Effect of a cutoff on pushed and bistable fronts of the reaction-diffusion equation

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We give an explicit formula for the change of speed of pushed and bistable fronts of the reaction-diffusion equation when a small cutoff is applied to the reaction term at the unstable or metastable equilibrium point. The results are valid for arbitrary reaction terms and include the case of density-dependent diffusion.

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I. INTRODUCTION

The effect of a cutoff of the reaction term on the speed of reaction-diffusion fronts has received much attention since it was observed by Brunet and Derrida [1] that fluctuations in propagating fronts which arise due as a result of the discreteness in the number N of propagating particles can be modeled by introducing a small cutoff ϵ on the reaction term in the deterministic reaction-diffusion equation. The cutoff parameter ϵ is inversely proportional to the number of diffusing particles. Additional studies on front propagation up a reaction rate gradient provide additional numerical evidence that confirms the validity of representing fluctuations by a cutoff in the reaction term [2]. We refer to [3] for a very complete review and references on this topic. The effect of fluctuations is particularly important for pulled fronts and for front propagation into a metastable state. In this last case the speed of the front without a cutoff may be quite small, zero in fact at an isolated point [4], in which case the fluctuations contribute significantly to the speed of propagation of the front [5].

The effect of a cutoff on the speed of pulled fronts has been studied extensively [1,3,6,7]; less attention has been paid to the effect of a cutoff on pushed and bistable fronts. For pulled fronts, with or without cutoff, the speed can be calculated by a linear analysis at the edge of the front. The effect of a cutoff on pushed and bistable fronts has been studied for the exactly solvable case of the Nagumo reaction term $f(u)=u(1-u)(u-a)$ which, for different values of the parameter a, describes bistable, pushed and pulled fronts (this is also known as the Schlögl reaction term when written in the variable $\rho=2u-1$). It was shown in [7,8] that the shift in the speed has a power-law dependence on the cutoff parameter ϵ in contrast to the inverse-squared logarithmic dependence on the cutoff parameter for pulled fronts found by Brunet and Derrida. It has also been shown using a variational approach [7] that a cutoff slows down pulled and pushed fronts, but speeds up bistable fronts.

The purpose of this work is to provide an explicit expression for the shift of the speed of pushed and bistable fronts with a cutoff for arbitrary reaction terms including the case of density-dependent diffusion. We show that the shift in the speed is given by

$$\delta c = -Kf'(0)\epsilon^{1+\lambda},$$

where

$$\begin{cases} -1<\lambda<0 & \text{for pushed fronts,} \\ 0<\lambda<1 & \text{for bistable fronts,} \end{cases}$$

where the constants K and λ are independent of ϵ. When the slope of the reaction term vanishes at $u=0$ we find

$$\delta c = -Kf(\epsilon) \quad \text{when } f'(0)=0.$$

We give explicit expressions for the constants λ and K which depend only on the front without a cutoff. The shift in the speed with density-dependent diffusion is contained in this last case $f'(0)=0$, as we indicate below.

Before proceeding with the actual derivation we need to recall some known results on the speed of fronts of the reaction-diffusion equation. We consider the reaction-diffusion equation

$$u_t = u_{xx} + f(u) \quad \text{with } f(0)=f(1)=0,$$

where the reaction term $f(u)$ satisfies additional conditions depending on the physical problem under consideration. We shall consider two generic classes. The first, type I, satisfies $f>0$ in (0,1). To this category belong pulled and pushed fronts. Type-II bistable reaction terms satisfy $f(u)<0$ for u in $(0,a)$ and $f>0$ in $(a,1)$ with $\int_0^1 f(u)du>0$.

For both types of reaction terms, sufficiently localized initial conditions evolve into a monotonic front [9]. For reaction terms of type I the system evolves into the monotonic front of minimal speed. If this minimal speed is that obtained from the linear analysis at the edge of the front [10]—that is, $c_{\min}\equiv c_{KPP}=2\sqrt{f'(0)}$—the front is called pulled. If this minimal speed is greater than c_{KPP}, the front is called pushed. For reaction terms of type II there is a unique speed for which a monotonic front exists. It has been shown [11] that the asymptotic speed of pushed and bistable fronts is given by the variational formula

$$c^2 = \max\left(2\frac{\int_0^1 fg\,du}{\int_0^1 (-g^2/g')du}\right), \tag{1}$$

where the maximum is taken over all positive decreasing functions $g(u)$ in (0,1) for which the integrals exist. The maximum is attained for a trial function $g=\hat{g}$ (unique, up to a multiplicative constant) which, close to $u=0$, diverges as

$$\hat{g} \approx \frac{1}{u^{c/m}}, \tag{2}$$

where

$$m = \frac{1}{2}[c + \sqrt{c^2 - 4f'(0)}].$$

For pulled fronts the maximum in (1) is not attained and the speed is given instead by the supremum of $2\int_0^1 fg\, du / \int_0^1 (-g^2/g')du$ over the class of functions mentioned above.

