Analytic QCD – a Short Review

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Analytic versions of QCD are those whose coupling $\alpha_s(Q^2)$ does not have the unphysical Landau singularities on the space-like axis ($-q^2 = Q^2 > 0$). The coupling is analytic in the entire complex plane except the time-like axis ($Q^2 < 0$). Such couplings are thus suitable for application of perturbative methods down to energies of order GeV. We present a short review of the activity in the area which started with a seminal paper of Shirkov and Solovtsov ten years ago. Several models for analytic QCD coupling are presented. Strengths and weaknesses of some of these models are pointed out. Further, for such analytic couplings, constructions of the corresponding higher order analytic couplings (the analogs of the higher powers of the perturbative coupling) are outlined, and an approach based on the renormalization group considerations is singled out. Methods of evaluation of the leading-twist part of space-like observables in such analytic frameworks are described. Such methods are applicable also to the inclusive time-like observables. Two analytic models are outlined which respect the ITEP Operator Product Expansion philosophy, and thus allow for an evaluation of higher-twist contributions to observables.

Keywords: Analytic coupling; Truncated analytic series (TAS); ITEP-OPE philosophy

I. INTRODUCTION

Perturbative QCD calculations involve coupling $a(Q^2) \equiv \alpha_s(Q^2)/\pi$ which has Landau singularities (poles, cuts) on the space-like semiaxis $0 \le Q^2 \le \Lambda^2$ ($q^2 \equiv -Q^2$). These lead to Landau singularities for the evaluated space-like observables $\mathcal{D}(Q^2)$ at low $Q^2 \lesssim \Lambda^2$. The existence of such singularities is in contradiction with the general principles of the local quantum field theories [1]. Further, lattice simulations [2] confirm that such singularities are not present in $a(Q^2)$.

An analytized coupling $\mathcal{A}_1(Q^2)$, which agrees with the perturbative $a(Q^2)$ at $Q^2 \to \infty$ and is analytic in the Euclidean part of the Q^2-plane ($Q^2 \in \mathcal{C}$, $Q^2 \not\le 0$), addresses this problem, and has been constructed by Shirkov and Solovtsov about ten years ago [3].

Several other analytic QCD (anQCD) models for $\mathcal{A}_1(Q^2)$ can be constructed, possibly satisfying certain additional constraints at low and/or at high Q^2.

Another problem is the analytization of higher power terms $a^n \mapsto \mathcal{A}_n$ in the truncated perturbation series (TPS) for $\mathcal{D}(Q^2)$. Also here, several possibilities appear.

Application of the Operator Product Expansion (OPE) approach, in the ITEP sense, to inclusive space-like observables appears to make sense only in a restricted class of such anQCD models.

This is a short and incomplete review of the activity in the area; relatively large space is given to the work of the review's authors. For an earlier and more extensive review, see e. g. Ref. [4].

Section II contains general aspects of analytization of the Euclidean coupling $a(Q^2) \mapsto \mathcal{A}_1(Q^2)$, and the definition of the time-like (Minkowskian) coupling $\mathfrak{A}_1(s)$. Further, in Sec. II we review the minimal analytization (MA) procedure developed by Shirkov and Solovtsov [3], and a variant thereof developed by Nesterenko [5]. In Sec. III we present various approaches of going beyond the MA procedure, i.e., various models for $\mathfrak{A}_1(s)$, and thus for $\mathcal{A}_1(Q^2)$ [6–11]. In Sec. IV, analytization procedures for the higher powers $a^n(Q^2) \mapsto \mathcal{A}_n(Q^2)$ in MA model are presented [12–14], and an alternative approach which is applicable to any model of analytic $\mathcal{A}_1(Q^2)$ [10, 11] is presented. In Sec. V, an analytization of noninteger powers $a^\nu(Q^2)$ is outlined [15]. In Sec. VI, methods of evaluations of space-like and of inclusive time-like observables in models with analytic $\mathcal{A}_1(Q^2)$ are described, and some numerical results are presented for semihadronic τ decay rate ratio r_τ, Adler function $d_v(Q^2)$ and Bjorken polarized sum rule (BjPSR) $d_b(Q^2)$ [10–14, 16]. In Sec. VII, two sets of models are presented [17, 18] whose analytic couplings $\mathcal{A}_1(Q^2)$ preserve the OPE-ITEP philosophy, i.e., at high Q^2 they fulfill: $|\mathcal{A}_1(Q^2) - a(Q^2)| < (\Lambda^2/Q^2)^k$ for any $k \in \mathcal{N}$. Section VIII contains a summary of the presented themes.

II. ANALYTIZATION $a(Q^2) \mapsto \mathcal{A}_1(Q^2)$

In perturbative QCD (pQCD), the beta function is written as a truncated perturbation series (TPS) of coupling a. Therefore, the renormalization group equation (RGE) for $a(Q^2)$ has the form

$$\frac{\partial a(\ln Q^2; \beta_2, \ldots)}{\partial \ln Q^2} = -\sum_{j=2}^{j_{\max}} \beta_{j-2}\, a^j(\ln Q^2; \beta_2, \ldots). \quad (1)$$

The first two coefficients [$\beta_0 = (1/4)(11 - 2n_f/3)$, $\beta_1 = (1/16)(102 - 38n_f/3)$] are scheme-independent in mass-independent schemes. The other coefficients ($\beta_2, \beta_3, \ldots$) characterize the renormalization scheme (RSch). The solution of

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perturbative RGE (1) can be written in the form

$$a(Q^2) = \sum_{k=1}^{\infty} \sum_{\ell=0}^{k-1} K_{k\ell} \, \frac{(\ln L)^{\ell}}{L^k}, \qquad (2)$$

where $L = \ln(Q^2/\Lambda^2)$ and $K_{k\ell}$ are constants depending on β_j's. In $\overline{\rm MS}$: $\Lambda = \overline{\Lambda} \sim 10^{-1}$ GeV.

The pQCD coupling $a(Q^2)$ is nonanalytic on $-\infty < Q^2 \leq \overline{\Lambda}^2$. Application of the Cauchy theorem gives the dispersion relation

$$a(Q^2) = \frac{1}{\pi} \int_{\sigma=-\Lambda^2-\eta}^{\infty} \frac{d\sigma \rho_1^{\rm (pt)}(\sigma)}{(\sigma+Q^2)}, \quad (\eta \to 0), \qquad (3)$$

where $\rho_1^{\rm (pt)}(\sigma)$ is the (pQCD) discontinuity function of a along the cut axis in the Q^2-plane: $\rho_1^{\rm (pt)}(\sigma) = {\rm Im}\, a(-\sigma - i\varepsilon)$. The MA procedure of Shirkov and Solovtsov [3] removes the pQCD contribution of the unphysical cut $0 < -\sigma \leq \Lambda^2$, keeping the discontinuity elsewhere unchanged ("minimal analytization" of a)

$$\mathcal{A}_1^{\rm (MA)}(Q^2) = \frac{1}{\pi} \int_{\sigma=0}^{\infty} \frac{d\sigma \rho_1^{\rm (pt)}(\sigma)}{(\sigma+Q^2)} \; . \qquad (4)$$

In general:

$$\mathcal{A}_1(Q^2) = \frac{1}{\pi} \int_{\sigma=0}^{\infty} \frac{d\sigma \rho_1(\sigma)}{(\sigma+Q^2)} \; , \qquad (5)$$

where $\rho_1(\sigma) = {\rm Im}\mathcal{A}_1(-\sigma - i\varepsilon)$. Relation (5) defines an analytic coupling in the entire Euclidean complex Q^2-plane, i.e., excluding the time-like semiaxis $-s = Q^2 \leq 0$. On this semiaxis, it is convenient to define the time-like (Minkowskian) coupling $\mathfrak{A}_1(s)$ [12–14]

$$\mathfrak{A}_1(s) = \frac{i}{2\pi} \int_{-s+i\varepsilon}^{-s-i\varepsilon} \frac{d\sigma'}{\sigma'} \mathcal{A}_1(\sigma') \; . \qquad (6)$$

The following relations hold between $\mathcal{A}_1$, $\mathfrak{A}_1$ and ρ_1:

$$\begin{aligned} \mathfrak{A}_1(s) &= \frac{1}{\pi} \int_s^{\infty} \frac{d\sigma}{\sigma} \rho_1(\sigma) \; , & (7) \\ \mathcal{A}_1(Q^2) &= Q^2 \int_0^{\infty} \frac{ds \mathfrak{A}_1(s)}{(s+Q^2)^2} \; , & (8) \\ \frac{d}{d\ln\sigma} \mathfrak{A}_1(\sigma) &= -\frac{1}{\pi} \rho_1(\sigma) \; . & (9) \end{aligned}$$

