degradation and strain-softening behavior caused by micro-crack propagation (Mazars, 1984; J. Simo & Ju, 1987; Faria, Oliver, & Cervera, 1998; Wu, Li, & Faria, 2006; Voyiadjis, Taqieddin, & Kattan, 2008); (iii) plastic-damage models, which combine plasticity and CDM theories (Lubliner, Oliver, Oller, & Oñate, 1989; J. Lee & Fenves, 1998; Armero & Oller, 2000; Wu et al., 2006; Taqieddin, Voyiadjis, & Almasri, 2012); (iv) fracture models, based on the nonlinear fracture mechanics theory to simulate the anisotropic behavior through crack planes of degradation (Rashid, 1968; Bažant, 1982; Rots, 1988; Cervera & Chiumenti, 2006); and (v) mixed models, which are a combination of the latter models (Červenka & Papanikolaou, 2008; Behbahani, Barros, & Ventura-Gouveia, 2015).

Most of these concrete models require a numerical implementation in a FE software at the integration point element level using shell and solid type elements. These models are commonly used to simulate complex geometries and multi-axial loading conditions. Moreover, the elaboration of a robust, reliable, and efficient numerical algorithm is key to correctly simulate the behavior of more complex RC structures like the ones that failed in 2010. For strain-driven models, two steps are required for the numerical implementation: (i) the algorithm to evaluate the updated stress tensor; and (ii) the construction of a consistent stiffness matrix according to the equations used in the updated stress. Several

numerical algorithms exist in the literature for the numerical implementation of concrete models (e.g. [J. C. Simo & Hughes, 1998](#); [de Souza Neto, Peric, & Owen, 2008](#)). It is also well-known that local concrete models are susceptible of numerical convergence problems and spurious mesh responses due to the strain-localization phenomenon ([Pijaudier-Cabot & Bažant, 1987](#)). Enhancement and robustness of these models can be improved by different methods, such as higher-order gradients models ([Peerlings, de Borst, Brekelmans, & de Vree, 1996](#)), non-local integral models ([Pijaudier-Cabot & Bažant, 1987](#)), and viscous-regularization models ([Needleman, 1988](#)). The latter approach includes a numerical viscosity in the constitutive equations, which significantly improves the convergence properties, and is broadly used for its relative implementation simplicity.

Plane-stress concrete models are commonly used with shell elements to model RC walls, slabs and membranes, where one element dimension is much smaller than the others, and the out-of-plane stress of the element is negligible. Moreover, multi-layered shell elements are adequate to simulate an accurate distribution of in-plane and out-of-plane concrete stresses with a considerable reduction in CPU time relative to solid elements (e.g. [Chacón, de la Llera, Hube, Marques, & Lemnitzer, 2017](#)). The plane-stress formulation and its numerical implementation for a material is very different from the 3D-case, due to the additional constraint imposed to satisfy the condition of zero normal stress. Indeed, to account for plastic effects, the radial return-mapping algorithm used in the 3D-case case is not valid for the plane-stress condition, and hence, the consistent plastic operator cannot be obtained explicitly. Consequently, the use of specific formulations and algorithms are required for plane-stress models.

0.3. Main questions

According to the provided background and literature review, several questions drive this research:

- Which are the most critical assumptions in modeling and computing the seismic response of RC free-plan buildings?; how significant are these sources of epistemic uncertainty?
- It is possible to unify inputs and notation of different stress-strain constitutive models for concrete used in FE software?
- Can numerical convergence of these models be improved without a significant loss of accuracy in the estimation of the response?
- What is the epistemic uncertainty generated by these different stress-strain constitutive models in simple benchmark examples?
- Are the 3D and plane-stress formulations compatible?

0.4. Hypothesis

The main hypotheses of this research are that:

- The epistemic uncertainty inherent in RC free-plan buildings may lead to significant discrepancies in the structural responses computed, which, in turn, may lead to unsafe building designs.
- Modeling assumptions lead to significant epistemic uncertainty in the seismic response and design of RC free-plan buildings.
- It is possible to recast five well-known local continuum stress-strain constitutive concrete models using consistent notations and algorithms to successfully implement them in existing FE software.
- The numerical convergence properties of these concrete models may be improved by using a smooth tangent stiffness operator and by adding numerical damping without sacrificing significant numerical accuracy.
- The epistemic uncertainty in the numerical response of an example of a simple concrete prism is useful to identify the sensitivity of the adopted nonlinear stress-strain constitutive models.

- The response and convergence properties of the selected inelastic stress-strain concrete models can be made essentially independent of the FE formulation adopted.

0.5. Objective and organization of the thesis

The main objective of this thesis is to evaluate the epistemic uncertainty associated with structural and modeling assumptions in RC structures at different model scales, ranging from the complete structure to the finite element level. This is done in two parts: **(i)** uncertainty of linear building models of free-plan buildings (Part I); **(ii)** uncertainty of inelastic stress-strain constitutive concrete models using a 3D formulation (Part II); and **(iii)** uncertainty of inelastic stress-strain constitutive concrete models, but employing a plane-stress condition (Part II). This research is organized in the following chapters.

Chapter 1 deals with the epistemic uncertainty in the seismic response of RC free-plan buildings using linear models. Thereby, the following modeling aspects are evaluated: **(1)** the type of finite elements used; **(2)** the in-plane (axial) and out-of-plane (bending) stiffness of the diaphragm; **(3)** the simplified SSI model; and **(4)** the building connection at the basement level. The rationale behind the selection of these four modelling aspects is predominantly based on true assumptions made in engineering practice, which are known to generate controversies during the review process of building projects. Six existing free-plan buildings, located in Santiago, Chile were considered. For each building, a detailed FE model was built using the software packages ETABS and ANSYS. Additionally, a Response Spectrum Analysis (RSA) was carried out to estimate the following response parameters: vibration periods, shear stresses, overturning moment to shear stress ratio, dynamic eccentricity, lateral displacements, and lateral and torsional inter-story drifts. The model uncertainty is estimated from a relative comparison using the mean and standard deviations of results using predicted ratios between *variant* models and *reference* models. This part also includes a comparison between measured and estimated building periods for the first four vibration modes.

Because in the non-linear case the building response is strongly controlled by the finite element selection, Chapter 2 is dedicated to compare at the finite-element scale the response of five different stress-strain constitutive concrete models and provide all the details necessary for an effective numerical implementation of the 3D formulation. The five concrete models studied are: **(i)** the hyperbolic Drucker-Prager (DPH) plastic model; **(ii)** the Lubliner-Lee-Fenves (LLF) plastic-damage model; **(iii)** the Wu-Li-Fara (WLF) model; **(iv)** the Fara-Oliver-Cervera (FOC) model; and **(v)** the total strain rotating (ROT) smeared-crack model. A complete description of these models is presented using a consistent tensorial notation, which by itself is a relevant result. Also, numerical convergence issues and their solution strategies are also presented for these models. Moreover, detailed algorithms for the numerical implementation of the updated stress tensor, and new explicit expressions for the algorithmic consistent tangent stiffness tensors of the models are developed and described. Also, a consistency check between models of input parameters, such as uniaxial laws and fracture energy, is presented. Furthermore, numerical examples using basic benchmark tests subject to monotonic and cyclic loading conditions under uniaxial, biaxial, and triaxial stress states are presented to demonstrate the capabilities of the proposed implementations. The unilateral effect, the strain-rate effect, the mesh size influence, and the strain-localization phenomena are discussed between the different models. Additionally, the numerical model for the compression failure mode of a test specimen is illustrated as an example of application. Finally, the epistemic uncertainty in the uniaxial, biaxial, and triaxial stress loads, in the unilateral effect, and in the strain-rate cases are evaluated with a set of response parameters with respect to the experimental benchmark tests.

Furthermore, Chapter 3 develops the numerical implementation of the five concrete models mentioned above for the plane-stress formulation, considering a consistent notation and a vectorized format. Similar to the 3D-case, numerical algorithms for the updated stress vector, and new expressions for the algorithmic consistent tangent stiffness matrix of the models are derived. Moreover, the same experimental benchmark tests considered

for the 3D-case are presented to demonstrate the capabilities of the proposed algorithms and implementations.

Finally, Chapter 4 presents a summary of the most important conclusions obtained relative to the epistemic uncertainty in each of the two parts of this work, focused on the building and the element scale.

1. EPISTEMIC UNCERTAINTY IN THE SEISMIC LINEAR RESPONSE OF RC FREE-PLAN BUILDINGS

During past decades reinforced concrete (RC) free-plan buildings have become a common structural layout in seismically prone countries such as Chile. Typical lateral force resisting systems in these buildings consist of a combination of core shear walls, a RC moment-resisting perimeter frame, and a post tensioned floor slab that couples the core and perimeter frame, which essentially works as an in-plane or out-of-plane diaphragm (Encina & de la Llera, 2013). Typical story heights (N) range from 18 to 25 stories above ground, and 4 to 8 stories below ground. Fundamental periods for free-plan buildings usually exceed the rule of thumb for frame structures $N/10$. Prior to the Maule, Chile earthquake, in 2010 ($M_w = 8.8$), little or no information about the seismic performance of these structures was available in the literature. Despite the large magnitude of this earthquake and the severe shaking records in Santiago, these buildings showed good performance, and essentially remained in the linear range without major structural or non-structural damage (Naeim et al., 2011; Lemnitzer et al., 2014). This performance can be attributed to good structural design, a detailed structural review process, and favorable local soil conditions.

A variety of building models have been proposed to evaluate building response parameters of structures under dynamic loading. One example is a simplified model that represents the building as a single beam with shear deformations, warping, and a diaphragm with bending stiffness, the latter being essential to adequately represent the seismic behavior of these structures (Encina & de la Llera, 2013; Sepúlveda et al., 2012). On the other extreme, complex Finite Elements Models (FEM) have been used to assess medium-rise buildings (Zekioglu, Willford, Jin, & Melek, 2007; Shin, Kang, & Grossman, 2010) and super-tall buildings (Besjak et al., 2010; Lu et al., 2011). Current standards and technical documents provide guidelines on how to create structural models for tall buildings, e.g., PEER/ATC-72 (ATC-72, 2010) and LATBSDC (LATBSDC, 2014) with a focus on Performance-Based Seismic Design (PBSD), which principally establishes different categories of behavior for different earthquake intensity levels.

Current literature however, is scarce on the quality of the prediction capabilities of these models, their inherent epistemic uncertainty and their effect on building design and loading responses. Free-plan buildings are particularly sensitive to this epistemic uncertainty given their simplicity and low redundancy. In order to quantify epistemic uncertainty at least three methodologies are identified: (i) stochastic FEM where variables distribute according to a Probability Density Function (PDF) ([Hardyniec & Charney, 2012](#)); (ii) sensitivity analyses, where some assumed variables lie on a range of possible discrete values ([Sousa et al., 2012](#)); and (iii) empirical data and reduction of uncertainty through model calibrations using real data ([Brownjohn, Pan, & Deng, 2000](#); [H. Liu, Goel, Bai, Scott, & Kono, 2006](#)). The primary objective of the first part of this thesis is the assessment of epistemic uncertainty inherent to modelling assumptions rather than parametric variations. Modelling assumptions intrinsically yield larger response variations and typically generate most of the discussion in the review process of building projects since there is little information and guidelines in practice on how to consider them in building design. Uncertainty resulting from small variations within a parameter (e.g. Young's modulus, damping, element dimensions, live loads, mass, and soil stiffness) have been studied by other researchers ([LATBSDC, 2014](#); [ASCE/SEI-7-10, 2013](#); [ATC-83, 2012](#)) and should be routinely evaluated during parametric sensitivity studies within the design process.

Recent studies ([Encina & de la Llera, 2013](#)) as well as empirical evidence after the 2010 Maule earthquake have validated the significance of floor diaphragms in the behavior of free-plan buildings. In common practice diaphragms are modeled with infinite in-plane rigidity and a reduced out-of-plane (bending) flexibility. This assumption allows an important reduction in the number of Degrees Of Freedom (DOF) of the model as well as in computational time. Several studies ([Ju & Lin, 1999](#); [Saffarini & Qudaimat, 1992](#)) have examined the implications of this modeling assumption and demonstrated that this assumption mainly affects low-rise buildings with short periods and small out-of-plane diaphragm stiffness relative to the stiffness of the lateral-load resisting system. By considering the in-plane deformation of the floor slab, the periods and displacements increase, and the seismic stresses decrease ([Fouad, Ali, & Mustapha, 2012](#)). Conversely, when the

rigid diaphragm assumption is applied at levels with abrupt changes in lateral stiffness, such as the transition zone between the first level and the basements, a significant shear stress is generated in the core walls; also known as *back-stay* effect (Rad & Adebar, 2009). For high-rise buildings, out-of-plane diaphragm stiffness becomes significant (Fouad et al., 2012; D.-G. Lee, Kim, & Chun, 2002).

Another important parameter when assessing the dynamic response of free-plan buildings is the constraint of the surrounding soil and the interaction thereof with the basement floors of the structure. Generally, Soil-Structure Interaction (SSI) increases internal damping, lengthens the vibration periods, increases the lateral displacements of the structure, and changes the stresses at the base depending on the frequency content of the seismic motion as well as the dynamic characteristics of the soil and structure (Tabatabaiefar & Massumi, 2010; Moehle, 2015; Mylonakis & Gazetas, 2000). Several approximations have been made for SSI models in high-rise buildings (Naeim, Tileylioglu, Alimoradi, & Stewart, 2010; Li, Lu, Lu, & Ye, 2014); most of which use simplified models, i.e. the soil is represented by a discrete arrangement of springs and dampers to provide computational efficiency with reasonable accuracy.

Current seismic codes do not provide explicit recommendations on how to model basements, the number of levels to include in the structural model or how to connect the model to the ground. This leads to discretionary interpretations on “how and where” to apply the minimum shear requirements for building design. Incorporating basements in the model usually generates an increase in building periods and displacements, as well as a reduction in seismic stresses for elements above ground level (D.-G. Lee & Kim, 2001).

This chapter presents in Section 1.1 the structural configuration, geometrical description, and vibration periods and modes of six real RC free-plan buildings, located in Santiago, Chile. Moreover, the metric considered to quantify the epistemic uncertainty for these buildings is illustrated in Section 1.2. Further, Section 1.3 summarizes the seismic response of these class of buildings through a response spectrum analysis elaborated by FE models in ETABS and ANSYS. Also, this section estimates the epistemic uncertainty

considering different finite element types. Moreover, Sections 1.4 to 1.6 quantifies the epistemic uncertainty of the diaphragm stiffness, the type of soil-structure constraints, and the building fixity level, respectively.

It should be noted that the recommendation of a “most accurate” model is beyond the scope of this thesis, as the selection of modeling techniques influences the building response and the selection of a “most suitable approach” depends on the specific needs and allowances of the respective project. Hence, quantitative comparisons will enable the reader in making proper case-based decisions.

1.1. Selected Buildings

Figs. 1.1.1 and 1.1.2 depict photographs, elevations and floor plans of all six buildings selected for this study and referred to hereafter as Buildings A through F. All buildings have RC cores of shear walls, a RC perimeter frame, post tensioned RC slabs and are founded on firm soil (ASCE site class C ([ASCE/SEI-7-10, 2013](#))). Basic geometric data are summarized in Tables 1.1.1 and 1.1.2. All buildings were designed according to the Chilean code NCh433 ([NCh433, 1996](#)) and ACI-318 ([ACI-318, 2005](#)). The materials specified are concrete H35 ($f'_c = 30$ MPa) and reinforcement steel of type A630-420H ($f_y = 420$ MPa). Four of the investigated buildings (i.e. A, C, E and F) were occupied and operating at the time of the earthquake, the other two buildings were close to construction completion. None of the structures suffered any relevant structural or content damage ([Naeim et al., 2011](#); [Lemnitzer et al., 2014](#)).

The selected buildings have between 19 and 24 stories and between 4 and 8 basements. The total building height (H_t) varies from 73 m to 105 m. The plan aspect ratio (B_x/B_y) varies between 1.0 and 3.3. With the exception of Building D, all basements have aspect ratio (B_{bx}/B_{by}) smaller than for that of the superstructure. The slenderness ratio (H_t/B) varies between 2.4 and 5.0. The typical floor area (A) varies between 299 m² and 2826 m²; the basement floor area (A_b) varies between 1023 m² and 7361 m²; and the floor area ratio

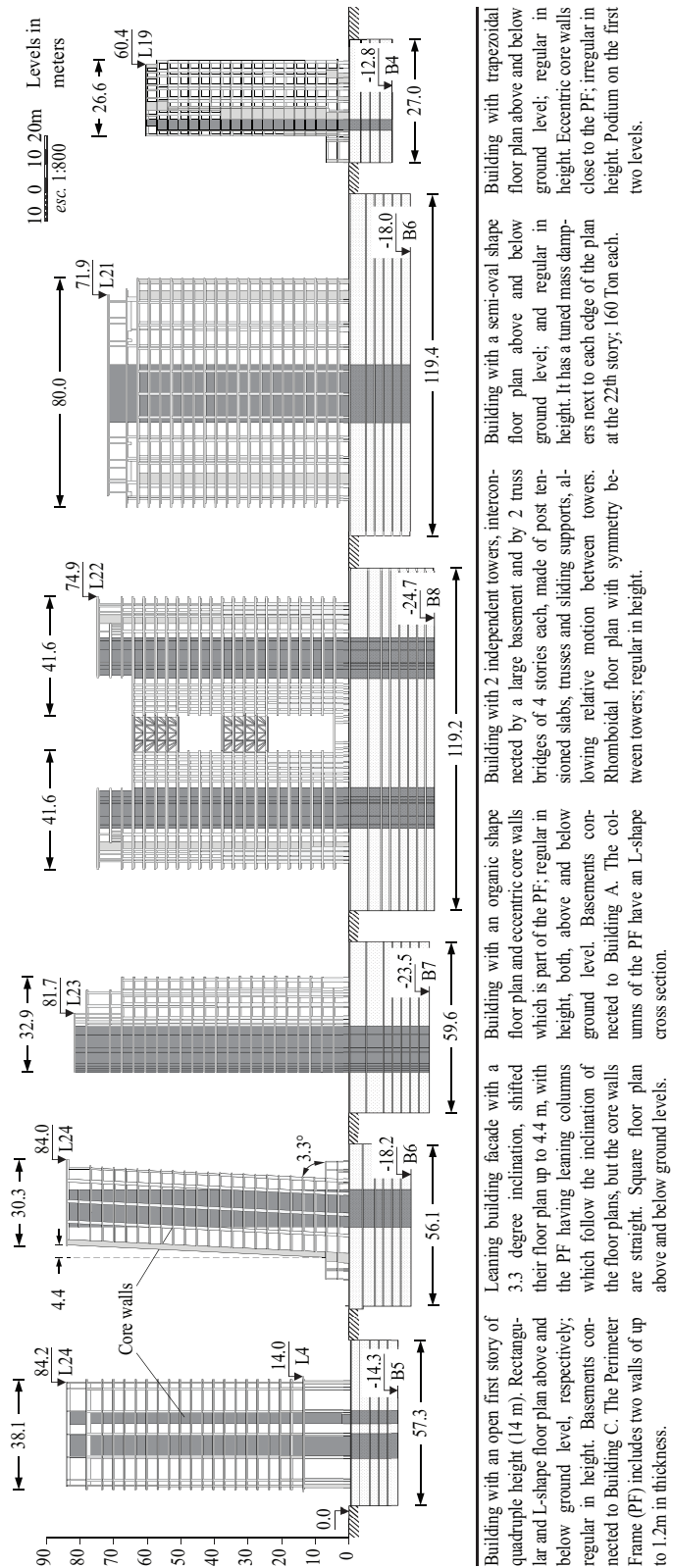
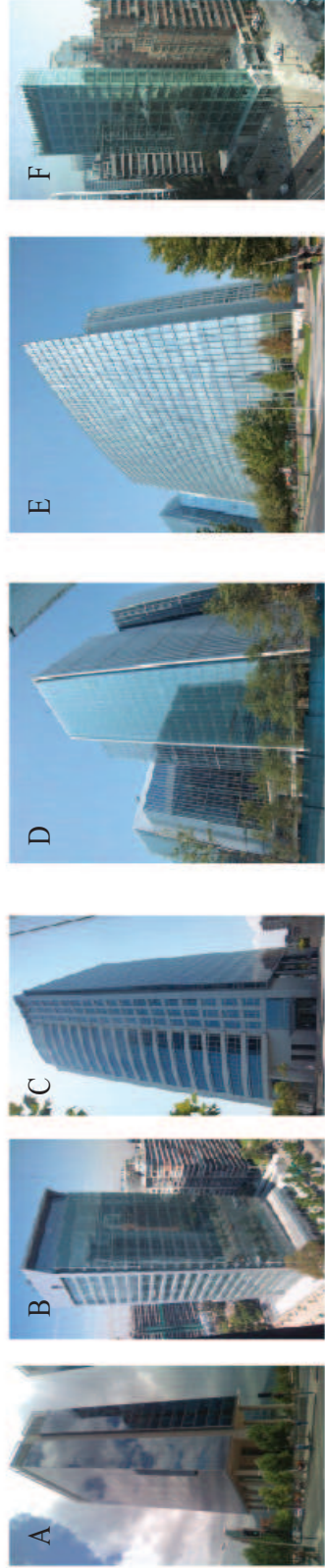


Figure 1.1.1. Overview, elevation, and brief description of the buildings.

(A/A_b) ranges from 0.26 to 0.38. The thickness of RC floor slabs range from 16 cm to 28 cm. Considering the core area (A_c) as the space used by elevators and staircases, the space efficiency ($\eta_A = 1 - A_c/A$) in all buildings is over 84%. The thicknesses of the RC core walls (e_c) range between 20 cm in the top stories to 130 cm in the first stories. The measured wall density with respect to the floor area in each direction in a typical story (ρ_w^x, ρ_w^y) varies between 0.68% and 2.82%, but is usually less than 1.5%, i.e., about half the amount of the 1985 Chilean shear wall buildings, which is 2.8% on average (Jünemann, de la Llera, Hube, Cifuentes, & Kausel, 2015). In addition, the vertical density in the typical story (ρ_v), defined as the total area of vertical structural elements divided by floor area, varies between 3.1% and 6.2%.

Table 1.1.1. Geometric parameters of analyzed buildings: N and N_b = number of stories above ground and below ground level, respectively; H = building height above ground level; H_t = total building height; h and h_b = story height above ground and below ground level, respectively; B_x, B_y = typical building plan dimensions; B_{xb}, B_{yb} = typical basement plan dimensions; $B = \min(B_x, B_y)$.

Building	Number of stories	Height (m)		Story height (m)		Floor plan dimension (m)		Aspect ratio		Slender ratio
		$N+N_b$	H	H_t	h	h_b	$B_x \times B_y$	$B_{xb} \times B_{yb}$	B_x/B_y	
A	24 + 5	84.2	98.5	3.3	2.6	38.1×23.2	57.3×67.2	1.6	1.2	4.2
B	24 + 6	84.0	102.2	3.5	2.6	33.4×30.3	53.8×56.1	1.1	1.0	3.3
C	23 + 7	81.7	105.2	3.5	2.6	32.9×48.2	59.6×65.9	1.5	1.1	3.2
D	22 + 8	74.9	99.6	3.3	2.6	41.6×41.7	119.2×63.3	1.0	1.9	2.4
E	21 + 6	71.9	89.9	3.3	2.6	80.0×24.5	119.4×46.0	3.3	2.6	3.7
F	19 + 4	60.4	73.2	3.2	2.8	30.2×14.5	48.3×29.4	2.1	1.6	5.0

Table 1.1.2. Geometric parameters of analyzed buildings (continuation of Table 1.1.1).

Building	Floor area (m^2)			Floor ratio	Space efficiency, η_A	Core wall thickness, e_c	Shear wall density (%)		Vertical density, ρ_v	Core wall width (m)
	A	A_c	A_b				ρ_w^x	ρ_w^y		
				A/A_b	(%)	(cm)			(%)	$l_c^x \times l_c^y$
A	874	137	3378	0.26	84	25-100	0.68	1.09	3.06	15.8×8.9
B	929	148	2963	0.31	84	25-130	1.56	2.82	6.18	11.9×12.4
C	1131	174	3932	0.29	85	20-110	0.71	0.80	3.24	15.3×15.0
D	2826	394	7361	0.38	86	25-120	0.99	1.19	3.87	14.1×20.0
E	1553	174	5053	0.31	89	25-85	0.93	0.98	4.01	19.2×9.8
F	299	34	1023	0.29	89	20-65	2.03	1.52	5.15	7.2×6.3

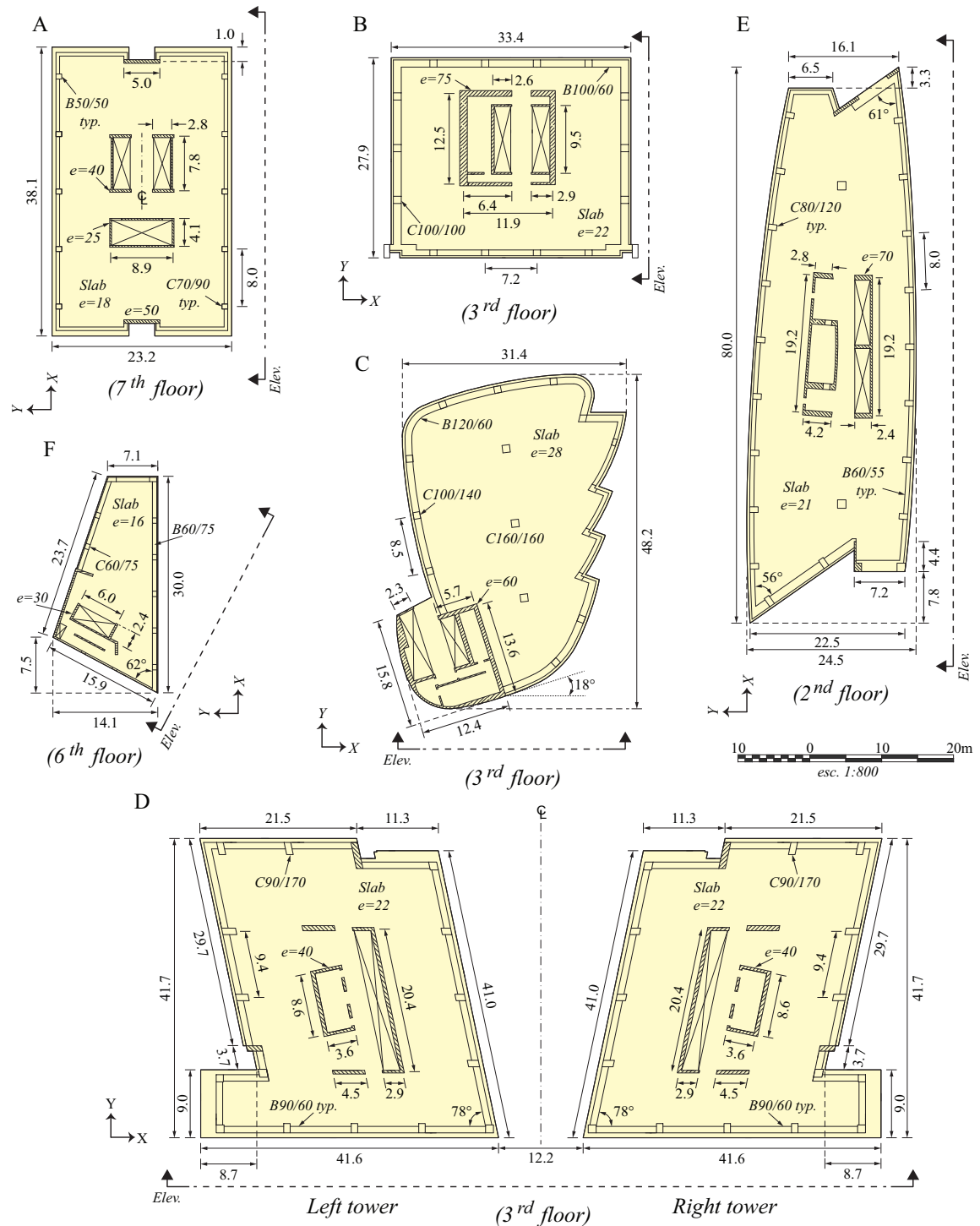


Figure 1.1.2. Schematic layout of the typical floor plans of the buildings; plans and general dimensions of the main structural elements (dimensions in meters and thicknesses of walls, slabs and beams in centimeters). Note: the X - and Y -axis directions were randomly selected and the X -direction does not coincide with the longest dimensions of the buildings.

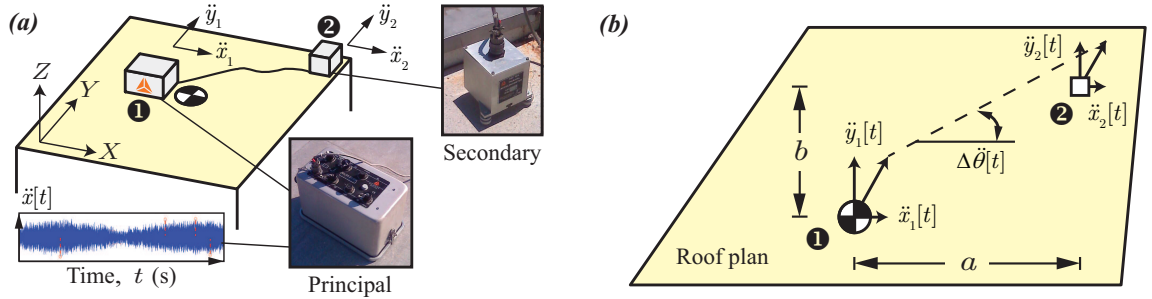


Figure 1.1.3. Diagram of the instrumentation used: (a) configuration of the sensors; (b) estimation of the torsional component.

In order to evaluate the uncertainty in estimating vibration periods and modes of the structural models, instruments were deployed to record ambient vibrations in all six buildings. Two tri-axial accelerometers were installed at the roof of each building: a principal accelerometer (1) placed at the approximate geometric center of the floor plan and a secondary (2) accelerometer placed at the farthest corner. The configuration shown in Fig. 1.1.3a captures the three accelerations ($\ddot{x}_1[t]$, $\ddot{y}_1[t]$, $\ddot{z}_1[t]$) at the geometric center of the plan layout as well as the torsional acceleration of the floor diaphragm ($\ddot{\theta}[t]$). The torsional acceleration is obtained using small displacement approximation (Fig. 1.1.3b) from:

$$\ddot{\theta}[t] = \frac{1}{a^2 + b^2} \{a(\ddot{y}_2[t] - \ddot{y}_1[t]) - b(\ddot{x}_2[t] - \ddot{x}_1[t])\} \quad (1.1.1)$$

where a and b are the sensor distances as defined in Fig. 1.1.3b, and were estimated according to building plans. It is completely true that any measurement has its own uncertainty. However, like with any laboratory experiment, the period measurements are assumed to be the true value, and such experimental uncertainty is considered to be of much smaller magnitude than the one associated with the studied modelling assumptions.

Recorded signals were processed as follows: (i) base line correction in the time domain; (ii) double integration of accelerations to displacements; (iii) conversion to frequency domain; and (iv) application of a Butterworth filter type 2. Following this procedure, the Power Spectral Density (PSD) is estimated in each direction, i.e.: $\mathbf{G}_x[w] = |X[w]|^2 t_r$, where $|X[w]|$ is the modulus of discrete Fourier transform of the signal $x[t]$ at

Table 1.1.3. Instrumentally measured periods (T) and directions of the first 4 modes, periods ratio T_2/T_1 , T_3/T_1 , and ratio of first torsional period to first lateral period $T_{\theta 1}/T_{lat 1}$, ratio of building height above ground level to first lateral period ($H/T_{lat 1}$), ratio of total building height to first lateral period ($H_t/T_{lat 1}$), and estimation of the fundamental period according to nominal frame building rule ($N/10$), for the 6 buildings.

Building	Period (s)				Period ratio			Ratio (m/s)		Period frame bldg. $N/10$ (s)
	T_1	T_2	T_3	T_4	T_2/T_1	T_3/T_1	$T_{\theta 1}/T_{lat 1}$	$H/T_{lat 1}$	$H_t/T_{lat 1}$	
A ¹	2.831 Y	2.310 X	2.182 Θ	0.737 Θ	0.82	0.77	0.77	29.7	34.8	2.4
A ²	2.778 Y	2.381 X	2.174 Y	0.758 X	0.86	0.78	—	30.3	35.5	2.4
B	2.263 X	1.652 Y	1.383 Θ	0.619 X	0.73	0.61	0.61	37.1	45.2	2.4
C	2.507 X	1.708 Y	0.755 X	0.557 Θ	0.68	0.30	0.22	32.6	42.0	2.3
D-Right	2.416 X	1.683 Y	1.230 Θ	0.737 Z	0.70	0.51	0.51	31.0	41.2	2.2
D-Left	2.430 X	1.641 Y	1.234 X	0.732 X	0.68	0.51	—	30.8	41.0	2.2
E	2.356 Y	1.959 Θ	1.307 X	0.700 Y	0.83	0.56	0.83	30.5	38.2	2.1
F	1.638 Y	1.305 X	0.875 Θ	0.457 Θ	0.80	0.53	0.53	36.9	44.7	1.9

¹ Ambient vibration measurements (November 2010 - November 2012); ² Aftershock measurements (March 2010).

a frequency w , and t_r is the total duration of the record. In order to identify the predominant direction, and to enable comparisons between measurements in different directions, a normalized PSD was used: $\langle \mathbf{G} \rangle_x[w] = \mathbf{G}_x[w] / \sum_j^N \mathbf{G}_x[w_j]$, where $\sum_j^N \mathbf{G}_x[w_j]$ consider the sum of N discrete terms of $\mathbf{G}_x[w]$.

Table 1.1.3 shows the back-calculated vibration periods (T) and the predominant direction of the first four modes for all buildings. The first building periods range between 1.638 s and 2.831 s. Furthermore, except for Buildings C and F which have very asymmetric floor plans, the first three periods of all buildings are larger than 1 s. This structural flexibility is apparent when comparing the building periods with the reference periods of frame buildings with similar height ($N/10$) as shown in Table 1.1.3 (last column). A general similarity between the first building periods of each structure can be seen when comparing the period ratios T_2/T_1 and T_3/T_1 as depicted in Table 1.1.3. Except for Building C, ratios vary between 0.51 and 0.86. Additionally, among the first four building modes, at least one of the modes shows strong lateral-torsional coupling, with ratios of ($T_{\theta 1}/T_{lat 1}$) varying between 0.22 and 0.77. This observation confirms that these buildings are torsionally stiff. Building D for example, was equipped with instrumentation in

each tower and yielded very similar periods. However, despite the building symmetry, the direction of modes matches only for the first two modes.

The ratio of building height above ground level to first lateral period (H/T_{lat1}) typically ranges between 30 and 35 for frame buildings and between 60 and 70 for wall buildings and is used as a proxy of the building stiffness. In Buildings A-F, the ratio H/T_{lat1} varies between 29.7 m/s and 37.1 m/s (Table 1.1.3)), which according to a previously proposed classification (Guendelman, Guendelman, & Lindenberg, 1997); classifies these structures as flexible ($H/T_{lat1} < 40$ m/s). However, if the total building height is considered (H_t), the ratio H_t/T_{lat1} increases to 34.8 m/s and 45.2 m/s, which makes Buildings A and E as flexible structures, while the rest as structures with normal stiffness ($40 < H_t/T_{lat1} < 70$ m/s).

In addition to ambient vibration measurements carried out by the authors in Building A, previously recorded aftershock measurements (Lemnitzer et al., 2014) were used to verify the first four periods of the building. Data were obtained via tri-axial accelerometers placed in the roof corners of the building. Measurements collected over four continuous days captured two aftershocks, one of them being a $M_w = 5.1$ earthquake on 18/03/2010. Aftershock data are labeled A² in Table 1.1.3 and are compared with the ambient vibration measurements. The ratio between the first four periods obtained for the two records varies between $T_{DYN}/T_{AMB} = 0.98$ and 1.03, and match the predominant direction of the first two modes. The similitude of these periods validates the use of ambient vibrations.

1.2. Estimation of the epistemic uncertainty

In order to estimate and compare the epistemic uncertainties of the structural models considered in this analysis (i.e. referred to as *variant* models), a *reference* model was defined by selecting common seismic design assumptions (ATC-72, 2010; LATBSDC, 2014) and following Chilean design practice, i.e.: **(1)** slabs have finite in-plane and out-of-plane stiffness; **(2)** SSI effects are not included, i.e., model are fixed at the base; **(3)** basements

are included in the model; **(4)** RC is assumed to behave elastic, isotropic, and remains uncracked; **(5)** uncracked section stiffnesses (gross cross section properties) are assumed for the structural elements, where the contribution of the reinforcement is not accounted for; **(6)** the mass of the building includes 25% of live loads; and **(7)** a dynamic concrete Young's modulus $E_{dyn} = 1.2E_c$ was assumed (Lydon & Balendran, 1986), where E_c is the static Young's modulus according to ACI-318: $E_c = 4700\sqrt{f'_c}$ with f'_c in MPa. With these assumptions, vibration periods similar to measured periods are obtained, as described later.

The seismic response of the buildings is estimated using modal RSA using the elastic design spectrum of the Chilean seismic base-isolation code, NCh2745 (NCh2745, 2003), corresponding to a firm soil with a PGA of $0.41g$, a maximum pseudo-acceleration of $1.2g$, and a 5% damping ratio. This spectrum was used for two reasons: (i) fits very well the ground motion data generated during the Maule, Chile earthquake; and (ii) the peer structural design review process of all these buildings was done using this design spectrum. The seismic response is computed with at least 80% of the cumulative effective modal mass (C_m) in each lateral direction (X and Y). The spectrum in both directions is taken into account independently, and the modal responses are combined by Complete Quadratic Combination (CQC) method. The building responses were obtained by adding the responses from gravitational and seismic actions. Seismic masses were used in the models to obtain their dynamic properties (periods and mode shapes), which are then used in RSA.

The response parameters considered in this study are (Fig. 1.2.4): **(1)** periods of the first four modes (T); **(2)** story shear (V_t) and shear carried by the core walls (V_c), both expressed as a percentage of the total seismic weight (W_t) (Fig. 1.2.4a); **(3)** the ratio of overturning moment to story shear, $\lambda_t = M_t/V_t l_t$, and the corresponding ratio for the core walls, $\lambda_c = M_c/V_c l_c$, where M_t is the building overturning moment using a horizontal axis passing through the vertical projection of the centroid of the accumulated story masses above the considered level (CM_t), M_c is similar to M_t , but passes through the

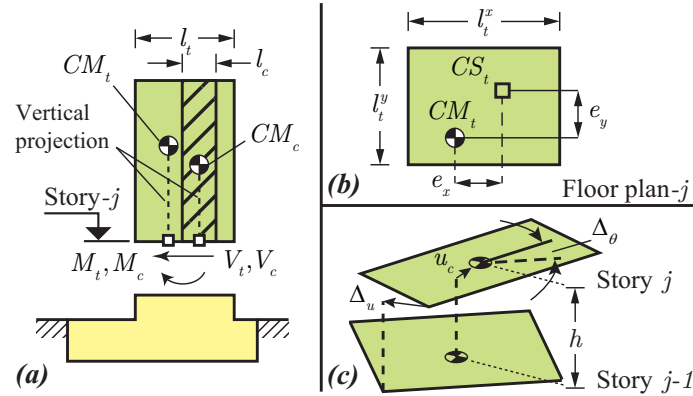


Figure 1.2.4. Definition of response parameters to measure epistemic uncertainty: (a) story shears and overturning moment; (b) eccentricity; and (c) floor displacements and inter-story drifts.

vertical projection of the centroid of the accumulated masses of core walls above the considered level (CM_c) (Fig. 1.2.4a)—the X - and Y -coordinates of these projections changes in different stories if the geometry (i.e. wall mass) changes—, l_t is the typical building width (B_x, B_y : above ground level and B_{xb}, B_{yb} : below ground level, respectively, Table 1.1.1) in the direction of analysis, and l_c is the typical wall width in the direction of analysis (l_c^x, l_c^y Table 1.1.2); (4) normalized eccentricity, $\bar{e} = (T_t/V_t)/l_t$, where T_t is the torque with respect to a vertical axis passing through the centroid CM_t , and is equivalent to the distance between the centroid CM_t and the centroid of accumulated stiffness (CS_t) in each story (Fig. 1.2.4b), divided by the plan width l_t ; (5) displacement of the geometric center of the diaphragm (u_c) in each floor; (6) lateral inter-story drift, $\delta_u = \Delta_u/h$, where Δ_u is the maximum relative displacement of each floor in the direction of analysis and h is the inter-story height (Fig. 1.2.4c); and (7) torsional inter-story drift, $\delta_\theta = \Delta_\theta/h$, where Δ_θ is the relative rotation between consecutive floors (Fig. 1.2.4c), and is determined as the average rotation between the center of the diaphragm and each of the four (or more) corners of the floor (Eq. (1.1.1)).

The uncertainty of the response parameters for each modeling assumption is evaluated by analyzing the ratio of respective *variant* models (R_v) and *reference* model (R_0) results. These ratios are grouped by building as well as analysis directions (X and Y). Shear force (V_t and V_c) uncertainties are evaluated at four levels: at mid-height of the tower ($H/2$), at

the base of the first story (L1), at the base of the first basement (B1), and at the base of the foundation level (BF). For all other responses, the uncertainty is evaluated only at levels where extreme values occur. The uncertainty of the ratios R_v/R_0 is characterized by its minimum and maximum values, and the standard deviation (σ). Hereby graphical results are provided for the buildings that show the largest uncertainties to demonstrate the scatter of data due to the input assumptions.

1.3. Effect of the finite element type and characterization of the seismic response

This section quantifies the epistemic uncertainty with respect to different finite element types and summarizes the seismic response of the free-plan buildings. All six buildings were modeled in ETABS (ET) (ETABS, 2013), with results being considered *reference* results due to the popularity and usability of this software in engineering firms. Two *variant* models were developed in ANSYS (ANSYS, 2018): four buildings (A, C, D and F) in ANSYS Parametric Design Language mode (AP), and four buildings (A, B, E and F) in ANSYS Workbench mode (AW). The finite elements used to model beams, columns, slabs and walls for each FEM model are shown in Fig. 1.3.5. In addition to the modeling assumptions described previously, each model considers the following:

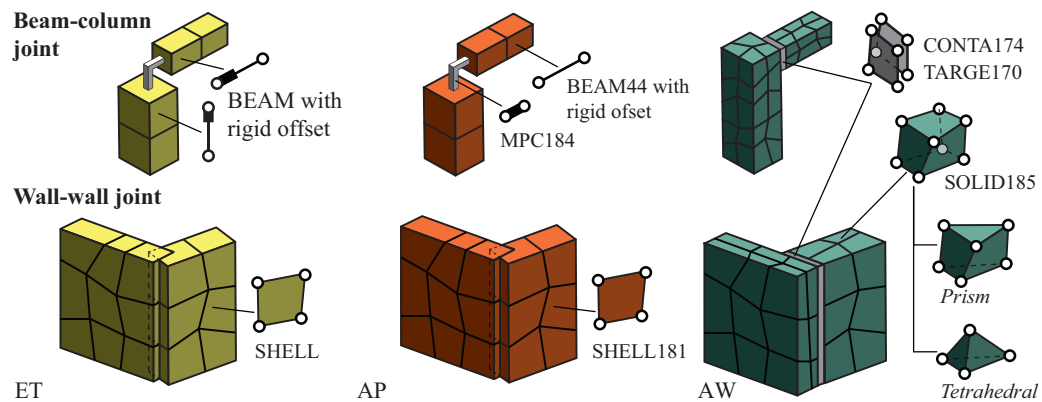


Figure 1.3.5. Finite element type and connection among structural elements: (a) ET; (b) AP; and (c) AW models, respectively.

(i) For ET models (Fig. 1.3.5a), the beams and columns were modeled using 2-node Timoshenko frame type elements with six DOFs per node, and rigid offset elements in beam-column joints. The walls and slabs were modeled using 4-node shell type elements with six DOFs per node and the Mindlin-Reissner formulation. All structural element connections considered mass overlap as well as a compatible mesh. Masses in the elements were assigned to the horizontal DOFs of the nodes. For all beams embedded in slabs, the moment of inertia of rectangular beams was multiplied by a β -coefficient to account for the real position of the neutral axis and includes the correct effective width of the slab. Therefore, the β -coefficient leads to the correct moment of inertia of the T-shaped or L-shaped cross sections. The effective slab width was estimated following ACI-318 (ACI-318, 2005) recommendations, and the β -coefficient was calculated as: $\beta = I_{comp}/I_o$, where I_{comp} is the moment of inertia of the composite cross section, and I_o is the moment of inertia of the rectangular section.

(ii) For AP models (Fig. 1.3.5b), beams and columns were modeled similarly to the ET models (BEAM44). The walls and slabs were represented via 4-node shell type elements with six DOFs per node using the Bathe-Dvorkin formulation (Bathe & Dvorkin, 1986) (SHELL181). All structural element connections consider mass overlap as well as a compatible mesh. Masses for the elements in this case were assigned to the six DOFs of the nodes, and similarly to the ET models, the moment of inertia of beams embedded in the slab was corrected by the β -coefficient.

(iii) For AW models (Fig. 1.3.5c), all structural elements were modeled using 8-node brick elements (SOLID185) with three DOFs per node and a Simplified Enhanced Strain formulation (J. Simo, Armero, & Taylor, 1993). The mesh was generated independently for each structural element and contact elements (CONTA174 and TARGE170) were used to connect the nodes of two elements, since meshes of adjacent elements were incompatible, as shown in Fig. 1.3.5c. Consequently, mass overlap was not generated in the connection of two elements. Masses in the elements were assigned to the three DOFs of the nodes.

Table 1.3.4 compares the number of nodes and elements in each FEM model based on a maximum mesh size of 1.5 m. Compared with the ET models, the AP models present 20% more nodes and 80% more elements, and the AW models between 9 and 14 times more nodes and between 10 to 17 times more elements. The total seismic weight (W_t) of the ET models varies between 130.2 MN (Building F) and 1431.1 MN (Building D). In general, the weight of the super-structure ranges between 50% and 61% of the total weight of the structure. The maximum difference of the seismic weight between the ET-AP models is less than 2.2%, and for the ET-AW models less than 4.3%. The AW models are always lighter than the ET models, since the AW models do not consider mass overlap.

Table 1.3.4. Comparison between FEM models.

Building	Nodes		Elements		Total seismic weight ¹ , W_t (MN) Dif. (%)			Computation time ² (hrs)	
	ET	ET:AP:AW	ET	ET:AP:AW	ET	ET-AP	ET-AW	ET	ET:AP:AW
A	43,652	1 : 1.1 : 11	46,416	1 : 1.1 : 10	364.2	1e-3	2.6	3.2	45 : 1 : 22
B	59,494	1 : - : 10	63,119	1 : - : 12	532.5	-	4.3	3.1	1.2 : - : 1
C	77,730	1 : 1.0 :-	82,568	1 : 1.6 : -	640.5	-8e-4	-	6.0	15 : 1 : -
D	90,132	1 : 1.1 :-	103,895	1 : 1.8 : -	1431.1	2.2	-	21.6	39 : 1 : -
E	90,912	1 : - : 9	95,456	1 : - : 11	680.8	-	3.3	32.3	9 : - : 1
F	17,690	1 : 1.2 : 14	19,330	1 : 1.2 : 17	130.2	8e-4	2.8	0.4	6 : 1 : 11

¹: Includes 25% of live load; ' - ': model not built; ²: Using a computer Intel Xeon® 3.33 GHz processor with 47.9 Gb RAM

For the ET models, 100 modes were calculated using eigenvalues and eigenvectors, and for the AP and AW models, 250 modes were obtained using the Block Lanczos algorithm (Grimes, Lewis, & Simon, 1994). Because the AP and AW models consider the vertical masses, a larger amount of modes relative to ET are required to obtain similar effective modal masses C_m in the horizontal directions. Computer limitations impede the use of more modes in the AW models. The computational time to obtain the vibration modes in the ET models is 6 to 45 times greater than for the AP models, and 1.2 to 9 times greater than for the AW models, with the exception of Building F, where the AW model is twice slower than the ET model (Table 1.3.4). This difference in computation

time is attributed to the different methods used to calculate the modal coordinates in each software. The ANSYS algorithm is significantly faster than that of ETABS.

Table 1.3.5 summarizes the periods and predominant direction of the first four modes in all three models. The fundamental periods of the buildings range from 1.61 s to 3.13 s, and the first three periods of all models are larger than 1 s, except for Building F. The directions of the translational modes also coincide among models. However, this is not the case for the torsional modes, which differ between models.

Table 1.3.5. Periods (T) and corresponding predominant direction of FEM models. Units: second

Building	ET				AP				AW			
	T_1	T_2	T_3	T_4	T_1	T_2	T_3	T_4	T_1	T_2	T_3	T_4
A	2.9258 Y	2.4144 X	2.0752 Θ	0.7383 Y	3.1262 Y	2.4471 X	2.2110 Y	0.7840 Y	2.7119 Y	2.2768 X	1.9035 Y	0.6926 Y
B	2.2652 X	1.8146 Y	1.3959 Θ	0.6367 X	-	-	-	-	2.1597 X	1.7686 Y	1.3099 Y	0.6104 X
C	2.7227 X	1.8742 Y	1.0654 Θ	0.8213 X	2.7127 X	1.8604 Y	1.0767 X	0.8264 X	-	-	-	-
D	2.8212 X	1.9564 Θ	1.7905 Y	0.8271 X	2.7765 X	1.9410 X	1.7736 Y	0.8205 X	-	-	-	-
E	2.6084 Y	1.9056 Θ	1.2981 X	0.6887 Y	-	-	-	-	2.3723 Y	1.7754 Θ	1.2600 X	0.6488 Y
F	1.6169 Y	1.2173 X	0.8552 Θ	0.4364 Y	1.6610 Y	1.2836 X	0.9227 X	0.4604 Y	1.6053 Y	1.2332 X	0.8267 X	0.4368 Y

Note: Values in parenthesis associated to the number of the mode; '—': model not built.

The uncertainty of the vibration periods is evaluated using Fig. 1.3.6. Fig. 1.3.6a compares the first four periods identified in Table 1.3.5 for the three models of Building A. The AP model predicts the longest periods, and the AW model the shortest. The largest difference between the estimated periods is 15.3% for the first mode, and 16.2% for mode three. Fig. 1.3.6b shows the ratio between the periods of the AP and AW models normalized with respect to the ET models for the first four modes of all six buildings. For mode one, a close-up with eight period ratios obtained with the available models is presented, summarized by a box-plot on the left of Fig. 1.3.6b. The box-plots used hereafter have a rectangle whose length is the difference between the first and third quartile, a mean $\bar{x}$ represented by an intermediate horizontal line, a median represented by a rhombus, whiskers equivalent in width to two standard deviations (2σ), and outliers which fall outside the range ($\bar{x} \pm \sigma$). For the first normalized period there is an estimated uncertainty of the models that varies from 0.91 to 1.07 with $\sigma = 5.2\%$. For modes one through four, Fig. 1.3.6b also shows the envelope of normalized periods. It is observed that the periods of the AP

models are up to 8% longer than the ET models, and the AW periods are up to 9.1% shorter than the ET models.

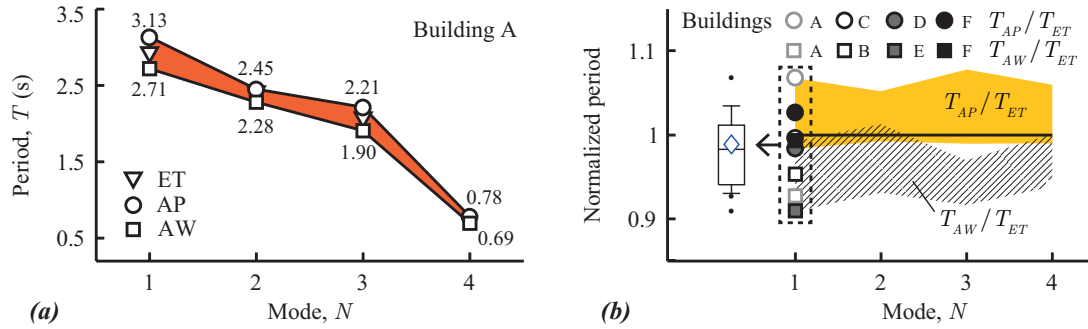


Figure 1.3.6. Period variation according to finite element type: (a) periods of Building A; (b) periods of the AP and AW models normalized by the ET model values for the 6 buildings.

Comparisons between analytical periods and those obtained from ambient vibrations for all six buildings are shown in Fig. 1.3.7. The AW models is the one that leads to smallest error in the prediction of vibration periods, which for the first four modes yield errors of 13% or less, and can be attributed to the advantage of using solid finite elements. Contrary, the ET and AP models generally overestimate the periods for the first four modes, and errors vary from 7% shorter to 48% longer than the measured values. For the first two modes, the theoretical models may differ up to 17% with respect to measured values. Considering all models, Building F represents the structure with the smallest error in predicting the period of the first four modes. The largest difference occurs in the ET and AP models of Buildings C and D, with errors up to 48% in the third and fourth mode. The complexity associated with modeling the bridges connecting the towers in Building D significantly affects the accuracy of the FEM model. In general, the models predict the predominant direction of the first two modes in good agreement with the instrumentation, with the sole exception of Building D in which only the first mode matches.

A comparison of the vertical distribution of selected response parameters for all three models is shown in Figs. 1.3.8 and 1.3.9 for Building A. The story shear V_t at level BF in the Y -direction predicted by the three models varies between 17.9% and 19.3% of the seismic weight (Fig. 1.3.8a). The mean difference between the three models considering

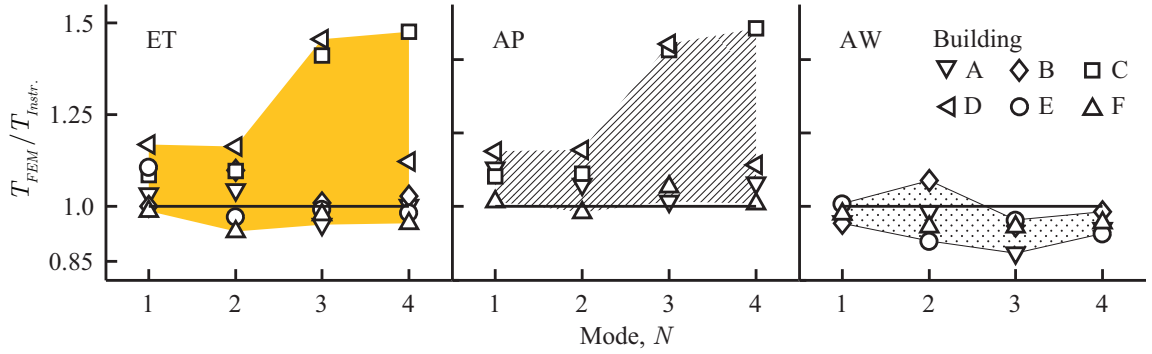


Figure 1.3.7. Periods of the ET, AP and AW models normalized with respect to ambient vibration instrument values for the 6 buildings.

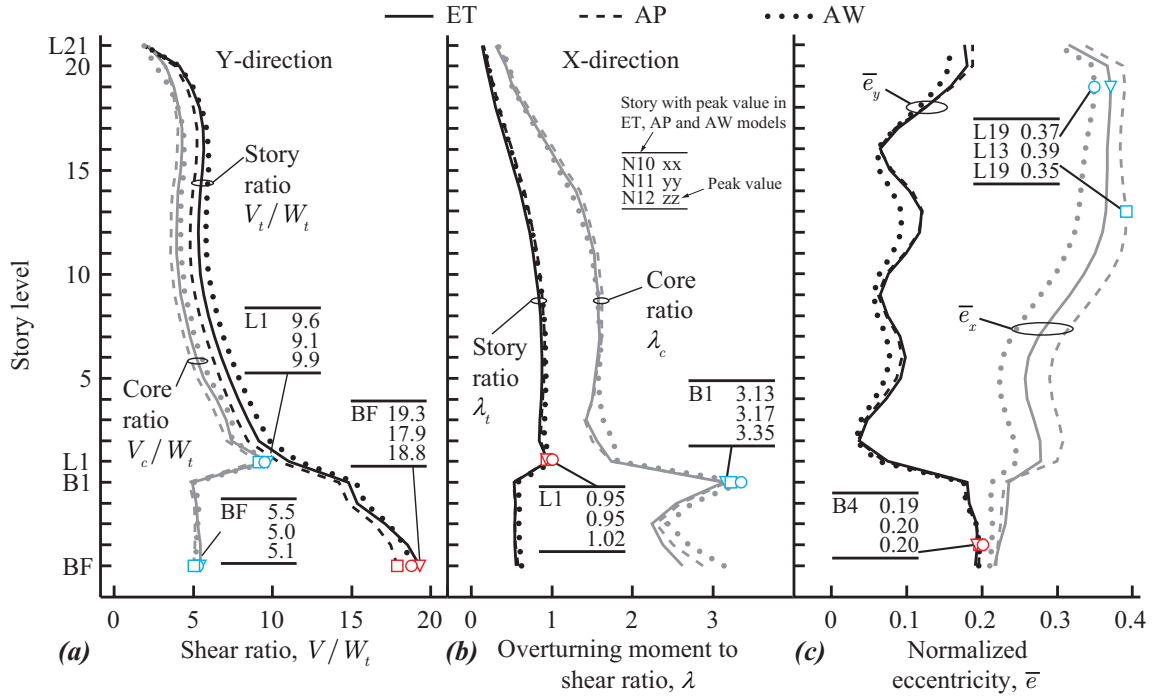


Figure 1.3.8. Vertical distribution of responses parameters for the ET, AP and AW models of Building A: (a) story shear ratio V_t/W_t and core shear ratio V_c/W_t ; (b) overturning moment to story shear ratio (λ_t) and the corresponding core ratio (λ_c); and (c) normalized eccentricity ($\bar{e}$). Black and grey lines in plots (a) and (b) represent total story and core wall responses, respectively. Black and grey lines in plot (c) represent eccentricity in X- and Y-direction, respectively.

all stories is 14.3%. The core shear V_c at level L1 in the Y-direction varies between 9.1% and 9.9% of the seismic weight, which is equivalent to a ratio V_c/V_t between 0.85 and 0.88. The same ratio V_c/V_t at level BF is considerably smaller (about 0.27) due to the transfer

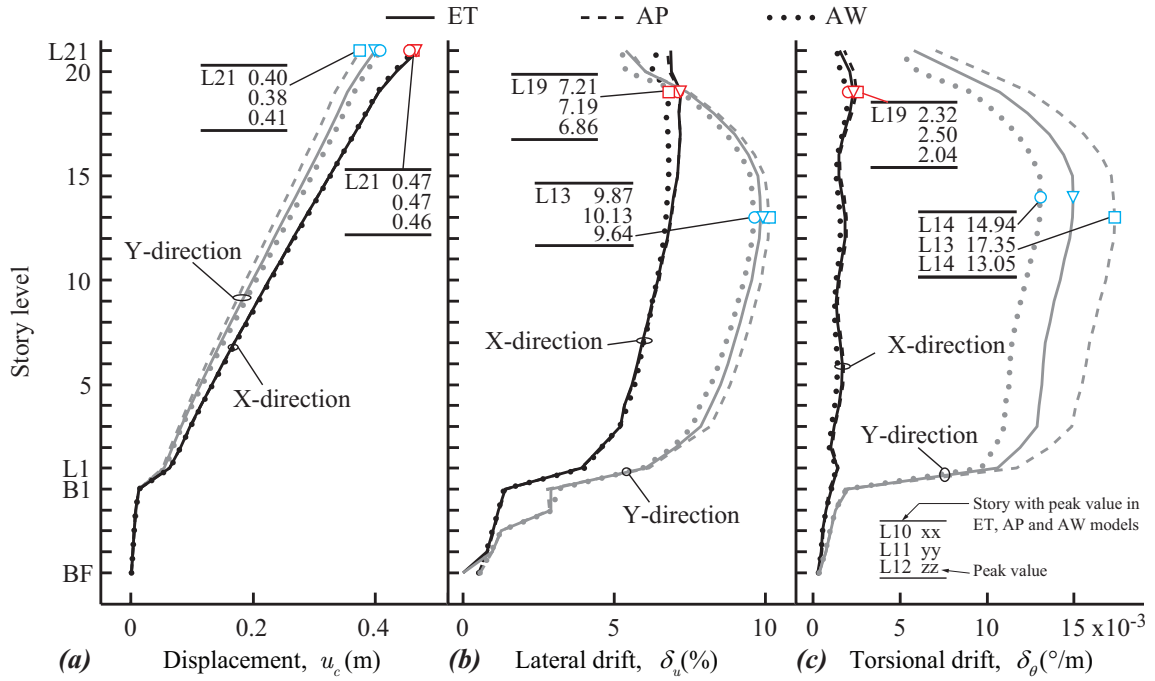


Figure 1.3.9. Vertical distribution of responses parameters for the ET, AP and AW models of Building A: (a) displacement of the geometric center of the diaphragm (u_c); (b) lateral inter-story drift (δ_u); and (c) torsional inter-story drift (δ_θ). Black and grey lines represent responses in X- and Y-direction, respectively.

of shear forces from the core to the perimeter walls. The ratio of overturning moment to story shear λ_t at level L1 in the X-direction varies between 0.95 and 1.02 as shown in Fig. 1.3.8b. For the core walls this ratio (λ_c), at level B1, varies between 3.13 and 3.35. The normalized eccentricity $\bar{e}$ is shown in Fig. 1.3.8c and varies between 0.03 and 0.20 as well as 0.28 and 1.26 in the X-direction ($\bar{e}_y$) and Y-direction ($\bar{e}_x$), respectively. The Y-direction shows the largest differences among the three models, with a mean difference in all stories of 18.4%.

A comparison of displacements, drifts, and plan rotations between the three models is shown in Fig. 1.3.9. Again, the Y-direction shows the largest differences among models. The mean difference between the model predictions at all evaluated stories is 7.7%, 5.8% and 30.0% for u_c , δ_u and δ_θ , respectively, in the Y-direction, and 1.5%, 3.4% and 17.7% for these same responses in the X-direction.

Similar results were obtained for the other five buildings though details are omitted for the sake of brevity. The values of response parameters for all ET models in the X - and Y -directions are: **(i)** the base story shear of the six buildings range between 19% and 30% of the seismic weight, and the peak shear ratio V_c/V_t occurs between the first and fourth story, and ranges between 35% and 99%; **(ii)** the peak overturning moment to story shear ratio varies between 0.6 and 2.1 and the corresponding peak of core wall ratio varies between 2.5 and 6.6; **(iii)** the peak normalized eccentricity varies between 0.1 and 0.5, which is indicative of large lateral-torsional coupling; **(iv)** the roof displacement of the center of the diaphragm varies between 20 cm and 50 cm; **(v)** the peak inter-story lateral drift varies between 5 ‰ and 11 ‰, and is predicted at about 3/4 of the height above ground level ($3/4H$); and **(vi)** the peak torsional inter-story drifts varies between $2/1000^\circ/\text{m}$ and $19/1000^\circ/\text{m}$, and also occur at $3/4H$.

The variability of the predicted seismic response for the six buildings, using the three models is shown in Fig. 1.3.10. The responses of *variant* models (AP and AW) are normalized with respect to the *reference* models (ET), and uncertainty is shown for the two directions of analysis (X and Y). The standard deviation (σ) and the range between maximum and minimum of these ratios are shown in the accompanying table. The standard deviation of all the normalized parameters is less than 11%. Larger uncertainty is obtained for core shear V_c than for the story shear V_t , especially for basements (B1 and BF), where the normalized shear V_c varies between 0.81 and 1.28. Also, largest uncertainty is identified for λ_c , $\bar{e}$ and δ_θ , with standard deviations of 8.8%, 8.6%, and 10.7%, respectively.

1.4. Effect of diaphragm stiffness

In order to evaluate the uncertainty generated by modeling the diaphragm stiffness of the free-plan buildings, the ET models were used to study four different diaphragm assumptions in all buildings as shown in Fig. 1.4.11: **(i)** a semi-rigid diaphragm (DS), which considers the in-plane and out-of-plane bending stiffness of the shell elements of the slab at each floor (*reference* model); **(ii)** a semi-rigid diaphragm, which is identical to DS but

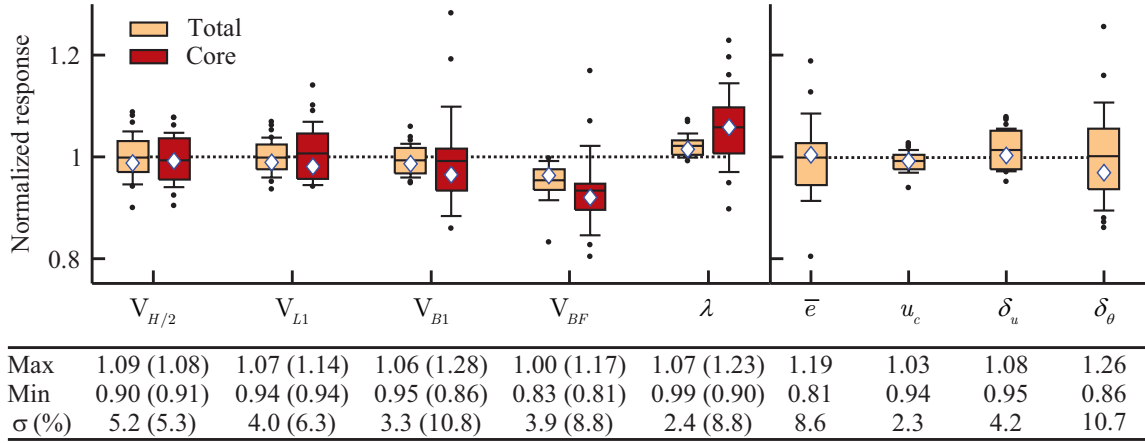


Figure 1.3.10. Response parameters of the AP and AW models normalized by the ET model results in the 6 buildings: box-plot diagram (top); and maximum, minimum and standard deviation σ (%) (bottom) (Values in parenthesis associated with the core walls.)

without bending stiffness (DSo); **(iii)** a rigid in-plane diaphragm (DR), which considers an infinite in-plane stiffness but includes the out-of-plane bending stiffness of the shell elements at each floor; and **(iv)** a rigid in-plane diaphragm, but without bending stiffness (DRo).

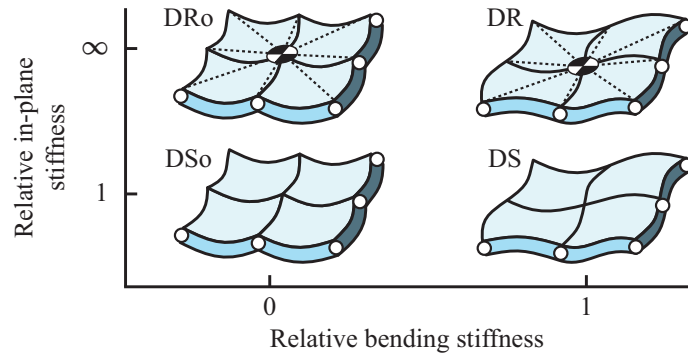


Figure 1.4.11. Models considered in the study of diaphragm stiffness.

The advantage of the imposed in-plane constraint in the DR and DRo models is that it reduces the number of DOFs by 1.4 and 1.7 times compared to the DS and DSo models, and consequently the computational time of the periods and vibration modes is reduced by 1.5 and 7.4 times, respectively. Fig. 1.4.12a compares the first four periods obtained for the four diaphragm assumptions for Building A. For the first four periods, the DSo

model consistently predicts longest periods and the DR model predicts the shortest. The difference among models in the estimation of the first period is 18.5%, with a maximum of 23.6% for the first four periods. Fig. 1.4.12b shows the periods of the DSo, DR and DRo models normalized with respect to the DS values. For mode one, a box-plot is shown for the data. In this case, the ratio of the periods varies between 0.95 and 1.27 with $\sigma = 10.4\%$. Using the DS models as a reference, the first four periods of the DR models are up to 10% shorter. On the contrary the first four periods of the DSo models are between 4% to 27% longer. Also, the first period of the DRo models are up to 19% longer. It is interesting to note that the difference of the four diaphragm models can result in 37% difference for the first four modes, especially if the bending stiffness of the diaphragm is ignored. Moreover, the smallest errors occur for T_{DR}/T_{DS} , which somewhat supports the historical assumption of using the DR model in practice.

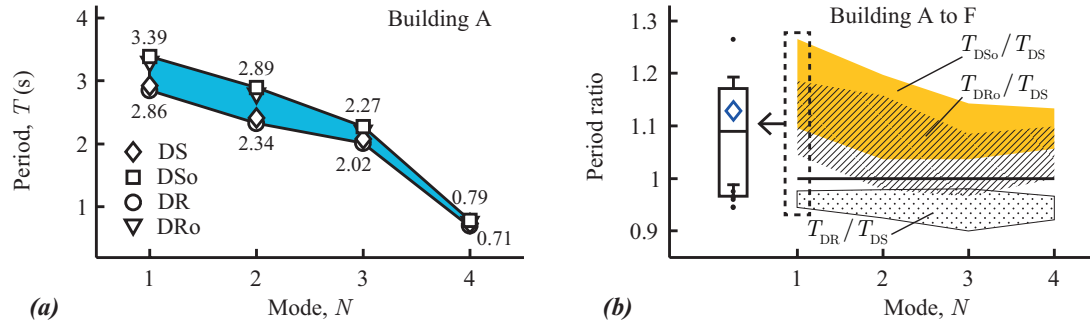


Figure 1.4.12. Period variation according to stiffness diaphragm models: (a) periods of Building A; (b) periods of the DSo, DR, DRo models normalized by the DS model results for the six buildings.

Fig. 1.4.13a shows the vertical distribution of story shear V_t and core shear V_c in X -direction of Building B using the four diaphragm modeling assumptions described above. The estimation of shear V_t among models showed a mean difference for all stories of 26.2%. Similarly, V_c has a mean difference of 34.1% for all stories above ground level, and up to 47.9% if basements are included. With respect to the effects of in-plane stiffness, shears V_t and V_c in the DR and DRo models are consistently bigger for all floors, compared to those yielded by the DS and DSo models. This increase is evident at the basements, and may reach up to 11.1% between the DRo and DSo models at level BF. In addition,

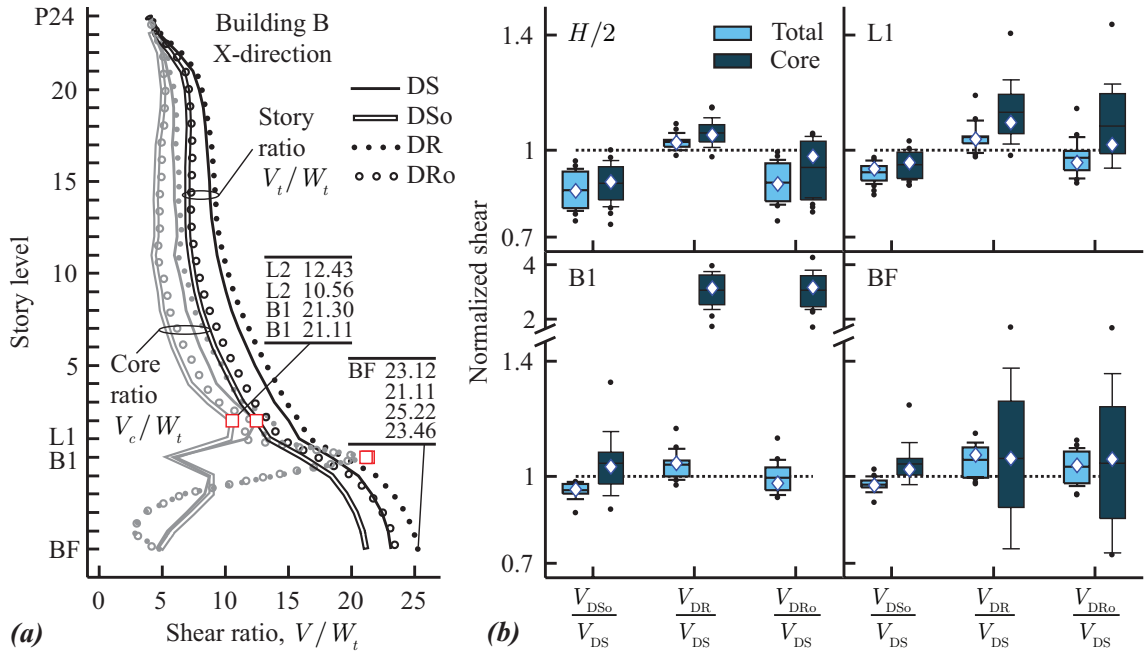


Figure 1.4.13. Story and normalized shears for the four different diaphragm stiffness: (a) Vertical distribution of story shear ratio V_t/W_t and core shear ratio V_c/W_t for Building B in the X-direction; (b) box-plot of the X- and Y-directions shears V_t and V_c for the DSo, DR and DRo models, normalized by the DS model results in the six buildings and at levels $H/2$, L1, B1, and BF, respectively. Black and grey lines in plot (a) represent total story and core wall shear, respectively.

core shear V_c observed to be 3.6 and 3.7 times larger in the DR and DRo models at level B1 than those of the DS and DSo models, respectively. This leads to V_c being up to 22% greater than V_t . Recall that this abrupt increase in force for the core walls can be attributed to the *back-stay* effect, as identified in the literature (Moehle, 2015). With respect to the bending stiffness, shears V_t and V_c of the DSo and DRo models are smaller than the DS and DR models, respectively (at all floors) with an observed maximum of 23.3% and 18.4%, respectively. The exception occurs at the basements, where the DS-DSo models and the DR-DRo look very similar.

The effect caused by each diaphragm assumption in the estimation of shears V_t and V_c is evaluated by normalizing the results by the DS results. Fig. 1.4.13b shows these ratios in a box-plot format for all six buildings at four different levels ($H/2$, L1, B1, and BF) by taking into account the results in both directions (X and Y). At building mid-height level ($H/2$), the normalized shears V_t and V_c of the DSo and DRo models average 0.86 and

0.89, respectively, with a minimum of 0.75. In addition, for the DR model, the normalized shears V_t and V_c are 1.06 on average, with a maximum of 1.15. Consequently, for the higher levels, the effect of the bending stiffness in the shear forces is more relevant than the in-plane stiffness. For the other levels (L1, B1, and BF), the mean of the normalized shears V_t and V_c of the DSo models varies between 0.92 and 1.05. The normalized shears V_t of the DR and DRo models increase mainly at level BF with a mean of 1.06 and a maximum of 1.15. It is apparent that a high degree of variability exists in the normalized shear V_c at levels L1, B1, and BF for the DR and DRo models due to the *back-stay* effect. Therefore, level L1 shows a mean of 1.13 and $\sigma = 14.6\%$; level B1 a mean of 3.07, with a maximum of 4.29, and $\sigma = 73.2\%$; and level BF a range between 0.57 and 1.71 with $\sigma = 31.4\%$.

Fig. 1.4.14 shows all normalized response parameters for the six buildings in X - and Y -directions in box-plot format. The standard deviation (σ) and the range between maximum and minimum of these ratios are shown in the accompanying table. Due to the change in bending stiffness of the diaphragm, the normalized shears V_t and V_c above ground level ($H/2$ and L1) vary between 0.75 and 1.44 with $\sigma = 13.3\%$. Contrary, minor uncertainty is observed in normalized shear V_t due to change in in-plane stiffness of the diaphragm, which varies at the basements B1 and BF between 0.88 and 1.17 with $\sigma = 6.4\%$. However, the *back-stay* effect causes the normalized shear V_c at the basements (B1 and BF) to vary much more significantly, namely between 0.57 and 4.29 with $\sigma = 112.2\%$. The same effect applies to the normalized ratio λ_c , which varies between 0.71 and 2.24 with $\sigma = 30.6\%$. In terms of standard deviation the uncertainty of the normalized forces of the core walls (V_c and λ_c), indicated in parenthesis in Fig. 1.4.14, is greater than for the normalized total forces (V_t and λ_t). The uncertainty of normalized displacement u_c and normalized drift δ_u is smaller, with $\sigma = 9.3\%$, unlike the normalized drift δ_θ which has $\sigma = 32.1\%$.

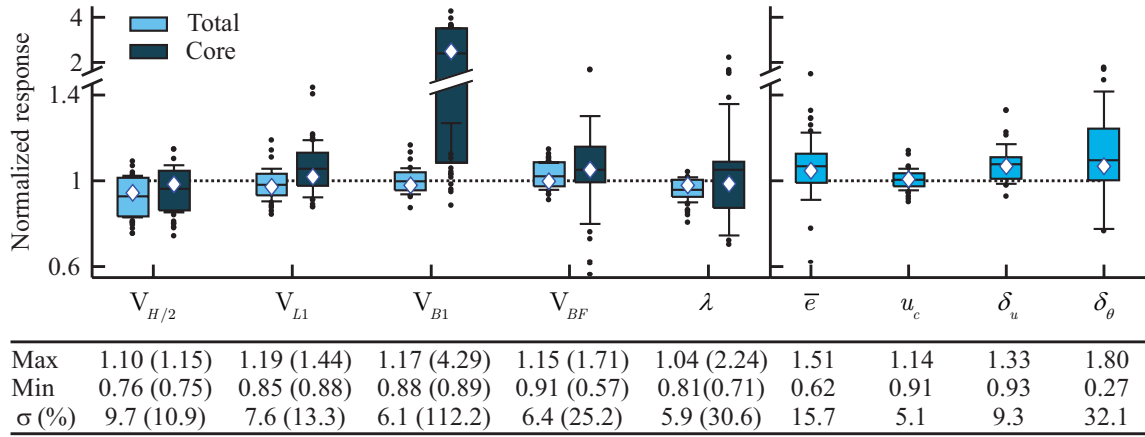


Figure 1.4.14. Response parameters of the DSo, DR and DRo models normalized by the DS model results in the six buildings: box-plot diagram (top); and maximum, minimum and standard deviation σ (%) (bottom). (Values in parenthesis associated with the core walls.)

1.5. Effect of the soil constraints

To study the uncertainty associated with the type of soil-structure constraints, the AP models of four buildings were used (A, C, D and F), and combined with five soil modeling assumptions as shown in Fig. 1.5.15: **(i)** fixed support (SF) of the structure to the base (*reference* model); **(ii)** vertical support (SV)—structure is supported elastically at the base in the vertical direction only; **(iii)** horizontal support (SH)—analogous to SF but including the lateral flexibility of the soil in contact with the basement perimeter walls; **(iv)** lateral and vertical support (SS)—combination of SV and SH; and **(v)** complete fixity (SB)—with all embedded elements fixed to the ground. In all models lateral displacements at the base level are assumed to be fixed.

To generate the SV, SH and SS soil models, a common soil profile was developed as each of the buildings is located on soil profiles with relatively similar properties. The stratigraphy summarized in Table 1.5.6 shows a rather soft layer of surface soil in the upper 1.5 m, followed by a very dense sandy gravel with increasing relative density (D_r) in depth. The sites can be categorized as site class C per ASCE-7 ([ASCE/SEI-7-10, 2013](#)). The shear wave velocity (V_s) for all soil strata is greater than 500 m/s. The modulus of vertical subgrade reaction (K_v) was taken as 174.1 N/cm³ and assumed to

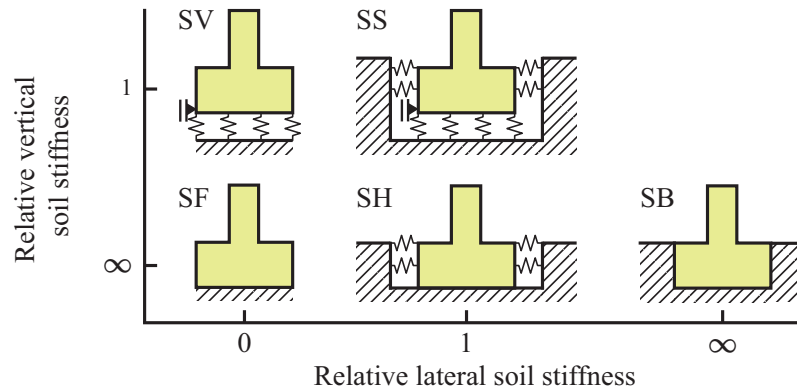


Figure 1.5.15. Soil-structure interaction assumptions in the studied models.

be constant. The modulus of horizontal subgrade reaction (K_h) for this non-cohesive soil is proportional to depth (z) and follows existing recommendations (MOP, 2014), i.e.: $K_h(z) = 2.2f(1 + 3.3d_s/D)(z/D)$, where $D = H_b - d_s$; H_b is the height of basements; d_s is the depth of the soil between ground surface and the start of the soil stratum; and f is a coefficient for each soil stratum with values 0, 24.5 and 34.3 N/cm^3 , respectively. Moreover, as a response spectrum is carried out, each soil moduli is amplified by a dynamic factor $F_{sis} = 1.9$.

Table 1.5.6. Typical soil stratigraphy characteristic in the building studied.

Depth (m)	Soil class	Relative density, D_r (%)
0.0 - 1.5	Poor soil	—
1.5 - 5.5	Dense sandy gravel	<65
+5.5	Dense sandy gravel	>80

An important parameter controlling the soil-structure inertial effects in tall buildings is the structure-to-soil stiffness ratio ($H_t/V_s T_1$) (ATC-83, 2012). Inertial effect should be considered if this ratio is greater than 0.1 (Tabatabaiefar & Massumi, 2010). As the soil in these buildings is stiff ($V_s > 500 \text{ m/s}$) and the ratio H_t/T_1 varies between 34.8 m/s and 60.4 m/s, the parameter $H_t/V_s T_1$ varies between 0.07 and 0.09, which is less than 0.1. Therefore, $H_t/V_s T_1$ can be considered insignificant and the analysis is dominated by the soil stiffness only. Please consider that the soil damping was neglected.

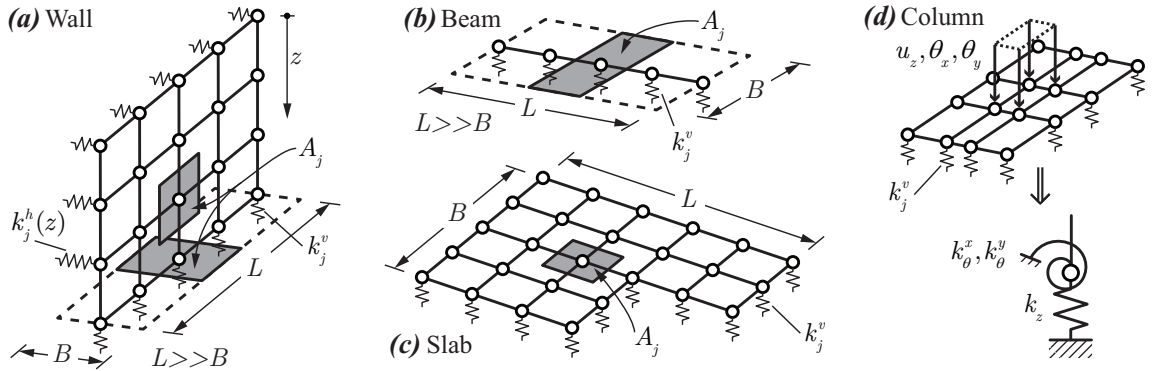


Figure 1.5.16. SSI modeling assumptions: (a) basements walls in contact with the soil; (b) foundation beams; (c) foundation slabs; and (d) isolated column footing.

