# CORRECTED IMMERSED BOUNDARY FINITE ELEMENT METHODS FOR MOVING STOKES INTERFACE PROBLEM 

## GENARO LAYMUNS RICHARD

Thesis submitted to the Office of Research and Graduate Studies in partial fulfillment of the requirements for the degree of Master of Science in Engineering

Advisor:
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Santiago de Chile, July 2021
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To my family.

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#### Abstract

Fluid-structure interaction problems for a Stokes flow have a wide variety of applications, especially in biomechanics. The presence of an immersed elastic interface in the domain produces a local force on the fluid which causes discontinuities in its velocity field and pressure. Because the solution of the problem is discontinuous, standard finite element methods need to be modified in order to achieve optimal convergence rates. In this thesis we develop a high-order finite element method where the mesh does not need to be modified to match the interface, and thus can easily be extended to moving-interface problems. We show that our method recovers the optimal convergence rate of standard finite element methods for non-interface Stokes problems by precalculating correction functions that hold all discontinuities across the interface. We finally show numerical examples that suggest that our method is more stable than the standard immersed boundary finite element method schemes, in the sense that it preserves physical quantities like energy and mass for different initial configurations of the interface.


Keywords: Stokes flow, immersed boundary, moving interface, finite element method, correction functions, Fourier interpolation.


#### Abstract

RESUMEN

Los problemas de interacción fluido-estructura en flujos de Stokes tienen una gran cantidad de aplicaciones, especialmente en biomecánica. La presencia de una interfaz elástica sumergida en el dominio genera una fuerza local en el fluido, lo cual produce discontinuidades en el comportamiento de la velocidad y presión del fluido a través de ella. Debido a que la solución del problema es discontinua, los métodos de elementos finitos tradicionales deben ser modificados para alcanzar la tasa de convergencia óptima. En esta tesis presentamos un método de elementos finitos de alto orden donde la malla no necesita ser modificada para coincidir con la interfaz, lo que nos permite extender fácilmente el método a problemas temporales con una interfaz en movimiento. El método desarrollado recupera la convergencia óptima de los métodos de elementos finitos tradicionales debido a que es capaz de precalcular funciones correctoras que cumplen todas las discontinuidades de la solución a través de la interfaz. Finalmente se muestran ejemplos numéricos para problemas temporales donde nuestro método es más estable que los métodos tradicionales de elementos finitos de frontera inmersa, en el sentido que conserva ciertas cantidades físicas del problema como la energía del sistema o la cantidad de masa para distintas configuraciones iniciales de la interfaz.


Palabras Claves: flujo de Stokes, frontera inmersa, interfaz en movimiento, método de elementos finitos, funciones correctoras, interpolación de Fourier.

## 1. INTRODUCTION

### 1.1. Motivation

Moving interface problems have many applications in engineering and physics, in fields such as fluid mechanics, gas dynamics, biomechanics, among others. In the particular case of laminar flow (low Reynolds number), i.e. where the fluid velocity is low, viscosity is large or the length-scales of the flow are small, the Stokes flow equations provide a good approximation to the flow field (Kim \& Karrila, 1991). This is the case, for example, of the dynamics of swimming microorganisms (Lauga \& Powers, 2009) and blood flow through the veins (Siddiqui, Sohail, Naqvi, \& Haroon, 2017). If we consider an immersed elastic interface in a Stokes flow, the presence of such interface produces local forces on the fluid causing it to move, while at the same time the velocity of the fluid determines how the interface moves.

Solving the fluid-structure interaction on a Stokes flow has many applications in biomechanics, which makes the problem not only interesting and challenging mathematically, but also a powerful tool for designing prosthetic valves or membranes, since it would be possible to approximate the flow field and shear stress for a specific design (Yang \& Wang, 1983). In order to study the fluid dynamics of heart valves, Peskin developed the Immersed Boundary method (Peskin, 1977) to solve the equations of motion of a fluid in presence of cardiac valves, and it has been applied to model both natural and prosthetic cardiac valves. Another example is the vibration of the basilar membrane within the cochlea (inner ear), which can be modeled by considering both rigid (surrounding bone) and flexible (basilar membrane) parts (Beyer, 1992). The main function of the cochlea is to convert mechanical vibrations induced by sound waves into electrical nerve impulses. A two-dimensional model of the cochlea is studied in (Beyer, 1992) and is solved using the Immersed Boundary Method to model the vibration of the basilar membrane. Solving
this problem can help us understand the mechanical impulses that cause hearing and thus design prosthetic cochleas to fix hearing loss.

Because of the presence of an immersed interface, local forces are applied in the fluid causing discontinuities on the solution. Different approaches have been followed to consider this local force using both finite difference and finite element methods. The Immersed Boundary method proposed by (Peskin, 1977) uses a finite difference approach to approximate the Dirac delta function, but have shown to be highly unstable on explicit schemes and stable on implicit schemes that are too expensive for practical computations (Tu \& Peskin, 1992). For a finite element method approach we need to consider triangulations of the domain. Standard finite element methods usually give continuous approximations of the solution and are only able to consider discontinuities across the faces of the elements. Since the solution is discontinuous across the interface, these methods need to be modified either matching the mesh to the interface or changing the finite element spaces. Boundary integral equation methods have also been proposed to approximate the solution of this problem (Mori, 2008; Mori, Rodenberg, \& Spirn, 2019), which are based on approximating the Green function for the interface Stokes problem, giving a solution for the velocity and pressure fields in $\mathbb{R}^{2}$.

In this thesis we will develop a high-order finite element method to approximate the solution of a steady-state Stokes interface problem. Our method is able to consider interfaces that not necessarily match the mesh, and thus it is easy to extend the steady-state method to approximate the solution of a moving Stokes interface problem.

### 1.2. Stokes flow physics

The Navier-Stokes equations are a set of partial differential equations that describe the motion of a viscous fluid by conserving certain physics properties. If we consider that mass should be conserved at any point of the domain, the following equation must hold in
the fluid

$$
\partial_{t} \rho+\nabla \cdot(\rho \boldsymbol{u})=0,
$$

or equivalently,

$$
\begin{equation*}
D_{t} \rho=-\rho \nabla \cdot \boldsymbol{u} \tag{1.1}
\end{equation*}
$$

where the nonlinear operator $D_{t}=\partial_{t}+\boldsymbol{u} \cdot \nabla$ is the material derivative. Here $\rho$ and $\boldsymbol{u}$ are the density and velocity field of the fluid, respectively.