The MA is equivalent to the minimal analytization of the TPS form of the $\beta(a) = \partial a(Q^2)/\partial \ln Q^2$ function [19]

$$\frac{\partial \mathcal{A}_1{}^{\rm (MA)}(\ln Q^2; \beta_2, \ldots)}{\partial \ln Q^2} = \frac{1}{\pi} \int_{\sigma=0}^{\infty} \frac{d\sigma \rho_\beta^{\rm (pt)}(\sigma)}{(\sigma+Q^2)}, \qquad (10)$$

where $\rho_\beta^{\rm (pt)}(\sigma) = {\rm Im}\beta(a)(-\sigma - i\varepsilon)$, and

$$\beta(a) = -\sum_{j=2}^{j_{\max}} \beta_{j-2} \, a^j (\ln Q^2; \beta_2, \ldots) \; . \qquad (11)$$

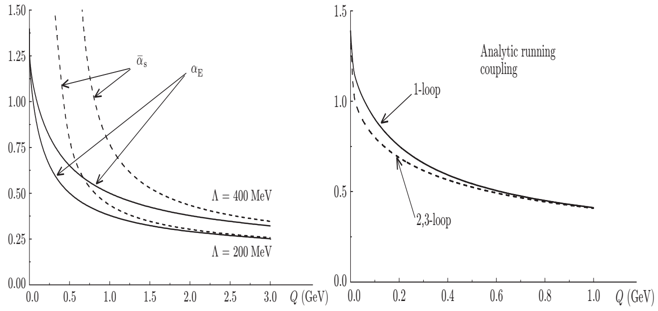

FIG. 1: Left: one-loop MA $\alpha_{\rm E}(Q) = \pi\mathcal{A}_1(Q^2)$ and its one-loop perturbative counterpart $\overline{\alpha}_s(Q^2)$ in $\overline{\rm MS}$, for $n_f = 3$ and $\Lambda = \overline{\Lambda} = 0.2$ and 0.4 GeV. Right: stability of the MA $\alpha_{\rm E}(Q) = \pi\mathcal{A}_1(Q^2)$ under the loop-level increase. Both figures from: Shirkov and Solovtsov, 1997 [3].

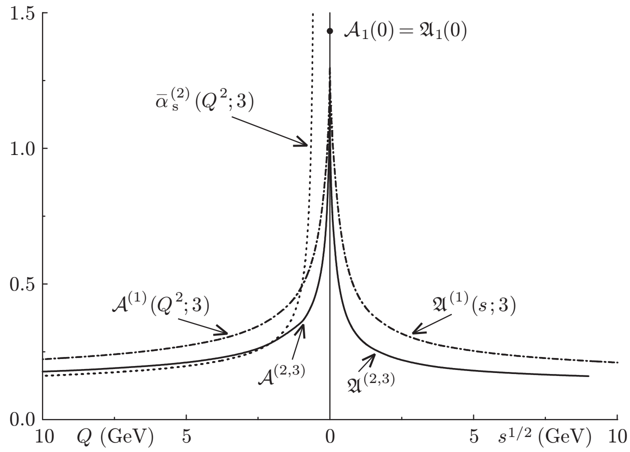

FIG. 2: The MA time-like and space-like couplings $\mathfrak{A}_1(s^{1/2})$ and $\mathcal{A}_1(Q)$ at 1-loop, 2-loop (3-loop) level; in $\overline{\rm MS}$ for $n_f = 3$ and $\overline{\Lambda} = 0.35$ GeV [$\mathfrak{A}_1$ and $\mathcal{A}_1$ in figure are $\pi\mathfrak{A}_1$ and $\pi\mathcal{A}_1$ in our normalization convention]. Figure from: Shirkov and Solovtsov, 2006 [16].

The MA couplings $\mathcal{A}_1(Q^2)$ and $\mathfrak{A}_1(s)$ are finite in the IR (with the value $1/\beta_0$ at $Q^2 = 0$, or $s = 0$) and show strong stability under the increase of the loop-level $n_{\rm m} = j_{\max} - 1$ (see Figs. 1, 2), and under the change of the renormalization scale (RScl) and scheme (RSch). Another similar pQCD-approach is to analytize minimally $\beta(a)/a = \partial \ln a(Q^2)/\partial \ln Q^2$ [5, 20, 21]. This leads to an IR-divergent analytic $\widetilde{\rm (MA)}$ coupling, $\mathcal{A}_1(Q^2) \sim (\Lambda^2/Q^2)(\ln(\Lambda^2/Q^2))^{-1}$ when $Q^2 \to 0$. At one-loop:

$$\mathcal{A}_1(Q^2) = \frac{1}{\beta_0} \frac{(Q^2/\Lambda^2) - 1}{(Q^2/\Lambda^2)\ln(Q^2/\Lambda^2)} \; . \qquad (12)$$

Also this coupling has improved stability under the loop-level change, and under the RScl and RSch changes (see Figs. 3, 4). Numerical predictions of this model, at the one-loop level, for various observables, were performed in Ref. [21], and they agree with the experimental results within the experimental uncertainties and the theoretical uncertainties of the one-loop approximation.

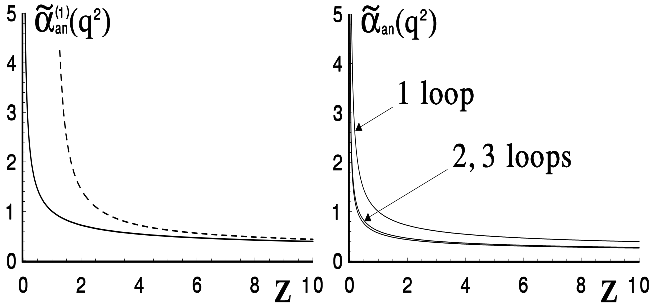

FIG. 3: Left: one-loop $\overline{\mathrm{MA}}$ $\tilde{\alpha}_{\mathrm{an}}(Q) = \beta_0 \mathcal{A}_1(Q^2)$ and its one-loop perturbative counterpart, as a function of $Z = Q^2/\Lambda^2$ (Figure from: Nesterenko, 2000 [5]). Right: stability of the $\overline{\mathrm{MA}}$ $\tilde{\alpha}_{\mathrm{an}}(Q) = \beta_0 \mathcal{A}_1(Q^2)$ under the loop-level increase, as a function of $Z = Q^2/\Lambda^2$ (Figure from: Nesterenko, 2001 [20]).

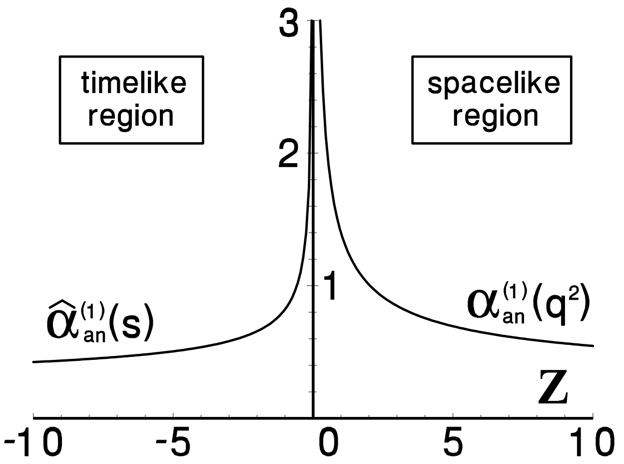

FIG. 4: One-loop time-like and space-like $\overline{\mathrm{MA}}$ couplings $\hat{\alpha}_{\mathrm{an}}(s) = \pi \mathfrak{A}_1(s)$ and $\alpha_{\mathrm{an}}(Q^2) = \pi \mathcal{A}_1(Q^2)$ as a function of $Z = -s/\Lambda^2$ or $Z = Q^2/\Lambda^2$, respectively. Figure from: Nesterenko, 2003 [21].

III. BEYOND THE MA

The idea to make the QCD coupling IR finite phenomenologically is an old one, by the substitution $\ln(Q^2/\Lambda^2) \mapsto \ln[(Q^2 + 4{m_g}^2)/\Lambda^2]$ where m_g is an effective gluon mass, cf. Refs. [22–24].

On the other hand, the analytic MA, or $\overline{\mathrm{MA}}$, couplings can be modified at low energies, bringing in additional parameter(s) such that there is a possibility to reproduce better a wide set of low energy QCD experimental data.