In the SV, SH and SS models, the soil is modeled with uncoupled Winkler springs (COMBIN4) and Terzaghi's criterion to estimate spring stiffnesses (Terzaghi, 1955). For perimeter walls, slabs, and foundation beams, axial springs are considered as shown in Fig. 1.5.16a-c. For isolated column footings, rotational springs are added (Fig. 1.5.16d). For vertical springs in walls, beams, columns, and slabs, the stiffness of the structural elements is larger than that of the soil, thus the spring stiffness in the j -th node is based on the overall dimensions of the foundation element, i.e.: $k_j^v = F_{sis} K_v \phi_B \phi_R A_j$, where $\phi_B = (B + 0.3)^2 / 4B^2$ and $\phi_R = (2 + B/L)/3$ are adjustment factors for non-cohesive soils; A_j is the tributary area of the j -th node; and L and B are the length and width of the foundation element, respectively. For lateral springs associated with perimeter walls, the spring stiffness in the j -th node is: $k_j^h(z) = F_{sis} K_h(z) \phi_j A_j$, where $\phi_j = (a_j + 0.3)^2 / 4a_j^2$ and $a_j = \sqrt{A_j}$ the equivalent width of the tributary area. Finally, for isolated column footings, the vertical and two rotational spring stiffness is calculated with a FEM model for each footing (Fig. 1.5.16d).

Fig. 1.5.17a compares the first four building periods obtained with the five different soil models for Building A. The SV models were found to always predict the longest periods, and the SB models the shortest. The first period estimates yielded by all SSI models showed the largest variance of 12.9%. Fig. 1.5.17b shows the first four periods of the SV, SH, SS and SB models normalized with respect to the SF values in Buildings A, C,

D and F. For mode one, the normalized ratio varies between 0.97 and 1.18 with $\sigma = 6.0\%$. Using the SF models as a reference, the first periods of the SV and SS models are up to 18% and 14% longer, respectively; on the contrary, the first four periods of the SH and SB models are between 3% and 10% shorter, respectively.

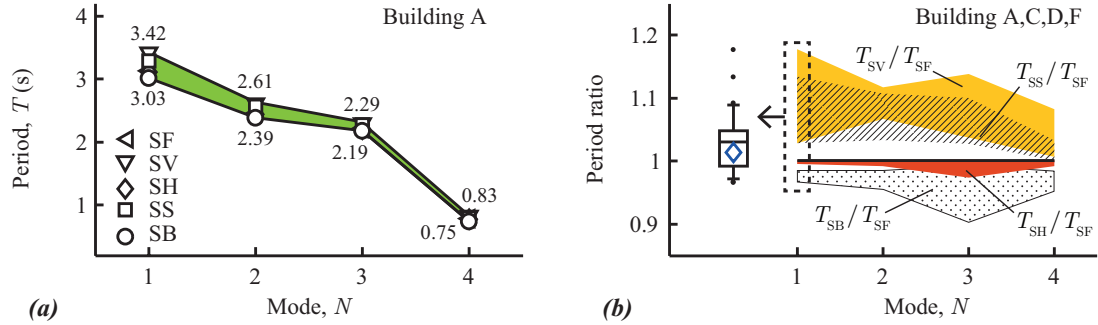


Figure 1.5.17. Period variation according to the soil-structure model used: (a) periods of Building A; (b) periods of the SV, SH, SS, SB models normalized by the SF model values for Buildings A, C, D and F.

Fig. 1.5.18a shows the vertical distribution of story shear V_t in the X -direction of Building F for the five soil models. In all models, the story shears above ground level are very similar with a maximum difference of 3.4%. However, the SH, SS, and SB models reduce the shear V_t at level BF, with a minimum of 3.1% of the seismic weight, i.e. 7.7 times less than the value of the SF model. On the other hand, the SV model generates a slight (6%) increase in the base story shear V_t relative to the SF model.

The variations in estimating shears V_t and V_c resulting from the different soil models are analyzed by normalizing the results by the SF results. Fig. 1.5.18b shows a box-plot of the shear ratios for Buildings A, C, D and F at four levels ($H/2$, L1, B1 and BF), and both directions (X and Y). While the normalized shears V_t and V_c above ground level ($H/2$ and L1) are practically constant (0.8-1.1) in all models, the distribution of these shears at basements (B1 and BF) depends on the soil model. Normalized shears V_t for the SH, SS and SB models decrease at lower levels. At level B1, it drops to a mean of 0.53-0.90 and at level BF to a mean of 0.23-0.53. This effect is not observed in the SV model, which shows up to 10% increase at level BF. Normalized shear V_c increases at the basements for the SV and SS models, reaching at level B1 a mean of 1.17-1.22 and at level BF a mean of

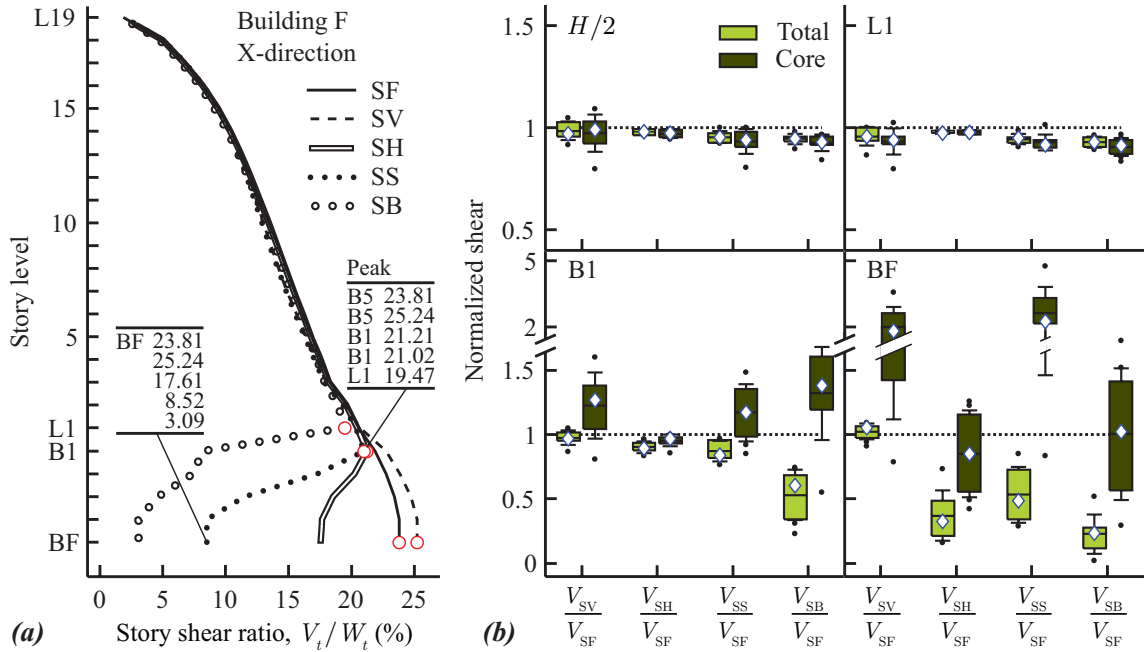


Figure 1.5.18. Story and normalized shears for the five different soil-structure models: (a) vertical distribution of story shear ratio V_t/W_t for Building F in the X-direction; (b) box-plot of the X- and Y-direction shears V_t and V_c for the SV, SH, SS and SB models, normalized by the SF model results in the Buildings A, C, D and F, and at four levels $H/2$, L1, B1 and BF, respectively.

2.02-2.64 (with a maximum of 3.6-4.8), respectively. Additionally, the normalized shear V_c of the SH and SB models reach more variability at level BF ranges between 0.3 and 1.7.

Fig. 1.5.19 shows all normalized response parameters in both directions (X and Y) in box-plot format. The standard deviation (σ) and the range between maximum and minimum of these ratios are shown in the accompanying table. Normalized shears V_t and V_c above ground level ($H/2$ and L1) vary between 0.80 and 1.10 with $\sigma = 6.1\%$; these normalized shears show greater variability in the lower basements; where shears vary between 0.24 and 1.64 with $\sigma = 27.3\%$ at level B1, and between 0.03 and 4.79 with $\sigma = 106.8\%$ at level BF. Analogously, normalized overturning moment to shear ratios λ_t and λ_c vary between 0.37 and 4.73 with $\sigma = 94.2\%$, and the normalized eccentricity $\bar{e}$ varies between 0.49 and 3.16 with $\sigma = 42.8\%$. The standard deviation of the normalized shear V_c is greater

than that for V_t at all levels. The origin of the high variability of the shear and overturning moments at basements stems from the use of different soil elements and stiffnesses considered in each model, which in turn modifies the forces and reactions of the structure. Independently of this observation, the normalized parameters u_c , δ_u and δ_θ have low variability with maximum $\sigma = 16.7\%$.

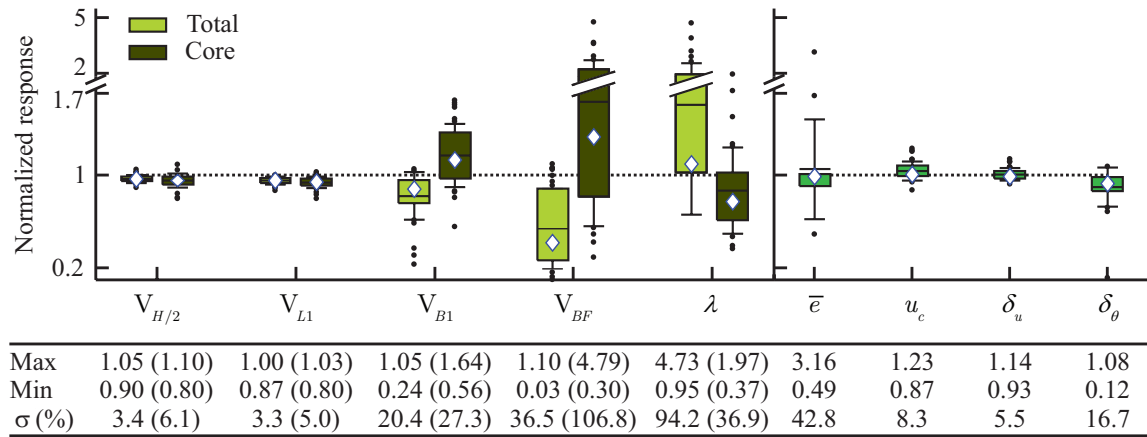


Figure 1.5.19. Responses parameters of the SV, SH, SS and SB models normalized by the SF model results in Buildings A, C, D and F: box-plot diagram (top); and maximum, minimum and standard deviation σ (%) (bottom). (Values in parenthesis associated with the core walls.)

1.6. Uncertainty associated to assumed building fixity level

To investigate the uncertainty associated with “where” the building is assumed to be fixed, the AP (Buildings A, C, D and F) and AW (Buildings A, B, E and F) models were used with different number of basements levels ranging from a model without basements to modeling all underground levels. A model with n -basements will be called U_n and will be fixed to the ground at the bottom of n -th level at depth z_n . To isolate this effect, SSI effects have been omitted in the analysis. In this section U_0 is the *reference* model.

Fig. 1.6.20a compares the first four building periods obtained by modifying the amount of basements and the fixity level of Building A. As expected, the U_5 model predicts the longest periods, and the U_0 model the shortest. The difference in the estimation of the first period is 10.9% and the maximum difference for the first four periods is 15.5%.

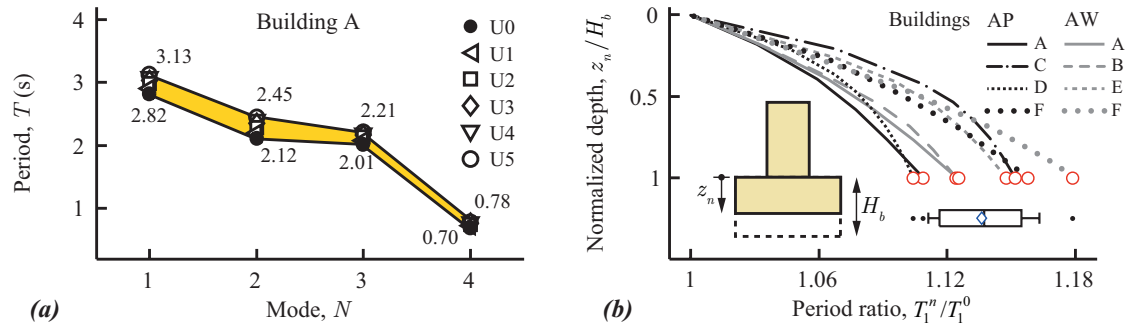


Figure 1.6.20. Variation in periods depending on the amount of basements considered in the models: (a) Building A periods for the AP models; (b) elongation of the first period (T_1^n/T_1^0) depending on the normalized depth of the basements (z_n/H_b) for the six buildings.

Fig. 1.6.20b shows the elongation of the first period (T_1^n/T_1^0) for the AP and AW building models as a function of the normalized depth of the basement (z_n/H_b); this ratio varies between 10% and 18% with $\sigma = 2.6\%$.

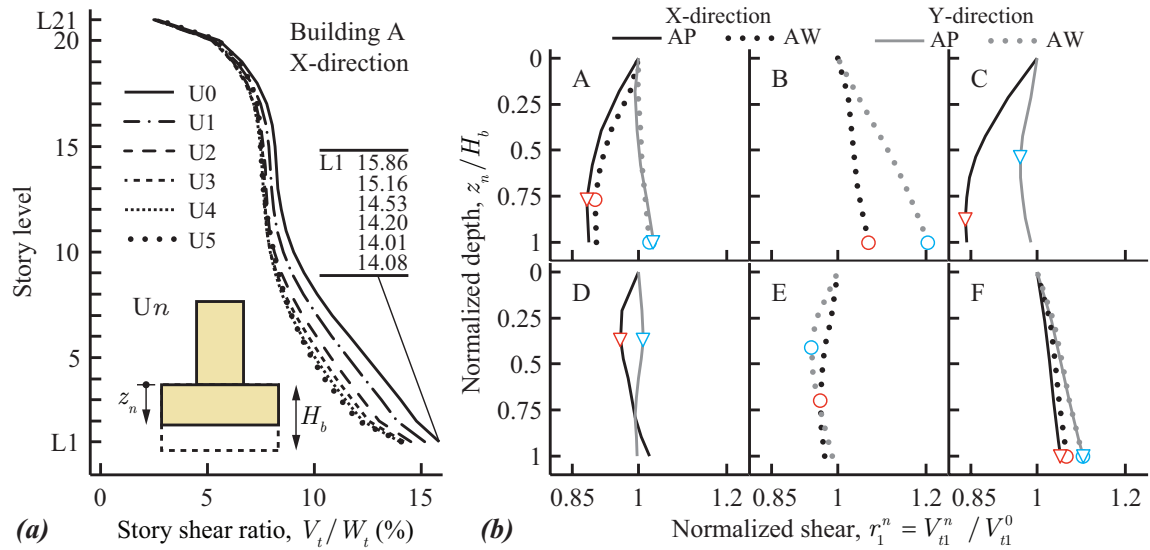


Figure 1.6.21. Building response for different number of basements in the models: (a) story shear ratio V_t/W_t of Building A in X-direction (AP models); (b) normalized depth of basements (z_n/H_b) versus normalized story shear at level L1 ($r_1^n = V_{t1}^n/V_{t1}^0$) in all six buildings and the X- and Y-directions, respectively.

Analogously, Fig. 1.6.21a shows the story shear V_t in Building A in the X-direction as a function of the number of basements. V_t varies between 14.0% (U4) and 15.9% (U0) of

the seismic weight, i.e. a maximum reduction of 11.7% with respect to U0. Fig. 1.6.21b-c show the story shear V_t at level L1 of model U_n (V_{t1}^n) normalized with respect to U0 (V_{t1}^0) expressed as a ratio $r_1^n = V_{t1}^n / V_{t1}^0$, for all six buildings in the X - and Y -direction, respectively. A value r_1^n less than one implies that the story shear at level L1 is reduced when the building model is fixed at n -th level. In other words, if the minimum code design shear is imposed at the n -th level, the base shear at level L1 could be less than the minimum base design shear. The shear V_t at level L1 may in principle increase or decrease as the number of basements is added depending on the model and the direction of analysis. For example, the ratio r_1^n in Buildings C and E in the X -direction reach a minimum of 0.84 and 0.96, respectively. Furthermore, r_1^n in Buildings B and F in the Y -direction reach 1.2 and 1.1, respectively. In all cases, these peak values of the ratio r_1^n occur for z_n/H_b between 0.37 and 1.

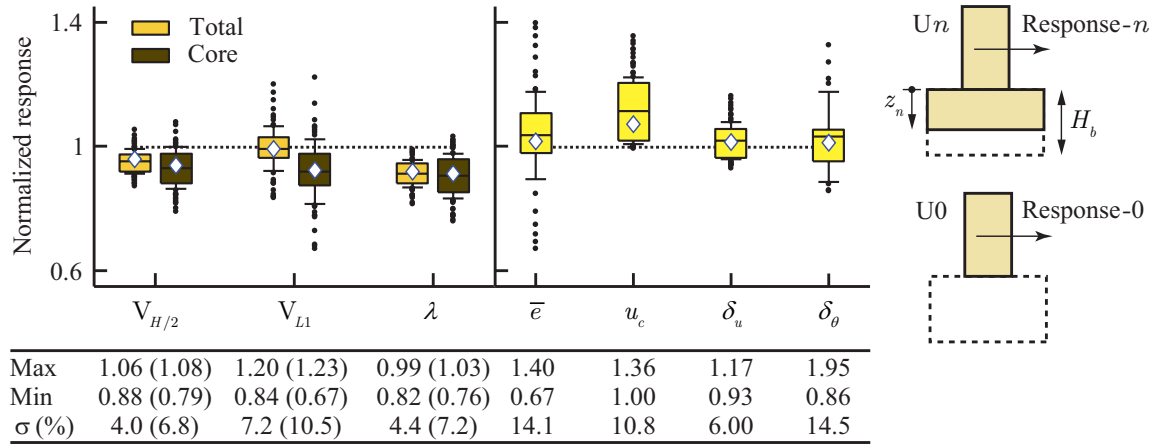


Figure 1.6.22. Response parameters of the U_n models normalized with respect to the U_0 models for the six buildings: box-plot diagram (top); and maximum, minimum and standard deviation σ (%) (bottom). (Values in parenthesis associated with the core walls.)

Fig. 1.6.22 shows all normalized response parameters for the six buildings in both directions (X and Y) in box-plot format. As before, normalization is performed by dividing all U_n responses by the reference U_0 response. The standard deviation (σ) and the range between maximum and minimum of these ratios are shown in the accompanying table. The variability of the normalized shears V_t and V_c ranges at level L1 between 0.67 and 1.23

with $\sigma = 10.5\%$. Both responses, V_t and V_c , show similar variability. In terms of standard deviation and range, the uncertainty of the normalized core wall shear and overturning moment at levels $H/2$ and L1, shown in parenthesis, is greater than that the normalized total quantities. On the other hand, large variability exists for the normalized parameters $\bar{e}$ and δ_θ , ranging between 0.67 and 1.95 with $\sigma = 14.1\%$ and 14.5% , respectively. Analogously, the normalized displacement u_c varies between 1.00 and 1.36 with $\sigma = 10.8\%$, and the normalized drift δ_u has $\sigma = 6\%$. The standard deviation of all normalized responses is less than 15%.

1.7. Summary and main results

This chapter evaluates the epistemic uncertainty of four major modelling assumptions, which typically generate debate during the structural design review process of building projects. These assumptions are: **(1)** the type of finite elements used; **(2)** the type of floor diaphragm considered; **(3)** the soil-structure interaction model used at the basements and foundation levels; and **(4)** the correct level of fixity for the model. In this quantification of epistemic uncertainty, aftershock and ambient vibration measurements together with the predicted elastic response of six reinforced concrete buildings located in Santiago, Chile were considered. The uncertainty of response parameters for each modeling assumption was evaluated by analyzing the ratio of predicted results from the *variant* models relative to the *reference* models. The main results obtained from this part are:

- The AW models with solid elements provided the best estimates of the first four building periods, with errors smaller than 13% relative to measured periods. For the three models considered (ET, AP and AW), a maximum error of 17% and 48% was found for the first two and four vibration period predictions, respectively. Although large, these errors are common when compared to experimental validation cases.

- The standard deviation of the different parameter response ratios obtained for the three models (ET, AP and AW) was less than 11%. Consequently, considering the higher computational cost involved in the AW model, and the relatively low values of epistemic uncertainty, the ET and AP models with shell and unidimensional beam elements are recommended for estimating dynamic responses in these free-plan buildings.
- The assumed diaphragm stiffness was found to be a relevant source of epistemic uncertainty. Variations in the diaphragm stiffness for the first four buildings periods may reach values up to 10% and down to 27%, respectively. These variations are measured relative to the *reference* model, which considers in-plane and out-of-plane diaphragm stiffness (DS model). The variation of the in-plane stiffness of the diaphragm generated a large variation of the predicted shear forces in the core walls. Normalized shear for the core, varied between 0.57 and 4.29 times for the first basement, and the standard deviation of this ratio was 112%. This large variation is attributed to the *back-stay* effect. Since current computer software allows modeling of the diaphragm stiffness, it is recommended to consider the in-plane stiffness of the diaphragm in the basements to reduce this *back-stay* effect in FEM models. For shear forces in higher stories, the effect of the bending stiffness of the diaphragm becomes larger than the effect of the in-plane stiffness. The normalized story shear and core shear at mid-height of the buildings ($H/2$) varied between 0.75 and 1.44, and was mainly influenced by the out-of-plane stiffness.
- Shear forces at the basement levels were found to be strongly dependent on the type of soil-structure interaction model used. Normalized story shear at the basement varied between 0.03 and 4.79 times, with a standard deviation of 106.8%. Above the ground level, the normalized story shears and core shears were found to be similar among models. Due to this large uncertainty, it is recommended to do sensitivity analysis of the building model including and neglecting the contribution of lateral soil stiffness to obtain an envelope of the expected responses.

Please recall also that in many cases the lateral soil stiffness physically disappears during the lifespan of the building due to a neighbour construction, and the sensitivity analysis is in such case mandatory.

- The influence of the level at which the structural model is considered fixed to the ground leads to changes in the first vibration period of the buildings from +10% to +18% relative to the *reference* model without basements. No clear trend was observed for the story shear value at the first story as more basements were added into the structural model and the fixity levels moves down. Normalized base shear varied between 0.84 and 1.2 times relative to the reference model, and depending at which level the code minimum design base shear is imposed into the model, different fixity assumptions may lead to conservative or non-conservative designs. In all buildings cases, peak responses occurred as the fixity level was imposed at intermediate underground levels. Furthermore, the standard deviation of all normalized responses was less than 14.5%. Due to the epistemic uncertainty associated with the building fixity level, it recommended to elaborate at least two models with different fixed levels and generate an envelope of story shears and element forces. The code minimum design shear should consider the envelope of these two models to avoid under design of the superstructure.
- Finally, from the studied responses it is concluded that larger uncertainty was identified for core wall forces (shear and overturning moment) than for story forces. Additionally, larger uncertainty was identified for story and core shear at the basements (B1 and BF) than for shears in the upper levels ($H/2$ and L1). Since free-plan buildings are characterized by having limited structural elements, the epistemic uncertainty of these quantities are relevant and should be accounted for in building design. One possible option to include these uncertainties is to consider the ranges and the standard deviations of the responses ratios presented herein, or otherwise, by considering the envelope of different buildings models.

The epistemic uncertainties evaluated in this article are limited to the linear elastic response of free-plan buildings and do not necessarily carry over to other structure types or inelastic behavior of these systems.

2. EPISTEMIC UNCERTAINTY OF 3D CONTINUUM STRESS-STRAIN CONCRETE MODELS AND CONSISTENT NUMERICAL IMPLEMENTATION

The use of more sophisticated inelastic stress-strain constitutive models in finite element (FE) analysis of structures and systems is becoming more common today in engineering practice. However, such models lead in many cases to different results, which is a concern for a designing structures. Indeed, quantifying this epistemic uncertainty inherent in these models is one of the objectives of the second part of this thesis, since it may lead to practical recommendations that increase trust in the obtained results. However, this is not a simple task since all available models use different parameters, notations and assumptions. This work also aims to provide a consistent notation and computational implementation for these models.

The quasi-brittle material behavior of concrete exhibits a pronounced nonlinear behavior associated with cracking in tension and crushing in compression. Tensile behavior is characterized by an elastic response until the tensile strength. For larger strains, strength softening occur due to crack propagation. This crack opening process is also followed by shear stress transfer degradation due to deterioration of the aggregate interlock. Thus, cracking induces damage anisotropy characterized by a non-symmetrical behavior between tensile and compressive regimes with an irreversible strength and stiffness degradation due to the propagation of micro-crack nucleation ([Krajcinovic, 1996](#)). Moreover, it is observed that the energy dissipated to form a unit area of crack surface (G_f) is relatively constant, and can be considered as a material parameter ([Hillerborg, Mod  er, & Petersson, 1976](#); [van Vliet & van Mier, 1995](#); [Nakamura & Higai, 2001](#)).

In contrast, concrete in compression exhibits the formation of considerable irreversible inelastic strains, which increase with the confinement and inelastic volumetric expansion (dilatancy). Under uniaxial compressive stress, nonlinear hardening is present at the pre-peak stage followed by a strength-softening stage. The complexity of this concrete behavior increases for multiaxial stress conditions. On the one hand, the compressive strength increases with lateral confinement, showing that concrete is a pressure-sensitive material.

Furthermore, a material densification (or compaction) due to collapse of micro-porosities is observed under high-confining stresses. And on the other hand, under cyclic loading conditions, the micro-cracks close under load reversals from tensile to compression; thus showing a partial stiffness recovery (unilateral effect). It is also well known the dependence of concrete strength with strain rates, due to the fact that growth of internal microcracking is delayed at high strain rates.

In the past, several two-dimensional (2D) and three-dimensional (3D) constitutive models have been proposed to describe the mechanical behavior of concrete under multi-axial stress paths. Together, the definition of a robust model and its correct computational implementation, are key aspects to correctly simulate the behavior of complex reinforced concrete (RC) structures. These models respond to the taxonomy of plastic, damage, plastic-damage, fracture, and mixed models.

Plastic concrete models are based on the plastic flow theory, which describes the behavior of irreversible plastic strains and hardening under multi-axial stress conditions. For concrete, these models use a non-associated flow rule to describe the dilatancy, kinematic or isotropic hardening, and load path-dependence. They also, include a single- or multi-surface yield criterion to describe the limit compressive regime (Mohr-Coulomb, ([Drucker & Prager, 1952](#); [Willam & Warnke, 1975](#); [DiMaggio & Sandler, 1971](#); [Bigoni & Piccolroaz, 2004](#))), which include pressure-sensitive behavior, and for other hand the tensile regimes, such as the commonly used Rankine (tension cut-off) criterion. Comprehensive overviews and comparison of plastic concrete models are available in ([Chen, 1982](#)). However, these models do not consider the damage process phenomenon associated with the stiffness degradation, the unilateral effect and the strain-softening.

Damage concrete models are based on continuum damage mechanics (CDM) theory ([Kachanov, 1958](#); [Mazars & Pijaudier-Cabot, 1989](#); [Krajcinovic, 1996](#)), which is based on the thermodynamics of irreversible processes, where the Helmholtz free energy (HFE) is defined to establish the constitutive relation using internal variables. Damage models can predict the degradation of the elastic stiffness tensor and the strain-softening behavior

caused by the irreversible propagation of micro-cracks. Critical to these models are the appropriate selection of the damage criteria, and the damage variables, which serve as a macroscopic approximation to describe the micro-cracking process (Voyiadjis & Kattan, 2005). Several damage criteria have been proposed depending on the relationship between nominal (damaged) and effective (undamaged) configurations. For instance, these criteria are: (i) equivalent strain-based (Mazars, 1984; Mazars & Pijaudier-Cabot, 1989); (ii) stress-based (Ortiz, 1985; J. Simo & Ju, 1987); (iii) energy-based (Carol, Rizzi, & Willam, 2001); and (iv) Damage Energy Release Rate-based (DERR)-based (Faria et al., 1998; Wu et al., 2006). According to the damage variable adopted, CDM models can be classified as: (i) scalar, where one or more scalars are used to characterize the isotropic damage process

As the name suggests, plastic-damage models combine the plasticity and CDM theories. Usually, the combination is based on isotropic hardening plasticity with either isotropic (scalar), or anisotropic (tensor) damage. Isotropic damage is widely used due to its simplicity to combine different types of plastic models, and can be classified according to the type of relation between the plastic and damage component. A first group assumes a plasticity formulation on the effective (undamaged) space (Lubliner et al., 1989; Yazdani & Schreyer, 1990; Faria et al., 1998; J. Lee & Fenves, 1998; Comi & Perego, 2001; Wu et al., 2006; Contrafatto & Cuomo, 2006; Cicekli, Voyiadjis, & Abu Al-Rub, 2007; Voyiadjis et al., 2008; Taqieddin et al., 2012). A second group adopts a strong-coupling approach in which plasticity is formulated in the nominal (damaged) stress space (Luccioni, Oller, & Danesi, 1996; Voyiadjis et al., 2008; Armero & Oller, 2000). In general, coupled relations are more complex than decoupled, and their implementation is not straightforward. Moreover, plastic-damage models formulated in the effective space are numerically more stable and attractive (Abu Al-Rub & Voyiadjis, 2009).

Fracture concrete models are based on the nonlinear fracture mechanics theory, where cracking can be simulated either by a discrete or a smeared crack approach. In discrete

crack models, the discontinuity of the strain field becomes explicit in simulating the initiation and propagation of dominant cracks. In contrast, in the smeared-crack models, cracks are smoothed in certain portions of the structure (inducing a length scale in the equations) to capture the deterioration process through a constitutive law (Bažant, 1982; Cervera & Chiumenti, 2006). Within the smeared-crack models, they are classified according to the formation of a crack planes of degradation, or axes of orthotropy, such as fixed crack models (Rashid, 1968; Gupta & Akbar, 1984; de Borst, 1986), rotating crack models (Cope, Rao, Clark, & Norris, 1980; Rots, 1988; TNO DIANA, 2018), multi-directional models (e.g. multi-fixed orthogonal and non-orthogonal crack models, (Maekawa, Pimanmas, & Okamura, 2003; Ventura-Gouveia, 2011), microplane model (Bažant, 1984; Caner & Bažant, 2013), among other theories.

Finally, mixed models combine more than one of the previous models. As an example, plastic-damage smeared crack, or fracture-plastic models, where plasticity and fracture mechanics are respectively used to describe compression and cracking-tension regimes (de Borst, 1986; Červenka & Papanikolaou, 2008; Behbahani et al., 2015).

Typically, concrete models are implemented on finite elements (FE) softwares, which requires the evaluation of the constitutive equations at every integration point of each element. Shell and solid elements are used for a best representation of strain and stress field distributions through complex geometries. Accuracy in these elements is strongly dependent on the algorithmic implementation and the integration techniques adopted (Krieg & Krieg, 1977; J. C. Simo & Taylor, 1985). For strain-driven models, two main algorithmic steps are needed: (i) the integration of an updated stress tensor given a strain increment; and (ii) the elaboration of a stiffness matrix according to the equations involved in the updated stress. The use of implicit integration schemes with return-mapping algorithms (RMA) is usual for plastic and plastic-damage models, whereas explicit integration schemes are used for damage and smeared crack models. A broad variety of algorithms for numerical implementation of concrete models are available in the literature (e.g. J. C. Simo & Hughes, 1998; de Souza Neto et al., 2008).

Most of these algorithms are implemented using local models, where the stress at each integration point depends only on the respective strain. However, it is well known that convergence problems are common in local models of materials with softening behavior and stiffness degradation due to strain-localization (Bažant, 1976). More specifically, when the uniaxial laws exhibits a negative slope, or more generally, when the stiffness matrix is no longer positive-definite, damage and strain localize in a zone of vanishing volume and the FE solutions exhibit spurious mesh sensitivity to size and alignment, giving unreliable results (Pijaudier-Cabot & Bažant, 1987). Thus, local models require the incorporation of an intrinsic length scale in the continuum equations to properly account for the strain-localization phenomenon. A useful and simple technique to correct this in current FE softwares is the fracture energy FE-regularization, originally proposed by (Bažant, 1982). This technique assumes that the energy dissipation takes place in a band of a certain width, which is irrespective of the element size. Thus, the uniaxial laws at the integration points are modified such that the energy dissipated by a completely degraded FE equals a constant value, which depends on the fracture energy of the material and the element size. In each element, the width of the fracture zone is referred as the characteristic length l_c . Mesh-objective responses at post-peak regimes when strain-localization occurs are obtained with this method (de Borst, 1986; Bažant, 1982). However, this technique is inadequate to overcome the ill-posed solutions present at the post-peak regime (Bažant & Jirásek, 2002).

Well-posed numerical solutions can be obtained by enhancing the local models using several techniques, so-called localization limiters: (i) higher-order continua, where additional kinematic variables are added to displacement field, (e.g. Cosserat theory and micropolar model, Tejchman & Wu, 1993; Eringen, 1999); (ii) higher-order gradients, which incorporates the gradient of strain field, (e.g. gradient-enhanced models, Peerlings et al., 1996; Abu Al-Rub & Voyiadjis, 2009); (iii) a non-local media with the stresses as a function of the mean strain measured in a certain representative volume of the material centered at that point, (e.g. non-local integral models, Pijaudier-Cabot & Bažant, 1987; Comi, 2001); and (iv) incorporation of rate-dependent terms (e.g. viscous-regularization

method, Needleman, 1988; Wang, Sluys, & de Borst, 1998; J. Liu, Lin, Fu, & Zhong, 2011). An extensive review of nonlocal models in the literature is presented in (Bažant & Jirásek, 2002; Jirásek, 1998). Further, special technique analyses are suggested to overcome the nonlinear issues with bifurcation points present in materials with softening regimes, such as explicit dynamic analysis (LS-DYNA, 2018) and arc-length techniques (Riks, 1979; M. Crisfield, 1981).

Among possibilities, viscous-regularization approach is the most adequate and easier to implement for plastic and damage concrete models such as models studied in this thesis. Visco-elastic and/or visco-plastic models (e.g. (Duvaut & Lions, 1972; Perzyna, 1966)) were developed to describe strain-dependent material behavior and help in regularizing rate-independent plastic or damage models. The basic idea of this approach is to add a numerical viscosity into the numerical integration of the equations, which converts the stiffness operator into a positive-definite matrix even in a strain-softening regime. This technique improves greatly the convergence at the expense of an over-stress condition that depends on a strain-rate increment.

The objective of this chapter is to compare the response of five different 3D continuum constitutive concrete models and provide all the details necessary for a correct numerical implementation. Because we aim to evaluate the epistemic uncertainty implicit in these models, several tests are run to compute differences between models. Hence, a second objective of this work is to try to bound this uncertainty and cast it in a form useful to the design engineering profession.

This chapter presents in Section 2.1 a complete description of the five continuum concrete models using a consistent notation. Section 2.2 is devoted to explain numerical convergence issues and their solution strategies for these models. Detailed algorithms for numerical implementation of the updated stress tensor are provided in Section 2.3. Moreover, new analytical explicit expressions for the algorithmic consistent tangent stiffness tensors of the models are described in Section 2.4. Also, a consistency check of input model parameters, such as uniaxial laws and fracture energy definition is presented in

Section 2.5. Further, numerical examples using basic benchmarks tests subject to monotonic and cyclic loading conditions under uniaxial, biaxial and triaxial stress states are presented in Section 2.6 to demonstrate the capabilities of the proposed implementations. Moreover, the unilateral effect, the strain-rate effect, the mesh size influence and strain-localization phenomena are discussed among models. Also, the compression failure mode of a test specimen is illustrated as an example of application. Additionally, Section 2.7 evaluate the epistemic uncertainty associated to nonlinear response of inelastic constitutive concrete models with a set of response parameters with respect to the experimental benchmark tests mentioned above. Finally, appendix B provides some useful conversion rules of tensors and their operations to a vectorized form for the computational implementation of models.

2.1. Description of concrete models

This section summarized the equations of the five continuum concrete models considered in this article. Also, include some modifications adequate to improve the convergence of models.

2.1.1. Drucker-Prager Hyperbolic (DPH) model

This plastic model, so-called the "Extended Drucker-Prager" model was defined by (Drucker & Prager, 1952) and modified by (ANSYS, 2018; ABAQUS, 2018). Is a simplification of Mohr-Coulomb model and have been used to simulate soil or cohesive materials, like concrete. First, the strain tensor ε is decomposed additively into its elastic, ε^e , and plastic part, ε^p as follow

$$\varepsilon = \varepsilon^e + \varepsilon^p. \quad (2.1.1)$$

Then, for the case of linear elasticity, they can be related to the Cauchy stress tensor σ by

$$\sigma = \mathcal{D}_e : (\varepsilon - \varepsilon^p), \quad (2.1.2)$$

where $\mathcal{D}_e$ is the fourth-order linear-elastic tensor (see appendix 1 for their definition). The yield criterion is defined as

$$F(\boldsymbol{\sigma}, \alpha) := \eta p + \sqrt{3J_2} - \xi c(\alpha), \quad (2.1.3)$$

where the hydrostatic stress p is included to simulate the pressure-dependent behavior and the asymmetric tensile/compressive strength of concrete; η and ξ are material parameters chosen according to the required approximation to the Mohr-Coulomb criterion or fitted to uniaxial/biaxial tensile and compressive strength of concrete; and $c(\alpha)$ is the cohesion hardening law, which is taken as function of the equivalent plastic strain α . The later variable is defined as $\alpha := \int_0^t \|\dot{\epsilon}^p\| dt$. Its assumed an exponential relation for the cohesion hardening law $c(\alpha)$ as

$$c(\alpha) := c_u + (c_y - c_u)e^{(-\alpha/\alpha_o)}, \quad (2.1.4)$$

where $c_u = Rc_y$ and $\alpha_o = c_u/E_o$, with $R > 1$ an experimental fitted parameter. Discussion of parameters η and ξ are detailed in Section 2.5. Fig. 2.1.1 shown the shape of DPH yield surface represented in different views. In addition, the figure include the initial yield surface for the LLF and WLF model.

For other hand, a hyperbolic shape is adopted for the flow potential, and is defined as

$$G(\boldsymbol{\sigma}) := \bar{\eta} p + \sqrt{3J_2 + \epsilon^2}, \quad (2.1.5)$$

where $\bar{\eta}$ is a constant that depends of the dilatancy angle and ϵ is a eccentricity parameter that controls the shape of surface near of tensile regime, generally used less than 0.001 (ABAQUS, 2018). Observe that this flow potential is a smoothed surface (C^2 -class) that avoid the singularity at the cone's apex present in the classical Drucker-Prager model, giving an unique flow direction in this region. Also, note that this flow potential converts to the classical Drucker-Prager model when $\epsilon = 0$. Then, the non-associated flow rule for

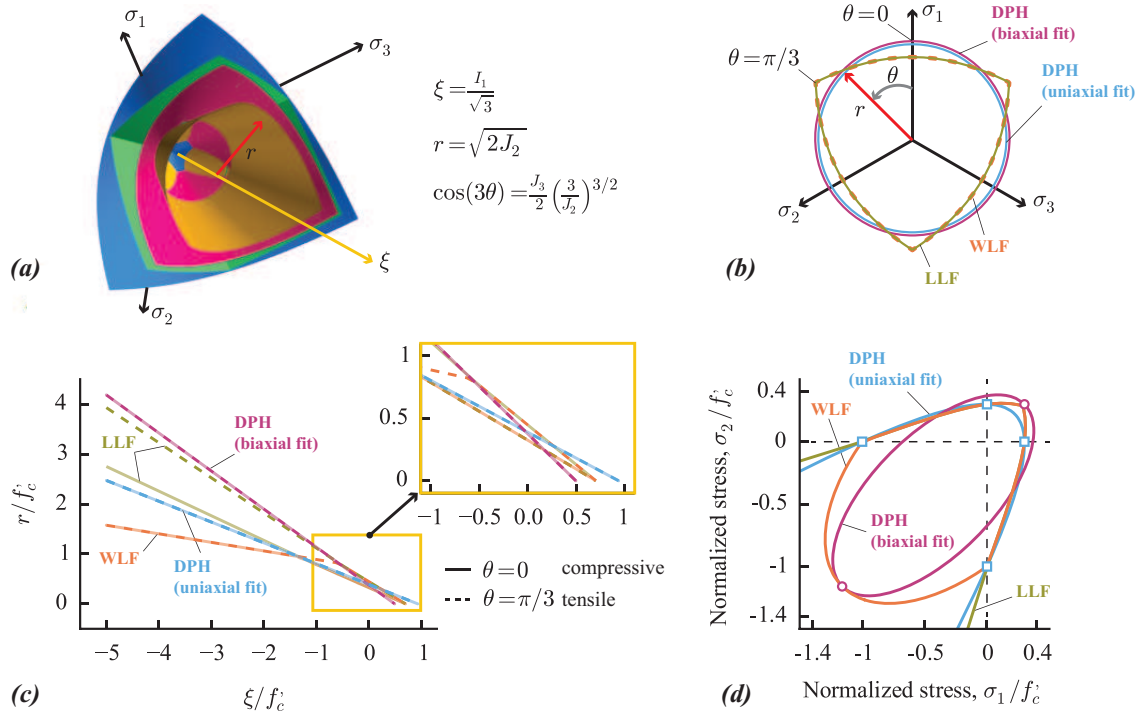


Figure 2.1.1. Yield criterion of DPH, LLF and WLF models: (a) 3D view in principal stress; (b) deviatoric π -plane; (c) tensile/compressive meridians in Rendulic plane; and (d) biaxial-stress plane. The following parameters are used. Common for all $f'_t=6$ MPa, $f'_c=20$ MPa, $f'_b = 1.16f'_c$. For the LLF and WLF models $\bar{\sigma}^+ = f'_t$, $\bar{\sigma}^- = f'_c$, $\omega^\pm = 0$ and $K_c=0.7$.

the plastic strain tensor is given by

$$\dot{\epsilon}^p := \dot{\gamma} \mathbf{N}, \quad (2.1.6)$$

where γ is the plastic operator and $\mathbf{N}$ denotes the flow tensor expressed as

$$\mathbf{N} := \frac{\partial G}{\partial \boldsymbol{\sigma}} = \frac{3}{2r} \mathbf{s} + \frac{\bar{\eta}}{3} \mathbf{I}, \quad (2.1.7)$$

with $r = \sqrt{q^2 + \epsilon^2}$ and $q = \sqrt{3J_2}$. Hence, using Eq. (A.1.5) and due that $\text{tr}(\mathbf{s}) = 0$, it follows that the volumetric strain rate can be estimated as

$$\dot{\epsilon}_v := \epsilon_v^e + \dot{\epsilon}_v^p = K^{-1} p + \dot{\gamma} \bar{\eta}, \quad (2.1.8)$$

where ε_v^e and ε_v^p are the elastic and plastic volumetric strain, respectively. It can be observed that $\bar{\eta}$ controls the inelastic volumetric strain rate (dilatancy). For the other hand, the evolution law for the equivalent plastic strain is stated as

$$\dot{\alpha} := \dot{\gamma}\xi. \quad (2.1.9)$$

Finally, the loading-unloading Karush-Kuhn-Tucker (KKT) and the consistency condition, respectively, are expressed as

$$\dot{\gamma} \geq 0, \quad F(\boldsymbol{\sigma}, \gamma) \leq 0, \quad \dot{\gamma}F(\boldsymbol{\sigma}, \gamma) = 0, \quad (2.1.10)$$

$$F(\boldsymbol{\sigma}, \gamma) = \dot{F}(\boldsymbol{\sigma}, \gamma) = 0. \quad (2.1.11)$$

2.1.2. Lubliner-Lee-Fenves (LLF) model

This plastic-damage model, so-called "Barcelona" model, was first developed by (Lubliner et al., 1989) and later improved by (J. Lee & Fenves, 1998). First, using Lemaitre's strain equivalent hypothesis (Lemaitre, 1989), the nominal stress tensor $\boldsymbol{\sigma}$ associated with the damage state is related to the effective stress $\bar{\boldsymbol{\sigma}}$ corresponding to the undamaged state as follows

$$\boldsymbol{\sigma} := (1 - \omega)\bar{\boldsymbol{\sigma}}, \quad (2.1.12)$$

where ω is the isotropic damage variable, with $\omega \in [0, 1]$.

Plastic component

To calculate this component, it is assumed the so-called *effective stress space plasticity*, which is related to the effective stress tensor $\bar{\boldsymbol{\sigma}}$ and is dependent (coupled) of damage component (Wu et al., 2006). First, two hardening scalar variables $\kappa^\pm$ are stated to control the positive/negative part of plastic-damage component, respectively. (Lubliner et al.,

1989) define normalized variables for uniaxial case as follows

$$\kappa^\pm := \frac{1}{g^\pm} \int_0^{\alpha^\pm} \sigma^\pm(\alpha^\pm) d\alpha^\pm, \quad (2.1.13)$$

which correspond to accumulated area under positive/negative uniaxial stress-equivalent plastic strain law $(\sigma^\pm - \alpha^\pm)$, respectively, with $\kappa^\pm \in [0, 1]$ and $g^\pm = \int_0^\infty \sigma^\pm(\alpha^\pm) d\alpha^\pm$ are the total area under their respective stress law. Note that positive values are used for $\sigma^\pm$. Moreover, the positive/negative equivalent plastic strain $\alpha^\pm$ are defined as

$$\alpha^\pm := \int |\dot{\varepsilon}_\pm^p| dt, \quad (2.1.14)$$

where $\dot{\varepsilon}_+^p = \dot{\varepsilon}_{max}^p$ and $\dot{\varepsilon}_-^p = -\dot{\varepsilon}_{min}^p$, with $\dot{\varepsilon}_{max,min}^p$ are the maximum and minimum eigenvalues ratio of principal plastic strain tensor ε^p , respectively. Then, the Eq. (2.1.13) can be written in an incremented format as

$$\dot{\kappa}^\pm = \frac{1}{g^\pm} \sigma^\pm(\alpha^\pm) \dot{\alpha}^\pm. \quad (2.1.15)$$

Moreover, in case for multi-axial condition, the evolution law of variables $\kappa^\pm$ in a vectorized format $\underline{\kappa} = [\kappa^+, \kappa^-]^T$ is defined as

$$\dot{\underline{\kappa}} := \underline{W}(\hat{\underline{\sigma}}, \underline{\kappa}) \cdot \dot{\underline{\varepsilon}}^p \cdot \underline{1}, \quad (2.1.16)$$

$$\underline{W}(\hat{\underline{\sigma}}, \underline{\kappa}) := \begin{bmatrix} \phi(\hat{\underline{\sigma}}) \sigma^+(\kappa^+)/g^+ & 0 & 0 \\ 0 & 0 & (\phi(\hat{\underline{\sigma}}) - 1) \sigma^-(\kappa^-)/g^- \end{bmatrix},$$

where $\hat{\underline{\sigma}}$ is the principal effective stress tensor; $\dot{\underline{\varepsilon}}^p = \text{diag}(\dot{\varepsilon}_1^p, \dots, \dot{\varepsilon}_N^p)$ is the ratio of principal plastic strain tensor, which is filled in an algebraic order (e.g. $\dot{\varepsilon}_1^p > \dots > \dot{\varepsilon}_N^p$); $\underline{1}$ is a vector filled of ones of length N ; and $\phi(\hat{\underline{\sigma}})$ is a weight factor $\in [0, 1]$, defined as

$$\phi(\hat{\underline{\sigma}}) := \begin{cases} 0, & \hat{\sigma}_i = 0 \\ \frac{\sum_{i=1}^N \langle \hat{\sigma}_i \rangle^+}{\sum_{i=1}^N |\hat{\sigma}_i|}, & \text{otherwise} \end{cases}. \quad (2.1.17)$$

An adequate conversion of uniaxial stress laws from the relation $\sigma^\pm - \alpha^\pm$ to $\sigma^\pm - \kappa^\pm$, using Eq. (2.1.13) is necessary to generate. Detail of this conversion is discussed in Section 2.5.1. For the other hand, similar to stated in the DPH model, the hyperbolic Drucker-Prager criterion defined by Eq. (2.1.5) is used for the flow potential. Moreover, due that any isotropic material satisfy the relation $G(\bar{\sigma}) = \hat{G}(\hat{\sigma})$ and that p , J_2 and r are invariants in the effective stress space $((\bar{\cdot}) = (\hat{\cdot}))$, the flow potential in the principal effective space can be rewritten as

$$\hat{G}(\hat{\sigma}) = \bar{\eta}\bar{p} + \sqrt{\bar{J}_2 + \epsilon^2}. \quad (2.1.18)$$

Then, the non-associated flow rule satisfy the relation in the principal space as

$$\dot{\hat{\epsilon}}^p = \dot{\gamma} \hat{N}, \quad (2.1.19)$$

$$\hat{N} := \frac{\partial \hat{G}}{\partial \hat{\sigma}} = \frac{3}{2\bar{r}} \hat{s} + \frac{\bar{\eta}}{3} \hat{I}, \quad (2.1.20)$$

where $\hat{N}$ is the principal effective flow tensor. Thus, Eq. (2.1.16) can be rewritten as

$$\dot{\underline{\kappa}} = \dot{\gamma} \underline{H}(\hat{\sigma}, \underline{\kappa}), \quad (2.1.21)$$

where $\underline{H}(\hat{\sigma}, \underline{\kappa}) = \underline{W}(\hat{\sigma}, \underline{\kappa}) \cdot \hat{N} \cdot \underline{1}$. For the other hand, the yield criterion is first established by (Lubliner et al., 1989) in the effective space as

$$\bar{F}(\hat{\sigma}) := \bar{\eta}\bar{p} + \sqrt{3\bar{J}_2} + \beta \langle \hat{\sigma}_{max} \rangle^+ - (1 - \alpha)c, \quad (2.1.22)$$

where $\alpha = (f'_b - f'_c) / (2f'_b - f'_c)$, $\beta = (1 - \alpha)f'_c / f'_t - (1 + \alpha)$ and c is the cohesion parameter (constant). Typical experimental values of the ratio f'_b / f'_c for concrete ranges from 1.10 to 1.16, yielding values of α between 0.08 and 0.12. Later, (J. Lee & Fenves, 2001) modify this function, adjusting the parameters to distinguish the different evolution

of strength under tension and compression as follows

$$\beta(\underline{\kappa}) := (1 - \alpha) \frac{\bar{\sigma}^-(\kappa^-)}{\bar{\sigma}^+(\kappa^+)} - (1 + \alpha), \quad c(\kappa^-) := \bar{\sigma}^-(\kappa^-), \quad (2.1.23)$$

where $\bar{\sigma}^\pm$ are the positive/negative uniaxial effective stress law, respectively. Additionally, (J. Lee & Fenves, 2001) include in the yield criterion a parameter δ to account the triaxial compression behavior. Thus, the yield criterion can be redefined finally as

$$\bar{F}(\hat{\boldsymbol{\sigma}}, \underline{\kappa}) := \eta \bar{p} + \sqrt{3} \bar{J}_2 + \beta(\underline{\kappa}) \langle \hat{\sigma}_{max} \rangle^+ - \delta \langle \hat{\sigma}_{max} \rangle^- - (1 - \alpha) c(\kappa^-), \quad (2.1.24)$$

where $\delta = 3(1 - K_c)/(2K_c - 1)$ denotes the ratio of corresponding values of $\sqrt{J_2}$ under tensile meridian and compressive meridian stress states for any given value of hydrostatic pressure I_1 and its assumed constant (Lubliner et al., 1989). Experimental values of K_c ranges in the interval $[2/3, 1]$, which gives a value of $\delta \in [0, 3]$.

Damage component

(J. Lee & Fenves, 1998) define the damage variable ω as follows

$$\omega := 1 - [1 - s(\hat{\boldsymbol{\sigma}}) \omega^+(\kappa^+)] [1 - \omega^-(\kappa^-)], \quad (2.1.25)$$

where $s(\hat{\boldsymbol{\sigma}}) = s_o + (1 - s_o) \phi(\hat{\boldsymbol{\sigma}})$ is a variable to represent the stiffness recovery from compression to tensile load state and $\omega^\pm(\kappa^\pm)$ are uniaxial positive/negative damage laws, respectively, which are in function of hardening variables $\kappa^\pm$. These damage laws are fitted experimentally and are generally known in terms of equivalent plastic strain $\alpha^\pm$, e.g. the common exponential relation is used as

$$\omega^\pm(\alpha^\pm) = 1 - \exp(-C^\pm \alpha^\pm), \quad (2.1.26)$$

with $C^\pm$ an experimental parameter that control the unloading branch of response. Due this, its required an adequate conversion from $\omega^\pm - \alpha^\pm$ to $\omega^\pm - \kappa^\pm$ laws as explained in

Section 2.5.1. Moreover, (ABAQUS, 2018) redefine the damage variable ω as follow

$$\omega := 1 - [1 - s^-(\hat{\sigma})\omega^+(\kappa^+)] [1 - s^+(\hat{\sigma})\omega^-(\kappa^-)], \quad (2.1.27)$$

with $s^\pm(\hat{\sigma})$ are the stiffness recovery functions defined as

$$s^+(\hat{\sigma}) := 1 - z_c^+ \phi(\hat{\sigma}), \quad s^-(\hat{\sigma}) := 1 - z_c^- (1 - \phi(\hat{\sigma})), \quad (2.1.28)$$

with $z_c^\pm \in [0, 1]$ are a stiffness recovery factor from tensile to compression load state and vice versa. Empirical evidence shown that compressive stiffness is recovered upon crack closure as the load changes from tension to compression ($z_c^+ \approx 1$). However, tensile stiffness is not recovered as the load changes from compression to tension once crushing micro-cracks have developed ($z_c^- \approx 0$). Thus, the uniaxial positive/negative stress $\sigma^\pm$ laws can be related to respective effective stress $\bar{\sigma}^\pm$ laws as follows

$$\sigma^\pm(\kappa^\pm) = [1 - \omega^\pm(\kappa^\pm)] \bar{\sigma}^\pm(\kappa^\pm). \quad (2.1.29)$$

Viscous component

Additionally, the model can include strain-rate dependency with a visco-plastic model, which improve the convergence in strain-softening regimes. To this, the nominal stress tensor σ is now converted to their respective viscous component σ^v , and is defined as

$$\sigma^v := (1 - \omega^v) \bar{\sigma}^v, \quad (2.1.30)$$

where ω^v is the viscous damage variable and $\bar{\sigma}^v$ is the effective viscous stress tensor. (J. Lee & Fenves, 2001) calculate this component using the (Duvaut & Lions, 1972) visco-plastic model, which is stated in the effective stress space as

$$\dot{\epsilon}^{vp} := \frac{1}{\mu_v} \mathcal{C}_e : (\bar{\sigma}^v - \bar{\sigma}), \quad (2.1.31)$$

$$\bar{\sigma}^v := \mathcal{D}_e : (\epsilon - \epsilon^{vp}), \quad (2.1.32)$$

with $\boldsymbol{\varepsilon}^{vp}$ is the visco-plastic strain tensor and μ_v is the numerical viscosity parameter and is equivalent to the relaxation time. Thus, combining both expressions gives the follow relation

$$\dot{\boldsymbol{\varepsilon}}^{vp} = -\frac{1}{\mu_v}(\boldsymbol{\varepsilon}^{vp} - \boldsymbol{\varepsilon}^p). \quad (2.1.33)$$

Moreover, the evolution law of viscous-damage variable ω^v is defined as

$$\dot{\omega}^v := -\frac{1}{\mu_v}(\omega^v - \omega). \quad (2.1.34)$$

2.1.3. Wu-Li-Fara (WLF) model

This plastic-damage model, was first developed by (Faria et al., 1998) and later modified by (Wu et al., 2006). Two variants are developed for this model: one aproach that include the plastic and damage components (WLF) and other one with pure damage behavior (WLF₀). First, assume that the effective stress tensor $\bar{\boldsymbol{\sigma}}$ are splitted into positive $\bar{\boldsymbol{\sigma}}^+$ and negative $\bar{\boldsymbol{\sigma}}^-$ parts, to account separately the cracking (tension) and shear (compression) damage mechanisms for degradation of concrete (Ladeveze, 1983; Ortiz, 1985), using the follow decomposition

$$\bar{\boldsymbol{\sigma}}^\pm := \sum_{i=1}^N \langle \hat{\sigma}_i \rangle^\pm \mathbf{E}_\sigma^{ii} = \mathcal{P}^\pm : \bar{\boldsymbol{\sigma}}, \quad (2.1.35)$$

$$\mathcal{P}^\pm := \sum_{i=1}^N H^\pm(\hat{\sigma}_i) (\mathbf{E}_\sigma^{ii} \otimes \mathbf{E}_\sigma^{ii}), \quad (2.1.36)$$

where $\mathcal{P}^\pm$ are the fourth-order projection tensors, with symbol ' $\pm$ ' denoting '+' or '-' as appropriate, $\hat{\sigma}_i$ denote the i -th eigenvalue of tensor $\bar{\boldsymbol{\sigma}}$ and $\mathbf{E}_\sigma^{ii}$ is the i -th eigenprojector tensor associated to $\bar{\boldsymbol{\sigma}}$ (see Eq. (A.1.9)). This decomposition satisfy the relations $\bar{\boldsymbol{\sigma}} = \bar{\boldsymbol{\sigma}}^+ + \bar{\boldsymbol{\sigma}}^-$ and $\mathcal{P}^+ + \mathcal{P}^- = \mathcal{I}$. Next, in order to establish the intended constitutive

law, (Wu et al., 2006) define the total elasto-plastic HFE ψ potential as follows

$$\psi(\boldsymbol{\varepsilon}^e, \underline{\boldsymbol{\omega}}, \underline{\boldsymbol{\kappa}}) := \psi^e(\boldsymbol{\varepsilon}^e, \underline{\boldsymbol{\omega}}) + \psi^p(\underline{\boldsymbol{\kappa}}, \underline{\boldsymbol{\omega}}), \quad (2.1.37)$$

$$\psi^e(\boldsymbol{\varepsilon}^e, \underline{\boldsymbol{\omega}}) = (1 - \omega^+) \psi_o^{e+}(\boldsymbol{\varepsilon}^e) + (1 - \omega^-) \psi_o^{e-}(\boldsymbol{\varepsilon}^e), \quad (2.1.38)$$

$$\psi^p(\underline{\boldsymbol{\kappa}}, \underline{\boldsymbol{\omega}}) = (1 - \omega^+) \psi_o^{p+}(\underline{\boldsymbol{\kappa}}) + (1 - \omega^-) \psi_o^{p-}(\underline{\boldsymbol{\kappa}}), \quad (2.1.39)$$

where $\omega^\pm = \omega^\pm(r^\pm)$ are positive/negative scalar damage variables $\in [0, 1]$, respectively, which are in function of the damage thresholds $r^\pm$ that controls the size of damage surfaces; $\underline{\boldsymbol{\omega}} = [\omega^+, \omega^-]^T$ denotes the damage vector; $\psi_o^{e\pm}$ are the undamaged elastic HFE potential and are equals to the strain energy per unity of volume, i.e. $\psi_o^{e\pm} = \frac{1}{2} \bar{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}^e$; and $\psi_o^{p\pm}$ are the undamaged plastic HFE potential. Moreover, the Eq. (2.1.37) can be reordered as

$$\psi(\boldsymbol{\varepsilon}^e, \omega^+, \omega^-, \underline{\boldsymbol{\kappa}}) := (1 - \omega^+) \psi_o^+(\boldsymbol{\varepsilon}^e, \underline{\boldsymbol{\kappa}}) + (1 - \omega^-) \psi_o^-(\boldsymbol{\varepsilon}^e, \underline{\boldsymbol{\kappa}}), \quad (2.1.40)$$

where $\psi_o^\pm$ is the positive/negative total undamaged elasto-plastic HFE potential and are written as

$$\psi_o^\pm = \psi_o^{e\pm} + \psi_o^{p\pm}. \quad (2.1.41)$$

For the other hand, the nominal Cauchy stress tensor can be defined as

$$\boldsymbol{\sigma} := \frac{\partial \psi^e}{\partial \boldsymbol{\varepsilon}^e}. \quad (2.1.42)$$

Then, using the relation $\frac{\partial \psi_o^{e\pm}}{\partial \boldsymbol{\varepsilon}^e} = \bar{\boldsymbol{\sigma}}^\pm$ and Eqs. (2.1.41), (2.1.35) and (2.1.38), this stress tensor is expressed as

$$\boldsymbol{\sigma} := [(1 - \omega^+) \mathcal{P}^+ + (1 - \omega^-) \mathcal{P}^-] : \bar{\boldsymbol{\sigma}} = \left(\sum_{\aleph} (1 - \omega^\aleph) \mathcal{P}^\aleph \right) : \bar{\boldsymbol{\sigma}}, \quad (2.1.43)$$

where $\aleph$ denote index summation for '+' and '-' part as appropriate hereafter.

Plastic component

Similar to the LLF model, its assumed the *effective stress space plasticity*, where the plastic component is calculated in terms of the effective stress tensor $\bar{\sigma}$ and in this case, is independent (decoupled) of damage component (Wu et al., 2006). Due to this condition, they can include the plastic component as an option, conversely to the LLF model.

(Wu et al., 2006) assume the Lee-Fenves yield criterion as stated in Eq. (2.1.24), with the parameter $\delta = 0$, and the classical Drucker-Prager flow potential criterion ($\epsilon = 0$). However, its recommend the use of a flow potential as defined in Eq. (2.1.5). For the other hand, similar to the LLF model, two hardening parameters $\kappa^\pm$ are proposed to control the positive/negative plastic component which are assumed as the positive/negative equivalent plastic strain $\alpha^\pm$ defined as $\alpha^\pm = \int |\dot{\hat{\epsilon}}_\pm| dt$. Then, for multi-axial condition, these hardening parameters are stated as $\kappa^+ = \phi(\hat{\sigma})\alpha^+$ and $\kappa^- = -(1 - \phi(\hat{\sigma}))\alpha^-$, with $\phi(\hat{\sigma})$ defined in Eq. (2.1.17). Then, the rate of hardening vector $\underline{\kappa} = [\kappa^+, \kappa^-]^T$ is defined similar to the Eq. (2.1.21), but with the matrix $\underline{W}(\hat{\sigma})$ given by

$$\underline{W}(\hat{\sigma}) := \begin{bmatrix} \phi(\hat{\sigma}) & 0 & 0 \\ 0 & 0 & \phi(\hat{\sigma}) - 1 \end{bmatrix}. \quad (2.1.44)$$

In addition, the positive/negative effective uniaxial stress $\bar{\sigma}^\pm(\kappa^\pm)$ laws are required. (Wu et al., 2006) assume a linear relation as follows

$$\bar{\sigma}^\pm(\kappa^\pm) = f_o^\pm + \bar{J}_\kappa^\pm \kappa^\pm, \quad (2.1.45)$$

where $f_o^\pm$ is the positive/negative initial stress, which are chosen for convenience in the range $f_o^+ \in [0, f'_t]$ and $f_o^- \in [0, f'_c]$, respectively, and $\bar{J}_\kappa^\pm = E_t^\pm E_o / (E_o - E_t^\pm)$, with $E_t^\pm$ are the hardening slope.

Damage component

For the damage component, its required a specific definition for the undamaged elasto-plastic HFE potential $\psi_o^\pm$. For one hand, (J. Simo & Ju, 1987) assume that $\psi_o^\pm$ can be as the positive/negative elastic strain energy per unit of volume and expressed as

$$\psi_o^\pm(\boldsymbol{\varepsilon}^e) := \frac{1}{2} (\bar{\boldsymbol{\sigma}}^\pm : \boldsymbol{\mathcal{C}}_e : \bar{\boldsymbol{\sigma}}) . \quad (2.1.46)$$

However, this HFE potential is more adequate in tensile regimes where contribution of plastic part is much smaller than the compression ones. Hence, for compressive regimes, (Wu et al., 2006) define the following HFE potential that include the biaxial and triaxial compression effects as follow

$$\psi_o^-(\boldsymbol{\varepsilon}^e) := b_o \left(\eta \bar{p} + \sqrt{3\bar{J}_2} - \delta \langle \hat{\bar{\sigma}}_{max} \rangle^- \right)^2 , \quad (2.1.47)$$

where b_o is a material parameter (defined in (Wu et al., 2006)) and $\eta = 3\alpha$. Next, the tensile and shear thermodynamic forces or Damage Energy Release Rate-based (DERR), $Y^\pm$, can be defined as

$$Y^\pm := -\frac{\partial \psi^\pm}{\partial \omega^\pm} = \psi_o^\pm \quad (2.1.48)$$

Then, the positive/negative damage criteria are defined as

$$F_d^\pm(Y^\pm, r^\pm) := g_d^\pm(Y^\pm) - g_d^\pm(r^\pm) \leq 0, \quad (2.1.49)$$

where $g_d^\pm(\cdot)$ can be any monotonically increasing scalar function. Using the Eqs. (2.1.46) and (2.1.47), these functions can be postulated as convenience as $g_d(\cdot)^\pm = \sqrt{2E_o(\cdot)}$ and $g_d(\cdot)^- = \sqrt{(\cdot)/b_o}$, respectively. Thus, the positive/negative DEERs can be rewritten as

$$Y^\pm := \sqrt{2E_o\psi_o^\pm} = \sqrt{E_o(\bar{\boldsymbol{\sigma}}^\pm : \boldsymbol{\mathcal{C}}_e : \bar{\boldsymbol{\sigma}})}, \quad (2.1.50)$$

$$Y^- := \sqrt{\frac{\psi_o^-}{b_o}} = \eta \bar{p} + \sqrt{3\bar{J}_2} - \delta \langle \hat{\bar{\sigma}}_{max} \rangle^-. \quad (2.1.51)$$

Moreover, the evolution damage laws can be defined analogously to the classical plasticity, where the flow rule, the loading-unloading and the consistency conditions of damage component are defined, respectively, as

$$\dot{\omega}^{\pm} = \dot{\gamma}_d^{\pm} \frac{\partial g_d^{\pm}}{\partial Y^{\pm}}, \quad (2.1.52)$$

$$\dot{\gamma}_d^{\pm} = \dot{r}^{\pm} \geq 0, \quad F_d^{\pm}(Y^{\pm}, r^{\pm}) \leq 0, \quad \dot{\gamma}_d^{\pm} F_d^{\pm}(Y^{\pm}, r^{\pm}) = 0, \quad (2.1.53)$$

$$F_d^{\pm}(Y^{\pm}, r^{\pm}) = \dot{F}_d^{\pm}(Y^{\pm}, r^{\pm}) = 0. \quad (2.1.54)$$

Its follow using Eqs. (2.1.53) and (2.1.54), that the damage thresholds $r^{\pm}$ are non-decreasing functions that satisfy the relations

$$r^{\pm} = \max \left(r_o^{\pm}, \max_{[0,t]}(Y^{\pm}) \right), \quad (2.1.55)$$

$$\dot{r}^{\pm} = \dot{Y}^{\pm}, \quad (2.1.56)$$

where $r_o^{\pm}$ are the initial damage thresholds. Assuming an uniaxial behavior and using Eqs. (2.1.50) and (2.1.51), these values can be calculated as $r_o^{\pm} = \sigma_o^{\pm}$ and $r_o^{-} = (1 - \alpha + \delta)\sigma_o^{-}$, respectively, where $\sigma_o^{\pm}$ are stress onset the nonlinear behavior. Although, its adequate adjust the negative initial threshold as $r_o^{-} = (1 - \alpha)\sigma_o^{-}$.

Finally, the positive/negative damage $\omega^{\pm}(r^{\pm})$ laws are generally derived of experimental cracking process. (Mazars, 1984) define an exponential relation for the positive/negative component, respectively, given by

$$\omega^{+}(r^{+}) := 1 - \frac{1}{\bar{z}^{+}} \left(1 - A^{+} + A^{+} e^{B^{+}(1-\bar{z}^{+})} \right), \quad (2.1.57)$$

$$\omega^{-}(r^{-}) := 1 - \frac{1}{\bar{z}^{-}} \left(1 - A^{-} + A^{-} \bar{z}^{-} e^{B^{-}(1-\bar{z}^{-})} \right), \quad (2.1.58)$$

where $\bar{z}^{\pm} = r^{\pm}/r_o^{\pm}$ and $A^{\pm}$ and $B^{\pm}$ are experimental parameters fitted with the fracture energy FE-regularization method explained in Section 2.5. This damage laws can be converted to an equivalent stress-strain $\sigma^{\pm}(\varepsilon^{\pm})$ relation and vice versa, being these last commonly more known and used than the respective damage laws.

Viscous component

Additionally, the model can include rate-dependent viscous regularization. Originally, (Faria et al., 1998) propose the use of Perzyna viscous model to the damage component of model, which involve an iterative process to solve the updated solution. In order to get a simplified solution, its proposed the use of (Duvaut & Lions, 1972) viscous model in the plastic and damage components of model. Thus, the nominal viscous stress tensor σ^v is defined as

$$\sigma^v := \sum_{\aleph} (1 - \omega^{\aleph}) \bar{\sigma}^{v\aleph}, \quad (2.1.59)$$

$$\bar{\sigma}^{v\pm} = \mathcal{P}_v^{\pm} : \bar{\sigma}^v, \quad \mathcal{P}_v^{\pm} := \sum_{i=1}^N H_0^{\pm} (\hat{\sigma}_i^v) (\mathbf{E}_{\bar{\sigma}_v}^{ii} \otimes \mathbf{E}_{\bar{\sigma}_v}^{ii}), \quad (2.1.60)$$

where $\bar{\sigma}^v$ is the effective viscous stress tensor given by Eq. (2.1.32) and $\mathcal{P}_v^{\pm}$ are their positive/negative projected tensors, respectively. Moreover, for the damage component, the evolution law of damage thresholds variables $r^{\pm}$ are defined as

$$\dot{r}^{\pm} := -\frac{1}{\mu_v} (r^{\pm} - Y^{\pm}). \quad (2.1.61)$$

2.1.4. Faría-Oliver-Cervera (FOC) model

This plastic-damage model was proposed by (Faria et al., 1998). Take identical assumptions than the WLF model for the damage and viscous components, and use a simplified representation for the plastic component, explained as follows.

Plastic component

Although, the formulation of the WLF model provides a strict framework to represent the evolution of plastic strain tensor, numerical implementation gives time consuming solving process. (Faria et al., 1998) proposed a simplified evolution law for the plastic

strain tensor as follow

$$\dot{\epsilon}^p := \dot{\gamma} \bar{\sigma}, \quad (2.1.62)$$

$$\dot{\gamma} = E_o \chi \frac{\langle \epsilon^e : \dot{\epsilon} \rangle^+}{(\bar{\sigma} : \bar{\sigma})}, \quad (2.1.63)$$

where $\chi = B^+ H^+(\dot{\omega}^+) + B^- H^+(\dot{\omega}^-) \geq 0$ is a material parameter to control the rate intensity of plastic deformation, with $B^\pm$ parameters associated to positive/negative component of stress, respectively; Heaviside function $H(\cdot)^+$ is used for active progressive damage rate of respective stress component; and McAulay $\langle \cdot \rangle^+$ function enable one to set a non-negative value for the product $(\bar{\epsilon}^e : \dot{\epsilon})$ required to ensure positive dissipation.

2.1.5. Total strain rotating crack (ROT) model

This smeared-crack model was developed by (Cope et al., 1980; Gupta & Akbar, 1984) and enhanced by (Rots, 1988; TNO DIANA, 2018). We proposed a simple and robust formulation than past.

Damage component

First, assume the so-called the "total strain" formulation present in the hypo-elastic materials, i.e. that stress tensor σ depends only of total strain tensor ϵ . Next, its assumed that a set of orthogonal crack planes rotates according to direction of principal strain tensor $\hat{\epsilon}$. Then, using a spectral decomposition of strain tensor ϵ (Eq. (A.1.8)), satisfy the relation

$$\epsilon = \underline{V}_\epsilon \hat{\epsilon} \underline{V}_\epsilon^T = \sum_{i=1}^N \hat{\epsilon}_i \underline{E}_\epsilon^{ii}, \quad (2.1.64)$$

where $\underline{V}_\epsilon$ is the orthogonal normalized eigenvectors matrix, $\hat{\epsilon}_i$ is the i -th eigenvalue and $\underline{E}_\epsilon^{ii}$ the i -th eigen-projector tensor (Eq. (A.1.9)).

According only to this condition, the model lack of memory for the damage evolution, where the loading and unloading follows the same path (hypo-elastic). Thus, in order to

add an irreversible damage process, a i -th positive/negative damage strain variables $\alpha_i^\pm$ are defined for respective principal strain direction $\hat{\varepsilon}_i$. Then, the evolution law for these damage variables satisfy the relation

$$\dot{\alpha}_i^\pm := z_i^\pm \dot{\varepsilon}_i, \quad (2.1.65)$$

where $z_i^\pm = 1 - r_i^\pm$ and $r_i^\pm = H_0^\pm(\alpha_i^\pm - \hat{\varepsilon}_i)$ are the damage threshold variables. Now, calling the damages strain vector as follow $\underline{\alpha} = [\underline{\alpha}^+, \underline{\alpha}^-]^T$, with $\underline{\alpha}^\pm = [\alpha_1^\pm, \dots, \alpha_N^\pm]^T$, Eq. (2.1.65) can be rewritten in a vectorized format as

$$\dot{\underline{\alpha}} = \underline{Z}(\hat{\varepsilon}, \underline{\alpha}) \cdot \dot{\hat{\varepsilon}} \cdot \underline{1}, \quad (2.1.66)$$

$$\underline{Z}(\hat{\varepsilon}, \underline{\alpha}) = \begin{bmatrix} \underline{Z}^+ \\ \underline{Z}^- \end{bmatrix}, \quad \underline{Z}^\pm = \text{diag}(z_1^\pm, \dots, z_N^\pm).$$

It should be noted the similarity of this expression with Eq. (2.1.16). For the other hand, the i -th principal stress $\hat{\sigma}_i$ evaluated in their respective principal strain direction is given by

$$\hat{\sigma}_i := m_i^+ h_i^+ + m_i^- h_i^-, \quad (2.1.67)$$

where $m_i^\pm = H_{1/2}^\pm(\hat{\varepsilon}_i)$ and $h_i^\pm = \sigma^\pm(\alpha_i^\pm) g_i^\pm$, with $\sigma^\pm(\alpha_i^\pm)$ are the uniaxial positive/negative stress laws, respectively, and $g_i^\pm$ are variables to control the loading/unloading stress. Assuming a secant unloading to origin (no plastic strains), the variables $g_i^\pm$ can be defined as

$$g_i^\pm := 1 - \frac{\alpha_i^\pm - \hat{\varepsilon}_i}{\alpha_i^\pm} = \frac{\hat{\varepsilon}_i}{\alpha_i^\pm}, \quad (2.1.68)$$

with $g_i^\pm \in [0, 1]$, where $g_i^\pm = 1$ in case of loading and $g_i^\pm < 1$ for unloading. Finally, the model assume the principle of co-axiality (Bažant, 1983), i.e. the principal stress directions coincide with the principal strain directions, for which satisfy the relation

$$\boldsymbol{\sigma} = \sum_{i=1}^N \hat{\sigma}_i \mathbf{E}_{ii}^\varepsilon. \quad (2.1.69)$$

Viscous component

Additionally, its suggested include a viscous model to improve the convergence of model. For this, the Duvaut-Lions viscous model can be incorporated as follows

$$\dot{\alpha}_i^{v\pm} := -\frac{1}{\mu_v}(\alpha_i^{v\pm} - \alpha_i^{\pm}), \quad (2.1.70)$$

where $\alpha_i^{v\pm}$ are the i -th viscous damage strain variable. Then, the i -th principal viscous-stress $\hat{\sigma}_i^v$ is expressed as

$$\hat{\sigma}_i^v = m_i^+ h_i^{v+} + m_i^- h_i^{v-}, \quad (2.1.71)$$

$$h_i^{v\pm} = \sigma^{\pm}(\alpha_i^{v\pm}) g_i^{v\pm}, \quad g_i^{v\pm} = \frac{\hat{\varepsilon}_i}{\alpha_i^{v\pm}}. \quad (2.1.72)$$

Thus, the viscous-stress tensor $\boldsymbol{\sigma}^v$ is given by

$$\boldsymbol{\sigma}^v := \sum_{i=1}^N \hat{\sigma}_i^v \mathbf{E}_{\varepsilon}^{ii}. \quad (2.1.73)$$

It should be noted that, this model can be extended to simulate the biaxial effects, such as biaxial strength in compression-compression (CC) regime or compression softening in tension-compression one. In both cases, it can be extended by means of modify the uniaxial stress-strain law as function of complete principal stress/strain tensor, i.e. $\sigma^{\pm} = \sigma^{\pm}(\hat{\varepsilon}, \hat{\boldsymbol{\sigma}})$. Complex derivatives involve this process and is beyond the scope of this work.

2.1.6. Resume of concrete models

Table 2.6.7 shown the main capabilities for described concrete models. Classification of models (plastic, damage, plastic-damage), strain-softening behavior, stress state effects (biaxial or triaxial), unilateral effect and strain-rate effect are mentioned. Also, the table lists the inelastic inputs parameters.

Table 2.1.1. Properties of concrete models and their input parameters.

Model	Class	Strain	Biaxial	Triaxial	Unilateral	Strain-rate	Inelastic inputs	
		softening	effect	effect	effect	effect	Scalar	Uniaxial laws
DPH	plastic		✓				$\eta, \bar{\eta}, \xi, \epsilon$	$c(\alpha)$
LLF	plastic-damage	✓	✓	✓	✓	✓	$f'_c, f'_b, K_c, \bar{\eta}, \epsilon, \mu_v, z_c^\pm$	$\sigma^\pm(\kappa^\pm), \omega^\pm(\kappa^\pm)$
WLF	plastic-damage	✓	✓	✓	✓	✓	$f'_c, f'_b, K_c, \bar{\eta}, \epsilon, \mu_v$	$\bar{\sigma}^\pm(\kappa^\pm), \omega^\pm(r^\pm)$
WLF ₀	damage	✓	✓	✓	✓	✓	f'_c, f'_b, K_c, μ_v	$\omega^\pm(r^\pm)$
FOC	plastic-damage	✓	✓	✓	✓	✓	$f'_c, f'_b, K_c, B^\pm$	$\omega^\pm(r^\pm)$
ROT	damage	✓	✓			✓	μ_v	$\sigma^\pm(\varepsilon^\pm), \sigma_{max}^-(\varepsilon)$

2.2. Convergence issues and solution strategies

Inner the possibilities of local models, three strategies are suggested and probed by authors to achieve a good convergence.

2.2.1. Stress updated algorithm

Determination of an adequate stress updated algorithm is necessary to give robust convergence for numerical models (Krieg & Krieg, 1977; J. C. Simo & Taylor, 1985). The following suggestions are proposed.

- Avoid singularities in the range of solution for all variables to be solved.
- Choice of an adequate initial value and no-null derivatives in the variables to be solved is a key issue in the Newton's method to give a correct convergence. Also, its better solve an unique scalar variable rather than a system of equations, specially when their magnitudes are very different. An example of this, its recommended to solve the scalar variable q_{n+1} rather than the deviatoric stress tensor s_{n+1} for the plastic component of the DPH, LLF and WLF models.
- Its highly recommended to avoid zero slope stages in the uniaxial stress laws (e.g. perfectly elasto-plastic) to give an unique plastic/damage consistency operator γ or γ_d , respectively. Zero slope is typically present in the residual stress under post-peak stage. To fix this, include a small value for the slope, say $10^{-5} \times E_o$.

2.2.2. Tangent stiffness operator

Special attention are given to the tangent stiffness operator, due to sensitivity of this operator in the convergence of models at a finite element level. We propose the following recommendations for this operator.

- Its recommended that all variables involved in this operator be of C^1 -class (continuous derivative), taking special attention in strain-softening regimes. One way to remedy this in the plastic component of models, is avoid singularities in the yield function and especially in the flow potential function by means of smoothed C^2 -class functions. Example of this occur in the DPH model, where the flow potential function has been modified by a smooth hyperbolic shape surface (Eq. (2.1.5)) to give unique derivatives at the apex's zone of cone (tensile regime). Another example happens in the LLF and WLF models, where the Heaviside function $H^\pm$ is present in the yield criterion (Eqs. (2.1.22) and (2.1.24)). To improve the convergence of model, its recommended replace this stepped function by a C^1 -class approximated function $\tilde{H}^\pm(\cdot)$ expressed by Eq. (A.1.22). In addition, the use of C^1 -class functions its recommended also for the uniaxial laws ($\sigma - \varepsilon$, $\sigma - \kappa$ or $\omega - r$). To this purpose, we recommend to replace a portion of the uniaxial law by a smoothed function in all breaks points, as shown in Fig. 2.2.2a. Inner the possibilities, the Hermite polynomial interpolation, cubic spline curves or any three-order polynomial can be used as a smoothed function. For the sake of simplicity, it can use a three-order polynomial $p(x)$ given by

$$p(x) := a_0 + a_1x + a_2x^2 + a_3x^3, \quad (2.2.74)$$

where the constants a_0 to a_3 are given by

$$\begin{aligned} a_0 &= f_1 - x_1[E_1 + x_1(c_1 + 2)], & a_1 &= E_1 + x_1(2c_1 + 3c_2), \\ a_2 &= -(c_1 + 3c_2), & a_3 &= \frac{c_2}{x_1}, \end{aligned}$$

with c_1 and c_2 are expressed as

$$c_1 = \frac{(2E_1 + E_2)\Delta x - 3\Delta f}{(\Delta x)^2}, \quad c_2 = \frac{(E_1 + E_2)\Delta x - 2\Delta f x_1}{(\Delta x)^3},$$

where $\Delta x = x_2 - x_1$ is the length of portion of the uniaxial law replaced by the smoothed function, with x_1 and x_2 are the abscissa before and after of respective break point, $\Delta f = f_2 - f_1$ and (f_1, E_1) and (f_2, E_2) are the values of uniaxial law and their derivative evaluated in x_1 and x_2 , respectively. An adequate portion Δx is key to gives a correct smooth function. Thus, for positive uniaxial laws, a value of $\Delta x = 5 \times 10^{-1}x_o$ is recommended, where x_o is the abscissa onset the non-linear behavior and for negative ones a value of $\Delta x = 5 \times 10^{-2}x_y$ is adequate, where x_y is the abscissa associated at the peak response.