The conservation of momentum can be expressed using Newton's second law, which result in the Cauchy momentum equation

$$
\begin{equation*}
\rho D_{t} \boldsymbol{u}=\nabla \cdot \boldsymbol{\sigma}+\boldsymbol{f} \tag{1.2}
\end{equation*}
$$

where $\sigma$ is the stress tensor and $f$ are all external forces per unit of volume acting on the fluid. For example, if we consider the gravitational acceleration $\boldsymbol{g}$, the external force is given by $\boldsymbol{f}=\rho \boldsymbol{g}$. The Cauchy stress tensor can be split into two separate terms,

$$
\boldsymbol{\sigma}=-p I+\boldsymbol{T}
$$

with $\boldsymbol{T}$ being the so-called deviatoric stress tensor. This can be interpreted as two separate forces acting on the fluid. The force caused by the pressure is given by $\nabla \cdot(-p I)=-\nabla p$, while the viscous forces of the fluid are given by $\nabla \cdot \boldsymbol{T}$. In the case of a compressible Newtonian fluid, it is possible to write the deviatoric stress tensor as

$$
\boldsymbol{T}=\mu\left(\nabla \boldsymbol{u}+(\nabla \boldsymbol{u})^{\top}\right)+\lambda(\nabla \cdot \boldsymbol{u}) I,
$$

where $\mu$ is the dynamic viscosity and $\lambda$ is the second viscosity coefficient, which is defined such that $\operatorname{trace}(\boldsymbol{T})=0$. Replacing this in (1.2) yields the Navier-Stokes equations for compressible Newtonian fluids

$$
\rho\left(\partial_{t} \boldsymbol{u}+\boldsymbol{u} \cdot \nabla \boldsymbol{u}\right)=-\nabla p+\nabla \cdot\left(\mu\left(\nabla \boldsymbol{u}+(\nabla \boldsymbol{u})^{\top}\right)+\lambda(\nabla \cdot \boldsymbol{u})\right)+\boldsymbol{f}
$$

Stokes flow equations can be easily obtained as a linearization of the Navier-Stokes equations. The Reynolds number $R e$ is defined as the ratio of inertial forces to viscous forces caused on a fluid (Kim \& Karrila, 1991),

$$
R e=\frac{\rho \boldsymbol{u} L}{\mu}
$$

where $L$ is a characteristic linear dimension of the domain, for example, for flow in a pipe $L$ is the diameter of the pipe. The Reynolds number quantifies the relative importance of both forces, and can be used to predict whether a certain flow will be turbulent or not. When viscous forces are dominant, i.e. $R e \ll 1$, it is possible to observe a laminar flow, while when inertial forces are dominant, i.e. $R e \gg 1$, turbulent flow can occur. The Reynolds number is small in situations where the fluid velocity is low enough, viscosity is high enough or the length-scale of the problem is small, like in blood flow through veins. In these cases the non-linear part of the Navier-Stokes equations can be neglected. If we also consider a stationary flow $\left(\partial_{t} u=0\right)$, we obtain the Stokes flow equations

$$
\begin{equation*}
-\nabla \cdot\left(\mu\left(\nabla \boldsymbol{u}+(\nabla \boldsymbol{u})^{\top}\right)+\lambda(\nabla \cdot \boldsymbol{u})\right)+\nabla p=\boldsymbol{f} \tag{1.3}
\end{equation*}
$$

In the case where the density $\rho$ is constant, conservation of mass (1.1) becomes a condition of an incompressible flow,

$$
\nabla \cdot \boldsymbol{u}=0
$$

By replacing this in (1.3) and assuming a constant viscosity $\mu$ we obtain the equations describing a Stokes flow for an incompressible fluid

$$
\begin{equation*}
-\mu \Delta \boldsymbol{u}+\nabla p=\boldsymbol{f} \tag{1.4}
\end{equation*}
$$

where we used the fact that $\nabla \cdot(\nabla \boldsymbol{u})^{\top}=\nabla(\nabla \cdot \boldsymbol{u})=\mathbf{0}$.

### 1.3. Fluid-structure interaction

In this section we will derive the model and introduce the strong formulation of the problem following a similar procedure as outlined in (Dunn, Lui, \& Sarkis, 2021). Consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^{d}$. Throughout this thesis we assume $d=2$, but the results can be easily extended to $d=3$.

Inside $\Omega$ there is a closed, smooth curve $\Gamma$ representing a massless, elastic interface, which the fluid cannot go through. This interface splits the domain $\Omega$ in two, which we will denote by $\Omega^{-}$for the region enclosed by $\Gamma$ and by $\Omega^{+}$for the exterior region, with boundaries defined as $\partial \Omega^{-}=\Gamma$ and $\partial \Omega^{+}=\Gamma \cup \partial \Omega$, respectively. At a given time $t \geq 0$, we denote the cartesian coordinates of the interface in parametric form by $\boldsymbol{X}(s, t)$, such that $\Gamma(t)=\{\boldsymbol{X}(s, t), s \in[0,2 \pi]\}$. We assume that this parametrization is known at $t=0$. This is not necessarily an arc-length parametrization, which allows us to define an initial elastic interface with both bending and stretching. Physically, we can interpret the point $\boldsymbol{X}(s, t)$ as the material point on the elastic membrane that moves from an initial position $\boldsymbol{X}(s, 0)$.

The unit tangent vector to the interface is defined as

$$
\boldsymbol{\tau}=\partial_{s} \boldsymbol{X} /\left\|\partial_{s} \boldsymbol{X}\right\|
$$

The boundary tension $T(s, t)$ is modeled using a generalized Hooke's law

$$
T(s, t)=\sigma\left(\left\|\partial_{s} \boldsymbol{X}\right\|, s, t\right)
$$

where $\sigma\left(\left\|\partial_{s} \boldsymbol{X}\right\|, s, t\right)$ is a function defining the stress caused on a point $\boldsymbol{X}(s, t)$ of the interface given its local deformation at any time $t$. To compute the elastic force between two points $a$ and $b$, we note that

$$
(T \boldsymbol{\tau})(b, t)-(T \boldsymbol{\tau})(a, t)=\int_{a}^{b} \partial_{s}(T \boldsymbol{\tau})(s, t) \mathrm{d} s=\int_{a}^{b} \frac{\partial_{s}(T \boldsymbol{\tau})(s, t)}{\left\|\partial_{s} \boldsymbol{X}\right\|} \cdot\left\|\partial_{s} \boldsymbol{X}\right\| \mathrm{d} s
$$

Since this holds for any open interval $(a, b)$, we can write the density force $\boldsymbol{\beta}$ acting on $\Gamma$ per unit length as

$$
\boldsymbol{\beta}(s, t)=\frac{\partial_{s}(T \boldsymbol{\tau})}{\left\|\partial_{s} \boldsymbol{X}\right\|}
$$

If we consider $\sigma\left(\left\|\partial_{s} \boldsymbol{X}\right\|, s, t\right)$ to be proportional to $\left\|\partial_{s} \boldsymbol{X}\right\|$, i.e., the deformation of the interface is withing elastic range, then the density force is given by

$$
\boldsymbol{\beta}(s, t)=-\kappa \frac{\partial_{s}^{2} \boldsymbol{X}}{\left\|\partial_{s} \boldsymbol{X}\right\|},
$$

where $\kappa>0$ is a constant depending on the material and the negative sign is chosen so the energy of the system dissipates (see Section 4.1).

The deformed and stretched interface applies a local force when computing the fluid velocity and pressure, which can be incorporated into equation (1.4) as a force $\boldsymbol{B}$

$$
\begin{aligned}
-\mu \Delta \boldsymbol{u}+\nabla p & =\boldsymbol{f}+\boldsymbol{B} \\
\boldsymbol{B}(\boldsymbol{x}) & =\int_{\Gamma} \boldsymbol{\beta} \cdot \delta(\boldsymbol{x}-\boldsymbol{X}) \mathrm{d} S .
\end{aligned}
$$

From its definition, we can easily check that $\boldsymbol{B}=0$ when $\boldsymbol{x} \in \Omega \backslash \Gamma$.
Two additional constraints must be added on the interface. We impose a no-slip condition between the fluid and the interface. This condition is considered by taking

$$
\begin{aligned}
\llbracket \boldsymbol{u} \rrbracket & =\mathbf{0} \quad \text { on } \Gamma \\
\partial_{t} \boldsymbol{X} & =\boldsymbol{u}(\boldsymbol{X}, t) .
\end{aligned}
$$

where $\llbracket \phi \rrbracket$ denotes the jump of a function $\phi$ across $\Gamma$.