Among the recent proposed analytic couplings are:

1. Synthetic coupling proposed by Alekseev [6]:

$$\alpha_{\mathrm{syn}}(Q^2) = \alpha^{(\mathrm{MA})}(Q^2) + \frac{\pi}{\beta_0}\left[\frac{c\Lambda^2}{Q^2} - \frac{d\,\Lambda^2}{Q^2 + {m_g}^2}\right], \tag{13}$$

where the three new parameters c, d and gluon mass m_g were determined by requiring $\alpha_{\mathrm{syn}}(Q^2) - \alpha_{\mathrm{pt}}(Q^2) \sim (\Lambda^2/Q^2)^3$ (for the convergence of the gluon condensate) and by the string condition $V(r) \sim \sigma r$ ($r \to \infty$) with $\sigma \approx 0.42^2 \mathrm{GeV}^2$. This coupling is IR-divergent.

2. The coupling by Sriwastawa *et al.* [7]:

$$\frac{1}{\alpha^{(1)}_{\mathrm{SPPW}}(Q^2)} = \frac{1}{\alpha^{(1)}_{\mathrm{SPPW}}(\Lambda^2)} + \frac{\beta_0}{\pi}\int_0^\infty \frac{(z-1)\,z^p}{(\sigma + z - i\varepsilon)(\sigma+1)(1+z^p)}\,d\sigma, \tag{14}$$

where $z = Q^2/\Lambda^2$ and $0 < p \le 1$. This formula coincides with Nesterenko's (one-loop) $\overline{\mathrm{MA}}$ coupling when $p = 1$.

3. An IR-finite coupling proposed by Webber [8]:

$$\alpha^{(1)}_W(Q^2) = \frac{\pi}{\beta_0}\left[\frac{1}{\ln z} + \frac{1}{1-z}\,\frac{z+b}{1+b}\left(\frac{1+c}{z+c}\right)^p\right], \tag{15}$$

where $z = Q^2/\Lambda^2$ and specific values are chosen for parameters $b = 1/4$, $c = 4$, and $p = 4$; $\alpha^{(1)}_W(0) \simeq \pi/(2\beta_0)$.

4. "Massive" $\overline{\mathrm{MA}}$ or MA couplings $\mathcal{A}_1(Q^2)$ and $\mathfrak{A}_1(s)$ proposed by Nesterenko and Papavassiliou [9]:

$$\begin{aligned} {\mathfrak{A}_1}^{(\mathrm{m})}(s) &= \Theta(s - 4m^2)\mathfrak{A}_1(s)\,, \\ {\mathcal{A}_1}^{(\mathrm{m})}(Q^2) &= \frac{Q^2}{Q^2 + 4m^2}\int_{4m^2}^\infty \rho_1(\sigma)\frac{\sigma - 4m^2}{\sigma + Q^2}\frac{d\sigma}{\sigma}\,, \end{aligned} \tag{16}$$

where $m \sim \overline{\Lambda}$; and $\rho_1(\sigma) = {\rho_1}^{(\mathrm{pt})}(\sigma)$ in the MA case. In this case: ${\mathcal{A}_1}^{(\mathrm{m})}(0) = {\mathfrak{A}_1}^{(\mathrm{m})}(0) = 0$. The mass m is some kind of threshold, and can be expected to be $\sim m_\pi$.

5. Two specific models of IR-finite analytic coupling [10, 11]: on the time-like axis $s \equiv -Q^2 > 0$, the parturbative discontinuity function $\rho_1(s)$, or equivalently ${\mathfrak{A}_1}^{(\mathrm{MA})}(s)$, was modified in the in the IR regime ($s \sim \overline{\Lambda}^2$). A first possibility (model 'M1'):

$$\begin{aligned} \mathfrak{A}_1^{(\mathrm{M1})}(s) &= c_f \overline{M}_r^2 \delta(s - \overline{M}_r^2) \\ &\quad + k_0 \Theta(\overline{M}_0^2 - s) + \Theta(s - \overline{M}_0^2)\mathfrak{A}_1^{(\mathrm{MA})}(s)\,, \end{aligned}$$

where c_f, k_0, $c_r = \overline{M}_r^2/\overline{\Lambda}^2$, $c_0 = \overline{M}_0^2/\overline{\Lambda}^2$ are four dimensionless parameters of the model, all ~ 1. One of them (k_0) can be eliminated by requiring the (approximate) merging of M1 with MA at large Q^2:

$$|{\mathcal{A}_1}^{(\mathrm{M1})}(Q^2) - {\mathcal{A}_1}^{(\mathrm{MA})}(Q^2)| \sim (\overline{\Lambda}^2/Q^2)^2.$$

The Euclidean $\mathcal{A}_1^{(\mathrm{M1})}(Q^2)$ is

$$\begin{aligned} \mathcal{A}_1^{(\mathrm{M1})}(Q^2) &= \mathcal{A}_1^{(\mathrm{MA})}(Q^2) + \Delta\mathcal{A}_1^{(\mathrm{M1})}(Q^2)\,, \\ \Delta\mathcal{A}_1^{(\mathrm{M1})}(Q^2) &= -\frac{1}{\pi}\int_{\sigma=0}^{\overline{M}_0^2} \frac{d\sigma \rho_1^{(\mathrm{pt})}(\sigma)}{(\sigma + Q^2)} + c_f \frac{\overline{M}_r^2 Q^2}{\left(Q^2 + \overline{M}_r^2\right)^2} \\ &\quad - d_f \frac{\overline{M}_0^2}{\left(Q^2 + \overline{M}_0^2\right)}\,, \end{aligned} \tag{17}$$

where the constant d_f is

$$d_f \equiv -k_0 + \frac{1}{\pi}\int_{\overline{M}_0^2}^\infty \frac{d\sigma}{\sigma}\rho_1^{(\mathrm{pt})}(\sigma)\,.$$

Another, simpler, possibility is (model 'M2'):

$$\mathfrak{A}_1^{(\mathrm{M1})}(s) = \mathfrak{A}_1^{(\mathrm{MA})}(s) + c_v \Theta(\overline{M}_p^2 - s) \ , \tag{18}$$

$$\mathcal{A}_1^{(\mathrm{M1})}(Q^2) = \mathcal{A}_1^{(\mathrm{MA})}(Q^2) + c_v \frac{\overline{M}_p^2}{(Q^2+\overline{M}_p^2)} \ , \tag{19}$$

where c_v and $c_p = \overline{M}_p^2/\overline{\Lambda}^2$ are the model parameters.

6. Those anQCD models which respect the OPE-ITEP condition are presented in Sec. VII.

IV. ANALYTIZATION OF HIGHER POWERS $a^k \mapsto \mathcal{A}_k$

In MA model, the construction is [3, 12–14] (MSSSh: Milton, Solovtsov, Solovtsova, Shirkov):

$$a^k(Q^2) \mapsto \mathcal{A}_k^{(\mathrm{MA})}(Q^2) = \frac{1}{\pi}\int_0^\infty \frac{d\sigma}{\sigma+Q^2}\rho_k^{(\mathrm{pt})}(\sigma) \ , \tag{20}$$

where $k=1,2,\ldots$; $\rho_k^{(\mathrm{pt})}(\sigma) = \mathrm{Im}[a^k(-\sigma - i\varepsilon)]$; and a is given, e.g., by Eq. (2). In other words, "minimal analytization" (MA) is applied to each power a^k.

As a consequence, in MA we have [19]

$$\begin{aligned}
\frac{\partial \mathcal{A}_1^{(\mathrm{MA})}(\mu^2)}{\partial \ln\mu^2} &= -\beta_0 \mathcal{A}_2^{(\mathrm{MA})}(\mu^2) - \beta_1 \mathcal{A}_3^{(\mathrm{MA})}(\mu^2) - \cdots ,\\
\frac{\partial^2 \mathcal{A}_1^{(\mathrm{MA})}(\mu^2)}{\partial (\ln\mu^2)^2} &= 2\beta_0^2 \mathcal{A}_3^{(\mathrm{MA})} + 5\beta_0\beta_1 \mathcal{A}_4^{(\mathrm{MA})} + \cdots ,
\end{aligned}$$

etc. This is so because a^k, and consequently $\rho_k^{(\mathrm{pt})}(\sigma)$, fulfill analogous RGE's.