- For non-symmetric stiffness matrices, its required the use of a unsymmetric Newton-Rapson solver method to get an adequate response. Its observed that the LLF and WLF models are specially sensitive to this condition. Moreover, when is forced a symmetrization of the consistent stiffness matrix ($\mathcal{D}_{n+1} = \frac{1}{2}(\mathcal{D}_{n+1} + \mathcal{D}_{n+1}^T)$) in this models, a "saw-tooth" shape response are generated, specially in the softening regimes and in biaxial and triaxial load states.
- Its well known that strain-softening behavior can generate a loss of positive-definite value of stiffness tensor and consequently a non-convergence of FE model. One way to remedy this, is to include a numerical viscous-plastic model in damage or plastic-damage models. The [Duvaut & Lions, 1972](#) model is more appropriate for the regularization of the rate-independent damage and plastic-damage models, because the Perzyna model fails to converge to the rate-independent backbone model in some cases ([J. Lee & Fenves, 1998](#)). The LLF, WLF, FOC and ROT models can include a viscous-regularization in their formulation using Duvaut-Lions model.

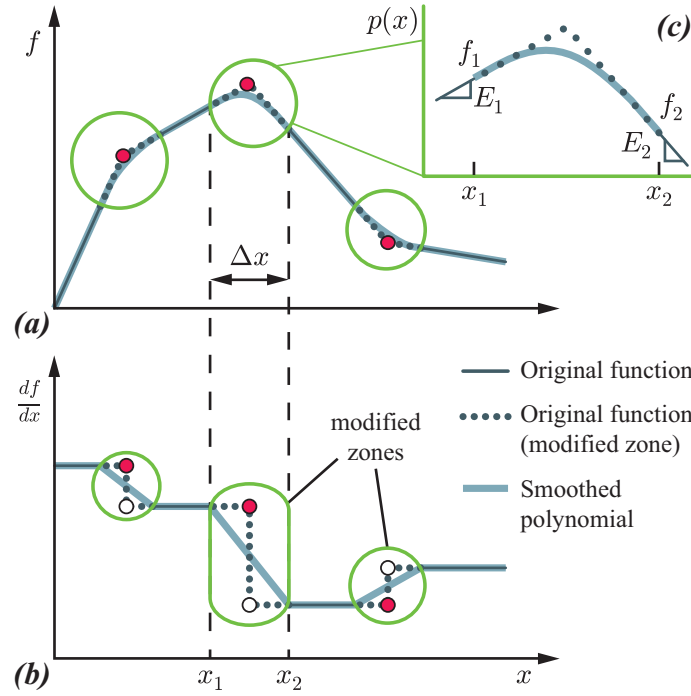


Figure 2.2.2. Smoothed function for uniaxial laws: (a) generic uniaxial law $f(x)$; (b) derivative of $f(x)$ and (c) smoothed polynomial used.

2.2.3. Additional recommendations

- We recommend the use of a linear algebra software (MATLAB, PYTHON) to check the adequate implementation and response of models.
- Its necessary to check the calculation of derivatives involved in the stiffness operator, e.g. compare the exact derivatives with their first-order approximation such as $\frac{\partial f}{\partial x} \approx (f[x_{j+1}] - f[x_j]) / \Delta x$, with $\Delta x = x_{j+1} - x_j$.
- Finally, for a correct computational implementation of numerical algorithms, its required that all tensors and their operations must be converted into adequate vector or matrix representation (vectorization and matricization). Thus, the second-order tensors are vectorized using Voigt's notation, whereas four-order tensors are converted into matrix standard format. Details of these conversions are explained in appendix B.

2.3. Stress updating algorithms

Numerical integration of constitutive equations requires of an algorithm to update the stress tensor and internal state variables at each integration point given a known strain increment. More specifically, given a (pseudo-) time increment $\Delta t = t_{n+1} - t_n$, it is assumed that at time t_n the strain tensor $\boldsymbol{\varepsilon}_n$, the stress tensor $\boldsymbol{\sigma}_n$ and the internal state variables $\boldsymbol{\alpha}_n$ are known. Then, the algorithm determine the updated stress tensor $\boldsymbol{\sigma}_{n+1}$ at time t_{n+1} for a given strain increment $\Delta \boldsymbol{\varepsilon} = \Delta t \dot{\boldsymbol{\varepsilon}}$.

Thereby, for one hand, the plastic component of models is commonly evaluated with a backward Euler (implicit) scheme. Return-mapping algorithms are the most used, where a trial elastic-predictor step and a plastic-corrector step are required (J. C. Simo & Hughes, 1998). Generally, this method lead implicit non-linear equations which are solved by means of an iterative Newton's method. For the other hand, the damage component of models is generally evaluated with an explicit scheme, with the exception of coupled plastic-damage models, which require the simultaneous solution of both components.

2.3.1. Trial elastic-predictor step

The elastic-trial step assume that the strain increment produces purely elastic deformation, where plastic deformation and evolution internal variables $\mathbf{q}$ are frozen ($\boldsymbol{\varepsilon}_{n+1}^{p \text{ tr}} = \boldsymbol{\varepsilon}_n^p$ and $\mathbf{q}_{n+1}^{\text{tr}} = \mathbf{q}_n$). Thus, the trial elastic strain and trial stress tensor are given by

$$\boldsymbol{\varepsilon}_{n+1}^{e \text{ tr}} = \boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}_n^p, \quad (2.3.75)$$

$$\boldsymbol{\sigma}_{n+1}^{\text{tr}} = \mathcal{D}_e : (\boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}_n^p) = \boldsymbol{\sigma}_n + \mathcal{D}_e : \Delta \boldsymbol{\varepsilon}_{n+1}, \quad (2.3.76)$$

where $\Delta \boldsymbol{\varepsilon}_{n+1} = \boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}_n$. Next, the trial state can be converted into the update solution if satisfy the condition

$$F_{n+1}^{\text{tr}} = F(\boldsymbol{\sigma}_{n+1}^{\text{tr}}, \mathbf{q}_{n+1}^{\text{tr}}) \leq 0. \quad (2.3.77)$$

This means that trial state lies within the elastic domain on the yield surface. In this case, the stress and internal variables are updated as $(\cdot)_{n+1} = (\cdot)_{n+1}^{\text{tr}}$. Otherwise, the trial step is not admissible, causing plastic response, being required any plastic-corrector step or a return-mapping algorithm to determine the update state.

2.3.2. Plastic-corrector step

The plastic-corrector step adjust the trial elastic-predictor step to give a correct updated stress. First, the updated plastic strain tensor ϵ_{n+1}^p is derived from linearization of flow rule as stated in Eq. (2.1.6)

$$\epsilon_{n+1}^p = \epsilon_n^p + \Delta\gamma \mathbf{N}_{n+1}. \quad (2.3.78)$$

Then, inserting this relation into Eq. (2.3.76), the updated stress tensor σ_{n+1} is written as

$$\sigma_{n+1} = \sigma_{n+1}^{\text{tr}} - \Delta\gamma \mathcal{D}_e : \mathbf{N}_{n+1}. \quad (2.3.79)$$

Thus, the only variable necessary to be solved is the discrete consistent operator $\Delta\gamma$, which is calculated according to their respective equations for each numerical model.

2.3.3. DPH model

The numerical stress integration of this model is based by the classical elastic-predictor (Section 2.3.1) and plastic-corrector step, the later explained as follow. First, substituting Eq. (2.1.7) into Eq. (2.3.79) and using Eq. (A.1.7), the updated stress tensor σ_{n+1} is given by

$$\sigma_{n+1} = \sigma_{n+1}^{\text{tr}} - \Delta\gamma \left(\frac{3\mu}{r_{n+1}} \mathbf{s}_{n+1} + \bar{\eta} K \mathbf{I} \right), \quad (2.3.80)$$

where $r_{n+1} = \sqrt{q_{n+1}^2 + \epsilon^2}$ and $q_{n+1} = \sqrt{3J_{2n+1}}$, with $J_{2n+1} = \frac{1}{2}\|\boldsymbol{\sigma}_{n+1}\|^2$. Then, the deviatoric and hydrostatic parts of this expression can be decomposed as

$$\mathbf{s}_{n+1} = \mathbf{s}_{n+1}^{\text{tr}} - \frac{3\mu\Delta\gamma}{r_{n+1}}\mathbf{s}_{n+1}, \quad (2.3.81)$$

$$p_{n+1} = p_{n+1}^{\text{tr}} - \bar{\eta}K\Delta\gamma. \quad (2.3.82)$$

Its easy to see that the updated deviatoric stress $\mathbf{s}_{n+1}$ is proportional, or geometrically parallel, to their respective trial stress $\mathbf{s}_{n+1}^{\text{tr}}$. This condition obeys to the radial return-mapping scheme (J. C. Simo & Hughes, 1998), i.e. equivalently expressed as

$$\frac{\mathbf{s}_{n+1}}{\|\mathbf{s}_{n+1}\|} = \frac{\mathbf{s}_{n+1}^{\text{tr}}}{\|\mathbf{s}_{n+1}^{\text{tr}}\|}, \quad (2.3.83)$$

$$\frac{\mathbf{s}_{n+1}}{q_{n+1}} = \frac{\mathbf{s}_{n+1}^{\text{tr}}}{q_{n+1}^{\text{tr}}}. \quad (2.3.84)$$

Substituting Eq. (2.3.84) into Eq. (2.3.81), the updated deviatoric stress tensor reads as

$$\mathbf{s}_{n+1} = f_{dev}\mathbf{s}_{n+1}^{\text{tr}}, \quad (2.3.85)$$

where $f_{dev} = 1 - 3\mu\Delta\gamma w_{n+1}/q_{n+1}^{\text{tr}}$, with $w_{n+1} = q_{n+1}/r_{n+1}$. Then, replacing Eq. (2.3.84) into Eq. (2.3.85), the variable q_{n+1} can be written as

$$q_{n+1} = q_{n+1}^{\text{tr}} - 3\mu w_{n+1}\Delta\gamma. \quad (2.3.86)$$

On the other hand, the updated equivalent plastic strain is obtained from the discrete version of Eq. (2.1.9) as

$$\alpha_{n+1} = \alpha_n + \xi\Delta\gamma. \quad (2.3.87)$$

Moreover, the updated cohesion law can be called as $c_{n+1} = c(\alpha_{n+1})$. Then, substituting Eqs. (2.3.86) and (2.3.82) into updated version of Eq. (2.1.3), the consistency

condition can be written as

$$\begin{aligned} F_{n+1} &= \eta p_{n+1} + q_{n+1} - \xi c_{n+1} = 0 \\ &= \eta(p_{n+1}^{\text{tr}} - \bar{\eta}K\Delta\gamma) + q_{n+1}^{\text{tr}} - 3\mu w_{n+1}\Delta\gamma - \xi c_{n+1}. \end{aligned} \quad (2.3.88)$$

Thus, the discrete consistency operator $\Delta\gamma$ can be computed in a partially closed form as

$$\Delta\gamma = \frac{q_{n+1}^{\text{tr}} + \eta p_{n+1}^{\text{tr}} - \xi c_{n+1}}{3\mu w_{n+1} + \eta \bar{\eta}K} = \frac{f_{1_{n+1}}}{f_{2_{n+1}}}, \quad (2.3.89)$$

where $f_{1_{n+1}} = f_1(\Delta\gamma)$ and $f_{2_{n+1}} = f_2(q_{n+1})$. Even so, note that its required an iterative process to calculate $\Delta\gamma$, e.g. Newton's method. For this, its convenient assume that the discrete consistency operator $\Delta\gamma$ is in function of variable q_{n+1} , i.e. $\Delta\gamma = \Delta\gamma(q_{n+1})$. Then, it required solve first the variable q_{n+1} and then obtain the consistency operator $\Delta\gamma$. Box 1 shown the algorithm suggested to solve the variable $\Delta\gamma$ for this model. The residual function and their total derivative are given by

$$R(q_{n+1}, \Delta\gamma(q_{n+1})) = -q_{n+1} + q_{n+1}^{\text{tr}} - 3\mu w_{n+1}\Delta\gamma, \quad (2.3.90)$$

$$\frac{dR}{dq_{n+1}} = \frac{\partial R}{\partial q_{n+1}} + \frac{\partial R}{\partial \Delta\gamma} \frac{\partial \Delta\gamma}{\partial q_{n+1}}, \quad (2.3.91)$$

where the derivatives involved in this expression are

$$\frac{\partial R}{\partial q_{n+1}} = -1 - 3\mu a_0 \Delta\gamma, \quad \frac{\partial R}{\partial \Delta\gamma} = -3\mu w_{n+1}, \quad (2.3.92)$$

$$\frac{\Delta\gamma}{\partial q_{n+1}} = -\frac{3\mu a_0 \Delta\gamma}{(f_{2_{n+1}} + J_\alpha \xi^2)}, \quad (2.3.93)$$

with $a_0 = \epsilon^2/r_{n+1}^3$ and $J_\alpha := \frac{\partial c_{n+1}}{\partial \alpha_{n+1}}$ the hardening modulus. Also, the recommended values for the number of iterations and tolerances are: $N_{iter} = 20$, $Tol_1 = 10^{-20}$, $Tol_2 = 10^{-5}$ and $Tol_3 = 10^{-2}$.

Box 1 :Algorithm to solve $\Delta\gamma$ for the DPH model

```

 $q_{n+1}^0 = q_{n+1}^{\text{tr}}, \quad c_{n+1}^0 = c(\alpha_n), \quad \Delta\gamma^0 = 0$  ▷ Set initial value
for  $j \leq N_{\text{iter}}$  do
     $r_{n+1}^j, w_{n+1}^j, \alpha_{n+1}^j, c_{n+1}^j, \Delta\gamma^j$  ▷ Use Eqs. (2.3.87) and (2.3.89)
     $R^j = R(q_{n+1}^j, \Delta\gamma^j)$  ▷ Residual function (Eq. (2.3.90))
     $dR^j = \frac{dR^j}{dq_{n+1}^j}(q_{n+1}^j, \Delta\gamma^j)$  ▷ Total derivative (Eq. (2.3.91))
     $dq^j = -R^j / dR^j$ 
     $q_{n+1}^{j+1} = q_{n+1}^j + dq^j$  ▷ Update solution
     $q_{n+1}^{j+1} = \max(q_{n+1}^{j+1}, Tol_1)$  ▷ Adjust solution
if  $(|R^j| < Tol_2 \text{ and } |dq^j| < Tol_3 q_{n+1}^j)$  or  $(dq^j \leq Tol_1)$  then
    exit

```

2.3.4. LLF model

The numerical stress integration of this model is composed by three steps: (i) an elastic-predictor step (Section 2.3.1); (ii) a plastic-corrector step with an implicit scheme to evaluate the updated effective stress tensor $\bar{\sigma}_{n+1}$; and (iii) a damage-corrector step with an explicit scheme to evaluate the updated damage variables ω_{n+1} and the nominal stress tensor σ_{n+1} . The development of plastic and damage steps are explained as follow.

Plastic component

First, due that the DPH and LLF share identical flow potential criterion, Eqs. (2.3.80) to (2.3.86) are also valid for this model, but expressed in the effective space ($\bar{\cdot}$). For the other hand, due that yield criterion is defined in terms of invariants and principal stresses, its convenient and efficient the use of Spectral Return Mapping Algorithm (SRMA) (J. Lee & Fenves, 1998). SRMA assume four conditions: (1) the effective stress tensor can be decomposed as $\bar{\sigma}_{n+1} = \underline{V} \hat{\bar{\sigma}}_{n+1} \underline{V}^T$, where $\hat{\bar{\sigma}}_{n+1}$ and $\underline{V}$ is the eigenvalue diagonal matrix and the eigenvector matrix of updated stress tensor $\bar{\sigma}_{n+1}$, respectively; (2) any eigenvector of trial effective stress tensor is also an eigenvector of updated effective stress tensor, i.e. $\bar{\sigma}_{n+1}^{\text{tr}} = \underline{V} \hat{\bar{\sigma}}_{n+1}^{\text{tr}} \underline{V}^T$; (3) any isotropic material satisfy the relation $G(\sigma) = \hat{G}(\hat{\sigma})$, which imply that $\bar{N}_{n+1} = \underline{V} \hat{N}_{n+1} \underline{V}^T$; and (4) substituting these expressions into the effective

expression of Eq. (2.3.80), the updated principal effective stress tensor is given by

$$\hat{\boldsymbol{\sigma}}_{n+1} = \hat{\boldsymbol{\sigma}}_{n+1}^{\text{tr}} - \Delta\gamma \mathcal{D}_e : \hat{\mathbf{N}}_{n+1}. \quad (2.3.94)$$

Moreover, using this expression, is easy to obtain the relation $\Delta\hat{\boldsymbol{\varepsilon}}_{n+1}^p = \Delta\gamma \hat{\mathbf{N}}_{n+1}$. It should be noted that variables $\bar{p}$, $\bar{q}$, $\bar{r}$ and $\bar{w}$ are invariants in effective space, i.e. $(\bar{\cdot}) = (\hat{\cdot})$. Also note that, due that yield criterion and hardening variables $\kappa^\pm$ are expressed in terms of maximum and minimum effective principal stresses, its necessary reordering the eigenvalues and their respective eigenvectors in a descending order ($\hat{\sigma}_1 \geq \dots \geq \hat{\sigma}_N$). Jacobi's method is recommended to calculate the eigenvalues and eigenvectors of any symmetric real tensor (Golub & van der Vorst, 2000).

Analogously to the DPH model, Eq. (2.3.83) is also valid, but now expressed in the principal effective space as

$$\hat{\mathbf{s}}_{n+1} = \frac{\bar{q}_{n+1}}{\bar{q}_{n+1}^{\text{tr}}} \hat{\mathbf{s}}_{n+1}^{\text{tr}}. \quad (2.3.95)$$

Using this expression, the updated principal effective flow tensor $\hat{\mathbf{N}}$, given by Eq. (2.1.20), can be written as

$$\hat{\mathbf{N}}_{n+1} = \frac{3}{2} \bar{w}_{n+1} \hat{\mathbf{t}}_{n+1}^{\text{tr}} + \frac{\bar{\eta}}{3} \mathbf{I}, \quad (2.3.96)$$

where $\hat{\mathbf{t}}_{n+1}^{\text{tr}} = \hat{\mathbf{s}}_{n+1}^{\text{tr}} / \bar{q}_{n+1}^{\text{tr}}$. Moreover, their positive/negative part are denoted as

$$\hat{n}_{n+1}^\pm = \frac{3}{2} \bar{w}_{n+1} \hat{t}_{\pm n+1}^{\text{tr}} + \frac{\bar{\eta}}{3}, \quad (2.3.97)$$

with $\hat{t}_{\pm n+1}^{\text{tr}} = \hat{\mathbf{I}} : \hat{\mathbf{t}}_{n+1}^{\text{tr}}$. Next, introducing Eq. (2.3.96) into Eq. (2.3.94) and using Eq. (A.1.7), the principal effective stress tensor can be written as

$$\hat{\boldsymbol{\sigma}}_{n+1} = \hat{\boldsymbol{\sigma}}_{n+1}^{\text{tr}} - \Delta\gamma \hat{\mathbf{B}}_{0n+1}, \quad (2.3.98)$$

where $\hat{\mathbf{B}}_{0_{n+1}} = 3\mu\bar{w}_{n+1}\hat{\mathbf{t}}_{n+1}^{\text{tr}} + \bar{\eta}K\mathbf{I}$. Also, the maximum updated principal effective stress $\hat{\sigma}_{+n+1} = \hat{\mathbf{I}}^+ : \hat{\boldsymbol{\sigma}}_{n+1}$, is expressed as

$$\hat{\sigma}_{+n+1} = \hat{\sigma}_{+n+1}^{\text{tr}} - \Delta\gamma\hat{b}_{0_{n+1}}, \quad (2.3.99)$$

where $\hat{\sigma}_{+n+1}^{\text{tr}} = \hat{\mathbf{I}}^+ : \hat{\boldsymbol{\sigma}}_{n+1}^{\text{tr}}$ and $\hat{b}_{0_{n+1}} = 3\mu\bar{w}_{n+1}\hat{t}_{+n+1}^{\text{tr}} + \bar{\eta}K$. In addition, Eqs. (2.3.86) and (2.3.82) can be rewritten in the effective space as

$$\bar{p}_{n+1} = \bar{p}_{n+1}^{\text{tr}} - \bar{\eta}K\Delta\gamma, \quad (2.3.100)$$

$$\bar{q}_{n+1} = \bar{q}_{n+1}^{\text{tr}} - 3\mu\bar{w}_{n+1}\Delta\gamma. \quad (2.3.101)$$

For the other hand, linearization of updated hardening variable $\underline{\kappa}_{n+1}$, given by Eq. (2.1.21), can be expressed as

$$\underline{\kappa}_{n+1} = \underline{\kappa}_n + \Delta\gamma\underline{\mathbf{H}}_{n+1}(\hat{\sigma}_{n+1}, \underline{\kappa}_{n+1}). \quad (2.3.102)$$

Although, its convenient take their positive and negative part as

$$\kappa_{n+1}^{\pm} = \kappa_n^{\pm} + \Delta\gamma h_{n+1}^{\pm}, \quad (2.3.103)$$

where $h_{n+1}^{\pm} = \hat{n}_{n+1}^{\pm}\varphi_{n+1}^{\pm}$, with $\hat{n}_{n+1}^{\pm} = \hat{\mathbf{I}}^{\pm} : \hat{\mathbf{N}}_{n+1}$ and the variables $\varphi_{n+1}^{\pm}$ defined as

$$\varphi_{n+1}^{\pm} := \theta_{1_{n+1}}^{\pm}\theta_{2_{n+1}}^{\pm}, \quad (2.3.104)$$

and with $\theta_{1_{n+1}}^{\pm}$ and $\theta_{2_{n+1}}^{\pm}$ defined as

$$\theta_{1_{n+1}}^+ := \phi(\hat{\boldsymbol{\sigma}}_{n+1}), \quad \theta_{1_{n+1}}^- := -[1 - \phi(\hat{\boldsymbol{\sigma}}_{n+1})], \quad (2.3.105)$$

$$\theta_{2_{n+1}}^+ := \sigma^+(\kappa_{n+1}^+)/g^+, \quad \theta_{2_{n+1}}^- := \sigma^-(\kappa_{n+1}^-)/g^-. \quad (2.3.106)$$

In addition, the updated parameters $\beta(\underline{\kappa})$ and $c(\kappa^-)$, given by Eq. (2.1.12), can be expressed as

$$\beta_{n+1} = (1 - \alpha) \frac{\bar{\sigma}^-(\kappa_{n+1}^-)}{\bar{\sigma}^+(\kappa_{n+1}^+)} - (1 + \alpha), \quad c_{n+1} = \bar{\sigma}^-(\kappa_{n+1}^-). \quad (2.3.107)$$

Finally, substituting Eqs. (2.3.99), (2.3.100) and (2.3.101) into Eq. (2.1.24), the consistency condition is written as

$$\begin{aligned} F_{n+1} &= \eta \bar{p}_{n+1} + \bar{q}_{n+1} + \beta_{n+1} \langle \hat{\sigma}_{+n+1} \rangle^+ - \delta \langle \hat{\sigma}_{+n+1} \rangle^- - (1 - \alpha) c_{n+1} = 0 \\ &= \eta (\bar{p}_{n+1}^{\text{tr}} - \bar{\eta} K \Delta \gamma) + \bar{q}_{n+1}^{\text{tr}} - 3\mu \bar{w}_{n+1} \Delta \gamma \\ &\quad + \hat{\rho}_1 \left(\hat{\sigma}_1^{\text{tr}} - \Delta \gamma \hat{b}_{0+n+1} \right) - (1 - \alpha) c_{n+1} = 0, \end{aligned} \quad (2.3.108)$$

where $\hat{\rho}_1 = \hat{\beta}_1 + \hat{\delta}_1$, with $\hat{\beta}_1 = \beta_{n+1} \tilde{H}^+(\hat{\sigma}_{+n+1})$, $\hat{\delta}_1 = \delta \tilde{H}^-(\hat{\sigma}_{+n+1})$ and $\tilde{H}^\pm(\cdot)$ a C^1 -class approximation of Heaviside function (see Eq. (A.1.22)). Thus, the discrete consistency operator $\Delta \gamma$ can be computed, similarly to the DPH model, in a partially closed form as

$$\Delta \gamma = \frac{\eta \bar{p}_{n+1}^{\text{tr}} + \bar{q}_{n+1}^{\text{tr}} + \hat{\rho}_1 \hat{\sigma}_{+n+1}^{\text{tr}} - (1 - \alpha) c_{n+1}}{\eta \bar{\eta} K + 3\mu \bar{w}_{n+1} + \hat{\rho}_1 \hat{b}_{0+n+1}} = \frac{\bar{f}_{1n+1}}{\bar{f}_{2n+1}}, \quad (2.3.109)$$

where $\bar{f}_{i_{n+1}} = \bar{f}_i(\underline{\kappa}_{n+1}, \Delta \gamma(\underline{\kappa}_{n+1}))$, with $i = 1, 2$. Its observed that a nested iterative process is required to obtain variables $\Delta \gamma$ and $\kappa_{n+1}^\pm$. Box 2 shown the algorithm used to calculate both variables. Three steps are involved: (i) set an initial value of variables κ , q and $\hat{\sigma}$ equal to the previous step; (ii) solve the consistency operator $\Delta \gamma$ using the algorithm described in Box 1, which is identical to the DPH model, but using the effective stress space in their expressions and the derivative $\frac{\partial \Delta \gamma}{\partial q_{n+1}}$ is expressed as

$$\frac{\partial \Delta \gamma}{\partial q_{n+1}} = \frac{(b_{8n+1} - b_{9n+1} - \Delta \gamma b_{11n+1}) \Delta \gamma}{(\bar{f}_{2n+1} - b_{7n+1} + \Delta \gamma b_{10n+1})}, \quad (2.3.110)$$

where b_{8n+1} to b_{11n+1} are scalar parameters. A detailed calculation of this derivative is explained in 1; and (iii) solve the hardening variables $\underline{\kappa}$ using the Newton's method. For

this, Eq. (2.3.102) is used as the residual function and rewritten as

$$\underline{Q}_{n+1}(\underline{\kappa}_{n+1}, \Delta\gamma, \hat{\underline{\sigma}}_{n+1}) = \underline{\kappa}_n + \Delta\gamma \underline{H}_{n+1}(\underline{\kappa}_{n+1}, \hat{\underline{\sigma}}_{n+1}) - \underline{\kappa}_{n+1}. \quad (2.3.111)$$

Thus, the total derivative of this residual function with respect to $\underline{\kappa}_{n+1}$ is given by

$$\begin{aligned} \frac{d\underline{Q}_{n+1}}{d\underline{\kappa}_{n+1}} &= \frac{\partial \underline{Q}_{n+1}}{\partial \underline{\kappa}_{n+1}} + \frac{\partial \underline{Q}_{n+1}}{\partial \Delta\gamma} \frac{\partial \Delta\gamma}{\partial \underline{\kappa}_{n+1}} \\ &\quad + \frac{\partial \underline{Q}_{n+1}}{\partial \underline{H}_{n+1}} \left(\frac{\partial \underline{H}_{n+1}}{\partial \hat{\underline{\sigma}}_{n+1}} \frac{\partial \hat{\underline{\sigma}}_{n+1}}{\partial \Delta\gamma} \frac{\partial \Delta\gamma}{\partial \underline{\kappa}_{n+1}} + \frac{\partial \underline{H}_{n+1}}{\partial \underline{\kappa}_{n+1}} \right) \\ &= -\underline{I}_2 + \left(\underline{H}_{n+1} + \Delta\gamma \frac{\partial \underline{H}_{n+1}}{\partial \hat{\underline{\sigma}}_{n+1}} \frac{\partial \hat{\underline{\sigma}}_{n+1}}{\partial \Delta\gamma} \right) \otimes \frac{\partial \Delta\gamma}{\partial \underline{\kappa}_{n+1}} + \Delta\gamma \frac{\partial \underline{H}_{n+1}}{\partial \underline{\kappa}_{n+1}}, \end{aligned} \quad (2.3.112)$$

where $\underline{I}_2 = \text{diag}(1, 1)$ and the derivatives involved are expressed as

$$\begin{aligned} \frac{\partial \underline{H}_{n+1}}{\partial \hat{\underline{\sigma}}_{n+1}} &= \left(\hat{\underline{Y}}_{n+1} \otimes \hat{\underline{\Phi}}_{n+1} \right) + a_3 \left(\hat{\underline{Z}}_{n+1} \otimes \hat{\underline{t}}_{n+1}^{\text{tr}} \right), \quad \frac{\partial \Delta\gamma}{\partial \underline{\kappa}_{n+1}} = \frac{1}{(f_{2_{n+1}} - L_{1_{n+1}})} \underline{l}_{0_{n+1}} \\ \frac{\partial \hat{\underline{\sigma}}_{n+1}}{\partial \Delta\gamma} &= -\hat{\underline{B}}_{0_{n+1}} + 9\mu^2 \bar{a}_2 \bar{w}_{n+1} \Delta\gamma \hat{\underline{t}}_{n+1}^{\text{tr}}, \quad \frac{\partial \underline{H}_{n+1}}{\partial \underline{\kappa}_{n+1}} = \hat{\underline{U}}_{n+1}. \end{aligned} \quad (2.3.113)$$

A detailed calculation of these derivatives are explained in 2. Also, to get an adequate convergence of model, is recommended use tolerances of $Tol_4 = 1 - 10^{-10}$ to adjust the solution values and $Tol_5 = 10^{-5}$ to check the residual function.

Damage component

An explicit evaluation of updated damage variable ω_{n+1} (Eq. (2.1.27)) are generated according to updated hardening variables $\kappa_{n+1}^{\pm}$ calculated in the plastic component of model.

Viscous component

First, assume that the rate of a generic variable x can be expressed as $\dot{x} = \Delta x / \Delta t$, with Δt is the load step increment. Then, using this relation in the linearization of Eqs. (2.1.33)

Box 2 :Algorithm to solve $\underline{\kappa}_{n+1}$ for the LLF model

$\underline{\kappa}_{n+1}^{\pm 0} = \underline{\kappa}_n^{\pm}, \quad \bar{q}_{n+1}^0 = \bar{q}_{n+1}^{\text{tr}}, \quad \hat{\sigma}_{n+1}^0 = \hat{\sigma}_{n+1}^{\text{tr}} \quad \triangleright$ Set initial value
for $j \leq N_{iter}$ **do**
 $\bar{\sigma}^{\pm}(\underline{\kappa}_{n+1}^{\pm j}), \beta_{n+1}^j, c_{n+1}^j, \tilde{H}^{\pm}(\hat{\sigma}_{n+1}^j) \quad \triangleright$ Use Eqs. (2.1.29), (2.3.99) and (2.3.107)
 $\bar{q}_{n+1}^j, \Delta\gamma^j, \hat{\sigma}_{n+1}^j \quad \triangleright$ Solve with Box 1 and Eq. (2.3.98)
 $\underline{Q}^j = Q_{n+1}(\underline{\kappa}_{n+1}^j, \Delta\gamma^j, \hat{\sigma}_{n+1}^j) \quad \triangleright$ Residual, Eq. (2.3.111)
 $d\underline{Q}^j = \frac{d\underline{Q}_{n+1}}{d\underline{\kappa}_{n+1}}(\underline{\kappa}_{n+1}^j, \Delta\gamma^j, \hat{\sigma}_{n+1}^j) \quad \triangleright$ Total derivative, Eq. (2.3.112)
 $d\underline{\kappa}^j = - (d\underline{Q}^j)^{-1} \underline{Q}^j$
 $\underline{\kappa}_{n+1}^{j+1} = \underline{\kappa}_{n+1}^j + d\underline{\kappa}^j \quad \triangleright$ Update solution
 $\underline{\kappa}_{n+1}^{j+1} = \min(\underline{\kappa}_{n+1}^j, Tol_4 \mathbf{1}) \quad \triangleright$ Adjust solution
if $(\|\underline{Q}^j\| \leq Tol_5)$ **then**
exit

and (2.1.34), the updated visco-plastic strain tensor $\underline{\epsilon}^{vp}$ and the viscous-damage variable ω^v can be expressed as

$$\underline{\epsilon}_{n+1}^{vp} = \zeta_v \underline{\epsilon}_n^{vp} + (1 - \zeta_v) \underline{\epsilon}_{n+1}^p \quad (2.3.114)$$

$$\omega_{n+1}^v = \zeta_v \omega_n^v + (1 - \zeta_v) \omega_{n+1}, \quad (2.3.115)$$

where $\zeta_v = (1 + \Delta t / \mu_v)^{-1}$. Then, substituting the Eq. (2.3.114) into Eq. (2.1.32) and with some algebraic manipulation, the updated effective viscous-stress tensor can be expressed in a convenient way as

$$\bar{\sigma}_{n+1}^v = \zeta_v (\bar{\sigma}_n^v + \mathcal{D}_e : \Delta \underline{\epsilon}_n) + (1 - \zeta_v) \bar{\sigma}_{n+1} \quad (2.3.116)$$

Finally, the updated viscous-stress tensor can be expressed as

$$\underline{\sigma}_{n+1}^v = (1 - \omega_{n+1}^v) \bar{\sigma}_{n+1}^v. \quad (2.3.117)$$

It should be noted that if $\mu_v / \Delta t \rightarrow 0$ ($\zeta_v = 0$) the solution relaxed to the rate-independent (or inviscid) response.

2.3.5. WLF model

Plastic component

The numerical stress integration of this model is identical to the LLF model, except for three considerations: (1) the parameter δ present in the yield criterion of Eq. (2.1.24) is null; (2) the matrix $\underline{\mathbf{H}}_{n+1}$ of Eq. (2.3.102) depends only of stress tensor $\hat{\sigma}$, for which the variables $\theta_2^\pm = 1$ and the derivative $\frac{\partial \mathbf{H}}{\partial \kappa}$, given by Eq. (2.3.113), is null; and (3) its observed that a tolerance to check the residual function of $Tol_5 = 10^{-10}$ can be used without convergence troubles.

Damage component

Giving the updated effective stress tensor $\bar{\sigma}_{n+1}$ calculated in the plastic component, the positive/negative part of effective stress tensor $\bar{\sigma}_{n+1}^\pm$ are evaluated using Eq. (2.1.35). Next, evaluating the DERRs, $Y^\pm$, according to their definition established by Eq. (2.1.50) or Eq. (2.1.51), and assuming an active damage process (Eq. (2.1.55)), the updated damage threshold are stated. Finally, and explicit evaluation of damage variables $\omega_{n+1}^\pm(r_{n+1}^\pm)$ is generated.

Viscous component

The updated viscous stress vector σ_{n+1}^v is calculated using Eq. (2.1.59), where the effective viscous stress vector $\bar{\sigma}_{n+1}^v$ is evaluated using Eq. (2.3.116). Also, the viscoplastic strain vector ϵ_{n+1}^{vp} is evaluated with Eq. (2.3.114). Moreover, the updated damage variables depends of updated damage thresholds variables $r_{n+1}^\pm$, which are obtained using a linearization of Eq. (2.1.61) as follows

$$r_{n+1}^\pm = \zeta_v r_n^\pm + (1 - \zeta_v) Y_{n+1}^\pm. \quad (2.3.118)$$

2.3.6. FOC model

Plastic component

First, the discretization of Eqs. (2.1.62) and (2.1.63) gives

$$\boldsymbol{\varepsilon}_{n+1}^p = \boldsymbol{\varepsilon}_n^p + \Delta\gamma \bar{\boldsymbol{\sigma}}_{n+1}, \quad (2.3.119)$$

$$\Delta\gamma = \frac{E_o \chi_{n+1}}{\|\bar{\boldsymbol{\sigma}}_{n+1}\|^2} \mathcal{C}_e : \langle \bar{\boldsymbol{\sigma}}_{n+1} : \Delta\boldsymbol{\varepsilon}_{n+1} \rangle^+, \quad (2.3.120)$$

where $\chi_{n+1} = B^+ H^+(\Delta\omega_{n+1}^+) + B^- H^+(\Delta\omega_{n+1}^-)$ and $\Delta\boldsymbol{\varepsilon}_{n+1} = \boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}_n$, with $\Delta\omega_{n+1}^\pm = \omega_{n+1}^\pm - \omega_n^\pm$. Next, using the relation of Eq. (2.3.79), with $\mathbf{N}_{n+1} = \bar{\boldsymbol{\sigma}}_{n+1}$, the updated effective stress tensor is given by

$$\bar{\boldsymbol{\sigma}}_{n+1} = \bar{\boldsymbol{\sigma}}_{n+1}^{\text{tr}} - \frac{E_o \chi_{n+1}}{\|\bar{\boldsymbol{\sigma}}_{n+1}\|^2} \langle \bar{\boldsymbol{\sigma}}_{n+1} : \Delta\boldsymbol{\varepsilon}_{n+1} \rangle^+ \bar{\boldsymbol{\sigma}}_{n+1} \quad (2.3.121)$$

It should be noted that $\bar{\boldsymbol{\sigma}}_{n+1}$ is proportional, or geometrically parallel, to $\bar{\boldsymbol{\sigma}}_{n+1}^{\text{tr}}$, analogically to deviatoric stress tensor as in the classical plasticity models, e.g. Eq. (2.3.83). Thus, satisfy the following relation

$$\frac{\bar{\boldsymbol{\sigma}}_{n+1}}{\|\bar{\boldsymbol{\sigma}}_{n+1}\|} = \frac{\bar{\boldsymbol{\sigma}}_{n+1}^{\text{tr}}}{\|\bar{\boldsymbol{\sigma}}_{n+1}^{\text{tr}}\|}. \quad (2.3.122)$$

Replacing this expression into Eq. (2.3.121), the updated effective stress tensor can be rewritten as

$$\bar{\boldsymbol{\sigma}}_{n+1} = m_{n+1}^{\text{tr}} \bar{\boldsymbol{\sigma}}_{n+1}^{\text{tr}}, \quad (2.3.123)$$

$$m_{n+1}^{\text{tr}} = 1 - \frac{E_o \chi_{n+1}}{n_{0n+1}} \langle n_{1n+1} \rangle^+, \quad (2.3.124)$$

where $n_{0n+1} = (\bar{\boldsymbol{\sigma}}_{n+1}^{\text{tr}} : \bar{\boldsymbol{\sigma}}_{n+1}^{\text{tr}})$ and $n_{1n+1} = (\bar{\boldsymbol{\sigma}}_{n+1}^{\text{tr}} : \Delta\boldsymbol{\varepsilon}_{n+1})$. It should be noted that, as the Heaviside function is present in the variable χ_{n+1} , it required an iterative process to solve $\bar{\boldsymbol{\sigma}}_{n+1}$. Box 3 shown an efficient and robust algorithm to solve the updated effective stress tensor $\bar{\boldsymbol{\sigma}}_{n+1}$.

Box 3 :Algorithm to solve $\bar{\sigma}_{n+1}$ for the FOC model

```

 $\underline{v}_1 = [0, 1, 0, 1]^T, \quad \underline{v}_2 = [0, 0, 1, 1]^T$  ▷ Set combinatory vectors
for  $j \leq 4$  do
     $h_1^j = \underline{v}_1[j], \quad h_2^j = \underline{v}_2[j]$  ▷ Trial Heaviside values
     $\chi_{n+1}^j = B^+ h_1^j + B^- h_2^j, \quad m_{n+1}^{\text{tr } j}$  ▷ Use Eq. (2.3.124)
     $\bar{\sigma}^j = m_{n+1}^{\text{tr } j} \bar{\sigma}_{n+1}^{\text{tr}}$  ▷ Trial effective stress
     $\bar{\sigma}^{j \pm} = \mathcal{P}^{\pm} : \bar{\sigma}^j$  ▷ Positive/negative effective stress (Eq. (2.1.36))
     $Y_j^{\pm}$  ▷ DEER's, according to Eq. (2.1.50) or Eq. (2.1.51)
     $F_d^{\pm j} = Y_j^{\pm} - r_n^{\pm}$  ▷ Positive/negative damage criteria (Eq. (2.1.49))
    if  $\left(h_1^j = H^+ \left(F_d^{+j}\right)\right)$  and  $\left(h_2^j = H^+ \left(F_d^{-j}\right)\right)$  then
        exit
     $\bar{\sigma}_{n+1} = \bar{\sigma}^j$  ▷ Update effective stress

```

Finally, replacing Eq. (2.3.122) into Eq. (2.3.119), the updated plastic strain tensor is derived as

$$\epsilon_{n+1}^p = \epsilon_n^p + (1 - m_{n+1}^{\text{tr}}) \mathcal{C}_e : \bar{\sigma}_{n+1}^{\text{tr}}. \quad (2.3.125)$$

Also, note that as Eq. (2.3.122) is valid either in 3D as in plane stress condition, this algorithm can be used in both cases.

2.3.7. ROT model

Damage component

Assuming an explicit integration scheme for the linearization of Eq. (2.1.65), the updated positive/negative i -th damage strain variable $\alpha_i^{\pm}$ is expressed as

$$\alpha_{i_{n+1}}^{\pm} = \alpha_{i_n}^{\pm} + z_{i_{n+1}}^{\pm} \Delta \hat{\epsilon}_{i_{n+1}} \quad (2.3.126)$$

where $z_{i_{n+1}}^{\pm} = 1 - r_{i_{n+1}}^{\pm}$ and $\Delta \hat{\epsilon}_{i_{n+1}} = \hat{\epsilon}_{i_{n+1}} - \hat{\epsilon}_{i_n}$, with $r_{i_{n+1}}^{\pm} = H_0^{\pm}(\alpha_{i_n}^{\pm} - \hat{\epsilon}_{i_{n+1}})$. Note that the term $\alpha_{i_{n+1}}^{\pm}$ inner the Heaviside function is used to get an explicit scheme. So, the evaluation of updated stress tensor σ is explicit (Eq. (2.1.69)) using the relations of Eqs. (2.1.67) and (2.1.68), where $m_{i_{n+1}}^{\pm} = H_{1/2}^{\pm}(\hat{\epsilon}_{i_{n+1}})$ and the variables $h_{i_{n+1}}^{\pm}$ and $g_{i_{n+1}}^{\pm}$

are written, respectively, as

$$h_{i_{n+1}}^{\pm} = \sigma^{\pm}(\alpha_{i_{n+1}}^{\pm})g_{i_{n+1}}^{\pm}, \quad g_{i_{n+1}}^{\pm} = \frac{\hat{\varepsilon}_{i_{n+1}}}{\alpha_{i_{n+1}}^{\pm}}. \quad (2.3.127)$$

Viscous component

Taking the linearization of Eq. (2.1.70), the updated positive/negative i -th viscous-damage strain $\alpha_i^{v\pm}$ can be expressed as

$$\alpha_{i_{n+1}}^{v\pm} = \zeta_v \alpha_{i_n}^{v\pm} + (1 - \zeta_v) \alpha_{i_{n+1}}^{\pm}. \quad (2.3.128)$$

Finally, the evaluation of updated viscous-stress tensor σ^v (Eq. (2.1.73)) is explicit using the relations of Eqs. (2.1.71) and (2.1.72).

2.4. Consistent tangent tensors

Additionally to the algorithm necessary to calculate the updated stress tensor, a material stiffness tensor is required for the solution. Continuum tangent stiffness tensor is derived for material models according to derivation of continuum constitutive equations as stated in Section 3. However, for numerical integration of model, is necessary to calculate the algorithmic consistent tangent tensor $\frac{d\sigma_{n+1}}{d\varepsilon_{n+1}}$, which are developed by computing the derivatives of equations involved in the stress updated algorithm. Complex derivatives involve this operator, but are necessary to achieve a second-order convergence at the structural level, rather than continuum tangent stiffness ([J. C. Simo & Hughes, 1998](#)). Only in explicit schemes, continuum and consistent stiffness tensors are identical. For the developed models, all these derivatives can be obtained analytically. Therefore, the consistent tangent operator can be written in an explicit expression. For sake the of simplicity of the presentation, is omitted the subscript $n+1$ in all updated variables.

2.4.1. Trial-predictor step

Using Eqs. (2.3.75) and (2.3.76), the differential of the trial elastic strain $\epsilon^{e\text{tr}}$ and the stress tensor σ are, respectively, given by

$$d\epsilon^{e\text{tr}} = d\epsilon, \quad (2.4.129)$$

$$d\sigma^{\text{tr}} = \mathcal{D}_e : d\epsilon. \quad (2.4.130)$$

It follow that in the derivation of consistent tangent stiffness tensor all trial variables $(\cdot)^{\text{tr}}$ have a no-null differential, contrary as in the calculation of stress updated algorithm, where their derivatives are neglected.

2.4.2. DPH model

First, using Eqs. (2.3.83) and (A.1.4) and the relation $q = \sqrt{3J_2}$, with $J_2 = \frac{1}{2}\|\mathbf{s}\|^2$, the unitary tensor of updated deviatoric stress can be rewritten as

$$\mathbf{M} = \frac{\mathbf{s}^{\text{tr}}}{\|\mathbf{s}^{\text{tr}}\|} = \frac{\sqrt{6}\mu}{q^{\text{tr}}} \boldsymbol{\theta}^{e\text{tr}}. \quad (2.4.131)$$

Then, inserting this expression into Eq. (A.1.14) and using Eq. (2.4.129), the differential of trial equivalent stress q^{tr} is given by

$$dq^{\text{tr}} = \frac{6\mu^2}{q^{\text{tr}}} \boldsymbol{\theta}^{\text{tr}} : d\epsilon^{\text{tr}} = \sqrt{6}\mu \mathbf{M} : d\epsilon^{\text{tr}}. \quad (2.4.132)$$

Next, the differential of relations $r = \sqrt{q^2 + \epsilon^2}$ and $w = q/r$ are given by $dr = w dq$ and $dw = a_0 dq$, with $a_0 = \epsilon^2/r^3$. Then, inserting this expressions into the differential of Eq. (2.3.86) and with some algebraic manipulation, the differential of updated variable q can be written as

$$dq = a_1 (dq^{\text{tr}} - 3\mu w d\Delta\gamma), \quad (2.4.133)$$

where $a_1 = (1 + 3\mu a_0 \Delta\gamma)^{-1}$. Moreover, using these relations, the differential of the variable $f_{dev} = 1 - 3\mu\Delta\gamma w/q$ is given by

$$df_{dev} = -\frac{3\mu}{q^{tr}} (a_3 d\Delta\gamma + a_4 \Delta\gamma dq^{tr}), \quad (2.4.134)$$

where $a_3 = w(1 - 3\mu a_2 \Delta\gamma)$ and $a_4 = a_2 - w/q^{tr}$, with $a_2 = a_0 a_1$. Then, using this relation and Eq. (2.4.131), the differential of updated deviatoric stress tensor (Eq. (2.3.85)) can be expressed as

$$d\mathbf{s} = 2\mu f_{dev} d\boldsymbol{\theta}^{e\,tr} - \sqrt{6}\mu \mathbf{M} (a_3 d\Delta\gamma + a_4 \Delta\gamma dq^{tr}). \quad (2.4.135)$$

For the other hand, using the relation of Eq. (A.1.15), the differential of updated hydrostatic stress p (Eq. (2.3.82)) is given by

$$dp = K (\mathbf{I} : d\boldsymbol{\varepsilon}^{e\,tr} - \bar{\eta} d\Delta\gamma). \quad (2.4.136)$$

In addition, using Eq. (2.3.87) and the chain rule, the differential of updated cohesion law can be written as

$$dc = \frac{\partial c}{\partial \alpha} d\alpha = J_\alpha \xi d\Delta\gamma, \quad (2.4.137)$$

where $J_\alpha := \frac{\partial c}{\partial \alpha}$ is the cohesion hardening modulus. Then, using Eqs. (2.4.133), (2.4.136) and (2.4.137), the differential of the yield criterion at consistency condition, given by Eq. (2.3.88), can be expressed as

$$\begin{aligned} dF &= \eta dp + dq - \xi dc = 0 \\ &= \eta K (\mathbf{I} : d\boldsymbol{\varepsilon}^{e\,tr} - \bar{\eta} d\Delta\gamma) + a_1 (dq^{tr} - 3\mu w d\Delta\gamma) - J_\alpha \xi^2 d\Delta\gamma. \end{aligned} \quad (2.4.138)$$

Thus, using Eq. (2.4.132), an explicit expression for the differential of consistency operator $\Delta\gamma$ can be obtained as

$$d\Delta\gamma = a_6 \left(a_1 \sqrt{6}\mu \mathbf{M} + \eta K \mathbf{I} \right) : d\boldsymbol{\varepsilon}^{e\,tr} = \mathbf{G} : d\boldsymbol{\varepsilon}^{e\,tr}, \quad (2.4.139)$$

where $a_6 = (a_5 + K\eta\bar{\eta} + J_\alpha\xi^2)^{-1}$, with $a_5 = 3\mu a_1 w$. Hence, using Eqs. (2.4.135), (2.4.136), (2.4.139) and (A.1.12), the differential of updated stress tensor is expressed as

$$\begin{aligned} d\boldsymbol{\sigma} = & \left[2\mu f_{dev} \boldsymbol{\mathcal{I}}_d - \sqrt{6}\mu a_3 (\boldsymbol{M} \otimes \boldsymbol{G}) - 6\mu^2 a_4 \Delta\gamma (\boldsymbol{M} \otimes \boldsymbol{M}) \right. \\ & \left. + K(\boldsymbol{I} \otimes \boldsymbol{I}) - \bar{\eta}K(\boldsymbol{I} \otimes \boldsymbol{G}) \right] : d\boldsymbol{\epsilon}^{e\text{tr}}. \end{aligned} \quad (2.4.140)$$

Finally, after some straightforward manipulation, an explicit expression for elasto-plastic consistent tangent operator is written as

$$\boldsymbol{\mathcal{D}}_{ep} = c_1 \boldsymbol{\mathcal{I}}_d + c_2 (\boldsymbol{M} \otimes \boldsymbol{M}) + c_3 (\boldsymbol{M} \otimes \boldsymbol{I}) + c_4 (\boldsymbol{I} \otimes \boldsymbol{M}) + c_5 (\boldsymbol{I} \otimes \boldsymbol{I}), \quad (2.4.141)$$

where the constants c_1 to c_6 are given by

$$\begin{aligned} c_1 &= 2\mu f_{dev}, & c_2 &= -6\mu^2 (a_1 a_3 a_6 + a_4 \Delta\gamma), & c_3 &= -\sqrt{6}\mu K \eta a_3 a_6, \\ c_4 &= -\sqrt{6}\mu K \bar{\eta} a_1 a_6, & c_5 &= K (1 - \eta \bar{\eta} K a_6). \end{aligned}$$

2.4.3. LLF model

Plastic component

The plastic component of the consistent tangent stiffness tensor is calculated from differential of the effective stress tensor. First, due that the LLF and DPH model share identical flow potential, Eqs. (2.4.131) to (2.4.136) are valid for this model, but expressed in the principal effective space ($\hat{\cdot}$). Next, the differential of the principal effective flow tensor, given by Eq. (2.3.96), can be expressed as

$$d\hat{\boldsymbol{N}} = \frac{3}{2} \left(\bar{a}_4 \hat{\boldsymbol{t}}^{\text{tr}} d\bar{q}^{\text{tr}} + \bar{u} d\hat{\boldsymbol{s}}^{\text{tr}} + \bar{a}_5 \hat{\boldsymbol{t}}^{\text{tr}} d\Delta\gamma \right), \quad (2.4.142)$$

where $\bar{a}_4 = \bar{a}_2 - \bar{u}$ and $\bar{a}_5 = -3\mu\bar{a}_2\bar{w}$, with $\bar{a}_2 = \bar{a}_0\bar{a}_1$, $\bar{a}_1 = 1 + 3\mu\bar{a}_0\Delta\gamma$, $\bar{a}_0 = \epsilon^2/\bar{r}^3$ and $\bar{u} = \bar{w}/\bar{q}^{\text{tr}}$. Then, the positive/negative part of flow tensor can be expressed as $\hat{n}^\pm =$

$\mathbf{I}^\pm : \hat{\mathbf{N}}$. Thus, their differential are given by

$$d\hat{n}^\pm = \frac{3}{2} \left(\bar{a}_4 \hat{t}_\pm^{\text{tr}} d\bar{q}^{\text{tr}} + \bar{u} \hat{\mathbf{I}}^\pm : d\hat{\mathbf{s}}^{\text{tr}} + \bar{a}_5 \hat{t}_\pm^{\text{tr}} d\Delta\gamma \right), \quad (2.4.143)$$

where $\hat{t}_\pm^{\text{tr}} = \hat{\mathbf{I}}^\pm : \hat{\mathbf{t}}^{\text{tr}}$. Next, the differential of tensor $\hat{\mathbf{B}}_0$, given by Eq. (2.3.98), is expressed as $d\hat{\mathbf{B}}_0 = 3\mu d(\bar{w}\hat{\mathbf{t}}) = 2\mu d\hat{\mathbf{N}}$. Then, using this relation and Eq. (2.3.98), the differential of the updated principal effective stress tensor $\hat{\boldsymbol{\sigma}}$ is given by

$$\begin{aligned} d\hat{\boldsymbol{\sigma}} &= d\hat{\boldsymbol{\sigma}}^{\text{tr}} - \hat{\mathbf{B}}_0 d\Delta\gamma - \Delta\gamma d\hat{\mathbf{B}}_0 \\ &= d\hat{\boldsymbol{\sigma}}^{\text{tr}} + \mathbf{A}_6 d\Delta\gamma + \mathbf{A}_7 d\bar{q}^{\text{tr}} + \bar{a}_8 d\hat{\mathbf{s}}^{\text{tr}}, \end{aligned} \quad (2.4.144)$$

where $\mathbf{A}_6 = -\hat{\mathbf{B}}_0 - 3\mu\bar{a}_5\Delta\gamma\hat{\mathbf{t}}^{\text{tr}}$, $\mathbf{A}_7 = -3\mu\bar{a}_4\Delta\gamma\hat{\mathbf{t}}^{\text{tr}}$ and $\bar{a}_8 = -3\mu\bar{u}\Delta\gamma$. Moreover, the differential of the maximum principal effective stress $\hat{\sigma}_+ = \hat{\mathbf{I}}^+ : \hat{\boldsymbol{\sigma}}$ is given by

$$d\hat{\sigma}_+ = \hat{\mathbf{I}}^+ : d\hat{\boldsymbol{\sigma}}^{\text{tr}} + a_{6+} d\Delta\gamma + a_{7+} d\bar{q}^{\text{tr}} + \bar{a}_8 \hat{\mathbf{I}}^+ : d\hat{\mathbf{s}}^{\text{tr}}, \quad (2.4.145)$$

where $a_{6+} = \hat{\mathbf{I}}^+ : \mathbf{A}_6$ and $a_{7+} = \hat{\mathbf{I}}^+ : \mathbf{A}_7$. For the other hand, the differential of variable $\phi(\hat{\boldsymbol{\sigma}})$ (Eq. (2.1.17)) is written as $d\phi = \hat{\boldsymbol{\Phi}} : d\hat{\boldsymbol{\sigma}}$, with $\hat{\boldsymbol{\Phi}}$ defined as

$$\hat{\boldsymbol{\Phi}} := \frac{\partial\phi}{\partial\hat{\boldsymbol{\sigma}}} = \text{diag} \left(\frac{\partial\phi}{\partial\hat{\sigma}_1}, \dots, \frac{\partial\phi}{\partial\hat{\sigma}_N} \right), \quad (2.4.146)$$

being their i -th component $\frac{\partial\phi}{\partial\hat{\sigma}_i}$ expressed as

$$\frac{\partial\phi}{\partial\hat{\sigma}_i} = [H_0^+(\hat{\sigma}_i) - \phi(\hat{\boldsymbol{\sigma}}) (2H_0^+(\hat{\sigma}_i) - 1)] \frac{1}{\sum_{i=1}^n |\hat{\sigma}_i|}$$

It should be noted, that this expression considered the stepped Heaviside function, due that variable $\phi \in [0, 1]$. It can observed that this condition not cause convergence troubles in the model. Then, the differential of variables $\theta_1^\pm$ and $\theta_2^\pm$ (Eqs. (2.3.105) and (2.3.106), respectively) are given by $d\theta_1^\pm = d\phi$ and $d\theta_2^\pm = \frac{J_\kappa^\pm}{g^\pm} d\kappa^\pm$, with $J_\kappa^\pm := \frac{\partial\sigma^\pm}{\partial\kappa^\pm}$ are the positive/negative hardening modulus, respectively. Hence, the differential of variables $\varphi^\pm$,

defined in Eq. (2.3.104), are given by

$$d\varphi^\pm = \theta_2^\pm \hat{\Phi} : d\hat{\sigma} + \frac{1}{g^\pm} \theta_1^\pm J_\kappa^\pm d\kappa^\pm. \quad (2.4.147)$$

Moreover, using this relation, the differential of variables $h^\pm$, defined in Eq. (2.3.103), are expressed as

$$dh^\pm = \theta_2^\pm \hat{n}^\pm \hat{\Phi} : d\hat{\sigma} + \hat{b}_{10}^\pm d\kappa^\pm + \varphi^\pm d\hat{n}^\pm, \quad (2.4.148)$$

where $\hat{b}_{10}^\pm = \frac{1}{g^\pm} \theta_1^\pm J_\kappa^\pm \hat{n}^\pm$. Thus, using Eq. (2.4.143), Eq. (2.4.144) and Eq. (2.4.148), the differential of positive/negative hardening variables $\kappa^\pm$, given by Eq. (2.3.103), can be written as

$$d\kappa^\pm = \hat{c}_1^\pm d\Delta\gamma + \Delta\gamma \left(\hat{c}_2^\pm d\bar{q}^{\text{tr}} + \hat{C}_3^\pm : d\hat{s}^{\text{tr}} + \hat{C}_4^\pm : d\hat{\sigma}^{\text{tr}} + \hat{b}_{10}^\pm d\kappa^\pm \right), \quad (2.4.149)$$

with $\hat{c}_1^\pm$, $\hat{c}_2^\pm$, $\hat{C}_3^\pm$ and $\hat{C}_4^\pm$ are expressed as

$$\begin{aligned} \hat{c}_1^\pm &= h^\pm + \Delta\gamma \left(\hat{C}_4^\pm : \mathbf{A}_6 + \frac{3}{2} \bar{a}_5 \varphi^\pm \hat{t}_\pm^{\text{tr}} \right), & \hat{c}_2^\pm &= \hat{C}_4^\pm : \mathbf{A}_7 + \frac{3}{2} \bar{a}_4 \varphi^\pm \hat{t}_\pm^{\text{tr}}, \\ \hat{C}_3^\pm &= \bar{a}_8 \hat{C}_4^\pm + \frac{3}{2} \bar{u} \varphi^\pm \hat{\mathbf{I}}^\pm, & \hat{C}_4^\pm &= \theta_2^\pm \hat{n}^\pm \hat{\Phi}. \end{aligned}$$

Hence, solving this linear equation for the differential of variable $\kappa^\pm$ gives

$$d\kappa^\pm = c_1^\pm d\Delta\gamma + \Delta\gamma \left(c_2^\pm d\bar{q}^{\text{tr}} + \mathbf{C}_4^\pm : d\hat{\sigma}^{\text{tr}} + \mathbf{C}_3^\pm : d\hat{s}^{\text{tr}} \right), \quad (2.4.150)$$

where $c_1^\pm$, $c_2^\pm$, $\mathbf{C}_3^\pm$ and $\mathbf{C}_4^\pm$ are multiple of their respective variables $\hat{c}_1^\pm$, $\hat{c}_2^\pm$, $\hat{C}_3^\pm$ and $\hat{C}_4^\pm$ by a factor of $\hat{b}_{20}^\pm = (1 - \Delta\gamma \hat{b}_{10}^\pm)^{-1}$. In addition, the differential of uniaxial positive/negative effective stress law $\bar{\sigma}^\pm$ are expressed as

$$d\bar{\sigma}^\pm = \bar{J}_\kappa^\pm d\kappa^\pm, \quad (2.4.151)$$

where $\bar{J}_\kappa^\pm := \frac{\partial \bar{\sigma}^\pm}{\partial \kappa^\pm}$ denotes the positive/negative effective hardening modulus, respectively. Taking the derivative of Eq. (2.1.29) with respect to the hardening variables $\kappa^\pm$, gives a

expression for this modulus

$$\bar{J}_\kappa^\pm = \frac{J_\kappa^\pm + \Omega_\kappa^\pm \bar{\sigma}^\pm}{1 - \omega^\pm}, \quad (2.4.152)$$

with $J_\kappa^\pm := \frac{\partial \sigma^\pm}{\partial \kappa^\pm}$ and $\Omega_\kappa^\pm := \frac{\partial \omega^\pm}{\partial \kappa^\pm}$. Moreover, the differential of variable c (Eq. (2.3.107)) is given by $dc = d\bar{\sigma} = \bar{J}_\kappa^- d\kappa^-$. Hence, using Eq. (2.4.151), the differential of updated variable β (Eq. (2.3.107)) can be written as

$$d\beta = c_4 \Delta\gamma + \Delta\gamma (c_5 d\bar{q}^{\text{tr}} + \mathbf{C}_6 : d\hat{\boldsymbol{\sigma}}^{\text{tr}} + \mathbf{C}_7 : \hat{\mathbf{s}}^{\text{tr}}), \quad (2.4.153)$$

where c_4 , c_5 , $\mathbf{C}_6$ and $\mathbf{C}_7$ are expressed as

$$\begin{aligned} c_4 &= m^+ c_1^- - m^- c_1^+, & c_5 &= m^+ c_2^- - m^- c_2^+, \\ \mathbf{C}_6 &= m^+ \mathbf{C}_4^- - m^- \mathbf{C}_4^+, & \mathbf{C}_7 &= m^+ \mathbf{C}_3^- - m^- \mathbf{C}_3^+, \end{aligned}$$

with $m^\pm = (1 - \alpha) \bar{J}_\kappa^\mp \frac{\bar{\sigma}^\pm}{(c^+)^2}$. Next, using Eqs. (2.4.133) and (2.4.136) in the effective space an Eqs. (2.4.150), (2.4.151), (2.4.153) and (A.1.24), the differential of yield criterion at consistency condition, given by Eq. (2.3.108), is written as

$$\begin{aligned} d\bar{F} &= \eta d\bar{p} + d\bar{q} + d\beta \langle \hat{\sigma}_1 \rangle^+ + \beta d\langle \hat{\sigma}_1 \rangle^+ - \delta d\langle \hat{\sigma}_1 \rangle^- - (1 - \alpha) dc = 0 \\ &= \eta K \mathbf{I} : d\boldsymbol{\epsilon}^{e\text{tr}} - g_0 d\Delta\gamma + g_1 d\bar{q}^{\text{tr}} + \mathbf{G}_2 : d\hat{\boldsymbol{\sigma}}^{\text{tr}} + \mathbf{G}_3 : d\hat{\mathbf{s}}^{\text{tr}}, \end{aligned} \quad (2.4.154)$$

where g_0 , g_1 , $\mathbf{G}_2$ and $\mathbf{G}_3$ are expressed as

$$\begin{aligned} g_0 &= \eta \bar{\eta} K + 3\mu \bar{a}_1 \bar{w} - \langle \hat{\sigma}_+ \rangle^+ c_4 + (1 - \alpha) \bar{J}_\kappa^- c_1^- - \hat{\rho}_3 a_{6+}, \\ g_1 &= \bar{a}_1 + \Delta\gamma [\langle \hat{\sigma}_+ \rangle^+ c_5 - (1 - \alpha) \bar{J}_\kappa^- c_2^-] + \hat{\rho}_3 a_{7+}, \\ \mathbf{G}_2 &= \Delta\gamma [\langle \hat{\sigma}_+ \rangle^+ \mathbf{C}_6 - (1 - \alpha) \bar{J}_\kappa^- \mathbf{C}_4^-] + \hat{\rho}_3 \hat{\mathbf{I}}^+, \\ \mathbf{G}_3 &= \Delta\gamma [\langle \hat{\sigma}_+ \rangle^+ \mathbf{C}_7 - (1 - \alpha) \bar{J}_\kappa^- \mathbf{C}_3^-] + \bar{a}_8 \hat{\rho}_3 \hat{\mathbf{I}}^+, \end{aligned}$$

where $\hat{\rho}_3 = \hat{\rho}_2 \hat{\sigma}_1 + \hat{\rho}_1$, with $\hat{\rho}_1 = \beta \tilde{H}^+(\hat{\sigma}_+) - \delta \tilde{H}^-(\hat{\sigma}_+)$ and $\hat{\rho}_2 = \beta \frac{\partial \tilde{H}^+}{\partial \hat{\sigma}_+} - \delta \frac{\partial \tilde{H}^-}{\partial \hat{\sigma}_+}$. Then, using Eqs. (2.4.132), (A.1.19) and (A.1.20), the differential of discrete consistency

operator $\Delta\gamma$ can be solved of Eq. (2.4.154) as

$$\begin{aligned} d\Delta\gamma &= \frac{1}{g_0} \left(\eta K \mathbf{I} : d\boldsymbol{\varepsilon}^{e\text{tr}} + g_1 d\bar{q}^{\text{tr}} + \mathbf{G}_2 : d\hat{\boldsymbol{\sigma}}^{\text{tr}} + \mathbf{G}_3 : d\hat{\mathbf{s}}^{\text{tr}} \right) \\ &= \frac{1}{g_0} \left(\eta K \mathbf{I} + g_1 \sqrt{6} \mu \bar{\mathbf{M}} + \mathbf{G}_4 + 2\mu \mathbf{G}_5 \right) : d\boldsymbol{\varepsilon}^{e\text{tr}} = \bar{\mathbf{G}} : d\boldsymbol{\varepsilon}^{e\text{tr}}, \end{aligned} \quad (2.4.155)$$

where $\mathbf{G}_4 = \mathbf{G}_2 : \mathcal{F}_{\bar{\sigma}} : \mathbf{D}_e$ and $\mathbf{G}_5 = \mathbf{G}_3 : \mathcal{F}_{\bar{\sigma}} : \mathbf{I}_d$ are second-order tensors. Thus, using Eqs. (2.4.135) and (2.4.136) in effective space and Eq. (2.4.155), the differential of updated effective stress tensor can be expressed as

$$\begin{aligned} d\bar{\boldsymbol{\sigma}} &= \left[2\mu f_{dev} \mathbf{I}_d + \sqrt{6} \mu a_3 (\bar{\mathbf{M}} \otimes \bar{\mathbf{G}}) + 6\mu^2 a_4 \Delta\gamma (\bar{\mathbf{M}} \otimes \bar{\mathbf{M}}) \right. \\ &\quad \left. + K (\mathbf{I} \otimes \mathbf{I}) - \bar{\eta} K (\mathbf{I} \otimes \bar{\mathbf{G}}) \right] : d\boldsymbol{\varepsilon}^{e\text{tr}}. \end{aligned} \quad (2.4.156)$$

Finally, introducing the left side of Eq. (2.4.155) into this relation, the effective elasto-plastic consistent tangent tensor is written as

$$\begin{aligned} \bar{\mathcal{D}}_{ep} &= c_1 \mathbf{I}_d + c_2 (\bar{\mathbf{M}} \otimes \bar{\mathbf{M}}) + c_3 (\bar{\mathbf{M}} \otimes \mathbf{I}) + c_4 (\mathbf{I} \otimes \bar{\mathbf{M}}) \\ &\quad + c_5 (\mathbf{I} \otimes \mathbf{I}) + c_6 (\bar{\mathbf{M}} \otimes \mathbf{G}_4) + c_7 (\mathbf{I} \otimes \mathbf{G}_4) \\ &\quad + c_8 (\bar{\mathbf{M}} \otimes \mathbf{G}_5) + c_9 (\mathbf{I} \otimes \mathbf{G}_5), \end{aligned} \quad (2.4.157)$$

where c_1 to c_9 are constants given by

$$\begin{aligned} c_1 &= 2\mu f_{dev}, & c_2 &= -6\mu^2 (\bar{a}_3 + \bar{a}_4 \Delta\gamma), & c_3 &= -\sqrt{6} \mu \bar{a}_3 g_0, \\ c_4 &= -\sqrt{6} \mu \bar{\eta} K g_1, & c_5 &= K (1 - \bar{\eta} g_0), & c_6 &= -\sqrt{6} \mu \bar{a}_3, \\ c_7 &= -\bar{\eta} K, & c_8 &= -2\sqrt{6} \mu^2 \bar{a}_3, & c_9 &= -2\mu \bar{\eta} K. \end{aligned}$$

Damage component

First, calling the variables $t_c^+ = -z_c^+$ and $t_c^- = z_c^-$, the differential of stiffness recovery functions $s^\pm$, defined in Eq. (2.1.28), are expressed as $ds^\pm = t_c^\pm d\theta_1^\pm = t_c^\pm d\phi$. Also, the differential of uniaxial damage laws $\omega^\pm$ are given by $d\omega^\pm = \Omega_\kappa^\pm d\kappa^\pm$, where $\Omega_\kappa^\pm :=$

$\frac{\partial \omega^\pm}{\partial \kappa^\pm}$. Then, expressing both relations in a vectorized format as $\underline{s} = [s^+, s^-]^T$ and $\underline{\omega} = [\omega^+, \omega^-]^T$, their differentials are written as

$$d\underline{s} = \hat{\underline{M}}_1 : d\hat{\underline{\sigma}}, \quad d\underline{\Omega} = \underline{\underline{M}}_2 d\underline{\kappa}, \quad (2.4.158)$$

where $\hat{\underline{M}}_1$ is a three-order tensor and $\underline{\underline{M}}_2$ a matrix, both expressed as

$$\hat{\underline{M}}_1 = \begin{bmatrix} t_c^+ \\ t_c^- \end{bmatrix} \otimes \hat{\underline{\Phi}}, \quad \underline{\underline{M}}_2 = \text{diag}(\Omega_\kappa^+, \Omega_\kappa^-).$$

For other hand, introducing Eqs. (2.4.132), (A.1.19) and (A.1.20), but expressed in the effective space, the differential of hardening variables $\kappa^\pm$ in a vectorized format $d\underline{\kappa} = [d\kappa^+, d\kappa^-]^T$ can be written as

$$\begin{aligned} d\underline{\kappa} &= \underline{c}_1 d\Delta\gamma + \Delta\gamma (\underline{c}_2 d\bar{q}^{\text{tr}} + \underline{C}_4 : d\hat{\underline{\sigma}}^{\text{tr}} + \underline{C}_3 : d\hat{\underline{s}}^{\text{tr}}) \\ &= \underline{c}_1 d\Delta\gamma + \Delta\gamma \underline{K} : d\epsilon^{\text{e tr}}, \end{aligned} \quad (2.4.159)$$

where $\underline{K} = \sqrt{6}\mu (\underline{c}_2 \otimes \bar{\underline{M}}) + \underline{C}_4 : \underline{\mathcal{F}}\bar{\underline{\sigma}} : \underline{\mathcal{D}}_e + 2\mu \underline{C}_3 : \underline{\mathcal{F}}\bar{\underline{\sigma}} : \underline{\mathcal{I}}_d$ is a three-order tensor; $\underline{c}_1$ and $\underline{c}_2$ are vectors and $\underline{C}_3$ and $\underline{C}_4$ are three-order tensors, both expressed as

$$\underline{c}_1 = \begin{bmatrix} c_1^+ \\ c_1^- \end{bmatrix}, \quad \underline{c}_2 = \begin{bmatrix} c_2^+ \\ c_2^- \end{bmatrix}, \quad \underline{C}_4 = [\underline{C}_4^+, \underline{C}_4^-], \quad \underline{C}_3 = [\underline{C}_3^+, \underline{C}_3^-].$$

Next, the differential of damage variable ω , given by Eq. (2.1.27), can be expressed as

$$d\omega = \underline{u}_1 \cdot d\underline{s} + \underline{u}_2 \cdot d\underline{\Omega}, \quad (2.4.160)$$

where $\underline{u}_1$ and $\underline{u}_2$ are vectors given by

$$\underline{u}_1 = \begin{bmatrix} s_1 \omega^- \\ s_2 \omega^+ \end{bmatrix}, \quad \underline{u}_2 = \begin{bmatrix} s_2 s^- \\ s_1 s^+ \end{bmatrix}.$$

In addition, substituting Eqs. (2.4.158), (2.4.159) and (A.1.19) (this later expressed in the effective space) and the relation $d\bar{\sigma} = \bar{\mathcal{D}}_{ep} : d\epsilon^{e\text{tr}}$, with $\bar{\mathcal{D}}_{ep}$ given by Eq. (2.4.157), into Eq. (2.4.160), the differential of damage variable ω can be rewritten as

$$\begin{aligned} d\omega &= \mathbf{V}_1 : d\hat{\bar{\sigma}} + \underline{v}_2 \cdot d\underline{\kappa} \\ &= [\mathbf{V}_1 : \mathcal{F}_{\bar{\sigma}} : \bar{\mathcal{D}}_{ep} + \underline{v}_2 \cdot (\underline{c}_1 \otimes \bar{\mathbf{G}} + \Delta\gamma \mathbf{K})] : \epsilon^{e\text{tr}}, \end{aligned} \quad (2.4.161)$$

where $\mathbf{V}_1 = \underline{u}_1 \cdot \hat{\mathbf{M}}_1$ is a second-order tensor and $\underline{v}_2 = \hat{\mathbf{M}}_2 \cdot \underline{u}_2$ a vector. For the other hand, the differential of updated stress tensor, given by Eq. (2.1.12), is expressed as

$$d\sigma = -\bar{\sigma}d\omega + (1 - \omega)d\bar{\sigma}. \quad (2.4.162)$$

Finally, introducing them Eq. (2.4.161) and the relation $d\bar{\sigma} = \bar{\mathcal{D}}_{ep} : d\epsilon^{e\text{tr}}$ (Eq. (2.4.157)) into this relation, the elasto-plastic-damage consistent tangent stiffness tensor is written as

$$\mathcal{D}_{epd} = [(1 - \omega)\mathcal{I} - (\bar{\sigma} \otimes \mathbf{V}_1) : \mathcal{F}_{\bar{\sigma}}] : \bar{\mathcal{D}}_{ep} - (\bar{\sigma} \otimes \underline{v}_2) \cdot (\underline{c}_1 \otimes \bar{\mathbf{G}} + \Delta\gamma \mathbf{K}). \quad (2.4.163)$$

Viscous component

Using Eq. (2.3.116) and the relation $d\bar{\sigma} = \bar{\mathcal{D}}_{ep} : d\epsilon^{e\text{tr}}$ into this relation (Eq. (2.4.157)), the differential of updated effective viscous-stress tensor σ^v can be expressed as

$$d\bar{\sigma}^v = (\zeta_v \bar{\mathcal{D}}_e + (1 - \zeta_v) \bar{\mathcal{D}}_{ep}) : d\epsilon^{e\text{tr}}. \quad (2.4.164)$$

Moreover, using Eq. (2.3.115), the differential of the visco-damage variable ω^v is given by

$$d\omega^v = (1 - \zeta_v)d\omega. \quad (2.4.165)$$

Moreover, the differential of updated viscous stress tensor σ^v (Eq. (2.3.117)) can be written as

$$d\sigma^v = (1 - \omega^v)d\bar{\sigma}^v - \bar{\sigma}^v d\omega^v. \quad (2.4.166)$$

Finally, substituting Eqs. (2.4.164), (2.4.165) and (2.4.161) into this relation, the viscoplastic-damage consistent tangent stiffness tensor is expressed as

$$D_{vpd} = \zeta_v(1 - \omega^v)\mathcal{D}_e + (1 - \zeta_v) \left\{ [(1 - \omega^v)\mathcal{I} - (\bar{\boldsymbol{\sigma}}^v \otimes \mathbf{V}_1) : \mathcal{F}_{\bar{\boldsymbol{\sigma}}}] : \bar{\mathcal{D}}_{ep} - (\bar{\boldsymbol{\sigma}}^v \otimes \underline{\mathbf{v}}_2) \cdot (\underline{\mathbf{c}}_1 \otimes \bar{\mathbf{G}} + \Delta\gamma\mathbf{K}) \right\}. \quad (2.4.167)$$

2.4.4. WLF model

Plastic component

This component is identical to the LLF model, with the exception that $\theta_2^\pm = 1$ and $\delta = 0$ (Eqs. (2.3.106) and (2.3.108), respectively). Thus, $d\theta_2^\pm = 0$, $\hat{b}_{10}^\pm = 0$ and $\hat{\rho}_i = \hat{\beta}_i$, with $i = 1, 3$.

Damage component

Using Eq. (A.1.18), the differential of positive/negative part of the effective stress tensor (Eq. (2.1.35)) are given by

$$\begin{aligned} d\bar{\boldsymbol{\sigma}}^\pm &= \sum_{i=1}^N H_0^\pm(\hat{\sigma}_i) \mathbf{E}_{\bar{\boldsymbol{\sigma}}}^{ii} d\hat{\sigma}_i + \sum_{i=1}^N \langle \hat{\sigma}_i \rangle^\pm d\mathbf{E}_{\bar{\boldsymbol{\sigma}}}^{ii} \\ &= \left(\sum_{i=1}^N H_0^\pm(\hat{\sigma}_i) (\mathbf{E}_{\bar{\boldsymbol{\sigma}}}^{ii} \otimes \mathbf{E}_{\bar{\boldsymbol{\sigma}}}^{ii}) + 2 \sum_{i=1, j>i}^N g_{ij}^\pm (\mathbf{E}_{\bar{\boldsymbol{\sigma}}}^{ij} \otimes \mathbf{E}_{\bar{\boldsymbol{\sigma}}}^{ij}) \right) : d\bar{\boldsymbol{\sigma}} = \mathcal{S}^\pm : d\bar{\boldsymbol{\sigma}}, \end{aligned} \quad (2.4.168)$$

where $g_{ij}^\pm$ is defined as

$$g_{ij}^\pm := \begin{cases} \frac{\langle \hat{\sigma}_i \rangle^\pm - \langle \hat{\sigma}_j \rangle^\pm}{(\hat{\sigma}_i^\pm - \hat{\sigma}_j^\pm)}, & \hat{\sigma}_i \neq \hat{\sigma}_j \\ H_0^\pm(\hat{\sigma}_i), & \hat{\sigma}_i = \hat{\sigma}_j. \end{cases}$$

So, during an active damage process, the Eq. (2.1.56) is satisfied. Then, using the chain rule, the differential of updated positive/negative damage law $\omega^\pm$ are expressed as

$$d\omega^\pm = \frac{\partial \omega^\pm}{\partial r^\pm} dr^\pm = \Omega_r^\pm dY^\pm, \quad (2.4.169)$$

where $\Omega_r^\pm := \frac{\partial \omega^\pm}{\partial r^\pm}$. Then, the differential of positive/negative DEER, $Y^\pm$, are calculated according to their definition. Thus, using Eqs. (2.1.50) to (2.4.168) and using the stepped Heaviside function $H_0^\pm(\cdot)$, their respective differentials are given by

$$dY^\pm = \frac{E_o}{2Y^\pm} (\bar{\boldsymbol{\sigma}} : \mathbf{C}_e : \mathbf{S}^\pm + \bar{\boldsymbol{\sigma}}^\pm : \mathbf{C}_e) : d\bar{\boldsymbol{\sigma}} = \mathbf{L}^\pm : d\bar{\boldsymbol{\sigma}}, \quad (2.4.170)$$

$$dY^- = \left(\alpha \mathbf{1} + \frac{3}{2\bar{q}} \bar{\mathbf{s}} + \delta H_0^-(\hat{\sigma}_1) \mathbf{E}_{\bar{\boldsymbol{\sigma}}}^+ \right) : d\bar{\boldsymbol{\sigma}} = \mathbf{L}^- : d\bar{\boldsymbol{\sigma}}, \quad (2.4.171)$$

where $\mathbf{E}_{\bar{\boldsymbol{\sigma}}}^+$ is the eigen-projector associated to the maximum principal effective stress $\hat{\sigma}_+$. Next, using Eq. (2.1.43), the differential of updated stress tensor is given by

$$d\boldsymbol{\sigma} = \sum_{\aleph} [(1 - \omega^\aleph) d\bar{\boldsymbol{\sigma}}^\aleph - \bar{\boldsymbol{\sigma}}^\aleph d\omega^\aleph]. \quad (2.4.172)$$

Finally, introducing Eqs. (2.4.169), (2.4.168) and (2.4.170) (or Eq. (2.4.171)) into this expression and using the relation $\bar{\boldsymbol{\sigma}} = \bar{\mathcal{D}}_{ep} : \boldsymbol{\varepsilon}^{e\text{tr}}$ (Eq. (2.4.157)), which considers the observations mentioned in the plastic component of this model, the plastic-damage consistent tangent stiffness tensor is written as

$$\mathcal{D}_{pd} = \left[\mathcal{I} - \sum_{\aleph} (\mathcal{W}^\aleph + \mathcal{R}^\aleph) \right] : \bar{\mathcal{D}}_{ep}, \quad (2.4.173)$$

where $\mathcal{W}^\pm = \omega^\pm \mathbf{S}^\pm$ and $\mathcal{R}^\pm = \Omega_r^\pm (\bar{\boldsymbol{\sigma}}^\pm \otimes \mathbf{L}^\pm)$ are fourth-order tensors.

Viscous component

First, using Eq. (2.3.118), the differential of threshold variable $r^\pm$ is given by

$$dr^\pm = (1 - \zeta_v) dY^\pm. \quad (2.4.174)$$

Then, using this relation, Eq. (2.4.170) or Eq. (2.4.171) and the chain rule, the differential of positive/negative damage variables $\omega^\pm$ are given by

$$d\omega^\pm = \frac{\partial \omega^\pm}{\partial r^\pm} dr^\pm = \Omega_r^\pm (1 - \zeta_v) \mathbf{L}^\pm : d\bar{\boldsymbol{\sigma}}. \quad (2.4.175)$$

Moreover, the differential of positive/negative viscous stress tensor, given by Eq. (2.1.60), is expressed as

$$d\bar{\boldsymbol{\sigma}}^{v\pm} = \mathcal{S}_v^\pm : d\bar{\boldsymbol{\sigma}}^v, \quad (2.4.176)$$

where $\mathcal{S}_v^\pm$ are the derivative of positive/negative projector tensor of $\bar{\boldsymbol{\sigma}}^v$. Then, using this relation, the differential of viscous-stress tensor $\boldsymbol{\sigma}^v$ is expressed as

$$d\boldsymbol{\sigma}^v = \sum_{\aleph} [(1 - \omega^\aleph) \mathcal{S}_v^\aleph : d\bar{\boldsymbol{\sigma}}^v - \bar{\boldsymbol{\sigma}}^{v\aleph} d\omega^\aleph]. \quad (2.4.177)$$

Finally, substituting Eqs. (2.4.164) and (2.4.175) and the relation $d\bar{\boldsymbol{\sigma}} = \bar{\mathcal{D}}_{ep} : d\boldsymbol{\varepsilon}$ (Eq. (2.4.157)) into Eq. (2.4.177) and with some straightforward manipulation, the viscoplastic-damage consistent tangent stiffness tensor can be expressed as

$$D_{vpd} = \alpha_\mu \left(\mathcal{I} - \sum_{\aleph} \mathcal{W}_v^\aleph \right) : \mathcal{D}_e + (1 - \zeta_v) \left(\mathcal{I} - \sum_{\aleph} (\mathcal{W}_v^\aleph + \mathcal{R}_v^\aleph) \right) : \bar{\mathcal{D}}_{ep}, \quad (2.4.178)$$

where $\mathcal{W}_v^\pm = \omega^\pm \mathcal{S}_v^\pm$ and $\mathcal{R}_v^\pm = \Omega_r^\pm (\bar{\boldsymbol{\sigma}}^{v\pm} \otimes \mathbf{L}^\pm)$. It should be noted that the tensors $\mathbf{L}^\pm$ are calculated using inviscid variables.