### 1.4. Previous work

To solve this problem, Peskin introduced the Immersed Boundary Method (Peskin, 1977) to approximate the fluid dynamics of blood flow through heart valves, a finite difference method with a specific smoothness of the $\delta$ function. See also the review paper (Peskin, 2002). The stiffness which arises from the singular force and the time-step limit in computations showed a high instability on explicit schemes and suggests the use of a fully implicit method for solving the Stokes equations (Tu \& Peskin, 1992), however the method proposed by Tu and Peskin was too expensive for practical computations (much higher computational time and storage space required when compared to the explicit method). The analysis of a simplified model for the Stokes equation is done in (Mori, 2008) and in (Mori et al., 2019) they proved the well-posedness and global behaviour. Subsequently, several finite difference methods were proposed to integrate the natural discontinuity of the equations and hence obtain higher order approximations, see for instance the immersed interface method (LeVeque \& Li, 1994, 1997; Li \& Lai, 2001; Li \& Ito, 2006) and the corrected method (Marques, Nave, \& Rosales, 2011).

As we will see in Section 3.1, the presence of a local force $\boldsymbol{B}$ on $\Gamma$ causes a discontinuity on $p$ and the derivatives of both $\boldsymbol{u}$ and $p$ across the interface. It is possible to strongly include the discontinuity conditions to the Stokes equations, which allow us to use different finite element methods to solve the problem. The basis functions used on the approximation by finite element can be easily replaced by higher order functions to obtain a better convergence rate, thus suggesting that a finite element method may be used to obtain more scalable methods.

If standard finite element methods were used to solve the problem without taking the discontinuities into account, the convergence rate of the methods have proven to be nonoptimal (see Section 3.6). These methods usually consider continuous solutions and only are able to consider discontinuities across the element edges. Because of this, a natural


Figure 1.1. Schematic depiction of a deforming-mesh method (Scott, 2015). approach to recover optimal convergence rates in the steady-state Stokes interface problem is to define a mesh that matches the elements with the interface $\Gamma$. However, this approach is not efficient when solving moving interface problems, because it would be necessary to obtain a new mesh every time the interface moves to a new position, forcing us to recalculate integrals between basis functions on the elements that were changed, and invert the new matrix of the linear system.

Figures 1.1, 1.2 and 1.3 (Scott, 2015) show how different approaches may be used to consider discontinuous solutions in different finite element methods applied to immersed boundary problems. In deforming-mesh methods, the nodes of the domain evolve along with the points of the interface. This method may give really good results when the mesh is adapted to more complex geometries of the interface, for example, if curved elements are considered for triangles touching the interface $\Gamma$. This may seem like a good approach at first, but it is possible that the deformations lead to meshes with a poor aspect ratio, thus resulting in much worse convergence rates.

On the other hand, universal meshes only adapt a small number of triangles near the interface by maintaining (or slightly changing) the aspect ratio of the original mesh and


Figure 1.2. Schematic depiction of a universal mesh (Scott, 2015).
matching the interface $\Gamma$ to the element boundaries. On both deforming-mesh and universal mesh methods, it is necessary at every time step to adapt the mesh to the domain depending on the position of the interface. Even a little movement of the interface forces us to obtain a new mesh and calculate integrals over each modified element again, which can be computationally expensive if we want to solve the problem for several different interfaces.

Fixed-mesh methods do not modify the mesh at any time step. The main advantage of this method is that it only needs to obtain a good-quality mesh once. This eases the calculations on integrals between basis functions on each triangle, since they are always the same except possibly for triangles which the interface intersects due to the local elastic force, but need to consider additional constraints on those triangles in order to recover optimal accuracy. Finite element methods with fixed mesh have been comprehensively studied for elliptic interface problem. For instance, the corrected schemes (Guzmán, Sánchez, \& Sarkis, 2016a, 2016b), the immersed interface finite element method (Li, 1998; Gong, Li, \& Li, 2008; Adjerid, Chaabane, \& Lin, 2015; He, Lin, \& Lin, 2011; Guzmán, Sánchez, \& Sarkis, 2017) and unfitted Nitsche scheme or cut finite element methods (Burman \&


Figure 1.3. Schematic depiction of a fixed-mesh method (Scott, 2015).
Hansbo, 2010; Burman \& Zunino, 2011; Burman \& Hansbo, 2012; Burman, Guzmán, Sánchez, \& Sarkis, 2018)

Throughout this thesis we will derive a fixed-mesh method that is able to compute the discontinuities across the interface before solving the equations in the domain $\Omega$. This method is based on obtaining local correction functions that capture the discontinuities of the solution (of any arbitrary order) across the interface, which allows us to consider discontinuous solutions while preserving the convergence rate of different finite element methods without modifying the mesh nor the finite element spaces at any time step.

## 2. PRELIMINARIES

### 2.1. Notation

First we introduce some definitions

DEFINITION 1. Let $\boldsymbol{u}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ be a sufficiently smooth vector field such that $\boldsymbol{u}=\left(u_{1}, \ldots, u_{n}\right)^{\top}$ and let $A, B: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n \times n}$ be sufficiently smooth tensors. We will use the following operations.

$$
\begin{aligned}
\nabla \boldsymbol{u} & =\left[\partial_{x_{j}} u_{i}\right]_{i, j=1}^{n} \\
\Delta \boldsymbol{u} & =\left(\Delta u_{1}, \ldots, \Delta u_{n}\right)^{\top} \\
\nabla \cdot A & =\left[\sum_{j=1}^{n} \partial_{x_{j}} A_{i j}\right]_{i=1}^{n} \\
A: B & =\sum_{i, j=1}^{n} A_{i, j} B_{i, j}
\end{aligned}
$$

Note that $\nabla \boldsymbol{u}$ is a matrix which $i$-th row is given by $\nabla u_{i}$, while $\nabla \cdot A$ is a vector which $i$-th component is obtained by taking divergence of the $i$-th row of $A$, and thus $\Delta \boldsymbol{u}=\nabla \cdot \nabla \boldsymbol{u}$ just like for scalar functions.

DEFINITION 2. Let $\boldsymbol{u}, \boldsymbol{v}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ be sufficiently smooth vector fields and let $p, q:$ $\mathbb{R}^{n} \rightarrow \mathbb{R}$ be scalar functions. Let $\Omega$ be any open subset of $\mathbb{R}^{n}$ and let $\Gamma$ be any $(n-1)$ dimensional surface in $\mathbb{R}^{n}$. We will use the following notation

$$
\begin{aligned}
(p, q)_{\Omega} & =\int_{\Omega} p q d x \\
(\boldsymbol{u}, \boldsymbol{v})_{\Omega} & =\int_{\Omega} \boldsymbol{u} \cdot \boldsymbol{v} d x \\
(\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_{\Omega} & =\int_{\Omega} \nabla \boldsymbol{u}: \nabla \boldsymbol{v} d x
\end{aligned}
$$

We define the bilinear forms $\langle\cdot, \cdot\rangle_{\Gamma}$ the same way as integrals $(\cdot, \cdot)_{\Omega}$ but integrating over $\Gamma$.