The approach (20) of constructing $\mathcal{A}_k$'s ($k \geq 2$) can be applied to a specific model only (MA). In other anQCD models (i.e., for other $\mathcal{A}_1(Q^2)$), the discontinuity functions ρ_k ($k \geq 2$) are not known. We present an approach [10, 11] that is applicable to any anQCD model, and reduces to the above approach in the MA model. We proposed to maintain the scale (RScl) evolution of these (truncated) relations for any version of anQCD

$$\begin{aligned}
\frac{\partial \mathcal{A}_1(\mu^2;\beta_2,\ldots)}{\partial \ln\mu^2} &= -\beta_0 \mathcal{A}_2 - \cdots - \beta_{n_\mathrm{m}-2}\mathcal{A}_{n_\mathrm{m}} \ ,\\
\frac{\partial^2 \mathcal{A}_1(\mu^2;\beta_2,\ldots)}{\partial (\ln\mu^2)^2} &= 2\beta_0^2 \mathcal{A}_3 + 5\beta_0\beta_1\mathcal{A}_4 + \cdots + \kappa_{n_\mathrm{m}}^{(2)}\mathcal{A}_{n_\mathrm{m}} ,
\end{aligned} \tag{21}$$

etc. Eqs. (21) define the couplings $\mathcal{A}_k(Q^2)$ ($k \geq 2$). Further, the evolution under the scheme (RSch) changes will also be maintained as in the MA case (and in pQCD):

$$\frac{\partial \mathcal{A}_1(\mu^2;\beta_2,\ldots)}{\partial \beta_2} \approx \frac{1}{\beta_0}\mathcal{A}_3 + \frac{\beta_2}{3\beta_0^2}\mathcal{A}_5 + \cdots + k_{n_\mathrm{m}}^{(2)}\mathcal{A}_{n_\mathrm{m}} , \tag{22}$$

analogously for $\partial\mathcal{A}_1/\partial\beta_3$, etc. In our approach, the basic space-like quantities are $\mathcal{A}_1(\mu^2)$ of a given anQCD model (e.g., MA, M1, M2) and its logarithmic derivatives

$$\widetilde{\mathcal{A}}_n(\mu^2) \equiv \frac{(-1)^{n-1}}{\beta_0^{n-1}(n-1)!}\frac{\partial^{n-1}\mathcal{A}_1(\mu^2)}{\partial(\ln\mu^2)^{n-1}} \ , \quad (n=1,2,\ldots), \tag{23}$$

whose pQCD analogs are

$$\widetilde{a}_n(\mu^2) \equiv \frac{(-1)^{n-1}}{\beta_0^{n-1}(n-1)!}\frac{\partial^{n-1}a(\mu^2)}{\partial(\ln\mu^2)^{n-1}} \ , \quad (n=1,2,\ldots). \tag{24}$$

At loop-level three ($n_\mathrm{m}=3$), where we include in RGE (1) term with $j_{\max}=4$ (thus β_2), relations (21) are

$$\widetilde{\mathcal{A}}_2(\mu^2) = \mathcal{A}_2(\mu^2) + \frac{\beta_1}{\beta_0}\mathcal{A}_3(\mu^2), \quad \widetilde{\mathcal{A}}_3(\mu^2) = \mathcal{A}_3(\mu^2), \tag{25}$$

implying

$$\mathcal{A}_2(\mu^2) = \widetilde{\mathcal{A}}_2(\mu^2) - \frac{\beta_1}{\beta_0}\widetilde{\mathcal{A}}_3(\mu^2), \quad \mathcal{A}_3(\mu^2) = \widetilde{\mathcal{A}}_3(\mu^2). \tag{26}$$

The RSch (β_2) dependence is obtained from the truncated Eqs. (22) and (21)

$$\frac{\partial\widetilde{\mathcal{A}}_j(\mu^2;\beta_2)}{\partial\beta_2} \approx \frac{1}{2\beta_0^3}\frac{\partial^2\widetilde{\mathcal{A}}_j(\mu^2;\beta_2)}{\partial(\ln\mu^2)^2} \ , \tag{27}$$

where ($j=1,2,\ldots$) and $\widetilde{\mathcal{A}}_1 \equiv \mathcal{A}_1$.

At loop-level four ($n_\mathrm{m}=4$), where we include in RGE (1) term with $j_{\max}=5$ (thus β_3), relations analogous to (26)-(27) can be found [11].

It turns out that there is a clear hierarchy in magnitudes $|\mathcal{A}_1(Q^2)| > |\mathcal{A}_2(Q^2)| > |\mathcal{A}_3(Q^2)| > \cdots$ at all Q^2, in all or most of the anQCD models (cf. Fig. 5 for MA, M1, M2; and Fig. 9 in Sec. VII for another model).

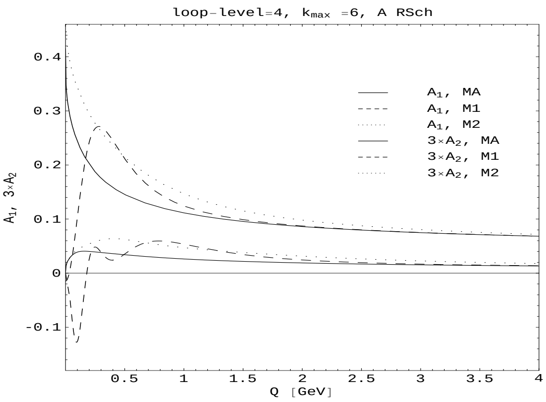

FIG. 5: $\mathcal{A}_1$ and $\mathcal{A}_2$ for various models (M1, M2 and MA) with specific model parameters: $c_0 = 2.94$, $c_r = 0.45$, $c_f = 1.08$ for M1; $c_v = 0.1$, $c_p = 3.4$ for M2; $n_f = 3$, $\overline{\Lambda}_{(n_f=3)} = 0.4$ GeV in all three models. The upper three curves are $\mathcal{A}_1$, the lower three are $3\times\mathcal{A}_2$. All couplings are in v-scheme (see Subsec. VI A). $\mathcal{A}_2$ is constructed with our approach. Figure from: Ref. [11].

We recall that the perturbation series of a space-like observable $\mathcal{D}(Q^2)$ ($Q^2 \equiv -q^2 > 0$) can be written as

$$\begin{aligned}
\mathcal{D}(Q^2)_{\mathrm{pt}} &= a + d_1 a^2 + d_2 a^3 + \cdots , &(28)\\
&= \widetilde{a}_1 + d_1\widetilde{a}_2 + \left(d_2 - \frac{\beta_1}{\beta_0}d_1\right)\widetilde{a}_3 + \cdots , &(29)
\end{aligned}$$

where the second form (29) is the reorganization of the perturbative power expansion (28) into a perturbation expansion in terms of $\widetilde{a}_n$'s (24) (note: $\widetilde{a}_1 \equiv a$). The basic analytization rule we adopt is the replacement

$$\widetilde{a}_n \mapsto \widetilde{\mathcal{A}}_n \qquad (n = 1, 2, \ldots) , \tag{30}$$

term-by-term in expansion (29), and this is equivalent to the analytization rule $a^n \mapsto \mathcal{A}_n$ term-by-term in expansion (28). However, in principle, other analytization procedures could be adopted, e.g. $a^n \mapsto \mathcal{A}_1^n$, or $a^n \mapsto \mathcal{A}_1 \mathcal{A}_{n-1}$, etc. The described analytization $a^n \mapsto \mathcal{A}_n$ reduces to the MSSSh analytization in the case of the MA model (i.e., in the case of $\mathcal{A}_1 = \mathcal{A}_1^{(\mathrm{MA})}$), because the aforementioned RGE-type relations hold also in the MA case.

Let's denote by $\mathcal{D}^{(n_\mathrm{m})}(Q^2)$ the TPS of (28) with terms up to (and including) the term $\sim a^{n_\mathrm{m}}$, and by $\mathcal{D}^{(n_\mathrm{m})}_{\mathrm{an.}}(Q^2)$ the corresponding truncated analytic series (TAS) obtained from the previous one by the term-by-term analytization $a^n \mapsto \mathcal{A}_n$. The evolution of $\mathcal{A}_k(Q^2)$ under the changes of the RSch was truncated in such a way that $\partial \mathcal{D}^{(n_\mathrm{m})}_{\mathrm{an.}}(Q^2)/\partial \beta_j \sim \mathcal{A}_{n_\mathrm{m}+1}$ (where $j \geq 2$). Further, our definition of $\mathcal{A}_k$'s ($k \geq 2$) via Eqs. (21) [cf. Eqs. (26)] involves truncated series which, however, still ensure the "correct" RScl-dependence $\partial \mathcal{D}^{(n_\mathrm{m})}_{\mathrm{an.}}(Q^2)/\partial \mu^2 \sim \mathcal{A}_{n_\mathrm{m}+1}$. This is all in close analogy with the pQCD results for TPS's: $\partial \mathcal{D}^{(n_\mathrm{m})}(Q^2)/\partial \beta_j \sim a^{n_\mathrm{m}+1}$, and $\partial \mathcal{D}^{(n_\mathrm{m})}(Q^2)/\partial \mu^2 \sim a^{n_\mathrm{m}+1}$. In conjunction with the mentioned hierarchy depicted in Fig. 5, this means that the evaluated TAS will have increasingly weaker RSch and RScl dependence when the number of TAS terms increases, at all values of Q^2.