2.4.5. FOC model

Plastic component

First, the differential of variables n_0 and n_1 , given by Eq. (2.3.124), are expressed as

$$dn_0 = 2 (\bar{\boldsymbol{\sigma}}^{\text{tr}} : d\bar{\boldsymbol{\sigma}}^{\text{tr}}), \quad dn_1 = \Delta \boldsymbol{\varepsilon} : d\bar{\boldsymbol{\sigma}}^{\text{tr}} + \bar{\boldsymbol{\sigma}}^{\text{tr}} : d\boldsymbol{\varepsilon}. \quad (2.4.179)$$

Also, assuming that the variable χ is constant during the plastic process and using Eq. (2.3.123), the differential of updated effective stress tensor $\bar{\sigma}$ is given by

$$d\bar{\sigma} = d\bar{\sigma}^{\text{tr}} - \frac{E_o\chi}{n_0} [n_0\bar{\sigma}^{\text{tr}}d\langle n_1 \rangle^+ + n_0\langle n_1 \rangle^+d\bar{\sigma}^{\text{tr}} - \langle n_1 \rangle^+\bar{\sigma}^{\text{tr}}dn_0]. \quad (2.4.180)$$

Then, using Eqs. (2.3.124), (A.1.1) and (A.1.24), taking the stepped Heaviside function $H_0^+(\cdot)$ and with some straightforward manipulation, the effective component of the consistent tangent stiffness tensor is given by

$$\bar{\mathcal{D}}_{ep} = [c_1\mathcal{I} + c_2(\bar{\sigma}^{\text{tr}} \otimes \Delta\epsilon) + (\bar{\sigma}^{\text{tr}} \otimes \bar{\sigma}^{\text{tr}}) : (c_3\mathcal{I} + c_2\mathcal{C}_e)] : \mathcal{D}_e, \quad (2.4.181)$$

where $c_1 = m^{\text{tr}}$, $c_2 = -(1 - m^{\text{tr}})/n_1$ and $c_3 = 2(1 - m^{\text{tr}})/n_0$.

2.4.6. ROT model

Damage component

First, the differential of i -th updated damage variable $\alpha_i^\pm$, stated in Eq. (2.3.126), can be expressed as

$$d\alpha_i^\pm = dz_i^\pm \Delta\hat{\epsilon}_i + z_i^\pm d\hat{\epsilon}_i = z_i^\pm d\hat{\epsilon}_i. \quad (2.4.182)$$

Next, the tangent and secant slope of positive/negative uniaxial stress-strain law can be defined as $K_i^\pm := \frac{\partial\sigma_i^\pm}{\partial\alpha_i^\pm}$ and $S_i^\pm := \sigma_i^\pm/\alpha_i^\pm$, respectively. Then, using this relation, the differential of updated variables $h_i^\pm$ and $g_i^\pm$, given by Eq. (2.3.127), are written as

$$dh_i^\pm = g_i^\pm K_i^\pm z_i^\pm d\hat{\epsilon}_i + \sigma_i^\pm dg_i^\pm \quad dg_i^\pm = \frac{1}{\alpha_i^\pm} (1 - g_i^\pm z_i^\pm) d\hat{\epsilon}_i. \quad (2.4.184)$$

Thus, using all these relations, the differential of i -th updated principal stress $\hat{\sigma}_i$, stated in Eq. (2.1.67), can be written as

$$d\hat{\sigma}_i = \left(\sum_{\mathbb{N}} m_i^{\mathbb{N}} [K_i^{\mathbb{N}} p_i^{\mathbb{N}} + S_i^{\mathbb{N}} (1 - p_i^{\mathbb{N}})] \right) d\hat{\epsilon}_i = J_{ii} d\hat{\epsilon}_i, \quad (2.4.185)$$

where $p_i^\pm = g_i^\pm z_i^\pm$. Moreover, using Eq. (A.1.17) and the relation $d\hat{\sigma}_i = \frac{\partial \hat{\sigma}_i}{\partial \hat{\epsilon}} : d\hat{\epsilon}$, this differential can be expressed as

$$\begin{aligned} d\hat{\sigma}_i &= \frac{\partial \hat{\sigma}_i}{\partial \hat{\epsilon}} : \frac{\partial \hat{\epsilon}}{\partial \epsilon} : d\epsilon = \mathbf{J}_i : \mathcal{F}_\epsilon : d\epsilon, \\ \mathbf{J}_i &= \text{diag} \left(\frac{\partial \hat{\sigma}_i}{\partial \hat{\epsilon}_1}, \dots, \frac{\partial \hat{\sigma}_i}{\partial \hat{\epsilon}_N} \right) = \text{diag} (J_{i1}, \dots, J_{iN}). \end{aligned}$$

Then, using this relation and Eq. (A.1.18), the differential of updated stress tensor σ , defined in Eq. (2.1.69), can be expressed as

$$d\sigma = \sum_{i=1}^N \left[(\mathbf{E}_\epsilon^{ii} \otimes \mathbf{J}_i) : \mathcal{F}_\epsilon + 2\hat{\sigma}_i \sum_{j \neq i}^N \frac{1}{(\hat{\epsilon}_i - \hat{\epsilon}_j)} (\mathbf{E}_\epsilon^{ij} \otimes \mathbf{E}_\epsilon^{ij}) \right] : d\epsilon. \quad (2.4.186)$$

Finally, with some algebraic manipulation, the damage consistent tangent stiffness tensor is written as

$$\mathcal{D}_d = \left(\sum_{i=1}^N (\mathbf{E}_\epsilon^{ii} \otimes \mathbf{J}_i) \right) : \mathcal{F}_\epsilon + 2 \sum_{i=1, j>i}^N g_{ij}^\epsilon (\mathbf{E}_\epsilon^{ij} \otimes \mathbf{E}_\epsilon^{ij}), \quad (2.4.187)$$

where g_{ij}^ϵ is defined as

$$g_{ij}^\epsilon := \begin{cases} \frac{(\hat{\sigma}_i - \hat{\sigma}_j)}{(\hat{\epsilon}_i - \hat{\epsilon}_j)}, & \hat{\epsilon}_i \neq \hat{\epsilon}_j \\ \frac{\partial \hat{\sigma}_i}{\partial \hat{\epsilon}_i} = J_{ii}, & \hat{\epsilon}_i = \hat{\epsilon}_j. \end{cases}$$

Note that the first term of right hand side is associated to local principal stiffness and the second term arises from rotation of principal strains. It can be demonstrated that, neglecting the damage variables, this expression is identical to obtained by (M. A. Crisfield & Wills, 1989). Also, note the similitude of the second term of this expression with Eq. (2.4.168).

Viscous component

First, using Eq. (2.4.182), the differential of i -th updated viscous damage strain, given by Eq. (2.3.128), can be expressed as

$$d\alpha_i^{v\pm} = (1 - \zeta_v) z_i^{\pm} d\hat{\varepsilon}_i. \quad (2.4.188)$$

Moreover, using this relation, the differential of updated variables $h_i^{v\pm}$ and $g_i^{v\pm}$ (Eq. (2.1.72)) are given by

$$dh_i^{v\pm} = g_i^{v\pm} K_i^{v\pm} z_i^{v\pm} d\hat{\varepsilon}_i + \sigma^{\pm} dg_i^{v\pm} \quad (2.4.189)$$

$$dg_i^{v\pm} = \frac{1}{\alpha_i^{v\pm}} [1 - (1 - \zeta_v) g_i^{v\pm} z_i^{v\pm}] d\hat{\varepsilon}_i, \quad (2.4.190)$$

where $K_i^{v\pm} := \frac{\partial \sigma^{v\pm}}{\partial \alpha_i^{v\pm}}$. Thus, using this relation and with some straightforward manipulation, the differential of i -th updated principal viscous stress, stated in Eq. (2.4.189), can be written as

$$\begin{aligned} d\hat{\sigma}_i^v &= \left(\zeta_v \sum_{\mathbb{N}} m_i^{\mathbb{N}} S_i^{v\mathbb{N}} + (1 - \zeta_v) \sum_{\mathbb{N}} m_i^{\mathbb{N}} [K_i^{v\mathbb{N}} p_i^{v\mathbb{N}} + S_i^{v\mathbb{N}} (1 - p_i^{v\mathbb{N}})] \right) d\hat{\varepsilon}_i \\ &= J_{ii}^v d\hat{\varepsilon}_i, \end{aligned} \quad (2.4.191)$$

where $p_i^{v\pm} = g_i^{v\pm} z_i^{\pm}$ and $S_i^{v\pm} := \sigma^{\pm} / \alpha_i^{v\pm}$. Finally, using this relation and Eq. (A.1.18), the viscous-damage consistent tangent stiffness tensor is given by

$$\mathcal{D}_{vd} = \left(\sum_{i=1}^N (\mathbf{E}_{\varepsilon}^{ii} \otimes \mathbf{J}_i^v) \right) : \mathcal{F}_{\varepsilon} + 2 \sum_{i=1, j>i}^N g_{ij}^{v\varepsilon} (\mathbf{E}_{\varepsilon}^{ij} \otimes \mathbf{E}_{\varepsilon}^{ij}), \quad (2.4.192)$$

where $\mathbf{J}_i^v = \text{diag}(J_{i1}^v, \dots, J_{iN}^v)$, with their j -th diagonal component given by $J_{ij}^v = \frac{\partial \hat{\sigma}_i}{\partial \hat{\varepsilon}_j}$, and $g_{ij}^{v\varepsilon}$ is expressed as

$$g_{ij}^{v\varepsilon} := \begin{cases} \frac{(\hat{\sigma}_i - \hat{\sigma}_j)}{(\hat{\varepsilon}_i - \hat{\varepsilon}_j)}, & \hat{\varepsilon}_i \neq \hat{\varepsilon}_j \\ \frac{\partial \hat{\sigma}_i}{\partial \hat{\varepsilon}_i} = J_{ii}^v, & \hat{\varepsilon}_i = \hat{\varepsilon}_j, \end{cases}$$

In addition, Table 2.4.2 shown the conditions for which the consistent tangent stiffness tensor is non-symmetrical in each one of models developed.

Table 2.4.2. Conditions for a non-symmetrical consistent tangent stiffness tensor.

Model	Non-symmetric condition
DPH	$\eta \neq \bar{\eta}$
LLF	always
WLF	$\nu > 0$
FOC	$\nu > 0$
ROT	never ¹

¹ Is non-symmetrical if the model is extended to include the biaxial effects.

2.5. Consistency check of input material parameters

This section are devoted to generate an conversion of input parameters among the concrete models described. Two key aspects are studied: conversion of uniaxial laws and conversion of fracture energy for the FE-regularization.

2.5.1. Conversion of uniaxial laws

Basically, the tensile/compression uniaxial stress-strain laws $\sigma^\pm - \varepsilon^\pm$ are the most known and adequate to fit with experimental concrete tests. However, for each concrete model used in this work a specific uniaxial law it required as input, as explained in Section 2.1 and summarized in Table 2.6.7. Thus, there is necessary generate a conversion from the uniaxial stress-strain laws to the required uniaxial law of each model. In addition, a conversion from $\omega - \alpha$ to $\omega - \kappa$ is explained.

Relation between $\sigma^\pm - \varepsilon^\pm$ and $\sigma^\pm - \alpha^\pm$

The uniaxial stress-strain $\sigma^\pm - \varepsilon^\pm$ laws can be converted to stress-equivalent plastic strain $\sigma^\pm - \alpha^\pm$ relation using the following expression

$$\varepsilon^\pm = \alpha^\pm + \frac{\sigma^\pm(\varepsilon^\pm)}{E_o}. \quad (2.5.193)$$

A simple conversion can be obtained in the case of piecewise-linear stress laws. In other case, an implicit expression exists for $\sigma^\pm(\alpha^\pm)$, in which given a value of $\alpha^\pm$, its necessary find their corresponding value of $\varepsilon^\pm$ that satisfy the relation of Eq. (2.5.193). Newton's method is suggested to solve this problem, where the residual function and their total derivative are expressed as

$$R(\varepsilon^\pm) = \varepsilon^\pm - \frac{\sigma^\pm(\varepsilon^\pm)}{E_o} - \alpha^\pm, \quad \frac{dR}{d\varepsilon^\pm} = 1 - \frac{J_\varepsilon^\pm}{E_o}, \quad (2.5.194)$$

with $J_\varepsilon^\pm := \frac{d\sigma^\pm}{d\varepsilon^\pm}$. An initial value of $\alpha^\pm + \sigma_o^\pm$ and a correction step of $\varepsilon^\pm = |\varepsilon^\pm|$ are necessary to achieve a good convergence of solution, where $\sigma_o^\pm$ is the stress onset non-linear behavior. Also, a tolerance of $Tol_1 = 10^{-10}$ is recommend to check the residual function. In addition, using Eq. (2.5.193), the derivative $\frac{d\sigma^\pm}{d\alpha^\pm}$ can be obtained as

$$\frac{d\sigma^\pm}{d\alpha^\pm} = \frac{E_o J_\varepsilon^\pm}{(E_o - J_\varepsilon^\pm)}. \quad (2.5.195)$$

Relation between $\sigma^\pm - \varepsilon^\pm$ and $\sigma^\pm - \kappa^\pm$

For the LLF model, this relation can be obtained according to the definition of hardening variables $\kappa^\pm$, where given a value of $\kappa^\pm$, the corresponding value of $\varepsilon^\pm$ is calculated. Thus, an implicit expression exist for $\sigma^\pm(\kappa^\pm)$ and can be solved by the Newton's method. If the LLF model is used, Eq. (2.1.13) gives the relation between $\kappa^\pm$ and $\alpha^\pm$ and Eq. (2.5.193) between $\alpha^\pm$ and $\varepsilon^\pm$, respectively. Hence, the residual function and their total

derivative are expressed as

$$\begin{aligned} R(\varepsilon^\pm) &= \frac{1}{g^\pm} \int_0^{\alpha^\pm} \sigma^\pm(\alpha^\pm) d\alpha^\pm - \kappa^\pm \\ &= \frac{1}{g^\pm} \left\{ \int_{\varepsilon_o^\pm}^{\varepsilon^\pm} \sigma^\pm(\varepsilon^\pm) d\varepsilon^\pm - \frac{1}{2E_o} [(\sigma^\pm(\varepsilon^\pm))^2 - (\sigma_o^\pm)^2] \right\} - \kappa^\pm \end{aligned} \quad (2.5.197)$$

$$\begin{aligned} &= \frac{1}{g^\pm} F(\varepsilon^\pm) - \kappa^\pm, \\ \frac{dR}{d\varepsilon^\pm} &= \frac{1}{g^\pm} \frac{dF}{d\varepsilon^\pm} = \frac{\sigma^\pm}{g^\pm} \left(1 - \frac{J_\varepsilon^\pm}{E_o} \right), \end{aligned} \quad (2.5.198)$$

where $\varepsilon_o^\pm = \sigma_o^\pm / E_o$. An initial value of $\varepsilon_o^\pm$ is suggested. For the other hand, Eq. (2.1.15) can be rewritten as $\frac{d\alpha^\pm}{d\kappa^\pm} = \frac{g^\pm}{\sigma^\pm}$. Hence, using this relation, Eq. (2.5.195) and the chain rule, the derivative $\frac{d\sigma^\pm}{d\kappa^\pm}$ is given by

$$\frac{d\sigma^\pm}{d\kappa^\pm} = \frac{d\sigma^\pm}{d\alpha^\pm} \frac{d\alpha^\pm}{d\kappa^\pm} = \frac{E_o J_\varepsilon^\pm g^\pm}{(E_o - J_\varepsilon^\pm) \sigma^\pm}. \quad (2.5.199)$$

Relation between $\sigma^\pm - \varepsilon^\pm$ and $\omega^\pm - r^\pm$

In the WLF and FOC models, this relation can be established according to the definition of DEERs $Y^\pm$ used. Considering an active damage process ($Y^\pm = r^\pm$), neglecting the plastic strains and under an uniaxial behavior, a linear relation for the positive/negative effective stress tensor can be derived as $\bar{\sigma}^\pm = E_o \varepsilon$, where $\bar{\sigma}^\pm$ is the effective uniaxial law and ε the uniaxial total strain. Then, using Eqs. (2.1.50) and (2.1.51), the DEERs $Y^\pm$ can be expressed, respectively, as

$$Y^\pm = |\bar{\sigma}^\pm| = E_o x^\pm = r^\pm, \quad (2.5.200)$$

$$Y^- = \alpha \bar{\sigma}^- + |\bar{\sigma}^-| = E_o x^- (1 - \alpha + \delta) = r^-, \quad (2.5.201)$$

where $x^\pm = |\varepsilon|$. Moreover, both expressions can be rewritten in a compact format as $r^\pm = E_o x^\pm \eta^\pm$, where $\eta^\pm = 1$ and $\eta^- = 1 - \alpha + \delta$ if Eqs. (2.1.50) and (2.1.51) are used, respectively. Hence, using the relation Eq. (2.1.29), the damage laws $\omega^\pm$ and their

derivative $\frac{d\omega^\pm}{dr^\pm}$ are expressed, respectively, as

$$\omega^\pm(r^\pm) = 1 - \frac{\eta^\pm}{r^\pm} \sigma^\pm \left(\frac{r^\pm}{\eta^\pm E_o} \right), \quad \frac{d\omega^\pm}{dr^\pm} = \left[\sigma^\pm \left(\frac{r^\pm}{\eta^\pm E_o} \right) - \frac{r^\pm J_\varepsilon}{\eta^\pm E_o} \right] \frac{\eta^\pm}{(r^\pm)^2}. \quad (2.5.202)$$

Relation between $\sigma^\pm - \varepsilon^\pm$ and $c - \alpha$

To elaborate this conversion, first is necessary relate the parameters ξ and η with the uniaxial/biaxial compression/tensile strength of concrete. Table 2.5.3 list two options to fit this parameters, where $f_y^\pm$ and $f_{yb}^\pm$ are the stress onset non-linear behavior of tension/compression uniaxial/biaxial stress-strain law, respectively.

For one hand, if an elasto-plastic stress-strain law relationship exist, for the Case A, the parameters $f_y^\pm$ are equals to the uniaxial tensile/compressive $f_p^\pm$ strength of concrete ($=f_t'$ or f_c' , respectively), whereas for the Case B, the parameters $f_{yb}^\pm$ are equals to their respective biaxial $f_{pb}^\pm$ strength ($=f_{tt}'$ or f_b' , respectively). Then, the conversion is trivial, i.e. $c = c_y$. Conversely, in a nonlinear stress-strain relation, only the pre-peak branch of stress-strain law can be converted to respective cohesion law due that this last relation must be something non-decreasing function ($\frac{\partial c}{\partial \alpha} > 0$).

Table 2.5.3. Analytical expressions of parameters η , ξ and c_y of Drucker-Prager model fitted with different approximations.

Case	Inputs	η	ξ	c_y
A	f_y^+, f_y^-	$3 \sin(\phi)^{(*)}$	$2 \cos(\phi)$	$f_y^+ f_y^- \tan(\phi) / (f_y^- - f_y^+)$
B	f_{yb}^+, f_{yb}^-	$\frac{3}{2} \sin(\phi)^{(**)}$	$2 \cos(\phi)$	$f_{yb}^+ f_{yb}^- \tan(\phi) / (f_{yb}^- - f_{yb}^+)$

Cases= **A**: fitted with the stress onset non-linear behavior of uniaxial tension/compression stress-strain relation of concrete f_y^+ and f_y^- , respectively and **B**: fitted with the stress onset non-linear behavior of biaxial tension/compression stress-strain relation of concrete f_{yb}^+ and f_{yb}^- , respectively. $(*)$: $\phi = \sin^{-1}((f_y^- - f_y^+) / (f_y^+ + f_y^-))$, $(**)$: $\phi = \sin^{-1}((f_{yb}^- - f_{yb}^+) / (f_{yb}^+ + f_{yb}^-))$.

Relation between $\omega^\pm - \alpha^\pm$ and $\omega^\pm - \kappa^\pm$

This relation can be solved using the same methodology as used for the relation $\sigma^\pm - \varepsilon^\pm / \sigma^\pm - \kappa^\pm$. First, the variable $\varepsilon^\pm$ is calculated given a value of $\kappa^\pm$ using the above methodology. Hence, the variable $\alpha^\pm$ can be evaluated as $\alpha^\pm = \varepsilon^\pm - \frac{\sigma^\pm(\varepsilon^\pm)}{E_o}$ and then obtain the respective value of the damage variable $\omega^\pm(\alpha^\pm)$. Finally, using the relation $\frac{d\alpha^\pm}{d\kappa^\pm} = \frac{g^\pm}{\sigma^\pm}$ and the chain rule, the derivative $\frac{d\omega^\pm}{d\kappa^\pm}$ is given by

$$\frac{d\omega^\pm}{d\kappa^\pm} = \frac{d\omega^\pm}{d\alpha^\pm} \frac{d\alpha^\pm}{d\kappa^\pm} = \frac{\Omega_\alpha^\pm g^\pm}{\sigma^\pm}, \quad (2.5.203)$$

where $\Omega_\alpha^\pm := \frac{d\omega^\pm}{d\alpha^\pm}$. Finally, Table 2.5.4 resume the steps necessities to do all these conversions, with the exception of cohesion hardening law.

2.5.2. Conversion of fracture energy

Fracture energy FE-regularisation is an common technique that induce a length scale in the constitutive equations and that is able to remove the spurious mesh-dependency observed in the numerical simulations when strain-localization occurs. This method is based in the experimental evidence, where the energy dissipated to form a unit area of crack surface G_f are considered as a material property (Hillerborg et al., 1976; van Vliet & van Mier, 1995; Nakamura & Higai, 2001). This dissipated energy can be distinguished into tensile fracture energy G_f^+ (cracking) and compression fracture energy G_f^- (crushing).

(Bažant, 1982; Hillerborg et al., 1976) shown that the tensile/compression fracture energy $G_f^\pm$ are related with the FE-regularized energy per unit of volume $g_f^\pm = G_f^\pm / l_c$, where l_c is the characteristic length of FE element. The characteristic length l_c vary according the size and type of finite element used. Thus, for linear shell elements $l_c = \sqrt{2A}$, complex shell elements $l_c = \sqrt{A}$ and solid brick elements $l_c = \sqrt[3]{V}$, being A the area and V the volume of the finite element, respectively.

Table 2.5.4. Steps necessities for the conversion from uniaxial $\sigma - \varepsilon$ to $\sigma - \alpha$, $\sigma - \kappa$ and $\omega - r$, and from $\omega - \alpha$ to $\omega - \kappa$ laws and the respective transformation for their derivatives.

Unknown relation to be solved					
Known relation	$\sigma^\pm - \alpha^\pm$	$\sigma^\pm - \kappa^\pm$	$\omega^\pm - r^\pm$	$\omega^\pm - \kappa^\pm$	
Variable to solve, x	$\varepsilon^\pm$ (iterative)	$\varepsilon^\pm$ (iterative)	$\varepsilon^\pm = \frac{r^\pm}{E_o \eta^\pm}$	$\omega^\pm - \alpha^\pm$	$\varepsilon^\pm$ (iterative)
Initial value, x_0	$\alpha^\pm + \sigma_o^\pm$	$\varepsilon_o^\pm$	-		$\varepsilon_o^\pm$
Residual, R_j	$\varepsilon_j^\pm - \frac{\sigma_{\varepsilon j}^\pm}{E_o} - \alpha^\pm$	$\frac{1}{g^\pm} F_j^\pm - \kappa^\pm$	-		$\frac{1}{g^\pm} F_j^\pm - \kappa^\pm$
Total derivative, dR_j	$1 - \frac{J_{\varepsilon j}^\pm}{E_o}$	$\frac{\sigma_{\varepsilon j}^\pm}{g^\pm} \left(1 - \frac{J_{\varepsilon j}^\pm}{E_o}\right)$	-		$\frac{\sigma_{\varepsilon j}^\pm}{g^\pm} \left(1 - \frac{J_{\varepsilon j}^\pm}{E_o}\right)$
Update solution, x_{j+1}	$ \varepsilon_j^\pm - R_j/dR_j $	$ \varepsilon_j^\pm - R_j/dR_j $	-		$ \varepsilon_j^\pm - R_j/dR_j $
Evaluate	$\sigma^\pm(\varepsilon_{j+1}^\pm)$	$\sigma^\pm(\varepsilon_{j+1}^\pm)$	$1 - \frac{\eta^\pm}{r^\pm} \sigma_{rj}^\pm$		$\omega^\pm \left(\varepsilon_{j+1}^\pm - \frac{\sigma^\pm}{E_o}\right)$
Known relation	$\frac{d\sigma^\pm}{d\varepsilon^\pm}$	$\frac{d\sigma^\pm}{d\kappa^\pm}$	$\frac{d\omega^\pm}{dr^\pm}$		
$J_\varepsilon^\pm = \frac{d\sigma^\pm}{d\varepsilon^\pm}$	$\frac{E_o J_\varepsilon^\pm}{(E_o - J_\varepsilon^\pm)}$	$\frac{E_o J_\varepsilon^\pm g^\pm}{(E_o - J_\varepsilon^\pm) \sigma^\pm}$	$\left(\sigma_{rj}^\pm - \frac{r^\pm J_\varepsilon^\pm}{\eta^\pm E_o}\right) \frac{\eta^\pm}{(r^\pm)^2}$	$\Omega_\alpha^\pm = \frac{d\omega^\pm}{d\alpha^\pm}$	$\frac{\Omega_\alpha^\pm g^\pm}{\sigma^\pm}$

$$\sigma_{\varepsilon j}^\pm = \sigma^\pm(\varepsilon_j^\pm), \quad \sigma_{j\tau}^\pm = \sigma^\pm\left(\frac{r^\pm}{\eta^\pm E_o}\right), \quad J_{\varepsilon j}^\pm := \frac{d\sigma^\pm(\varepsilon_j^\pm)}{d\varepsilon_j^\pm}, \quad F_j = \int_{\varepsilon_o^\pm}^{\varepsilon_j^\pm} \sigma^\pm(x) dx - \frac{1}{2E_o} [(\sigma_{\varepsilon j}^\pm)^2 - (\sigma_o^\pm)^2],$$

(1) $\sigma_o^\pm$ is the stress onset non-linear behavior, (2) $\varepsilon_o^\pm = \sigma_o^\pm/E_o$.

(3) The parameter $\eta^\pm$ depends of DEER, $Y^\pm$, adopted in the model, where $\eta^\pm = 1$ and $\eta^- = 1 - \alpha + \delta$ if Eqs. (2.1.50) and (2.1.51) are used, respectively.

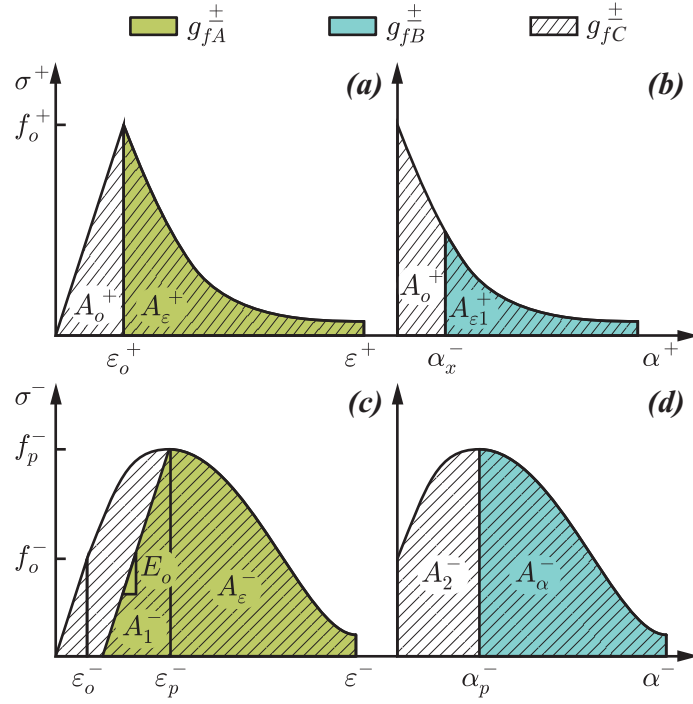


Figure 2.5.3. Schematic definition of FE-regularized concrete fracture energy $g_f^\pm = G_f^\pm / l_c$ in uniaxial stress laws: (a) $\sigma^+ - \varepsilon^+$; (b) $\sigma^+ - \alpha^+$; (c) $\sigma^- - \varepsilon^-$ and (d) $\sigma^- - \alpha^-$.

Three definitions of fracture energy are discussed: (i) tensile regimes; (ii) compression ones; and (iii) thermodynamically dissipation. Fig. 2.5.3 shown the graphical representation of these definitions in uniaxial stress laws, whose are explained in the following sections.

Tensile fracture energy

For a tensile regime, exist a broad consensus, available with several studies realized in the past (Hillerborg et al., 1976; Bažant, 1982), that shown that the energy dissipated per unit of volume in the post-peak stress-displacement relation G_f^+ is the adequate to be included in the FE-regularization of concrete models, i.e.

$$g_{fA}^+ := \int_{\varepsilon_o^+}^{\infty} \sigma^+(\varepsilon^+) d\varepsilon^+ = A_\varepsilon^+. \quad (2.5.204)$$

and which correspond to the depicted area in Fig. 2.5.3a. Moreover, its possible convert this definition using a stress-equivalent plastic strain $\sigma^+ - \alpha^+$ law as follows. First, using Eq. (2.5.193), the total energy dissipated under the $\sigma^+ - \alpha^+$ relation is given by $A_\alpha^+ = \int_0^\infty \sigma^+(\alpha^+) d\alpha^+ = A_o^+ + A_\varepsilon^+$, where $A_o^+ = \frac{1}{2E_o}(\sigma_o^+)^2$, with σ_o^+ the stress onset non-linear behavior. Then, an equivalent energy to g_{fA}^+ can be defined as

$$g_{fB}^+ := \int_0^\infty \sigma^+(\alpha^+) d\alpha^+ - \frac{1}{2E_o}(\sigma_o^+)^2 = \int_{\alpha_x^+}^\infty \sigma^+(\alpha^+) d\alpha^+, \quad (2.5.205)$$

which correspond to depicted area of Fig. 2.5.3b, with α_x^+ an unknown positive value. Note that this expression is different from the stated by (Lubliner et al., 1989).

Compression fracture energy

In contrast, the definition of compression fracture energy is scarce (Vonk, 1992; van Vliet & van Mier, 1995; Jansen & Shah, 1997; Nakamura & Higai, 2001), being matter of discussion. For one hand, (Nakamura & Higai, 2001) define the compression fracture energy dissipated per unit of area as

$$g_{fN}^- := \sigma_p^-(\varepsilon_p^- - \varepsilon_1^-) + \int_{\varepsilon_p^-}^\infty \sigma^-(\varepsilon^-) d\varepsilon^-, \quad (2.5.206)$$

where $\varepsilon_1^- = \frac{\varepsilon_p^-}{2}$. However, its convenient redefined slightly this expression as follows

$$g_{fA}^- := \frac{(\sigma_p^-)^2}{2E_o} + \int_{\varepsilon_p^-}^\infty \sigma^-(\varepsilon^-) d\varepsilon^- = A_1^- + A_\varepsilon^-, \quad (2.5.207)$$

which correspond to coloured area of Fig. 2.5.3c. Similar to tensile regime, its possible convert this definition using a $\sigma^- - \alpha^-$ law. Thus, the energy dissipated in the post-peak $\sigma^- - \alpha^-$ law is defined as

$$g_{fB}^- := \int_{\alpha_p^-}^\infty \sigma^-(\alpha^-) d\alpha^- = A_\alpha^-, \quad (2.5.208)$$

and is associated to depicted area of Fig. 2.5.3d. Then, using Eq. (2.5.193), this expression can be related to Eq. (2.5.207) as follows $A_{\alpha}^{-} = A_1^{-} + A_{\varepsilon}^{-}$, i.e. $g_{fA}^{-} = g_{fB}^{-}$. It should be noted that this definition is agree with the stated by (Lubliner et al., 1989).

Thermodynamical dissipation energy

In damage models, the fracture energy can be defined as the total dissipation energy as follow

$$g_{fC}^{\pm} = \int_{t_0}^{\infty} \dot{\gamma}^{\pm} dt, \quad (2.5.209)$$

where $\dot{\gamma}$ denotes the ratio of total dissipated energy and is evaluated according to HFE potential established. According to the second principle of thermodynamics, any irreversible process satisfies the Clausius-Duhem inequality, whose reduced form is expressed as

$$\dot{\gamma} := -\dot{\psi} + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} \geq 0. \quad (2.5.210)$$

Next, assume that the WLF or FOC model are used. Then, using Eqs. (2.1.41), (2.1.42), (2.1.38) and (2.1.39), the differentiation of Eq. (2.1.37) with respect to time yields

$$\dot{\gamma} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \left(\frac{\partial \psi^e}{\partial \boldsymbol{\varepsilon}^e} : \dot{\boldsymbol{\varepsilon}}^e + \frac{\partial \psi^p}{\partial \underline{\boldsymbol{\kappa}}} \cdot \dot{\underline{\boldsymbol{\kappa}}} + \frac{\partial \psi}{\partial \omega^+} \dot{\omega}^+ + \frac{\partial \psi}{\partial \omega^-} \dot{\omega}^- \right) \quad (2.5.211)$$

$$= \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p - \frac{\partial \psi^p}{\partial \underline{\boldsymbol{\kappa}}} \cdot \dot{\underline{\boldsymbol{\kappa}}} + \psi_o^+ \dot{\omega}^+ + \psi_o^- \dot{\omega}^-. \quad (2.5.212)$$

Now, if only the damage behavior is assumed (non-plastic strains), the positive/negative part of the ratio of dissipation energy can be reduced to $\dot{\gamma}^{\pm} = \psi_e^{\pm} \dot{\omega}^{\pm}$. Also, assuming an positive/negative uniaxial behavior with an undamaged or effective stress $\bar{\sigma}^{\pm}$, the undamaged energy is given by $\psi_e^{\pm} = \frac{(\bar{\sigma}^{\pm})^2}{2E_o}$. Moreover, using the chain rule, the rate of damage variable can be expressed as $\dot{\omega}^{\pm} = \frac{d\omega^{\pm}}{dr^{\pm}} \frac{dr^{\pm}}{dt}$. Thus, using these relations and Eq. (2.5.203),

the dissipation energy can be rewritten as

$$\begin{aligned} g_{fC}^{\pm} &:= \frac{1}{2E_o} \int_{r_o^{\pm}}^{\infty} (\bar{\sigma}^{\pm})^2 \frac{d\omega^{\pm}}{dr^{\pm}} dr^{\pm} = \frac{1}{2E_o\eta^{\pm}} \int_{r_o^{\pm}}^{\infty} \left(\sigma^{\pm} - r^{\pm} \frac{d\sigma^{\pm}}{dr^{\pm}} \right) dr^{\pm} \\ &= \frac{1}{2} \int_{x_0^{\pm}}^{\infty} \left(\sigma^{\pm} - x \frac{d\sigma^{\pm}}{dx^{\pm}} \right) dx^{\pm}. \end{aligned} \quad (2.5.213)$$

Finally, assuming that the uniaxial $\sigma^{\pm} - \varepsilon^{\pm}$ and their respective $\sigma^{\pm} - \alpha^{\pm}$ laws depicts closed regions. Then, applying the Green's theorem into this relation, it can demonstrated that

$$g_{fC}^{\pm} = \int_0^{\infty} \sigma^{\pm}(\varepsilon^{\pm}) d\varepsilon^{\pm} = \int_0^{\infty} \sigma^{\pm}(\alpha^{\pm}) d\alpha^{\pm}, \quad (2.5.214)$$

which represents the total area under $\sigma^{\pm} - \varepsilon^{\pm}$ and $\sigma^{\pm} - \alpha^{\pm}$ laws, respectively, and are depicted in dashed area as shown in each plot of Fig. 2.5.3. It should be noted that this result are agree with mentioned in (Oliver et al., 1990; J. Lee & Fenves, 1998). This imply that in the LLF model the variable $g^{\pm}$ satisfy the relation $g^{\pm} = g_{fC}^{\pm}$.

In addition, to include the plastic component of the dissipated energy in the plastic-damage models, its necessary consider the plastic terms of Eq. (2.5.212). Complex equations are involved in this process due to sofistication of concrete models developed and are beyond the scope of this article. A complete development of this terms can be encountered in (Cervera, Tesei, & Ventura, 2018).

2.5.3. Example of application

An example is elaborated to shown the conversion among uniaxial laws and the different definitions of fracture energies stated. Exponential relation of (Mazars, 1984; Oliver et al., 1990) is used both for tensile as the compressive regime. Table 2.5.5 shown the analytical expressions for stress-strain $\sigma - \varepsilon$ laws, their derivatives $\frac{\partial \sigma}{\partial \varepsilon}$ and the accumulated area under $\sigma - \varepsilon$ law $F(\varepsilon)$. Additionally, table contains the damage laws $\omega(r)$ and their derivatives $\frac{\partial \omega}{\partial r}$ converted according to Eq. (2.5.203).

Calibration of inputs parameters $A^\pm$ and $B^\pm$ are generated according to different definition of tensile/compression fracture energy stated. Table 2.5.6 shown analytical expressions for this parameters. In addition, to avoid the snap-back in the uniaxial stress-strain laws, a maximum value for l_c is determined. Fig. 2.5.4 shown the uniaxial $\sigma^\pm - \varepsilon^\pm$ laws and the equivalent $\sigma^\pm - \alpha^\pm$, $\sigma^\pm - \kappa^\pm$ and $\omega^\pm - r^\pm$ laws applying their respective conversions.

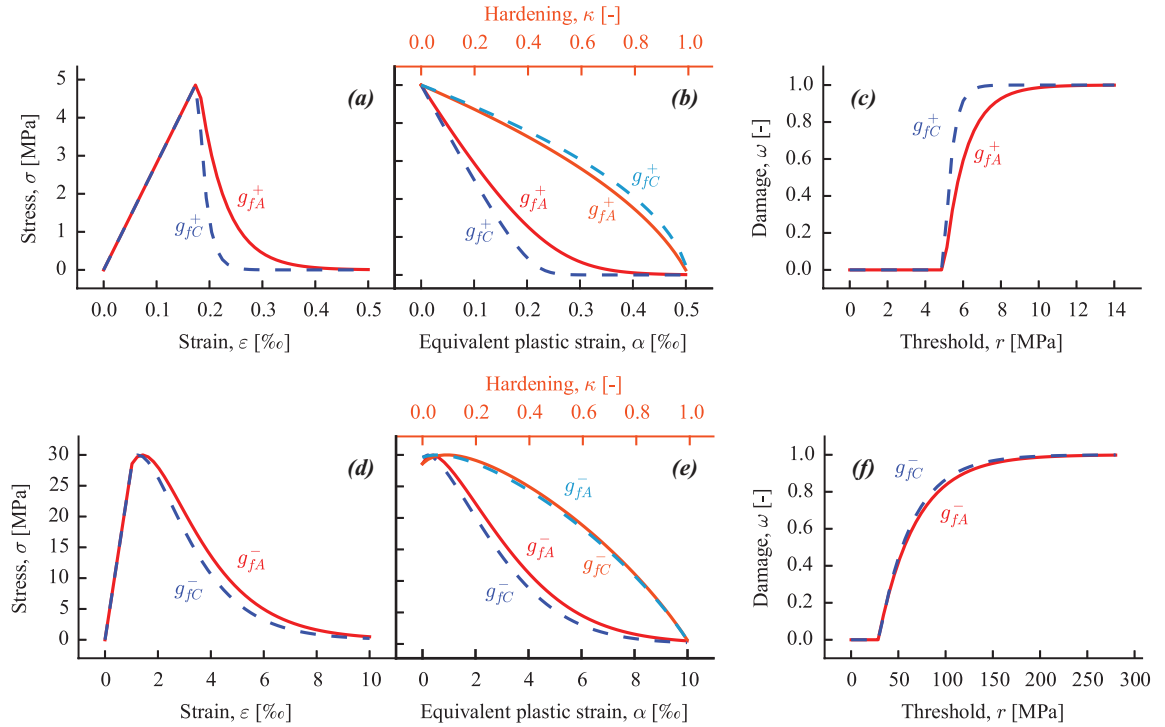


Figure 2.5.4. Example of conversion of uniaxial laws among $\sigma - \varepsilon$, $\sigma - \alpha$, $\sigma - \kappa$ and $\omega - r$ relations for the exponential model of (Mazars et al., 1990; Oliver et al., 1990): (a-c) tensile regime and (d-f) compressive regime. The following parameters are used: $E_o=30$ GPa, $f_t'=5$ MPa, $f_c'=30$ MPa, $l_c=500$ mm, $C^+=6000$ and $C^-=100$.

2.6. Validation examples

In this section, a set of numerical examples are used to validate the capabilities of the constitutive concrete models described in Section 2.1. Taking the numerical algorithms

Table 2.5.5. Conversion of uniaxial tensile/compressive laws for the exponential relation of (Mazars, 1984; Oliver et al., 1990), respectively.

	ExpoP		ExpoN	
	Elastic	Non-linear	Elastic	Non-linear
$\sigma(\varepsilon)$	$f_t z^+$	$f_t \left(1 - A^+ + A^+ e^{B^+(1-z^+)}\right)$	$f_o z^-$	$f_o \left(1 - A^- + A^- z^- e^{B^-(1-z^-)}\right)^\dagger$
$\frac{d\sigma}{d\varepsilon}$	E_o	$-E_o A^+ B^+ e^{B^+(1-z^+)}$	E_o	$E_o A^- \left(1 - B^- z^-\right) e^{B^-(1-z^-)}$
$F(\varepsilon)$	$\frac{f_t^2 z^2}{E_o}$	$\frac{f_t^2}{E_o} \left[\frac{1}{2} \left(1 - e^{2B^+(1-z^+)}\right) + \frac{1}{B^+} \left(1 - e^{B^+(1-z^+)}\right) \right]$	$\frac{f_o^2 z^2}{E_o}$	$\frac{f_o^2}{E_o} \left[\frac{1}{2} \left(1 - (z^-)^2 e^{2B^-(1-z^-)}\right) + \frac{1}{B^-} \left(1 + B^- - (1 + B^- z^-) e^{B^-(1-z^-)}\right) \right]$
$\omega(r)$	0	$1 - \frac{1}{\bar{z}^+} \left(1 - A^+ + A^+ e^{B^+(1-\bar{z}^+)}\right)$	0	$1 - \frac{1}{\bar{z}^-} \left(1 - A^- + A^- \bar{z}^- e^{B^-(1-\bar{z}^-)}\right)$
$\frac{d\omega}{dr}$	0	$\frac{1}{r_o^+ (\bar{z}^+)^2} \left(1 - A^+ + A^+ (1 + B^+ \bar{z}^+) e^{B^+(1-\bar{z}^+)}\right)$	0	$\frac{1}{r_o^- (\bar{z}^-)^2} \left(1 - A^- + A^- B^- (\bar{z}^-)^2 e^{B^-(1-\bar{z}^-)}\right)$

$z^\pm(\varepsilon) = \frac{\varepsilon}{\varepsilon_o^\pm}$, $\bar{z}^\pm(r) = \frac{r^\pm}{r_o^\pm}$, $\varepsilon_o^+ = \frac{f_t}{E_o}$, $\varepsilon_o^- = \frac{f_o}{E_o}$, $r_o^\pm = E_o \varepsilon_o^\pm$. The partial area is defined as $F(\varepsilon) = \int_0^\varepsilon \sigma(\varepsilon) d\varepsilon$. To this variable, its assuming a value $A^\pm = 1$. Then, the total area if defined as $g = F(\infty)$, where for these models are given by $g^+ = \frac{f_t^2(B^++2)}{2E_o B^+}$ and $g^- = \frac{f_o^2((B^-)^2+2B^-+2)}{2E_o (B^-)^2}$, respectively. [†] Parameters f_o^- and B^- are calculated in an iterative process (see Appendix E). For σ , $\frac{d\sigma}{d\varepsilon}$ and $F(\varepsilon)$ the elastic and non-linear stage is limited at the range $[0, \varepsilon_o^\pm]$ and $[\varepsilon_o^\pm, \infty]$, respectively. Also, for ω and $\frac{d\omega}{dr}$ the elastic and non-linear stage is limited at the range $[0, r_o^\pm]$ and $[r_o^\pm, \infty]$, respectively

Table 2.5.6. Calibration of inputs parameters and upper limit of characteristic length l_c according to fracture energy definition.

Model	Parameter	Fracture energy	
		$g_{fA}^{\pm} = g_{fB}^{\pm}$	$g_{fC}^{\pm}$
ExpoP	A^+	1	1
	B^+	$\hat{J}_t^{-1}$	$\left(\hat{J}_t - \frac{1}{2}\right)^{-1}$
	$l_{c,max}$	-	$2J_t$
ExpoN	A^-	1	1
	$B^{-\dagger}$	$\left(\frac{1}{2} + \sqrt{\hat{J}_c + \frac{1}{4}}\right) / \hat{J}_c$	$\left(\frac{1}{2} + \sqrt{\hat{J}_c - \frac{1}{4}}\right) \left(\hat{J}_c - \frac{1}{2}\right)^{-1}$
	$l_{c,max}$	$\frac{1}{2}\eta J_c$	$\frac{2}{5}\eta J_c$

$J_t = \frac{G_f^+ E_o}{f_t^2}$, $J_c = \frac{G_f^- E_o}{f_o^2}$, $\hat{J}_t = \frac{J_t}{l_c}$, $\hat{J}_c = \frac{J_c}{l_c}$, $\dagger$ Parameters f_o and B^- are calculated in an iterative process (see Appendix F). ³ Values calculated according to condition $\frac{d\sigma}{d\varepsilon} \in [0, E_o] \rightarrow B^- \in [0, 1]$. An tolerance of $\eta = 0.9$ is chosen for convenience.

presented in Sections 2.3 and 2.4, the five concrete models were implemented in the software (ANSYS, 2018) through user-material FORTRAN77 routines (USERMAT.f). These material routines works at Gauss integration point level of each finite element.

Five class of experimental benchmark test are simulated with a single-element according to loading conditions: (i) uniaxial cyclic tension and compression; (ii) biaxial monotonic; (iii) triaxial monotonic; (iv) uniaxial cyclic tension-compression and (v) strain-rate effect and numerical viscosity. Also, the strain-localization and fracture-energy FE-regularization are discussed with a fictitious example. In addition, the compression failure mode of a test specimen varying their slenderness is illustrated as an example of application.

All examples were modeled using 8-node isoparametric solid element (SOLID185) with three Degree Of Freedom (DOF) at each node using 2x2x2 Gauss integration scheme and $\bar{B}$ -formulation (selective reduced integration method) (Hughes, 1980). All models, except the DPH model, assume an exponential relation for the positive/negative uniaxial stress laws given by Eqs. (2.1.57) and (2.1.58), respectively. An adequate conversion

among uniaxial laws required for each concrete model is generated, as explained in Table 2.5.4. Table 2.6.7 list the material parameters adopted for each benchmark test. Additional parameters are listed in the figure of each example.

Table 2.6.7. List of parameters used in the concrete models.

Author	Test	B mm	H mm	E_o GPa	ν -	f'_t MPa	f'_c MPa	$G_f^{+\dagger}$ N/mm	$G_f^{-\dagger}$ N/mm	K_c -
Gopalaratnam & Shah, 1985	uniaxial tension	82.6	82.6	31.0	0.18	3.48	27.6	0.04	11.38	1.0
Karsan & Jirsa, 1969	uniaxial compression	82.6	82.6	31.7	0.2	3.48	27.6	0.04	11.38	1.0
Kupfer et al., 1969	biaxial	200	50	31.0	0.15	3.5	32.06	2.0	80.0	1.0
Imran & Pantazopoulou, 1996	triaxial	54	115	19.0	0.22	5.0	47.4	1.0	50.0	0.7
Mazars et al., 1990	unilateral effect	80	80	16.4	0.2	1.4	18.1	0.011	7.0	1.0
Suaris & Shah, 1985	strain-rate effect	100	100	34.0	0.22	5.37	46.8	0.5	20.0	1.0
-	strain-localization	100	600	32.0	0.0	5.0	39.0	4.0	40.0	1.0
van Vliet & van Mier, 1995	slenderness of specimen	100	50-200	27.8	0.2	6.0	36.34	2.0	5.7	(¹)

[†] values used in the WLF₀ as reference. (¹) see Fig. 2.6.16. For all cases: $f'_b = 1.16f'_c$, $\epsilon = 0.001$, $z_c^+ = 0$, $z_c^- = 1$, $f_o^+ = f'_t$ and $\mu_v = 0$, unless otherwise indicated.

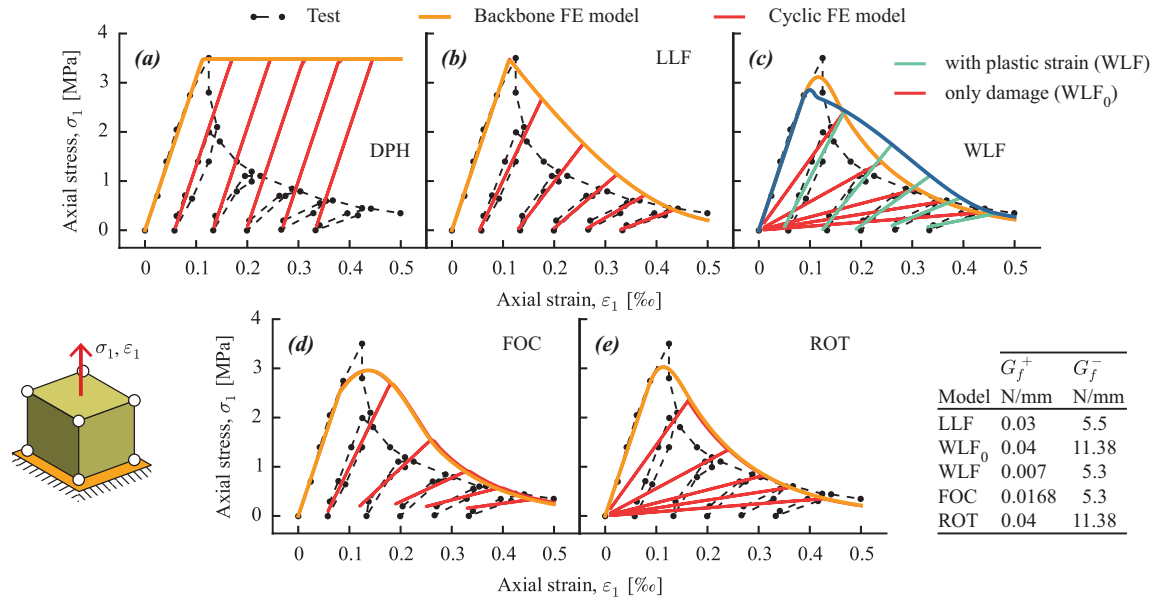


Figure 2.6.5. Validation of concrete models under uniaxial cyclic tension test of (Gopalaratnam & Shah, 1985): (a) DPH model; (b) LLF model; (c) WLF₀ and WLF models; (d) FOC model; and (e) ROT model. The following additional parameters are used. For the DPH model: $f_y^+ = 3.48$ MPa, $f_y^- = 12$ MPa, $a_0 = 3c_u/E_o$, $R=1$; LLF model: $C^+ = 6500$, $C^- = 7500$; and WLF model: $f_o^- = 20$ MPa, $E_t^+ = 0.16E_o$, $E_t^- = 0.48E_o$.

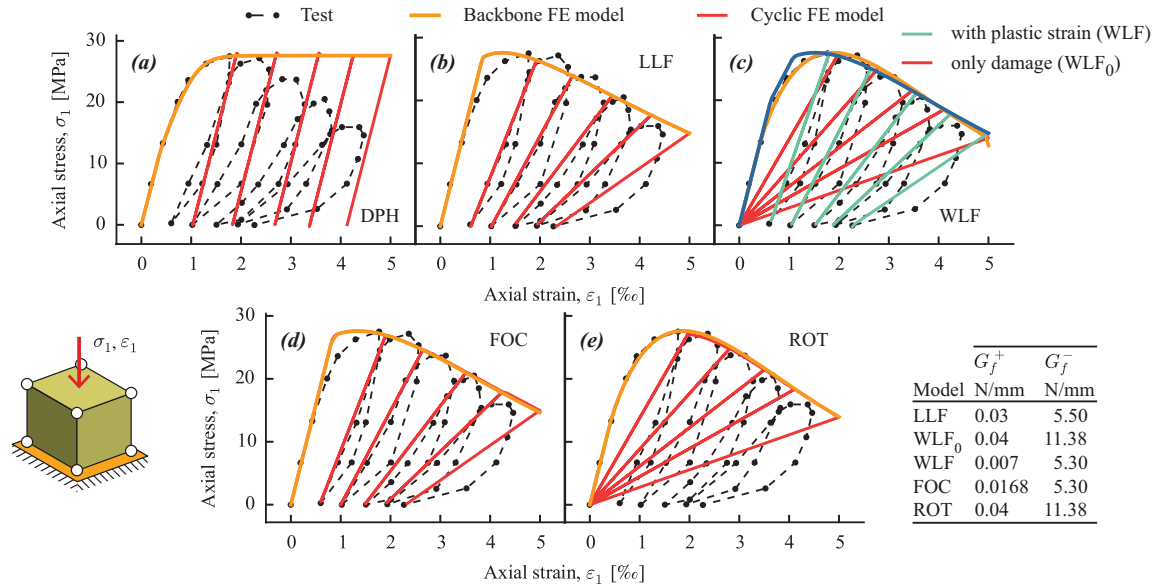


Figure 2.6.6. Validation of concrete models under uniaxial cyclic compression test of (Karsan & Jirsa, 1969): (a) DPH model; (b) LLF model; (c) WLF model; (d) FOC model; and (e) ROT model. The following additional parameters are used. For the DPH model: $f_y^+ = 3.48$ MPa, $f_y^- = 12$ MPa, $a_0 = 3c_u/E_o$, $R=1$; LLF model: $C^+ = 6500$, $C^- = 7500$; and WLF model: $f_o^- = 20$ MPa, $E_t^+ = 0.16E_o$, $E_t^- = 0.48E_o$.

2.6.1. Uniaxial cyclic tests

Numerical concrete models are compared with uniaxial cyclic tensile and compressive loading-unloading and reloading experimental data reported by (Gopalaratnam & Shah, 1985) and by (Karsan & Jirsa, 1969), respectively. Figs. 2.6.5 and 2.6.6 shown the response of the five concrete models under tensile and compressive loads, respectively. FE models are elaborated with a single-element cube of 82.6 mm. Its assumed a characteristic length of $l_c = 82.6$ mm and a pure uniaxial stress state for the boundary constraints.

In general, it can be observed that in all models, except for the DPH model, fits well with the post-peak backbone response of experimental tests, where the WLF₀ and ROT models give the best approximation. Although, both models fail in the unloading branch, due to that they neglect plastic strains (pure damage only). Also note that these models have identical responses, although they are elaborated with formulations completely different.

In contrast, the unloading branch of the LLF, WLF and FOC models fits close to experimental response due that incorporate the plastic and damage components in their formulations. In plastic-damage models, its required adjust the parameters to fit simultaneously the backbone curve and the unloading branch. Thus, the first half of residual backbone is mainly influenced by the parameters of the plastic component and the last half by the fracture energy $G_f^\pm$. Moreover, the parameters $C^\pm$, $E_t^\pm$ and $B^\pm$ for the LLF, WLF and FOC models, respectively, controls the backbone and slope of unloading branch in a coupled manner, i.e. when their values are reduced cause an increase in the slope of unloading branch and consequently reduce the backbone response.

The follows additional observations are considered. For the tensile regime, in all models, except the LLF model, the peak stress do not fit exactly with the experimental value due to the incorporation of smoothed polynomial function in the uniaxial laws (see Section 2.2). For other hand, the value of fracture energy $G_f^\pm$ used in the plastic-damage models to fit the experimental tests is less than in the damage models. This is due to that the plastic component induce an additional dissipation of energy that is not taken account in the FE-regularization (Section 2.5).

For the FOC model, it has observed the influence of strain increment size $\Delta\varepsilon$ in the response, where an gradual over-stress response is caused with a relative large strain increments. In the same way, its observed a difference between cyclic and backbone responses, gradually incremented over the last unloading/loading cycles, but that disappear with a relative small $\Delta\varepsilon$. Both conditions are due to explicit integration scheme used in the numerical algorithm to calculate the plastic strain tensor. In addition, it can observed the influence of parameter B^- in the tensile response.

2.6.2. Biaxial monotonic tests

All the concrete models, except the ROT model, are compared with biaxial monotonic test of (Kupfer et al., 1969). This test is performed with a constant biaxial loading ratio of $a = \sigma_1/\sigma_2$, where σ_1 and σ_2 are the stresses imposed. FE models are elaborated with a

single-element of 200×50 mm of base and 200 mm of height. Its assumed a characteristic length of $l_c = 200$ mm and a pure biaxial stress state for the boundary constraints, as observed in the experimental test. A stress-controlled test are performed up to reach the peak stress, with the exception of the uniaxial case ($a = 0$) simulated with displacement-controlled. The inputs parameters are chosen by means to fit the cases $a=0, 1$ and 0.52 simultaneously.

Fig. 2.6.7 shown the axial stress σ_1 vs axial ε_1 and the lateral strains ε_2 and ε_3 , respectively, for the WLF model and using a loading ratio of $a=0, 1$ and 0.52 . This model use a DEER given by Eq. (2.1.51) to include the biaxial strength. It can observed a good fit with the pre-peak stress response of experimental test and a relatively good adjust exist in the lateral strains, especially when $a=0.52$. In general, the same observations are concluded in all models. Fig. 2.6.8 shown the biaxial peak strength surface for the DPH,

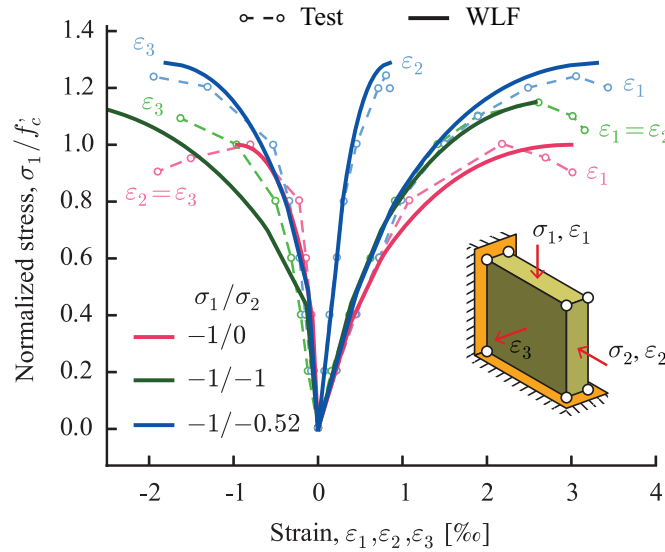


Figure 2.6.7. Validation for the WLF model under biaxial test of (Kupfer et al., 1969). The following additional parameters are used: $G_f^+ = 0.5$ N/mm, $G_f^- = 35$ N/mm, $f_o^- = 12$ MPa, $E_t^+ = 0.3E_o$ and $E_t^- = 0.65E_o$.

WLF₀, WLF, FOC and LLF models under different combination of biaxial loading ratios $a = \sigma_1/\sigma_2$. For the DPH model, the parameters η and ξ are fitted with tension/compression biaxial strength of concrete. Also, for the WLF₀ and WLF models, the DEER given by

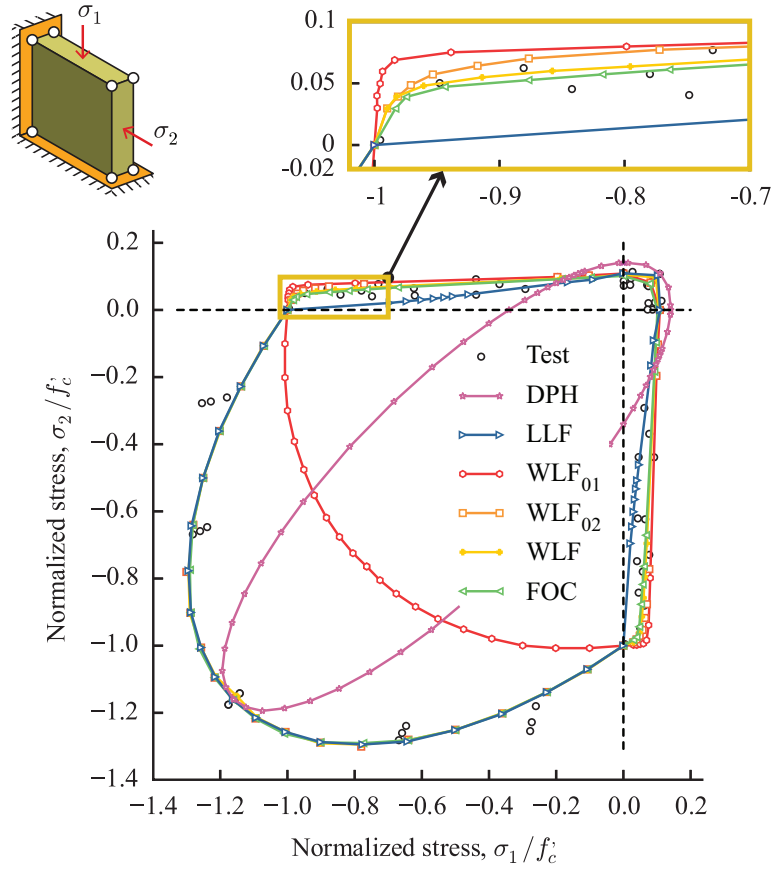


Figure 2.6.8. Biaxial peak strength surface for the DPH, WLF₀, WLF, FOC and LLF models and the biaxial test results of (Kupfer et al., 1969). For the DPH model the following parameters are used $f_y^+ = 3.5$ MPa and $f_y^- = f'_b$.

Eq. (2.1.51) is used to include the biaxial strength. In addition, the response of the WLF₀ model using the Eq. (2.1.50) is compared.

It can be observed, that all models fit close with the experimental results, specially in compression-compression (C-C) regime, where are influenced by the Drucker-Prager yield criterion. The major differences among models occur in the tension-compression (T-C) regimes. The exception occurs with the DPH and WLF₀₁ model. For the first, it fits well only in the equal biaxial loading ratio $a = 1$ and the second one fits well in the T-C regime, but with a reduced strength in the C-C regime. Both observations are obtained such as expected in the literature (de Souza Neto et al., 2008; Mazars, 1984; J. Simo & Ju, 1987).

Additionally, it can be observed, similar to uniaxial case, a less value of fracture energy is required in the plastic-damage models than damage models to fit with experimental results. Conversely to the uniaxial case, under certain conditions, an increment in the value of compression fracture energy cause a reduction in the backbone response.

2.6.3. Triaxial monotonic tests

The LLF, WLF₀, WLF and FOC models are compared with monotonic lateral confining triaxial test of (Imran & Pantazopoulou, 1996). A cylindrical specimen of $D=54$ mm of diameter and $H=115$ mm of height was tested. Moreover, for the sake of simplicity, a single-element prism of base $B \times B$ of equal area than the cylinder is simulated (i.e. $B = \sqrt{\pi D/2}$). Its assumed a characteristic length of $l_c = \sqrt[3]{B^2 h}$ and a pure triaxial stress state for the boundary constraints. Seven confining pressure levels are applied: 0 (uniaxial), 2.15, 4.3, 8.6, 17.2, 30.1 and 43.0 MPa. The loading paths protocol used was the follow. A first phase during which the confining pressure p_o was gradually increased to a specified level while the specimen was unrestrained in the axial direction (hydrostatic pressure). Beyond that stage, an axial compressive stress $\bar{\sigma}_3$ is gradually applied under displacement-control, while the level of confining pressure was maintained constant. The input material parameters are chosen in order to fit simultaneously the seven levels of confining pressure. Some convergence trouble are observed in the simulations, for which a numerical viscosity of $\mu_v/\Delta t = 0.001$ is incorporated in all models.

Fig. 2.6.9a-b shown the normalized total axial compressive stress σ_3/f'_c ($\sigma_3 = \bar{\sigma}_3 + p_o$) vs the axial ε_3 and the lateral ε_{lat} strain for the WLF and LLF models, respectively. For the WLF model, it can be observed a gradual strength and ductility level as it increases the confining pressure, such as observed experimentally. A relatively good correlation exist in all cases for the pre-peak response and the peak strength. The best correlation occur in the cases $p_o=8.6$ MPa and $p_o=17.2$ MPa (medium-level pressure). Conversely, the largest difference between the estimated peak strength is 6.94% for the case $p_o=43.0$ MPa. Also, lateral strain values are similar to experimental ones. Similar observations are obtained for

the LLF model (Fig. 2.6.9b), with the exception of a considerable over-strength in the pre-peak stage for the cases $p_o=30.1$ MPa and $p_o=43.0$ MPa (high-level pressure). Although, the peak strength is increased only in a 6.9%. Analogous results with the LLF model are observed for the WLF₀ and FOC models.

Fig. 2.6.9c resume the normalized peak stress $\sigma_{3_{max}}/f'_c$ vs the confining pressure p_o applied for the four models. Note the similitude with the compression meridian of Fig. 2.1.1. A well fit correlation is observed, specially at low to medium-confining levels. However, its noted a over-strength of 3.1% and 6.9% in the cases $p_o=30.1$ and 43 MPa, respectively (high-confining), whereas a relatively lower values (up to 4.3%) exists for the cases $p_o=4.3$, 8.6 and 17.2 MPa (low- to medium-confining). In both cases, this is due to incorporation of triaxial confinement in the equations throughout the constant parameter δ (Eqs. (2.1.24) and (2.1.51)). More sophisticated models are required to fit close with experimental results in overall range of confining pressures (Zhang, Zhang, & Chen, 2010).

Fig. 2.6.9d-e shown the normalized total stress σ_3/f'_c vs the volumetric strain ε_v ($= \varepsilon_1 + 2\varepsilon_{lat}$) for the WLF and LLF models, respectively. For the WLF model, its observed a low level of dilatancy upon the post-peak regime in all confining levels, such as observed experimentally. Contrary, for the LLF model, a reduction in the volume in all range of confining pressures are measured. This last condition is noticed also in the WLF₀ and FOC models.

2.6.4. Uniaxial cyclic tension-compression test

To validate the unilateral effect, the LLF, WLF₀, WLF, FOC and ROT models are compared with the uniaxial cyclic test of (Mazars et al., 1990). This test was first subjected to uniaxial tension followed by uniaxial compression in parallel directions. FE models are elaborated with a single-element cube of 80 mm of width. Its assumed a characteristic length of $l_c=80$ mm and a pure uniaxial stress state for the boundary constraints.

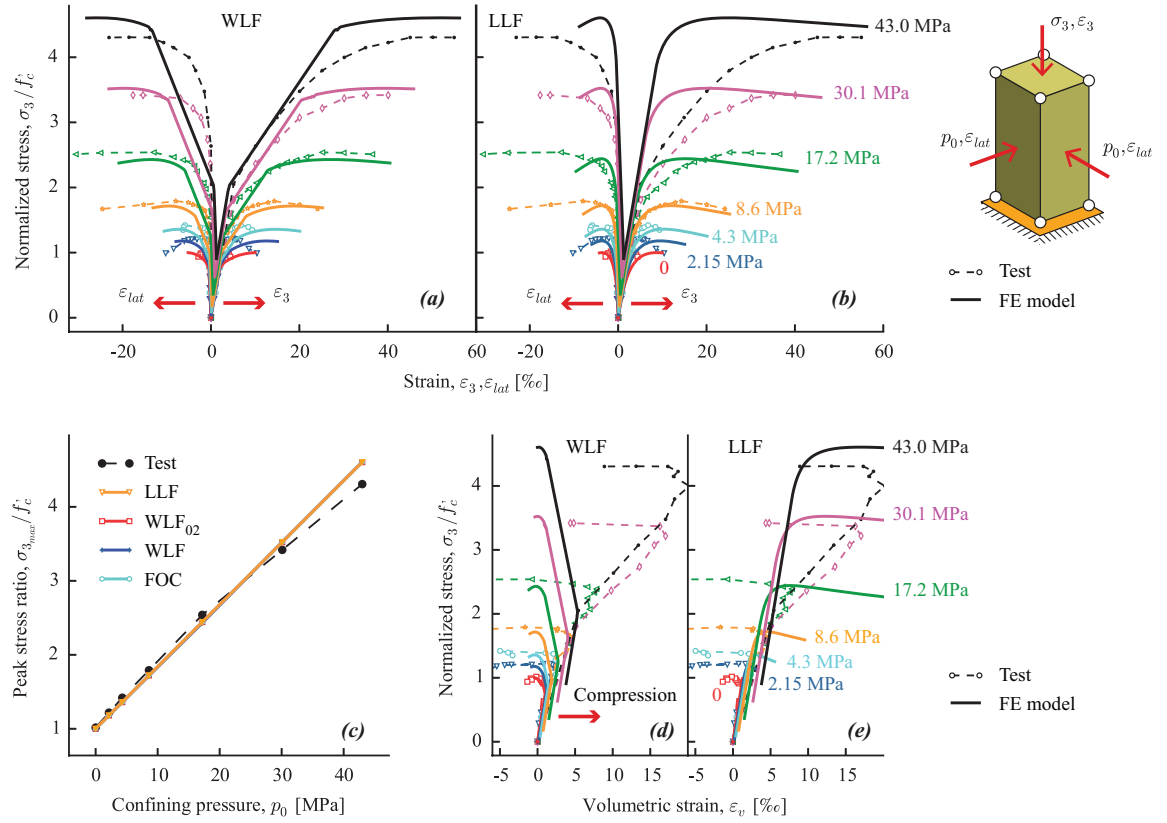


Figure 2.6.9. Validation of concrete models under triaxial test of (Imran & Pantazopoulou, 1996): (a-b) normalized total stress σ_3/f'_c vs axial ε_1 and lateral ε_{lat} strain for the WLF and LLF model, respectively; (c) normalized peak stress $\sigma_{3,max}/f'_c$ vs confining pressure p_o for LLF, WLF₀, WLF and FOC models; and (d-e) normalized total stress σ_3/f'_c vs volumetric strain ε_v for the WLF and LLF models. The following additional parameters are used. For the LLF model: $C^+=1000$, $C^-=200$ and the WLF model: $f_o^-=35$ MPa, $E_t^+ = 0.5E_o$ and $E_t^- = 0.25E_o$.

Fig. 2.6.10 shown the axial stress σ_1 vs axial strain ε_1 of this models. It also included the response of the LLF model with three values of stiffness recovery factor z_c^- (0, 0.5 and 1). It can noticed that all models recovery the initial elastic stiffness once the load goes into the compression state (step 2 and 4). The exception occur, obviously, in the LLF model when $z_c^-=0.5$ and 0, due that this parameter controls the value of recovery compression stiffness. Moreover, its observed that all models, with the exception of the WLF₀ and ROT models, take the compression backbone branch close to experimental data (step 4), due that include plastic strain in their formulations. In addition, its observed that the LLF, WLF and FOC models recovery the damaged stiffness obtained in the last cycle

of tension (step 3) when the load goes from compression to tension state (step 6). This condition is also shared by the WLF_0 and ROT models (not shown in the plot) and is so-called that the models have "damage memory", which is agree with the thermodynamic of irreversible process.

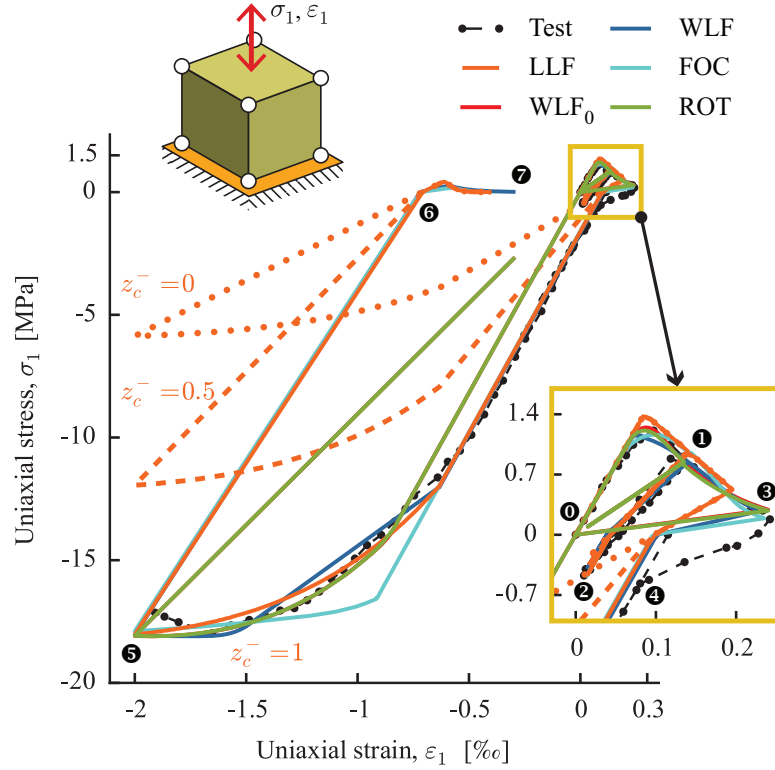


Figure 2.6.10. Validation of the LLF, WLF_0 , WLF, FOC and ROT models under uniaxial cyclic tension-compression test of (Mazars et al., 1990). The following additional parameters are used. For the LLF model: $C^+=12000$, $C^-=200$; WLF model: $f_o^-=12$ MPa, $E_t^+ = 0.3E_o$ and $E_t^- = 0.4E_o$; and FOC model: $B^+=0.54$ and $B^-=0.75$.

2.6.5. Strain-rate tests

Experimentally, the strain-rate effect is important under impulsive loading (impacts or explosions), but already important under earthquake loading, with rates of straining $\dot{\epsilon}$ ranges between $10^{-6}/s$ to $10^{-1}/s$. Then, due that the all models, except the DPH model, can

simulate the rate-dependent behavior through of incorporation of a visco-elastic/visco-plastic model, they are compared with the strain-rate test of (Suaris & Shah, 1985). FE models are elaborated with a single-element cube of 100 mm of width. Its assumed a characteristic length of $l_c=100$ mm and a pure uniaxial stress state for the boundary constraints. Two uniaxial tests are performed, one for tension and other for compression, both with a range of straining rates $\dot{\epsilon}$ between $10^{-6}/s$ to $1/s$. The material parameters are fitted with the tests loaded with a strain-rate of $\dot{\epsilon}=10^{-6}/s$ (pseudo-static). For the sake of simplicity, a numerical viscosity $\mu_v = 2 \times 10^{-3}$ s is used in all cases. Also, a constant number of steps $N_s=150$ and a maximum displacement of $\delta_{max}=0.25$ mm for tension and -0.55 mm for compression are used, for which the time increment used is given by $\Delta t = \frac{|\delta_{max}|}{N_s \dot{\epsilon}}$.

Fig. 2.6.11a-b shown the normalized uniaxial tension/compression viscous stress $\sigma_1^v/\sigma_{1max}^0$ vs uniaxial strain ϵ_1 , respectively, for the WLF₀ model, where σ_{1max}^0 denotes the peak inviscid stress (f'_t and f'_c , respectively). In both plots, for high straining rates, an increment of up to 3.4 and 1.1 times respect to the inviscid case ($\dot{\epsilon}=10^{-6}/s$) is observed for tension and compression, respectively. Moreover, its denoted an over-estimation of 59.4% in the tensile peak stress respect to experimental test, whereas a lower-estimation of 12.4% exist for the compression peak stress. Similar observations are derived using the other models.

Fig. 2.6.11c shown the peak stress ratio $\sigma_{1max}^v/\sigma_{1max}^0$ or Dynamic Increase Factor (DIF) vs the applied strain-rate $\dot{\epsilon}$ for all models, where σ_{1max}^0 denotes the peak stress at inviscid response. As can observed, peaks strengths grow continuously as straining rates are increased, becoming clearly distinguishable from the inviscid response upon a strain-rate value of $10^{-2}/s$. Also noted, that the tensile response is largest than the compressive one in overall range of straining rates analyzed, growing up to 6 times respect to the inviscid response. In addition, the FE results shown that the DIF is underestimated as compared to the both experiments for the small strain-rates $\dot{\epsilon} < 10^{-1}/s$ and overestimated for the large strain rates $\dot{\epsilon} \geq 2.5 \times 10^{-1}/s$. To get a best estimation with respect to the experimental tests, its required modify the visco-plastic model used, e.g the modified Perzyna model proposed by (Faria & Oliver, 1993; Faria et al., 1998).

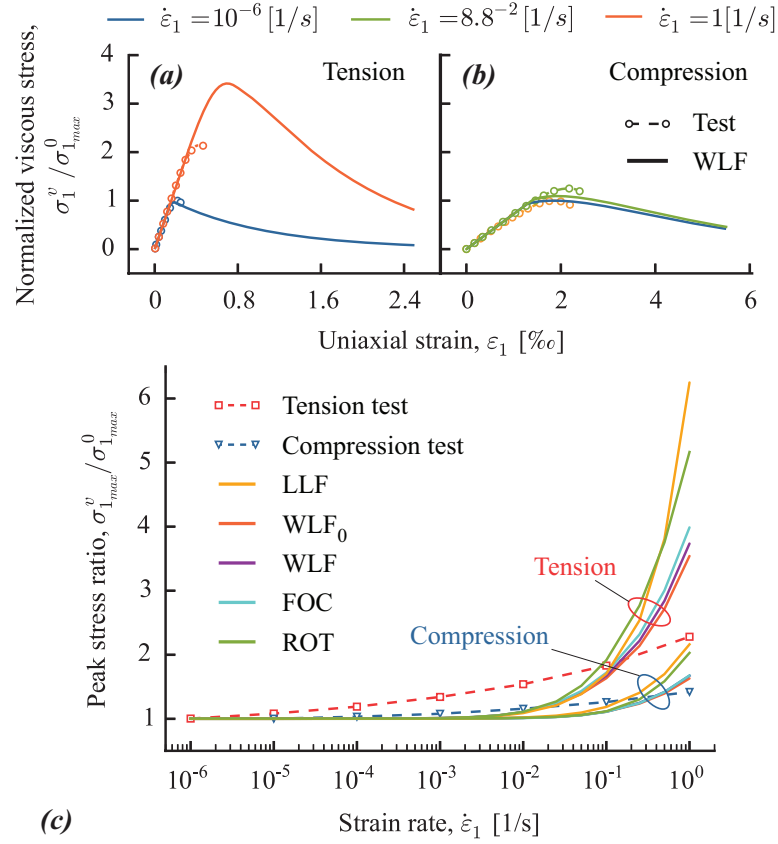


Figure 2.6.11. Validation of strain-rate effect in the concrete models under monotonic uniaxial tests of (Suaris & Shah, 1985): (a-b) normalized uniaxial tensile/compressive stress $\sigma_1^v / \sigma_{1,max}^0$ vs uniaxial strain ε_1 for the WLF₀ model, respectively; and (c) peak stress ratio $\sigma_{1,max}^v / \sigma_{1,max}^0$ or Dynamic Increase Factor (DIF) vs the applied strain-rate $\dot{\varepsilon}$ for the LLF, WLF₀, WLF, FOC and ROT model under tensile and compressive loads.

2.6.6. Effect of the numerical viscosity

In order to investigate the effects of numerical viscosity in the response, a numerical test are generated varying the numerical viscosity-time increment ratio $\mu_v / \Delta t$ for the WLF model. This adimensional parameter is related to the variable $\zeta_v = (1 + \Delta t / \mu_v)^{-1}$ (Eq. (2.3.114)) required for the stress updated algorithms of models. For the sake of simplicity, the material parameters used are the same than in the strain-rate effect simulation. Uniaxial tensile load is applied in a single-element varying the relation $\mu_v / \Delta t$ in a range between 10^{-6} (inviscid) to 50.

Fig. 2.6.12a shown the uniaxial viscous stress-strain $\sigma_1^v - \varepsilon_1$ response with different values of $\mu_v/\Delta t$. Similar to Fig. 2.6.11c its observed an gradual over-stress response proportional to the increased value of the numerical viscosity. Moreover, Fig. 2.6.12b shown the respective axial stiffness-strain $\frac{\partial \sigma_1^v}{\partial \varepsilon_1} - \varepsilon_1$ response for one integration point of the FE model. Similar to the stress response, a gradual increment of axial stiffness is presented as increasing the value of $\mu_v/\Delta t$, up to get a positive value although a strain-softening regimes exists. This key advantage can convert into a positive-definite the consistent tangent stiffness tensor and is demonstrated that expand the range of convergence of the models in strain-softening regimes.

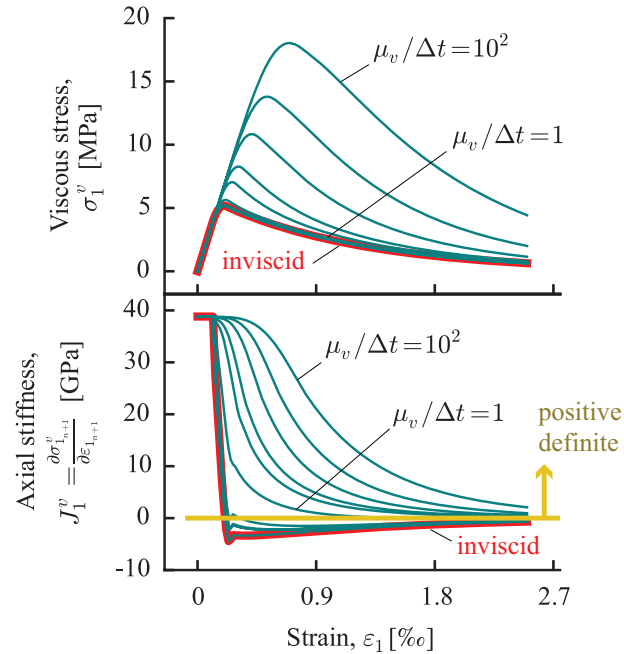


Figure 2.6.12. Variation of uniaxial response using different values of the ratio numerical viscosity/time increment $\mu_v/\Delta t$ for the WLF model: (a) uniaxial viscous stress-strain $\sigma_1^v - \varepsilon_1$ relation and (b) axial stiffness-strain $\frac{\partial \sigma_1^v}{\partial \varepsilon_1} - \varepsilon_1$ relation for one integration point of the FE model.