DEFINITION 3. We will use the following vector spaces for scalar functions

$$
\begin{aligned}
L^{2}(\Omega) & =\left\{\varphi: \Omega \rightarrow \mathbb{R}, \int_{\Omega}|\varphi|^{2} d \boldsymbol{x}<\infty\right\} \\
L_{0}^{2}(\Omega) & =\left\{\varphi: \varphi \in L^{2}(\Omega), \int_{\Omega} \varphi d \boldsymbol{x}=0\right\} \\
H^{1}(\Omega) & =\left\{\varphi: \varphi \in L^{2}(\Omega), \partial_{x_{i}} \varphi \in L^{2}(\Omega), \text { for } i=1, \ldots, n\right\} \\
H_{0}^{1}(\Omega) & =\left\{\varphi: \varphi \in H^{1}(\Omega), \gamma_{0} \varphi=0\right\}
\end{aligned}
$$

For any vector field $\boldsymbol{u}=\left(u_{1}, \ldots, u_{n}\right)^{\top}$ and any vector space $V$ of scalar functions defined on $\Omega \subset \mathbb{R}^{n}$, we say that $\boldsymbol{u} \in[V]^{n}$ if $u_{i} \in V$ for $i=1, \ldots, n$. The functional $\gamma_{0}: H^{1}(\Omega) \rightarrow H^{1 / 2}(\partial \Omega)$ is called the Dirichlet trace and restricts the function to the domain boundary $\partial \Omega$. See (Brezis, 2010) for a definition of fractional Sobolev spaces $H^{s}(\Omega)$ for $s \in(0,1)$.

We also introduce some standard definitions for domain and triangulations to be used in a finite element setting. We follow (Ern \& Guermond, 2013). See also (Brenner, Scott, \& Scott, 2008), (Braess, 2007).

Definition 4. A domain $\Omega \subset \mathbb{R}^{n}$ is an open, bounded, connected set with Lipschitz continuous boundary $\partial \Omega$. We say that $\Omega$ is of class $C^{m}$, for $m \geq 1$, if the respective local Lipschitz maps are of class $C^{m}$.

We define a triangulation $\mathcal{T}_{h}$ of the domain $\Omega \in \mathbb{R}^{n}$ as the union of a finite number of non-empty interior simplices $K$ (triangles for $n=2$ ) such that $\mathcal{T}_{h}$ forms a partition of $\Omega$, i.e.

$$
\bar{\Omega}=\bigcup_{K \in \mathcal{T}_{h}} \bar{K} \quad \text { and } \quad \operatorname{int}(K) \cap \operatorname{int}\left(K^{\prime}\right)=\emptyset \quad \text { for } K, K^{\prime} \in \mathcal{T}_{h}, K \neq K^{\prime}
$$

The subsets $K$ are called elements. Moreover, for all $K, K^{\prime} \in \mathcal{T}_{h}$ the intersection $\bar{K} \cap \bar{K}^{\prime}$ is:

- Either empty or a common vertex in dimension $n=1$,
- Either empty, or a common vertex, or a common edge in dimension $n=2$,
- Either empty, or a common vertex, or a common edge, or a common face in dimension $n=3$.

The subscript $h$ refers to the level of refinement of the mesh. Defining the diameter of an element $K \in \mathcal{T}_{h}$ as

$$
h_{K}=\operatorname{diam}(K)=\max _{x_{1}, x_{2} \in K}\left\|x_{1}-x_{2}\right\|_{2}
$$

where $\|\cdot\|_{2}$ is the Euclidean norm in $\mathbb{R}^{n}$, the parameter $h$ (mesh-size) is defined by

$$
h=\max _{K \in \mathcal{T}_{h}} h_{K}
$$

We will denote a sequence of triangulations by $\left\{\mathcal{T}_{h}\right\}_{h>0}$.

### 2.2. Stokes Problem

In this section we derive a mixed weak formulation for the standard steady-state Stokes problem and state its well-posedness.

Let $\Omega$ be an open, bounded, connected set in $\mathbb{R}^{2}$ with Lipschitz boundary and assume that the flow is stationary, $\partial_{t} \boldsymbol{u}=0$. Given a function $\boldsymbol{f}: \Omega \rightarrow \mathbb{R}^{2}$, a body force acting on the fluid, we seek for the velocity $\boldsymbol{u}: \Omega \rightarrow \mathbb{R}^{2}$ and pressure $p: \Omega \rightarrow \mathbb{R}^{2}$ solution of

$$
\begin{array}{rlrl}
-\mu \Delta \boldsymbol{u}+\nabla p & =\boldsymbol{f} & & \text { in } \Omega, \\
\nabla \cdot \boldsymbol{u} & =0 & & \text { in } \Omega, \\
\boldsymbol{u} & =\mathbf{0} & & \text { on } \partial \Omega, \\
\int_{\Omega} p \mathrm{~d} \boldsymbol{x} & =0 . & \tag{2.1d}
\end{array}
$$

These equations are termed as the momentum equation (2.1a) and the mass conservation equation (2.1b). We first derive the weak formulation for (2.1a) multiplying by a vector
field test function $\boldsymbol{v} \in\left[H_{0}^{1}(\Omega)\right]^{n}$ and integrating over $\Omega$. We obtain

$$
-(\mu \Delta \boldsymbol{u}, \boldsymbol{v})_{\Omega}+(\nabla p, \boldsymbol{v})_{\Omega}=(\boldsymbol{f}, \boldsymbol{v})_{\Omega}
$$

Integrating by parts the two terms on the left-hand side and imposing the zero boundary condition of the test function we obtain

$$
(\mu \nabla \boldsymbol{u}, \nabla \boldsymbol{v})_{\Omega}-(p, \nabla \cdot \boldsymbol{v})_{\Omega}=(\boldsymbol{f}, \boldsymbol{v})_{\Omega}
$$

Similarly, we obtain the weak form of the second equation (2.1b) multiplying by a scalar test function $q \in L_{0}^{2}$

$$
(\nabla \cdot \boldsymbol{u}, q)_{\Omega}=0
$$

Thus, the mixed weak formulation reads as follows. Find $\boldsymbol{u} \in\left[H_{0}^{1}(\Omega)\right]^{n}$ and $p \in$ $L_{0}^{2}(\Omega)$ such that

$$
\begin{align*}
(\mu \nabla \boldsymbol{u}, \nabla \boldsymbol{v})_{\Omega}-(p, \nabla \cdot \boldsymbol{v})_{\Omega} & =(\boldsymbol{f}, \boldsymbol{v})_{\Omega} & & \forall \boldsymbol{v} \in\left[H_{0}^{1}(\Omega)\right]^{n},  \tag{2.2a}\\
(q, \nabla \cdot \boldsymbol{u})_{\Omega} & =0 & & \forall q \in L_{0}^{2}(\Omega) . \tag{2.2b}
\end{align*}
$$

Observe that these equations are well-defined for $\boldsymbol{u} \in\left[H_{0}^{1}(\Omega)\right]^{n}$ and $p \in L_{0}^{2}(\Omega)$, and the given force $\boldsymbol{f} \in\left[L^{2}(\Omega)\right]^{n}$.