On the other hand, if the analytization of powers were performed by another rule, for example, by the simple rule $a^n \mapsto \mathcal{A}_1^n$, the above RScl&RSch-dependence of the TAS would not be valid any more. An increasingly weaker RScl&RSch-dependence of TAS (when the number of TAS terms is increased) would not be guaranteed any more.

V. CALCULATION OF $\mathcal{A}_\nu$ FOR ν NONINTEGER

Analytization of noninteger powers in MA model was performed and used in Refs. [15], representing a generalization of results of Ref. [25]. The approach was motivated by a previous work [26] where MA-type of analytization of expressions for hadronic observables was postulated, these being integrals linear in $a(tQ^2)$ [similar to the dressed gluon approximation expressions, cf. Eq. (44) and the first line of Eq. (48)]. Analytization of noninteger powers a^ν or $a^\nu \ln a$, is needed in calculations of pion electromagnetic form factor, and in some resummed expressions for Green functions or observables, calculated within an anQCD model.

In the mentioned approach, use is made of the Laplace transformation $(f)_L$ of function f

$$f(z) \mapsto (f)_L(t): \quad f(z) = \int_0^\infty dt e^{-zt} (f)_L(t) ,$$

where $z \equiv \ln(Q^2/\Lambda^2)$. Using notations (24) and (23), it can be shown

$$(\widetilde{a}_n)_L(t) = \frac{t^{n-1}}{\beta_0^{n-1}(n-1)!} (a)_L(t) , \tag{31}$$

$$(\widetilde{\mathcal{A}}_n)_L(t) = \frac{t^{n-1}}{\beta_0^{n-1}(n-1)!} (\mathcal{A}_1)_L(t) . \tag{32}$$

Therefore, it is natural to define for any real ν the following Laplace transforms:

$$(\widetilde{a}_\nu)_L(t) = \frac{t^{\nu-1}}{\beta_0^{\nu-1}\Gamma(\nu)} (a)_L(t) ; \tag{33}$$

$$(\widetilde{\mathcal{A}}_\nu)_L(t) = \frac{t^{\nu-1}}{\beta_0^{\nu-1}\Gamma(\nu)} (\mathcal{A}_1)_L(t) . \tag{34}$$

In MA model, at one-loop level, $(a)_L(t)$ and $(\mathcal{A}_1)_L(t)$ are known

$$a(z) = \frac{1}{\beta_0 z} \Rightarrow (a)_L(t) = \frac{1}{\beta_0} . \tag{35}$$

$$\mathcal{A}_1(z) = \frac{1}{\beta_0}\left(\frac{1}{z} - \frac{1}{e^z - 1}\right) \Rightarrow$$

$$(\mathcal{A}_1)_L(t) = \frac{1}{\beta_0}\left(1 - \sum_{k=1}^{\infty} \delta(t-k)\right) . \tag{36}$$

Since at one-loop $\widetilde{\mathcal{A}}_\nu = \mathcal{A}_\nu$, it follows in one-loop MA model

$$\mathcal{A}_\nu(z) = \int_0^\infty dt e^{-zt} \frac{t^{\nu-1}}{\beta_0^\nu \Gamma(\nu)} \left(1 - \sum_{k=1}^{\infty} \delta(t-k)\right) . \tag{37}$$

Similarly, since

$$a^\nu(z) \ln a(z) = \frac{d}{d\nu} a^\nu(z) ,$$

it can be defined

$$\left[\frac{d}{d\nu} a^\nu(z)\right]_{\mathrm{MA}} \equiv \frac{d}{d\nu} \mathcal{A}_\nu(z) . \tag{38}$$

To calculate higher (two-)loop level $\mathcal{A}_\nu(z)$ in MA model, the authors of Refs. [15] expressed the two-loop $a_{(2)}(z)$ in terms of one-loop powers ${a_{(1)}}^m(z) \ln^n a_{(1)}(z)$ and then followed the above procedure.

VI. EVALUATION METHODS FOR OBSERVABLES

In pQCD, the most frequent method of evaluation of the leading-twist part of a space-like physical quantity is the evaluation of the available (RG-improved) truncated perturbation series (TPS) in powers of perturbative coupling a. Within the anQCD models, an analogous method is the aforementioned replacement $a^n \mapsto \mathcal{A}_n$ in the TPS (where $\mathcal{A}_n$ are constructed in Sec. IV), and the evaluation thereof. More specifically, consider an observable $\mathcal{D}(Q^2)$ depending on a single space-like physical scale $Q^2(\equiv -q^2) > 0$. Its usual perturbation series has the form (28), where $a = a(\mu^2;\beta_2,\beta_3,\ldots)$, with $\mu^2 \sim Q^2$. For each TPS $\mathcal{D}(Q^2)_{\rm pt}^{(N)}$ of order N, in the minimal anQCD (MA) model, the authors MSSSh [12–14] introduced the aforementioned replacement $a^n \mapsto \mathcal{A}_n^{(\rm MA)}$:

$$\mathcal{D}(Q^2)_{\rm an}^{(N)(\rm MSSSh)} = \mathcal{A}_1{}^{(\rm MA)} + d_1 \mathcal{A}_2{}^{(\rm MA)} + \cdots d_{N-1} \mathcal{A}_N{}^{(\rm MA)} . \tag{39}$$

This method of evaluation (via $a^n \to \mathcal{A}_n$) was extended to any anQCD model in [10, 11] (cf. Sec. IV). Further, in the case of inclusive space-like observables, the evaluation was extended to the resummation of the large-β_0 terms:

A. Large-β_0-motivated expansion of observables

We summarize the presentation of Ref. [11]. We work in the RSch's where each β_k ($k \ge 2$) is a polynomial in n_f of order k; in other words, it is a polynomial in β_0:

$$\beta_k = \sum_{j=0}^{k} b_{kj} \beta_0^j , \qquad k = 2,3,\ldots \tag{40}$$

The $\overline{\rm MS}$ belongs to this class of schemes. In such schemes, the coefficients d_n of expansion (28) have the following specific form in terms of β_0:

$$\mathcal{D}(Q^2)_{\rm pt} = a + (c_{11}\beta_0 + c_{10})a^2 + (c_{22}\beta_0^2 + c_{21}\beta_0 + c_{20} + c_{2,-1}\beta_0^{-1})a^3 + \cdots . \tag{41}$$

We can construct a separation of this series into a sum of two RScl-independent terms – the leading-β_0 (Lβ_0), and beyond-the-leading-β_0 (BLβ_0)

$$\mathcal{D}_{\rm pt} = \mathcal{D}_{\rm pt}^{(\rm L\beta_0)} + \mathcal{D}_{\rm pt}^{(\rm BL\beta_0)} , \tag{42}$$

where

$$\mathcal{D}_{\rm pt}^{(\rm L\beta_0)} = a + a^2\left[\beta_0 c_{11}\right] + a^3\left[\beta_0^2 c_{22} + \beta_1 c_{11}\right] + a^4\left[\beta_0^3 c_{33} + \frac{5}{2}\beta_0\beta_1 c_{22} + \beta_2 c_{11}\right] + O(\beta_0^4 a^5). \tag{43}$$

Expression (43) is not the standard leading-β_0 contribution, since it contains also terms with β_j ($j \ge 1$), but only in a minimal way to ensure that the expression contains all the leading-β_0 terms and at the same time remains RScl-independent. It can be shown that, for inclusive observables, all the coefficients in this Lβ_0 contribution can be obtained, and can be expressed in the integral form [27]

$$\mathcal{D}^{(\rm L\beta_0)}(Q^2)_{\rm pt} = \int_0^\infty \frac{dt}{t} F_{\mathcal{D}}^{\mathcal{E}}(t)\, a(te^{\mathcal{C}} Q^2) , \tag{44}$$

where $F_{\mathcal{D}}^{\mathcal{E}}(t)$ is the (Euclidean) Lβ_0 -characteristic function. In $\overline{\rm MS}$ scheme, $\Lambda = \overline{\Lambda}$ which corresponds here to $\mathcal{C} = \overline{\mathcal{C}} \equiv -5/3$. No RScl μ^2 appears in (44). Expression (44) is referred to in the literature sometimes as dressed gluon approximation.