2.6.7. Strain-localization and FE-regularization

Strain-localization phenomena is present in local models with strain-softening behavior. Imperfection of material properties, irregularities in the geometry and non-symmetrical

boundary/load conditions can induce the formation of this phenomena. The fracture energy FE-regularization is an popular technique that introduce a length scale in the constitutive equations and that is able to remove the spurious mesh-dependency observed when strain-localization exists. It should be noted that, ignoring the FE-regularization, local models with strain-softening behavior can correctly describe the damage only when remain uniformly distributed (perfect material). In order to study this phenomena in the

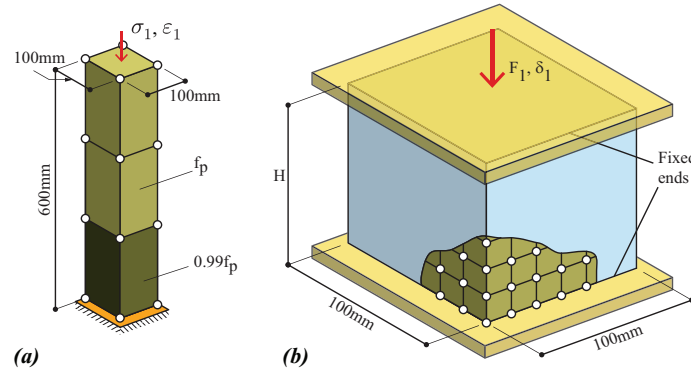


Figure 2.6.13. Description of FE models used in two tests: (a) strain-localization and (b) compression of a specimen test.

concrete models developed, two uniaxial tests are performed, one for tension and other for compression, varying the number of finite elements (i.e. varying their characteristic length l_c). For the sake of simplicity, a prism of 100×100 mm of base and 600 mm of height is divided into 2, 3 and 4 elements. Also, its assumed a pure uniaxial stress state for the boundary constraints (Fig. 2.6.13a). Table 2.6.7 list the material parameter used. The election of parameters E_o , f'_c and $G_f^\pm$ are chosen in order to satisfy the range of characteristic length l_c admissible by the uniaxial compression stress law given by Eq. (2.3.77). In order to induce the localization phenomena, one of elements (shaded element) has been reduced slightly their uniaxial tension/compression strength (f'_t/f'_c) than others elements (0.99 times), for tensile/compressive load case, respectively. In addition, due that some convergence trouble are observed in the simulations, a numerical viscosity of $\mu_v/\Delta t = 0.05$ is incorporated in all models.

Fig. 2.6.14a-b shown the normalized uniaxial tensile stress σ_1/σ_{1max} vs post-peak displacement δ_{1pp} for the WLF_0 and WLF model, respectively, varying the mesh size of model, whereas Fig. 2.6.14c-d shown the respective compressive response for the WLF_0 and LLF, respectively. Additionally, the figure shown the failure mode of their respective specimens, through the field of damage variable $\omega^\pm$. The post-peak displacement is defined as $\delta_{1pp} := \delta - \delta_o$, where δ is the total displacement of specimen and δ_o the displacement at peak response.

Its observed in all models with imperfection a mesh-objectivity response and the damage zone occur only in the modified element, such as expected in literature. However, in the case without imperfection, two kinds of response are observed. For one hand, the response for the WLF_0 model is mesh-dependent with an uniform strain field, either in tension as in compression. This condition is due that the FE-regularization modify the uniaxial stress-strain law despite exist an uniform strain field in the model. Then, its concluded that this technique is only necessary when the damage zone localize. For other hand, the LLF and WLF models (with the exception of one case $l_c=300$ mm) gives a mesh-objectivity response. This atypical condition can be attributed first to the non-symmetric consistent tangent stiffness tensor and largely to numerical errors induced in the iterative process to calculate the plastic component.

Similar observations can be concluded in the other cases as explained as follows. All models gives a mesh-objectivity response and the damage zone is localized in one element (modified element) when a perturbation exists in the material. In contrast, not all the models have an uniform strain field in the case without imperfections. Its observed that the WLF and FOC models localize with a tensile load, whereas the LLF model localize both in the tension as in the compression case. In contrast, the WLF_0 and ROT models not localize using a perfect material.

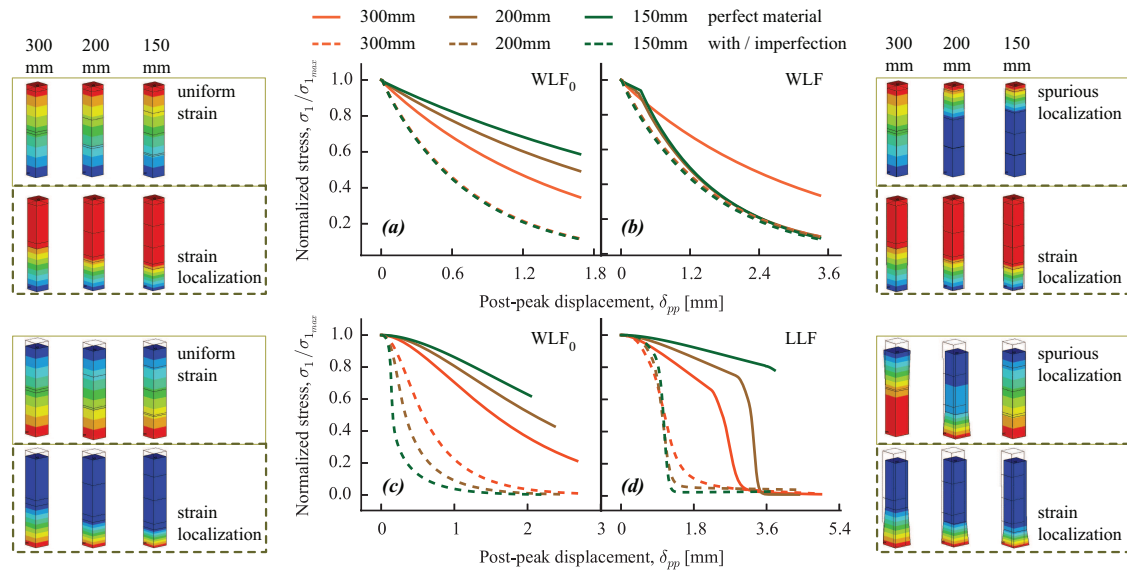


Figure 2.6.14. Comparison the normalized uniaxial stress $\sigma_1 / \sigma_{1_{max}}$ vs post-peak displacement $\delta_{1_{pp}}$ using three FE mesh sizes: 150 mm, 200 mm and 300 mm: (a-b) tensile response for the WLF₀ and WLF models, respectively; and (c-d) compressive response for the WLF₀ and LLF models, respectively. The following additional parameter are used. For the LLF model: $C^+ = 6000$, $C^- = 500$; and WLF model: $f_o^- = 20$ MPa, $E_t^+ = 0.5E_o$ and $E_t^- = 0.5E_o$.

2.6.8. Variation of slenderness specimen in a compression test

Experimentally, the compressive strength of a concrete test specimen is influenced by several factors. Avoiding effects associated to the concrete mixture characteristic (normal or high strength concrete) or the strain-rate effects, three main factors affects in the response: the specimen slenderness or height-to-width ratio H/B ; the election of boundary constrains imposed in the loading platens used in the experiments; and geometrical effects (cylinder/prism and specimen size). Round-robin tests were performed in the past to guess this problem (van Mier et al., 1997).

Its observed by several authors (van Vliet & van Mier, 1995; Vonk, 1992), in compression test where a reduced friction exist between the test specimen and the loading platens (e.g. steel brushes, thin layers of Teflon and grease or stearic acid), their compressive strength is relatively independent of slenderness of specimen tested. Conversely, in test where a high friction (e.g. dry steel platens) is present, the compressive strength is

inversely proportional to the slenderness of specimen, i.e, the larger the height, the lower the strength. For a slenderness values upper 2 the peak strength tends to the case with low friction (pure uniaxial). This condition is due that the lateral shear stresses presents in the loading platens induce triaxially confining pressure at the boundary of the specimen.

Numerical simulation of experimental test performed by (van Vliet & van Mier, 1995) are generated to shown the response and failure mode. Four prisms specimens of 100×100 mm of base, with different slenderness H/B are simulated, three of them $H=50$, 100 and 200 mm of height are identical to the experimental test and one additional fictitious specimen of 400 mm of height is considered.

Fig. 2.6.13b shown a schematic representation of the FE model generated. A mesh of 16.6 mm is used. Fixed boundary restraints are imposed in both ends to simulated the high friction provided by dry steel platens. The material input parameters of models are calibrated with a specimen of $100 \times 100 \times 50$ mm and with low friction (uniaxial case) that was included as part of the experimental program.

Fig. 2.6.15 shown of compressive response of four FE models varying the height of specimen and using the WLF model. It can observed a good correlation in the post-peak response with the experimental results. The highest value in the compressive strength occur for the specimen of 50 mm of height. Also, its observed a over-estimation of resistance for the specimen of 50 mm, whereas an under-estimation exists for the specimens of 100 and 200 mm. In both cases, similar to concluded in the triaxial test simulation, is due to incorporation of triaxial confinement parameter δ in the equations. Similar observations occurs with the other models. Fig. 2.6.15 shown the variation of compressive peak strength with respect to the slenderness of a test specimen for the LLF, WLF₀, WLF and FOC models. It can observed a good correlation with experimental results and with a similar response among them. In all cases, the compressive strength tends to the uniaxial peak strength f'_c (low friction case) with a slenderness value over two, such as recommended in ACI code. In addition, Fig. 2.6.17 shown the failure mode of experimental specimens and

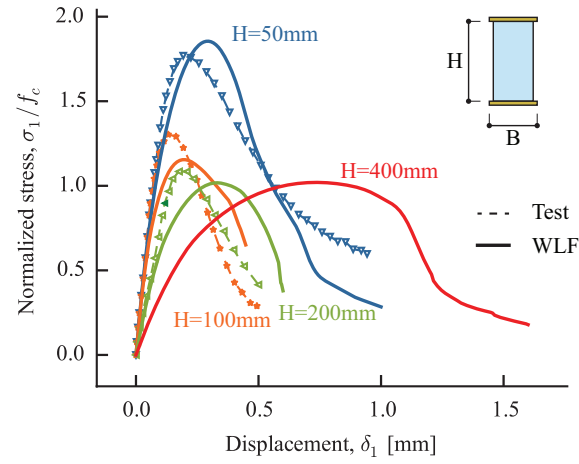


Figure 2.6.15. Simulation of compressive response of a test specimen varying their slenderness for the WLF concrete model using the experimental test of (van Vliet & van Mier, 1995). The following additional parameters are used: $K_c=0.74$, $f_o^-=30$ MPa, $E_t^+ = 0.8E_o$ and $E_t^- = 0.8E_o$.

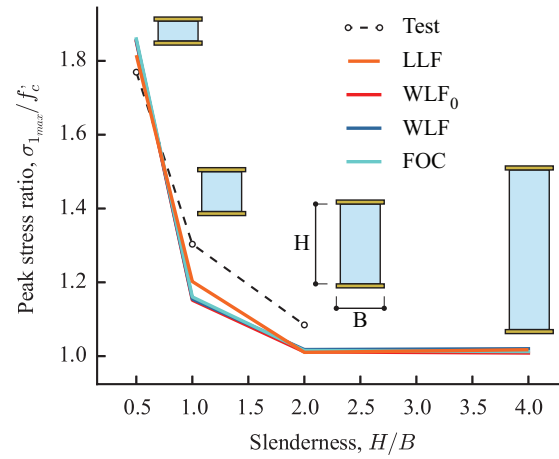


Figure 2.6.16. Comparison of compressive peak strength vs slenderness of a test specimen for the LLF, WLF₀, WLF and FOC models.

their respective numerical simulations for the LLF, WLF₀, WLF and FOC models varying their slenderness.

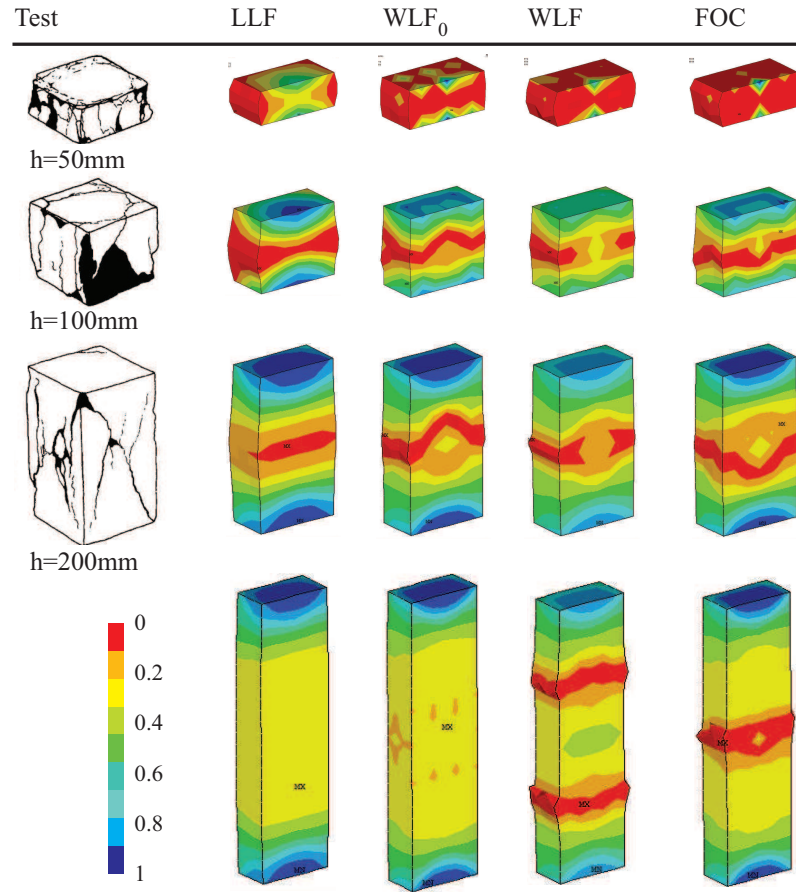


Figure 2.6.17. Comparison of failure mode of test specimen varying their slenderness for the LLF, WLF_0 , WLF and FOC models under the monotonic test of (van Vliet & van Mier, 1995).

2.7. Estimation of the epistemic uncertainty

To measure the epistemic uncertainties on inelastic constitutive concrete models studied, five types of response parameters were considered: **(1)** peak stress σ_p and respective strain ε_p of the monotonic stress-strain curve; **(2)** dissipated energy of the monotonic $\bar{G}_m$ stress-strain curve; **(3)** dissipated energy of the first $\bar{G}_{c1}$, last $\bar{G}_{c\infty}$, and total $\bar{G}_c$ loading-unloading cycle of stress-strain curve, respectively; **(4)** linearized least square stiffness of hardening branch $\bar{K}_h$, and softening branch $\bar{K}_s$, of monotonic stress-strain curve, respectively; and **(5)** first, $\bar{K}_{c1}$, and last $\bar{K}_{c\infty}$ linearized least square stiffnesses in loading-unloading cycle, respectively. Fig. 2.7.18 illustrates these parameters for clarification.

The uncertainty of inelastic concrete models is measured in only six experimental tests cases: (i) the uniaxial cyclic tension test; (ii) the uniaxial cyclic compression test; (iii) a biaxial monotonic test; (iv) a triaxial monotonic test; (v) the uniaxial cyclic tension-compression test; and (vi) a strain-rate case, for both tensile and compressive loading. For all these cases, the uncertainty is measured as the ratio of simulated concrete models R_{num} relative to experimental test results R_{exp} . The uncertainty of the ratios $R_{\text{num}}/R_{\text{exp}}$ is characterized by its minimum, maximum values, and the standard deviation σ .

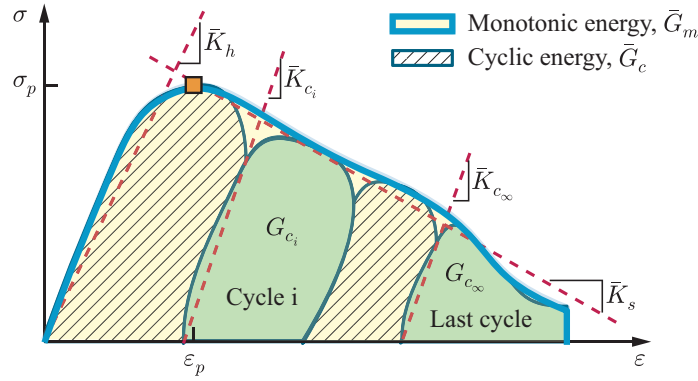


Figure 2.7.18. Definition of response parameters to measure epistemic uncertainty in inelastic concrete models.

Fig. 2.7.19 summarize a box-plot of the uniaxial cyclic tension and compression simulation with the five class of output parameters defined. The box-plots considered hereafter contains a rectangle whose length is the difference between the first and third quartile, a median $\bar{x}$ represented by an intermediate horizontal line, a mean represented by a rhombus, whiskers equivalent in width to two standard deviations (2σ), and outliers which fall outside the range $(\bar{x} \pm \sigma)$.

It can be observed up to 47% less amount of energy dissipated by the first loading-unloading cycle $\bar{G}_{c_1}$ of numerical simulations than experimental test, with a considerable uncertainty of up to $\sigma = 20.6\%$, both in compression as in tensile regime. Similarly, up to 40% more flexible is the first loading-unloading stiffness $\bar{K}_{c_1}$, mainly due to WLF₀ and ROT models, which unloads to origin. Conversely, there are up to 2.68 and 2.41 times

more energy dissipated by monotonic $\bar{G}_m$, and cyclic stress-strain curve $\bar{G}_c$, respectively (excepting in compressive load) by numerical simulations than in the experimental test, both in tensile as compressive case, mainly due to DPH plastic model. Furthermore, a high variability exist for the energy dissipated by the last cycle $\bar{G}_{c\infty}$, with $\sigma = 112.2\%$ for tensile case, and by loading-unloading stiffness of last cycle $\bar{K}_{c\infty}$, both in tension as in compression regime, with $\sigma = 262.6\%$ and 175.6% , respectively.

In contrast, for tensile regime, the variables σ_p , ε_p , and $\bar{K}_h$ gives a good fit adjustment in all concrete models, with a standard deviation less than 10%, whereas for compressive regime, the variables σ_p , $\bar{G}_m$, and $\bar{G}_c$ gives a $\sigma \leq 15\%$. Finally, its concluded that the most important source of epistemic uncertainty in tensile regime is observed by the energy dissipated $\bar{G}_{c\infty}$ and linearized stiffness $\bar{K}_{c\infty}$ in the last loading-unloading cycle, with a standard deviation of 112.2% and 262.6%, respectively, whereas for tensile regime the variable $\bar{K}_{c\infty}$ gives a considerable uncertainty, with $\sigma = 175.6\%$. The main reason of this high uncertainty is due to the differences in the taxonomy of stress-strain constitutive concrete models considered (plastic, damage or plastic-damage).

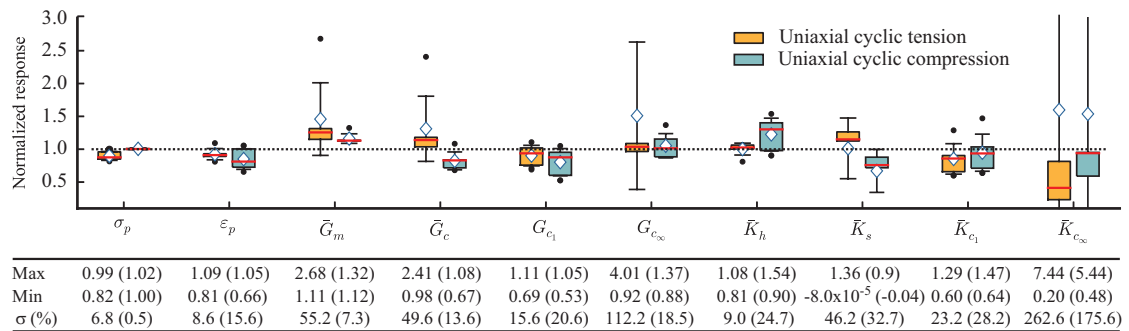


Figure 2.7.19. Response parameters of the numerical concrete models normalized by the experimental benchmark test results in the uniaxial cyclic tension and respective compression test: box-plot diagram (top); and maximum, minimum and standard deviation σ (%) (bottom). (Values in parenthesis associated with the uniaxial cyclic compression simulation.)

Fig. 2.7.20a-b shown the box-plot of response parameters for the biaxial and triaxial monotonic simulations, respectively. For the biaxial case, the box-plot for each variable correspond to combination of all stress ratios simulated $a = \sigma_1/\sigma_2$ with $a=0, 1$, and

0.52, whereas for triaxial case, each box-plot combine all confining pressure simulated, p_o from 0 to 43 MPa. For one hand, a good fit adjust exist for all response variables measured, with a medium-level of uncertainty less than 30%. For other hand, a low-level of uncertainty is observed for the peak stress σ_p , with $\sigma = 4\%$, whereas a high uncertainty is observed in the variables ε_p , $\bar{G}_m$, and $\bar{K}_h$, with a standard deviation of 83.7%, 58.9% and 55.2%, respectively. Later, this uncertainty is due to simplicity of term considered in the constitutive concrete equations to simulate the triaxial effect.

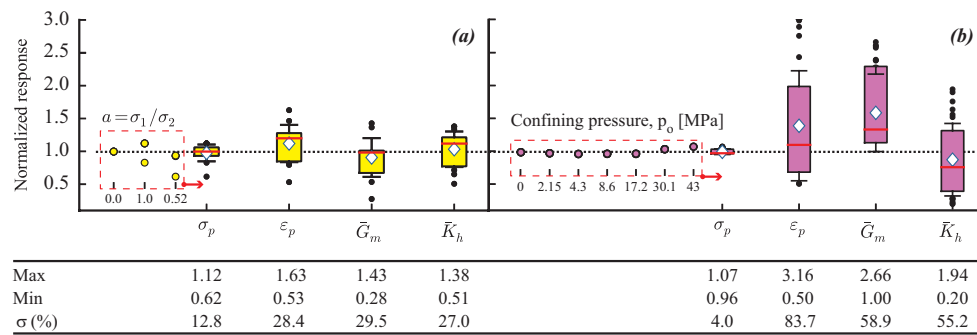


Figure 2.7.20. Response parameters of the numerical concrete models normalized by the experimental benchmark test results: (a) biaxial monotonic; and (b) triaxial monotonic. Box-plot diagram (top); and maximum, minimum and standard deviation σ (%) (bottom).

Fig. 2.7.21a-b shown the box-plot for the uniaxial cyclic tension-compression (unilateral effect) and strain-rate case, respectively. For one hand, the standard deviation of all parameters is less than 10%, with the exception of the variables $\bar{G}_m$ and $\bar{K}_{c\infty}$, where $\sigma=25.9\%$ and 38.3% , respectively. For other hand, a good fit correlation exist for strain-rate simulations less than $\dot{\varepsilon} < 10^{-1}/s$, in all concrete models, as both in tensile as in compressive regimes, with values that ranges between 0.71 and 1.05 times the experimental tests results. Conversely, for a strain-rate over $1/s$, higher values are observed of up to 2.74 and 1.52 times, for tensile and compressive load, respectively. Thus, combining all strain-rate cases in an unique box-plot, gives a standard deviation of 44.0% and 14.2%, for tensile and compressive load, respectively. Both observations demonstrates that there are more uncertainty in tensile regime than in compressive one.

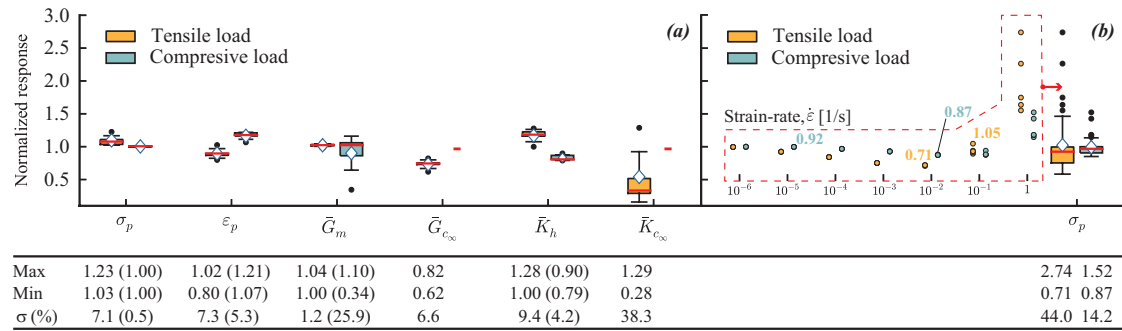


Figure 2.7.21. Response parameters of the numerical concrete models normalized by the experimental benchmark test results: (a) uniaxial cyclic tension-compression; and (b) strain-rate test. Box-plot diagram (top); and maximum, minimum and standard deviation σ (%) (bottom). (In both cases, values in parenthesis associated with the compressive load case.)

2.8. Summary and main results

This chapter study the epistemic uncertainty in five continuum stress-strain local constitutive concrete models for the three dimensional finite element formulation. The models considered are the most commonly used in the literature for plastic, plastic-damage and fracture mechanics of concrete. Convergence problems were encountered under certain conditions, especially in strain-softening regimes. Herein, a complete description of these models in a common notation was presented, providing all the necessary steps required to ensure adequate convergence and a consistent numerical implementation. Analytical expressions for the updated stress algorithms and new explicit expressions for the algorithmic consistent tangent stiffness tensor were developed. Also, a consistency check of input model parameters, such as uniaxial laws and fracture energy definition is discussed. The conversion from tensors and tensor operations to the vectorized format are provided for computational convenience. Numerical examples of benchmark tests under uniaxial, biaxial, and triaxial stresses demonstrated the capabilities of the proposed implementations. Moreover, the unilateral and strain-rate effects, the mesh size influence, and the strain-localization phenomenon are evaluated for each model. Further, the compression failure mode of a test specimen is illustrated as an example of application. The main results obtained from these part are:

- The construction of a robust updated stress algorithm consistent with the exact linearization of the evolution laws is necessary to get an adequate response of the models. Implicit schemes with return-mapping algorithms are preferred for the plastic component of models, whereas explicit schemes are sufficient for the damage ones. A counterexample of this happens with the FOC model, since its response is sensitive to the load step-size due to the explicit integration scheme used to compute the plastic component. Also, the election of an adequate initial value, non-zero derivatives, and a unique scalar variable to be solved rather than a system of equations, are critical for the convergence of Newton's method used. The latter is critical to solve the plastic component of models. Examples of this occur in the solution of the consistency operator of the DPH, LLF and WLF models.
- The correct derivation of the consistent tangent stiffness tensor is also critical in achieving convergence of the models. For the sake of numerical convergence we recommend the use of continuous and smooth derivatives (C^1 -class) for this operator. Analogously, we recommend the use of smooth C^2 -class functions for the flow potential of the DPH model; the C^1 -class approximated Heaviside function in the yield criterion of the LLF and WLF models; and the use of C^1 -class functions for the uniaxial laws ($\sigma - \varepsilon$, $\sigma - \kappa$ or $\omega - r$) in all models. In addition, the use of any asymmetric Newton-Raphson solver is mandatory if the stiffness matrix is non-symmetric. It is apparent that the LLF and WLF models are very sensitive to this condition under biaxial and triaxial loadings.
- Including a viscous model in the constitutive equations is a simple and robust technique to overcome convergence problems caused by the strain-localization phenomena in local models. This technique includes an artificial numerical viscosity in the equations to convert the stiffness tensor into a positive-definite matrix despite of the existence of strain-softening regimes. However, this comes at the expense of a gradual strength over-estimation depending on the strain-rate increment. This method has been used in the plastic and damage components of

the LLF, WLF, FOC and ROT models using the Duvaut-Lions model. It is recommended to use a ratio of numerical viscosity to load step increment between 0.001 and 1.0 to get an adequate convergence without compromising accuracy in the response.

- All models, with the exception of the DPH model, can simulate the strain-softening behavior correctly. Also, the LLF, WLF and FOC models can predict the inelastic strains and stiffness degradation, whereas the WLF model without plastic strains (WLF_0) and the ROT model both unload to the origin, i.e. they are pure damage models. It should be noted that the WLF_0 and ROT models give identical responses in all cases, although they work with formulations that are completely different.
- All models, except the ROT model, incorporate the biaxial effect adequately since they include the Drucker-Prager yield criterion in their equations. In contrast, only the LLF, WLF and FOC models can simulate the triaxial effect correctly. Moreover, these models have been validated with a confining pressure up to 43 MPa, giving a good correlation with experimental results. More complex models may be required to simulate higher confinement pressure levels. In addition, the volumetric expansion (dilatancy) is only simulated by the WLF model, whereas the other two models present a reduction in volume (compaction) for all ranges of confining pressures.
- With the exception of the DPH model, all models can simulate the unilateral effect correctly, in which the unloading compression stiffness is recovered once the load goes from tension to compression (crack-closure), and the unloading tension stiffness is obtained in the reverse case (crack-opening). Both conditions denote that the models have "damage memory", which agrees with the thermodynamics of irreversible processes.
- Excepting the DPH model, all models can simulate the strain-rate effect by means of the Duvaut-Lions visco-plastic model. It is observed that the tensile response is more sensitive to the strain-rate increments than the compression one

for all models. However, a poor fit is obtained relative to experimental test. In that case, the use of other models such as the Perzyna model is required.

- All models give a mesh-objective response with a localized damaged zone if a perturbation exists in the material of a FE model. Conversely, not all of the models have an uniform strain field in cases without imperfections, where the WLF and FOC models localize under tensile loads, whereas the LLF model localizes both in tension and compression. In contrast, the WLF₀ and ROT models do not localize if we have a perfect material.
- It can be observed that the unloading-loading linearized stiffness of the last cycle $\bar{K}_{c\infty}$, both for the uniaxial cyclic tension as well as in compressive test, is the most important source of epistemic uncertainty, with a standard deviation of the normalized results of 262.6% and 175.6%, respectively. Moreover, a considerable level of uncertainty is observed in the energy dissipated by the last unloading-loading uniaxial tensile cycle $\bar{G}_{c\infty}$, with $\sigma=112.2\%$. This high uncertainty is due to the different concrete models considered, i.e. plastic, damage and plastic-damage types.
- Furthermore, a standard deviation of 83.7%, 58.9% and 55.2% was estimated for the strain at peak stress ε_p , energy dissipated by monotonic stress-strain curve $\bar{G}_m$, and linearized stiffness of hardening branch of monotonic stress-strain curve $\bar{K}_h$, respectively, in triaxial monotonic tests, considering a confining pressure of up to 43 MPa. Also, a significant source of uncertainty occurs in the peak stresses σ_p for the strain-rate case with strain-rates $\dot{\varepsilon}$ over 1/s, both in tension as in compression loads, where the simulations give values of up to 2.74 times respect to experimental tests. A standard deviation of 44.0% and 14.2% was also obtained in this case, for tensile and compressive load, respectively, leading more uncertainty in tensile loads than in compressive ones. This is due to the use of the visco-plastic Duvaut-Lions model, used herein as a first approximation to simulate strain-rate effects given its simpler numerical implementation.

- In contrast, low uncertainty is observed in the peak stress σ_p , in the six test cases, excepting the strain-rate test, with a standard deviation less than 12.8%. Moreover, the dissipated energy by the monotonic curve $\bar{G}_m$, for the uniaxial cyclic compression and tensile branch of uniaxial tension-compressive test is less than 7.3%.

3. CONTINUUM STRESS-STRAIN CONCRETE MODELS AND CONSISTENT NUMERICAL IMPLEMENTATION FOR PLANE-STRESS CONDITION

Plane-stress constitutive models are widely used with shell elements to model RC walls, slabs and membranes. Because concrete is a brittle material that exhibits a strongly nonlinear response associated with the propagation of cracking, its correct modeling in practice is difficult and requires substantial expertise and robust numerical algorithms to achieve convergence. These analyses are critical given the brittle performance observed in reinforced concrete shear wall buildings during recent earthquakes ([Jünemann et al., 2015](#); [Jünemann, de la Llera, Hube, Vásquez, & Chacón, 2016](#)). The plane-stress condition arises in structural elements in which one dimension is much smaller than the others and the element is subjected to loads perpendicular to the thickness. The proper formulation and numerical implementation of plane stress is very different from the full 3D-case, and hence, this chapter describes in detail this formulation and implementation.

Shell finite elements are commonly used to simulate plane-stress conditions. Indeed, multi-layered shell type elements are best suited to represent an accurate distribution of in-plane and out-of-plane concrete stresses, with a considerable reduction of computational cost relative to the use of solid finite elements ([Chacón et al., 2017](#)). Accuracy in these elements is strongly dependent on the algorithmic implementation and the integration techniques adopted ([Krieg & Krieg, 1977](#); [J. C. Simo & Taylor, 1985](#)). For strain-driven models, these algorithms seek to: (i) the integration of the updated stress vector given the strain increment; and (ii) the computation of the stiffness matrix according to the updated stresses. The use of implicit integration schemes with return-mapping algorithms (RMA) is typical for plastic and plastic-damage models, whereas explicit integration schemes are used for damage and smeared crack models. A large variety of algorithms for numerical implementation of these models are available in the literature ([J. C. Simo & Hughes, 1998](#); [de Souza Neto et al., 2008](#)), but this article focuses on the plane-stress case, which also had some numerical features different from the 3D-case.

Specifically, if a strain-driven model is used, an additional constraint is imposed to satisfy the condition of zero normal stress. Moreover, to account for plastic effects, the radial return-mapping algorithm used in the 3D-case formulation is not valid for the plane-stress condition, and hence, the consistent operator cannot be obtained explicitly. Consequently, the use of specific algorithms for plane-stress is mandatory. Three approaches are commonly adopted to solve this problem: (i) to include the plane-stress constraint within the 3D algorithm equations considering a nonlinear iterative solver at the Gauss point level (Aravas, 1987; Dodds, 1987; Klinkel & Govindjee, 2002); (ii) the use of standard 3D algorithms at the Gauss point level with the plane-stress condition added as a structural constraint at global level (de Borst, 1991); or (iii) the use of plane stress-projected equations, in which the plane-stress constraint is enforced within the equations at the Gauss point level (Schreyer, Kulak, & Kramer, 1979; Jetteur, 1986; J. C. Simo & Taylor, 1986). In general, the first two options are more adequate for complex plastic models at the expense of an additional computational cost. In contrast, the projected plane-stress method is a direct, exact, and efficient computationally procedure that involves only the in-plane stress and strain components. However, more complex equations are involved in the numerical integration of the projected plane-stress approach, that can only be formulated for relatively simple models. The latter method is more popular than the two former ones (J. Lee & Fenves, 1998; Lourenço, de Borst, & Rots, 1998; J. C. Simo & Hughes, 1998; de Souza Neto et al., 2008; Valoroso & Rosati, 2009).

The objective of this chapter is to provide, in a common vectorized notation, the numerical implementation of the plane-stress formulation for the same five concrete models considered in the past chapter.

This chapter presents in Section 3.1 a complete description of the set concrete models considered. A detailed development of the algorithms for numerical implementation of the updated stress vector for the plane-stress condition is provided in Section 3.2. Moreover, new explicit analytical expressions for the algorithmic consistent tangent stiffness matrix of such models are presented in Section 3.3. Furthermore, numerical examples using

basic benchmarks tests subject to monotonic and cyclic loading conditions under uniaxial and biaxial stresses are presented in Section 3.4 to demonstrate the capabilities of these concrete models. In addition, the unilateral effect, strain-rate effects, mesh size influence and strain-localization phenomena are discussed for each model.

3.1. Description of concrete models

This section summarized the equations of the five continuum concrete models considered in this article. Also, include some modifications adequate to improve the convergence of models. For the sake of simplicity, all equations are described in a vectorized and matrix format according to an adequate conversion of their respective tensors, considering only the components associated to in-plane behavior $((\cdot)_{11}, (\cdot)_{22}, (\cdot)_{12})$, unless otherwise stated. Details of this conversion can be founded in Appendix B.

3.1.1. Drucker-Prager Hyperbolic (DPH) model

This plastic model, so-called the "Extended Drucker-Prager" model was defined by (Drucker & Prager, 1952) and modified by (ANSYS, 2018; ABAQUS, 2018). Is a simplification of Mohr-Coulomb model and have been used to simulate soil or cohesive materials, like concrete. First, the strain vector $\underline{\epsilon}$ is decomposed additively into its elastic, $\underline{\epsilon}^e$, and plastic part, $\underline{\epsilon}^p$, as follow

$$\underline{\epsilon} = \underline{\epsilon}^e + \underline{\epsilon}^p. \quad (3.1.1)$$

Then, for the case of linear elasticity, they can be related to the Cauchy stress vector $\underline{\sigma}$ by

$$\underline{\sigma} = \underline{D}_e(\underline{\epsilon} - \underline{\epsilon}^p), \quad (3.1.2)$$

where $\underline{D}_e$ is the linear-elastic stiffness matrix (see Eq. (A.2.6) for their definition). The yield criterion is defined as

$$F(\underline{\sigma}, \alpha) := \eta p + \sqrt{3J_2} - \xi c(\alpha), \quad (3.1.3)$$

where the hydrostatic stress p is included to simulate the pressure-dependent behavior and the asymmetric tensile/compressive strength of concrete; η and ξ are material parameters chosen according to the required approximation to the Mohr-Coulomb criterion or fitted to uniaxial/biaxial tensile and compressive strength of concrete; and $c(\alpha)$ is the cohesion hardening law, which is taken as function of the equivalent plastic strain α . The later variable is defined as $\alpha := \int_0^t \|\dot{\epsilon}^p\| dt$. It is assumed an exponential relation for the cohesion hardening law $c(\alpha)$ as

$$c(\alpha) = c_u + (c_y - c_u)e^{(-\alpha/a_o)}, \quad (3.1.4)$$

where $c_u = Rc_y$ and $\alpha_o = c_u/E_o$, with $R > 1$ an experimental fitted parameter. Discussion of parameters η and ξ are detailed in Section 2.5.

For other hand, a hyperbolic shape is adopted for the flow potential, and is defined as

$$G := \bar{\eta}p + \sqrt{q^2 + \epsilon^2}, \quad (3.1.5)$$

where $\bar{\eta}$ is a constant that depends of the dilatancy angle, ϵ is a eccentricity parameter that controls the shape of surcafe near of tensile regimes, generally used less than 0.001 (ABAQUS, 2018) and $q = \sqrt{3J_2}$. For the plane stress condition, the invariant J_2 can be correctly expressed as $J_2 = \frac{1}{2}\underline{\sigma}^T \underline{P}\underline{\sigma}$ (see Eq. (A.2.6)). Observe that this flow potential is a smoothed surface (C^2 -class) that avoid the singularity at the cone's apex present in the classical Drucker-Prager model, giving an unique flow direction in this region. Moreover, the projected region of the plane-stress condition for this flow potential gives always a smoothed function (C^2 -class) whatever value of eccentricity adopted. Then, the non-associated flow rule for the plastic strain vector is given by

$$\underline{\dot{\epsilon}}^p := \dot{\gamma} \underline{n}, \quad (3.1.6)$$

where γ is the plastic operator and $\underline{n}$ denotes the flow vector expressed as

$$\underline{n} := \frac{\partial G}{\partial \underline{\sigma}} = \frac{3}{2r} \underline{P\sigma} + \frac{\bar{\eta}}{3} \underline{1}, \quad (3.1.7)$$

with $r = \sqrt{q^2 + \epsilon^2}$. Also, the evolution of equivalent plastic strain α is stated as

$$\dot{\alpha} := \dot{\gamma} \xi. \quad (3.1.8)$$

Finally, the loading-unloading Karush-Kuhn-Tucker (KKT) and consistency conditions, respectively, are expressed as

$$\dot{\gamma} \geq 0, \quad F(\underline{\sigma}, \gamma) \leq 0, \quad \dot{\gamma} F(\underline{\sigma}, \gamma) = 0 \quad (3.1.9)$$

$$F(\underline{\sigma}, \gamma) = \dot{F}(\underline{\sigma}, \gamma) = 0 \quad (3.1.10)$$

In addition, the out-of-plane plastic strain ε_{33}^p , can be derived considering all components of strain tensor (in-plane and out-of-plane). Then, using Eq. (A.2.8), the second-order flow tensor $\underline{N}_3$ is expressed as

$$\underline{N}_3 := \frac{\partial G}{\partial \underline{\sigma}_3} = \frac{3}{2r} \underline{s}_3 + \frac{\bar{\eta}}{3} \underline{I}_3, \quad (3.1.11)$$

where subscripts "3" denotes the 3D second-order tensor (see appendix 2), $\underline{s}_3$ is the deviatoric stress tensor (Eq. (A.2.5)) and $\underline{I}_3 = \text{diag}(1, 1, 1)$ the second-order unitary tensor. Thus, the evolution law of out-of-plane plastic strain is written as

$$\dot{\varepsilon}_{33}^p = \dot{\gamma} \left[-\frac{1}{2r} (\sigma_{11} + \sigma_{22}) + \frac{\bar{\eta}}{3} \right] \quad (3.1.12)$$

Moreover, using Eq. (A.2.9) and due that $\text{tr}(\underline{s}_3) = 0$, it follows that the volumetric strain rate can be estimated as

$$\dot{\varepsilon}_v = \dot{\varepsilon}_v^e + \dot{\varepsilon}_v^p = K^{-1} p + \dot{\gamma} \bar{\eta}, \quad (3.1.13)$$

where ε_v^e and ε_v^p are the elastic and plastic volumetric strain, respectively, and K the Bulk moduli. It can be observed that $\bar{\eta}$ controls the inelastic volumetric strain rate (dilatancy).

3.1.2. Lubliner-Lee-Fenves (LLF) model

This plastic-damage model, so-called "Barcelona" model, was first developed by (Lubliner et al., 1989) and later improved by (J. Lee & Fenves, 1998). First, using Lemaitre's strain equivalent hypothesis (Lemaitre, 1989), the nominal stress vector $\underline{\sigma}$ associated with the damage state is related to the effective stress $\bar{\sigma}$ corresponding to the undamaged state as follows

$$\underline{\sigma} := (1 - \omega)\bar{\sigma}, \quad (3.1.14)$$

where ω is the isotropic damage variable, with $\omega \in [0, 1]$.

Plastic component

To calculate this component, it is assumed the so-called *effective stress space plasticity*, which are related to the effective stress vector and is dependent (coupled) of damage component (Wu et al., 2006). First, two hardening scalar variables $\kappa^\pm$ are stated to control the positive/negative part of plastic-damage component, respectively. (Lubliner et al., 1989) define normalized variables for uniaxial case as follows

$$\kappa^\pm := \frac{1}{g^\pm} \int_0^{\alpha^\pm} \sigma^\pm(\alpha^\pm) d\alpha^\pm, \quad \dot{\kappa}^\pm = \frac{1}{g^\pm} \sigma^\pm(\alpha^\pm) \dot{\alpha}^\pm, \quad (3.1.15)$$

which correspond to accumulated area under positive/negative uniaxial stress-equivalent plastic strain law ($\sigma^\pm - \alpha^\pm$), respectively, with $\kappa^\pm \in [0, 1]$, $g^\pm = \int_0^\infty \sigma^\pm(\alpha^\pm) d\alpha^\pm$ are the total area under their respective stress law. Note that positive values are used for $\sigma^\pm$. Moreover, the positive/negative equivalent plastic strain $\alpha^\pm$ are defined as

$$\alpha^\pm = \int |\dot{\varepsilon}_\pm^p| dt, \quad (3.1.16)$$

where $\dot{\varepsilon}_+^p = \dot{\varepsilon}_{max}^p$ and $\dot{\varepsilon}_-^p = -\dot{\varepsilon}_{min}^p$, with $\dot{\varepsilon}_{max,min}^p$ are the maximum and minimum eigenvalues ratio of principal plastic strain vector $\underline{\varepsilon}^p$, respectively. Moreover, in case for multi-axial condition, the evolution law of variables $\kappa^\pm$ in a vectorized format $\underline{\kappa} = [\kappa^+, \kappa^-]^T$ is defined as

$$\dot{\underline{\kappa}} := \underline{W}(\hat{\underline{\sigma}}, \underline{\kappa}) \dot{\underline{\varepsilon}}^p \quad (3.1.17)$$

$$\underline{W}(\hat{\underline{\sigma}}, \underline{\kappa}) := \text{diag} \left[\phi(\hat{\underline{\sigma}}) \frac{\sigma^+(\kappa^+)}{g^+}, (\phi(\hat{\underline{\sigma}}) - 1) \frac{\sigma^-(\kappa^-)}{g^-} \right],$$

where $\dot{\underline{\varepsilon}}^p = [\dot{\varepsilon}_1^p, \dot{\varepsilon}_2^p]^T$ is the ratio of principal plastic strain vector, which is filled in an algebraic order (e.g. $\dot{\varepsilon}_1^p > \dot{\varepsilon}_2^p$); and $\phi(\hat{\underline{\sigma}})$ is a weight factor $\in [0, 1]$, defined as

$$\phi(\hat{\underline{\sigma}}) := \begin{cases} 0, & \hat{\sigma}_i = 0 \\ \frac{\sum_{i=1}^N \langle \hat{\sigma}_i \rangle^+}{\sum_{i=1}^N |\hat{\sigma}_i|}, & \text{otherwise} \end{cases} \quad (3.1.18)$$

An adequate conversion of uniaxial stress laws from the relation $\sigma^\pm - \alpha^\pm$ to $\sigma^\pm - \kappa^\pm$, using Eq. (3.1.15) is necessary to generate. Detail of this conversion is discussed in Section 2.5.1. For the other hand, similar to stated in the DPH model, the hyperbolic Drucker-Prager criterion as defined by Eq. (3.1.5) is used for the flow potential. Moreover, due that any isotropic material satisfy the relation $G(\bar{\underline{\sigma}}) = \hat{G}(\hat{\underline{\sigma}})$ and that p , J_2 , and r are invariants in the effective stress space $((\cdot) = (\hat{\cdot}))$, the flow potential in the principal effective space can be rewritten as

$$\hat{G}(\hat{\underline{\sigma}}) = \bar{\eta} \bar{p} + \sqrt{\bar{J}_2 + \epsilon^2} \quad (3.1.19)$$

Then, the non-associated flow rule satisfy the relation in the principal space as

$$\dot{\underline{\varepsilon}}^p = \dot{\gamma} \hat{\underline{n}}, \quad (3.1.20)$$

$$\hat{\underline{n}} := \frac{\partial \hat{G}}{\partial \hat{\underline{\sigma}}} = \frac{3}{2r} \hat{\underline{P}} \hat{\underline{\sigma}} + \frac{\bar{\eta}}{3} \hat{\underline{1}}, \quad (3.1.21)$$

where $\hat{\underline{n}}$ is the principal effective flow vector. Thus, Eq. (3.1.17) can be rewritten as

$$\dot{\underline{\kappa}} = \dot{\gamma} \underline{H}(\hat{\underline{\sigma}}, \underline{\kappa}), \quad (3.1.22)$$

where $\underline{H}(\hat{\underline{\sigma}}, \underline{\kappa}) = \underline{W}(\hat{\underline{\sigma}}, \underline{\kappa}) \hat{\underline{n}}$. Finally, the yield criterion is first established by (Lubliner et al., 1989) and later modified by (J. Lee & Fenves, 2001) as follow

$$\bar{F}(\hat{\underline{\sigma}}, \underline{\kappa}) := \eta \bar{p} + \sqrt{3 \bar{J}_2} + \beta(\underline{\kappa}) \langle \hat{\sigma}_{max} \rangle^+ - (1 - \alpha) c(\kappa^-), \quad (3.1.23)$$

where $\alpha = (f'_b - f'_c)/(2f'_b - f'_c)$ and $\beta(\underline{\kappa})$ and $c(\kappa^-)$ are parameters to distinguish the different evolution of strength under tension and compression given by

$$\beta(\underline{\kappa}) := (1 - \alpha) \frac{\bar{\sigma}^-(\kappa^-)}{\bar{\sigma}^+(\kappa^+)} - (1 + \alpha), \quad c(\kappa^-) := \bar{\sigma}^-(\kappa^-), \quad (3.1.24)$$

where $\bar{\sigma}^\pm$ are the positive/negative uniaxial effective stress law, respectively. Typical experimental values of the ratio f'_b/f'_c for concrete ranges from 1.10 to 1.16, yielding values of α between 0.08 and 0.12. It should be noted that this yield function do not include the triaxial effect as proposed by (J. Lee & Fenves, 2001) due to plane-stress condition.

In addition, similar to the DPH model, using Eqs. (3.1.12) and (A.2.7) expressed in the effective stress space, the evolution law for the plastic ε_{33}^p and elastic ε_{33}^e out-of-plane strain are given, respectively, by

$$\varepsilon_{33}^p = \dot{\gamma} \left[-\frac{1}{2\bar{r}} (\bar{\sigma}_{11} + \bar{\sigma}_{22}) + \frac{\bar{\eta}}{3} \right], \quad (3.1.25)$$

$$\varepsilon_{33}^e = -\frac{\nu}{E_o} (\bar{\sigma}_{11} + \bar{\sigma}_{22}). \quad (3.1.26)$$

Moreover, using this relations and Eq. (A.2.9), the volumetric strain rate can be estimated as

$$\dot{\varepsilon}_v = \varepsilon_v^e + \varepsilon_v^p = K^{-1} \bar{p} + \dot{\gamma} \bar{\eta} \quad (3.1.27)$$

Damage component

(ABAQUS, 2018) define the damage variable ω as

$$\omega := 1 - [1 - s^-(\hat{\underline{\sigma}})\omega^+(\kappa^+)] [1 - s^+(\hat{\underline{\sigma}})\omega^-(\kappa^-)], \quad (3.1.28)$$

where $s^\pm(\hat{\underline{\sigma}})$ are stiffness recovery functions and $\omega^\pm(\kappa^\pm)$ positive/negative damage laws, respectively. For one hand, the stiffness recovery functions $s^\pm$ can be defined as

$$s^+ := 1 - z_c^+ \phi(\hat{\underline{\sigma}}), \quad s^- := 1 - z_c^-(1 - \phi(\hat{\underline{\sigma}})), \quad (3.1.29)$$

where $z_c^\pm \in [0, 1]$ are a stiffness recovery factor from tensile to compression load state and viceversa. Empirical evidence shown that compressive stiffness is recovered upon crack closure as the load changes from tension to compression ($z_c^+ \approx 1$). However, tensile stiffness is not recovered as the load changes from compression to tension once crushing micro-cracks have developed ($z_c^- \approx 0$). For other hand, the positive/negative damage laws $\omega^\pm(\kappa^\pm)$ laws are fitted experimentaly and generally known in terms of equivalent plastic strain $\alpha^\pm$, e.g. a common exponential relation is used as

$$\omega^\pm(\alpha^\pm) = 1 - \exp(-C^\pm \alpha^\pm), \quad (3.1.30)$$

with $C^\pm$ an experimental parameter that control the unloading branch of response. Due this, its required an adequate conversion from $\omega^\pm - \alpha^\pm$ to $\omega^\pm - \kappa^\pm$ laws as explained in Section 2.5. Thus, the uniaxial positive/negative stress $\sigma^\pm$ laws can be related to respective effective stress $\bar{\sigma}^\pm$ laws as follows

$$\sigma^\pm(\kappa^\pm) = [1 - \omega(\kappa^\pm)] \bar{\sigma}^\pm(\kappa^\pm). \quad (3.1.31)$$

Viscous component

Additionally, the model can include strain-rate dependency with a visco-plastic model, which improve the convergence in strain-softening regimes. To this, the nominal stress

vector $\underline{\sigma}$ is now converted to their respective viscous component $\underline{\sigma}^v$, and is defined as

$$\underline{\sigma}^v = (1 - \omega^v) \bar{\underline{\sigma}}^v, \quad (3.1.32)$$

where ω^v is the viscous damage variable and $\bar{\underline{\sigma}}^v$ is the effective viscous stress vector. (J. Lee & Fenves, 2001) calculate this component using the (Duvaut & Lions, 1972) visco-plastic model, which is stated in the effective stress space as

$$\dot{\underline{\epsilon}}^{vp} := \frac{1}{\mu_v} \underline{C}_e (\bar{\underline{\sigma}}^v - \bar{\underline{\sigma}}), \quad (3.1.33)$$

$$\bar{\underline{\sigma}}^v := \underline{D}_e (\underline{\epsilon} - \underline{\epsilon}^{vp}), \quad (3.1.34)$$

with $\underline{\epsilon}^{vp}$ is the visco-plastic strain vector and μ_v is the numerical viscosity parameter and is equivalent to the relaxation time. Thus, combining both relation gives

$$\dot{\underline{\epsilon}}^{vp} = -\frac{1}{\mu} (\underline{\epsilon}^{vp} - \underline{\epsilon}^p). \quad (3.1.35)$$

Moreover, the evolution law of viscous-damage variable ω^v is defined as

$$\dot{\omega}^v := -\frac{1}{\mu_v} (\omega^v - \omega). \quad (3.1.36)$$

3.1.3. Wu-Li-Fara (WLF) model

This plastic-damage model, was first developed by (Faria et al., 1998) and modified by (Wu et al., 2006). Two variants are developed for this model: one approach that include the plastic and damage components (WFL) and other one with pure damage behavior (WFL₀). First, assume that the effective stress vector $\bar{\underline{\sigma}}$ are splitted into positive $\bar{\underline{\sigma}}^+$ and negative $\bar{\underline{\sigma}}^-$ parts, to account separately the cracking (tension) and shear (compression) damage mechanisms for degradation of concrete (Ladeveze, 1983; Ortiz, 1985), using the follow

decomposition

$$\underline{\bar{\sigma}}^{\pm} := \sum_{i=1}^N \langle \hat{\sigma}_i \rangle^{\pm} \underline{e}_{\bar{\sigma}}^{ii} = \underline{P}^{\pm} \underline{\bar{\sigma}}, \quad (3.1.37)$$

$$\underline{P}^{\pm} := \sum_{i=1}^N H^{\pm}(\hat{\sigma}_i) (\underline{e}_{\bar{\sigma}}^{ii} \otimes \underline{e}_{\bar{\sigma}}^{ii}) \underline{R}, \quad (3.1.38)$$

where $\underline{P}^{\pm}$ are the projection matrices, with symbol ' $\pm$ ' denoting '+' or '-' as appropriate, $\hat{\sigma}_i$ denote the i -th eigenvalue of mapped tensor $\bar{\sigma}_2$ and $\underline{e}_i$ is the i -th eigen-projector vector associated to $\bar{\sigma}$ (see appendix A). This decomposition satisfy the relations $\underline{\bar{\sigma}} = \underline{\bar{\sigma}}^+ + \underline{\bar{\sigma}}^-$ and $\underline{P}^+ + \underline{P}^- \neq \underline{I}$. Next, in order to establish the intended constitutive law, (Wu et al., 2006) define the total elasto-plastic HFE potential as follows

$$\psi(\underline{\epsilon}^e, \underline{\omega}, \underline{\kappa}) := \psi^e(\underline{\epsilon}^e, \underline{\omega}) + \psi^p(\underline{\kappa}, \underline{\omega}), \quad (3.1.39)$$

$$\psi^e(\underline{\epsilon}^e, \underline{\omega}) = (1 - \omega^+) \psi_o^{e+}(\underline{\epsilon}^e) + (1 - \omega^-) \psi_o^{e-}(\underline{\epsilon}^e), \quad (3.1.40)$$

$$\psi^p(\underline{\kappa}, \underline{\omega}) = (1 - \omega^+) \psi_o^{p+}(\underline{\kappa}) + (1 - \omega^-) \psi_o^{p-}(\underline{\kappa}), \quad (3.1.41)$$

where $\omega^{\pm} = \omega^{\pm}(r^{\pm})$ are positive/negative scalar damage variables $\in [0, 1]$, respectively, which are in function of the damage thresholds $r^{\pm}$, that controls the size of damage surfaces; $\underline{\omega} = [\omega^+, \omega^-]^T$ denotes the damage vector; $\psi_o^{e\pm}$ are the undamaged elastic HFE potential and are equals to the strain energy per unity of volume. Thus $\psi_o^{e\pm} = \frac{1}{2} \underline{\bar{\sigma}}^T \underline{\epsilon}^e$; and $\psi_o^{p\pm}$ are the undamaged plastic HFE potential. Moreover, the Eq. (3.1.39) can be reordered as

$$\psi(\underline{\epsilon}^e, \omega^+, \omega^-, \underline{\kappa}) := (1 - \omega^+) \psi_o^+(\underline{\epsilon}^e, \underline{\kappa}) + (1 - \omega^-) \psi_o^-(\underline{\epsilon}^e, \underline{\kappa}) \quad (3.1.42)$$

where $\psi_o^{\pm}$ is the positive/negative total undamaged elasto-plastic HFE potential and are written as

$$\psi_o^{\pm} = \psi_o^{e\pm} + \psi_o^{p\pm} \quad (3.1.43)$$

For the other hand, the nominal Cauchy stress vector can be defined as

$$\underline{\sigma} := \frac{\partial \psi^e}{\partial \underline{\epsilon}^e} \quad (3.1.44)$$

Then, using the relation $\frac{\partial \psi_o^{e\pm}}{\partial \underline{\epsilon}^e} = \bar{\underline{\sigma}}^\pm$ and Eqs. (3.1.44), (3.1.37) and (3.1.40), this stress vector is expressed as

$$\underline{\sigma} := [(1 - \omega^+) \underline{P}^+ + (1 - \omega^-) \underline{P}^-] \bar{\underline{\sigma}} = \left(\sum_{\aleph} (1 - \omega^{\aleph}) \underline{P}^{\aleph} \right) \bar{\underline{\sigma}} \quad (3.1.45)$$

where $\aleph$ denote index summation for '+' and '-' part as appropriate hereafter.

Plastic component

Similar to the LLF model, the *effective stress space plasticity*, which are related to the effective stress vector, but in this case is independent (decoupled) of damage component (Wu et al., 2006). Although, the damage component depends of the variables stated in the plastic component. Due to this condition, they can include the plastic component as an option, conversely to the LLF model.

(Wu et al., 2006) assumed a Lee-Fenves yield criterion as stated in Eq. (3.1.23). Also, its assumed a flow potential criterion as defined in Eq. (3.1.5). For the other hand, similar to the LLF model, two hardening parameters $\kappa^\pm$ are proposed to control the positive/negative plastic component and are defined as the positive/negative equivalent plastic strain $\alpha^\pm$, where $\alpha^\pm = \int |\dot{\epsilon}_\pm| dt$. Then, for multi-axial condition, these hardening parameters are stated as $\kappa^+ = \phi(\hat{\underline{\sigma}}) \alpha^+$ and $\kappa^- = -[1 - \phi(\hat{\underline{\sigma}})] \alpha^-$, with $\phi(\hat{\underline{\sigma}})$ defined in Eq. (3.1.18). Then, the rate of hardening vector $\underline{\kappa} = [\kappa^+, \kappa^-]^T$ is defined similar to the Eq. (3.1.17), but with the matrix $\underline{W}(\hat{\underline{\sigma}})$ given by

$$\underline{W}(\hat{\underline{\sigma}}) := \text{diag} [\phi(\hat{\underline{\sigma}}), \phi(\hat{\underline{\sigma}}) - 1] . \quad (3.1.46)$$

In addition, the effective uniaxial stress $\bar{\sigma}^\pm(\kappa^\pm)$ laws are required. (Wu et al., 2006) assume a linear relation as follows

$$\bar{\sigma}^\pm(\kappa^\pm) = f_o^\pm + \bar{J}_\kappa^\pm \kappa^\pm, \quad (3.1.47)$$

where $f_o^\pm$ is the positive/negative initial stress, which are chosen for convenience in the range $f_o^+ \in [0, f_t']$ and $f_o^- \in [0, f_c']$, respectively, and $\bar{J}_\kappa^\pm = E_t^\pm E_o / (E_o - E_t^\pm)$ with $E_t^\pm$ are the hardening slope.

Damage component

For the damage component, its required a specific definition for the undamaged elasto-plastic HFE potential. For one hand, (J. Simo & Ju, 1987) assume that $\psi_o^\pm$ can be as the positive/negative elastic strain energy per unit of volume and expressed as

$$\psi_o^\pm(\underline{\epsilon}^e) := \frac{1}{2} (\bar{\underline{\sigma}}^+{}^T \underline{C}_e \bar{\underline{\sigma}}). \quad (3.1.48)$$

However, this HFE potential is more adequate in tensile regimes where contribution of plastic part is much smaller than the compression ones. Hence, for compressive regimes, Wu et al. define the following HFE potential that include the biaxial compression effect as follow

$$\psi_o^-(\underline{\epsilon}^e) := b_o \left(\eta \bar{p} + \sqrt{3 \bar{J}_2} \right)^2, \quad (3.1.49)$$

where b_o is a material parameter (defined in (Wu et al., 2006)) and $\eta = 3\alpha$. Next, the tensile and shear thermodynamic forces or Damage Energy Release Rate-based (DERR), $Y^\pm$, can be defined as

$$Y^\pm := -\frac{\partial \psi^\pm}{\partial \omega^\pm} = \psi_o^\pm \quad (3.1.50)$$

Then, the positive/negative damage criteria are defined as

$$F_d := g_d^\pm(Y^\pm) - g_d^\pm(r^\pm) \leq 0, \quad (3.1.51)$$

where $g_d(\cdot)$ can be any monotonically increasing scalar function. Using the Eqs. (3.1.48) and (3.1.49), these functions can be postulated as convenience as $g_d(\cdot)^\pm = \sqrt{2E_o(\cdot)}$ and $g_d(\cdot)^- = \sqrt{(\cdot)/b_o}$, respectively. Thus, the positive/negative DEERs can be rewritten as

$$Y^\pm := \sqrt{2E_o\psi_o^\pm} = \sqrt{E_o(\bar{\boldsymbol{\sigma}}^\pm{}^T \underline{\mathbf{C}}_e \bar{\boldsymbol{\sigma}})} \quad (3.1.52)$$

$$Y^- := \sqrt{\frac{\psi_o^-}{b_o}} = \eta\bar{p} + \sqrt{3\bar{J}_2}. \quad (3.1.53)$$

Moreover, the evolution damage law can be defined analogously to the classical plasticity, where the flow rule, the loading-unloading and the consistency conditions of damage component are defined, respectively, as

$$\dot{\omega}^\pm = \dot{\gamma}_d^\pm \frac{\partial g_d^\pm}{\partial Y^\pm} \quad (3.1.54)$$

$$\dot{\gamma}_d^\pm = \dot{r}^\pm \geq 0, \quad F_d^\pm \leq 0, \quad \dot{\gamma}_d^\pm F_d^\pm = 0, \quad (3.1.55)$$

$$F_d^\pm = \dot{F}_d^\pm = 0. \quad (3.1.56)$$

Its follow using Eqs. (3.1.55) and (3.1.56), that the damage thresholds $r^\pm$ are non-decreasing functions that satisfy the relations

$$r^\pm = \max \left(r_o^\pm, \max_{[0,t]}(Y^\pm) \right), \quad (3.1.57)$$

$$\dot{r}^\pm = \dot{Y}^\pm, \quad (3.1.58)$$

where $r_o^\pm$ are the initial damage thresholds. Assuming an uniaxial behavior and using Eqs. (3.1.52) and (3.1.53), these values can be calculated as $r_o^\pm = \sigma_o^\pm$ and $r_o^- = (1 - \alpha)\sigma_o^-$, respectively, where $\sigma_o^\pm$ are stress onset the nonlinear behavior. Finally, the positive/negative damage $\omega^\pm(r^\pm)$ laws are generally derived of experimental cracking process. (Mazars, 1984) define an exponential relation for the positive/negative component,

respectively, given by

$$\omega^+(r^+) := 1 - \frac{1}{\bar{z}^+} \left(1 - A^+ + A^+ e^{B^+(1-\bar{z}^+)} \right), \quad (3.1.59)$$

$$\omega^-(r^-) := 1 - \frac{1}{\bar{z}^-} \left(1 - A^- + A^- \bar{z}^- e^{B^-(1-\bar{z}^-)} \right) \quad (3.1.60)$$

where $\bar{z}^\pm = r^\pm / r_o^\pm$ and $A^\pm$ and $B^\pm$ are experimental parameters fitted with the fracture energy FE-regularization method explained in Section 2.5. This damage laws can be converted to an equivalent stress-strain $\sigma^\pm(\varepsilon^\pm)$ relation and viceversa, being these last commonly more known and used than the respective damage laws.

Viscous component

Additionally, the model can include rate-dependent viscous regularization. Its proposed the use of (Duvaut & Lions, 1972) viscous model in the plastic and damage components of model. Thus, the nominal viscous stress vector $\underline{\sigma}^v$ is defined as

$$\underline{\sigma}^v := \sum_{\aleph} (1 - \omega^\aleph) \bar{\underline{\sigma}}^{v\aleph}, \quad (3.1.61)$$

$$\bar{\underline{\sigma}}^{v\pm} = \underline{P}_v^\pm \underline{\sigma}^v, \quad \underline{P}_v^\pm := \sum_{i=1}^N H_0^\pm (\hat{\sigma}_i^v) (\underline{e}_{\bar{\sigma}_v}^{ii} \otimes \underline{e}_{\bar{\sigma}_v}^{ii}), \quad (3.1.62)$$

where $\bar{\underline{\sigma}}^v$ is the effective viscous stress vector given by Eq. (3.1.34) and $\underline{P}_v^\pm$ are their positive/negative projected tensors, respectively. Moreover, for the damage component, the evolution law of damage thresholds variables $r^\pm$ are defined as

$$\dot{r}^\pm := -\frac{1}{\mu_v} (r^\pm - Y^\pm). \quad (3.1.63)$$

3.1.4. Faría-Oliver-Cervera (FOC) model

This plastic-damage model was proposed by (Faria et al., 1998). Take identical assumptions than the WLF model for the damage and viscous components, and use a simplified representation for the plastic component, explained as follows.

Plastic component

Although, the formulation of WLF model provides a strict framework to represent the evolution of plastic strain, numerical implementation gives time consuming solving process. (Faria et al., 1998) proposed a simplified evolution law for the plastic strain vector as follow

$$\dot{\underline{\varepsilon}}^p := \dot{\gamma} \underline{\sigma}, \quad (3.1.64)$$

$$\dot{\gamma} = E_o \chi \frac{\langle \underline{\varepsilon}^e T \dot{\underline{\varepsilon}} \rangle^+}{(\underline{\sigma}^T \underline{R} \underline{\sigma})}, \quad (3.1.65)$$

where $\chi = B^+ H^+(\dot{\omega}^+) + B^- H^+(\dot{\omega}^-) \geq 0$ is a material parameter to control the rate intensity of plastic deformation, with $B^\pm$ a parameter associated to positive/negative component of stress, Heaviside function $H(\cdot)^+$ is used for active progressive damage rate of respective stress component, and McAulay $\langle \cdot \rangle^+$ function enable one to set a non-negative value for the product $(\underline{\varepsilon}^e \cdot \dot{\underline{\varepsilon}})$ required to ensure positive dissipation.

In addition, due that the flow rule is proportional to the stress vector, its follow that $\varepsilon_{33}^p = 0$. Also, the elastic part of out-of-plane strain is expressed by Eq. (3.1.26). Hence, using Eq. (A.2.9), the volumetric strain rate can be estimated as

$$\dot{\varepsilon}_v = \varepsilon_v^e + \varepsilon_v^p = (K^{-1} + 3\dot{\gamma})\bar{p} \quad (3.1.66)$$

3.1.5. Total strain rotating crack (ROT) model

This smeared-crack model was developed by (Cope et al., 1980; Gupta & Akbar, 1984) and enhanced by (Rots, 1988; TNO DIANA, 2018). We proposed a simple and robust formulation than past.

Damage component

First, assume the so-called the "total strain" formulation present in the hypo-elastic materials, i.e. that stress vector $\underline{\sigma}$ depends only of total strain vector $\underline{\varepsilon}$. Next, its assumed

that a set of orthogonal crack planes rotates according to direction of principal strain vector $\underline{\hat{\epsilon}}$. Then, using a spectral decomposition of strain vector $\underline{\epsilon}$, satisfy the relation

$$\underline{\epsilon} = \sum_{i=1}^N \hat{\epsilon}_i \underline{e}_{\epsilon}^{ii} = \underline{E}_{\epsilon} \underline{\hat{\epsilon}} \quad (3.1.67)$$

where $\hat{\epsilon}_i$ is the i -th eigenvalue, $\underline{e}_{\epsilon}^{ii}$ the i -th eigen-projector vector and $\underline{E}_{\epsilon}$ the eigen-projector matrix (Eq. (A.2.11)).

According only to this condition, the model lack of memory for the damage evolution, where the loading and unloading follows the same path (hypo-elastic). Thus, in order to add an irreversible damage process, a i -th positive/negative damage strain variables $\alpha_i^{\pm}$ are defined for respective principal strain direction $\hat{\epsilon}_i$. Then, the evolution law for these damage variables satisfy the relation

$$\dot{\alpha}_i^{\pm} := z_i^{\pm} \dot{\hat{\epsilon}}_i \quad (3.1.68)$$

where $z_i^{\pm} = 1 - r_i^{\pm}$ and $r_i^{\pm} = H_0^{\pm}(\alpha_i^{\pm} - \hat{\epsilon}_i)$ are the damage threshold variables. Now, calling the damage strain vector as follow $\underline{\alpha} = [\underline{\alpha}^+, \underline{\alpha}^-]^T$, with $\underline{\alpha}^{\pm} = [\alpha_1^{\pm}, \alpha_2^{\pm}]^T$, Eq. (3.1.68) can be rewritten in a vectorized format as

$$\dot{\underline{\alpha}} = \underline{Z}(\underline{\hat{\epsilon}}, \underline{\alpha}) \dot{\underline{\hat{\epsilon}}}, \quad (3.1.69)$$

$$\underline{Z}(\underline{\hat{\epsilon}}, \underline{\alpha}) = \begin{bmatrix} \underline{Z}^+ \\ \underline{Z}^- \end{bmatrix}, \quad \underline{Z}^{\pm} = \text{diag}(z_1^{\pm}, z_2^{\pm})$$

It should be noted the similarity of this expression with Eq. (3.1.17). For the other hand, the i -th principal stress $\hat{\sigma}_i$ in their respective principal strain $\hat{\epsilon}_i$ is given by

$$\hat{\sigma}_i := m_i^+ h_i^+ + m_i^- h_i^- \quad (3.1.70)$$

where $m_i^\pm = H_{1/2}^\pm(\hat{\varepsilon}_i)$ and $h_i^\pm = \sigma^\pm(\alpha_i^\pm)g_i^\pm$, with $\sigma^\pm(\alpha_i^\pm)$ are the uniaxial positive/negative stress laws, respectively, and $g_i^\pm$ are variables to control the loading/unloading stress. Assuming a secant unloading to origin (no plastic strains), the variables $g_i^\pm$ can be defined as

$$g_i^\pm := 1 - \frac{\alpha_i^\pm - \hat{\varepsilon}_i}{\alpha_i^\pm} = \frac{\hat{\varepsilon}_i}{\alpha_i^\pm} \quad (3.1.71)$$

with $g_i^\pm \in [0, 1]$, where $g_i^\pm = 1$ in case of loading and $g_i^\pm < 1$ for unloading. Finally, the model assume the principle of co-axiality, that is, the principal stress directions coincide with the principal strain directions i.e. satisfy the relation

$$\underline{\sigma} := \sum_{i=1}^N \hat{\sigma}_i \underline{e}_\varepsilon^{ii} = \underline{E}_\varepsilon \hat{\sigma} \quad (3.1.72)$$

Viscous component

Additionally, its suggested include a viscous model to improve the convergence of model. For this, the Duvaut-Lions viscous model can be incorporated as follows

$$\dot{\alpha}_i^{v\pm} := -\frac{1}{\mu_v}(\alpha_i^{v\pm} - \alpha_i^\pm) \quad (3.1.73)$$

where $\alpha_i^{v\pm}$ are the i -th viscous damage strain variable. Then, the i -th principal viscous-stress $\hat{\sigma}_i^v$ is expressed as

$$\hat{\sigma}_i^v = m_i^+ h_i^{v+} + m_i^- h_i^{v-}, \quad (3.1.74)$$

$$h_i^{v\pm} = \sigma^\pm(\alpha_i^{v\pm})g_i^{v\pm}, \quad g_i^{v\pm} = \frac{\hat{\varepsilon}_i}{\alpha_i^{v\pm}} \quad (3.1.75)$$

Thus, the viscous-stress tensor σ^v is given by

$$\underline{\sigma}^v := \sum_{i=1}^N \hat{\sigma}_i^v \underline{e}_\varepsilon^{ii} = \underline{E}^\varepsilon \hat{\sigma}^v \quad (3.1.76)$$

It should be noted that, this model can be extended to simulate the biaxial effects, such as biaxial strength in compression-compression (CC) regime or compression softening

in tension-compression one. In both cases, it can be extended by means of modify the uniaxial stress-strain law as function of complete principal stress/strain vector, i.e. $\sigma^\pm = \sigma^\pm(\underline{\hat{\epsilon}}, \underline{\hat{\sigma}})$. Complex derivatives involve this process and is beyond the scope of this work.

3.2. Stress updating algorithms

Numerical integration of constitutive equations requires of an algorithm to update the stress vector and internal state variables at each integration point given a known strain increment. More specifically, given a (pseudo-) time increment $\Delta t = t_{n+1} - t_n$, it is assumed that at time t_n the strain vector $\underline{\epsilon}_n$, the stress vector $\underline{\sigma}_n$ and the internal state variables $\underline{\alpha}_n$ are known. Then, the algorithm determine the updated stress vector $\underline{\sigma}_{n+1}$ at time t_{n+1} for a given strain increment $\Delta \underline{\epsilon} = \Delta t \dot{\underline{\epsilon}}$.

Thereby, for one hand, the plastic component of models is commonly evaluated with a backward Euler (implicit) scheme. Return-mapping algorithms are the most used, where a trial elastic-predictor step and a plastic-corrector step are required (J. C. Simo & Hughes, 1998). Generally, this method lead implicit non-linear equations which are solved by means of an iterative Newton's method. Specifically, for the plane stress condition, the projected-return mapping algorithm is adopted as solution for plastic and plastic-damage models. Thus, an enforcement of the consistency condition is used to reduce the solution to a simple nonlinear equation. For the other hand, the damage component of models is generally evaluated with an explicit scheme, with the exception of coupled plastic-damage models, which require the simultaneous solution of both components.

3.2.1. Trial elastic-predictor step

The elastic-trial step assume that the strain increment produces purely elastic deformation, where plastic deformation and evolution internal variables $\underline{q}$ are frozen ($\underline{\epsilon}_{n+1}^{p \text{ tr}} = \underline{\epsilon}_n^p$

and $\underline{q}_{n+1}^{\text{tr}} = \underline{q}_n$). Thus, the trial elastic strain and trial stress vector are given by

$$\underline{\epsilon}_{n+1}^{e \text{ tr}} = \underline{\epsilon}_{n+1} - \underline{\epsilon}_n^p, \quad (3.2.77)$$

$$\underline{\sigma}_{n+1}^{\text{tr}} = \underline{D}_e(\underline{\epsilon}_{n+1} - \underline{\epsilon}_n^p) = \underline{\sigma}_n + \underline{D}_e \Delta \underline{\epsilon}_{n+1}, \quad (3.2.78)$$

where $\Delta \underline{\epsilon}_{n+1} = \underline{\epsilon}_{n+1} - \underline{\epsilon}_n$. Next, the trial state can be converted into the update solution if satisfy the condition

$$F_{n+1}^{\text{tr}} = F(\underline{\sigma}_{n+1}^{\text{tr}}, \underline{q}_{n+1}^{\text{tr}}) \leq 0. \quad (3.2.79)$$

This means that trial state lies within the elastic domain on the yield surface. In this case, the stress and internal variables are updated as $(\cdot)_{n+1} = (\cdot)_{n+1}^{\text{tr}}$. Otherwise, the trial step is not admissible, causing plastic response, being required any plastic-corrector step or a return-mapping algorithm to determine the update state.

3.2.2. Plastic-corrector step

The plastic-corrector step adjust the trial elastic-predictor step to give a correct updated stress. First, the updated plastic strain vector $\underline{\epsilon}_{n+1}^p$ is derived from linearization of flow rule as stated in Eq. (3.1.6)

$$\underline{\epsilon}_{n+1}^p = \underline{\epsilon}_n^p + \Delta \gamma \underline{n}_{n+1}. \quad (3.2.80)$$

Then, inserting this relation into Eq. (3.1.1), the updated stress vector $\underline{\sigma}_{n+1}$ is written as

$$\underline{\sigma}_{n+1} = \underline{\sigma}_{n+1}^{\text{tr}} - \Delta \gamma \underline{D}_e \underline{n}_{n+1}. \quad (3.2.81)$$

Thus, the only variable necessary to be solved is the discrete consistent operator $\Delta \gamma$, which is calculated according to their respective equations for each numerical model.

3.2.3. DPH model

The numerical stress integration of this model is based by the classical elastic-predictor (Section 3.2.1) and plastic-corrector step, the later explained as follow. First, the updated expression of flow vector $\underline{n}$, given by Eq. (3.1.7), is given by

$$\underline{n}_{n+1} = \frac{3}{2r_{n+1}} \underline{P} \underline{\sigma}_{n+1} + \frac{\bar{\eta}}{3} \underline{1}, \quad (3.2.82)$$

where $r_{n+1} = \sqrt{q_{n+1}^2 + \epsilon^2}$, with $q_{n+1} = \sqrt{\frac{3}{2} z_{n+1}}$ and $z_{n+1} = \underline{\sigma}_{n+1}^T \underline{P} \underline{\sigma}_{n+1}$. Moreover, multiplying both sides of Eq. (3.2.81) by the compliance stiffness matrix $\underline{C}_e$ (Eq. (A.2.4)) and introducing them Eq. (3.2.82), gives the updated stress vector as follow

$$\underline{\sigma}_{n+1} = \underline{A}_{n+1} \underline{\sigma}_{n+1}^{\text{tr}} - \frac{\bar{\eta}}{3} \Delta \gamma \underline{\Xi}_{n+1} \underline{1}, \quad (3.2.83)$$

where $\underline{A}_{n+1} = \underline{\Xi}_{n+1} \underline{C}_e$, with $\underline{\Xi}_{n+1}$ is the modified (algorithmic) elastic tangent matrix given by

$$\underline{\Xi}_{n+1} = (\underline{C}_e + t_{n+1} \Delta \gamma \underline{P})^{-1}, \quad (3.2.84)$$

and $t_{n+1} = \frac{3}{2r_{n+1}}$. Now, it can be probed that the matrices $\underline{P}$ and $\underline{C}_e$ share identical eigenvectors, for which they can be decomposed in a spectral format as

$$\underline{P} = \underline{Q}^T \bar{\underline{P}} \underline{Q}, \quad \underline{C}_e = \underline{Q}^T \bar{\underline{C}}_e \underline{Q}, \quad (3.2.85)$$

where $\bar{\underline{P}}$ and $\bar{\underline{C}}_e$ are the eigenvalues matrices defined as

$$\bar{\underline{P}} := \text{diag} \left(\frac{1}{3}, 1, 2 \right), \quad \bar{\underline{C}}_e := \text{diag} \left(\frac{1-\nu}{E}, \frac{1}{2\mu}, \frac{1}{\mu} \right), \quad (3.2.86)$$

and $\underline{Q}$ is the orthogonal ($\underline{Q}^{-1} = \underline{Q}^T$) eigenvector matrix given by

$$\underline{Q} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}. \quad (3.2.87)$$

Then, its follow that the matrices $\underline{\Xi}_{n+1}$ and $\underline{A}_{n+1}$ can be decomposed in their spectral representation as $\underline{\Xi}_{n+1} = \underline{Q}^T \bar{\underline{\Xi}}_{n+1} \underline{Q}$ and $\underline{A}_{n+1} = \underline{Q}^T \bar{\underline{A}}_{n+1} \underline{Q}$, respectively, where $\bar{\underline{\Xi}}_{n+1}$ and $\bar{\underline{A}}_{n+1}$ are expressed as

$$\begin{aligned} \bar{\underline{\Xi}}_{n+1} &= (\bar{\underline{C}}_e + t_{n+1} \Delta \gamma \bar{\underline{P}})^{-1} \\ &= \text{diag} \left[\left(\frac{1-\nu}{E} + \frac{\Delta \gamma t_{n+1}}{3} \right)^{-1}, \left(\frac{1}{2\mu} + \Delta \gamma t_{n+1} \right)^{-1}, \left(\frac{1}{2\mu} + \Delta \gamma t_{n+1} \right)^{-1} \right], \\ \bar{\underline{A}}_{n+1} &= \text{diag} (\bar{a}_{1n+1}, \bar{a}_{2n+1}, \bar{a}_{2n+1}) \\ &= \text{diag} [(1 + \lambda t_{n+1} \Delta \gamma)^{-1}, (1 + 2\mu t_{n+1} \Delta \gamma)^{-1}, (1 + 2\mu t_{n+1} \Delta \gamma)^{-1}], \end{aligned} \quad (3.2.88)$$

with $\lambda = \frac{E}{3(1-\nu)}$. Now, using the stress vector transformation $\underline{\tau}_{n+1}^{\text{tr}} = \underline{Q} \underline{\sigma}_{n+1}^{\text{tr}}$ and replacing the relation Eq. (3.2.88) into Eq. (3.2.83), gives a final expression for the updated stress vector as follow

$$\underline{\sigma}_{n+1} = \underline{Q}^T \left(\bar{\underline{A}}_{n+1} \underline{\tau}_{n+1}^{\text{tr}} - \frac{\bar{\eta}}{3} \Delta \gamma \bar{\underline{\Xi}}_{n+1} \underline{Q} \underline{1} \right) = \underline{Q}^T \bar{\underline{B}}_{n+1} \underline{\tau}_{n+1}^{\text{tr}} \quad (3.2.89)$$

$$= \underline{B}_{n+1} \underline{\sigma}_{n+1}^{\text{tr}}, \quad (3.2.90)$$

where $\underline{B}_{n+1} = \underline{Q}^T \bar{\underline{B}}_{n+1} \underline{Q}$, with $\bar{\underline{B}}_{n+1}$ given by

$$\bar{\underline{B}}_{n+1} = \text{diag} (\bar{b}_{1n+1}, \bar{b}_{2n+1}, \bar{b}_{2n+1}) = \text{diag} \left(\frac{g_{1n+1} \bar{a}_{1n+1}}{\tau_{11n+1}^{\text{tr}}}, \bar{a}_{2n+1}, \bar{a}_{2n+1} \right), \quad (3.2.91)$$

and $g_{1n+1} = \tau_{11n+1}^{\text{tr}} - \sqrt{2} \lambda \bar{\eta} \Delta \gamma$. On the other hand, the updated equivalent plastic strain is obtained from the discrete version of Eq. (3.1.7) as

$$\alpha_{n+1} = \alpha_n + \xi \Delta \gamma. \quad (3.2.92)$$

Moreover, the updated cohesion law can be called as $c_{n+1} = c(\alpha_{n+1})$. It should be noted that the discrete consistency operator $\Delta \gamma$ can not be expressed in a explicit form such as in 3D formulation Eq. (2.3.89). Then, it required solve this operator in an iterative process e.g. Newton's method. Box 1 shown the algorithm suggested to solve the variable

$\Delta\gamma$ for this model. In this case, the residual function is the yield criterion, which enforced the consistency condition at the solution, and is expressed, using Eq. (A.2.6), as

$$\begin{aligned} F_{n+1} &= \eta p_{n+1} + q_{n+1} - \xi c_{n+1} \\ &= \frac{\eta}{3} \mathbf{1}^T \underline{\sigma}_{n+1} + \sqrt{\frac{3}{2}} \sqrt{z_{n+1}} - \xi c_{n+1}, \end{aligned} \quad (3.2.93)$$

and their derivative with respect to $\Delta\gamma$ is given by

$$\frac{\partial F_{n+1}}{\partial \Delta\gamma} = \frac{\sqrt{2}\eta}{3} \frac{\partial \bar{b}_{1n+1}}{\partial \Delta\gamma} \tau_{11n+1}^{\text{tr}} + \frac{3}{4q_{n+1}} \frac{\partial z_{n+1}}{\partial \Delta\gamma} - \xi^2 J_\alpha. \quad (3.2.94)$$

A detailed calculation of this expression is explained in 1. Additionally, the recommended values for the number of iterations and tolerances are: $N_{iter} = 20$, $Tol_1 = 10^{-20}$, $Tol_2 = 10^{-5}$ and $Tol_3 = 10^{-2}$.