Equivalently, it is possible to rewrite (2.2) in a more general setting introducing the bilinear forms $a:\left[H_{0}^{1}(\Omega)\right]^{n} \times\left[H_{0}^{1}(\Omega)\right]^{n} \rightarrow \mathbb{R}$ and $b:\left[H_{0}^{1}(\Omega)\right]^{n} \times L_{0}^{2}(\Omega) \rightarrow \mathbb{R}$ defined by

$$
a(\boldsymbol{\varphi}, \boldsymbol{\psi})=(\mu \nabla \boldsymbol{\varphi}, \nabla \boldsymbol{\psi})_{\Omega}, \quad b(\boldsymbol{\varphi}, \phi)=-(\phi, \nabla \cdot \boldsymbol{\varphi})
$$

for all functions $\boldsymbol{\varphi}, \boldsymbol{\psi} \in\left[H_{0}^{1}(\Omega)\right]^{n}$ and $\phi \in L_{0}^{2}(\Omega)$, obtaining the formulation: Given $\boldsymbol{f} \in\left[L^{2}(\Omega)\right]^{n}$, find $\boldsymbol{u} \in\left[H_{0}^{1}(\Omega)\right]^{n}$ and $p \in L_{0}^{2}(\Omega)$ such that

$$
\begin{aligned}
a(\boldsymbol{u}, \boldsymbol{v})+b(\boldsymbol{v}, p) & =(\boldsymbol{f}, \boldsymbol{v})_{\Omega} & & \forall \boldsymbol{v} \in\left[H_{0}^{1}(\Omega)\right]^{n}, \\
b(\boldsymbol{u}, q)_{\Omega} & =0 & & \forall q \in L_{0}^{2}(\Omega) .
\end{aligned}
$$

Note that if $\boldsymbol{f}$ and $\partial \Omega$ are sufficiently smooth then the solution $(\boldsymbol{u}, p)$ of the weak formulation (2.2) is also smooth and solves the Stokes problem (2.1).

For the sake of completeness, we state the well-posedness of problem (2.2). See (Girault \& Raviart, 1981; Ern \& Guermond, 2013) for a more detailed proof.

THEOREM 1. The mixed weak form Stokes problem (2.2) is well posed, i.e., there exists a unique solution $(\boldsymbol{u}, p) \in\left[H_{0}^{1}(\Omega)\right]^{n} \times L_{0}^{2}(\Omega)$ and there is a constant $C>0$ such that for $\boldsymbol{f} \in\left[L^{2}(\Omega)\right]^{n}$

$$
\|\boldsymbol{u}\|_{H^{1}(\Omega)}+\|p\|_{L^{2}(\Omega)} \leq C\|\boldsymbol{f}\|_{L^{2}(\Omega)}
$$

Proof. To prove existence and uniqueness of this problem, it is convenient to consider the vector space $H_{0}(\operatorname{div}, \Omega)=\left\{\boldsymbol{v} \in\left[H_{0}^{1}(\Omega)\right]^{n}, \nabla \cdot \boldsymbol{v}=0\right\}$, with the norm $\|\cdot\|_{H^{1}(\Omega)}=|\cdot|_{1, \Omega}$, where we used that in $H_{0}^{1}(\Omega)$ the semi-norm $|\cdot|_{1, \Omega}=\|\nabla \cdot\|_{L^{2}(\Omega)}$ is an equivalent norm.

From the definition of the bilinear form $a(\boldsymbol{\varphi}, \boldsymbol{\psi})$, we can easily see that it is continuous in $H_{0}(\operatorname{div}, \Omega)$ and that it is $H_{0}(\operatorname{div}, \Omega)$-elliptic, since $a(\boldsymbol{v}, \boldsymbol{v})=\mu|\boldsymbol{v}|_{1, \Omega}^{2}=\mu\|\boldsymbol{v}\|_{H^{1}(\Omega)}$. From Lemma 3.2 from (Girault \& Raviart, 1981), we know that $\nabla \cdot(\cdot)$ is an isomorphism from $H_{0}(\operatorname{div}, \Omega)^{\top}$ to $L_{0}^{2}(\Omega)$. Using this result and the open mapping theorem, we obtain that for any function $q \in L_{0}^{2}(\Omega)$ there is a unique function $\boldsymbol{v}_{q} \in H_{0}(\operatorname{div}, \Omega)^{\top}$ such that $q=\nabla \cdot \boldsymbol{v}_{q}$ and $\left|\boldsymbol{v}_{q}\right|_{1, \Omega} \leq c\|q\|_{H^{1}(\Omega)}$ for some constant $c>0$ independent of $q$. Therefore $b$ satisfies the inf-sup condition

$$
\inf _{q \in L_{0}^{2}(\Omega)} \sup _{\boldsymbol{v} \in\left[H_{0}^{1}(\Omega)\right]^{n}} \frac{(q, \nabla \cdot \boldsymbol{v})_{\Omega}}{\|q\|_{L^{2}(\Omega)}\|\boldsymbol{v}\|_{H^{1}(\Omega)}} \geq \inf _{q \in L_{0}^{2}(\Omega)} \frac{\left(q, \nabla \cdot \boldsymbol{v}_{q}\right)_{\Omega}}{\|q\|_{L^{2}(\Omega)}\left\|\boldsymbol{v}_{q}\right\|_{H^{1}(\Omega)}} \geq \frac{1}{c}
$$

Since both $a$ and $b$ are continuous, $a$ is $H_{0}(\operatorname{div}, \Omega)$-elliptic and $b$ meets the inf-sup condition, by using the Ladyzhenskaya-Babuška-Brezzi Theorem (see Theorem 4.1, Chapter I, from (Girault \& Raviart, 1981)), we conclude that problem (2.2) has a unique solution, and the solution depends continuously on the data.

REMARK 1. We observe that the two key components in the proof of Theorem 1 are the coercivity of the bilinear form a and the surjectivity of the divergence differential operator from the space for the velocity onto the space of the pressure. The last one in particular is a major restriction in order to define conforming finite element spaces for the Stokes problem.

REMARK 2. The results of Theorem 1 can be extended to the nonzero Dirichlet boundary condition $\boldsymbol{u}=\boldsymbol{g}$ on $\partial \Omega$, where $\boldsymbol{g} \in\left[H^{1 / 2}(\partial \Omega)\right]^{n}$. Note that for this we have the compatibility condition $\int_{\partial \Omega} \boldsymbol{g} \cdot \boldsymbol{n} d S=0$, because

$$
0=\int_{\Omega} \nabla \cdot \boldsymbol{u} d \boldsymbol{x}=\int_{\partial \Omega} \boldsymbol{u} \cdot \boldsymbol{n} d S=\int_{\partial \Omega} \boldsymbol{g} \cdot \boldsymbol{n} d S
$$

It is possible to write $\boldsymbol{u}=\boldsymbol{u}_{0}+\boldsymbol{u}_{\boldsymbol{g}}$ where $\boldsymbol{u}_{0} \in\left[H_{0}^{1}(\Omega)\right]^{n}$ and $\boldsymbol{u}_{\boldsymbol{g}}$ is an extension to $\left[H^{1}(\Omega)\right]^{n}$ of $\boldsymbol{g}$. This extension $\boldsymbol{u}_{\boldsymbol{g}}$ is not unique, but for every such extension there is a unique $\boldsymbol{u}_{0}$ that solves the problem. If two different extensions are considered, we obtain two possibly different solutions, namely $(\hat{\boldsymbol{u}}, \hat{p})$ and $(\tilde{\boldsymbol{u}}, \tilde{p})$. By subtracting the weak formulations for both extensions of $\boldsymbol{g}$, we obtain

$$
\begin{aligned}
(\mu \nabla(\hat{\boldsymbol{u}}-\tilde{\boldsymbol{u}}), \nabla \boldsymbol{v})_{\Omega}-(\hat{p}-\tilde{p}, \nabla \cdot \boldsymbol{v})_{\Omega}=0 & \forall \boldsymbol{v} \in\left[H_{0}^{1}(\Omega)\right]^{n}, \\
-(q, \nabla \cdot(\hat{\boldsymbol{u}}-\tilde{\boldsymbol{u}}))_{\Omega}=0 & \forall q \in L^{2}(\Omega) .
\end{aligned}
$$

Since $\hat{\boldsymbol{u}}=\tilde{\boldsymbol{u}}=\boldsymbol{g}$ on $\partial \Omega$, then $\hat{\boldsymbol{u}}-\tilde{\boldsymbol{u}} \in\left[H_{0}^{1}(\Omega)\right]^{n}$. Thus, we obtain that $(\hat{\boldsymbol{u}}-\tilde{\boldsymbol{u}}, \hat{p}-\tilde{p})$ must be a solution of problem (2.2) for $\boldsymbol{f}=0$, therefore it must be zero, concluding that the Dirichlet boundary problem also has a unique solution.