The BLβ_0 contribution is usually known only to $\sim a^3$ or $\sim a^4$. For it, we can use an arbitrary RScl $\mu^2 \equiv Q^2 e^{\mathcal{C}} \sim Q^2$. Further, the powers a^k can be reexpressed in terms of $\widetilde{a}_n(\mu^2)$ (24):

$$a^2 = \widetilde{a}_2 - (\beta_1/\beta_0)\widetilde{a}_3 + \cdots , \quad a^3 = \widetilde{a}_3 + \cdots . \tag{45}$$

Therefore,

$$\mathcal{D}(Q^2)_{(\rm TPS)} = \mathcal{D}^{(\rm L\beta_0)}(Q^2)_{\rm pt} + \widetilde{t}_2\, \widetilde{a}_2(Q^2 e^{\mathcal{C}}) + \widetilde{t}_3\, \widetilde{a}_3(Q^2 e^{\mathcal{C}}) + \widetilde{t}_4\, \widetilde{a}_4(Q^2 e^{\mathcal{C}}), \tag{46}$$

where $\widetilde{t}_2 = c_{10}$ is scheme-independent, and coefficients $\widetilde{t}_3$ and $\widetilde{t}_4$ have a scheme dependence (depend on β_2, β_3 – i.e., on b_{2j} and b_{3j}). We note that expression (46) is not really a pure TPS, because its Lβ_0 contribution (43) is not truncated. An observable-dependent scheme (D-scheme) can be chosen such that $\widetilde{t}_3 = \widetilde{t}_4 = 0$. For the Adler function $\mathcal{D} = d_v$, such a scheme will be called v-scheme. The analytization of the obtained $\mathcal{D}(Q^2)_{(\rm TPS)}$ (46) is performed by the substitution $\widetilde{a}_n \mapsto \widetilde{\mathcal{A}}_n$, Eq. (30), leading to the truncated analytic series (TAS)

$$\mathcal{D}(Q^2) = \mathcal{D}(Q^2)_{(\rm TAS)} + O(\beta_0^3 \widetilde{\mathcal{A}}_5) , \tag{47}$$

$$\mathcal{D}(Q^2)_{(\rm TAS)} = \int_0^\infty \frac{dt}{t} F_{\mathcal{D}}^{\mathcal{E}}(t)\, \mathcal{A}_1(te^{\mathcal{C}} Q^2) + c_{10}\widetilde{\mathcal{A}}_2(Q^2 e^{\mathcal{C}}) + \widetilde{t}_3 \widetilde{\mathcal{A}}_3(Q^2 e^{\mathcal{C}}) + \widetilde{t}_4 \widetilde{\mathcal{A}}_4(Q^2 e^{\mathcal{C}}). \tag{48}$$

In the D-scheme, the last two terms disappear. Eq. (48) is a method that one can use to evaluate any inclusive space-like QCD observable in any anQCD model. As argued in Sec. IV, the scale and scheme dependence of the TAS is very suppressed

$$\frac{\partial \mathcal{D}(Q^2)_{(\rm TAS)}}{\partial X} \sim \beta_0^3 \widetilde{\mathcal{A}}_5 \sim \beta_0^3 \mathcal{A}_5 \quad (X = \ln\mu^2, \beta_j) . \tag{49}$$

If the BLβ_0 perturbative contribution is known exactly only up to (and including) $\sim a^3$, then no $\widetilde{t}_4$ term appears in Eq. (48) and the precision in Eqs. (47) and (49) is diminished: $O(\beta_0^3 \mathcal{A}_5) \mapsto O(\beta_0^2 \mathcal{A}_4)$.

It is interesting to note that the Taylor expansion of $\mathcal{A}_1(te^{\mathcal{C}} Q^2)$ in $\mathcal{D}^{(\rm L\beta_0)}(Q^2)_{\rm an}$ in (48) around a chosen RScl $\ln(\mu^2)$ reveals just the aforementioned $a^n \mapsto \mathcal{A}_n$ analytization of the large-β_0 part (43), in any anQCD:

$$\begin{aligned}\mathcal{D}_{\rm an}^{(\rm L\beta_0)} &= \int_0^\infty \frac{dt}{t} F_{\mathcal{D}}^{\mathcal{E}}(t)\, \mathcal{A}_1(te^{\mathcal{C}} Q^2) \\ &= \mathcal{A}_1 + \mathcal{A}_2\left[\beta_0 c_{11}\right] + \mathcal{A}_3\left[\beta_0^2 c_{22} + \beta_1 c_{11}\right] \\ &\quad + \mathcal{A}_4\left[\beta_0^3 c_{33} + \frac{5}{2}\beta_0\beta_1 c_{22} + \beta_2 c_{11}\right] + O(\beta_0^4 \mathcal{A}_5),\end{aligned}$$

where $\mathcal{A}_k = \mathcal{A}_k(\mu^2;\beta_2,\beta_3,\ldots)$. In other words, at the leading-β_0 level, the natural analytization $a \mapsto \mathcal{A}_1$ in integral (44) is equivalent to the term-by-term analytization $a^n \mapsto \mathcal{A}_n$ ($\Leftrightarrow \widetilde{a}_n \mapsto \widetilde{\mathcal{A}}_n$) in the corresponding perturbation series. This thus represents yet another motivation for the analytization $a^n \mapsto \mathcal{A}_n$ [$\Leftrightarrow$ Eq. (30) postulated in Sec. IV] of *all* the available perturbation terms in $\mathcal{D}$. For the first motivation, based on the systematic weakening of the RScl&RSch dependence of the truncated analytized $\mathcal{D}$, see the end of Sec. IV.

TABLE I: Various order contributions to observables within PT, and MSSSh (=APT) methods [14, 16]:

Process	Method	1st order	2nd	3rd
GLS	PT	65.1%	24.4%	10.5%
($Q \sim 1.76$GeV)	APT	75.7%	20.7%	3.6%
r_τ	PT	54.7%	29.5%	15.8%
($M_\tau = 1.78$GeV)	APT	87.9%	11.0%	1.1%

B. Applications in phenomenology

Evaluations in MA model, with the MSSSh-approach $a^n \mapsto \mathcal{A}_n^{(\mathrm{MA})}$ [12–14], are usually performed in $\overline{\mathrm{MS}}$ scheme. The only free parameter is Λ ($= \overline{\Lambda}$). Fitting the experimental data for Υ-decay, $Z \to$ hadrons, $e^+e^- \to$ hadrons, to the MSSSh approach for MA at the two- or three-loop level, they obtained $\Lambda_{n_f=5} \approx 0.26$-$0.30$ GeV, corresponding to: $\Lambda_{n_f=3} \approx 0.40$-$0.44$ GeV, and $\pi \mathcal{A}_1{}^{(\mathrm{MA})}(M_Z^2) \approx 0.124$, which is above the pQCD world-average value $\alpha_s(M_Z^2) \approx 0.119 \pm 0.001$. The apparent convergence of the MSSSh nonpower truncated series is also remarkable – see Table I.

In Refs. [10, 11], the aforementioned TAS evaluation method (48) in anQCD models MA (4), M1 (17) and M2 (19) was applied to the inclusive observables Bjorken polarized sum rule (BjPSR) $d_b(Q^2)$, Adler function $d_v(Q^2)$ and semihadronic τ decay ratio r_τ The exact values of coefficients d_1 and d_2 are known for space-like observables BjPSR $d_b(Q^2)$ [28] and (massless) Adler function $d_v(Q^2)$ [29, 30]. (The exact coefficient d_3 of d_v has been recently obtained [31], but was not included in the analysis of Ref. [11] that we present here; rather, an estimated value of d_3 was used.) In the v-scheme, the evaluated massless $d_v(Q^2)$ is

$$d_v(Q^2)_{(\mathrm{TAS})} = \int_0^\infty \frac{dt}{t} F_v^{\mathcal{E}}(t)\, \mathcal{A}_1(te^{\overline{C}}Q^2; \beta_2{}^{(\mathrm{x})}, \beta_3{}^{(\mathrm{x})}) + \frac{1}{12} \widetilde{\mathcal{A}}_2(e^{\overline{C}}Q^2) \, , \qquad (50)$$

while BjPSR $d_b(Q^2)_{(\mathrm{TAS})}$ has one more term $\widetilde{t}_3 \widetilde{\mathcal{A}}_3(e^{\overline{C}}Q^2)$. The difference between the (massless) true $d_x(Q^2)$ ($x = v, b$) and $d_x(Q^2)_{(\mathrm{TAS})}$ is $O(\beta_0^2 \widetilde{\mathcal{A}}_4)$. The semihadronic τ decay ratio r_τ is, on the other hand, a time-like quantity, but can be expressed as a contour integral involving the Adler function d_v:

$$r_\tau(\Delta S{=}0, m_q{=}0) = \frac{2}{\pi} \int_0^{m_\tau^2} \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 + 2\frac{s}{m_\tau^2}\right) \mathrm{Im}\Pi(s) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\phi\, (1 + e^{i\phi})^3 (1 - e^{i\phi}) d_v(Q^2 = m_\tau{}^2 e^{i\phi}). \qquad (51)$$