Box 4 :Algorithm to solve $\Delta\gamma$ for the DPH model

$\Delta\gamma^0 = 0, \quad q_{n+1}^0 = q_{n+1}^{\text{tr}}, \quad g_{2n+1}^0 = g_{2n+1}^{\text{tr}}$	$\triangleright$ Set initial value
for $j \leq N_{iter}$ do	
$z_{n+1}^j, r_{n+1}^j, p_{n+1}^j, \alpha_{n+1}^j, c_{n+1}^j$	$\triangleright$ Use Eqs. (3.2.82) and (3.2.92)
$R^j = F_{n+1}(\Delta\gamma^j)$	$\triangleright$ Residual function (Eq. (3.2.93))
$dR^j = \frac{\partial F_{n+1}}{\partial \Delta\gamma}(\Delta\gamma^j)$	$\triangleright$ Total derivative (Eq. (3.2.94))
$d\Delta\gamma^j = -R^j/dR^j$	
$\Delta\gamma^{j+1} = \Delta\gamma^j + d\Delta\gamma^j$	$\triangleright$ Update solution
$\Delta\gamma^{j+1} = \max(\Delta\gamma^{j+1}, Tol_1)$	$\triangleright$ Adjust solution
if $(R^j < Tol_2 \text{ and } d\Delta\gamma^j < Tol_3 \Delta\gamma^j) \text{ or } (d\Delta\gamma^j \leq Tol_1)$ then	
exit	

In addition, taking the linearization of Eq. (3.1.12), gives the updated out-of-plane plastic strain ε_{33}^p as

$$\varepsilon_{33n+1}^p = \varepsilon_{33n}^p + \Delta\gamma \left[-\frac{1}{2r_{n+1}} (\sigma_{11n+1} + \sigma_{22n+1}) + \frac{\bar{\eta}}{3} \right] \quad (3.2.95)$$

Also, using Eq. (3.1.13), the updated volumetric strain is given by

$$\varepsilon_{v_{n+1}} = \varepsilon_{v_n} + K^{-1} p_{n+1} + \Delta\gamma \bar{\eta} \quad (3.2.96)$$

3.2.4. LLF model

The numerical stress integration of this model is composed by three steps: (i) an elastic-predictor step (Section 3.2.1); (ii) a plastic-corrector step with an implicit scheme to evaluate the updated effective stress tensor $\bar{\underline{\sigma}}_{n+1}$; and (iii) a damage-corrector step with an explicit scheme to evaluate the updated damage variables ω_{n+1} and the nominal stress tensor $\underline{\sigma}_{n+1}$. The development of plastic and damage steps are explained as follow.

Plastic component

First, due that the DPH and LLF share identical flow potential criterion, Eqs. (3.2.82) to (3.2.94) are also valid for this model, but expressed in the effective space ($\bar{\cdot}$). For the other hand, due that yield criterion is defined in terms of invariants and principal stresses, its convenient and efficiency the use of Spectral Return Mapping Algorithm (SRMA) (J. Lee & Fenves, 1998). SRMA assume four conditions: (1) the effective stress vector can be decomposed as $\bar{\underline{\sigma}}_{n+1} = \underline{\underline{E}}_{\bar{\sigma}} \hat{\underline{\sigma}}_{n+1}$, where $\hat{\underline{\sigma}}_{n+1}$ and $\underline{\underline{E}}_{\bar{\sigma}}$ is the eigenvalue vector and the eigen-projector matrix of updated stress vector $\bar{\underline{\sigma}}_{n+1}$, respectively (see 2); (2) any eigenvector of trial effective stress vector is also an eigenvector of updated effective stress vector, i.e. $\bar{\underline{\sigma}}_{n+1}^{\text{tr}} = \underline{\underline{E}}_{\bar{\sigma}} \hat{\underline{\sigma}}_{n+1}^{\text{tr}}$; (3) any isotropic material satisfy the relation $G(\underline{\sigma}) = \hat{G}(\hat{\underline{\sigma}})$, which imply that $\bar{\underline{n}}_{n+1} = \underline{\underline{E}}_{\bar{\sigma}} \hat{\underline{n}}_{n+1}$; and (4) substituting these expressions into Eq. (3.2.81), the updated principal effective stress vector is given by

$$\hat{\underline{\sigma}}_{n+1} = \hat{\underline{\sigma}}_{n+1}^{\text{tr}} - \Delta\gamma \hat{\underline{D}}_e \hat{\underline{n}}_{n+1} \quad (3.2.97)$$

Moreover, using this expression, is easy to obtain the relation $\Delta\hat{\underline{\varepsilon}}_{n+1}^p = \Delta\gamma \hat{\underline{n}}_{n+1}$. It should be noted that variables $\bar{p}$, $\bar{q}$, $\bar{r}$, $\bar{z}$ and $\bar{t}$ are invariants in effective space, i.e. $(\bar{\cdot}) = (\hat{\cdot})$. Also note that, due that yield criterion and hardening variables $\kappa^{\pm}$ are expressed in terms of maximum and minimum effective principal stresses, its necessary reordering the eigenvalues and their respective eigenvectors in a descending order ($\hat{\sigma}_1 \geq \dots \geq \hat{\sigma}_N$).

Then, the updated expression of the effective principal flow vector, given by Eq. (3.1.21), is written as

$$\hat{\underline{n}}_{n+1} = \frac{3}{2\bar{r}_{n+1}} \hat{\underline{P}} \hat{\underline{\sigma}}_{n+1} + \frac{\bar{\eta}}{3} \hat{\underline{1}}, \quad (3.2.98)$$

where $\bar{r}_{n+1} = \sqrt{\bar{q}_{n+1}^2 + \epsilon^2}$, $\bar{q}_{n+1} = \sqrt{\frac{3}{2} \bar{z}_{n+1}}$ and $\bar{z}_{n+1} = \hat{\underline{\sigma}}_{n+1}^T \hat{\underline{P}} \hat{\underline{\sigma}}_{n+1}$. Next, multiplying both sides of Eq. (3.2.97) by matrix $\hat{\underline{C}}_e$ (see Eq. (A.2.16)) and introducing them Eq. (3.2.98), gives the updated principal effective stress vector as

$$\hat{\underline{\sigma}}_{n+1} = \hat{\underline{A}}_{n+1} \hat{\underline{\sigma}}_{n+1}^{\text{tr}} - \frac{\bar{\eta}}{3} \Delta \gamma \hat{\underline{\Xi}}_{n+1} \hat{\underline{1}}, \quad (3.2.99)$$

where $\hat{\underline{A}}_{n+1} = \hat{\underline{\Xi}}_{n+1} \hat{\underline{C}}_e$ with $\hat{\underline{\Xi}}_{n+1}$ is given by

$$\hat{\underline{\Xi}}_{n+1} = \left(\hat{\underline{C}}_e + \bar{t}_{n+1} \Delta \gamma \hat{\underline{P}} \right)^{-1}, \quad (3.2.100)$$

and $\bar{t}_{n+1} = \frac{3}{2\bar{r}_{n+1}}$. Similar to the matrices $\underline{P}$ and $\underline{C}_e$, the matrices $\hat{\underline{P}}$ and $\hat{\underline{C}}_e$ share identical eigenvectors and can be decomposed in their spectral format as

$$\hat{\underline{P}} = \hat{\underline{Q}}^T \hat{\underline{P}} \hat{\underline{Q}}, \quad \hat{\underline{C}}_e = \hat{\underline{Q}}^T \hat{\underline{C}}_e \hat{\underline{Q}}, \quad (3.2.101)$$

where $\hat{\underline{P}}$ and $\hat{\underline{C}}_e$ are the eigenvalues matrices given by

$$\hat{\underline{P}} = \text{diag} \left(\frac{1}{3}, 1 \right), \quad \hat{\underline{C}}_e = \text{diag} \left(\frac{1-\nu}{E}, \frac{1}{2\mu} \right), \quad (3.2.102)$$

and $\hat{\underline{Q}}$ is the orthogonal eigenvector matrix given by

$$\hat{\underline{Q}} := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}. \quad (3.2.103)$$

It follows that the matrices $\hat{\underline{\Xi}}_{n+1}$ and $\hat{\underline{A}}_{n+1}$ can be decomposed in their spectral format as $\hat{\underline{\Xi}}_{n+1} = \hat{\underline{Q}}^T \hat{\underline{\Xi}}_{n+1} \hat{\underline{Q}}$ and $\hat{\underline{A}}_{n+1} = \hat{\underline{Q}}^T \hat{\underline{A}}_{n+1} \hat{\underline{Q}}$, respectively, where $\hat{\underline{\Xi}}_{n+1}$ and $\hat{\underline{A}}_{n+1}$ are

expressed as

$$\begin{aligned}
\hat{\underline{\Xi}}_{n+1} &= \left(\hat{\underline{C}}_e + \bar{t}_{n+1} \hat{P} \right)^{-1} \\
&= \text{diag} \left[\left(\frac{1-\nu}{E} + \frac{\Delta\gamma \bar{t}_{n+1}}{3} \right)^{-1}, \left(\frac{1}{2\mu} + \Delta\gamma \bar{t}_{n+1} \right)^{-1} \right], \\
\hat{\underline{A}}_{n+1} &= \text{diag} (\hat{a}_{1n+1}, \hat{a}_{2n+1}) \\
&= \text{diag} [(1 + \lambda \bar{t}_{n+1} \Delta\gamma)^{-1}, (1 + 2\mu \bar{t}_{n+1} \Delta\gamma)^{-1}]. \tag{3.2.104}
\end{aligned}$$

Now, using the stress vector transformation $\hat{\underline{\tau}}_{n+1}^{\text{tr}} = \hat{\underline{Q}} \hat{\underline{\sigma}}_{n+1}^{\text{tr}}$ and replacing these relations into Eq. (3.2.99), gives a final expression for updated stress vector as

$$\hat{\underline{\sigma}}_{n+1} = \hat{\underline{Q}}^T \left(\hat{\underline{A}}_{n+1} \hat{\underline{\tau}}_{n+1}^{\text{tr}} - \frac{\bar{\eta}}{3} \Delta\gamma \hat{\underline{\Xi}}_{n+1} \hat{\underline{Q}} \hat{\underline{1}} \right) = \hat{\underline{Q}}^T \hat{\underline{B}}_{n+1} \hat{\underline{\tau}}_{n+1}^{\text{tr}} \tag{3.2.105}$$

$$= \hat{\underline{B}}_{n+1} \hat{\underline{\sigma}}_{n+1}^{\text{tr}}, \tag{3.2.106}$$

where $\hat{\underline{B}}_{n+1} = \hat{\underline{Q}}^T \hat{\underline{B}}_{n+1} \hat{\underline{Q}}$, with $\hat{\underline{B}}_{n+1}$ given by

$$\hat{\underline{B}}_{n+1} = \text{diag} (\hat{b}_{1n+1}, \hat{b}_{2n+1}) = \text{diag} \left(\frac{\hat{g}_{1n+1} \hat{a}_{1n+1}}{\hat{\tau}_{1n+1}^{\text{tr}}}, \hat{a}_{2n+1} \right), \tag{3.2.107}$$

and $\hat{g}_{1n+1} = \hat{\tau}_{1n+1}^{\text{tr}} - \sqrt{2}\lambda\bar{\eta}\Delta\gamma$. Moreover, the maximum updated principal effective stress is expressed as $\hat{\sigma}_{+n+1} = \hat{\underline{1}}_+^T \hat{\underline{\sigma}}_{n+1}$. For the other hand, linearization of updated hardening variable $\underline{\kappa}_{n+1}$ (Eq. (3.1.22)) can be expressed as

$$\underline{\kappa}_{n+1} = \underline{\kappa}_n + \Delta\gamma \underline{H}_{n+1}(\hat{\underline{\sigma}}_{n+1}, \underline{\kappa}_{n+1}). \tag{3.2.108}$$

Although, its convenient take their positive and negative part as

$$\kappa_{n+1}^{\pm} = \kappa_n^{\pm} + \Delta\gamma h_{n+1}^{\pm}, \tag{3.2.109}$$

where $h_{n+1}^{\pm} = \hat{n}_{n+1}^{\pm} \varphi_{n+1}^{\pm}$, with $\hat{n}_{n+1}^{\pm} = \hat{\mathbf{1}}_{\pm}^T \hat{\mathbf{n}}_{n+1}$ and the variable $\varphi_{n+1}^{\pm}$ are defined as

$$\varphi_{n+1}^{\pm} := \theta_{1_{n+1}}^{\pm} \theta_{2_{n+1}}^{\pm}, \quad (3.2.110)$$

and with $\theta_{1_{n+1}}^{\pm}$ and $\theta_{2_{n+1}}^{\pm}$ defined as

$$\theta_{1_{n+1}}^{+} := \phi(\hat{\boldsymbol{\sigma}}_{n+1}), \quad \theta_{1_{n+1}}^{-} := -[1 - \phi(\hat{\boldsymbol{\sigma}}_{n+1})], \quad (3.2.111)$$

$$\theta_{2_{n+1}}^{+} := \sigma^{+}(\kappa_{n+1}^{+})/g^{+}, \quad \theta_{2_{n+1}}^{-} := \sigma^{-}(\kappa_{n+1}^{-})/g^{-}. \quad (3.2.112)$$

In addition, the updated parameters β and c (Eq. (3.1.24)) can be expressed, respectively, as

$$\beta_{n+1} = (1 - \alpha) \frac{\bar{\sigma}^{-}(\kappa_{n+1}^{-})}{\bar{\sigma}^{+}(\kappa_{n+1}^{+})} - (1 + \alpha), \quad c_{n+1} = \bar{\sigma}^{-}(\kappa_{n+1}^{-}). \quad (3.2.113)$$

Finally, the yield criterion at consistency condition (Eq. (3.1.23)) is written as

$$F_{n+1} = \eta \bar{p}_{n+1} + \bar{q}_{n+1} + \beta_{n+1} \langle \hat{\sigma}_1 \rangle^{+} - (1 - \alpha) c_{n+1} = 0. \quad (3.2.114)$$

Its observed that a nested iterative process is required to obtain variables $\Delta\gamma$ and $\kappa_{n+1}^{\pm}$. Box 5 shown the algorithm used to calculate both variables. Three steps are involved: (i) set an initial value of variables κ , q and $\hat{\sigma}_{+}$ equal to the previous step; (ii) solve the consistency operator $\Delta\gamma$ using the algorithm described in Box 4 which is identical to the DPH model, but using the effective stress space in their expressions and the derivative $\frac{\partial \bar{F}_{n+1}}{\partial \Delta\gamma}$ is given by

$$\begin{aligned} \frac{\partial \bar{F}_{n+1}}{\partial \Delta\gamma} = & \frac{\sqrt{2}\eta}{3} \frac{\partial \bar{b}_{1_{n+1}}}{\partial \Delta\gamma} \hat{\tau}_{11_{n+1}}^{\text{tr}} + \frac{3}{4\bar{q}_{n+1}} \frac{\partial \bar{z}_{n+1}}{\partial \Delta\gamma} + \hat{\sigma}_{+n+1} \left(\hat{b}_7 + \Delta\gamma \hat{b}_8 \right) + \hat{\beta}_1 \frac{\partial \hat{\sigma}_{+n+1}}{\partial \Delta\gamma} \\ & - (1 - \alpha) \left(b_3^{-} + \Delta\gamma b_4^{-} \right). \end{aligned} \quad (3.2.115)$$

A detailed calculation of this derivative is explained in 2; and (iii) solve the hardening variables $\underline{\kappa}$ using the Newton's method. For this, Eq. (3.2.106) is used as the residual

function and rewritten as

$$\underline{Q}_{n+1}(\underline{\kappa}_{n+1}, \Delta\gamma, \hat{\underline{\sigma}}_{n+1}) = \underline{\kappa}_n + \Delta\gamma \underline{H}_{n+1}(\underline{\kappa}_{n+1}, \hat{\underline{\sigma}}_{n+1}) - \underline{\kappa}_{n+1}. \quad (3.2.116)$$

Thus, the total derivative of this residual function with respect to $\underline{\kappa}_{n+1}$ is given by

$$\begin{aligned} \frac{d\underline{Q}_{n+1}}{d\underline{\kappa}_{n+1}} &= \frac{\partial \underline{Q}_{n+1}}{\partial \underline{\kappa}_{n+1}} + \frac{\partial \underline{Q}_{n+1}}{\partial \Delta\gamma} \frac{\partial \Delta\gamma}{\partial \underline{\kappa}_{n+1}} \\ &\quad + \frac{\partial \underline{Q}_{n+1}}{\partial \underline{H}_{n+1}} \left(\frac{\partial \underline{H}_{n+1}}{\partial \hat{\underline{\sigma}}_{n+1}} \frac{\partial \hat{\underline{\sigma}}_{n+1}}{\partial \Delta\gamma} \frac{\partial \Delta\gamma}{\partial \underline{\kappa}_{n+1}} + \frac{\partial \underline{H}_{n+1}}{\partial \underline{\kappa}_{n+1}} \right) \\ &= -\underline{I}_2 + \left(\underline{H}_{n+1} + \Delta\gamma \frac{\partial \underline{H}_{n+1}}{\partial \hat{\underline{\sigma}}_{n+1}} \frac{\partial \hat{\underline{\sigma}}_{n+1}}{\partial \Delta\gamma} \right) \otimes \frac{\partial \Delta\gamma}{\partial \underline{\kappa}_{n+1}} + \Delta\gamma \frac{\partial \underline{H}_{n+1}}{\partial \underline{\kappa}_{n+1}}, \end{aligned} \quad (3.2.117)$$

where $\underline{I}_2 = \text{diag}(1, 1)$ and the derivatives involved are expressed as

$$\begin{aligned} \frac{\partial \underline{H}_{n+1}}{\partial \hat{\underline{\sigma}}_{n+1}} &= \left(\hat{\underline{y}}_{n+1} \otimes \hat{\underline{\Phi}}_{n+1} \right) + \left(\hat{\underline{z}}_{n+1} \otimes \frac{\partial \hat{\underline{n}}_{n+1}}{\partial \hat{\underline{\sigma}}_{n+1}} \right), & \frac{\partial \Delta\gamma}{\partial \underline{\kappa}_{n+1}} &= \frac{1}{L_1} \underline{l}_{0_{n+1}}, \\ \frac{\partial \hat{\underline{\sigma}}_{n+1}}{\partial \Delta\gamma} &= \hat{\underline{Q}}^T \frac{\partial \hat{\underline{B}}_{n+1}}{\partial \Delta\gamma} \hat{\underline{\tau}}_{n+1}^{\text{tr}}, & \frac{\partial \underline{H}_{n+1}}{\partial \underline{\kappa}_{n+1}} &= \hat{\underline{U}}_{n+1} \end{aligned} \quad (3.2.118)$$

A detailed calculation of these derivatives are explained in 3. Also, is recommended tolerances of $Tol_4 = 1 - 10^{-10}$ to adjust the solution values and $Tol_5 = 10^{-5}$ to check the residual function, giving an adequate convergence of model.

In addition, taking the linearization of Eq. (3.1.25) and using Eq. (3.1.26), the updated plastic ε_{33}^p and elastic ε_{33}^e out-of-plane strain are given, respectively, as

$$\varepsilon_{33_{n+1}}^p = \varepsilon_{33_n}^p + \Delta\gamma \left[-\frac{1}{2\bar{r}_{n+1}} (\bar{\sigma}_{11_{n+1}} + \bar{\sigma}_{22_{n+1}}) + \frac{\bar{\eta}}{3} \right], \quad (3.2.119)$$

$$\varepsilon_{33_{n+1}}^e = -\frac{\nu}{E_o} (\bar{\sigma}_{11_{n+1}} + \bar{\sigma}_{22_{n+1}}). \quad (3.2.120)$$

Also, using Eq. (A.2.9), the updated volumetric strain is given by

$$\varepsilon_{v_{n+1}} = \varepsilon_{v_n} + K^{-1} \bar{p}_{n+1} + \Delta\gamma \bar{\eta} \quad (3.2.121)$$

Box 5 :Algorithm to solve $\underline{\kappa}_{n+1}$ for the LLF model

$\underline{\kappa}_{n+1}^0 = \underline{\kappa}_n, \quad \bar{q}_{n+1}^0 = \bar{q}_{n+1}^{\text{tr}}, \quad \hat{\sigma}_{n+1}^0 = \hat{\sigma}_{n+1}^{\text{tr}}$ ▷ Set initial value
for $j \leq N_{\text{iter}}$ **do**
 $\bar{\sigma}_{n+1}^{\pm}(\kappa_{n+1}^{\pm j}), \beta_{n+1}^j, \tilde{H}^+(\hat{\sigma}_{n+1}^j)$ ▷ Use Eqs. (3.1.31) and (3.2.113)
 $q_{n+1}^j, \Delta\gamma^j, \hat{\sigma}_{n+1}^j, \phi$ ▷ Solve with Box 4 and Eq. (3.2.106)
 $\underline{Q}^j = Q_{n+1}(\underline{\kappa}_{n+1}^j, \Delta\gamma^j, \hat{\sigma}_{n+1}^j)$ ▷ Residual, Eq. (3.2.116)
 $d\underline{Q}^j = \frac{d\underline{Q}_{n+1}}{d\underline{\kappa}_{n+1}}(\underline{\kappa}_{n+1}^j, \Delta\gamma^j, \hat{\sigma}_{n+1}^j)$ ▷ Total derivative, Eq. (3.2.117)
 $d\underline{\kappa}^j = -(d\underline{Q}^j)^{-1}\underline{Q}^j$
 $\underline{\kappa}_{n+1}^{j+1} = \underline{\kappa}_{n+1}^j + d\underline{\kappa}^j$ ▷ Update solution
 $\underline{\kappa}_{n+1}^{j+1} = \min(\underline{\kappa}_{n+1}^j, Tol_4 \underline{1})$ ▷ Adjust solution
if $(\|\underline{Q}^j\| \leq Tol_5)$ **then**
exit

Damage component

An explicit evaluation of updated damage variable ω_{n+1} (Eq. (3.1.28)) are generated according to updated hardening variables $\kappa_{n+1}^{\pm}$ calculated in the plastic component of model.

Viscous component

Assuming that the rate of a generic variable x can be expressed as $\dot{x} = \Delta x / \Delta t$, with Δt is the load step increment. Then, using this relation in the linearization of Eqs. (3.1.35) and (3.1.36), the updated visco-plastic strain vector $\underline{\epsilon}_{n+1}^{vp}$ and the viscous-damage variable ω_{n+1}^v can be expressed, respectively, as

$$\underline{\epsilon}_{n+1}^{vp} = \zeta_v \underline{\epsilon}_n^{vp} + (1 - \zeta_v) \underline{\epsilon}_{n+1}^p, \quad (3.2.122)$$

$$\omega_{n+1}^v = \zeta_v \omega_n^v + (1 - \zeta_v) \omega_{n+1}, \quad (3.2.123)$$

where $\zeta_v = (1 + \Delta t / \mu_v)^{-1}$. Then, substituting the Eq. (3.2.122) into updated version of Eq. (3.1.34) and with some algebraic manipulation, the updated effective viscous-stress

vector can be expressed in a convenient way as

$$\bar{\underline{\sigma}}_{n+1}^v = \zeta_v(\bar{\underline{\sigma}}_n^v + \underline{D}_e \Delta \underline{\epsilon}_n) + (1 - \zeta_v)\bar{\underline{\sigma}}_{n+1}. \quad (3.2.124)$$

Finally, the updated viscous-stress vector can be expressed as

$$\underline{\sigma}_{n+1}^v = (1 - \omega_{n+1}^v)\bar{\underline{\sigma}}_{n+1}^v \quad (3.2.125)$$

It should be noted that if $\mu_v/\Delta t \rightarrow 0$ ($\zeta_v = 0$) the solution relaxed to the rate-independent (or inviscid) response.

3.2.5. WLF model

Plastic component

Numerical stress integration of this model is identical to the LLF model, except for two considerations: (1) the matrix $\underline{W}$ of Eq. (3.1.46) depends only of , the matrix $\underline{H}_{n+1}$ of Eq. (3.2.108) depends only of stress vector $\hat{\underline{\sigma}}$, for which the variables $\theta_2^\pm = 1$ and the derivative $\frac{\partial \underline{H}}{\partial \underline{\kappa}}$, given by Eq. (3.2.118), is null; and (2) its observed that a tolerance to check the residual function of $Tol_5 = 10^{-6}$ can be used without convergence troubles.

Damage component

Giving the updated effective stress vector $\bar{\underline{\sigma}}_{n+1}$ calculated in the plastic component, the positive/negative part of effective stress vector $\bar{\underline{\sigma}}_{n+1}^\pm$ are evaluated using Eq. (3.1.37). Next, evaluating the DERR $Y^\pm$ according to the definition established (Eq. (3.1.52) or Eq. (3.1.53)), and assuming an active damage process (Eq. (3.1.57)), the updated damage threshold are stated. Finally, and explicit evaluation of damage variables $\omega_{n+1}^\pm(r_{n+1}^\pm)$ is generated.

Viscous component

The updated viscous stress vector $\underline{\sigma}_{n+1}^v$ is calculated using Eq. (3.1.61), where the effective viscous stress tensor $\bar{\underline{\sigma}}_{n+1}^v$ is evaluated using Eq. (3.2.124). Also, the viscoplastic strain vector $\underline{\varepsilon}_{n+1}^{vp}$ is evaluated with Eq. (3.2.122). Moreover, the updated damage variables depends of updated damage thresholds variables $r_{n+1}^\pm$, which are obtained using a linearization of Eq. (3.1.63) as follows

$$r_{n+1}^\pm = \zeta_v r_n^\pm + (1 - \zeta_v) Y_{n+1}^\pm. \quad (3.2.126)$$

3.2.6. FOC model

Plastic component

First, the discretization of Eqs. (3.1.64) and (3.1.65) gives

$$\underline{\varepsilon}_{n+1}^p = \underline{\varepsilon}_n^p + \Delta\gamma \bar{\underline{\sigma}}_{n+1}, \quad (3.2.127)$$

$$\Delta\gamma = \frac{E_o \chi_{n+1}}{\|\bar{\underline{\sigma}}_{n+1}\|^2} \underline{C}_e \langle \bar{\underline{\sigma}}_{n+1}^T \Delta \underline{\varepsilon}_{n+1} \rangle^+, \quad (3.2.128)$$

where $\Delta \underline{\varepsilon}_{n+1} = \underline{\varepsilon}_{n+1} - \underline{\varepsilon}_n$ and $\chi_{n+1} = B^+ H^+(\Delta \omega_{n+1}^+) + B^- H^+(\Delta \omega_{n+1}^-)$, with $\Delta \omega_{n+1}^\pm = \omega_{n+1}^\pm - \omega_n^\pm$. Next, using the relation of Eq. (3.2.81) with $\underline{n}_{n+1} = \bar{\underline{\sigma}}_{n+1}$, the updated effective stress vector is given by

$$\bar{\underline{\sigma}}_{n+1} = \bar{\underline{\sigma}}_{n+1}^{\text{tr}} - \frac{E_o \chi_{n+1}}{\|\bar{\underline{\sigma}}_{n+1}\|^2} \langle \bar{\underline{\sigma}}_{n+1}^T \Delta \underline{\varepsilon}_{n+1} \rangle^+ \bar{\underline{\sigma}}_{n+1} \quad (3.2.129)$$

It should be noted that $\bar{\underline{\sigma}}_{n+1}$ is proportional, or geometrically parallel, to $\bar{\underline{\sigma}}_{n+1}^{\text{tr}}$. Thus, satisfy the following relation

$$\frac{\bar{\underline{\sigma}}_{n+1}}{\|\bar{\underline{\sigma}}_{n+1}\|} = \frac{\bar{\underline{\sigma}}_{n+1}^{\text{tr}}}{\|\bar{\underline{\sigma}}_{n+1}^{\text{tr}}\|}. \quad (3.2.130)$$

Replacing this expression into Eq. (3.2.129), the updated effective stress tensor is given as

$$\bar{\underline{\sigma}}_{n+1} = m_{n+1}^{\text{tr}} \bar{\underline{\sigma}}_{n+1}^{\text{tr}}, \quad (3.2.131)$$

$$m_{n+1}^{\text{tr}} = 1 - \frac{E_o \xi_{n+1}}{n_0} \langle n_1 \rangle^+, \quad (3.2.132)$$

where $n_0 = (\bar{\underline{\sigma}}_{n+1}^{\text{tr}} \cdot \bar{\underline{\sigma}}_{n+1}^{\text{tr}})$ and $n_1 = (\bar{\underline{\sigma}}_{n+1}^{\text{tr}} \cdot \Delta \underline{\epsilon}_{n+1})$. It should be noted that, as the Heaviside function is present in the variable χ_{n+1} , it required an iterative process to solve $\bar{\underline{\sigma}}_{n+1}$. Box 3 shown an efficient and robust algorithm to solve the updated effective stress tensor $\bar{\underline{\sigma}}_{n+1}$.

Box 6 :Algorithm to solve $\bar{\underline{\sigma}}_{n+1}$ for the FOC model

$\underline{v}_1 = [0, 1, 0, 1]^T, \quad \underline{v}_2 = [0, 0, 1, 1]^T$ for $j \leq 4$ do $h_1^j = \underline{v}_1[j], \quad h_2^j = \underline{v}_2[j]$ $\xi^j = B^+ h_1^j + B^- h_2^j, \quad m_{n+1}^j$ $\bar{\underline{\sigma}}^j = m_{n+1}^j \bar{\underline{\sigma}}_{n+1}^{\text{tr}}$ $\bar{\underline{\sigma}}^{j\pm} = \underline{P}^{\pm} \bar{\underline{\sigma}}^j$ $Y_j^{\pm}$ $F_d^{j\pm} = Y_j^{\pm} - r_n^{\pm}$ if $(h_1^j = H^+(F_d^{j+}))$ and $(h_2^j = H^+(F_d^{j-}))$ then exit $\bar{\underline{\sigma}}_{n+1} = \bar{\underline{\sigma}}^j$	$\triangleright$ Set combinatory vectors $\triangleright$ Trial Heaviside values $\triangleright$ Use Eq. (3.2.132) $\triangleright$ Trial effective stress $\triangleright$ Positive/negative effective stress $\triangleright$ DEER's, according to Eq. (3.1.52) or Eq. (3.1.53) $\triangleright$ Positive/negative damage criteria (Eq. (3.1.51)) $\triangleright$ Update effective stress
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Finally, replacing Eq. (3.2.130) into Eq. (3.2.127), the updated plastic strain vector is derived as

$$\underline{\epsilon}_{n+1}^p = \underline{\epsilon}_n^p + (1 - m_{n+1}^{\text{tr}}) \underline{\mathcal{C}}_e : \bar{\underline{\sigma}}_{n+1}^{\text{tr}}. \quad (3.2.133)$$

3.2.7. ROT model

Damage component

Assuming an implicit integration scheme for the linearization of Eq. (3.1.68), the updated positive/negative i -th damage strain variable $\alpha_i^\pm$ is expressed as

$$\alpha_{i_{n+1}}^\pm = \alpha_{i_n}^\pm + z_{i_{n+1}}(\alpha_{i_n}^\pm, \hat{\varepsilon}_{i_{n+1}})\Delta\hat{\varepsilon}_{i_{n+1}}, \quad (3.2.134)$$

where $z_{i_{n+1}} = 1 - r_{i_{n+1}}$ and $\Delta\hat{\varepsilon}_{i_{n+1}} = \hat{\varepsilon}_{i_{n+1}} - \hat{\varepsilon}_{i_n}$, with $r_{i_{n+1}}^\pm = H_0^\pm(\alpha_{i_n}^\pm - \hat{\varepsilon}_{i_{n+1}})$. Note that the term $\alpha_{i_{n+1}}^\pm$ inner the Heaviside function is used to get an explicit scheme. So, the evaluation of updated stress vector $\underline{\sigma}$ is explicit (Eq. (3.1.72)) using the relations of Eqs. (3.1.70) and (3.1.71), where $m_{i_{n+1}}^\pm = H_{1/2}^\pm(\hat{\varepsilon}_{i_{n+1}})$ and the variables $h_{i_{n+1}}^\pm$ and $g_{i_{n+1}}^\pm$ are written, respectively, as

$$h_{i_{n+1}}^\pm = \sigma^\pm(\alpha_{i_{n+1}}^\pm)g_{i_{n+1}}^\pm, \quad g_{i_{n+1}}^\pm = \frac{\hat{\varepsilon}_{i_{n+1}}}{\alpha_{i_{n+1}}^\pm}. \quad (3.2.135)$$

Viscous component

Taking the linearization of Eq. (3.1.73), the updated positive/negative i -th viscous-damage strain $\alpha_i^{v\pm}$ can be expressed as

$$\alpha_{i_{n+1}}^{v\pm} = \zeta_v \alpha_{i_n}^{v\pm} + (1 - \zeta_v) \alpha_{i_{n+1}}^\pm. \quad (3.2.136)$$

Finally, the evaluation of updated viscous-stress vector $\underline{\sigma}^v$ (Eq. (3.1.76)) is explicit using the relations of Eqs. (3.1.74) and (3.1.75).

3.3. Consistent tangent tensors

Additionally to the algorithm necessary to calculate the updated stress vector, a material stiffness matrix is required for the solution. Continuum tangent stiffness matrix is derived for material models according to derivation of continuum constitutive equations

as stated in Section 3.1. However, for numerical integration of model, is necessary to calculate the algorithmic consistent tangent matrix $\frac{d\boldsymbol{\sigma}_{n+1}}{d\boldsymbol{\varepsilon}_{n+1}}$, which are found by computing the derivatives of equations involved in the stress updated algorithm. Complex derivatives involve this operator, but are necessary to achieve a second-order convergence at the structural level, rather than continuum tangent stiffness (J. C. Simo & Hughes, 1998). For the developed models, all these derivatives can be obtained analytically. Therefore, the consistent tangent operator can be written in an explicit expression. For sake the of simplicity of the presentation, we omitted the subscript $n+1$ in all updated variables.

3.3.1. Trial-predictor step

Using Eqs. (3.2.77) and (3.2.78), the differential of the trial elastic strain $\underline{\boldsymbol{\varepsilon}}^{e\text{tr}}$ and the stress vector $\boldsymbol{\sigma}$ are, respectively, given by

$$d\underline{\boldsymbol{\varepsilon}}^{e\text{tr}} = d\underline{\boldsymbol{\varepsilon}}, \quad (3.3.137)$$

$$d\underline{\boldsymbol{\sigma}}^{\text{tr}} = \underline{\boldsymbol{D}}_e : d\underline{\boldsymbol{\varepsilon}} \quad (3.3.138)$$

It follow that in the derivation of consistent tangent stiffness matrix all trial variables $(\cdot)^{\text{tr}}$ have a no-null differential, contrary as in the calculation of stress updated algorithm, where their derivatives are neglected.

3.3.2. DPH model

First, the differential of variables $\bar{a}_1$, $\bar{a}_2$ and g_1 , given by Eqs. (3.2.91) and (3.2.88), are expressed, respectively, as

$$\begin{aligned} d\bar{a}_1 &= -\lambda\bar{a}_1^2(u\Delta\gamma dz + td\Delta\gamma), & d\bar{a}_2 &= -2\mu\bar{a}_2^2(u\Delta\gamma dz + td\Delta\gamma), \\ dg_1 &= d\tau_1^{\text{tr}} - \sqrt{2}\bar{\eta}\lambda d\Delta\gamma, \end{aligned} \quad (3.3.139)$$

with $u = -\frac{9}{8r^3}$. Then, taking this relations, the differential of matrix $\bar{\underline{B}}$, given by Eq. (3.2.91), is written as follows

$$\begin{aligned} d\bar{\underline{B}} &= \text{diag} (d\bar{b}_1, d\bar{b}_2, d\bar{b}_2) \\ &= \text{diag} \left\{ \frac{1}{(\tau_{11}^{\text{tr}})^2} [\tau_{11}^{\text{tr}}(\bar{a}_1 dg_1 + g_1 d\bar{a}_1) - g_1 \bar{a}_1 d\tau_{11}^{\text{tr}}], d\bar{a}_2, d\bar{a}_2 \right\} \\ &= \bar{\underline{B}}_1 d\tau_{11}^{\text{tr}} + \bar{\underline{B}}_2 d\Delta\gamma + \bar{\underline{B}}_3 dz, \end{aligned} \quad (3.3.140)$$

where $\bar{\underline{B}}_1$, $\bar{\underline{B}}_2$ and $\bar{\underline{B}}_3$ are given by

$$\begin{aligned} \bar{\underline{B}}_1 &= \text{diag} \left(\frac{\sqrt{2}\lambda\bar{a}_1\bar{\eta}\Delta\gamma}{(\tau_{11}^{\text{tr}})^2}, 0, 0 \right), \\ \bar{\underline{B}}_2 &= \text{diag} \left(-\frac{\bar{a}_1\lambda}{\tau_{11}^{\text{tr}}} (\bar{a}_1 tg_1 + \sqrt{2}\bar{\eta}), -2\mu\bar{a}_2^2 t, -2\mu\bar{a}_2^2 t \right), \\ \bar{\underline{B}}_3 &= \text{diag} \left(-\frac{\lambda\bar{a}_1^2 u g_1 \Delta\gamma}{\tau_{11}^{\text{tr}}}, -2\mu\bar{a}_2^2 u \Delta\gamma, -2\mu\bar{a}_2^2 u \Delta\gamma \right). \end{aligned}$$

Then, differential of matrix $\underline{B}$ (Eq. (3.2.90)) is expressed as

$$\begin{aligned} d\underline{B} &= \underline{Q}^T d\bar{\underline{B}} \underline{Q} \\ &= \underline{B}_1 d\tau_{11}^{\text{tr}} + \underline{B}_2 d\Delta\gamma + \underline{B}_3 dz, \end{aligned} \quad (3.3.141)$$

where $\underline{B}_i = \underline{Q}^T \bar{\underline{B}}_i \underline{Q}$, with $i = 1, 2, 3$. Moreover, using the relation Eq. (3.2.90), the variable $z = 2J_2$ (Eq. (A.2.6)) can be expressed as $z = \underline{\sigma}^{\text{tr}T} \underline{B} \underline{P} \underline{B} \underline{\sigma}^{\text{tr}}$. Then, using the relations $\underline{1}_+^T \underline{Q} = \frac{1}{\sqrt{2}} \underline{1}^T$, $d\tau_{11}^{\text{tr}} = \underline{1}_+^T \underline{Q} d\underline{\sigma}^{\text{tr}}$ and Eq. (3.3.141), the differential of this variable is given by

$$\begin{aligned} dz &= 2\underline{\sigma}^{\text{tr}T} \underline{B} \underline{P} (\underline{B} d\underline{\sigma}^{\text{tr}} + d\underline{B} \underline{\sigma}^{\text{tr}}) \\ &= 2 \left[\left(\underline{v}_{dev}^T \underline{B} + \frac{\xi_1}{\sqrt{2}} \underline{1}^T \right) d\underline{\sigma}^{\text{tr}} + \xi_2 d\Delta\gamma + \xi_3 dz \right], \end{aligned} \quad (3.3.142)$$

where $\underline{v}_{dev} = \underline{P}\underline{\sigma}$ and $\xi_i = \underline{v}_{dev}^T \underline{B}_i \underline{\sigma}^{tr}$, with $i = 1, 3$. Then, the differential of variable z can be solved of this expression as

$$dz = \xi_0 \left[\left(\underline{v}_{dev}^T \underline{B} + \frac{\xi_1}{\sqrt{2}} \underline{1}^T \right) d\underline{\sigma}^{tr} + \xi_2 d\Delta\gamma \right], \quad (3.3.143)$$

with $\xi_0 = (1/2 - \xi_3)^{-1}$. Then, using Eqs. (3.3.143) and (3.3.141), the differential of updated stress vector $\underline{\sigma}$, given by Eq. (3.2.90), can be expressed as

$$\begin{aligned} d\underline{\sigma} &= d\underline{B}\underline{\sigma}^{tr} + \underline{B}d\underline{\sigma}^{tr} \\ &= \underline{A}_5 d\underline{\sigma}^{tr} + \underline{a}_6 d\Delta\gamma, \end{aligned} \quad (3.3.144)$$

where $\underline{A}_5$ and $\underline{a}_6$ are given by

$$\begin{aligned} \underline{A}_5 &= \frac{1}{\sqrt{2}} (\underline{B}_1 + \xi_0 \xi_1 \underline{B}_3) (\underline{\sigma}^{tr} \otimes \underline{1}) + [\xi_0 \underline{B}_3 (\underline{\sigma}^{tr} \otimes \underline{v}_{dev}) + \underline{I}] \underline{B}, \\ \underline{a}_6 &= (\underline{B}_2 + \xi_0 \xi_2 \underline{B}_3) \underline{\sigma}^{tr}. \end{aligned}$$

In addition, using Eq. (3.2.92) and the chain rule, the differential of updated cohesion law can be written as

$$dc = \frac{\partial c}{\partial \alpha} d\alpha = J_\alpha \xi d\Delta\gamma, \quad (3.3.145)$$

where $J_\alpha := \frac{\partial c}{\partial \alpha}$ is the cohesive hardening modulus. Then, using Eqs. (3.3.143), (3.3.144) and (A.2.21), the differential of consistency condition for the yield criterion, given by Eq. (3.2.93), can be expressed as

$$\begin{aligned} dF &= \frac{\eta}{3} \underline{1}^T d\underline{\sigma} + \frac{3}{4q} dz - \xi^2 J_\alpha d\Delta\gamma = 0 \\ &= \underline{g}_0^T d\underline{\sigma}^{tr} + g_0 d\Delta\gamma, \end{aligned} \quad (3.3.146)$$

where $\underline{g}_0^T$ and g_0 are expressed as

$$\underline{g}_0^T = \frac{\eta}{3} \underline{1}^T \underline{A}_5 + \frac{3\xi_0}{4q} \left(\underline{v}_{dev}^T \underline{B} + \frac{\xi_1}{\sqrt{2}} \underline{1}^T \right), \quad g_0 = \frac{\eta}{3} \underline{1}^T \underline{a}_6 + \frac{3\xi_0 \xi_2}{4q} - \xi^2 J_\alpha. \quad (3.3.147)$$

Then, the differential of consistency operator $\Delta\gamma$ is solved as

$$d\Delta\gamma = -\frac{1}{g_0} \underline{\mathbf{g}}^T d\underline{\boldsymbol{\sigma}}^{\text{tr}} = \underline{\mathbf{g}}^T d\underline{\boldsymbol{\sigma}}^{\text{tr}}. \quad (3.3.148)$$

For other hand, using the relations Eqs. (3.3.143) and (3.3.144), the differential of flow vector $\underline{\mathbf{n}}$ is given by

$$\begin{aligned} d\underline{\mathbf{n}} &= \frac{3}{2r^3} \underline{\mathbf{P}} \left(r^2 d\underline{\boldsymbol{\sigma}} - \frac{3}{4} \underline{\boldsymbol{\sigma}} dz \right) \\ &= \underline{\mathbf{A}}_0 d\underline{\boldsymbol{\sigma}}^{\text{tr}} + \underline{\mathbf{a}}_1 d\Delta\gamma, \end{aligned} \quad (3.3.149)$$

where $\underline{\mathbf{A}}_0$ and $\underline{\mathbf{a}}_1$ are written as

$$\begin{aligned} \underline{\mathbf{A}}_0 &= \frac{3}{2r} \underline{\mathbf{P}} \underline{\mathbf{A}}_5 + \xi_0 u \left[(\underline{\mathbf{v}}_{dev} \otimes \underline{\mathbf{v}}_{dev}) \underline{\mathbf{B}} + \frac{\xi_1}{\sqrt{2}} (\underline{\mathbf{v}}_{dev} \otimes \underline{\mathbf{1}}) \right], \\ \underline{\mathbf{a}}_1 &= \frac{3}{2r} \underline{\mathbf{P}} \underline{\mathbf{a}}_6 + \xi_0 \xi_2 u \underline{\mathbf{v}}_{dev} \end{aligned}$$

For other hand, using Eq. (3.2.81), the differential of updated stress vector can be written in a convenient format as

$$d\underline{\boldsymbol{\sigma}} = \underline{\mathbf{D}}_e \left[\underline{\mathbf{C}}_e d\underline{\boldsymbol{\sigma}}^{\text{tr}} - \Delta\gamma d\underline{\mathbf{n}} - \underline{\mathbf{n}} d\Delta\gamma \right]. \quad (3.3.150)$$

Hence, inserting Eqs. (3.3.148) and (3.3.149) into this relation, gives a final expression for the differential of updated stress vector as

$$d\underline{\boldsymbol{\sigma}} = \underline{\mathbf{D}}_e \left[\underline{\mathbf{C}}_e - \Delta\gamma (\underline{\mathbf{A}}_0 + \underline{\mathbf{a}}_1 \otimes \underline{\mathbf{g}}) - (\underline{\mathbf{n}} \otimes \underline{\mathbf{g}}) \right] d\underline{\boldsymbol{\sigma}}^{\text{tr}}. \quad (3.3.151)$$

Finally, using Eq. (3.3.137), an explicit expression for the elasto-plastic consistent tangent stiffness matrix can be written as

$$\underline{\mathbf{D}}_{ep} = \underline{\mathbf{D}}_e \left[\underline{\mathbf{C}}_e - \Delta\gamma (\underline{\mathbf{A}}_0 + \underline{\mathbf{a}}_1 \otimes \underline{\mathbf{g}}) - (\underline{\mathbf{n}} \otimes \underline{\mathbf{g}}) \right] \underline{\mathbf{D}}_e. \quad (3.3.152)$$

In addition, the 1 include an alternative derivation of this operator considering the differential of updated stress vector directly, rather than use Eq. (3.2.90) as in this case.

3.3.3. LLF model

Plastic component

The plastic component of the consistent tangent stiffness matrix is calculated from the differential of the effective stress tensor. To this, firstly, the differential of variables $\hat{a}_1$, $\hat{a}_2$ and $\hat{g}_1$, given by Eqs. (3.2.104) and (3.2.106), are expressed as

$$\begin{aligned} d\hat{a}_1 &= -\lambda\hat{a}_1^2(\bar{u}\Delta\gamma d\bar{z} + \bar{t}d\Delta\gamma), & d\hat{a}_2 &= -2\mu\hat{a}_2^2(\bar{u}\Delta\gamma d\bar{z} + \bar{t}d\Delta\gamma), \\ d\hat{g}_1 &= d\hat{\tau}_1^{\text{tr}} - \sqrt{2}\bar{\eta}\lambda d\Delta\gamma, \end{aligned} \quad (3.3.153)$$

where $\bar{u} = -\frac{9}{8\bar{\epsilon}^3}$. Now, taking these relations, the differential of matrix $\hat{\underline{\underline{B}}}$, given by Eq. (3.2.107), is expressed as

$$\begin{aligned} d\hat{\underline{\underline{B}}} &= \text{diag} \left(d\hat{b}_1, d\hat{b}_2 \right) \\ &= \text{diag} \left\{ \frac{1}{(\hat{\tau}_1^{\text{tr}})^2} \left[(\hat{a}_1 d\hat{g}_1 + \hat{g}_1 d\hat{a}_1) \hat{\tau}_1^{\text{tr}} - \hat{g}_1 \hat{a}_1 d\hat{\tau}_1^{\text{tr}} \right], d\hat{a}_2 \right\} \\ &= \hat{\underline{\underline{B}}}_1 d\hat{\tau}_1^{\text{tr}} + \hat{\underline{\underline{B}}}_2 d\Delta\gamma + \hat{\underline{\underline{B}}}_3 dz, \end{aligned} \quad (3.3.154)$$

where $\hat{\underline{\underline{B}}}_1$ to $\hat{\underline{\underline{B}}}_3$ are expressed as

$$\begin{aligned} \hat{\underline{\underline{B}}}_1 &= \text{diag} \left(\frac{\sqrt{2}\hat{a}_1\bar{\eta}\lambda\Delta\gamma}{(\hat{\tau}_1^{\text{tr}})^2}, 0 \right), \\ \hat{\underline{\underline{B}}}_2 &= \text{diag} \left(-\frac{\hat{a}_1\lambda}{\hat{\tau}_1^{\text{tr}}} \left(\hat{a}_1\bar{t}\hat{g}_1 + \sqrt{2}\bar{\eta} \right), -2\mu\hat{a}_2^2\bar{t} \right), \\ \hat{\underline{\underline{B}}}_3 &= \text{diag} \left(-\frac{\hat{a}_1^2\lambda\hat{\theta}\hat{g}_1\Delta\gamma}{\hat{\tau}_1^{\text{tr}}}, -2\mu\hat{a}_2^2\hat{\theta}\Delta\gamma \right). \end{aligned}$$

Then, the differential of matrix $\hat{\underline{B}}$ (Eq. (3.2.107)) is given by

$$\begin{aligned} d\hat{\underline{B}} &= \hat{\underline{Q}}^T d\hat{\underline{B}} \hat{\underline{Q}} \\ &= \hat{\underline{B}}_1 d\hat{\tau}_1^{\text{tr}} + \hat{\underline{B}}_2 d\Delta\gamma + \hat{\underline{B}}_3 d\bar{z}, \end{aligned} \quad (3.3.155)$$

where $\hat{\underline{B}}_i = \hat{\underline{Q}}^T \hat{\underline{B}}_i \hat{\underline{Q}}$, with $i = 1, , 3$. Moreover, using the relation Eq. (3.2.106), the variable $\bar{z} = 2\bar{J}_2$ (Eq. (A.2.18)) can be expressed as $\bar{z} = \hat{\underline{\sigma}}^{\text{tr}} \hat{\underline{B}} \hat{\underline{P}} \hat{\underline{B}} \hat{\underline{\sigma}}^{\text{tr}}$. Thus, using the relations $d\hat{\tau}_1^{\text{tr}} = \hat{\underline{1}}_+^T \hat{\underline{Q}} d\hat{\underline{\sigma}}^{\text{tr}}$ and $\hat{\underline{1}}_+^T \hat{\underline{Q}} = \frac{1}{\sqrt{2}} \hat{\underline{1}}^T$, the differential of variable $\bar{z}$ is expressed as

$$\begin{aligned} d\bar{z} &= 2\hat{\underline{\sigma}}^{\text{tr}T} \hat{\underline{B}} \hat{\underline{P}} \left(\hat{\underline{B}} d\hat{\underline{\sigma}}^{\text{tr}} + d\hat{\underline{B}} \hat{\underline{\sigma}}^{\text{tr}} \right) \\ &= 2 \left[\left(\hat{\underline{v}}_{dev}^T \hat{\underline{B}} + \frac{\hat{\xi}_1}{\sqrt{2}} \hat{\underline{1}}^T \right) d\hat{\underline{\sigma}}^{\text{tr}} + \hat{\xi}_2 d\Delta\gamma + \hat{\xi}_3 d\bar{z} \right], \end{aligned} \quad (3.3.156)$$

where $\hat{\xi}_i = \hat{\underline{v}}_{dev}^T \hat{\underline{B}}_i \hat{\underline{\sigma}}^{\text{tr}}$, with $i = 1, , 3$. Then, the differential of variable $\bar{z}$ can be solved as

$$d\bar{z} = \hat{\xi}_0 \left[\left(\hat{\underline{v}}_{dev}^T \hat{\underline{B}} + \frac{\hat{\xi}_1}{\sqrt{2}} \hat{\underline{1}}^T \right) d\hat{\underline{\sigma}}^{\text{tr}} + \hat{\xi}_2 d\Delta\gamma \right], \quad (3.3.157)$$

with $\hat{\xi}_0 = (1/2 - \hat{\xi}_3)^{-1}$. Then, using Eqs. (3.3.157) and (3.3.155), the differential of the updated principal effective stress vector $\hat{\underline{\sigma}}$ is given by

$$\begin{aligned} d\hat{\underline{\sigma}} &= d\hat{\underline{B}} \hat{\underline{\sigma}}^{\text{tr}} + \hat{\underline{B}} d\hat{\underline{\sigma}}^{\text{tr}} \\ &= \hat{\underline{A}}_5 d\hat{\underline{\sigma}}^{\text{tr}} + \hat{\underline{a}}_6 d\Delta\gamma, \end{aligned} \quad (3.3.158)$$

where $\hat{\underline{A}}_5$ and $\hat{\underline{a}}_6$ are given by

$$\begin{aligned} \hat{\underline{A}}_5 &= \frac{1}{\sqrt{2}} \left(\hat{\underline{B}}_1 + \hat{\xi}_0 \hat{\xi}_1 \hat{\underline{B}}_3 \right) (\hat{\underline{\sigma}} \otimes \hat{\underline{1}}) + \left[\hat{\xi}_0 \hat{\underline{B}}_3 (\hat{\underline{\sigma}}^{\text{tr}} \otimes \hat{\underline{v}}_{dev}) \hat{\underline{P}} + \hat{\underline{I}} \right] \hat{\underline{B}}, \\ \hat{\underline{a}}_6 &= \left(\hat{\underline{B}}_2 + \hat{\xi}_0 \hat{\xi}_2 \hat{\underline{B}}_3 \right) \hat{\underline{\sigma}}^{\text{tr}}. \end{aligned}$$

Moreover, the differential of maximum principal effective stress $\hat{\sigma}_+ = \underline{\hat{\mathbf{1}}}_+^T d\underline{\hat{\sigma}}$ can be expressed as

$$d\hat{\sigma}_+ = \underline{\hat{\mathbf{a}}}_{5+}^T d\underline{\hat{\sigma}} + \hat{a}_{6+} d\Delta\gamma, \quad (3.3.159)$$

where $\underline{\hat{\mathbf{a}}}_{5+}^T = \underline{\hat{\mathbf{1}}}_+^T \underline{\hat{\mathbf{A}}}_5$ and $\hat{a}_{6+} = \underline{\hat{\mathbf{1}}}_+^T \underline{\hat{\mathbf{a}}}_6$. Next, using Eqs. (3.3.157) and (3.3.158), the differential of the principal effective flow vector, given by Eq. (3.2.98), can be expressed as

$$\begin{aligned} d\underline{\hat{\mathbf{n}}} &= \frac{3}{2\bar{r}^3} \underline{\hat{\mathbf{P}}} \left(\bar{r}^2 d\underline{\hat{\sigma}} - \frac{3}{4} \underline{\hat{\sigma}} d\bar{z} \right) \\ &= \underline{\hat{\mathbf{A}}}_0 d\underline{\hat{\sigma}}^{\text{tr}} + \underline{\hat{\mathbf{a}}}_1 d\Delta\gamma, \end{aligned} \quad (3.3.160)$$

where $\underline{\hat{\mathbf{A}}}_0$ and $\underline{\hat{\mathbf{a}}}_1$ are given by

$$\begin{aligned} \underline{\hat{\mathbf{A}}}_0 &= \frac{3}{2\bar{r}} \underline{\hat{\mathbf{P}}} \underline{\hat{\mathbf{A}}}_5 + \bar{u} \hat{\xi}_0 \left[(\underline{\hat{\mathbf{v}}}_{dev} \otimes \underline{\hat{\mathbf{v}}}_{dev}) \underline{\hat{\mathbf{B}}} + \frac{\hat{\xi}_1}{\sqrt{2}} (\underline{\hat{\mathbf{v}}}_{dev} \otimes \underline{\hat{\mathbf{1}}}) \right], \\ \underline{\hat{\mathbf{a}}}_1 &= \bar{t} \underline{\hat{\mathbf{P}}} \underline{\hat{\mathbf{a}}}_6 + \bar{u} \hat{\xi}_0 \hat{\xi}_2 \underline{\hat{\mathbf{v}}}_{dev}. \end{aligned}$$

Moreover, the differential of positive/negative part of effective principal flow vector $\hat{n}^\pm = \underline{\hat{\mathbf{1}}}_\pm^T \underline{\hat{\mathbf{n}}}$ are given by

$$d\hat{n}^\pm = \underline{\hat{\mathbf{a}}}_0^\pm{}^T d\underline{\hat{\sigma}}^{\text{tr}} + \hat{a}_1^\pm d\Delta\gamma, \quad (3.3.161)$$

where $\underline{\hat{\mathbf{a}}}_0^\pm = \underline{\hat{\mathbf{1}}}_\pm^T \underline{\hat{\mathbf{A}}}_0$ and $\hat{a}_1^\pm = \underline{\hat{\mathbf{1}}}_\pm^T \underline{\hat{\mathbf{a}}}_1$. In addition, using Eq. (3.3.157), the differential of effective flow vector $\underline{\hat{\mathbf{n}}}$ (Eq. (3.2.82) in effective space) is expressed as

$$\begin{aligned} d\underline{\hat{\mathbf{n}}} &= \frac{3}{2\bar{r}^3} \underline{\hat{\mathbf{P}}} \left(\bar{r}^2 d\underline{\hat{\sigma}} - \frac{3}{4} \underline{\hat{\sigma}} d\bar{z} \right), \\ &= \underline{\hat{\mathbf{A}}}_0 d\underline{\hat{\sigma}}^{\text{tr}} + \underline{\hat{\mathbf{a}}}_1 d\Delta\gamma + \underline{\hat{\mathbf{A}}}_2 d\underline{\hat{\sigma}}, \end{aligned} \quad (3.3.162)$$

where $\bar{\underline{\mathbf{A}}}_0$, $\bar{\underline{\mathbf{a}}}_1$ and $\bar{\underline{\mathbf{A}}}_2$ are given by

$$\begin{aligned}\bar{\underline{\mathbf{A}}}_0 &= \bar{u}\hat{\xi}_0 \left[(\bar{\underline{\mathbf{v}}}_{dev} \otimes \hat{\underline{\mathbf{v}}}_{dev}) \hat{\underline{\mathbf{B}}} + \frac{\hat{\xi}_1}{\sqrt{2}} (\bar{\underline{\mathbf{v}}}_{dev} \otimes \hat{\underline{\mathbf{1}}}) \right], \\ \bar{\underline{\mathbf{a}}}_1 &= \bar{u}\hat{\xi}_0\hat{\xi}_2\bar{\underline{\mathbf{v}}}_{dev}, \quad \bar{\underline{\mathbf{A}}}_2 = \bar{t}\underline{\mathbf{P}}.\end{aligned}$$

For the other hand, the differential of updated variable ϕ (Eq. (3.1.18)) is written as $d\phi = \hat{\underline{\Phi}}^T d\hat{\underline{\sigma}}$, with $\hat{\underline{\Phi}}$ defined as

$$\hat{\underline{\Phi}} = \frac{\partial\phi}{\partial\hat{\underline{\sigma}}} = \text{diag} \left(\frac{\partial\phi}{\partial\hat{\sigma}_1}, \frac{\partial\phi}{\partial\hat{\sigma}_2} \right), \quad (3.3.163)$$

being their i -th component $\frac{\partial\phi}{\partial\hat{\sigma}_i}$ expressed as

$$\frac{\partial\phi}{\partial\hat{\sigma}_i} = [H_o^+(\hat{\sigma}_i) - \phi(2H_o^+(\hat{\sigma}_i) - 1)] \frac{1}{\sum_{i=1}^N |\hat{\sigma}_i|} \quad (3.3.164)$$

It should be noted, that this expression considered the stepped Heaviside function, due that variable $\phi \in [0, 1]$. It can observed that this condition not cause convergence troubles in the model. Then, the differential of variables $\theta_1^\pm$ and $\theta_2^\pm$ (Eqs. (3.2.111) and (3.2.112), respectively) are given by $d\theta_1^\pm = d\phi$ and $d\theta_2^\pm = \frac{J_\kappa^\pm}{g^\pm} d\kappa^\pm$, with $J_\kappa^\pm := \frac{\partial\sigma^\pm}{\partial\kappa^\pm}$ are the positive/negative hardening modulus, respectively. Hence, the differential of variables $\varphi^\pm$, defined in Eq. (3.2.110), are given by

$$d\varphi^\pm = \theta_2^\pm \hat{\underline{\Phi}}^T d\hat{\underline{\sigma}} + \frac{1}{g^\pm} \theta_1^\pm J_\kappa^\pm d\kappa^\pm. \quad (3.3.165)$$

Moreover, using this relation, the differential of variables $h^\pm$, defined in Eq. (3.2.109), are expressed as

$$dh^\pm = \theta_2^\pm \hat{n}^\pm \hat{\underline{\Phi}}^T d\hat{\underline{\sigma}} + \hat{b}_{10}^\pm d\kappa^\pm + \varphi^\pm d\hat{n}^\pm, \quad (3.3.166)$$

where $\hat{b}_{10}^\pm = \frac{1}{g^\pm} \theta_1^\pm J_\kappa^\pm \hat{n}^\pm$. Next, using this relation and Eqs. (3.3.161) and (3.3.158), the differential of updated positive/negative hardening variables $\kappa^\pm$, given by Eq. (3.2.109),

can be written as

$$d\kappa^\pm = \hat{c}_1^\pm d\Delta\gamma + \Delta\gamma \left(\hat{\underline{c}}_2^{\pm T} d\hat{\underline{\sigma}}^{\text{tr}} + \hat{b}_{10}^\pm d\kappa^\pm \right), \quad (3.3.167)$$

with $\hat{c}_1^\pm$ and $\hat{\underline{c}}_2^\pm$ expressed as

$$\hat{c}_1^\pm = h^\pm + \Delta\gamma \left(\hat{\theta}_2^\pm \hat{n}^\pm \hat{\Phi}^T \hat{\underline{a}}_6 + \varphi^\pm \hat{a}_1^\pm \right), \quad \hat{\underline{c}}_2^{\pm T} = \hat{\theta}_2^\pm \hat{n}^\pm \hat{\Phi}^T \hat{\underline{A}}_5 + \varphi^\pm \hat{\underline{a}}_0^{\pm T}.$$

Hence, solving this linear equation for the differential of variable $\kappa^\pm$ gives

$$d\kappa^\pm = c_1^\pm d\Delta\gamma + \Delta\gamma \underline{c}_2^{\pm T} d\hat{\underline{\sigma}}^{\text{tr}}, \quad (3.3.168)$$

where $c_1^\pm$ and $\underline{c}_2^\pm$ are multiple of their respective variables $\hat{c}_1^\pm$ and $\hat{\underline{c}}_2^\pm$ by a factor of $\hat{b}_{20}^\pm = \left(1 - \Delta\gamma \hat{b}_{10}^\pm \right)^{-1}$. In addition, the differential of uniaxial positive/negative effective stress law $\bar{\sigma}^\pm (= c^\pm)$ are expressed as

$$d\bar{\sigma}^\pm = \bar{J}_\kappa^\pm d\kappa^\pm, \quad (3.3.169)$$

where $\bar{J}_\kappa^\pm := \frac{\partial \bar{\sigma}^\pm}{\partial \kappa^\pm}$ denotes the respective effective hardening modulus. Then, using this relation and Eq. (3.3.168), the differential of variable β (Eq. (3.2.113)) is given by

$$d\beta = c_4 \Delta\gamma + \Delta\gamma \underline{c}_6^T d\hat{\underline{\sigma}}^{\text{tr}}, \quad (3.3.170)$$

where c_4 and $\underline{c}_6$ are expressed as

$$c_4 = m^+ c_1^- - m^- c_1^+, \quad \underline{c}_6 = m^+ \underline{c}_2^- - m^- \underline{c}_2^+,$$

with $m^\pm = (1 - \alpha) \bar{J}_\kappa^\mp \frac{\bar{\sigma}^\pm}{(\bar{\sigma}^\pm)^2}$. It should be noted that, for this model only the uniaxial $\sigma^\pm$ and $\omega^\pm$ laws are known. Then, using the relation Eq. (3.1.31), the positive/negative effective hardening modulus $\bar{J}_\kappa^\pm$ can be derived as

$$\bar{J}_\kappa^\pm = \frac{J_\kappa^\pm + \Omega_\kappa^\pm \bar{\sigma}^\pm}{1 - \omega^\pm}, \quad (3.3.171)$$

with $J_\kappa^\pm := \frac{\partial \sigma^\pm}{\partial \kappa^\pm}$ and $\Omega_\kappa^\pm := \frac{\partial \omega^\pm}{\partial \kappa^\pm}$. Next, using Eqs. (3.3.159), (3.3.169), (3.3.170), (3.3.158), (A.1.24) and (A.2.24), the differential of yield criterion at consistency condition, given by Eq. (3.2.114), is written as

$$\begin{aligned} d\bar{F} &= \frac{\eta}{3} \hat{\mathbf{1}}^T d\hat{\underline{\sigma}} + \frac{3}{4\bar{q}} d\bar{z} + \langle \hat{\sigma}_+ \rangle^+ d\beta + \beta d\langle \hat{\sigma}_+ \rangle^+ - (1 - \alpha) d\bar{\sigma}^- = 0 \\ &= \hat{g}_0 d\Delta\gamma + \hat{\underline{g}}_0^T d\hat{\underline{\sigma}}^{\text{tr}}, \end{aligned} \quad (3.3.172)$$

where $\hat{g}_0$ and $\hat{\underline{g}}_0^T$ are expressed as

$$\begin{aligned} \hat{g}_0 &= \frac{\eta}{3} \hat{\mathbf{1}}^T \hat{\underline{a}}_6 + \frac{3\hat{\xi}_0\hat{\beta}_2}{4\bar{q}} + \langle \hat{\sigma}_+ \rangle^+ c_4 + \hat{\beta}_3 \hat{a}_{6+} - (1 - \alpha) \bar{J}_k^- c_1^-, \\ \hat{\underline{g}}_0^T &= \frac{\eta}{3} \hat{\mathbf{1}}^T \hat{\underline{A}}_5 + \frac{3\hat{\xi}_0}{4\bar{q}} \left(\hat{\underline{v}}_{dev}^T \hat{\underline{B}} + \frac{\hat{\xi}_1}{\sqrt{2}} \hat{\mathbf{1}}^T \right) + \hat{\beta}_3 \hat{\underline{a}}_{5+}^T \\ &\quad + \Delta\gamma \left[\langle \hat{\sigma}_+ \rangle^+ \hat{\underline{c}}_6^T - (1 - \alpha) \bar{J}_k^- \hat{\underline{c}}_2^T \right], \end{aligned}$$

with $\hat{\beta}_3 = \hat{\beta}_2 \hat{\sigma}_1 + \hat{\beta}_1$, $\hat{\beta}_1 = \beta \tilde{H}^+(\hat{\sigma}_1)$ and $\hat{\beta}_2 = \beta \frac{d\tilde{H}^+}{d\hat{\sigma}_1}$. Then, the differential of the discrete consistency operator $\Delta\gamma$ can be solved directly of Eq. (3.3.172) as

$$d\Delta\gamma = -\frac{1}{\hat{g}_0} \hat{\underline{g}}_0^T d\hat{\underline{\sigma}}^{\text{tr}} = \hat{\underline{g}}_0^T d\hat{\underline{\sigma}}^{\text{tr}}. \quad (3.3.173)$$

Now, the differential of updated effective stress vector can be derived in the same manner that Eq. (3.3.150), but expressed in the effective space as

$$d\bar{\underline{\sigma}} = d\bar{\underline{\sigma}}^{\text{tr}} - \underline{D}_e (\Delta\gamma d\bar{\underline{n}} - \bar{\underline{n}} d\Delta\gamma). \quad (3.3.174)$$

Then, using the relation $d\hat{\underline{\sigma}}^{\text{tr}} = \underline{F}_{\bar{\sigma}} d\bar{\underline{\sigma}}^{\text{tr}}$ (see Eq. (A.2.22)) and substituting Eqs. (3.3.173) and (3.3.162) into this expression, the differential of updated effective stress vector can be rewritten as

$$d\bar{\underline{\sigma}} = \underline{D}_e \left[\underline{C}_e - \Delta\gamma \left(\bar{\underline{a}}_1 \otimes \hat{\underline{g}}_0 + \bar{\underline{A}}_0 \right) \underline{F}_{\bar{\sigma}} - \left(\bar{\underline{n}} \otimes \hat{\underline{g}}_0 \right) \underline{F}_{\bar{\sigma}} \right] d\bar{\sigma}^{\text{tr}} - \Delta\gamma \underline{D}_e \bar{\underline{A}}_2 d\bar{\sigma}. \quad (3.3.175)$$

Finally, solving the updated effective stress vector of this expression and introducing the relation Eq. (3.3.138), the effective elasto-plastic consistent tangent matrix is written as

$$\underline{\bar{D}}_{ep} = \underline{\Xi}^* \left[\underline{C}_e - \Delta\gamma \left(\underline{\bar{a}}_1 \otimes \underline{\hat{g}}_0 + \underline{\bar{A}}_0 \right) \underline{F}_{\bar{\sigma}} - \left(\underline{\bar{n}} \otimes \underline{\hat{g}}_0 \right) \underline{F}_{\bar{\sigma}} \right] \underline{D}_e, \quad (3.3.176)$$

where $\underline{\Xi}^* = (\underline{C}_e + \bar{t}\Delta\gamma\underline{P})^{-1}$. It should be noted that, this matrix is well-posed independent of input material parameters used.

Damage component

First, calling the variables $t_c^+ = -z_c^+$ and $t_c^- = z_c^-$, the differential of stiffness recovery functions $s^\pm$, defined in Eq. (3.1.29), are expressed as $ds^\pm = t_c^\pm d\theta_1^\pm = t_c^\pm d\phi$. Also, the differential of uniaxial damage laws $\omega^\pm$ are given by $d\omega^\pm = \Omega_\kappa^\pm d\kappa^\pm$, where $\Omega_\kappa^\pm := \frac{\partial \omega^\pm}{\partial \kappa^\pm}$. Then, expressing both relations in a vectorized format as $\underline{s} = [s^+, s^-]^T$ and $\underline{\omega} = [\omega^+, \omega^-]^T$, respectively, their differentials are written as

$$d\underline{s} = \hat{\underline{M}}_1 d\hat{\underline{\sigma}}, \quad d\underline{\Omega} = \hat{\underline{M}}_2 d\underline{\kappa}, \quad (3.3.177)$$

where $\hat{\underline{M}}_1$ and $\hat{\underline{M}}_2$ are expressed as

$$\hat{\underline{M}}_1 = \begin{bmatrix} t_c^+ \\ t_c^- \end{bmatrix} \otimes \hat{\underline{\Phi}}, \quad \hat{\underline{M}}_2 = \begin{bmatrix} \Omega_\kappa^+ & 0 \\ 0 & \Omega_\kappa^- \end{bmatrix}.$$

For other hand, using Eq. (3.3.168), the differential of hardening vector $\underline{\kappa} = [\kappa^+, \kappa^-]^T$ are expressed as

$$d\underline{\kappa} = \underline{c}_1 d\Delta\gamma + \Delta\gamma \underline{C}_2^T d\hat{\underline{\sigma}}^{\text{tr}}, \quad (3.3.178)$$

where $\underline{c}_1$ and $\underline{C}_0$ are expressed as

$$\underline{c}_1 = \begin{bmatrix} c_1^+ \\ c_1^- \end{bmatrix}, \quad \underline{C}_2 = [\underline{C}_2^+, \underline{C}_2^-]$$

Next, the differential of damage variable ω (Eq. (3.1.28)) is given by

$$d\omega = \underline{\mathbf{u}}_1^T d\underline{\mathbf{s}} + \underline{\mathbf{u}}_2^T d\underline{\mathbf{\Omega}}, \quad (3.3.179)$$

where $\underline{\mathbf{u}}_1$ and $\underline{\mathbf{u}}_2$ are written as

$$\underline{\mathbf{u}}_1 = \begin{bmatrix} s_1 \omega^- \\ s_2 \omega^+ \end{bmatrix}, \quad \underline{\mathbf{u}}_2 = \begin{bmatrix} s_2 s^- \\ s_1 s^+ \end{bmatrix},$$

with $s_1 = 1 - s^- \omega^+$ and $s_2 = 1 - s^+ \omega^-$. In addition, substituting Eq. (3.3.177) and the relations $d\hat{\underline{\boldsymbol{\sigma}}} = \underline{\mathbf{F}}_{\bar{\boldsymbol{\sigma}}} d\bar{\underline{\boldsymbol{\sigma}}}$, $d\hat{\underline{\boldsymbol{\sigma}}}^{\text{tr}} = \underline{\mathbf{F}}_{\bar{\boldsymbol{\sigma}}}^{\text{tr}} = d\bar{\underline{\boldsymbol{\sigma}}}^{\text{tr}}$ (Eq. (A.2.22)) and $d\bar{\underline{\boldsymbol{\sigma}}} = \bar{\underline{\mathbf{D}}}_{ep} d\bar{\underline{\boldsymbol{\epsilon}}}^{\text{tr}}$ (Eq. (3.3.176)) into Eq. (3.3.179), the differential of damage variable ω can be rewritten as

$$\begin{aligned} d\omega &= \underline{\mathbf{v}}_1^T d\hat{\underline{\boldsymbol{\sigma}}} + \underline{\mathbf{v}}_2^T d\bar{\underline{\boldsymbol{\sigma}}} \\ &= \left[\underline{\mathbf{v}}_1^T \underline{\mathbf{F}}_{\bar{\boldsymbol{\sigma}}} \bar{\underline{\mathbf{D}}}_{ep} + \underline{\mathbf{v}}_2^T \left(\underline{\mathbf{c}}_1 \otimes \hat{\underline{\mathbf{g}}}_0 + \Delta\gamma \underline{\mathbf{C}}_2^T \right) \underline{\mathbf{F}}_{\bar{\boldsymbol{\sigma}}} \underline{\mathbf{D}}_e \right] d\bar{\underline{\boldsymbol{\epsilon}}}, \end{aligned} \quad (3.3.180)$$

where $\underline{\mathbf{v}}_1^T = \underline{\mathbf{u}}_1^T \hat{\underline{\mathbf{M}}}_1$ and $\underline{\mathbf{v}}_2^T = \underline{\mathbf{u}}_2^T \hat{\underline{\mathbf{M}}}_2$. For the other hand, the differential of updated stress vector (Eq. (3.1.14)) is given by

$$d\underline{\boldsymbol{\sigma}} = -\bar{\underline{\boldsymbol{\sigma}}} d\omega + (1 - \omega) d\bar{\underline{\boldsymbol{\sigma}}}. \quad (3.3.181)$$

Finally, introducing Eq. (3.3.180) and the relation $d\bar{\underline{\boldsymbol{\sigma}}} = \bar{\underline{\mathbf{D}}}_{ep} d\bar{\underline{\boldsymbol{\epsilon}}}^{\text{tr}}$ (Eq. (3.3.176)) into this relation, the elasto-plastic-damage consistent tangent matrix is expressed as

$$\underline{\mathbf{D}}_{epd} = [(1 - \omega) \underline{\mathbf{I}} - (\bar{\underline{\boldsymbol{\sigma}}} \otimes \underline{\mathbf{v}}_1) \underline{\mathbf{F}}_{\bar{\boldsymbol{\sigma}}}] \bar{\underline{\mathbf{D}}}_{ep} - (\bar{\underline{\boldsymbol{\sigma}}} \otimes \underline{\mathbf{v}}_2) \left(\underline{\mathbf{c}}_1 \otimes \hat{\underline{\mathbf{g}}}_0 + \Delta\gamma \underline{\mathbf{C}}_2^T \right) \underline{\mathbf{F}}_{\bar{\boldsymbol{\sigma}}} \underline{\mathbf{D}}_e. \quad (3.3.182)$$

Viscous component

Using Eq. (3.2.124) and the relation $d\bar{\underline{\sigma}} = \bar{\underline{D}}_{ep} d\bar{\underline{\epsilon}}^{tr}$ (Eq. (3.3.176)), the differential of updated effective viscous-stress vector can be expressed as

$$d\bar{\underline{\sigma}}^v = (\zeta_v \underline{\underline{D}}_e + (1 - \zeta_v) \bar{\underline{D}}_{ep}) d\bar{\underline{\epsilon}}. \quad (3.3.183)$$

Moreover, using Eqs. (3.2.122) and (3.2.123), the differential of visco-plastic strain vector $\underline{\underline{\epsilon}}^{vp}$ and visco-damage variable ω^v are given, respectively, by

$$d\underline{\underline{\epsilon}}^{vp} = (1 - \zeta_v) d\underline{\underline{\epsilon}}^p, \quad (3.3.184)$$

$$d\omega^v = (1 - \zeta_v) d\omega. \quad (3.3.185)$$

Finally, substituting these relations and Eqs. (3.3.183) and (3.3.180) into the differential of Eq. (3.2.125), the visco-plastic-damage consistent tangent matrix is expressed as

$$\begin{aligned} D_{vpd} = & \zeta_v (1 - \omega^v) \underline{\underline{D}}_e + (1 - \zeta_v) \left\{ [(1 - \omega^v) \underline{\underline{I}} - (\bar{\underline{\sigma}}^v \otimes \underline{\underline{v}}_1) \underline{\underline{F}}_{\bar{\underline{\sigma}}}] \bar{\underline{D}}_{ep} \right. \\ & \left. - (\bar{\underline{\sigma}}^v \otimes \underline{\underline{v}}_2) \left(\underline{\underline{c}}_1 \otimes \underline{\underline{g}}_0 + \Delta\gamma \underline{\underline{C}}_2^T \right) \underline{\underline{F}}_{\bar{\underline{\sigma}}} \underline{\underline{D}}_e \right\}. \end{aligned}$$

In addition, 2 include an alternative derivation of this operator considering the differential of updated stress vector directly, rather than use Eq. (3.2.106) as in this case.

3.3.4. WLF model

Plastic component

This component is identical to the LLF model, with the exception that $\theta_2^\pm = 1$ and $\hat{b}_{10}^\pm = 0$.

Damage component

Using Eqs. (A.2.22) and (A.2.23), the differential of positive/negative part of effective stress tensor (Eq. (3.1.37)) are given by

$$\begin{aligned} d\bar{\sigma}^{\pm} &= \sum_{i=1}^N H_0^{\pm}(\hat{\sigma}_i) \underline{e}_{\bar{\sigma}}^{ii} d\hat{\sigma}_i + \sum_{i=1}^N \langle \hat{\sigma}_i \rangle^{\pm} d\underline{e}_{\bar{\sigma}}^{ii} \\ &= \left(\sum_{i=1}^N H_0^{\pm}(\hat{\sigma}_i) (\underline{e}_{\bar{\sigma}}^{ii} \otimes \underline{e}_{\bar{\sigma}}^{ii}) + 2 \sum_{i=1, j>i}^N g_{ij}^{\pm} (\underline{e}_{\bar{\sigma}}^{ij} \otimes \underline{e}_{\bar{\sigma}}^{ij}) \right) \underline{R} d\bar{\sigma} = \underline{S}^{\pm} d\bar{\sigma}, \quad (3.3.186) \end{aligned}$$

where $\underline{S}^{\pm}$ are so-called the derivative of positive/negative projector effective stress vector, respectively, which satisfy the relations $\underline{S}^{+} + \underline{S}^{-} = \underline{I}$ and $\bar{\sigma}^{\pm} = \underline{S}^{\pm} \bar{\sigma}$, and $g_{ij}^{\pm}$ is defined as

$$g_{ij}^{\pm} := \begin{cases} \frac{\langle \hat{\sigma}_i \rangle^{\pm} - \langle \hat{\sigma}_j \rangle^{\pm}}{(\hat{\sigma}_i^{\pm} - \hat{\sigma}_j^{\pm})}, & \hat{\sigma}_i \neq \hat{\sigma}_j \\ H_0^{\pm}(\hat{\sigma}_i), & \hat{\sigma}_i = \hat{\sigma}_j. \end{cases}$$

So, during an active damage process, satisfy the relation Eq. (3.1.58). Then, using the chain rule, the differential of updated positive/negative damage law $\omega^{\pm}$ are expressed as

$$d\omega^{\pm} = \frac{\partial \omega^{\pm}}{\partial r^{\pm}} dr^{\pm} = \Omega_r^{\pm} dY^{\pm}, \quad (3.3.187)$$

where $\Omega_r^{\pm} := \frac{\partial \omega^{\pm}}{\partial r^{\pm}}$ and the differential of DEER, $Y^{\pm}$, are stated according their definition.

Thus, using Eqs. (3.1.52) and (3.1.53), their respective differentials are given by

$$dY^{\pm} = \frac{E_o}{2Y^{\pm}} (\bar{\sigma}^T \underline{C}_e \underline{S}^{\pm} + \bar{\sigma}^{\pm T} \underline{C}_e) d\bar{\sigma} = \underline{L}^{\pm T} d\bar{\sigma}, \quad (3.3.188)$$

$$dY^{-} = \left(\alpha \underline{1}^T + \frac{3}{2\bar{q}} \bar{\mathbf{v}}_{dev}^T \right) d\bar{\sigma} = \underline{L}^{-T} d\bar{\sigma}, \quad (3.3.189)$$

where $\bar{\mathbf{v}}_{dev} = \underline{P} \bar{\sigma}$. Next, using Eq. (3.1.45), the differential of updated stress vector is given by

$$d\bar{\sigma} = \sum_{\mathbb{N}} [(1 - \omega^{\mathbb{N}}) d\bar{\sigma}^{\mathbb{N}} - \bar{\sigma}^{\mathbb{N}} d\omega^{\mathbb{N}}]. \quad (3.3.190)$$

Finally, introducing Eqs. (3.3.187), (3.3.186) and (3.3.188) or Eq. (3.3.189) and $d\bar{\underline{\sigma}} = \bar{\underline{D}}_{ep}d\bar{\underline{\epsilon}}$ (Eq. (3.3.176)) into this expression, the plastic-damage consistent tangent matrix is written as

$$\underline{D}_{pd} = \left[\underline{I} - \sum_{\mathbb{N}} (\underline{W}^{\mathbb{N}} + \underline{R}^{\mathbb{N}}) \right] \bar{\underline{D}}_{ep}, \quad (3.3.191)$$

where $\underline{W}^{\pm} = \omega^{\pm} \underline{S}^{\pm}$ and $\underline{R}^{\pm} = \Omega_r^{\pm} (\bar{\underline{\sigma}}^{\pm} \otimes \underline{L}^{\pm})$. It should be noted that, the terms of this expression associated to plastic and damage component are decoupled.