REMARK 3. It is convenient to rewrite the zero integral condition on the pressure space of problem (2.2) by adding a Lagrange multiplier for the $\int_{\Omega} q d \boldsymbol{x}=0$ constraint.

Then the problem reads as: Find $(\boldsymbol{u}, p, \lambda) \in\left[H_{0}^{1}(\Omega)\right]^{n} \times L^{2}(\Omega) \times \mathbb{R}$ such that

$$
\begin{align*}
(\mu \nabla \boldsymbol{u}, \nabla \boldsymbol{v})_{\Omega}-(p, \nabla \cdot \boldsymbol{v})_{\Omega} & =(\boldsymbol{f}, \boldsymbol{v})_{\Omega} & & \forall \boldsymbol{v} \in\left[H_{0}^{1}(\Omega)\right]^{n},  \tag{2.3a}\\
-(q, \nabla \cdot \boldsymbol{u})_{\Omega}+\lambda(q, 1)_{\Omega} & =0 & & \forall q \in L^{2}(\Omega)  \tag{2.3b}\\
\tau(p, 1)_{\Omega} & =0 & & \forall \tau \in \mathbb{R} . \tag{2.3c}
\end{align*}
$$

It is easy to see that any solution of equations (2.2) will also yield a solution for equations (2.3) with $\lambda=0$. To prove this, let $(\boldsymbol{u}, p)$ be the solution of problem (2.2), then it holds the first condition of (2.3). For the second equation, consider any $q \in L^{2}(\Omega)$ and let $\bar{q} \in \mathbb{R}$ be its mean value over $\Omega$, then $q-\bar{q} \in L_{0}^{2}(\Omega)$ and thus

$$
\begin{aligned}
-(q, \nabla \cdot \boldsymbol{u})_{\Omega}+\lambda(q, 1)_{\Omega} & =-(q-\bar{q}, \nabla \cdot \boldsymbol{u})_{\Omega}-\bar{q}(1, \nabla \cdot \boldsymbol{u})_{\Omega}+\lambda(q, 1)_{\Omega} \\
& =-\bar{q}\langle 1, \boldsymbol{u} \cdot \boldsymbol{n}\rangle_{\partial \Omega}+\lambda(q, 1)_{\Omega} \\
& =\lambda(q, 1)_{\Omega}
\end{aligned}
$$

and since this must be zero for all $q \in L^{2}(\Omega)$, we conclude that $\lambda=0$. We can see that this is also true for the nonzero Dirichlet boundary condition $\boldsymbol{u}=\boldsymbol{g}$ on $\partial \Omega$, because of the compatibility condition $\int_{\partial \Omega} \boldsymbol{g} \cdot \boldsymbol{n} d S=0$.

The solution of problem (2.3) is also unique. Indeed, we consider the case $\boldsymbol{f}=\mathbf{0}$ and use test functions $\boldsymbol{v}=\boldsymbol{u}$ and $q=p$. The second equation now becomes $(p, \nabla \cdot \boldsymbol{u})_{\Omega}=0$. Replacing this in the first equation yields $\mu(\nabla \boldsymbol{u}, \nabla \boldsymbol{u})_{\Omega}=0$, thus giving $\boldsymbol{u}=$ constant. Since $\boldsymbol{u}=0$ in $\partial \Omega$, we conclude that $\boldsymbol{u}=0$. By replacing $\boldsymbol{u}=0$ in equations (2.3) with $f=0$, we obtain

$$
\begin{aligned}
(p, \nabla \cdot \boldsymbol{v})_{\Omega}=0 & \forall \boldsymbol{v} \in\left[H_{0}^{1}(\Omega)\right]^{n}, \\
\lambda(q, 1)_{\Omega}=0 & \forall q \in L^{2}(\Omega), \\
(p, 1)_{\Omega}=0 . &
\end{aligned}
$$

By choosing $\boldsymbol{v} \in\left[H_{0}^{1}(\Omega)\right]^{n}$ such that $\nabla \cdot \boldsymbol{v}=p$ (this is possible from Lemma 3.2 in (Girault \& Raviart, 1981), and because pintegrates 0), the first equation becomes $(p, p)_{\Omega}=0$ and thus $p=0$. From the second equation, we can easily see that $\lambda=0$. Since the only solution to the homogeneous problem is the trivial solution, we conclude that a solution for problem (2.3) exists and is unique, and such solution must give $\lambda=0$.

### 2.3. Finite Element Methods for Stokes flow

In this section we discretize the Stokes problem (2.2) using finite element methods. We first write the approximation in abstract setting for conforming finite-dimensional spaces $V_{h} \subset\left[H_{0}^{1}(\Omega)\right]^{n}$ and $Q_{h} \subset L_{0}^{2}(\Omega)$, and seek the approximations $\boldsymbol{u}_{h} \in V_{h}$ and $p_{h} \in Q_{h}$ to the velocity field $\boldsymbol{u}$ and the pressure $p$, respectively, solution of

$$
\begin{align*}
a\left(\boldsymbol{u}_{h}, \boldsymbol{v}\right)+b\left(\boldsymbol{v}, p_{h}\right) & =(\boldsymbol{f}, \boldsymbol{v})_{\Omega} & & \forall \boldsymbol{v} \in V_{h},  \tag{2.4a}\\
b\left(\boldsymbol{u}_{h}, q\right)_{\Omega} & =0 & & \forall q \in Q_{h} . \tag{2.4b}
\end{align*}
$$

As we mentioned in Remark 1 the finite element spaces for approximating the solution of the Stokes problem (2.2) must satisfy a so-called compatibility inf-sup conditions. We state this condition next

THEOREM 2. The discrete problem (2.4) is well-posed if and only if the spaces $V_{h}$ and $Q_{h}$ satisfy the discrete inf-sup compatibility condition, i.e., there is a constant $c_{h}>0$ such that

$$
\inf _{q \in Q_{h}} \sup _{\boldsymbol{v} \in V_{h}} \frac{(q, \nabla \cdot \boldsymbol{v})_{\Omega}}{\|q\|_{L^{2}(\Omega)}\|\boldsymbol{v}\|_{H^{1}(\Omega)}} \geq \frac{1}{c_{h}}
$$

Proof. See Proposition 4.13 in (Ern \& Guermond, 2013).

### 2.3.1. Finite element discretization

To define stable finite element approximation for the Stokes problem we need a pair of finite element spaces $\left(V_{h}, Q_{h}\right)$ satisfying the discrete inf-sup condition. Below we provide examples of these. Also, there are several documented examples of unstable finite element spaces such as the $\mathcal{P}_{1}-\mathcal{P}_{1}$ and the $\mathcal{P}_{1}-\mathcal{P}_{0}$ finite elements.