For pushed and bistable fronts, the existence of a variational principle allows one to use the Feynman-Hellman theorem to calculate the dependence of the speed on parameters of the reaction term. We shall use this theorem to study the effect of a cutoff of the reaction term on pushed and bistable fronts. Suppose that the reaction term f depends on a parameter α [i.e., $f=f(u,\alpha)$]. In the context of the variational principle (1), the Feynman-Hellman theorem reads as follows:

$$\frac{\partial c^2}{\partial \alpha} = 2\frac{\int_0^1 \frac{\partial f}{\partial \alpha}(u,\alpha)\hat{g}(u,\alpha)du}{\int_0^1 (-\hat{g}^2/\hat{g}_u)du}, \tag{3}$$

where $\hat{g}(u,\alpha)$ is the function (unique up to a multiplicative constant) that yields the maximum in (1) at the given parameter α. We use a subscript to denote the partial derivative with respect to the corresponding argument. Notice that the Feynman-Hellman theorem holds only if the maximum is attained, which is not the case for pulled fronts. In what follows we use the Feynman-Hellman theorem, taking the cutoff ϵ as the parameter.

Consider a reaction term with a cutoff of the form $f(u)\Theta(u-\epsilon)$, where the reaction term $f(u)$ without a cutoff gives rise to a pushed or bistable front [here, $\Theta(x)$ denotes the Heaviside step function]. The Feynman-Hellman theorem tells us that

$$\frac{\partial c^2}{\partial \epsilon} = 2\frac{\int_0^1 \frac{\partial f(u)\Theta(u-\epsilon)}{\partial \epsilon}\hat{g}(u,\epsilon)du}{\int_0^1 (-\hat{g}^2/\hat{g}_u)du} = -2\frac{f(\epsilon)\hat{g}(\epsilon,\epsilon)}{\int_0^1 (-\hat{g}^2/\hat{g}_u)du}. \tag{4}$$

In the expression above $\hat{g}(\epsilon,\epsilon)$ is the optimizing function for the speed of the front with the reaction term $f(u)\Theta(u-\epsilon)$. We are interested in the speed when ϵ is small. The ordinary differential equation for the traveling fronts $u_{zz}+cu_z+f(u)=0$ implies that u and u_z are continuous at $u=\epsilon$ and therefore $p(u)=-u_z(u)$ is also continuous. The optimizing g, i.e., $\hat{g}$, satisfies [11]

$$c\frac{\hat{g}_u}{\hat{g}} = -c\frac{1}{p(u)}.$$

From the formula above one can see that $\ln(\hat{g})$ is continuous and that both $\hat{g}$ and $\hat{g}_u$ are continuous as well. Therefore, in leading order we may approximate $\hat{g}(u,\epsilon)=\hat{g}(u,0)$. The function $\hat{g}(u,0)$ is the optimizing function for the reaction term $f(u)$ without the cutoff, which we call simply $\hat{g}_0(u)$. To leading order in ϵ we obtain then

$$c\left(\frac{dc}{d\epsilon}\right)_{\epsilon=0} = -\frac{f(\epsilon)\hat{g}_0(\epsilon)}{\int_0^1 [-\hat{g}_0^2(u)/\hat{g}_0'(u)]du}. \tag{5}$$

The shift in the speed is given by

$$\delta c = \epsilon\left(\frac{dc}{d\epsilon}\right)_{\epsilon=0} = -K\epsilon f(\epsilon)\hat{g}_0(\epsilon), \tag{6}$$

where the proportionality constant K, which is independent of ϵ, is given by

$$K = \left[c_0\int_0^1 (-\hat{g}_0^2/\hat{g}_0')du\right]^{-1} = \frac{c_0}{2\int_0^1 f\hat{g}_0\, du}. \tag{7}$$

In the formula above, c_0 is the speed of the front in the absence of cutoff. Using (2) we have that

$$\delta c = -Kf(\epsilon)\epsilon^{1-c_0/m}. \tag{8}$$

Replacing the value of c_0/m, we obtain, finally,

$$\delta c = \begin{cases} -Kf(\epsilon) & \text{if } f'(0)=0, \\ -Kf'(0)\epsilon^{1+\lambda} & \text{if } f'(0)\neq 0, \end{cases} \tag{9}$$

where

$$\lambda = \frac{\sqrt{1-4f'(0)/c_0^2}-1}{\sqrt{1-4f'(0)/c_0^2}+1}. \tag{10}$$

In the expression above for δc we used $f(\epsilon)\approx \epsilon f'(0)$ in leading order. For pushed fronts, $f'(0)>0$; therefore, $-1<\lambda<0$ and $\delta c<0$. For bistable fronts, $f'(0)<0$; therefore, $0<\lambda<1$ and $\delta c>0$.