Viscous component

First, using Eq. (3.2.126), the differential of positive/negative threshold variable $r^{\pm}$ are given by

$$dr^{\pm} = (1 - \zeta_v) dY^{\pm}. \quad (3.3.192)$$

Then, using this relation, Eq. (3.3.188) or Eq. (3.3.189) and the chain rule, the differential of positive/negative damage variables $\omega^{\pm}$ are given by

$$d\omega^{\pm} = \frac{\partial \omega^{\pm}}{\partial r^{\pm}} dr^{\pm} = \Omega_r^{\pm} (1 - \zeta_v) \underline{L}^{\pm T} d\bar{\underline{\sigma}}. \quad (3.3.193)$$

Moreover, using Eq. (3.1.62), the differential of positive/negative viscous stress vector is expressed as $d\bar{\underline{\sigma}}^{v\pm} = \underline{S}_v^{\pm} d\bar{\underline{\sigma}}^v$, where $\underline{S}_v^{\pm}$ are the derivative of positive/negative projector vector $\bar{\underline{\sigma}}^v$ (similar to Eq. (3.3.186)). Then, the differential of viscous-stress vector $\underline{\sigma}^v$, given by Eq. (3.1.61), is expressed as

$$d\underline{\sigma}^v = \sum_{\mathbb{N}} [(1 - \omega^{\mathbb{N}}) \underline{S}_v^{\mathbb{N}} d\bar{\underline{\sigma}}^v - \bar{\underline{\sigma}}^{v\mathbb{N}} d\omega^{\mathbb{N}}]. \quad (3.3.194)$$

Finally, substituting Eqs. (3.3.183) and (3.3.193) and the relation $d\bar{\underline{\sigma}} = \bar{\underline{D}}_{ep}d\bar{\underline{\epsilon}}$ (Eq. (3.3.176)) into Eq. (3.3.194), the visco-plastic-damage consistent tangent matrix can be expressed as

$$\underline{D}_{vpd} = \alpha_{\mu} \left(\underline{I} - \sum_{\mathbb{N}} \underline{W}_v^{\mathbb{N}} \right) \underline{D}_e + (1 - \zeta_v) \left[\underline{I} - \sum_{\mathbb{N}} (\underline{W}_v^{\mathbb{N}} + \underline{R}_v^{\mathbb{N}}) \right] \bar{\underline{D}}_{ep}, \quad (3.3.195)$$

where $\underline{W}_v^\pm = \omega^\pm \underline{S}_v^\pm$ and $\underline{R}_v^\pm = \Omega_r^\pm (\bar{\underline{\sigma}}^v \pm \otimes \underline{L}^\pm)$. It should be noted that vectors $\underline{L}^\pm$ are evaluated using inviscid variables ($\underline{\sigma}$).

3.3.5. FOC model

Plastic component

First, the differential of variables n_0 and n_1 of Eq. (3.2.132) are expressed as

$$dn_0 = 2\bar{\underline{\sigma}}^{\text{tr}T} d\bar{\underline{\sigma}}^{\text{tr}}, \quad dn_1 = \Delta \underline{\epsilon}^T d\bar{\underline{\sigma}}^{\text{tr}} + \bar{\underline{\sigma}}^{\text{tr}T} d\underline{\epsilon}. \quad (3.3.196)$$

Also, it is assumed that the variable ξ is constant during the plastic process. Thus, using Eq. (3.2.129), the differential of updated effective stress tensor is given by

$$d\bar{\underline{\sigma}} = d\bar{\underline{\sigma}}^{\text{tr}} - \frac{E_o \chi}{n_0} [n_0 \bar{\underline{\sigma}}^{\text{tr}} d\langle n_1 \rangle^+ + n_0 \langle n_1 \rangle^+ d\bar{\underline{\sigma}}^{\text{tr}} - \langle n_1 \rangle^+ \bar{\underline{\sigma}}^{\text{tr}} dn_0]. \quad (3.3.197)$$

Using Eq. (3.2.132) and Eq. (A.1.24), using a stepped Heaviside function, and with some straightforward manipulation, the effective component of consistent tangent stiffness is given by

$$\bar{\underline{D}}_{ep} = [c_1 \underline{I} + c_2 (\bar{\underline{\sigma}}^{\text{tr}} \otimes \Delta \underline{\epsilon}) + (\bar{\underline{\sigma}}^{\text{tr}} \otimes \bar{\underline{\sigma}}^{\text{tr}}) (c_3 \underline{I} + c_2 \underline{C}_e)] \underline{D}_e, \quad (3.3.198)$$

where $c_1 = m^{\text{tr}}$, $c_2 = -(1 - m^{\text{tr}})/n_1$ and $c_3 = 2(1 - m^{\text{tr}})/n_0$.

3.3.6. ROT model

Damage component

First, the differential of updated i -th positive/negative damage variable $\alpha_i^\pm$, stated in Eq. (3.2.134), can be expressed as

$$d\alpha_i^\pm = dz_i^\pm \Delta \hat{\epsilon}_i + w_i^\pm d\hat{\epsilon}_i = z_i^\pm d\hat{\epsilon}_i \quad (3.3.199)$$

Next, the tangent and secant slope of positive/negative uniaxial stress-strain law can be defined as $K_i^\pm := \frac{\partial \sigma^\pm}{\partial \alpha_i^\pm}$ and $S_i^\pm := \sigma_i^\pm / \alpha_i^\pm$, respectively. Then, using Eq. (3.2.135), the differential of variables $h_i^\pm$ and $g_i^\pm$ are written, respectively, as

$$dh_i^\pm = g_i^\pm K_i^\pm d\alpha_i^\pm + \sigma^\pm dg_i^\pm, \quad dg_i^\pm = \frac{1}{\alpha_i^\pm} (1 - g_i^\pm z_i^\pm) d\hat{\epsilon}_i. \quad (3.3.200)$$

Thus, using all these relations, the differential of i -th updated principal stress $\hat{\sigma}_i$, stated in Eq. (3.1.70), is written as

$$d\hat{\sigma}_i = \left(\sum_{\mathbb{N}} m_i^{\mathbb{N}} [K_i^{\mathbb{N}} p_i^{\mathbb{N}} + S_i^{\mathbb{N}} (1 - p_i^{\mathbb{N}})] \right) d\hat{\epsilon}_i = \underline{j}_{ii} d\hat{\epsilon}_i, \quad (3.3.201)$$

where $p_i^\pm = g_i^\pm z_i^\pm$. For the other hand, using Eq. (A.2.22), the differential of i -th principal stress can be expressed as

$$d\hat{\sigma}_i = \frac{\partial \hat{\sigma}_i}{\partial \underline{\hat{\epsilon}}} \frac{\partial \underline{\hat{\epsilon}}}{\partial \underline{\epsilon}} d\underline{\epsilon} = \underline{j}_i^T \underline{F}_\epsilon d\underline{\epsilon}, \quad \underline{j}_i = \left[\frac{\partial \hat{\sigma}_i}{\partial \hat{\epsilon}_1}, \frac{\partial \hat{\sigma}_i}{\partial \hat{\epsilon}_2} \right]^T. \quad (3.3.202)$$

Then, using this relation, the differential of updated stress vector $\underline{\sigma}$, given by Eq. (3.1.72), is written as

$$d\underline{\sigma} = \sum_{i=1}^N \left[(\underline{e}_{ii}^{\epsilon} \otimes \underline{j}_i) \underline{F}_\epsilon d\underline{\epsilon} + \hat{\sigma}_i d\underline{e}_{ii}^{\epsilon} \right]. \quad (3.3.203)$$

Then, using Eq. (A.2.23), it follows that damage consistent tangent matrix is written as

$$\underline{D}_d = \left(\sum_{i=1}^N (\underline{e}_{ii}^{\epsilon} \otimes \underline{j}_i) \right) \underline{F}_\epsilon + 2 \sum_{i=1, j>i}^N g_{ij}^{\epsilon} (\underline{e}_{ij}^{\epsilon} \otimes \underline{e}_{ij}^{\epsilon}), \quad (3.3.204)$$

where g_{ij}^{ϵ} is defined as

$$g_{ij}^{\epsilon} := \begin{cases} \frac{(\hat{\sigma}_i - \hat{\sigma}_j)}{(\hat{\epsilon}_i - \hat{\epsilon}_j)}, & \hat{\epsilon}_i \neq \hat{\epsilon}_j \\ \frac{\partial \hat{\sigma}_i}{\partial \hat{\epsilon}_i}, & \hat{\epsilon}_i = \hat{\epsilon}_j. \end{cases}$$

Note that the first term of right hand side is associated to local principal stiffness and the second term arises from rotation of principal strains. It can be demonstrated that this

expression, neglecting the damage variables, is identical to obtained by (M. A. Crisfield & Wills, 1989). Also, note the similitude of the second term of this expression with Eq. (3.3.186).

Viscous component

First, using Eqs. (3.2.136) and (3.3.199), the differential of updated i -th positive/negative viscous damage variable can be expressed as

$$d\alpha_i^{v\pm} = (1 - \zeta_v)w_i^{\pm}d\hat{\varepsilon}_i. \quad (3.3.205)$$

Moreover, the differential of variables $h_i^{v\pm}$ and $g_i^{v\pm}$ (Eqs. (3.1.74) and (3.1.75)) are given, respectively, by

$$dh_i^{v\pm} = g_i^{v\pm}K_i^{v\pm}d\alpha_i^{v\pm} + \sigma^{\pm}(\alpha_i^{v\pm})dg_i^{v\pm}, \quad (3.3.206)$$

$$dg_i^{v\pm} = \frac{1}{\alpha_i^{v\pm}} [1 - (1 - \zeta_v)g_i^{v\pm}z_i^{v\pm}] d\hat{\varepsilon}_i, \quad (3.3.207)$$

where $K_i^{v\pm} := \frac{\partial \sigma^{\pm}}{\partial \alpha_i^{v\pm}}$. Thus, using these relations and with some straightforward manipulation, the differential of updated i -th principal viscous stress, stated in Eq. (3.1.74), can be written as

$$\begin{aligned} d\hat{\sigma}_i^v &= \left\{ \zeta_v \sum_{\aleph} m_i^{\aleph} S_i^{v\aleph} + (1 - \zeta_v) \sum_{\aleph} m_i^{\aleph} [K_i^{v\aleph} p_i^{v\aleph} + S_i^{v\aleph} (1 - p_i^{v\aleph})] \right\} d\hat{\varepsilon}_i \\ &= J_{ii}^v d\hat{\varepsilon}_i, \end{aligned} \quad (3.3.208)$$

where $p_i^{v\pm} = g_i^{v\pm}z_i^{\pm}$ and $S_i^{v\pm} = \sigma^{\pm}/\alpha_i^{v\pm}$. Finally, substituting this relation and Eq. (A.2.23) into the differential of the viscous stress vector, given by Eq. (3.1.76), the viscous-damage consistent tangent tensor can be expressed as

$$\underline{\underline{D}}_{vd} = \left(\sum_{i=1}^N (\underline{\underline{e}}_{ii}^{\varepsilon} \otimes \underline{\underline{j}}_i^v) \right) \underline{\underline{F}}_{\varepsilon} + 2 \sum_{i=1, j>i}^N g_{ij}^v (\underline{\underline{e}}_{\varepsilon}^{ij} \otimes \underline{\underline{e}}_{\varepsilon}^{ij}), \quad (3.3.209)$$

where $\underline{j}_i^v = \left[\frac{\partial \hat{\sigma}_i^v}{\partial \hat{\varepsilon}_1}, \frac{\partial \hat{\sigma}_i^v}{\partial \hat{\varepsilon}_2} \right]^T$ and g_{ij}^v is defined as

$$g_{ij}^v := \begin{cases} \frac{(\hat{\sigma}_i^v - \hat{\sigma}_j^v)}{(\hat{\varepsilon}_i - \hat{\varepsilon}_j)}, & \hat{\varepsilon}_i \neq \hat{\varepsilon}_j \\ \frac{\partial \hat{\sigma}_i^v}{\partial \hat{\varepsilon}_i}, & \hat{\varepsilon}_i = \hat{\varepsilon}_j. \end{cases}$$

3.4. Validation examples

In this section, a set of numerical examples are used to validate the capabilities of the constitutive concrete models described in Section 3.1. Taking the numerical algorithms presented in Sections 3.2 and 3.3, the five concrete models were implemented in the software ([ANSYS, 2018](#)) through user-material FORTRAN77 routines (USERMAT.f). These material routines work at Gauss integration point level of each finite element.

Four class of experimental benchmark test are simulated with a single-element according to loading conditions: (i) uniaxial cyclic tension and compression; (ii) biaxial monotonic; (iii) uniaxial cyclic tension-compression; and (iv) strain-rate effect and numerical viscosity. Also, the strain-localization and fracture-energy FE-regularization are discussed with a fictitious example.

All examples were modeled using 4-node isoparametric shell element (SHELL181) with six Degree Of Freedom (DOF) at each node using 2x2 Gauss integration scheme. All models, except the DPH model, assume an exponential relation for the positive/negative uniaxial stress laws given by Eqs. (3.1.59) and (3.1.60), respectively. An adequate conversion among uniaxial laws required for each concrete model is generated, as explained in Table 2.5.4. Table 3.4.1 list the material parameters adopted for each benchmark test. It should be noted that identical parameters are used as in the 3D-case.

Table 3.4.1. List of parameters used in the concrete models.

Author	Test	B mm	H mm	E_o GPa	ν -	f'_t MPa	f'_c MPa	$G_f^{+\dagger}$ N/mm	$G_f^{-\dagger}$ N/mm	K_c -
Gopalaratnam & Shah, 1985	uniaxial tension	82.6	82.6	31.0	0.18	3.48	27.6	0.04	11.38	1.0
Karsan & Jirsa, 1969	uniaxial compression	82.6	82.6	31.7	0.2	3.48	27.6	0.04	11.38	1.0
Kupfer et al., 1969	biaxial	200	50	31.0	0.15	3.5	32.06	2.0	80.0	1.0
Mazars et al., 1990	unilateral effect	80	80	16.4	0.2	1.4	18.1	0.011	7.0	1.0
Suaris & Shah, 1985	strain-rate effect	100	100	34.0	0.22	5.37	46.8	0.5	20.0	1.0
-	strain-localization	100	600	32.0	0.0	5.0	39.0	4.0	40.0	1.0

$\dagger$ values used in the WLF₀ as reference. For all cases: $f'_b = 1.16f'_c$, $\epsilon=0.001$, $z_c^+=0$, $z_c^-=1$, $\bar{\sigma}_0^+ = f'_t$ and $\mu_v = 0$, unless otherwise indicated.

3.4.1. Uniaxial cyclic tests

Numerical concrete models are compared with uniaxial cyclic tension and compression loading-unloading and reloading experimental data reported by (Gopalaratnam & Shah, 1985) and by (Karsan & Jirsa, 1969), respectively. Figs. 3.4.1 and 3.4.2 shown the response of the five concrete models under tensile and compressive loads, respectively. FE

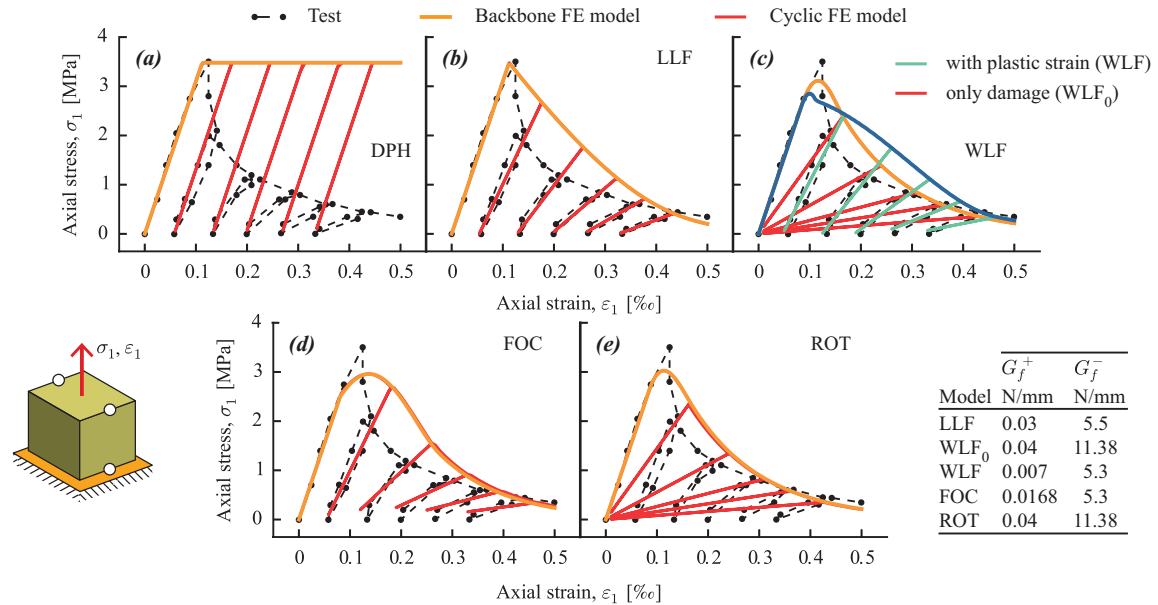


Figure 3.4.1. Validation of concrete models under uniaxial cyclic tension test of (Gopalaratnam & Shah, 1985): (a) DPH model; (b) LLF model; (c) WLF₀ and WLF models; (d) FOC model; and (e) ROT model. The following additional parameters are used. For the DPH model: $f_y^+=3.48$ MPa, $f_y^-=12$ MPa, $a_0 = 3c_u/E_o$, $R=1$; LLF model: $C^+=6500$, $C^-=7500$; and WLF model: $f_o^-=20$ MPa, $E_t^+=0.16E_o$, $E_t^-=0.48E_o$.

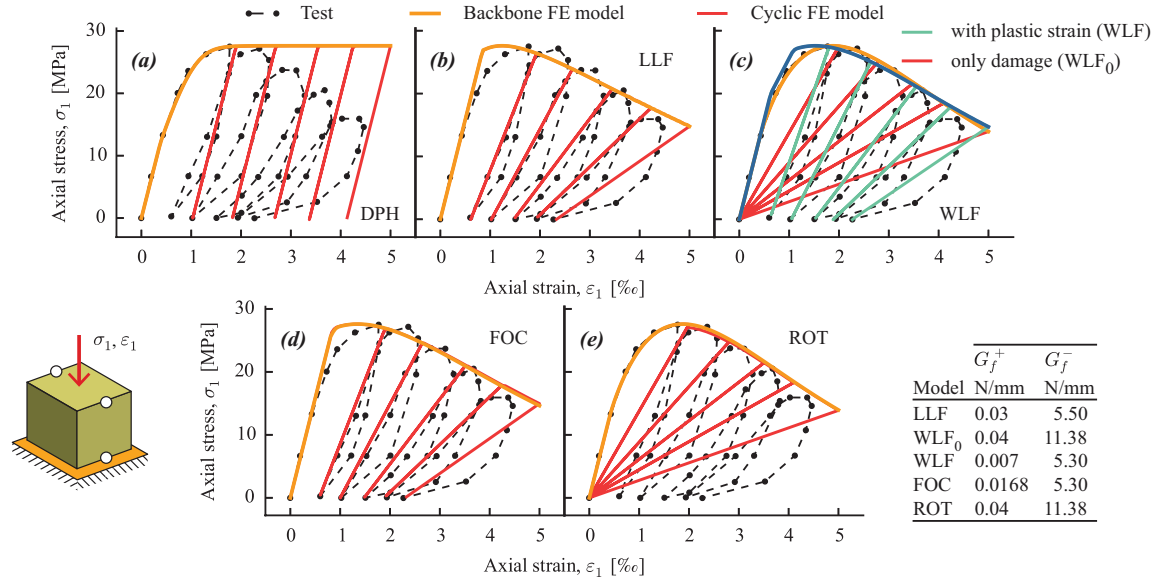


Figure 3.4.2. Validation of concrete models under uniaxial cyclic compression test of (Karsan & Jirsa, 1969): (a) DPH model; (b) LLF model; (c) WLF model; (d) FOC model; and (e) ROT model. The following additional parameters are used. For the DPH model: $f_y^+ = 3.48$ MPa, $f_y^- = 12$ MPa, $a_0 = 3c_u/E_o$, $R=1$; LLF model: $C^+ = 6500$, $C^- = 7500$; and WLF model: $f_o^- = 20$ MPa, $E_t^+ = 0.16E_o$, $E_t^- = 0.48E_o$.

models are elaborated with a single-element cube of 82.6 mm. Its assumed a characteristic length of $l_c = 82.6$ mm and a pure uniaxial stress state for the boundary constraints.

In general, it can be observed that in all models, except for the DPH model, fits well with the post-peak backbone response of experimental tests, where the WLF₀ and ROT models give the best approximation. Although, both models fail in the unloading branch, due that neglects the plastic strains (pure damage only). Also note that these models have identical responses, although they are elaborated with formulations completely different.

In contrast, the unloading branch of the LLF, WLF and FOC models fits close to experimental response due that incorporate the plastic and damage components in their formulations. In plastic-damage models, it is required to adjust the parameters to fit simultaneously the backbone curve and the unloading branch. Thus, the first half of residual backbone is mainly influenced by the parameters of the plastic component and the last half by the fracture energy $G_f^\pm$. Moreover, the parameters $C^\pm$, $E_t^\pm$ and $B^\pm$ for the LLF, WLF and FOC models, respectively, controls the backbone and slope of unloading branch in a coupled

manner, i.e. when their values are reduced cause an increase in the slope of unloading branch and consequently reduce the backbone response.

The follows additional observations are considered. For the tensile regime, in all models, except the LLF model, the peak stress do not fit exactly with the experimental value due to the incorporation of smoothed polynomial function in the uniaxial laws (Eq. (2.2.74)). For other hand, the value of fracture energy $G_f^\pm$ used in the plastic-damage models to fit the experimental tests is less than in the damage models. This is due to that the plastic component induce an additional dissipation of energy that is not taken account in the FE-regularization (Section 2.5).

For the FOC model, it has observed the influence of strain increment size $\Delta\varepsilon$ in the response, where an gradual over-stress response is caused with a relative large strain increments. In the same way, its observed a difference between cyclic and backbone responses, gradually incremented over the last unloading/loading cycles, but that disappear with a relative small $\Delta\varepsilon$. Both conditions are due to explicit integration scheme used in the numerical algorithm to calculate the plastic strain tensor. In addition, it can observed the influence of parameter B^- in the tensile response.

3.4.2. Biaxial monotonic tests

All the concrete models, except the ROT model, are compared with biaxial monotonic test of (Kupfer et al., 1969). This test is performed with a constant biaxial loading ratio of $a = \sigma_1/\sigma_2$, where σ_1 and σ_2 are the stresses imposed. FE models are elaborated with a single-element of 200×50 mm of base and 200 mm of height. Its assumed a characteristic length of $l_c = 200$ mm and a pure biaxial stress state for the boundary constraints, as observed in the experimental test. A stress-controlled test are performed up to reach the peak stress, with the exception of the uniaxial case ($a = 0$) simulated with displacement-controlled. The inputs parameters are chosen by means to fit the cases $a=0$, 1 and 0.52 simultaneously.

Fig. 3.4.3 shown the axial stress σ_1 vs axial ε_1 and the lateral strains ε_2 and ε_3 , respectively, of the WLF model using a loading ratio of $a=0, 1$ and 0.52 . This model use a DEER given by Eq. (3.1.55) to include the biaxial strength. It can observed a good fit with the pre-peak stress response of experimental test and a relatively good adjust exist in the lateral strains, especially when $a=0.52$. In general, the same observations are concluded in all models. Fig. 3.4.4 shown the biaxial peak strength surface for the DPH,

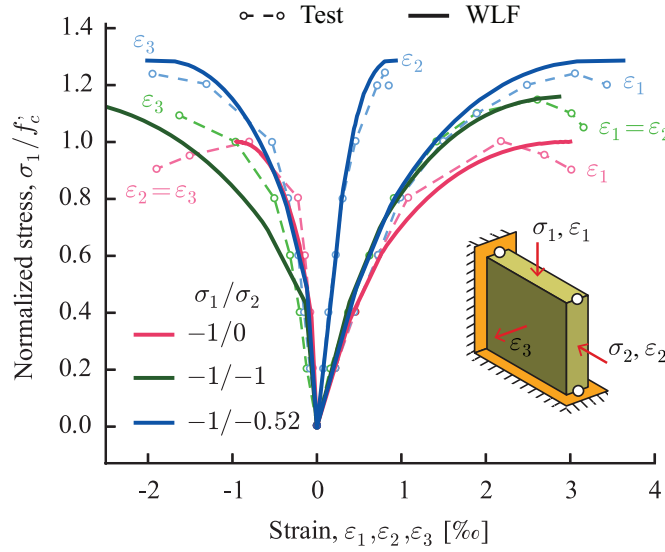


Figure 3.4.3. Validation for the WLF model under biaxial test of (Kupfer et al., 1969). The following additional parameters are used: $G_f^+=0.5$ N/mm, $G_f^-=35$ N/mm, $\bar{\sigma}_0^-=12$ MPa, $E_t^+=0.3E_o$ and $E_t^-=0.65E_o$.

WLF₀, WLF, FOC and LLF models under different combination of biaxial loading ratios $a = \sigma_1/\sigma_2$. For the DPH model, the parameters η and ξ are fitted with tension/compression biaxial strength of concrete. Also, for the WLF₀ and WLF models, the DEER given by Eq. (3.1.55) is used to include the biaxial strength. In addition, the response of the WLF₀ model using the Eq. (3.1.54) is compared.

It can observed, that all models fits close with the experimental results, specially in compression-compression (C-C) regime, where are influenced by the Drucker-Prager yield criterion. The major differences among models occur in the tension-compression (T-C) regimes. The exception occur with the DPH and WLF₀1 model. For the the first, fit well

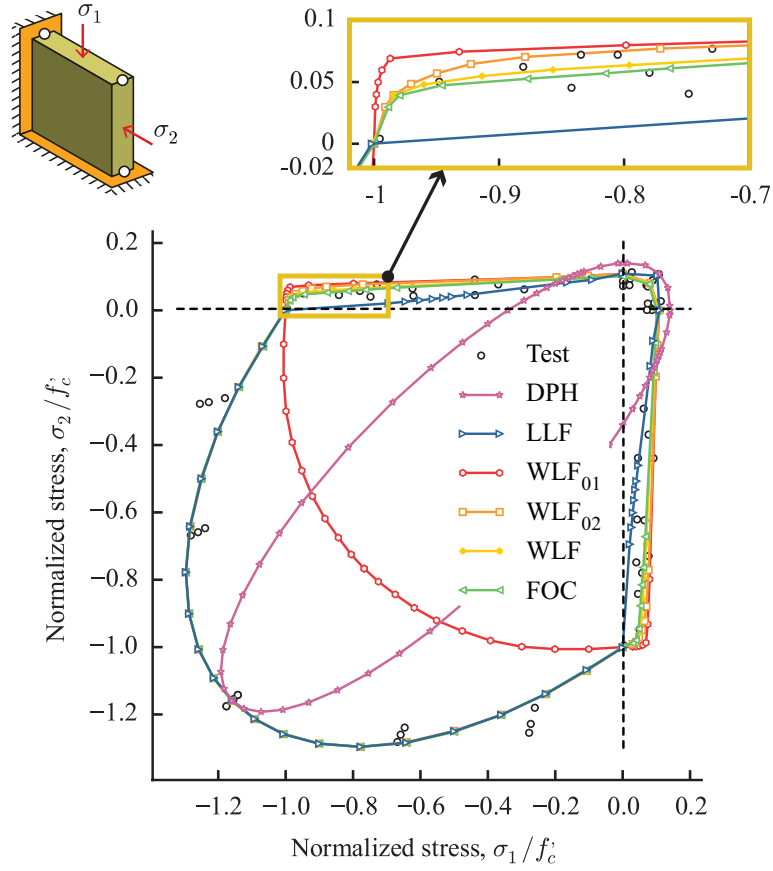


Figure 3.4.4. Biaxial peak strength surface for the DPH, WLF₀, WLF, FOC and LLF models and the biaxial test results of (Kupfer et al., 1969). For the DPH model the following parameters are used $f_y^+ = 3.5$ MPa and $f_y^- = f'_b$.

only in the equal biaxial loading ratio $a = 1$ and the second one fit well in the T-C regime, but with a reduced strength in the C-C regime. Both observations are obtained such as expect in the literature (de Souza Neto et al., 2008; Mazars, 1984; J. Simo & Ju, 1987).

Additionally, it can be observed, similar to uniaxial case, a less value of fracture energy is required in the plastic-damage models than damage models to fit with experimental results. Conversely to the uniaxial case, under certain conditions, an increment in the value of compression fracture energy cause a reduction in the backbone response.

3.4.3. Uniaxial cyclic tension-compression test

To validate the unilateral effect, the LLF, WLF₀, WLF, FOC and ROT models are compared with the uniaxial cyclic test of (Mazars et al., 1990). This test was first subjected to uniaxial tension followed by uniaxial compression in parallel directions. FE models are elaborated with a single-element cube of 80 mm of width. Its assumed a characteristic length of $l_c=80$ mm and a pure uniaxial stress state for the boundary constraints.

Fig. 3.4.5 shown the axial stress σ_1 vs axial strain ε_1 of this models. It also included the response of the LLF model with three values of stiffness recovery factor z_c^- (0, 0.5 and 1). It can noticed that all models recovery the initial elastic stiffness once the load goes into the compression state (step 2 and 4). The exception occur, obviously, in the LLF model when $z_c^-=0.5$ and 0, due that this parameter controls the value of recovery compression stiffness. Moreover, its observed that all models, with the exception of the WLF₀ and ROT models, take the compression backbone branch close to experimental data (step 4), due that include plastic strain in their formulations. In addition, its observed that the LLF, WLF and FOC models recovery the damaged stiffness obtained in the last cycle of tension (step 3) when the load goes from compression to tension state (step 6). This condition is also shared by the WLF₀ and ROT models (not shown in the plot) and is so-called that the models have "damage memory", which is agree with the thermodynamic of irreversible process.

3.4.4. Strain-rate tests

Experimentally, the strain-rate effect is important under impulsive loading (impacts or explosions), but already important under earthquake loading, with rates of straining $\dot{\varepsilon}$ ranges between $10^{-6}/s$ to $10^{-1}/s$. Then, due that the all models, except the DPH model, can simulate the rate-dependent behavior through of incorporation of a visco-elastic/visco-plastic model, they are compared with the strain-rate test of (Suaris & Shah, 1985). FE models are elaborated with a single-element cube of 100 mm of width. Its assumed a characteristic length of $l_c=100$ mm and a pure uniaxial stress state for the boundary constraints.

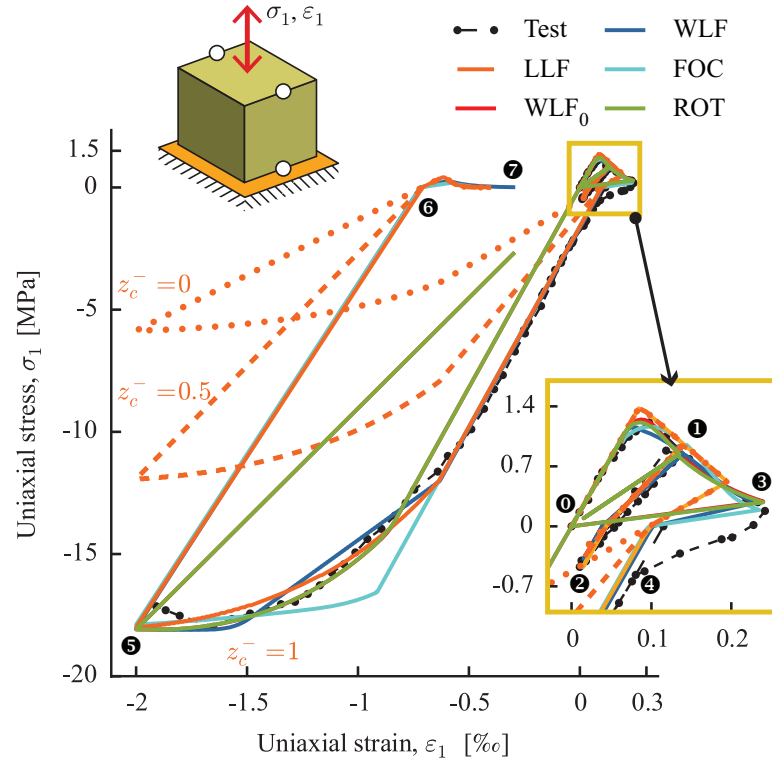


Figure 3.4.5. Validation of the LLF, WLF₀, WLF, FOC and ROT models under uniaxial cyclic tension-compression test of (Mazars et al., 1990). The following additional parameters are used. For the LLF model: $C^+=12000$, $C^-=200$; WLF model: $f_o^-=12$ MPa, $E_t^+ = 0.3E_o$ and $E_t^- = 0.4E_o$; and FOC model: $B^+=0.54$ and $B^-=0.75$.

Two uniaxial tests are performed, one for tension and other for compression, both with a range of straining rates $\dot{\epsilon}$ between 10^{-6} /s to 1/s. The material parameters are fitted with the tests loaded with a strain-rate of $\dot{\epsilon}=10^{-6}$ /s (pseudo-static). For the sake of simplicity, a numerical viscosity $\mu_v = 2 \times 10^{-3}$ s is used in all cases. Also, a constant number of steps $N_s=150$ and a maximum displacement of $\delta_{max}=0.25$ mm for tension and -0.55 mm for compression are used, for which the time increment used is given by $\Delta t = \frac{|\delta_{max}|}{N_s \dot{\epsilon}}$.

Fig. 3.4.6a-b shown the normalized uniaxial tension/compression viscous stress $\sigma_1^v / \sigma_{1max}^0$ vs uniaxial strain ϵ_1 , respectively, for the WLF₀ model, where σ_{1max}^0 denotes the peak inviscid stress (f_t' and f_c' , respectively). In both plots, for high straining rates, an increment of up to 3.4 and 1.1 times respect to the inviscid case ($\dot{\epsilon}=10^{-6}$ /s) is observed for tension and compression, respectively. Moreover, its denoted a over-estimation of 59.4% in the

tensile peak stress respect to experimental test, whereas a lower-estimation of 12.4% exist for the compression peak stress. Similar observations are derived using the other models.

Fig. 3.4.6c shown the peak stress ratio $\sigma_{1max}^v / \sigma_{1max}^0$ or Dynamic Increase Factor (DIF) vs the applied strain-rate $\dot{\epsilon}$ for all models, where σ_{1max}^0 denotes the peak stress at inviscid response. As can observed, peaks strengths grow continuously as straining rates are increased, becoming clearly distinguishable from the inviscid response upon a strain-rate value of $10^{-2}/s$. Also noted, that the tensile response is largest than the compressive one in overall range of straining rates analyzed, growing up to 6 times respect to the inviscid response. In addition, the FE results shown that the DIF is underestimated as compared to the both experiments for the small strain-rates $\dot{\epsilon} < 10^{-1}/s$ and overestimated for the large strain rates $\dot{\epsilon} \geq 2.5 \times 10^{-1}/s$. To get a best estimation with respect to the experimental tests, its required modify the visco-plastic model used, e.g the modified Perzyna model proposed by (Faria & Oliver, 1993; Faria et al., 1998).

3.4.5. Effect of the numerical viscosity

In order to investigate the effects of numerical viscosity in the response, a numerical test are generated varying the numerical viscosity-time increment ratio $\mu_v / \Delta t$ for the WLF model. This adimensional parameter is related to the variable $\zeta_v = (1 + \Delta t / \mu_v)^{-1}$ (Eq. (3.2.122)) required for the stress updated algorithms of models. For the sake of simplicity, the material parameters used are the same than in the strain-rate effect simulation. Uniaxial tensile load is applied in a single-element varying the relation $\mu_v / \Delta t$ in a range between 10^{-6} (inviscid) to 50.

Fig. 3.4.7a shown the uniaxial viscous stress-strain $\sigma_1^v - \epsilon_1$ response with different values of $\mu_v / \Delta t$. Similar to Fig. 3.4.6c its observed an gradual over-stress response proportional to the increased value of the numerical viscosity. Moreover, Fig. 3.4.7b shown the respective axial stiffness-strain $\frac{\partial \sigma_1^v}{\partial \epsilon_1} - \epsilon_1$ response for one integration point of the FE model. Similar to the stress response, a gradual increment of axial stiffness is presented as increasing the value of $\mu_v / \Delta t$, up to get a positive value although a strain-softening

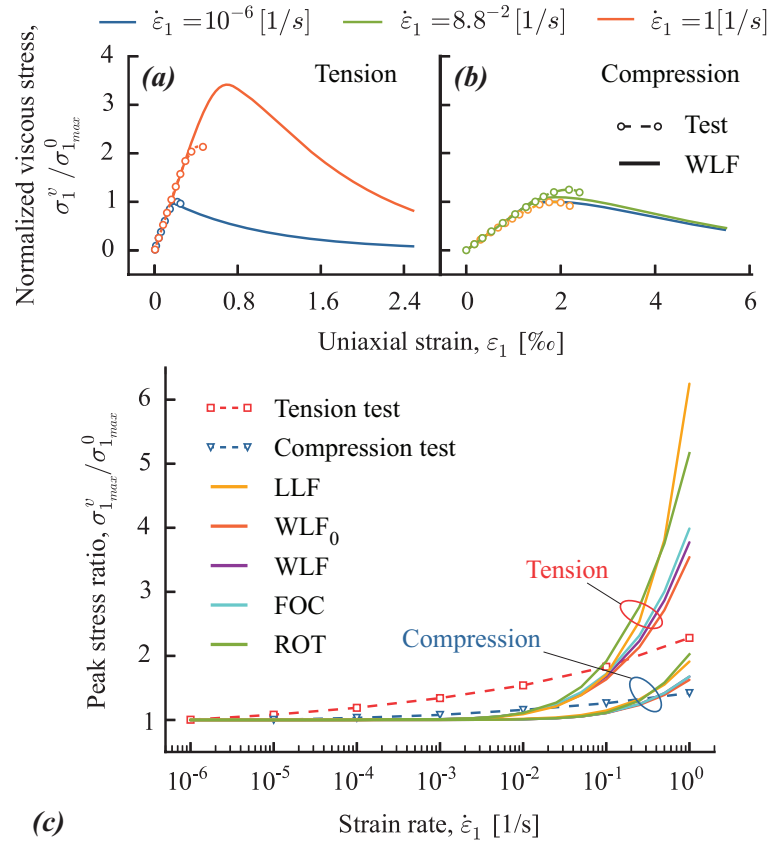


Figure 3.4.6. Validation of strain-rate effect in the concrete models under monotonic uniaxial tests of (Suaris & Shah, 1985): (a-b) normalized uniaxial tensile/compressive stress $\sigma_1^v / \sigma_{1,max}^0$ vs uniaxial strain ε_1 for the WLF₀ model, respectively; and (c) peak stress ratio $\sigma_{1,max}^v / \sigma_{1,max}^0$ or Dynamic Increase Factor (DIF) vs the applied strain-rate $\dot{\varepsilon}$ for the LLF, WLF₀, WLF, FOC and ROT model under tensile and compressive loads.

regimes exists. This key advantage can convert into a positive-definite the consistent tangent stiffness tensor and is demonstrated that expand the range of convergence of the models in strain-softening regimes.

3.4.6. Strain-localization and FE-regularization

Strain-localization phenomena is present in local models with strain-softening behavior. Imperfection of material properties, irregularities in the geometry and non-symmetrical boundary/load conditions can induce the formation of this phenomena. The fracture energy FE-regularization is an popular technique that introduce a length scale in the constitutive equations and that is able to remove the spurious mesh-dependency observed when

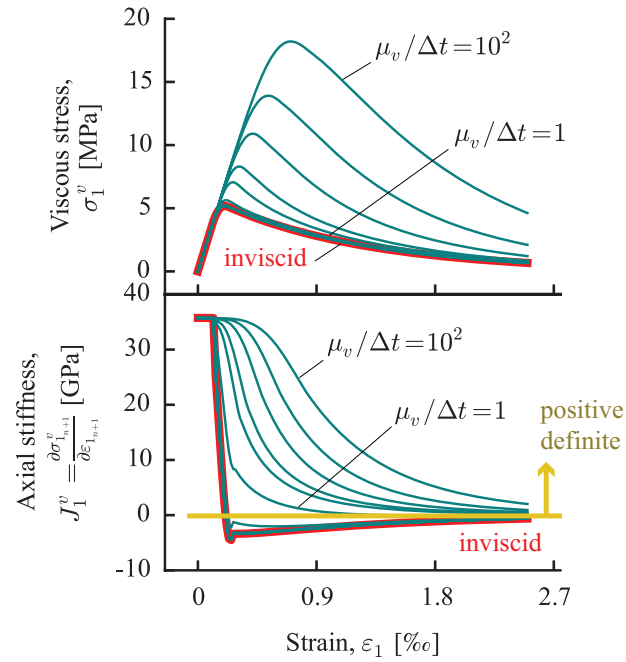


Figure 3.4.7. Variation of uniaxial response using different values of the ratio numerical viscosity/time increment $\mu_v/\Delta t$ for the WLF model: (a) uniaxial viscous stress-strain $\sigma_1^v - \varepsilon_1$ relation and (b) axial stiffness-strain $\frac{\partial \sigma_1^v}{\partial \varepsilon_1} - \varepsilon_1$ relation for one integration point of the FE model.

strain-localization exists. It should be noted that, ignoring the FE-regularization, local models with strain-softening behavior can correctly describe the damage only when remain uniformly distributed (perfect material). In order to study this phenomena in the

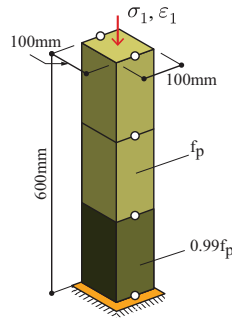


Figure 3.4.8. Description of FE model used in strain-localization test.

concrete models developed, two uniaxial tests are performed, one for tension and other for compression, varying the number of finite elements (i.e. varying their characteristic length

l_c). For the sake of simplicity, a prism of 100×100 mm of base and 600 mm of height is divided into 2, 3 and 4 elements. Also, its assumed a pure uniaxial stress state for the boundary constraints (Fig. 3.4.8a). Table 3.4.1 list the material parameter used. The election of parameters E_o , f'_c and $G_f^\pm$ are chosen in order to satisfy the range of characteristic length l_c admissible by the uniaxial compression stress law given by Eq. (3.2.79). In order to induce the localization phenomena, one of elements (shaded element) has been reduced slightly their uniaxial tension/compression strength (f'_t/f'_c) than others elements (0.99 times), for tensile/compressive load case, respectively. In addition, due that some convergence trouble are observed in the simulations, a numerical viscosity of $\mu_v/\Delta t = 0.05$ is incorporated in all models.

Fig. 3.4.9a-b shown the normalized uniaxial tensile stress $\sigma_1/\sigma_{1_{max}}$ vs post-peak displacement $\delta_{1_{pp}}$ for the WLF₀ and WLF model, respectively, varying the mesh size of model, whereas Fig. 3.4.9c-d shown the respective compressive response for the WLF₀ and LLF, respectively. Additionally, the figure shown the failure mode of their respective specimens, through the field of damage variable $\omega^\pm$. The post-peak displacement is defined as $\delta_{1_{pp}} := \delta - \delta_o$, where δ is the total displacement of specimen and δ_o the displacement at peak response.

Its observed in all models with imperfection a mesh-objectivity response and the damage zone occur only in the modified element, such as expected in literature. However, in the case without imperfection, two kinds of response are observed. For one hand, the response for the WLF₀ model is mesh-dependent with an uniform strain field, either in tension as in compression. This condition is due that the FE-regularization modify the uniaxial stress-strain law despite exist an uniform strain field in the model. Then, its concluded that this technique is only necessary when the damage zone localize. For other hand, the LLF and WLF models (with the exception of one case $l_c=300$ mm) gives a mesh-objectivity response. This atypical condition can be attributed first to the non-symmetric consistent tangent stiffness tensor and largely to numerical errors induced in the iterative process to calculate the plastic component.

Similar observations can be concluded in the other cases as explained as follows. All models gives a mesh-objectivity response and the damage zone is localized in one element (modified element) when a perturbation exists in the material. In contrast, not all the models have an uniform strain field in the case without imperfections. Its observed that the WLF and FOC models localize with a tensile load, whereas the LLF model localize both in the tension as in the compression case. In contrast, the WLF₀ and ROT models not localize using a perfect material.

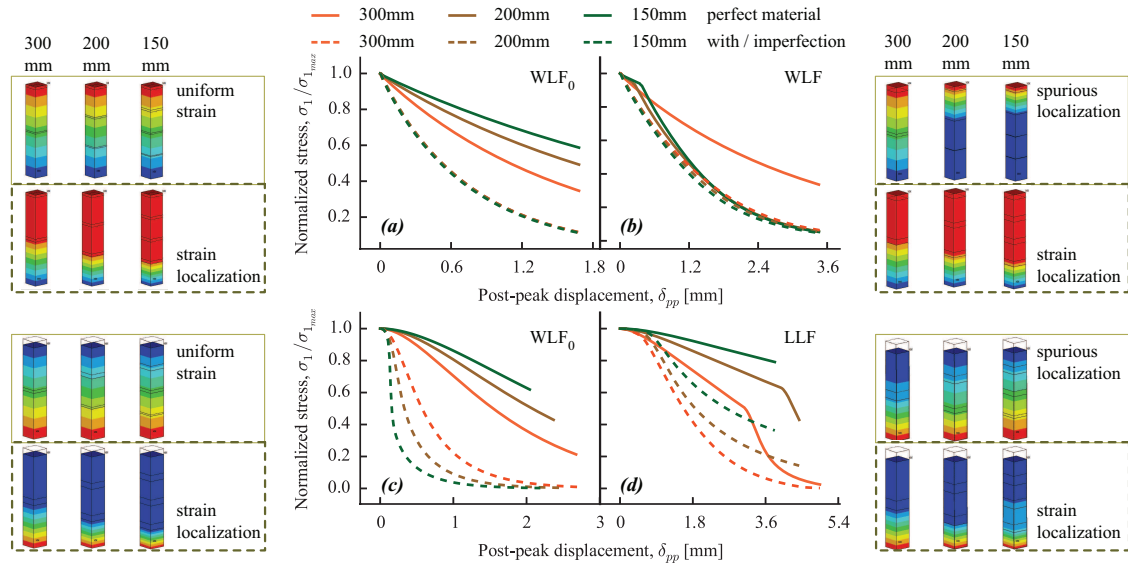


Figure 3.4.9. Comparison the normalized uniaxial stress $\sigma_1 / \sigma_{1_{max}}$ vs post-peak displacement $\delta_{1_{pp}}$ using three FE mesh sizes: 150 mm, 200 mm and 300 mm: (a-b) tensile response for the WLF₀ and WLF models, respectively; and (c-d) compressive response for the WLF₀ and LLF models, respectively. The following additional parameter are used. For the LLF model: $C^+=6000$, $C^-=500$; and WLF model: $f_o^-=20$ MPa, $E_t^+ = 0.5E_o$ and $E_t^- = 0.5E_o$.

In conclusion, identical results as in the 3D-case are obtained in all test simulated, with the exception for the mesh regularization test, where some differences in the responses of the LLF and WLF models are observed. It should be noted that completely different formulations are used for the plastic component of the DPH, LLF and WLF model respect to the 3D-case, for which it can concluded that the numerical implementation of these concrete models are correctly applied in the plane-stress case.

3.5. Summary and main results

This chapter study the epistemic uncertainty of five plane-stress continuum stress-strain local constitutive concrete models. As for the 3D-case, convergence problems are encountered in certain cases, especially in strain-softening regimes. Herein, a complete description of these models in a common vectorized and notation was presented, providing all the necessary steps required to ensure adequate convergence and a consistent numerical implementation. Analytical expressions for the updated stress algorithms and new explicit expressions for the algorithmic consistent tangent stiffness matrices were developed. Moreover, similar to 3D-case, numerical benchmark test examples are evaluated for each model. The main results obtained from these part are:

- The construction of a strong updated stress algorithm were necessary to get an adequate response of the models developed. Implicit schemes with the projected return-mapping algorithm were considered for the plastic component of models, whereas explicit schemes were used for the damage ones. Furthermore, the consistency plastic operator of the DPH, LLF and WLF models was solved with the iterative Newton's method, where a choice of adequate initial value and solution of an unique scalar variable rather than a system of equations were mandatory to ensure the convergence of this component. Moreover, continuous and smooth functions (C^1 -class) for the elaboration of the consistent tangent stiffness matrix were used to achieve the convergence of these models. Smooth yield criterion, flow potential and uniaxial laws are taken in the operator of these models. In addition, the incorporation of the Duvant-Lions viscous model in the constitutive equations of the LLF, WLF, FOC and ROT models was proved as a simple and robust technique to overcome convergence problems caused by strain-localizations in these models.
- All models, with the exception of the DPH model, simulates the strain-softening behavior, where the LLF, WLF and FOC models predicts the inelastic strains and stiffness degradation, whereas the WLF model without plastic strains (WLF_0)

and the ROT model both unload to the origin (pure damage models). Also, all models, except the ROT model, incorporate the biaxial effect adequately since they include the Drucker-Prager yield criterion in their equations. Moreover, with the exception of the DPH model, all models simulate the unilateral and strain-rate effects correctly. It is observed that the tensile response is more sensitive to strain-rate increments than the compression response for all models. However, a poor fit is obtained relative to experimental test.

- Moreover, all models give a mesh-objective response with a localized damaged zone if a perturbation exists in the material of a FE model. Conversely, not all of the models have a uniform strain field if there are no imperfections, where the WLF and FOC models localize under tensile loads, whereas the LLF model localizes both in tension and compression. In contrast, the WLF_0 and ROT models do not localize if we have a perfect material.

4. CONCLUSIONS

This research evaluates the epistemic uncertainty associated with computational modeling assumptions of reinforced concrete (RC) structures using different scales ranging from complete structures to the type of finite elements used. Because of their complexity, linear models were used to estimate the dynamic response of six RC free-plan buildings modelled in different softwares. For the inelastic models, finite element stress-strain constitutive concrete models were considered. This structure corresponds to the different chapters of this work: **(i)** uncertainty of linear building models; **(ii)** uncertainty of stress-strain constitutive concrete models using a three-dimensional (3D) formulation; and **(iii)** uncertainty in stress-strain concrete models using a plane-stress assumption. The main conclusions obtained in this thesis are:

- Building models with solid elements (AW) provide the best approximations to experimentally measured periods of RC free-plan buildings, with errors smaller than 13% for the first four periods. Models with beam and shell elements (ET and AP), as well as for the AW models, lead to a peak error of 17% for the predicted first two periods. In spite of this, the standard deviation of the errors to the different response parameter ratios obtained for the three models (ET, AP and AW) was smaller than 11%.
- The stiffness of the diaphragm is an important source of epistemic uncertainty. Indeed, the first four periods may reach values up to 10% and down to 27%, relative to models that consider either in-plane and out-of-plane stiffnesses. Moreover, the variation of the in-plane stiffness of the diaphragm generates large uncertainty in the shear forces of the wall mainly in the first basements due to the so-called *back-stay* effect. Normalized shear forces for the core vary between 0.57 and 4.29 times in the first basement, with a standard deviation of 112%. In contrast, the bending stiffness of the diaphragm affects story shear forces in higher stories more than variations of the in-plane stiffness of the diaphragm.

The normalized story and core shears at mid-height of the buildings ($H/2$) varied between 0.75 and 1.44.

- The soil-structure interaction model considered generates larger uncertainties in the story shears, leading to variations of the normalized story shear of the last basement between 0.03 and 4.79 times. Also, the influence of the level of building fixity leads to changes in the first building period from +10% to +18% relative to a model without basements. In all cases, peak responses occurred when the fixity was imposed at an intermediate basement level. This condition supports the observation that selecting an arbitrary basement level to impose the code minimum-design base shear is an incorrect practice and can generate incorrect designs. In conditions of uncertainty of the correct level of fixity, the designer should use the envelope of the designs generated by the different level of fixity.
- For all studied modelling assumptions, larger uncertainties were identified for forces at the shear wall core (shear and overturning moment) than for the story forces. Additionally, larger uncertainties were identified for story and core shears at the basements (B1 and BF) than for the upper levels ($H/2$ and L1).
- Due to the several sources of uncertainty, it is recommended to use the following assumptions in modeling RC free-plan buildings: (i) adopt a FE model with shell elements for walls and unidimensional frame elements for beams and columns rather than solids elements; (ii) consider the in-plane stiffness of the diaphragms at the basements to reduce the *back-stay* effect in FE models; (iii) develop two models, one that includes and other that neglects the contribution of lateral soil stiffness, and compute the envelope of story basement forces; (iv) generate at least two models with different levels of fixity at the basement level, and evaluate the envelope of story shears and element forces as well as the code minimum design shear for each case.

- Chapters 2 and 3 present a consistent notation description of five stress-strain concrete models together with all ingredients necessary for a correct numerical implementation of the models. This implies analytical expressions for the updated stress algorithms, explicit expressions for the consistent tangent stiffness tensor, consistency checks of input material parameters between models, and an adequate conversion from tensors and tensor operations to a vector format. Furthermore, numerical examples of benchmark tests were developed under uniaxial, biaxial, and triaxial stresses. The unilateral and strain-rate effects, the mesh size influence, and the strain-localization phenomena were analyzed for all models.
- A robust updated stress algorithm was developed to ensure an adequate convergence for all concrete models. Tensor notation was considered for the 3D-case, whereas vector notation for the in-plane components of tensors were used for the plane-stress formulation. Implicit schemes with a return-mapping algorithm were considered for the plastic component of models, while explicit integration schemes were used for the damage ones. Furthermore, continuous and smooth functions (C^1 -class) for the elaboration of the consistent tangent stiffness tensor were used to improve model convergence. Smooth functions for the yield criterion, flow potential, and uniaxial laws were also considered to calculate the stiffness operator of all models. In addition, the use of the Duvaut-Lions viscous model in the constitutive equations of the LLF, WLF, FOC and ROT models was tested as a simple and robust technique to overcome convergence problems caused by strain-localization phenomenon. It is highly recommended to use a ratio of numerical viscosity/load step increment between 0.001 and 1.0 to obtain adequate convergence without overshooting the stress response.
- Several benchmark tests were simulated to describe the main capabilities of the set of concrete models. It was observed that all models, with the exception of the DPH model, were capable of simulating the strain-softening behavior. The

LLF, WLF and FOC models predict the inelastic strains and stiffness degradation, whereas the WLF model without plastic strains (WLF_0) and the ROT model, both, unload to the origin of the stress-strain relationship (pure damage models). Moreover, all models, with the exception of the ROT model, incorporate the biaxial effect adequately due to the incorporation of the Drucker-Prager yield criterion in the equations. Further, for the 3D-case, only the LLF, WLF and FOC models can simulate the triaxial effect correctly, whereas the WLF model can correctly simulate also the volumetric expansion (dilatancy). In addition, with the exception of the DPH model, all models adequately simulate the unilateral and strain-rate effects; however, a poor fit is observed with respect to the benchmark experimental tests. All models give a mesh-objective response with a localized damaged zone if a perturbation is introduced in the material of the FE model. Conversely, some models lead to a uniform strain field for a nominally perfect model, with the exception of the WLF and FOC models, which localize under tensile loads, whereas the LLF model localizes both in tension as well as compression.

- The epistemic uncertainty observed in the response of a concrete prism is enough to assess correctly the sensitivity of these concrete models. Thus, it is concluded that the unloading-loading linearized stiffness of the last cycle, $\bar{K}_{c\infty}$, for the uniaxial cyclic tension and compression test, and the energy dissipated by the last loading-unloading cycle $\bar{G}_{c\infty}$ of the uniaxial tensile test, are the most important sources of epistemic uncertainty given the stress-strain constitutive concrete models considered. Moreover, a significant level of uncertainty was observed in some response variables for the triaxial monotonic tests due to the simplified term considered in the equations to simulate this effect. Also, a considerable source of uncertainty exist in the peak stresses for the strain-rate case for high strain-rates over $10^{-1}/s$, both in tension as well as in compression, with values up to 2.74 times the ones obtained through experimental tests, mainly due to the use of the viscoplastic Duvaut-Lions model. In contrast, low uncertainty

was observed in peak stresses σ_p , for all test simulations, with the exception of strain-rate case over $10^{-1}/s$. Standard deviation values reach up to 7.3%.

- Finally, identical results were obtained for the 3D and plane-stress formulations, despite the fact that the equations developed for the plastic component of the models are completely different. This serves as a validation of all of the algebra and computational implementation of the different models.

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APPENDICES

APPENDICES A. SOME USEFUL IDENTITIES

1. THREE DIMENSIONAL FORMULATION

1.1. Basic identities

First, let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ second-order tensors. Then, the following identity is satisfied

$$\mathbf{A}(\mathbf{B} : \mathbf{C}) = (\mathbf{A} \otimes \mathbf{B}) : \mathbf{C} = (\mathbf{A} \otimes \mathbf{C}) : \mathbf{B}. \quad (\text{A.1.1})$$

For the other hand, any second-order tensor can be decomposed in their deviatoric and hydrostatic part as follow

$$\mathbf{A}_{dev} = \mathcal{I}_d : \mathbf{A} = \mathbf{A} - \frac{1}{3}\text{tr}(\mathbf{A})\mathbf{I}, \quad (\text{A.1.2})$$

$$\mathbf{A}_{vol} = \frac{1}{3}\text{tr}(\mathbf{A})\mathbf{I} = \frac{1}{3}(\mathbf{I} : \mathbf{A})\mathbf{I}, \quad (\text{A.1.3})$$

where $\mathcal{I}_d$ is defined in Table II.0.1 and $\text{tr}(\cdot)$ denotes the trace of tensor. It can be probed that $\mathcal{I}_d : \mathcal{I}_d = \mathcal{I}_d$. Moreover, applying this decomposition to the stress tensor $\boldsymbol{\sigma} = \mathbf{s} + p\mathbf{I}$, it can be written as

$$\mathbf{s} = \mathcal{I}_d : \boldsymbol{\sigma} = 2G\boldsymbol{\theta}^e, \quad \boldsymbol{\theta}^e = \mathcal{I}_d : \boldsymbol{\varepsilon}^e, \quad (\text{A.1.4})$$

$$p = \frac{1}{3}(\mathcal{I} : \boldsymbol{\sigma}) = K\varepsilon_v^e, \quad \varepsilon_v^e = \mathcal{I} : \boldsymbol{\varepsilon}^e, \quad (\text{A.1.5})$$

where $\mathbf{s}$ is the deviator of stress (or deviatoric stress) tensor, p the hydrostatic stress, G the shear modulus, K the Bulk modulus, $\boldsymbol{\theta}^e$ the deviatoric elastic strain and ε_v^e the volumetric elastic strain. Moreover, the stress $\boldsymbol{\sigma}$ and elastic strain tensor $\boldsymbol{\varepsilon}^e$ can be related according to the relation $\boldsymbol{\sigma} = \mathcal{D}_e : \boldsymbol{\varepsilon}^e$, where $\mathcal{D}_e$ is the linear-elastic stiffness tensor given by

$$\mathcal{D}_e = 2\mu\mathcal{I}_d + K\mathbf{I} \otimes \mathbf{I} = 2\mu\mathcal{I}_s + \lambda\mathbf{I} \otimes \mathbf{I}, \quad (\text{A.1.6})$$

where μ and λ are the Lamé's constants ($\mu = G$). Hence, substituting Eqs. (A.1.2) and (A.1.3) into this relation, gives the follow expression

$$\mathcal{D}_e : \mathbf{A} = 2\mu \mathbf{A}_{dev} + K \mathbf{A}_{vol}. \quad (\text{A.1.7})$$

1.2. Spectral decomposition

The spectral decomposition of a second-order tensor $\mathbf{A}$ is defined as

$$\mathbf{A} := \sum_{i=1}^N \hat{a}_i \mathbf{E}_A^{ii}, \quad (\text{A.1.8})$$

where $\hat{a}_i$ is the i -th eigenvalue and $\mathbf{E}_A^{ii}$ the i -th eigen-projector tensor defined as

$$\mathbf{E}_A^{ii} = \underline{\mathbf{v}}^i \otimes \underline{\mathbf{v}}^i, \quad (\text{A.1.9})$$

with $\underline{\mathbf{v}}^i$ the i -th column of eigenvector matrix $\underline{\mathbf{V}}$.

1.3. Differentials

First, the differential of the norm of tensor $\|\mathbf{A}\| = (\mathbf{A} : \mathbf{A})$ is expressed as

$$d\|\mathbf{A}\| = \|\mathbf{A}\|^{-1} (\mathbf{A} : d\mathbf{A}). \quad (\text{A.1.10})$$

Next, let $c(\mathbf{A})$ a scalar variable that is in function of tensor $\mathbf{A}$, and $\mathbf{B}$ any other tensor. Then, the differential of product $\mathbf{B}c(\mathbf{A})$ is given by

$$d(\mathbf{B}c(\mathbf{A})) = c(\mathbf{A})d\mathbf{B} + \left(\mathbf{B} \otimes \frac{\partial c}{\partial \mathbf{A}} \right) : d\mathbf{A}. \quad (\text{A.1.11})$$

For the other hand, using Eq. (A.1.4) and the identity $\mathcal{I}_d : \mathcal{I}_d = \mathcal{I}_d$, the following relations can be derived for the deviatoric elastic strain tensor

$$d\boldsymbol{\theta}^e = \mathcal{I}_d : d\boldsymbol{\epsilon}^e, \quad \boldsymbol{\theta}^e : d\boldsymbol{\theta}^e = \boldsymbol{\theta}^e : d\boldsymbol{\epsilon}^e. \quad (\text{A.1.12})$$

Also, using this relation, the differential of invariants J_2 , $q = \sqrt{3J_2}$ and p can be written as

$$dJ_2 = \mathbf{s} : d\mathbf{s} = 4\mu^2(\boldsymbol{\theta} : d\boldsymbol{\varepsilon}), \quad (\text{A.1.13})$$

$$dq = \frac{3}{2q}\mathbf{s} : d\mathbf{s} = \frac{6\mu^2}{q}(\boldsymbol{\theta} : d\boldsymbol{\varepsilon}), \quad (\text{A.1.14})$$

$$dp = \frac{1}{3}(\mathbf{I} : d\boldsymbol{\sigma}). \quad (\text{A.1.15})$$

For other hand, (de Souza Neto et al., 2008) demonstrate that the differential of i -th eigenvalue of tensor A is given by

$$d\hat{a}_i = \mathbf{E}_A^{ii} : d\mathbf{A}. \quad (\text{A.1.16})$$

Moreover, the differential of eigenvalue tensor is written as

$$d\hat{A} = \mathcal{F}_A : d\mathbf{A}, \quad (\text{A.1.17})$$

where $\mathcal{F} = \frac{\partial \hat{A}}{\partial \mathbf{A}}$ is a fourth-order tensor. In addition, (Faria, Oliver, & Cervera, 2000) demonstrate that the differential of i -th eigenprojector $\mathbf{E}_A^{ii}$ is expressed as

$$d\mathbf{E}_A^{ii} = 2 \sum_{j \neq i}^N \frac{1}{(\hat{a}_i - \hat{a}_j)} (\mathbf{E}_A^{ij} \otimes \mathbf{E}_A^{ij}) : d\mathbf{A}, \quad (\text{A.1.18})$$

where $\mathbf{E}_A^{ij} = \frac{1}{2}(\underline{\mathbf{v}}^i \otimes \underline{\mathbf{v}}^j + \underline{\mathbf{v}}^j \otimes \underline{\mathbf{v}}^i)$. Hence, using Eqs. (A.1.4) and (A.1.17), the differential of the principal stress and deviatoric stress tensor can be written, respectively, as

$$d\hat{\boldsymbol{\sigma}} = \mathcal{F}_\sigma : \mathcal{D}_e : d\boldsymbol{\varepsilon}^e, \quad (\text{A.1.19})$$

$$d\hat{\mathbf{s}} = 2\mu\mathcal{F}_\sigma : \mathcal{I}_d : d\boldsymbol{\varepsilon}^e. \quad (\text{A.1.20})$$

1.4. Heaviside function and their approximation

The positive/negative Heaviside function $H^\pm(x, y_0)$ are defined as

$$H_{y_0}^\pm(x) = H(\pm x, y_0) = \begin{cases} 1, & \pm x > 0 \\ y_0, & x = 0 \\ 0, & \pm x < 0 \end{cases}, \quad (\text{A.1.21})$$

where y_0 is a arbitrary value $\in [0, 1]$ (usually assume a value of 0, $\frac{1}{2}$ or 1). Then, this stepped function can be approximated with several C^1 -class functions $\tilde{H}^\pm(x)$, e.g.

$$\tilde{H}^\pm(x) = \frac{1}{2} [1 \pm \tanh(kx)] = (1 \pm \exp(-2kx))^{-1}, \quad (\text{A.1.22})$$

with k is an arbitrary parameter such as $\lim_{k \rightarrow \infty} \tilde{H}^\pm(x) = H^\pm(x, 1)$. Then, its useful express the absolute and McAuley functions in terms of the approximated Heaviside function as follows

$$|x| = \left(2\tilde{H}^+(x) - 1\right)x, \quad \langle x \rangle^\pm = \pm \tilde{H}^\pm(x)x. \quad (\text{A.1.23})$$

Finally, taking this relation, the differential of absolute and McAuley function are expressed as

$$d|x| = \left[2 \left(\frac{d\tilde{H}^+}{dx}x + \tilde{H}^+\right) - 1\right]dx, \quad d\langle x \rangle^\pm = \pm \left(\frac{d\tilde{H}^\pm}{dx}x + \tilde{H}^\pm\right)dx, \quad (\text{A.1.24})$$

where $\frac{d\tilde{H}^-}{dx} = -\frac{d\tilde{H}^+}{dx}$.

2. PLANE-STRESS FORMULATION

2.1. Plane-stress relations

In plane-stress condition, the Cauchy stress and strain tensor are assumed as

$$\boldsymbol{\sigma}_3 = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\varepsilon}_3 = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{12} & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix}. \quad (\text{A.2.1})$$

Now, mapping this tensors onto plane stress subspace, i.e. considering only the in-plane stress and strain components for the respective tensors, gives

$$\boldsymbol{\sigma}_2 = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}, \quad \boldsymbol{\varepsilon}_2 = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} \end{bmatrix}. \quad (\text{A.2.2})$$

Then, converting this tensors to vectorized format using Voigt's notation, gives the respective vectors

$$\underline{\boldsymbol{\sigma}} = \text{Voigt}(\boldsymbol{\sigma}_2) = [\sigma_{11}, \sigma_{22}, \sigma_{12}]^T, \quad \underline{\boldsymbol{\varepsilon}} = \text{Voigt}(\boldsymbol{\varepsilon}_2) = [\varepsilon_{11}, \varepsilon_{22}, 2\varepsilon_{12}]^T. \quad (\text{A.2.3})$$

Moreover, the stress $\underline{\boldsymbol{\sigma}}$ and elastic strain vector $\underline{\boldsymbol{\varepsilon}}^e$ can be related with the relations $\underline{\boldsymbol{\sigma}} = \underline{\boldsymbol{D}}_e \underline{\boldsymbol{\varepsilon}}^e$ and $\underline{\boldsymbol{\varepsilon}}^e = \underline{\boldsymbol{C}}_e \underline{\boldsymbol{\sigma}}$, where $\underline{\boldsymbol{D}}_e$ and $\underline{\boldsymbol{C}}_e$ are the linear-elastic stiffness/compliance matrix, respectively and are expressed as

$$\underline{\boldsymbol{D}}_e = \frac{E_o}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1 - \nu}{2} \end{bmatrix}, \quad \underline{\boldsymbol{C}}_e = \frac{1}{E_o} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & \frac{E_o}{\nu} \end{bmatrix}. \quad (\text{A.2.4})$$

Moreover, the deviatoric stress tensor $\mathbf{s}_3$, defined as $\mathbf{s}_3 = \boldsymbol{\sigma}_3 - \frac{1}{3}\mathbf{I}_3$, is expressed as

$$\mathbf{s}_3 = \begin{bmatrix} s_{11} & s_{12} & 0 \\ s_{12} & s_{22} & 0 \\ 0 & 0 & s_{33} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2\sigma_{11} - \sigma_{22} & 3\sigma_{12} & 0 \\ 3\sigma_{12} & -\sigma_{11} + 2\sigma_{22} & 0 \\ 0 & 0 & -(\sigma_{11} + \sigma_{22}) \end{bmatrix}. \quad (\text{A.2.5})$$

Now, the invariants $I_1 = \boldsymbol{\sigma}_3 : \mathbf{I}_3$ and $J_2 = \frac{1}{2}\mathbf{s}_3 : \mathbf{s}_3$ are correctly defined in the plane stress condition using the follow relations

$$I_1 = \underline{\mathbf{1}}^T \underline{\boldsymbol{\sigma}}, \quad J_2 = \frac{1}{2} \underline{\boldsymbol{\sigma}}^T \underline{\mathbf{P}} \underline{\boldsymbol{\sigma}}, \quad (\text{A.2.6})$$

where $\underline{\mathbf{P}}$ is the projected matrix given by

$$\underline{\mathbf{P}} = \frac{1}{3} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Additionally, the elastic ε_{33}^e and plastic out-of-plane strain ε_{33}^p can be derived, respectively, as follow. First, the out-of-plane elastic strain ε_{33}^e is obtained from the three-dimensional elastic stiffness tensor as

$$\varepsilon_{33}^e = -\frac{\nu}{E_o}(\sigma_{11} + \sigma_{22}). \quad (\text{A.2.7})$$

In contrast, their respective plastic component ε_{33}^p depends of equations considered for each concrete model (see Section 3.1). Although, in all cases, considering a generic flow potential G_3 expressed in terms of 3D stress tensor $\boldsymbol{\sigma}_3$ and other hardening variable $\underline{\mathbf{q}}_3$. Then, the flow rule is given by

$$\dot{\varepsilon}_3^p := \dot{\gamma} \mathbf{N}_3, \quad (\text{A.2.8})$$

where $\mathbf{N}_3 := \frac{\partial G_3}{\partial \boldsymbol{\sigma}_3}$ is the flow tensor. Then, the evolution laws of out-of-plane plastic strain can be expressed as $\dot{\varepsilon}_{33}^p = \mathbf{I}_3 : \dot{\varepsilon}^p$, with $\mathbf{I}_3 = \text{diag}(0, 0, 1)$. In addition, the evolution law

for the volumetric strain is written as

$$\dot{\varepsilon}_v = \varepsilon_v^e + \dot{\varepsilon}_v^p = \text{tr}(\dot{\varepsilon}_3^e) + \text{tr}(\dot{\varepsilon}_3^p) = \frac{1}{K}p + \dot{\gamma}\text{tr}(\mathbf{N}_3). \quad (\text{A.2.9})$$

2.2. Spectral decomposition

The spectral decomposition of a second-order tensor $\mathbf{A}$ is defined as

$$\mathbf{A} = \sum_{i=1}^N \hat{a}_i \mathbf{E}_A^{ii}, \quad (\text{A.2.10})$$

where $\hat{a}_i$ is the i -th eigenvalue and $\mathbf{E}_A^{ii}$ the i -th eigen-projector tensor given by

$$\mathbf{E}_A^{ii} = \underline{\mathbf{v}}^i \otimes \underline{\mathbf{v}}^i, \quad (\text{A.2.11})$$

with $\underline{\mathbf{v}}^i$ the i -th column of eigenvector matrix $\mathbf{V}$. Then, this decomposition can be converted for a vectorized tensor $\underline{\mathbf{a}} = \text{Voigt}(\mathbf{A})$ as follow

$$\underline{\mathbf{a}} \sum_{i=1}^N \hat{a}_i \underline{\mathbf{e}}_a^{ii} = \underline{\mathbf{E}}_a \underline{\hat{\mathbf{a}}}, \quad (\text{A.2.12})$$

where $\underline{\hat{\mathbf{a}}} = \text{Voigt}(\hat{\mathbf{A}})$ is the vectorized eigenvalue tensor $\hat{\mathbf{A}}$, $\underline{\mathbf{e}}_a^{ii} = \text{Voigt}(\mathbf{E}_A^{ii})$ is the i -th vectorized eigen-projector tensor and $\underline{\mathbf{E}}_a$ is the eigen-projector matrix written as

$$\underline{\mathbf{E}}_a = [\underline{\mathbf{e}}_a^{11}, \underline{\mathbf{e}}_a^{22}, \dots, \underline{\mathbf{e}}_a^N]. \quad (\text{A.2.13})$$

Moreover, using Eqs. (A.2.4), (A.2.12) and (A.2.13), the follows identities are satisfied

$$\hat{\underline{\mathbf{I}}} = \underline{\mathbf{E}}_a^T \underline{\mathbf{R}} \underline{\mathbf{E}}_a, \quad \hat{\underline{\mathbf{P}}} = \underline{\mathbf{E}}_a^T \underline{\mathbf{P}} \underline{\mathbf{E}}_a, \quad (\text{A.2.14})$$

$$\hat{\underline{\mathbf{D}}}_e = \underline{\mathbf{E}}_a^T \underline{\mathbf{R}}^2 \underline{\mathbf{D}}_e \underline{\mathbf{E}}_a, \quad \hat{\underline{\mathbf{C}}}_e = \underline{\mathbf{E}}_a^T \underline{\mathbf{C}}_e \underline{\mathbf{E}}_a, \quad (\text{A.2.15})$$

where $\hat{\underline{I}} = \text{diag}(1, 1)$ and $\hat{\underline{D}}_e$, $\hat{\underline{C}}_e$ and $\hat{\underline{P}}$ are given by

$$\hat{\underline{D}}_e = \frac{E_o}{1 - \nu^2} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix}, \quad \hat{\underline{C}}_e = \frac{1}{E_o} \begin{bmatrix} 1 & -\nu \\ -\nu & 1 \end{bmatrix}, \quad \hat{\underline{P}} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}. \quad (\text{A.2.16})$$

Then, multiplying both sides of Eq. (A.2.12) by the matrix $\underline{E}_a^T \underline{R}$ and using the left side of Eq. (A.2.14), the eigenvalue vector and their respective i -th component, can be expressed as

$$\hat{\underline{a}} = \underline{F}_a \underline{a}, \quad \hat{a}_i = \underline{e}_{ii}^T \underline{R} \underline{a}, \quad (\text{A.2.17})$$

with $\underline{F}_a = \underline{E}_a^T \underline{R}$. In addition, using Eqs. (A.2.14) and (A.2.12), the invariants I_1 and J_2 can be expressed in the principal space as

$$\hat{I}_1 = \hat{\underline{1}}^T \hat{\underline{\sigma}}, \quad \hat{J}_2 = \frac{1}{2} \hat{\underline{\sigma}}^T \hat{\underline{P}} \hat{\underline{\sigma}}. \quad (\text{A.2.18})$$

2.3. Differentials

Next, let $c(\underline{A})$ a scalar variable that is in function of vector $\underline{A}$, and $\underline{B}$ any other vector. Then, the differential of product $\underline{B}c(\underline{A})$ is given by

$$d(c\underline{b}) = c d\underline{b} + \left(\underline{b} \otimes \frac{\partial c}{\partial \underline{a}} \right) d\underline{a}. \quad (\text{A.2.19})$$

For the other hand, the differential of variable $z = \underline{\sigma}^T \underline{P} \underline{\sigma} = 2J_2$ is given by

$$dz = (d\underline{\sigma}^T \underline{P} \underline{\sigma} + \underline{\sigma}^T \underline{P} d\underline{\sigma}) = 2\underline{v}_{dev}^T d\underline{\sigma}, \quad (\text{A.2.20})$$

where $\underline{v}_{dev} = \underline{P} \underline{\sigma}$. Then, the differential of invariants $q = \sqrt{3J_2} = \sqrt{\frac{3}{2}}z$ and $r = \sqrt{q^2 + \epsilon^2}$ can be written as

$$dq = \frac{3}{4q} dz = \frac{3}{2q} \underline{v}_{dev}^T d\underline{\sigma}, \quad dr = \frac{3}{4r} dz = \frac{3}{2r} \underline{v}_{dev}^T d\underline{\sigma}. \quad (\text{A.2.21})$$

Now, using the relation Eq. (A.2.17), the differential of eigenvalue vector and their i -th component are written, respectively, as

$$d\hat{\underline{\mathbf{a}}} = \underline{\mathbf{F}}_a d\underline{\mathbf{a}}, \quad d\hat{a}_i = \underline{\mathbf{e}}_a^{iiT} \underline{\mathbf{R}} d\underline{\mathbf{a}}. \quad (\text{A.2.22})$$

In addition, (Faria et al., 2000) demonstrate that the differential of i -th eigenprojector vector e_a^{ii} is expressed as

$$d\underline{\mathbf{e}}_a^{ii} = 2 \left[\sum_{j \neq i}^N \frac{1}{(\hat{a}_i - \hat{a}_j)} (\underline{\mathbf{e}}_a^{ij} \otimes \underline{\mathbf{e}}_a^{ij}) \right] \underline{\mathbf{R}} d\underline{\mathbf{a}}, \quad (\text{A.2.23})$$

where $\underline{\mathbf{e}}_{ij} = \frac{1}{2} \text{Voigt}(\underline{\mathbf{v}}^i \otimes \underline{\mathbf{v}}^j + \underline{\mathbf{v}}^j \otimes \underline{\mathbf{v}}^i)$, with $\underline{\mathbf{v}}^i$ the i -th column of eigenvector matrix $\underline{\mathbf{V}}$. In addition, using Eq. (A.2.18), the differential of invariants expressed in the principal space $\hat{q} = \sqrt{\frac{3}{2}\hat{z}}$ and $\hat{r} = \sqrt{\hat{q}^2 + \epsilon^2}$ can be written as

$$d\hat{q} = \frac{3}{4\hat{q}} d\hat{z} = \frac{3}{2\hat{q}} \hat{\underline{\mathbf{v}}}_{dev}^T d\hat{\underline{\boldsymbol{\sigma}}}, \quad d\hat{r} = \frac{3}{4\hat{r}} d\hat{z} = \frac{3}{2\hat{r}} \hat{\underline{\mathbf{v}}}_{dev}^T d\hat{\underline{\boldsymbol{\sigma}}}. \quad (\text{A.2.24})$$

APPENDICES B. CONVERSION OF TENSORS AND THEIR OPERATIONS TO VECTOR AND MATRIX FORMAT

This section detailed the conversion of some useful tensors and their operations to vector/matrix representation (vectorization and matricization) for the 3D case ($N = 3$) necessary to implement numerically the concrete models. This conversion is elaborated keeping identical results between tensors and vectors/matrix representation.

First, to elaborate this conversion, any symmetric second-order tensor $\mathbf{A} \equiv A_{ij}$ can be written as

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ & A_{22} & A_{23} \\ sym & & A_{33} \end{bmatrix}. \quad (\text{B.0.1})$$

Then, this tensor can be vectorized in a $\mathbb{R}^N$ vector using Voigt's notation as $\underline{\mathbf{a}} = \text{Voigt}(\mathbf{A})$, where $\text{Voigt}(\cdot)$ operation is given by

$$\text{Voigt}(\mathbf{A}) = [A_{11}, A_{22}, A_{33}, A_{12}, A_{23}, A_{13}]^T. \quad (\text{B.0.2})$$

Moreover, any third-order tensor $\mathbf{A}_3$ can be represented in a $\mathbb{R}^{2NxN}$ matrix as $\underline{\mathbf{A}}_3 = \text{Matr}_3(\mathbf{A})$, where $\text{Matr}_3(\cdot)$ operation is expressed as

$$\text{Matr}_3(\mathbf{A}) = [\underline{\mathbf{a}}^1, \underline{\mathbf{a}}^2, \dots, \underline{\mathbf{a}}^N], \quad (\text{B.0.3})$$

where $\underline{\mathbf{a}}^k = \text{Voigt}(\mathbf{A}^k)$ and $\mathbf{A}^k \equiv A_{ij}^k$. In contrast, any symmetric fourth-order tensor $\mathcal{A} \equiv A_{ijkl}$ that satisfy the condition $A_{ijkl} = A_{ijlk} = A_{jikl}$ can be matricized in a $\mathbb{R}^{2Nx2N}$

matrix as $\underline{\mathbf{A}} = \text{Matr}_4(\mathbf{A})$, where $\text{Matr}_4(\cdot)$ is given by

$$\text{Matr}_4(\mathbf{A}) = \begin{bmatrix} A_{1111} & A_{1122} & A_{1133} & A_{1123} & A_{1113} & A_{1112} \\ A_{2211} & A_{2222} & A_{2233} & A_{2223} & A_{2213} & A_{2212} \\ A_{3311} & A_{3322} & A_{3333} & A_{3323} & A_{3313} & A_{3312} \\ A_{2311} & A_{2322} & A_{2333} & A_{2323} & A_{2313} & A_{2312} \\ A_{1311} & A_{1322} & A_{1333} & A_{1323} & A_{1313} & A_{1312} \\ A_{1211} & A_{1222} & A_{1233} & A_{1223} & A_{1213} & A_{1212} \end{bmatrix}. \quad (\text{B.0.4})$$

This condition occur e.g. when a fourth-order tensor $\mathbf{A}$ is elaborated by two symmetrical second-order tensors $\mathbf{B}$ and $\mathbf{C}$, thus $\mathbf{A} = \mathbf{B} \otimes \mathbf{C}$. Moreover, if the fourth-order tensor satisfy the relation $A_{ijkl} = A_{klij}$, their representative matrix $\underline{\mathbf{A}}$ is symmetric. Example of this is when a fourth-order tensor $\mathbf{A}$ can be defined as the product of two identical second-order tensors $\mathbf{B}$, thus $\mathbf{A} = \mathbf{B} \otimes \mathbf{B}$.