Consider a shape-regular triangulation $\mathcal{T}_{h}$ of the domain $\Omega$. We now introduce the Taylor-Hood finite element by

$$
\begin{aligned}
V_{h} & =\left\{\boldsymbol{v}_{h} \in[C(\Omega)]^{2}:\left.\boldsymbol{v}_{h}\right|_{K} \in\left[\mathcal{P}^{k+1}(\Omega)\right]^{n}, \forall K \in \mathcal{T}_{h}, \boldsymbol{v}_{h}=\mathbf{0} \text { on } \partial \Omega\right\}, \\
Q_{0, h} & =\left\{q_{h} \in L_{0}^{2}(\Omega) \cap C(\Omega):\left.q_{h}\right|_{K} \in \mathcal{P}^{k}(\Omega), \forall K \in \mathcal{T}_{h}\right\} .
\end{aligned}
$$

Lemma 1. Assume that $n=2,3$ and that every element $K \in \mathcal{T}_{h}$ has at least $n$ edges in the interior of $\Omega$. Then, there is $C>0$ such that for the Taylor-Hood element spaces $V_{h}$ and $Q_{h, 0}$ we have

$$
\sup _{\boldsymbol{v} \in V_{h}} \frac{(q, \nabla \cdot \boldsymbol{v})_{\Omega}}{\|\boldsymbol{v}\|_{H^{1}(\Omega)}} \geq C\left(\sum_{K \in \mathcal{T}_{h}} h_{K}^{2}\|q\|_{L^{2}(K)}^{2}\right)^{1 / 2}
$$

and hence satisfy the discrete inf-sup condition uniformly with respect to $h$

Proof. See Lemma 4.23 and 4.24 in (Ern \& Guermond, 2013).

In the following chapters we will present our model problem under Stokes flow in two dimensions ( $n=2$ ). As we observed in Remark 3, for an easier implementation of the method it is convenient to rewrite problem (2.4) introducing a Lagrange multiplier imposing the zero integral condition of the pressure space. Thus, we obtain the following
system: Find $\left(\boldsymbol{u}_{h}, p_{h}, \lambda\right) \in V_{h} \times Q_{h} \times \mathbb{R}$ such that

$$
\begin{array}{rlrl}
\left(\mu \nabla \boldsymbol{u}_{h}, \nabla \boldsymbol{v}_{h}\right)_{\Omega}-\left(p_{h}, \nabla \cdot \boldsymbol{v}_{h}\right)_{\Omega} & =\left(\boldsymbol{f}, \boldsymbol{v}_{h}\right)_{\Omega} & & \forall \boldsymbol{v}_{h} \in V_{h}, \\
-\left(q_{h}, \nabla \cdot \boldsymbol{u}_{h}\right)_{\Omega}+\lambda\left(q_{h}, 1\right)_{\Omega} & =0 & \forall q_{h} \in Q_{h} \\
\tau\left(p_{h}, 1\right)_{\Omega} & =0 & & \forall \tau \in \mathbb{R}, \tag{2.5c}
\end{array}
$$

where

$$
Q_{h}=\left\{q_{h} \in C(\Omega):\left.q_{h}\right|_{K} \in \mathcal{P}^{k}(\Omega), \forall K \in \mathcal{T}_{h}\right\} .
$$

We solve these equations instead of problem (2.2) simply because it is easier to work with spaces $Q_{h} \subset L^{2}(\Omega)$ than spaces $Q_{0, h} \subset L_{0}^{2}(\Omega)$ where the basis functions should be continuous and integrate zero.

### 2.3.2. Some implementation details in 2D

To fix ideas, consider continuous piecewise $\mathcal{P}^{2}-\mathcal{P}^{1}$ functions, i.e., the lowest order Taylor-Hood element. Since we are working in $\mathbb{R}^{2}$, to obtain a linear function in each triangle $K \in \mathcal{T}_{h}$ we need 3 degrees of freedom, while for quadratic functions we need 6 . For a reference element $\hat{K}$ with nodes $\hat{\boldsymbol{v}}_{1}=(0,0)^{\top}, \hat{\boldsymbol{v}}_{2}=(1,0)^{\top}, \hat{\boldsymbol{v}}_{3}=(0,1)^{\top}$, we will use the basis functions

$$
\hat{\varphi}_{1}(\hat{x}, \hat{y})=1-\hat{x}-\hat{y}, \quad \hat{\varphi}_{2}(\hat{x}, \hat{y})=\hat{x}, \quad \hat{\varphi_{3}}(\hat{x}, \hat{y})=\hat{y}
$$

such that $\hat{\varphi}_{i}\left(\hat{\boldsymbol{v}}_{j}\right)=\delta_{i, j}$. Every linear function in $\hat{K}$ can be expressed as a linear combination of functions $\hat{\varphi}_{i}$. For quadratic functions we consider the Bernstein basis functions $\hat{\psi}_{1}=\hat{\varphi}_{1}^{2}, \hat{\psi}_{2}=\hat{\varphi}_{2}^{2}, \hat{\psi}_{3}=\hat{\varphi}_{3}^{2}, \hat{\psi}_{4}=2 \hat{\varphi}_{1} \hat{\varphi}_{2}, \hat{\psi}_{5}=2 \hat{\varphi}_{1} \hat{\varphi}_{3}, \hat{\psi}_{6}=2 \hat{\varphi}_{2} \hat{\varphi}_{3}$, such that any quadratic function in $\hat{K}$ can be expressed as a linear combination of functions $\hat{\psi}_{i}$. To define the basis functions in an arbitrary element $K \in \mathcal{T}_{h}$ with nodes $\boldsymbol{v}_{1}=\left(x_{1}, y_{1}\right)^{\top}$,
$\boldsymbol{v}_{2}=\left(x_{2}, y_{2}\right)^{\top}, \boldsymbol{v}_{3}=\left(x_{3}, y_{3}\right)^{\top}$, consider the function $F_{K}: \hat{K} \rightarrow K$, defined as

$$
F_{K}\binom{\hat{x}}{\hat{y}}=\left(\begin{array}{ll}
x_{2}-x_{1} & x_{3}-x_{1} \\
y_{2}-y_{1} & y_{3}-y_{1}
\end{array}\right)\binom{\hat{x}}{\hat{y}}+\binom{x_{1}}{y_{1}}=B_{K}\binom{\hat{x}}{\hat{y}}+\binom{x_{1}}{y_{1}}
$$

This is an affine function such that $F_{K}\left(\hat{\boldsymbol{v}}_{i}\right)=\boldsymbol{v}_{i}$ for all $i$. The basis functions in $K$ are defined as

$$
\begin{array}{ll}
\phi_{i}^{K}=\phi_{i} \circ F_{K}^{-1}, & i=1,2,3 \\
\psi_{i}^{K}=\psi_{i} \circ F_{K}^{-1}, & i=1, \ldots, 6 \tag{2.6b}
\end{array}
$$

Defining $\hat{\nabla}$ as the gradient operator in the coordinates of $\hat{K}$, and $\nabla$ as the gradient in the coordinates of $K$, we can now obtain the derivatives of $\phi_{i}^{K}$ by applying the chain rule in equations (2.6):

$$
\begin{align*}
& \nabla \phi_{i}^{K}=B_{K}^{-\top} \hat{\nabla} \hat{\phi}_{i} \circ F_{K}^{-1}, \quad i=1,2,3,  \tag{2.7a}\\
& \nabla \psi_{i}^{K}=B_{K}^{-\top} \hat{\nabla} \hat{\psi}_{i} \circ F_{K}^{-1}, \quad i=1, \ldots, 6 . \tag{2.7b}
\end{align*}
$$