This implies for the leading-β_0 term of r_τ

$$r_\tau(\Delta S{=}0, m_q{=}0)^{(\mathrm{L}\beta_0)} = \int_0^\infty \frac{dt}{t} F_r^{\mathcal{M}}(t)\, \mathfrak{A}_1(te^{\overline{C}} m_\tau{}^2), \qquad (52)$$

where $\mathfrak{A}_1$ is the time-like coupling appearing in Eqs. (6)-(9), and superscript $\mathcal{M}$ in the characteristic function indicates that it is Minkowskian (time-like). The latter was obtained by Neubert (second entry of Refs. [27]). The beyond-the-leading-β_0 (BLβ_0) contribution is the contour integral

$$r_\tau(\triangle S{=}0, m_q{=}0)^{(\mathrm{BL}\beta_0)} = \frac{1}{24\pi} \int_{-\pi}^{+\pi} d\phi\, (1{+}e^{i\phi})^3 (1{-}e^{i\phi}) \widetilde{\mathcal{A}}_2(e^{\overline{C}} m_\tau{}^2 e^{i\phi}). \qquad (53)$$

The parameters of anQCD models M1 (17) and M2 (19) were

TABLE II: Results of evaluation of $r_\tau(\triangle S = 0, m_q = 0)$ and of BjPSR $d_b(Q^2)$ ($Q^2 = 2$ and 1GeV^2), in various anQCD models, using TAS method (48). The experimental values are $r_\tau(\triangle S = 0, m_q = 0) = 0.204 \pm 0.005$, $d_b(Q^2 = 2\ \mathrm{GeV}^2) = 0.16 \pm 0.11$ and $d_b(Q^2 = 1\ \mathrm{GeV}^2) = 0.17 \pm 0.07$.

	r_τ	$d_b(Q^2 = 2)$	$d_b(Q^2 = 1)$
MA	0.141	0.137	0.155
M1	0.204	0.160	0.170
M2	0.204	0.189	0.219

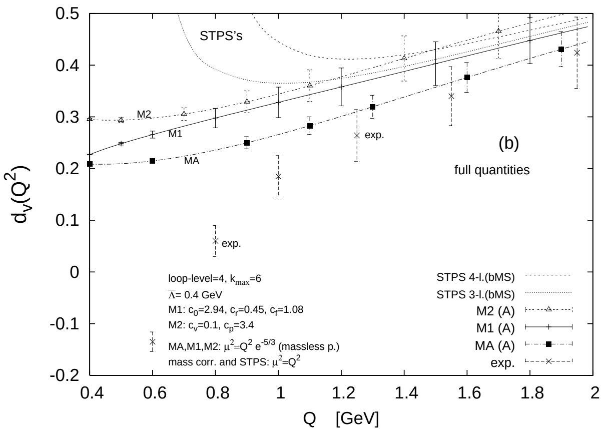

FIG. 6: Adler function as predicted by pQCD, and by our approach in several anQCD models: MA, M1, M2. The full quantity is depicted, with the contribution of massive quarks included. The experimental values are from [32]. Figure from: Ref. [11].

then determined [11] by fitting the evaluated observables to the experimental central values $r_\tau(\triangle S = 0, m_q = 0) = 0.204$ (for M1 and M2), and to $d_b(Q^2 = 1\mathrm{GeV}^2) = 0.17$ and $d_b(Q^2 = 2) = 0.16$ (for M1). For M1 we obtained: $c_f = 1.08$, $c_r = 0.45$, $c_0 = 2.94$. For M2 we obtained: $c_v = 0.1$ and $c_p = 3.4$.

The numerical results were then obtained [11]. In models MA, M1 and M2 they are given for r_τ in Table II, for Adler function $d_v(Q^2)$ in Fig. 6, and for BjPSR $d_b(Q^2)$ (in M1 and M2) in Figs. 7 and 8 (Table II and Figs. 6, 7, 8 are taken from Ref. [11]). All results were calculated in the v-scheme. For details, we refer to Ref. [11].

Analytic QCD models have been used also in the physics of mesons [33, 34], in calculating various meson masses by summing two contributions: that of the confining part and that of the (one-loop) perturbative part of the Bethe-Salpeter potential. In Refs. [33], the (one-loop) MA coupling [3] was used to calculate/predict the masses; in Refs. [34], the experimen-

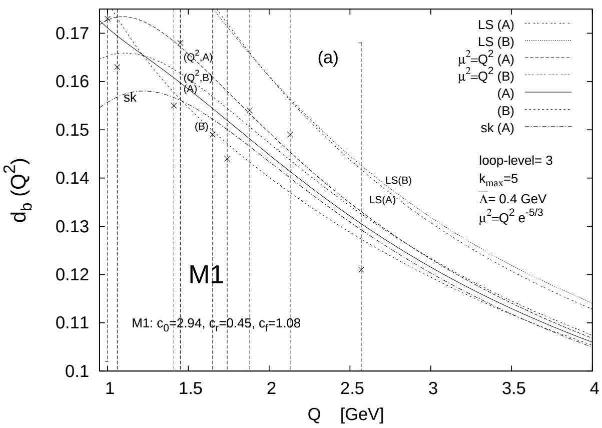

FIG. 7: Bjorken polarized sum rule (BjPSR) $d_b(Q^2)$ in model M1, in various RSch's and at various RScl's. The vertical lines represent experimental data, with errorbars in general covering the entire depicted range of values.

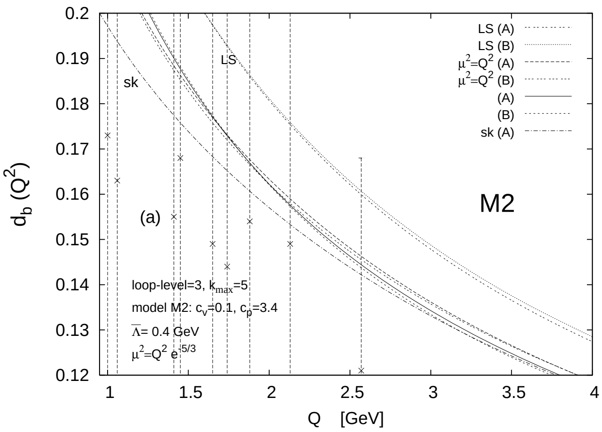

FIG. 8: As in the Fig. 7, but this time for model M2. Both figures from: Ref. [11].

tal mass spectrum was used to extract the approximate values of the (analytic) coupling $\mathcal{A}_1(Q^2)$ at low Q^2. In this formalism, the current quark masses were replaced by the constituent quark masses, accounting in this way approximately for the quark self-energy effects. The results by the authors of Ref. [34] indicate that $\mathcal{A}_1(Q^2)$ remains finite (and becomes possibly zero) when $Q^2 \to 0$.

VII. ANALYTIC QCD AND ITEP-OPE PHILOSOPHY

In general, the deviations of analytic $\mathcal{A}_1(Q^2)$ from the perturbative coupling $a_{\rm pt}(Q^2)$ at high $Q^2 \gg \Lambda^2$ are power terms

$$|\delta\mathcal{A}_1(Q^2)| \equiv |\mathcal{A}_1(Q^2) - a_{\rm pt}(Q^2)| \sim \left(\frac{\Lambda^2}{Q^2}\right)^k \quad (Q^2 \gg \Lambda^2),$$

where k is a given positive integer. Such a coupling introduces in the evaluation (of the leading-twist) of inclusive space-like observables $\mathcal{D}(Q^2)$, already at the leading-β_0 level, an UV contribution $\delta\mathcal{D}^{(\rm UV)}(Q^2)$ which behaves like a power term [18]

$$\delta\mathcal{D}^{(\rm UV)}(Q^2) \sim \left(\frac{\Lambda^2}{Q^2}\right)^{\min(k,n)} \qquad \text{if } k \neq n \;, \tag{54}$$

where $n \; \varepsilon \; \mathcal{N}$ is the position of the leading IR renormalon of the observable $\mathcal{D}(Q^2)$; if $k = n$, then the left-hand side of Eq. (54) changes to $(\Lambda^2/Q^2)^n \ln(\Lambda^2/Q^2)$ [18]. Such nonperturbative contributions coming from the UV sector contradict the ITEP Operator Product Expansion (OPE) philosophy (the latter saying that such terms can come only from the IR sector) [35].