Table II.0.1 shown the conversion for some useful tensors and their operations used in this article. Additionally, the following special cases are considered. The relation between stress $\boldsymbol{\varepsilon}$ and strain tensor $\boldsymbol{\varepsilon}$ (Hooke's law), can be converted as follows

$$\boldsymbol{\sigma} = \mathcal{D}_e : \boldsymbol{\varepsilon} \rightarrow \underline{\boldsymbol{\sigma}} = \underline{\mathcal{D}}_e \underline{\boldsymbol{\varepsilon}}, \quad (\text{B.0.5})$$

where $\underline{\boldsymbol{\sigma}} = \text{Voigt}(\boldsymbol{\sigma})$, $\underline{\mathcal{D}}_e = \text{Matr}_4(\mathcal{D}_e)$ and $\underline{\boldsymbol{\varepsilon}} = \underline{\mathbf{R}} \text{Voigt}(\boldsymbol{\varepsilon})$ is the strain vector with engineering strains, thus $\underline{\boldsymbol{\varepsilon}} = [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \gamma_{13}, \gamma_{23}, \gamma_{12}]^T$, which is commonly used in computer software. Moreover, using this strain vector, the inverse relation can be established as

$$\boldsymbol{\varepsilon} = \mathcal{C}_e : \boldsymbol{\sigma} \rightarrow \underline{\boldsymbol{\varepsilon}} = \underline{\mathcal{C}}_e \underline{\boldsymbol{\sigma}}, \quad (\text{B.0.6})$$

where $\underline{\mathcal{C}}_e = \text{Matr}_4(\mathcal{C}_e) \underline{\mathbf{R}}^2$. Similarly, the inner product between any second-order tensor $\mathbf{A}$ and the strain tensor is convert as $\mathbf{A} : \boldsymbol{\varepsilon} \rightarrow \underline{\mathbf{a}}^T \underline{\boldsymbol{\varepsilon}}$. Also, the double inner product among any two second-order tensors $\mathbf{A}$, $\mathbf{B}$ and the compliance tensor $\mathcal{C}_e$ is converted as $\mathbf{A} : \mathcal{C}_e : \mathbf{B} \rightarrow \underline{\mathbf{a}}^T \underline{\mathcal{C}}_e \underline{\mathbf{b}}$.

For the other hand, the inner product between two eigenvalue tensors $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ can be converted as $\hat{\mathbf{A}}: \hat{\mathbf{B}} \rightarrow \underline{\hat{\mathbf{a}}}^T \underline{\hat{\mathbf{b}}}$. Also, the relation $\mathcal{I}_d: \mathcal{I}_d = \mathcal{I}_d$ can be converted as $\rightarrow \underline{\mathbf{I}}_d \underline{\mathbf{R}} \underline{\mathbf{I}}_d = \underline{\mathbf{I}}_d$. Finally, the relations of Eqs. (A.1.19) and (A.1.20) can be converted as follows

$$d\underline{\hat{\boldsymbol{\sigma}}} = \underline{\mathbf{E}}^T \underline{\mathbf{R}} \underline{\mathbf{D}}_e d\underline{\boldsymbol{\varepsilon}} \quad d\underline{\hat{\mathbf{s}}} = 2\mu \underline{\mathbf{E}}^T \underline{\mathbf{R}} \underline{\mathbf{I}}_d d\underline{\boldsymbol{\varepsilon}}. \quad (\text{B.0.7})$$

Table II.0.1. Conversion of some useful tensors and their operations to vector/matrix representation.

Description	Tensorial		3D vector/matrix representation	
	Symbol	Definition	Index	Symbol Array
Second-order tensor	$\mathbf{A}$	-	A_{ij}	$\underline{\mathbf{a}}$ Voigt($\mathbf{A}$)
Third-order tensor	$\mathbf{A}_3$	-	A_{ijk}	$\underline{\mathbf{A}}_3$ Matr ₃ ($\mathbf{A}$)
Fourth-order tensor	$\mathbf{A}$	-	A_{ijkl}	$\underline{\mathbf{A}}$ Matr ₄ ($\mathbf{A}$)
Second-order identity tensor	$\mathbf{I}$	-	δ_{ij}	$\underline{\mathbf{i}}$ $[1, 1, 1, 0, 0, 0]^T$
Second-order +/- identity tensor	$\mathbf{I}^\pm$	-	δ_{1i}/δ_{Ni}	$\mathbf{1}^\pm$ $[1, 0, 0]^T/[0, 0, 1]^T$
Fourth-order symmetric identity tensor	$\mathcal{I}_s$	-	$\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$	$\underline{\mathbf{I}}_s$ $\text{diag}(1, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
Fourth-order deviatoric tensor	$\mathcal{I}_d$	$= \mathcal{I}_s - \frac{1}{3}\mathbf{I} \otimes \mathbf{I}$	$(I_s)_{ijkl} - \frac{1}{3}\delta_{ij}\delta_{kl}$	$\underline{\mathbf{I}}_d$ $\underline{\mathbf{I}}_s - \frac{1}{3}\underline{\mathbf{i}} \otimes \underline{\mathbf{i}}$
Outer (dyadic) product (2-2) ⁽¹⁾	$\mathbf{A} \otimes \mathbf{B}$	-	$A_{ij}B_{kl}$	$\underline{\mathbf{a}}\underline{\mathbf{b}}^T$
Inner product (2-2)	$\mathbf{A} : \mathbf{B}$	-	$A_{ij}B_{ij}$	$\underline{\mathbf{a}}^T \underline{\mathbf{R}}\underline{\mathbf{b}}^{(2)}$
Inner product (4-2)	$\mathbf{A} : \mathbf{B}$	-	$A_{ijkl}B_{kl}$	$\underline{\mathbf{A}}\underline{\mathbf{R}}\underline{\mathbf{b}}$
Inner product (4-4)	$\mathbf{A} : \mathbf{B}$	-	$A_{ijkl}B_{klmn}$	$\underline{\mathbf{A}}\underline{\mathbf{R}}\underline{\mathbf{B}}$
Double inner product (2-4-2)	$\mathbf{A} : \mathbf{B} : \mathbf{C}$	-	$A_{ij}B_{ijkl}C_{kl}$	$\underline{\mathbf{a}}^T \underline{\mathbf{B}}\underline{\mathbf{R}}^2 \underline{\mathbf{c}}$
Norm (2)	$\ \mathbf{A}\ $	-	$\sqrt{A_{ij}A_{ij}}$	$\sqrt{\underline{\mathbf{a}}^T \underline{\mathbf{R}}\underline{\mathbf{a}}}$
Eigenvalues/eigenvectors tensors	$\hat{\mathbf{A}}, \mathbf{V}$	$\mathbf{A} = \sum_{i=1}^N \hat{a}_i \mathbf{E}_A^{ii}$	-	$\hat{\mathbf{a}}, \underline{\mathbf{E}}_A$ $\underline{\mathbf{a}} = \underline{\mathbf{E}}_A \hat{\mathbf{a}}$
Second-order i -th eigen-projector tensor	$\mathbf{E}_A^{ii}$	$= \underline{\mathbf{v}}^i \otimes \underline{\mathbf{v}}^{i(3)}$	$v_k^i v_k^{i(4)}$	$\underline{\mathbf{e}}_A^{ii}$ Voigt($\mathbf{E}_A^{ii}$)
Fourth-order eigen-projector tensor	$\mathcal{E}_A$	-	-	$\underline{\mathbf{E}}_A$ $= [\underline{\mathbf{e}}_A^{11}, \underline{\mathbf{e}}_A^{22}, \dots, \underline{\mathbf{e}}_A^{NN}]$
Second-order derivative of i -th eigenvalue	$\frac{\partial \hat{a}_i}{\partial \mathbf{A}}$	$= \mathbf{E}_A^{ii}$	-	$\frac{\partial \hat{a}_i}{\partial \underline{\mathbf{A}}}$ $= \underline{\mathbf{e}}_A^{iT} \underline{\mathbf{R}}$
Fourth-order derivative eigenvalue tensor	$\frac{\partial \hat{\mathbf{A}}}{\partial \mathbf{A}}$	$= \mathcal{F}_A$	-	$\frac{\partial \hat{\mathbf{A}}}{\partial \underline{\mathbf{A}}}$ $= \underline{\mathbf{E}}_A^T \underline{\mathbf{R}}^{(5)}$

(1) parenthesis denotes the order of tensor considered; (2) $\underline{\mathbf{R}}$ is the Reiter's matrix, defined for the 3D case as $\underline{\mathbf{R}} = \text{diag}(1, 1, 1, 2, 2, 2)$; (3) $\underline{\mathbf{v}}^i$ denote the i -th column of eigenvector matrix $\mathbf{V}$; (4) superscripts represent no index summation; (5) the following identity is satisfied $\underline{\mathbf{E}}_A^T \underline{\mathbf{R}} \underline{\mathbf{E}}_A = \hat{\mathbf{I}}$, where $\hat{\mathbf{I}} = \text{diag}(1, 1, 1)$.

APPENDICES C. CALCULATION OF SOME DERIVATIVES OF CONCRETE MODELS FOR 3D FORMULATION

1. DETAILED CALCULATION OF DERIVATIVE $\frac{\partial \Delta\gamma}{\partial Q}$ FOR LLF AND WLF MODELS

This appendix present the steps necessaries to determinate the derivative $\frac{\partial \Delta\gamma}{\partial q}$ expressed in Eq. (2.3.110) required to calculate the consistency operator $\Delta\gamma$ for the LLF and WLF models. Specifically, they are developed the derivatives for the LLF model, due that is more general than the WLF model. For the sake of simplicity, is omitted the subscript ' $n + 1$ ' in all variables hereafter. Also, all derivatives are taken with respect to variable $\bar{q}$ and the variable $\Delta\gamma'$ is denoted as the derivative $\frac{\partial \Delta\gamma}{\partial q}$.

First, using the relations $\bar{r} = \sqrt{\bar{q}^2 + \epsilon^2}$ and $\bar{w} = \bar{q}/\bar{r}$, it follows that their derivatives are given by

$$\frac{\partial \bar{r}}{\partial \bar{q}} = \frac{\bar{q}}{\bar{r}}, \quad \frac{\partial \bar{w}}{\partial \bar{q}} = \bar{a}_0, \quad (\text{C.1.1})$$

with $\bar{a}_0 = \epsilon^2/\bar{r}^3$. Next, using the chain rule, the derivative of variable ϕ , given by Eq. (2.1.17), is expressed as $\frac{\partial \phi}{\partial \bar{q}} = \hat{\Phi} : \frac{\partial \hat{\sigma}}{\partial \bar{q}}$, where $\hat{\Phi}$ is defined in Eq. (2.4.146). Moreover, using Eq. (2.3.98) and Eq. (C.1.1), the derivative of updated principal effective stress tensor can be written as

$$\frac{\partial \hat{\sigma}}{\partial \bar{q}} = - \left(\Delta\gamma' \hat{\mathbf{B}}_0 + \Delta\gamma \hat{\mathbf{B}}_1 \right), \quad (\text{C.1.2})$$

where $\hat{\mathbf{B}}_0 = 3\mu\bar{w}\hat{\mathbf{t}}^{\text{tr}} + \bar{\eta}K\mathbf{I}$ and $\hat{\mathbf{B}}_1 = 3\mu\bar{a}_0\hat{\mathbf{t}}^{\text{tr}}$. Similarly, using Eq. (2.3.96), the derivative of updated principal effective flow tensor is expressed as

$$\frac{\partial \hat{\mathbf{N}}}{\partial \bar{q}} = \frac{3}{2}\bar{a}_0\hat{\mathbf{t}}^{\text{tr}}. \quad (\text{C.1.3})$$

Also, using the chain rule, the derivative of variables $\theta_1^\pm$ and $\theta_2^\pm$ (Eqs. (2.3.105) and (2.3.106)) are given by $\frac{\partial \theta_1^\pm}{\partial \bar{q}} = \frac{\partial \phi}{\partial \bar{q}}$ and $\frac{\partial \theta_2^\pm}{\partial \bar{q}} = \frac{1}{g^\pm} \frac{\partial \sigma^\pm}{\partial \kappa^\pm} \frac{\partial \kappa^\pm}{\partial \bar{q}}$. Then, the derivative of the variables $\varphi^\pm$

(Eq. (2.3.104)) are expressed as

$$\frac{\partial \varphi^\pm}{\partial \bar{q}} = \theta_2^\pm \frac{\partial \phi}{\partial \bar{q}} + \frac{1}{g^\pm} \theta_1^\pm J_\kappa^\pm \frac{\partial \kappa^\pm}{\partial \bar{q}}, \quad (\text{C.1.4})$$

where $J_\kappa^\pm := \frac{\partial \sigma^\pm}{\partial \kappa^\pm}$. Hence, using the obtained relations, the derivative of the variables $h^\pm = \hat{n}^\pm \varphi^\pm$ are given by

$$\frac{\partial h^\pm}{\partial \bar{q}} = -\hat{n}^\pm \theta_2^\pm \hat{\Phi} : \left(\Delta \gamma' \hat{\mathbf{B}}_0 + \Delta \gamma \hat{\mathbf{B}}_1 \right) + \hat{b}_{10}^\pm \frac{\partial \kappa^\pm}{\partial \bar{q}} + b_2^\pm, \quad (\text{C.1.5})$$

with $\hat{b}_{10}^\pm = \frac{1}{g^\pm} \hat{n}^\pm \theta_1^\pm J_\kappa^\pm$ and $b_2^\pm = \frac{3}{2} \bar{a}_0 \varphi^\pm \hat{t}_\pm^{\text{tr}}$. Consequently, using this relation, the derivatives of hardening variables $\kappa^\pm$ are expressed as

$$\frac{\partial \kappa^\pm}{\partial \bar{q}} = \hat{b}_{30}^\pm \Delta \gamma' + \hat{b}_{40}^\pm \Delta \gamma + \Delta \gamma \hat{b}_{10}^\pm \frac{\partial \kappa^\pm}{\partial \bar{q}}, \quad (\text{C.1.6})$$

where $\hat{b}_{30}^\pm = h^\pm - \Delta \gamma \hat{n}^\pm \theta_2^\pm (\Phi : \hat{\mathbf{B}}_0)$ and $\hat{b}_{40}^\pm = b_2^\pm - \Delta \gamma \hat{n}^\pm \theta_2^\pm (\Phi : \hat{\mathbf{B}}_1)$. Solving this equation, the derivative $\frac{\partial \kappa^\pm}{\partial \bar{q}}$ can be rewritten finally as

$$\frac{\partial \kappa^\pm}{\partial \bar{q}} = \hat{b}_3^\pm \Delta \gamma' + \hat{b}_4^\pm \Delta \gamma, \quad (\text{C.1.7})$$

with $\hat{b}_3^\pm = \hat{b}_{20}^\pm \hat{b}_{30}^\pm$ and $\hat{b}_4^\pm = \hat{b}_{20}^\pm \hat{b}_{40}^\pm$, with $\hat{b}_{20}^\pm = (1 - \Delta \gamma \hat{b}_{10}^\pm)^{-1}$. Then, using these relations and the chain rule, the derivatives for the positive/negative uniaxial effective stress $\bar{\sigma}^\pm$ law are expressed as

$$\frac{\partial \bar{\sigma}^\pm}{\partial \bar{q}} = \frac{\partial \bar{\sigma}^\pm}{\partial \kappa^\pm} \frac{\partial \kappa^\pm}{\partial \bar{q}} = b_3^\pm \Delta \gamma' + b_4^\pm \Delta \gamma, \quad (\text{C.1.8})$$

where $b_3^\pm = \bar{J}_\kappa^\pm \hat{b}_3^\pm$ and $b_4^\pm = \bar{J}_\kappa^\pm \hat{b}_4^\pm$, with $\bar{J}_\kappa^\pm := \frac{\partial \bar{\sigma}^\pm}{\partial \kappa^\pm}$ their hardening modulus. Moreover, the derivative of updated parameter β , given by Eq. (2.3.107), is written as

$$\frac{\partial \beta}{\partial q} = b_5 \Delta \gamma' + b_6 \Delta \gamma, \quad (\text{C.1.9})$$

where $b_5 = \frac{(1-\alpha)}{(\bar{\sigma}^-)^2}(\bar{\sigma}^+ b_3^- - \bar{\sigma}^- b_3^+)$ and $b_6 = \frac{(1-\alpha)}{(\bar{\sigma}^-)^2}(\bar{\sigma}^+ b_4^- - \bar{\sigma}^- b_4^+)$. Then, the derivative of the variable $\hat{\rho}_1 = \beta \tilde{H}^+(\hat{\sigma}_+) + \delta \tilde{H}^-(\hat{\sigma}_+)$ is expressed as

$$\frac{\partial \hat{\rho}_1}{\partial \bar{q}} = \hat{b}_7 \Delta \gamma' + \hat{b}_8 \Delta \gamma, \quad (\text{C.1.10})$$

where $\hat{b}_7 = b_5 \tilde{H}^+(\hat{\sigma}_+) - \hat{\rho}_2 \hat{b}_{0+}$ and $\hat{b}_8 = b_6 \tilde{H}^+(\hat{\sigma}_+) - \hat{\rho}_2 \hat{b}_{1+}$, with $\hat{\rho}_2 = \beta \frac{\partial \tilde{H}^+}{\partial \hat{\sigma}_+} + \delta \frac{\partial \tilde{H}^-}{\partial \hat{\sigma}_+}$, $\hat{b}_{0+} = 3\mu \bar{w} \hat{t}_+^{\text{tr}} + \bar{\eta} K$ and $\hat{b}_{1+} = 3\mu \bar{a}_0 \hat{t}_+^{\text{tr}}$.

Finally, the derivative of numerator and denominator $\bar{f}_1, \bar{f}_2$ of expression Eq. (2.3.109), can be written as

$$\frac{\partial \bar{f}_1}{\partial \bar{q}} = \frac{\partial \hat{\rho}_1}{\partial \bar{q}} \hat{\sigma}_1^{\text{tr}} - (1 - \alpha) \frac{\partial \bar{\sigma}^-}{\partial \bar{q}} = b_7 \Delta \gamma' + b_8 \Delta \gamma, \quad (\text{C.1.11})$$

$$\frac{\partial \bar{f}_2}{\partial \bar{q}} = 3\mu \frac{\partial \bar{w}}{\partial \bar{q}} + \frac{\partial \hat{\rho}_1}{\partial \bar{q}} \hat{b}_{0+} + \hat{\rho}_1 \frac{\partial \hat{b}_{0+}}{\partial \bar{q}} = b_9 + b_{10} \Delta \gamma' + b_{11} \Delta \gamma, \quad (\text{C.1.12})$$

where b_7 to b_{11} are constants expressed as

$$\begin{aligned} b_7 &= \hat{b}_7 \hat{\sigma}_+^{\text{tr}} - (1 - \alpha) b_3^-, & b_8 &= \hat{b}_8 \hat{\sigma}_+^{\text{tr}} - (1 - \alpha) b_4^-, \\ b_9 &= 3\mu \bar{a}_0 \left(1 + \hat{\rho}_1 \hat{t}_+^{\text{tr}}\right), & b_{10} &= \hat{b}_7 \hat{b}_{0+}, & b_{11} &= \hat{b}_8 \hat{b}_{0+}. \end{aligned}$$

Thus, the derivative of discrete consistency operator $\Delta \gamma$ can be expressed as

$$\Delta \gamma' = \frac{1}{\bar{f}_2} [b_7 \Delta \gamma' + b_8 \Delta \gamma - \Delta \gamma (b_9 + b_{10} \Delta \gamma' + b_{11} \Delta \gamma)]. \quad (\text{C.1.13})$$

Solving this linear algebraic expression for the variable $\Delta \gamma'$, gives the expression of Eq. (2.3.110). It should be noted that, setting the variables $\theta_2^\pm = 1$ and $\hat{b}_{10}^\pm = 0$, it can obtain the expressions adequate for the derivatives of the WLF model.

2. DETAILED CALCULATION OF DERIVATIVES TO SOLVE HARDENING VECTOR κ FOR LLF AND WLF MODELS

This appendix present the steps necessary to determinate the derivatives expressed in Eq. (2.3.113) required to calculate the hardening vector $\underline{\kappa} = [\kappa^+, \kappa^-]^T$. For simplicity of the presentation, there are omitted the subscript ' $n+1$ ' in all variables hereafter.

Derivative $\frac{\partial \underline{H}}{\partial \hat{\sigma}}$

In this subsection, the derivatives are taken with respect to principal effective stress tensor $\hat{\sigma}$. First, using the relation $\frac{\partial \bar{q}}{\partial \hat{\sigma}} = \frac{3}{2} \frac{\hat{s}}{\bar{q}}$, the Eq. (C.1.1) and the chain rule, the derivative of positive/negative part of principal effective flow tensor, given by Eq. (2.3.97), can be expressed as

$$\frac{\partial \hat{n}^\pm}{\partial \hat{\sigma}} = \frac{3}{2} \hat{t}_{\pm}^{\text{tr}} \frac{\partial \bar{w}}{\partial \bar{q}} \frac{\partial \bar{q}}{\partial \hat{\sigma}} = \bar{a}_3 \hat{t}_{\pm}^{\text{tr}} \hat{t}^{\text{tr}}, \quad (\text{C.2.1})$$

with $\bar{a}_3 = 9/4\bar{a}_0$. Next, using the relation $\frac{\partial \theta_1^\pm}{\partial \hat{\sigma}} = \hat{\Phi}$, the derivative of the variable $\varphi^\pm$ (Eq. (2.3.104)), is written as $\frac{\partial \varphi^\pm}{\partial \hat{\sigma}} = \theta_2^\pm \hat{\Phi}$. Thus, the derivative of the variables $h^\pm = \varphi^\pm \hat{n}^\pm$ are given by

$$\frac{\partial h^\pm}{\partial \hat{\sigma}} = \hat{n}^\pm \theta_2^\pm \hat{\Phi} + \bar{a}_3 \varphi^\pm \hat{t}_{\pm}^{\text{tr}} \hat{t}^{\text{tr}}. \quad (\text{C.2.2})$$

Hence, rewritten this relation in a matrix format, the derivative of matrix $\underline{H}$ can be expressed as

$$\frac{\partial \underline{H}}{\partial \hat{\sigma}} = \underline{\hat{Y}} \otimes \hat{\Phi} + \bar{a}_3 \underline{\hat{Z}} \otimes \hat{t}^{\text{tr}}, \quad (\text{C.2.3})$$

where $\underline{\hat{Y}}$ and $\underline{\hat{Z}}$ are matrices given by

$$\underline{\hat{Y}} = \begin{bmatrix} \hat{n}^+ \theta_2^+ & 0 & 0 \\ 0 & 0 & \hat{n}^- \theta_2^- \end{bmatrix}, \quad \underline{\hat{Z}} = \begin{bmatrix} \varphi^+ \hat{t}_+^{\text{tr}} & 0 & 0 \\ 0 & 0 & \varphi^- \hat{t}_-^{\text{tr}} \end{bmatrix}.$$

Derivative $\frac{\partial \bar{w}}{\partial \Delta \gamma}$ and $\frac{\partial \hat{\sigma}}{\partial \Delta \gamma}$

In this subsection, the derivatives are taken with respect to variable $\Delta \gamma$. First, using the relation $\bar{r} = \sqrt{\bar{q}^2 + \epsilon^2}$ and Eq. (2.3.101), the derivative of variables $\bar{r}$ and $\bar{q}$ are given by

$$\frac{\partial \bar{r}}{\partial \Delta \gamma} = \bar{w} \bar{q}, \quad \frac{\partial \bar{q}}{\partial \Delta \gamma} = -3\mu \left(\frac{\partial \bar{w}}{\partial \Delta \gamma} \Delta \gamma + \bar{w} \right). \quad (\text{C.2.4})$$

Moreover, the derivative of the relation $\bar{w} = \bar{q}/\bar{r}$ is given by $\frac{\partial \bar{w}}{\partial \Delta \gamma} = \bar{a}_0 \frac{\partial \bar{q}}{\partial \Delta \gamma}$, with $\bar{a}_0 = \epsilon^2/\bar{r}^3$. Thus, inserting Eq. (C.2.4) into this relation, the derivative $\frac{\partial \bar{w}}{\partial \Delta \gamma}$ can be solved as

$$\frac{\partial \bar{w}}{\partial \Delta \gamma} = -3\mu \bar{a}_2 \bar{w}, \quad (\text{C.2.5})$$

where $\bar{a}_2 = \bar{a}_0 \bar{a}_1$ with $\bar{a}_1 = (1 + 3\mu \bar{a}_0 \Delta \gamma)^{-1}$. Finally, using this relation and Eq. (2.3.98), the derivative of principal effective stress tensor $\hat{\sigma}$ is given by

$$\frac{\partial \hat{\sigma}}{\partial \Delta \gamma} = -\hat{\mathbf{B}}_0 + 9\mu^2 \bar{a}_2 \bar{w} \Delta \gamma \hat{\mathbf{t}}^{\text{tr}}. \quad (\text{C.2.6})$$

Derivative $\frac{\partial \Delta \gamma}{\partial \underline{\kappa}}$

In this subsection, the derivatives are taken with respect to hardening vector $\underline{\kappa} = [\kappa^+, \kappa^-]^T$. Also, the variable $\underline{\Delta \gamma}'$ is denoted as the derivative $\frac{\partial \Delta \gamma}{\partial \underline{\kappa}}$. First, the derivatives of uniaxial effective stress $\bar{\sigma}^\pm(\kappa^\pm)$ laws are expressed as

$$\frac{\partial \bar{\sigma}^\pm}{\partial \underline{\kappa}} := \underline{\mathbf{v}}^\pm, \quad \underline{\mathbf{v}}^+ = \begin{bmatrix} \bar{J}_\kappa^+ \\ 0 \end{bmatrix}, \quad \underline{\mathbf{v}}^- = \begin{bmatrix} 0 \\ \bar{J}_\kappa^- \end{bmatrix}, \quad (\text{C.2.7})$$

with $\bar{J}_\kappa^\pm := \frac{\partial \bar{\sigma}^\pm}{\partial \kappa^\pm}$. For other hand, using the chain rule, the derivative of the principal effective stress tensor can be expressed as $\frac{\partial \hat{\sigma}}{\partial \underline{\kappa}} = \frac{\partial \hat{\sigma}}{\partial \Delta \gamma} \otimes \underline{\Delta \gamma}'$, with $\frac{\partial \hat{\sigma}}{\partial \Delta \gamma}$ given by Eq. (C.2.6). Thus, the derivative of the maximum principal effective stress is written as

$$\frac{\partial \hat{\sigma}_+}{\partial \underline{\kappa}} = h_0 \underline{\Delta \gamma}', \quad (\text{C.2.8})$$

where $h_0 = -\hat{b}_{0+} + 9\mu^2\bar{a}_2\bar{w}\Delta\gamma\hat{t}_+^{\text{tr}}$, with $\hat{b}_{0+} = \hat{\mathbf{I}}: \hat{\mathbf{B}}_0$ and $\hat{t}_+^{\text{tr}} = \hat{\mathbf{I}}: \hat{\mathbf{t}}^{\text{tr}}$. Next, using Eq. (C.2.7), the derivative of variable β (Eq. (2.3.107)) is given by

$$\frac{\partial\beta}{\partial\mathbf{\underline{\kappa}}} = \frac{(1-\alpha)}{(\bar{\sigma}^+)^2}\mathbf{\underline{v}}_0, \quad \mathbf{\underline{v}}_0 = \bar{\sigma}^+\mathbf{\underline{v}}^- - \bar{\sigma}^-\mathbf{\underline{v}}^+ = \begin{bmatrix} -\bar{\sigma}^-\bar{J}_\kappa^+ \\ \bar{\sigma}^+\bar{J}_\kappa^- \end{bmatrix}. \quad (\text{C.2.9})$$

Then, the derivative of variable $\hat{\rho}_1 = \beta\tilde{H}^+(\hat{\sigma}_+) + \delta\tilde{H}^-(\hat{\sigma}_+)$ is expressed as

$$\frac{\partial\hat{\rho}_1}{\partial\mathbf{\underline{\kappa}}} = \tilde{H}^+(\hat{\sigma}_+)\frac{(1-\alpha)}{(\bar{\sigma}^+)^2}\mathbf{\underline{v}}_0 + \hat{\rho}_2h_0\mathbf{\underline{\Delta\gamma}}', \quad (\text{C.2.10})$$

where $\hat{\rho}_2 = \beta\frac{\partial\tilde{H}^+}{\partial\hat{\sigma}_+} + \delta\frac{\partial\tilde{H}^-}{\partial\hat{\sigma}_+}$. In addition, using Eq. (C.2.5), the derivative of variable $\hat{b}_{0+}$ (Eq. (2.3.99)) is written as $\frac{\partial\hat{b}_{0+}}{\partial\mathbf{\underline{\kappa}}} = -9\mu^2\bar{a}_2\bar{w}\hat{t}_+^{\text{tr}}\mathbf{\underline{\Delta\gamma}}'$.

Finally, using this last relation and Eqs. (C.2.5), (C.2.7) and (C.2.10), the derivatives of the numerator $\bar{f}_1$ and the denominator $\bar{f}_2$ of expression Eq. (2.3.109) can be written, respectively, as

$$\frac{\partial\bar{f}_1}{\partial\mathbf{\underline{\kappa}}} = (1-\alpha) \left(\frac{\hat{\sigma}_+^{\text{tr}}\tilde{H}^+(\hat{\sigma}_+)}{(\bar{\sigma}^+)^2}\mathbf{\underline{v}}_0 - \mathbf{\underline{v}}^- \right) + \hat{\sigma}_+^{\text{tr}}\hat{\rho}_2h_0\mathbf{\underline{\Delta\gamma}}', \quad (\text{C.2.11})$$

$$\frac{\partial\bar{f}_2}{\partial\mathbf{\underline{\kappa}}} = \left[-9\mu^2\bar{a}_2\bar{w}(1 + \hat{\rho}_1\hat{t}_+^{\text{tr}}) + b_{0+}\hat{\rho}_2h_0 \right] \mathbf{\underline{\Delta\gamma}}' + \hat{b}_{0+}\tilde{H}^+(\hat{\sigma}_+)\frac{(1-\alpha)}{(\bar{\sigma}^+)^2}\mathbf{\underline{v}}_0. \quad (\text{C.2.12})$$

Thus, using Eqs. (2.3.99) and (2.3.109), the derivative of discrete consistency operator $\Delta\gamma$ can be expressed as

$$\mathbf{\underline{\Delta\gamma}}' = \frac{1}{\bar{f}_2} \left(\frac{\partial\bar{f}_1}{\partial\mathbf{\underline{\kappa}}} - \Delta\gamma\frac{\partial\bar{f}_2}{\partial\mathbf{\underline{\kappa}}} \right) = \frac{1}{\bar{f}_2} (\mathbf{\underline{l}}_0 + L_1\mathbf{\underline{\Delta\gamma}}'), \quad (\text{C.2.13})$$

where $\mathbf{\underline{l}}_0$ and L_1 are given by

$$\mathbf{\underline{l}}_0 = (1-\alpha) \left(\frac{\langle\hat{\sigma}_+\rangle^+}{(\bar{\sigma}^+)^2}\mathbf{\underline{v}}_0 - \mathbf{\underline{v}}^- \right), \quad L_1 = \hat{\rho}_2h_0\hat{\sigma}_1 + 9\mu^2\bar{a}_2\bar{w}\Delta\gamma(1 + \hat{\rho}_1\hat{t}_+^{\text{tr}}). \quad (\text{C.2.14})$$

Finally, solving the linear algebraic expression of Eq. (C.2.13) for $\underline{\Delta\gamma}'$ gives

$$\underline{\Delta\gamma}' = \frac{1}{(\bar{f}_2 - L_1)} \underline{l}_0. \quad (\text{C.2.15})$$

Derivative $\frac{\partial \underline{H}}{\partial \underline{\kappa}}$

In this subsection, the derivatives are taken with respect to hardening vector κ . First, using Eq. (2.3.104), the derivative of relation $\varphi^\pm = \theta_1^\pm \theta_2^\pm$ is given by $\frac{\partial \varphi^\pm}{\partial \kappa^\pm} = \theta_1^\pm \frac{\partial \theta_2^\pm}{\partial \kappa^\pm} = \frac{1}{g^\pm} \theta_1^\pm J_\kappa^\pm$. Then, the derivative of the relations $h^\pm = \varphi^\pm \hat{n}^\pm$ are given by

$$\frac{\partial h^\pm}{\partial \kappa^\pm} = \hat{n}^\pm \frac{\partial \varphi^\pm}{\partial \kappa^\pm} = \frac{1}{g^\pm} \hat{n}^\pm \theta_1^\pm J_\kappa^\pm = \hat{b}_{10}^\pm. \quad (\text{C.2.16})$$

Thus, the derivative of matrix $\underline{H}$ is given by

$$\frac{\partial \underline{H}}{\partial \underline{\kappa}} = \underline{U} = \text{diag} \left(\hat{b}_{10}^+, \hat{b}_{10}^- \right). \quad (\text{C.2.17})$$

APPENDICES D. DETAILED CALCULATION OF SOME DERIVATIVES OF CONCRETE MODELS FOR PLANE STRESS FORMULATION

1. DETAILED CALCULATION OF DERIVATIVE $\frac{\partial F}{\partial \Delta\gamma}$ FOR THE DPH MODEL

This appendix present the steps necessities to determinate the derivative $\frac{\partial F}{\partial \Delta\gamma}$ expressed in Eq. (3.2.93) and required to calculate the consistency operator $\Delta\gamma$ for the DPH model. For the sake of simplicity, is omitted the subscript ' $_{n+1}$ ' in all variables hereafter.

Derivative $\frac{\partial z}{\partial \Delta\gamma}$ and $\frac{\partial \sigma}{\partial \Delta\gamma}$

In this subsection, the derivatives are taken with respect to variable $\Delta\gamma$. First, the derivative of variable $q = \sqrt{\frac{3}{2}}\sqrt{z}$ and $r = \sqrt{q^2 + \epsilon^2}$ are expressed as

$$\frac{\partial q}{\partial \Delta\gamma} = \frac{3}{4q} \frac{\partial z}{\partial \Delta\gamma}, \quad \frac{\partial r}{\partial \Delta\gamma} = \frac{3}{4r} \frac{\partial z}{\partial \Delta\gamma}. \quad (\text{D.1.1})$$

Next, using Eq. (3.2.89), the variable $z = \underline{\sigma}^T \underline{P} \underline{\sigma}$ can be expressed as

$$\begin{aligned} z &= \underline{\tau}^{\text{tr} T} \underline{\bar{B}}^2 \underline{\bar{P}} \underline{\tau}^{\text{tr}} \\ &= \frac{1}{3} \bar{b}_1^2 (\tau_{11}^{\text{tr}})^2 + \bar{b}_2^2 [(\tau_{22}^{\text{tr}})^2 + 2(\tau_{12}^{\text{tr}})^2] \\ &= \frac{1}{3} g_1^2 \bar{a}_1^2 + g_2 \bar{a}_2^2, \end{aligned} \quad (\text{D.1.2})$$

where $g_1 = \tau_{11}^{\text{tr}} - \sqrt{2}\lambda\bar{\eta}\Delta\gamma$ and $g_2 = (\tau_{22}^{\text{tr}})^2 + 2(\tau_{12}^{\text{tr}})^2$. Now, the derivative of variables $\bar{a}_1, \bar{a}_2$ (Eq. (3.2.88)), g_1 and g_2 are given by

$$\begin{aligned} \frac{\partial \bar{a}_1}{\partial \Delta\gamma} &= -\lambda \bar{a}_1^2 \left(u \Delta\gamma \frac{\partial z}{\partial \Delta\gamma} + t \right), & \frac{\partial \bar{a}_2}{\partial \Delta\gamma} &= -2\mu \bar{a}_2^2 \left(u \Delta\gamma \frac{\partial z}{\partial \Delta\gamma} + t \right), \\ \frac{\partial g_1}{\partial \Delta\gamma} &= -\sqrt{2}\bar{\eta}\lambda, & \frac{\partial g_2}{\partial \Delta\gamma} &= 0, \end{aligned} \quad (\text{D.1.3})$$

where $u = -\frac{9}{8r^3}$. Thus, using these expressions, the derivative of the variable z is written as

$$\begin{aligned}\frac{\partial z}{\partial \Delta\gamma} &= \frac{2}{3} \left(\bar{a}_1^2 g_1 \frac{\partial g_1}{\partial \Delta\gamma} + g_1^2 \bar{a}_1 \frac{\partial \bar{a}_1}{\partial \Delta\gamma} \right) + 2g_2 \bar{a}_2 \frac{\partial \bar{a}_2}{\partial \Delta\gamma} + \bar{a}_2^2 \frac{\partial g_2}{\partial \Delta\gamma} \\ &= -2 \left(\psi t + \frac{\sqrt{2}}{3} \bar{\eta} \lambda \bar{a}_1^2 g_1 + \frac{\partial z}{\partial \Delta\gamma} u \psi \Delta\gamma \right),\end{aligned}\quad (\text{D.1.4})$$

where $\psi = \frac{1}{3} \lambda \bar{a}_1^3 g_1^2 + 2\mu \bar{a}_2^3 g_2$. Then, solving for this expressoin the derivative of z gives

$$\frac{\partial z}{\partial \Delta\gamma} = -\frac{\psi t + \frac{\sqrt{2}}{3} \bar{\eta} \lambda \bar{a}_1^2 g_1}{\frac{1}{2} + \psi u \Delta\gamma}.\quad (\text{D.1.5})$$

Finally, using the relation Eq. (3.2.89), the derivative of stress vector is given by

$$\frac{\partial \underline{\sigma}}{\partial \Delta\gamma} = \underline{\mathbf{Q}}^T \frac{\partial \underline{\bar{\mathbf{B}}}}{\partial \Delta\gamma} \underline{\tau}_{n+1}^{\text{tr}},\quad (\text{D.1.6})$$

where $\frac{\partial \underline{\bar{\mathbf{B}}}}{\partial \Delta\gamma}$ is given by

$$\begin{aligned}\frac{\partial \underline{\bar{\mathbf{B}}}}{\partial \Delta\gamma} &= \text{diag} \left(\frac{\partial \bar{b}_1}{\partial \Delta\gamma}, \frac{\partial \bar{b}_2}{\partial \Delta\gamma}, \frac{\partial \bar{b}_2}{\partial \Delta\gamma} \right) \\ &= \text{diag} \left\{ -\frac{\lambda \bar{a}_1}{\tau_{11}^{\text{tr}}} \left[\sqrt{2} \bar{\eta} + g_1 \bar{a}_1 \left(u \Delta\gamma \frac{\partial z}{\partial \Delta\gamma} + t \right) \right], \frac{\partial \bar{a}_2}{\partial \Delta\gamma}, \frac{\partial \bar{a}_2}{\partial \Delta\gamma} \right\}.\end{aligned}$$

Derivative $\frac{\partial F}{\partial \Delta\gamma}$

First, using the relation Eq. (D.1.6), the derivative of hydrostatic pressure p is given by

$$\frac{\partial p}{\partial \Delta\gamma} = \frac{1}{3} \underline{\mathbf{1}}^T \frac{\partial \underline{\sigma}}{\partial \Delta\gamma} = \frac{\sqrt{2}}{3} \frac{\partial \bar{b}_1}{\partial \Delta\gamma} \tau_{11}^{\text{tr}}.\quad (\text{D.1.7})$$

For the other hand, using Eq. (3.2.92) and the chain rule, the derivative of cohesion law is given by

$$\frac{\partial c}{\partial \Delta\gamma} = \frac{\partial c}{\partial \alpha} \frac{\partial \alpha}{\partial \Delta\gamma} = J_\alpha \xi,\quad (\text{D.1.8})$$

where $J_\alpha \frac{\partial c}{\partial \alpha}$. Finally, using these relations and Eq. (D.1.1), the derivative of yield criterion F , given by Eq. (3.2.93), can be written as

$$\frac{\partial F}{\partial \Delta\gamma} = \frac{\sqrt{2}\eta}{3} \frac{\partial \bar{b}_1}{\partial \Delta\gamma} \tau_{11}^{\text{tr}} + \frac{3}{4q} \frac{\partial z}{\partial \Delta\gamma} - \xi^2 J_\alpha. \quad (\text{D.1.9})$$

2. DETAILED CALCULATION OF DERIVATIVE $\frac{\partial \bar{F}}{\partial \Delta\gamma}$ FOR THE LLF AND WLF MODELS

This appendix present the steps necessities to determinate the derivative $\frac{\partial \bar{F}}{\partial \Delta\gamma}$ expressed in Eq. (3.2.115) required to calculate the consistency operator $\Delta\gamma$ for the LLF and WLF models. Specifically, they are developed the derivatives for the LLF model, due that is more general than the WLF model. For the sake of simplicity, is omitted the subscript ' $_{n+1}$ ' in all variables hereafter.

Derivative $\frac{\partial \bar{z}}{\partial \Delta\gamma}$ and $\frac{\partial \hat{\sigma}}{\partial \Delta\gamma}$

In this subsection, the derivatives are taken with respect to variable $\Delta\gamma$. First, the derivative of variable $\bar{q} = \sqrt{\frac{3}{2}}\sqrt{\bar{z}}$ and $\bar{r} = \sqrt{\bar{q}^2 + \epsilon^2}$ are expressed as

$$\frac{\partial \bar{q}}{\partial \Delta\gamma} = \frac{3}{4\bar{q}} \frac{\partial \bar{z}}{\partial \Delta\gamma}, \quad \frac{\partial \bar{r}}{\partial \Delta\gamma} = \frac{3}{4\bar{r}} \frac{\partial \bar{z}}{\partial \Delta\gamma}. \quad (\text{D.2.1})$$

Next, using Eq. (3.2.105), its convenient to express the variable $\bar{z} = \hat{\sigma}^T \underline{\hat{P}} \hat{\sigma}$ as follows

$$\begin{aligned} \bar{z} &= \hat{\tau}^{\text{tr} T} \underline{\hat{B}}^2 \underline{\hat{P}} \hat{\tau}^{\text{tr}} \\ &= \frac{1}{3} \hat{b}_1^2 (\hat{\tau}_1^{\text{tr}})^2 + \hat{b}_2^2 (\hat{\tau}_2^{\text{tr}})^2 \\ &= \frac{1}{3} \hat{g}_1^2 \hat{a}_1^2 + \hat{g}_2^2 \hat{a}_2^2, \end{aligned} \quad (\text{D.2.2})$$

where $\hat{g}_1 = \hat{\tau}_1^{\text{tr}} - \sqrt{2}\lambda\bar{\eta}\Delta\gamma$ and $\hat{g}_2 = (\hat{\tau}_2^{\text{tr}})^2$. Now, the derivative of variables $\hat{a}_1, \hat{a}_2$ (Eq. (3.2.104)), $\hat{g}_1$ and $\hat{g}_2$ are expressed as

$$\begin{aligned}\frac{\partial \hat{a}_1}{\partial \Delta\gamma} &= -\lambda \hat{a}_1^2 \left(\bar{u} \Delta\gamma \frac{\partial \bar{z}}{\partial \Delta\gamma} + \bar{t} \right), & \frac{\partial \hat{a}_2}{\partial \Delta\gamma} &= -2\mu \hat{a}_2^2 \left(\bar{u} \Delta\gamma \frac{\partial \bar{z}}{\partial \Delta\gamma} + \bar{t} \right), \\ \frac{\partial \hat{g}_1}{\partial \Delta\gamma} &= -\sqrt{2}\bar{\eta}\lambda, & \frac{\partial \hat{g}_2}{\partial \Delta\gamma} &= 0,\end{aligned}\quad (\text{D.2.3})$$

where $\bar{u} = -\frac{9}{8r^3}$. Thus, using these relations, the derivative of the variable $\bar{z}$ is expressed as

$$\begin{aligned}\frac{\partial \bar{z}}{\partial \Delta\gamma} &= \frac{2}{3} \left(\hat{a}_1^2 \hat{g}_1 \frac{\partial \hat{g}_1}{\partial \Delta\gamma} + \hat{g}_1^2 \hat{a}_1 \frac{\partial \hat{a}_1}{\partial \Delta\gamma} \right) + 2\hat{g}_2 \hat{a}_2 \frac{\partial \hat{a}_2}{\partial \Delta\gamma} \\ &= -2 \left(\hat{\psi} \bar{t} + \frac{\sqrt{2}}{3} \bar{\eta} \lambda \hat{a}_1^2 \hat{g}_1 + \frac{\partial \bar{z}}{\partial \Delta\gamma} \bar{u} \hat{\psi} \Delta\gamma \right),\end{aligned}\quad (\text{D.2.4})$$

where $\hat{\psi} = \frac{1}{3} \lambda \hat{a}_1^3 \hat{g}_1^2 + 2\mu \hat{a}_2^3 \hat{g}_2$. Then, solving this expression for the derivative of $\bar{z}$ gives

$$\frac{\partial \bar{z}}{\partial \Delta\gamma} = -\frac{\hat{\psi} \bar{t} + \frac{\sqrt{2}}{3} \bar{\eta} \lambda \hat{a}_1^2 \hat{g}_1}{\frac{1}{2} + \hat{\psi} \bar{u} \Delta\gamma}, \quad (\text{D.2.5})$$

where $\frac{\partial \hat{\underline{\mathbf{B}}}}{\partial \Delta\gamma}$ is given by

$$\begin{aligned}\frac{\partial \hat{\underline{\mathbf{B}}}}{\partial \Delta\gamma} &= \text{diag} \left(\frac{\partial \hat{b}_1}{\partial \Delta\gamma}, \frac{\partial \hat{b}_2}{\partial \Delta\gamma} \right) \\ &= \text{diag} \left\{ -\frac{\lambda \hat{a}_1}{\hat{\tau}_1^{\text{tr}}} \left[\sqrt{2}\bar{\eta} + \hat{g}_1 \hat{a}_1 \left(\bar{u} \Delta\gamma \frac{\partial \bar{z}}{\partial \Delta\gamma} + \bar{t} \right) \right], \frac{\partial \hat{a}_2}{\partial \Delta\gamma} \right\}.\end{aligned}\quad (\text{D.2.6})$$

Then, using Eq. (3.2.105), the derivative of principal effective stress vector $\hat{\underline{\boldsymbol{\sigma}}}$ is given by

$$\frac{\partial \hat{\underline{\boldsymbol{\sigma}}}}{\partial \Delta\gamma} = \hat{\underline{\mathbf{Q}}}^T \frac{\partial \hat{\underline{\mathbf{B}}}}{\partial \Delta\gamma} \hat{\underline{\mathbf{r}}}^{\text{tr}}. \quad (\text{D.2.7})$$

Moreover, the derivative of maximum principal effective stress $\hat{\sigma}_+ = \hat{\mathbf{1}}^T \hat{\underline{\sigma}}$ is written as

$$\frac{\partial \hat{\sigma}_+}{\partial \Delta \gamma} = \frac{\sqrt{2}}{2} \left(\hat{\tau}_1^{\text{tr}} \frac{\partial \hat{b}_1}{\partial \Delta \gamma} - \hat{\tau}_2^{\text{tr}} \frac{\partial \hat{b}_2}{\partial \Delta \gamma} \right). \quad (\text{D.2.8})$$

Derivative $\frac{\partial F}{\partial \Delta \gamma}$

First, using Eqs. (3.2.105), (D.2.1) and (D.2.7), the derivative of principal effective flow vector $\hat{\underline{n}}$, given by Eq. (3.2.98), is expressed as

$$\frac{\partial \hat{\underline{n}}}{\partial \Delta \gamma} = \frac{3}{2\bar{r}^3} \hat{\underline{P}} \hat{\underline{Q}}^T \left(\bar{r}^2 \frac{\partial \hat{\underline{B}}}{\partial \Delta \gamma} - \frac{3}{4} \frac{\partial \bar{z}}{\partial \Delta \gamma} \hat{\underline{B}} \right) \hat{\underline{\tau}}^{\text{tr}}. \quad (\text{D.2.9})$$

Then, their positive/negative part can be expressed as $\frac{\partial \hat{n}^\pm}{\partial \Delta \gamma} = \hat{\mathbf{1}}_\pm^T \frac{\partial \hat{\underline{n}}}{\partial \Delta \gamma}$. For other hand, using the chain rule, the derivative of variable ϕ (Eq. (3.1.18)) is expressed as $\frac{\partial \phi}{\partial \Delta \gamma} = \hat{\underline{\Phi}}^T \frac{\partial \hat{\underline{\sigma}}}{\partial \Delta \gamma}$, where $\hat{\underline{\Phi}}$ is defined by Eq. (3.3.163). Also, using the chain rule, the derivative of variables $\theta_1^\pm$ and $\theta_2^\pm$ (Eqs. (3.2.111) and (3.2.112)) are given by $\frac{\partial \theta_1^\pm}{\partial \Delta \gamma} = \frac{\partial \phi}{\partial \Delta \gamma}$ and $\frac{\partial \theta_2^\pm}{\partial \Delta \gamma} = \frac{1}{g^\pm} \frac{\partial \sigma^\pm}{\partial \kappa^\pm} \frac{\partial \kappa^\pm}{\partial \Delta \gamma}$, respectively. Then, the derivative of the variable φ (Eq. (3.2.110)) are expressed as

$$\frac{\partial \varphi^\pm}{\partial \Delta \gamma} = \theta_2^\pm \hat{\underline{\Phi}}^T \frac{\partial \hat{\underline{\sigma}}}{\partial \Delta \gamma} + \frac{1}{g^\pm} \theta_1^\pm J_\kappa^\pm \frac{\partial \kappa^\pm}{\partial \Delta \gamma}, \quad (\text{D.2.10})$$

where $J_\kappa^\pm := \frac{\partial \sigma^\pm}{\partial \kappa^\pm}$. Hence, using these obtained relations, the derivative of the variable $h^\pm = \hat{n}^\pm \varphi^\pm$ (Eq. (3.2.109)) is given by

$$\frac{\partial h^\pm}{\partial \Delta \gamma} = \hat{n}^\pm \theta_2^\pm \hat{\underline{\Phi}}^T \frac{\partial \hat{\underline{\sigma}}}{\partial \Delta \gamma} + \hat{b}_{10}^\pm \frac{\partial \kappa^\pm}{\partial \Delta \gamma} + b_2^\pm, \quad (\text{D.2.11})$$

with $\hat{b}_{10}^\pm = \frac{1}{g^\pm} \hat{n}^\pm \theta_1^\pm J_\kappa^\pm$ and $b_2^\pm = \varphi^\pm \frac{\partial \hat{n}^\pm}{\partial \Delta \gamma}$. Consequently, using this relation, the derivatives of hardening variables $\kappa^\pm$ (Eq. (3.2.109)) are expressed as

$$\frac{\partial \kappa^\pm}{\partial \Delta \gamma} = h^\pm + \Delta \gamma \left(\hat{b}_{40}^\pm + \hat{b}_{10}^\pm \frac{\partial \kappa^\pm}{\partial \Delta \gamma} \right), \quad (\text{D.2.12})$$

with $\hat{b}_{40}^\pm = b_2^\pm + \hat{n}^\pm \theta_2^\pm \hat{\Phi}^T \frac{\partial \hat{\sigma}}{\partial \Delta\gamma}$. Solving this equation for the derivative $\frac{\partial \kappa^\pm}{\partial \Delta\gamma}$ gives

$$\frac{\partial \kappa^\pm}{\partial \Delta\gamma} = \hat{b}_3^\pm + \Delta\gamma \hat{b}_4^\pm, \quad (\text{D.2.13})$$

where $\hat{b}_3^\pm = \hat{b}_{20}^\pm h^\pm$ and $\hat{b}_4^\pm = \hat{b}_{20}^\pm \hat{b}_{40}^\pm$, with $\hat{b}_{20}^\pm = \left(1 - \Delta\gamma \hat{b}_{10}^\pm\right)^{-1}$. Then, using this relation and the chain rule, the derivatives for the positive/negative uniaxial effective stress law $\bar{\sigma}^\pm$ are expressed as

$$\frac{\partial \bar{\sigma}^\pm}{\partial \Delta\gamma} = \frac{\partial \bar{\sigma}^\pm}{\partial \kappa^\pm} \frac{\partial \kappa^\pm}{\partial \Delta\gamma} = b_3^\pm + \Delta\gamma b_4^\pm, \quad (\text{D.2.14})$$

where $b_3^\pm = \bar{J}_\kappa^\pm \hat{b}_3^\pm$ and $b_4^\pm = \bar{J}_\kappa^\pm \hat{b}_4^\pm$, with $\bar{J}_\kappa^\pm := \frac{\partial \bar{\sigma}^\pm}{\partial \kappa^\pm}$ their effective hardening modulus. Moreover, using this relation, the derivative of updated variable β (Eq. (3.2.113)) is written as

$$\frac{\partial \beta}{\partial \Delta\gamma} = b_5 + \Delta\gamma b_6, \quad (\text{D.2.15})$$

where $b_5 = \frac{(1-\alpha)}{(\bar{\sigma}^-)^2} (\bar{\sigma}^+ b_3^- - \bar{\sigma}^- b_3^+)$ and $b_6 = \frac{(1-\alpha)}{(\bar{\sigma}^-)^2} (\bar{\sigma}^+ b_4^- - \bar{\sigma}^- b_4^+)$. Then, the derivative of variable $\hat{\beta}_1 = \beta \tilde{H}^+(\hat{\sigma}_+)$ is given by

$$\frac{\partial \hat{\beta}_1}{\partial \Delta\gamma} = \hat{b}_7 + \Delta\gamma \hat{b}_8, \quad (\text{D.2.16})$$

where $\hat{b}_7 = b_5 \tilde{H}^+(\hat{\sigma}_+) + \hat{\beta}_2 \frac{\partial \hat{\beta}_1}{\partial \Delta\gamma}$ and $\hat{b}_8 = b_6 \tilde{H}^+(\hat{\sigma}_+)$, with $\hat{\beta}_2 = \beta \frac{d\tilde{H}^+}{d\hat{\sigma}_+}$. For the other hand, using the relation Eq. (D.2.7), the derivative of hydrostatic pressure $\bar{p}$ is given by

$$\frac{\partial \bar{p}}{\partial \Delta\gamma} = \frac{1}{3} \hat{\mathbf{1}}^T \frac{\partial \hat{\sigma}}{\partial \Delta\gamma} = \frac{\sqrt{2}}{3} \frac{\partial \hat{b}_1}{\partial \Delta\gamma} \hat{\tau}_1^{\text{tr}}. \quad (\text{D.2.17})$$

Finally, using the relations Eqs. (D.2.1), (D.2.8), (D.2.14), (D.2.16) and (D.2.17), the derivative of residual function $\bar{F}$ of expression Eq. (3.2.114), can be written as

$$\begin{aligned} \frac{\partial \bar{F}}{\partial \Delta\gamma} &= \sqrt{2} \hat{\tau}_1^{\text{tr}} \frac{\partial \hat{b}_1}{\partial \Delta\gamma} \left(\frac{\eta}{3} + \frac{\hat{\beta}_1}{2} \right) - \frac{\sqrt{2}}{2} \hat{\beta}_1 \hat{\tau}_2^{\text{tr}} \frac{\partial \hat{b}_2}{\partial \Delta\gamma} + \frac{3}{4\bar{q}} \frac{\partial \bar{z}}{\partial \Delta\gamma} \\ &\quad + \hat{\sigma}_+ (\hat{b}_7 + \Delta\gamma \hat{b}_8) - (1-\alpha) (b_3^- + \Delta\gamma b_4^-) \end{aligned} \quad (\text{D.2.18})$$

It should be noted that, setting the variables $\theta_2^\pm = 1$ and $\hat{b}_{10}^\pm = 0$, it can obtain the expressions for derivatives of the WLF model.

3. DETAILED CALCULATION OF DERIVATIVES TO SOLVE κ FOR THE LLF AND WLF MODELS

This appendix present the steps necessary to determinate the derivatives expressed in Eq. (3.2.118) and used to calculate the hardening vector $\underline{\kappa}_{n+1}$. For simplicity of the presentation, there are omitted the subscript ' $_{n+1}$ ' in all variables hereafter.

Derivative $\frac{\partial H}{\partial \hat{\underline{\sigma}}}$

In this subsection, the derivatives are taken with respect to updated principal effective stress vector $\hat{\underline{\sigma}}$. First, using Eq. (A.2.18), the derivative of variables $\bar{q} = \sqrt{\bar{J}_2}$ and $\bar{r} = \sqrt{\bar{q}^2 + \epsilon^2}$ can be expressed as

$$\frac{\partial \bar{q}}{\partial \hat{\underline{\sigma}}} = \frac{3}{2\bar{q}} \hat{\underline{v}}_{dev}, \quad \frac{\partial \bar{r}}{\partial \hat{\underline{\sigma}}} = \frac{3}{2\bar{r}} \hat{\underline{v}}_{dev}, \quad (\text{D.3.1})$$

with $\hat{\underline{v}}_{dev} = \hat{\underline{P}} \hat{\underline{\sigma}}$. Then, using these relations, the derivative of principal effective flow vector $\hat{\underline{n}}$ (Eq. (3.2.98)) is given by

$$\frac{\partial \hat{\underline{n}}}{\partial \hat{\underline{\sigma}}} = \frac{3}{2\bar{r}^3} \left(\bar{r}^2 \hat{\underline{P}} - \frac{3}{2} \hat{\underline{v}}_{dev} \otimes \hat{\underline{v}}_{dev} \right). \quad (\text{D.3.2})$$

Next, using the relation $\frac{\partial \theta_1^\pm}{\partial \hat{\underline{\sigma}}} = \frac{\partial \phi}{\partial \hat{\underline{\sigma}}} = \hat{\underline{\Phi}}$ (Eq. (3.3.163)) and $\frac{\partial \theta_2^\pm}{\partial \hat{\underline{\sigma}}} = 0$, the derivative of variable $\varphi^\pm$ (Eq. (3.2.110)), is written as $\frac{\partial \varphi^\pm}{\partial \hat{\underline{\sigma}}} = \theta_2^\pm \hat{\underline{\Phi}}$. Thus, the derivative of variable $h^\pm = \varphi^\pm \hat{n}^\pm$ (Eq. (3.2.109)) is given by

$$\frac{\partial h^\pm}{\partial \hat{\underline{\sigma}}} = \hat{n}^\pm \theta_2^\pm \hat{\underline{\Phi}} + \varphi^\pm \frac{\partial \hat{n}^\pm}{\partial \hat{\underline{\sigma}}}. \quad (\text{D.3.3})$$

Hence, rewritten this relation, the derivative of matrix $\underline{\underline{H}}$ can be expressed as

$$\frac{\partial \underline{\underline{H}}}{\partial \hat{\underline{\underline{\sigma}}}} = \hat{\underline{\underline{y}}} \otimes \hat{\underline{\underline{\Phi}}} + \hat{\underline{\underline{Z}}} \otimes \frac{\partial \hat{\underline{\underline{n}}}}{\partial \hat{\underline{\underline{\sigma}}}}, \quad (\text{D.3.4})$$

where $\hat{\underline{\underline{y}}}$ and $\hat{\underline{\underline{Z}}}$ are given by

$$\hat{\underline{\underline{y}}} = \begin{bmatrix} \hat{n}^+ \theta_2^+ \\ \hat{n}^- \theta_2^- \end{bmatrix}, \quad \hat{\underline{\underline{Z}}} = \text{diag}(\varphi^+, \varphi^-).$$

Derivative $\frac{\partial \Delta \gamma}{\partial \underline{\underline{\kappa}}}$

In this subsection, the derivatives are taken with respect to hardening vector $\underline{\underline{\kappa}}$. Also, the variable $\underline{\underline{\Delta \gamma'}}$ is denoted as the derivative $\frac{\partial \Delta \gamma}{\partial \underline{\underline{\kappa}}}$. First, the derivatives of uniaxial effective stress $\bar{\sigma}^\pm(\kappa^\pm)$ are expressed as

$$\frac{\partial \bar{\sigma}^\pm}{\partial \underline{\underline{\kappa}}} := \underline{\underline{v}}^\pm, \quad \underline{\underline{v}}^+ = \begin{bmatrix} \bar{J}_\kappa^+ \\ 0 \end{bmatrix}, \quad \underline{\underline{v}}^- = \begin{bmatrix} 0 \\ \bar{J}_\kappa^- \end{bmatrix}, \quad (\text{D.3.5})$$

with $\bar{J}_\kappa^\pm := \frac{\partial \bar{\sigma}^\pm}{\partial \kappa^\pm}$. Next, using the chain rule, the derivative of the principal effective stress tensor can be expressed as

$$\frac{\partial \hat{\underline{\underline{\sigma}}}}{\partial \underline{\underline{\kappa}}} = \frac{\partial \hat{\underline{\underline{\sigma}}}}{\partial \Delta \gamma} \otimes \underline{\underline{\Delta \gamma'}}. \quad (\text{D.3.6})$$

Hence, the derivative of the maximum principal effective stress is written as $\frac{\partial \hat{\sigma}_+}{\partial \underline{\underline{\kappa}}} = \frac{\partial \hat{\sigma}_+}{\partial \Delta \gamma} \underline{\underline{\Delta \gamma'}}$. In the same way, using Eqs. (D.2.1), (D.2.7) and (D.3.6), the derivative of variables $\bar{p}$ and $\bar{q}$ are written, respectively, as

$$\frac{\partial \bar{p}}{\partial \underline{\underline{\kappa}}} = \frac{1}{3} \hat{\underline{\underline{1}}}^T \frac{\partial \hat{\underline{\underline{\sigma}}}}{\partial \underline{\underline{\kappa}}} \otimes \underline{\underline{\Delta \gamma'}} = \frac{\sqrt{2}}{3} \frac{\partial \hat{b}_1}{\partial \Delta \gamma} \hat{r}^{\text{tr}} \underline{\underline{\Delta \gamma'}}, \quad (\text{D.3.7})$$

$$\frac{\partial \bar{q}}{\partial \underline{\underline{\kappa}}} = \frac{\partial \bar{q}}{\partial \Delta \gamma} \underline{\underline{\Delta \gamma'}} = \frac{3}{4\bar{q}} \frac{\partial \bar{z}}{\partial \Delta \gamma} \underline{\underline{\Delta \gamma'}} \quad (\text{D.3.8})$$

For other hand, using Eq. (D.3.5), the derivative of variable β (Eq. (3.2.113)) is given by

$$\frac{\partial \beta}{\partial \underline{\kappa}} = \frac{(1 - \alpha)}{(\bar{\sigma}^+)^2} \underline{v}_0, \quad \underline{v}_0 = \bar{\sigma}^+ \underline{v}^- - \bar{\sigma}^- \underline{v}^+ = \begin{bmatrix} -\bar{\sigma}^- \bar{J}_\kappa^+ \\ \bar{\sigma}^+ \bar{J}_\kappa^- \end{bmatrix} \quad (\text{D.3.9})$$

Then, the derivative of variable $\hat{\beta}_1 = \beta \tilde{H}^+(\hat{\sigma}_+)$ is given by

$$\frac{\partial \hat{\beta}_1}{\partial \underline{\kappa}} = \tilde{H}^+(\hat{\sigma}_+) \frac{(1 - \alpha)}{(\bar{\sigma}^+)^2} \underline{v}_0 + \hat{\beta}_2 \frac{\partial \hat{\sigma}_+}{\partial \Delta \gamma} \underline{\Delta \gamma}', \quad (\text{D.3.10})$$

where $\hat{\beta}_2 = \beta \frac{d\tilde{H}^+}{d\hat{\sigma}_+}$. Finally, using all these expressions and the relation $\beta \langle \hat{\sigma}_+ \rangle^+ = \hat{\beta}_1 \hat{\sigma}_+$, the derivative of the yield criterion $\bar{F}$ at consistency condition (Eq. (3.2.114)) can be written as

$$\frac{\partial \bar{F}}{\partial \underline{\kappa}} = L_1 \underline{\Delta \gamma}' + \underline{l}_0 = 0, \quad (\text{D.3.11})$$

where $\underline{l}_0$ and L_1 are given by

$$\begin{aligned} \underline{l}_0 &= (1 - \alpha) \left(\frac{\langle \hat{\sigma}_1 \rangle^+}{(\bar{\sigma}^+)^2} \underline{v}_0 - \underline{v}^- \right), \\ L_1 &= \sqrt{2} \hat{\tau}_1^{\text{tr}} \frac{\partial \hat{b}_1}{\partial \Delta \gamma} \left(\frac{\eta}{3} + \frac{\hat{\beta}_3}{2} \right) - \frac{\sqrt{2}}{2} \hat{\beta}_3 \hat{\tau}_2^{\text{tr}} \frac{\partial \hat{b}_2}{\partial \Delta \gamma} + \frac{3}{4\bar{q}} \frac{\partial \bar{z}}{\partial \Delta \gamma}. \end{aligned}$$

Thus, solving the derivative of discrete consistency operator $\underline{\Delta \gamma}'$ gives

$$\underline{\Delta \gamma}' = -\frac{1}{L_1} \underline{l}_0. \quad (\text{D.3.12})$$

Derivative $\frac{\partial \underline{H}}{\partial \underline{\kappa}}$

In this subsection, the derivatives are taken with respect to hardening vector $\underline{\kappa}$. First, using the chain rule, the derivative of variable $\varphi^\pm$ (Eq. (3.2.110)) is given by $\frac{\partial \varphi^\pm}{\partial \kappa^\pm} = \theta_1^\pm \frac{\partial \theta_2^\pm}{\partial \kappa^\pm} = \frac{1}{g^\pm} \theta_1^\pm J_\kappa^\pm$. Then, the derivative of relation $h^\pm = \varphi^\pm \hat{n}^\pm$ (Eq. (3.2.109)) is given by

$$\frac{\partial h^\pm}{\partial \kappa^\pm} = \hat{n}^\pm \frac{\partial \varphi^\pm}{\partial \kappa^\pm} = \frac{1}{g^\pm} \hat{n}^\pm \theta_1^\pm J_\kappa^\pm = \hat{b}_{10}^\pm. \quad (\text{D.3.13})$$

Thus, the derivative of matrix $\underline{\mathbf{H}}$ is given by

$$\frac{\partial \underline{\mathbf{H}}}{\partial \underline{\boldsymbol{\kappa}}} = \underline{\mathbf{U}} = \text{diag} \left(\hat{b}_{10}^+, \hat{b}_{10}^- \right). \quad (\text{D.3.14})$$

APPENDICES E. ALTERNATIVE DERIVATION OF CONSISTENT TANGENT STIFFNESS TENSOR FOR PLANE STRESS FORMULATION

1. ALTERNATIVE DERIVATION OF CONSISTENT TANGENT STIFFNESS MATRIX FOR THE DPH MODEL

This appendix present an alternative calculation for the consistent tangent stiffness matrix of the DPH model, which are based in the differential of updated stress vector. For simplicity of the presentation, there are omitted the subscript ' $_{n+1}$ ' in all variables hereafter.

First, using the relation of Eq. (A.2.21), the differential of updated flow vector, given by Eq. (3.2.82), is expressed as follows

$$d\mathbf{n} = \frac{3}{2r^3} \left[r^2 \mathbf{P} - \frac{3}{2} (\mathbf{v}_{dev} \otimes \mathbf{v}_{dev}) \right] d\mathbf{\sigma} = \mathbf{A}_0 d\mathbf{\sigma}. \quad (\text{E.1.1})$$

Then, using the relation $dp = \frac{1}{3} \mathbf{1}^T d\mathbf{\sigma}$ and Eqs. (3.3.145) and (A.2.21), the differential of yield criterion at consistency condition (Eq. (3.2.93)) is written as

$$dF = \frac{\eta}{3} \mathbf{1}^T d\mathbf{\sigma} + \frac{3}{2q} \mathbf{v}_{dev}^T d\mathbf{\sigma} - \xi^2 J_\alpha d\Delta\gamma = 0. \quad (\text{E.1.2})$$

Thus, the differential of discrete consistency operator $\Delta\gamma$ can be solved of this equation as follows

$$d\Delta\gamma = \frac{1}{\xi^2 J_\alpha} \left(\frac{3}{2q} \mathbf{v}_{dev}^T + \frac{\eta}{3} \mathbf{1}^T \right) d\mathbf{\sigma} = \frac{1}{\xi^2 J_\alpha} \mathbf{m}^T d\mathbf{\sigma}. \quad (\text{E.1.3})$$

Finally, substituting Eqs. (3.3.138) and (E.1.1) into Eq. (3.3.139), the differential of updated stress vector is written as

$$d\mathbf{\sigma} = \mathbf{D}_e \left\{ d\mathbf{\varepsilon}^{e\text{tr}} - \left[\Delta\gamma \mathbf{A}_0 + \frac{1}{\xi^2 J_\alpha} (\mathbf{n} \otimes \mathbf{m}) \right] d\mathbf{\sigma} \right\}. \quad (\text{E.1.4})$$

Then, solving of this expression the differential of stress vector, the consistent tangent stiffness matrix can be obtained as

$$\underline{\mathbf{D}}_{ep} = \left[\underline{\mathbf{C}}_{el} + \Delta\gamma \underline{\mathbf{A}}_0 + \frac{1}{\xi^2 J_\alpha} (\underline{\mathbf{n}} \otimes \underline{\mathbf{m}}) \right]^{-1}. \quad (\text{E.1.5})$$

It should be noted, that this matrix can be ill-conditioned when $J_\alpha = 0$ (perfectly elasto-plastic problem), being suggested the use of the consistent tangent operator as explained in section Section 3.3.

2. ALTERNATIVE CALCULATION OF CONSISTENT TANGENT STIFFNESS FOR THE LLF MODELS

This appendix present the steps necessary to determinate an alternative expression of the consistent tangent stiffness matrix for the LLF models. For simplicity of the presentation, there are omitted the subscript ' $n+1$ ' in all variables hereafter.

Plastic component

The plastic component of the consistent tangent stiffness matrix is calculated from differential of the effective stress vector. First, similar to Eq. (E.1.1), the differential of effective flow vector can be written as

$$d\bar{\underline{\mathbf{n}}} = \frac{3}{2\bar{r}^3} \left[\bar{r}^2 \underline{\mathbf{P}} - \frac{3}{2} (\bar{\underline{\mathbf{v}}}_{dev} \otimes \bar{\underline{\mathbf{v}}}_{dev}) \right] d\bar{\underline{\boldsymbol{\sigma}}} = \bar{\underline{\mathbf{A}}}_0 d\bar{\underline{\boldsymbol{\sigma}}}. \quad (\text{E.2.1})$$

Moreover, using Eq. (A.2.24), the differential of principal effective flow vector $\hat{\underline{\mathbf{n}}}$, given by Eq. (3.2.98), can be expressed as

$$d\hat{\underline{\mathbf{n}}} = \frac{3}{2\bar{r}^3} \left[\bar{r}^2 \hat{\underline{\mathbf{P}}} - \frac{3}{2} (\hat{\underline{\mathbf{v}}}_{dev} \otimes \hat{\underline{\mathbf{v}}}_{dev}) \right] d\hat{\underline{\boldsymbol{\sigma}}} = \hat{\underline{\mathbf{A}}}_0 d\hat{\underline{\boldsymbol{\sigma}}}. \quad (\text{E.2.2})$$

Then, their positive/negative part can be expressed as

$$d\hat{\underline{\mathbf{n}}}^\pm = \hat{\underline{\mathbf{a}}}_0^\pm d\hat{\underline{\boldsymbol{\sigma}}}, \quad (\text{E.2.3})$$

where $\hat{\underline{a}}_0^\pm = \hat{\underline{1}}_\pm^T \hat{\underline{A}}_0$. Next, using this relation and Eq. (3.3.166), the differential of hardening variables $\kappa^\pm$ (Eq. (3.2.109)) can be written as

$$d\kappa^\pm = h^\pm d\Delta\gamma + \Delta\gamma \left(\hat{\underline{c}}_0^\pm{}^T d\hat{\underline{\sigma}} + \hat{b}_{10}^\pm d\kappa^\pm \right), \quad (\text{E.2.4})$$

where $\hat{\underline{c}}_0^\pm = \hat{\theta}_2^\pm \hat{n}^\pm \hat{\underline{\Phi}} + \varphi^\pm \hat{\underline{a}}_0^\pm$. Hence, solving this linear equation for the differential of variable $\kappa^\pm$ gives

$$d\kappa^\pm = \Delta\gamma \hat{\underline{c}}_0^\pm{}^T d\hat{\underline{\sigma}}^{\text{tr}} + c_1^\pm d\Delta\gamma, \quad (\text{E.2.5})$$

where $\hat{\underline{c}}_0^\pm = \hat{b}_{20}^\pm \hat{\underline{c}}_0^\pm$ and $c_1^\pm = \hat{b}_{20}^\pm h^\pm$, with $\hat{b}_{20}^\pm = \left(1 - \Delta\gamma \hat{b}_{10}^\pm\right)^{-1}$. Then, using this relation, the differential of variable β (Eq. (3.2.113)) is given by

$$d\beta = c_{40} \Delta\gamma + \Delta\gamma \hat{\underline{c}}_{60}^T d\hat{\underline{\sigma}}^{\text{tr}}, \quad (\text{E.2.6})$$

where c_{40} and $\hat{\underline{c}}_{60}$ are expressed as

$$c_{40} = m^+ c_1^- - m^- c_1^+, \quad \hat{\underline{c}}_{60} = m^+ \hat{\underline{c}}_0^- - m^- \hat{\underline{c}}_0^+,$$

with $m^\pm = (1 - \alpha) \bar{J}_k^\mp \frac{\bar{\sigma}^\pm}{(\bar{\sigma}^\pm)^2}$. Next, using the relations $d\hat{\underline{1}}^T d\hat{\underline{\sigma}}, \beta \langle \hat{\sigma}_+ \rangle^+ = \hat{\beta}_1 \hat{\sigma}_+$, with $\hat{\beta}_1 = \beta \tilde{H}^+(\hat{\sigma}_+)$, and Eqs. (3.3.169), (A.2.18), (A.2.24), (E.2.5) and (E.2.8), the differential of yield criterion at consistency condition (Eq. (3.2.114)) is written as

$$\begin{aligned} dF &= \frac{\eta}{3} \hat{\underline{1}}^T d\hat{\underline{\sigma}} + \frac{3}{2\bar{q}} \hat{\underline{v}}_{dev} d\hat{\underline{\sigma}} + \hat{\sigma}_+ d\hat{\beta}_1 + \hat{\beta}_1 d\hat{\sigma}_+ - (1 - \alpha) d\bar{\sigma}^- = 0 \\ &= \hat{g}_1 d\Delta\gamma + \hat{\underline{g}}_1^T d\hat{\underline{\sigma}}, \end{aligned} \quad (\text{E.2.7})$$

where $\hat{g}_1$ and $\hat{\underline{g}}_1$ are expressed as

$$\begin{aligned} \hat{g}_1 &= \langle \hat{\sigma}_1 \rangle^+ c_{40} - (1 - \alpha) \bar{J}_k^- c_1^-, \\ \hat{\underline{g}}_1 &= \frac{\eta}{3} \hat{\underline{1}} + \frac{3}{2\bar{q}} \hat{\underline{v}}_{dev} + \Delta\gamma \left(\langle \hat{\sigma}_+ \rangle^+ \hat{\underline{c}}_{60} - (1 - \alpha) \bar{J}_k^- \hat{\underline{c}}_0^- \right) + \hat{\beta}_3 \hat{\underline{1}}_+, \end{aligned}$$

where $\hat{\beta}_3 = \hat{\beta}_2 \hat{\sigma}_+ + \hat{\beta}_1$, with $\hat{\beta}_2 = \beta \frac{d\hat{H}^+}{d\hat{\sigma}_+}$. Then, the differential of $\Delta\gamma$ can be solved as

$$d\Delta\gamma = -\frac{1}{\hat{g}_1} \hat{\underline{g}}_1^T d\hat{\underline{\sigma}} = \hat{\underline{g}}_1^T d\hat{\underline{\sigma}}. \quad (\text{E.2.8})$$

Thus, substituting this relation and $d\hat{\sigma} = F_{\bar{\sigma}} d\bar{\underline{\sigma}}$ (Eq. (A.2.22)) into Eq. (3.3.174), the differential of updated effective stress vector can be expressed as

$$d\bar{\underline{\sigma}} = \underline{\underline{D}}_e \left\{ \underline{\underline{C}}_e d\bar{\sigma}^{\text{tr}} - \left[\Delta\gamma \bar{\underline{A}}_0 + \left(\bar{\underline{n}} \otimes \hat{\underline{g}}_1 \right) \underline{\underline{F}}_{\bar{\sigma}} \right] d\bar{\sigma} \right\}. \quad (\text{E.2.9})$$

Finally, solving of this expression the updated effective stress vector and introducing Eq. (3.3.138), the effective elasto-plastic consistent tangent matrix can be written as

$$\bar{\underline{\underline{D}}}_{ep} = \left[\underline{\underline{C}}_e + \Delta\gamma \bar{\underline{A}}_0 + \left(\bar{\underline{n}} \otimes \hat{\underline{g}}_1 \right) \underline{\underline{F}}_{\bar{\sigma}} \right]^{-1}. \quad (\text{E.2.10})$$

Damage component

First, using Eq. (E.2.5), the differential of hardening vector $d\underline{\underline{\kappa}}$ is written as

$$d\underline{\underline{\kappa}} = \Delta\gamma \underline{\underline{C}}_0^T d\hat{\underline{\sigma}}^{\text{tr}} + \underline{\underline{c}}_1 d\Delta\gamma, \quad (\text{E.2.11})$$

where $\underline{\underline{C}}_0$ and $\underline{\underline{c}}_1$ are expressed as

$$\underline{\underline{C}}_0 = [\underline{\underline{c}}_0^+, \underline{\underline{c}}_0^-], \quad \underline{\underline{c}}_1 = \begin{bmatrix} c_1^+ \\ c_1^- \end{bmatrix}.$$

In addition, substituting Eqs. (3.3.177), (E.2.8) and (E.2.11), the relations $d\hat{\underline{\sigma}} = \underline{\underline{F}}_{\bar{\sigma}} d\bar{\underline{\sigma}}$, $d\bar{\sigma} = \bar{\underline{\underline{D}}}_{ep} : d\epsilon^{\text{tr}}$, with $\bar{\underline{\underline{D}}}_{ep}$ given by Eq. (E.2.10), into Eq. (3.3.179), the differential of damage variable ω can be rewritten as

$$d\omega = \left[\underline{\underline{v}}_1^T + \underline{\underline{v}}_2^T \left(\Delta\gamma \underline{\underline{C}}_0^T + \underline{\underline{c}}_1 \otimes \bar{\underline{g}}_1 \right) \right] \underline{\underline{F}}_{\bar{\sigma}} \bar{\underline{\underline{D}}}_{ep} d\bar{\underline{\epsilon}}^{\text{tr}} \quad (\text{E.2.12})$$

where $\underline{v}_1^T = \underline{u}_1^T \hat{\underline{M}}_1$ and $\underline{v}_2 = \underline{u}_2^T \hat{\underline{M}}_2$. Finally, introducing $d\bar{\underline{\sigma}} = \bar{\underline{D}}_{ep} d\bar{\underline{\epsilon}}^{e\text{ tr}}$ (Eq. (E.2.10)) into Eq. (3.3.181), the elasto-plastic-damage consistent tangent tensor is expressed as

$$\underline{D}_{epd} = \left[(1 - \omega) \underline{I} - (\bar{\underline{\sigma}} \otimes \underline{v}_1) \underline{F}_{\bar{\underline{\sigma}}} - (\bar{\underline{\sigma}} \otimes \underline{v}_2) \left(\Delta\gamma \underline{C}_0^T + \underline{c}_1 \otimes \hat{\underline{g}}_1 \right) \underline{F}_{\bar{\underline{\sigma}}} \right] \bar{\underline{D}}_{ep}. \quad (\text{E.2.13})$$

Viscous component

Substituting Eqs. (3.3.183), (3.3.185) and (E.2.12) into the differential of Eq. (3.2.125), the visco-plastic-damage consistent tangent matrix is expressed as

$$\begin{aligned} D_{vpd} = \zeta_v (1 - \omega^v) \underline{D}_e + (1 - \zeta_v) & \left[(1 - \omega^v) \underline{I} - (\bar{\underline{\sigma}}^v \otimes \underline{v}_1) \underline{F}_{\bar{\underline{\sigma}}} \right. \\ & \left. - (\bar{\underline{\sigma}}^v \otimes \underline{v}_2) \left(\Delta\gamma \underline{C}_0^T + \underline{c}_1 \otimes \hat{\underline{g}}_1 \right) \underline{F}_{\bar{\underline{\sigma}}} \right] \bar{\underline{D}}_{ep}. \end{aligned} \quad (\text{E.2.14})$$

APPENDICES F. MISCELANEOUS

1. CALCULATION OF PARAMETER F_O^- FOR THE UNIAXIAL COMPRESSION STRESS-STRAIN LAW OF MAZARS

This section detail the Newton's method used to calculate the parameters f_o^- and B^- of the exponential uniaxial compression stress-strain relation of (Mazars, 1984) given by Table Table 2.5.5. First, the peak strength of uniaxial stress law is given by

$$f_p = \frac{f_o^-}{B^-} e^{(B^- - 1)}, \quad (\text{F.1.1})$$

and their corresponding strain is expressed as $\varepsilon_p = \varepsilon_o / B^-$, with $\varepsilon_o = f_o^- / B^-$. Next, introducing the FE-regularization in uniaxial stress law according to compression fracture energy criterion stated in Section 2.5.2, the parameters B^- and the upper limit of characteristic length l_c are expressed in Table 2.5.6. Then, the parameter f_o^- is the variable to be solved, where the residual function and their total derivative are given by

$$R(f_o) = \frac{f_o^-}{B^-} e^{(B^- - 1)} - f_p, \quad \frac{dR}{df_o^-} = \frac{\partial R}{\partial f_o^-} + \frac{\partial R}{\partial B^-} \frac{\partial B^-}{\partial f_o^-}, \quad (\text{F.1.2})$$

where $\frac{\partial R}{\partial f_o^-}$, $\frac{\partial R}{\partial B^-}$ and $\frac{\partial B^-}{\partial f_o^-}$ are expressed as

$$\begin{aligned} \frac{\partial R}{\partial f_o^-} &= \frac{1}{B^-} e^{(B^- - 1)}, & \frac{\partial R}{\partial B^-} &= \frac{f_o^- (B^- - 1)}{(B^-)^2} e^{(B^- - 1)}, \\ \frac{\partial B^-}{\partial f_o^-} &= \begin{cases} -\frac{m_c \left(\hat{J}_c + \frac{1}{2} + \sqrt{\hat{J}_c + \frac{1}{4}} \right)}{2 \hat{J}_c^2 \sqrt{\hat{J}_c + \frac{1}{4}}}, & \text{if } g_{fA}^- \text{ is used} \\ -\frac{m_c \left(\hat{J}_c + \sqrt{\hat{J}_c - \frac{1}{4}} \right)}{2 \left(\hat{J}_c - \frac{1}{2} \right)^2 \sqrt{\hat{J}_c - \frac{1}{4}}}, & \text{if } g_{fC}^- \text{ is used,} \end{cases} \end{aligned}$$

with $m_c = -\frac{2G_f^- E_o}{l_c(f_o^-)^3}$ and $\hat{J}_c = \frac{G_f^- E_o}{f_t^2 l_c}$. Also, a correction step is added as $f_o^{-j+1} = \max(f_o^{-j+1}, 10^{-2}f_p)$ and a tolerance of $10^{-2}f_p$ is adequate to check the residual function. In addition, a minimum value is imposed in the characteristic length of $l_c^{min} = \max\left(l_c, 0.14\frac{G_f^- E_o}{f_p^2}\right)$ to get an adequate convergence of algorithm.