The precise value of the constant λ can be determined when the speed c_0 in the absence of the cutoff is known. For the determination of the constant K the maximizing function $\hat{g}_0(u)$ for the speed in the absence of the cutoff must also be known.

As an example, we may apply the above result to the Nagumo reaction term

$$f(u) = u(1-u)(u-a) \tag{11}$$

for which an exact solution is known. This is the case studied previously by other methods [3,7,8].

For $0<a<1/2$ this is a bistable reaction term. For negative values of a this is a reaction term of type I, which for $-1/2<a<0$ gives rise to a pushed front. The speed without the cutoff is given by

$$c_0 = \frac{1}{\sqrt{2}} - a\sqrt{2}, \tag{12}$$

which is obtained from the variational principle (1) with the trial function [11]

$$\hat{g}_0(u) = \left(\frac{1-u}{u}\right)^{1-2a}. \tag{13}$$

For this reaction term, $f'(0) = -a$. The value of K is

$$K = \left[c_0 \int_0^1 (-g_0^2/g_0')dq\right]^{-1} = \frac{\sqrt{2}\Gamma(4)}{\Gamma(1+2a)\Gamma(3-2a)}, \tag{14}$$

and $\lambda = 2a$. Therefore,

$$\delta c = \frac{\sqrt{2}\Gamma(4)}{\Gamma(1+2a)\Gamma(3-2a)} a\epsilon^{1+2a}. \tag{15}$$

The power dependence of the shift on ϵ is in agreement with previous results [8]; the magnitude has not been calculated elsewhere. Notice that near the value $a = 1/2$, where the speed of the front without cutoff vanishes, the magnitude of the shift induced by the cutoff becomes of comparable magnitude to the speed of the front itself.

When $a = 0$, $f'(0) = 0$, and in leading order, the shift of the speed is given by

$$\delta c = -6\sqrt{2}\epsilon^2. \tag{16}$$

As a second example, consider the Fisher-Kolmogorov equation with density-dependent diffusion,

$$u_t = [D(u)u_x]_x + u(1-u), \tag{17}$$

where the diffusion coefficient $D(u)$ satisfies $D(0) = 0$, $D'(u) > 0$. When the diffusion coefficient is not constant, it is not possible to determine the speed of the front from linear considerations. Moreover, the wave of minimal speed is sharp; that is, it does not decay exponentially at infinity. One of the most studied cases is a power-law dependence of the form $D(u) = u^m$. The exact solution is known for the value $m = 1$, the case for which the asymptotic speed of the front is given by $c = 1$ [12,13]. The speed of the traveling fronts of Eq. (17) is equal to the speed of the reaction-diffusion equation with constant diffusion, but with a reaction term $D(u)f(u) = D(u)u(1-u)$ [11], and it belongs to the case of reaction terms with vanishing derivative at the origin. Therefore, a cutoff ϵ produces a shift in the speed given by

$$\delta c = -KD(\epsilon)f(\epsilon) = -K\epsilon D(\epsilon), \tag{18}$$

in leading order. Again, the constant K is independent of ϵ and it can be determined if the exact solution without a cutoff is known.

In summary, we have established the effect of a cutoff on the speed of fronts of the reaction-diffusion equation for all fronts which are not pulled in the absence of a cutoff. This has been done in a simple, unified way, making use of a variational principle for the asymptotic speed of the fronts. We find not only the dependence on the cutoff, but an explicit expression for the shift. The method used to obtain these results is the Feynman-Hellman theorem, which enables one to determine the effect of varying any parameter of the reaction term. For pulled fronts, the speed is given by the supremum of an integral expression, not the maximum; hence, the Feynman-Hellman theorem is not valid for them. For pulled fronts the effect of the cutoff can also be calculated from the variational expression, but by directly solving the Euler-Lagrange equation in the linear approximation [14]. The approach used here to calculate the effect of the cutoff on the speed of pushed fronts can be used for the reaction convection-diffusion equation and for the hyperbolic reaction-diffusion equation, for which integral variational principles have been formulated [15,16].

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