Two specific sets of models of anQCD have been introduced in the literature so far such that they do not contradict the ITEP-OPE:

(A) a model set based on a modification of the $\beta(a)$ function [17];

(B) a model set obtained by a direct construction [18].

A. Set of models A

This is the set of models constructed in Refs. [17]. The TPS $\beta(a)$ used in pQCD is

$$\frac{\partial a}{\partial \ln Q^2} = \beta^{(N)}(a) = -\beta_0 a^2 \left(1 + \sum_{j=1}^{N} c_j a^j\right) . \tag{55}$$

This was then modified, $\beta^{(N)}(a) \mapsto \widetilde{\beta}^{(N)}(a)$, by fulfilling three main conditions:

1.) $\widetilde{\beta}^{(N)}(a)$ has the same expansion in powers of a as $\beta^{(N)}(a)$;

2.) $\widetilde{\beta}^{(N)}(a) \sim -\zeta a^p$ with $\zeta > 0$ and $p \leq 1$, for $a \gg 1$, in order to ensure the absence of Landau singularities;

3.) $\widetilde{\beta}^{(N)}(a)$ is analytic function at $a = 0$, in order to ensure $|a(Q^2) - a_{\rm pt}(Q^2)| < (\Lambda^2/Q^2)^k$ for any $k > 0$ at large Q^2 (thus respecting the ITEP-OPE approach).

This modification was performed by the substitution $a \mapsto u(a) \equiv a/(1+\eta a)$, $\eta > 0$ being a parameter, and

$$\widetilde{\beta}^{(N)}(a) \;=\; -\beta_0 \left[\kappa(a - u(a)) + \sum_{j=0}^{N} \widetilde{c}_j u(a)^{j+2}\right] , \tag{56}$$

and $\widetilde{c}_j$ are adjusted so that the first condition is fulfilled

$$\widetilde{c}_0 = 1 - \eta\kappa, \quad \widetilde{c}_1 = c_1 + 2\eta - \eta^2\kappa, \quad \text{etc.}$$

This procedure results in an analytic coupling $a(Q^2)$, with $p = 1$ and $\zeta = \beta_0\kappa$, and with two positive adjustable parameters κ and η. The QCD parameter Λ was taken the same as in the pQCD. Evaluation of observables was carried out in terms of power expansion, with the replacement $a_{\rm pt}^n \mapsto a^n$. Further, the couplings in this set are IR infinite: $a(Q^2) \sim 1/(Q^2)^{\beta_0\kappa} \to \infty$ when $Q^2 \to 0$. These new $a(Q^2)$'s are analytic ($a \equiv \mathcal{A}_1$). The RScl and RSch sensitivity of the modified TPS's of space-like observables turned out to be reduced. The author of Refs. [17] chose $\kappa = 1/\beta_0$; by fitting the predicted values of the static interquark potential to lattice results, he obtained $\eta \approx 4.1$.

B. Set of models B

This is the set of models for $\mathcal{A}_1$ constructed in Ref. [18]. A class of IR-finite analytic couplings which respect the ITEP-OPE philosophy can be constructed directly. The proposed class of couplings has three parameters (η, h_1, h_2). In the intermediate energy region ($Q \sim 1$ GeV), the proposed coupling has low loop-level and renormalization scheme dependence. We outline here the construction. We recall expansion (2) for the perturbative coupling $a(Q^2)$, where $L = \log Q^2/\Lambda^2$ and $K_{k\ell}$ are functions of the β-function coefficients. This expansion (sum) is in practice usually truncated in the index k ($k \le k_m$). The proposed coupling is obtained by modifying (the nonanalytic) L's to analytic quantities L_0 and L_1 that fall faster than any inverse power of Q^2 at large Q^2, and by adding to the truncated sum another quantity with such properties:

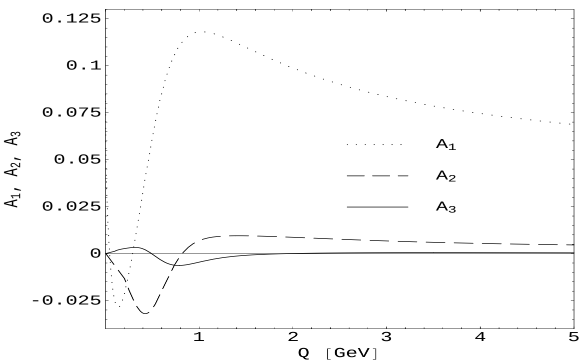

FIG. 9: The couplings $\mathcal{A}_2$ and $\mathcal{A}_3$, together with the corresponding coupling $\mathcal{A}_1$, are plotted as a function of Q, in the $\overline{\mathrm{MS}}$-scheme, with $\Lambda = 0.4$ GeV. The parameters used for the couplings are $\eta = 0.3$, $h_1 = 0.1$, and $h_2 = 0$. Figure from: Ref. [18].

$$\mathcal{A}_1{}^{(k_m)}(Q^2) = \sum_{k=1}^{k_m}\sum_{\ell=0}^{k-1} K_{l\ell}\,\frac{(\log L_1)^\ell}{L_0^k} + e^{-\eta\sqrt{x}} f(x), \qquad (57)$$

where $x = Q^2/\Lambda^2$. The second term is only relevant in the IR region, and the first term (double sum) plays, in the UV region, the role of the perturbative coupling. L_0 and L_1 are analytic and chosen aiming at a low k_m-dependence in the IR region.

$$\frac{1}{L_i} = \frac{1}{L} + \frac{e^{\nu_i(1-\sqrt{x})}}{1-x} g_i(x), \quad \nu_i > 0, \quad i = 0,1. \qquad (58)$$

Functions $g_i(x)$ are chosen in simple meromorphic form

$$g_0(x) = \frac{2x}{(1+\nu_0)+x(1-\nu_0)}, \quad 0 < \nu_0 < 1; \qquad (59)$$

$$g_1(x) = \frac{de^{-\nu_1}+x(d+1-de^{-\nu_1})}{d+x}, \quad d > 0, \qquad (60)$$

with the constants fixed at typical values $\nu_0 = 1/2$ and $\nu_1 = d = 2$. The additional expoinential term in (57) is chosen in a similar meromorphic form

$$e^{-\eta\sqrt{x}} f(x) = h_1\,\frac{1+h_2 x}{(1+x/2)^2}\,e^{-\eta\sqrt{x}}, \qquad (61)$$

Results for $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$, for specific typical values of parameters η, h_1 and h_2, are shown in Fig. 9. Couplings $\mathcal{A}_2$ and $\mathcal{A}_3$ are constructed via $\widetilde{\mathcal{A}}_2$ and $\widetilde{\mathcal{A}}_3$, according to the procedure described in Sec. IV, Eqs. (26).

A general remark: if $\mathcal{A}_1(Q^2)$ differs from the perturbative $a(Q^2)$ by less than any negative power of Q^2 at large Q^2 ($\gg \Lambda^2$), then the same is true for the difference between any $\widetilde{\mathcal{A}}_k(Q^2)$ and $\widetilde{a}_k(Q^2)$ ($k = 2,3,\ldots$).

VIII. SUMMARY

Various analytic (anQCD) models, i.e., analytic couplings $\mathcal{A}_1(Q^2)$, were reviewed, including some of those beyond the minimal analytization (MA) procedure.

Analytization of the higher powers $a^n \mapsto \mathcal{A}_n$ was considered; an RGE-motivated approach, which is applicable to any model of analytic $\mathcal{A}_1$, was described. Analytization of noninteger powers a^ν in MA model was outlined.

Evaluation methods for space-like and time-like observables in anQCD models were reviewed. A large-β_0-motivated expansion of space-like inclusive observables is proposed, with the resummed leading-β_0 part; on its basis, an evaluation of such observables in anQCD models is proposed: truncated analytic series (TAS). Several evaluated observables in various anQCD models were compared to the experimental data. We recall that evaluated expressions for space-like observables in anQCD respect the physical analyticity requirement even at low energy, in contrast to those in perturbative QCD (pQCD).

Finally, specific classes of analytic couplings $\mathcal{A}_1(Q^2)$ which preserve the OPE-ITEP philosophy were discussed, i.e., at high Q^2 they approach the pQCD coupling faster than any inverse power of Q^2. Such analytic couplings should eventually enable us to use the OPE approach in anQCD models.

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