To ensure continuity of the solution, in $Q_{h}$ we will consider one basis function $\varphi_{i}$ for each node $i$ of the triangulation. For every triangle $K \in \mathcal{T}_{h}$ that contains this node, letting $n_{i}$ be its local enumeration within $K$, then the basis function restricted to $K$ is given by $\left.\varphi_{i}\right|_{K}=\varphi_{n_{i}}^{K}$, while $\varphi_{i}=0$ for every other triangle $K$ that does not contain the node $i$. Similarly for quadratic functions, we will consider one basis function $\psi_{i}$ for every node and every edge of the domain. Letting $N N$ and $N e$ be the number of nodes and edges in $\mathcal{T}_{h}$ respectively, and because $V_{h}$ is defined in $\mathbb{R}^{2}$, we use basis functions $\boldsymbol{\psi}_{i}=\left(\psi_{i}, 0\right)^{\top}$ and $\boldsymbol{\psi}_{i+N N+N e}=\left(0, \psi_{i}\right)^{\top}$ for $V_{h}$ with $i=1, \ldots, N N+N e$. Thus, the spaces $V_{h}$ and $Q_{h}$ can
be expressed as

$$
\begin{align*}
V_{h} & =\left\{\boldsymbol{v}_{h} \in[C(\Omega)]^{2}: \boldsymbol{v}_{h}=\sum_{i=1}^{2(N N+N e)} \alpha_{i} \boldsymbol{\psi}_{i}, \alpha_{i} \in \mathbb{R}, \boldsymbol{v}_{h}=\mathbf{0} \text { on } \partial \Omega,\right\},  \tag{2.8a}\\
Q_{h} & =\left\{\boldsymbol{q}_{h} \in C(\Omega): q_{h}=\sum_{i=1}^{N N} \beta_{i} \phi_{i}, \beta_{i} \in \mathbb{R}\right\} . \tag{2.8b}
\end{align*}
$$

Using these spaces, the discrete formulation of problem (2.5) is as follows. Find $\boldsymbol{\alpha} \in$ $\mathbb{R}^{2(N N+N e)}, \boldsymbol{\beta} \in \mathbb{R}^{N N}$ and $\lambda \in \mathbb{R}$ such that

$$
\begin{array}{rlrl}
\sum_{j} \alpha_{j}\left(\mu \nabla \boldsymbol{\psi}_{j}, \nabla \boldsymbol{\psi}_{i}\right)_{\Omega}-\sum_{j} \beta_{j}\left(\phi_{j}, \nabla \cdot \boldsymbol{\psi}_{i}\right)_{\Omega} & =\left(\boldsymbol{f}, \boldsymbol{\psi}_{i}\right)_{\Omega} & & i=1, \ldots, 2(N N+N e), \\
-\sum_{j} \alpha_{j}\left(\phi_{i}, \nabla \cdot \boldsymbol{\psi}_{j}\right)_{\Omega}+\lambda\left(\phi_{i}, 1\right)_{\Omega} & =0 & i=1, \ldots, N N, \\
\sum_{j} \beta_{j}\left(\phi_{j}, 1\right)_{\Omega} & =0 . &
\end{array}
$$

It is possible to rewrite this problem as a linear system in $A \vec{x}=\vec{f}$. The components of $A$ are given by $\left(\mu \nabla \boldsymbol{\psi}_{i}, \nabla \boldsymbol{\psi}_{j}\right)_{\Omega},-\left(\nabla \cdot \boldsymbol{\psi}_{i}, \phi_{j}\right),\left(\phi_{i}, 1\right)$ or zero depending on the position that the integrals appear, resulting in a symmetric matrix, which also is positive-definite for the Taylor-Hood elements. Similarly the coordinates of $\vec{f}$ are given by $\left(\boldsymbol{f}, \boldsymbol{\psi}_{i}\right)_{\Omega}$ or zero, while $\vec{x}=\left(\boldsymbol{\alpha}^{\top} \boldsymbol{\beta}^{\top} \lambda\right)^{\top}$ is the solution of the linear system. By solving this, the finite element approximation of the solution would be

$$
\begin{aligned}
\boldsymbol{u}_{h} & =\sum_{j=1}^{2(N N+N e)} \alpha_{j} \boldsymbol{\psi}_{j}, \\
p_{h} & =\sum_{j=1}^{N N} \beta_{j} \boldsymbol{\psi}_{j} .
\end{aligned}
$$

### 2.3.3. Computing the integrals

When solving the linear system, each coordinate of $A$ represents an integral over $\Omega$ of combinations of basis functions. As we will see, it is not necessary to integrate each one of them separately. First of all, we can see that

$$
\begin{align*}
\left(\nabla \boldsymbol{\psi}_{i}, \nabla \boldsymbol{\psi}_{j}\right)_{\Omega} & =\sum_{K}\left(\nabla \boldsymbol{\psi}_{i}, \nabla \boldsymbol{\psi}_{j}\right)_{K}  \tag{2.9a}\\
\left(\phi_{i}, \nabla \cdot \boldsymbol{\psi}_{j}\right)_{\Omega} & =\sum_{K}\left(\phi, \nabla \cdot \boldsymbol{\psi}_{j}\right)_{K}  \tag{2.9b}\\
\left(\phi_{i}, 1\right)_{\Omega} & =\sum_{K}(\phi, 1)_{K} \tag{2.9c}
\end{align*}
$$

For the first integral from (2.9) we note that the base functions for $V_{h}$ are either zero on their first or second component. Therefore, we only need to compute this integral when both $\psi_{i}$ and $\psi_{j}$ are nonzero on the same coordinate. In this case, we can write $\left(\nabla \boldsymbol{\psi}_{i}, \nabla \boldsymbol{\psi}_{j}\right)_{K}=\left(\nabla \psi_{i}, \nabla \psi_{j}\right)_{K}$ for scalar functions $\psi_{i}$ and $\psi_{j}$. By making a change of variables $\hat{\boldsymbol{x}}=F_{K}^{-1}(\boldsymbol{x})$ and using identity (2.7) we obtain

$$
\begin{aligned}
\left(\nabla \psi_{i}, \nabla \psi_{j}\right)_{K} & =\left|\operatorname{det} B_{K}\right| \cdot\left(B_{K}^{-\top} \hat{\nabla} \hat{\psi}_{i}, B_{K}^{-\top} \hat{\nabla} \hat{\psi}_{j}\right)_{\hat{K}} \\
& =\left|\operatorname{det} B_{K}\right| \cdot\left(\hat{\nabla} \hat{\psi}_{i}, B_{K}^{-1} B_{K}^{-\top} \hat{\nabla} \hat{\psi}_{j}\right)_{\hat{K}}
\end{aligned}
$$

Now we only need to compute the integrals $\left(\partial_{x} \hat{\psi}_{i}, \partial_{x} \hat{\psi}_{j}\right)_{\hat{K}},\left(\partial_{x} \hat{\psi}_{i}, \partial_{y} \hat{\psi}_{j}\right)_{\hat{K}},\left(\partial_{y} \hat{\psi}_{i}, \partial_{x} \hat{\psi}_{j}\right)_{\hat{K}}$ and $\left(\partial_{y} \hat{\psi}_{i}, \partial_{y} \hat{\psi}_{j}\right)_{\hat{K}}$, and then multiply their values by the corresponding coordinates of $B_{K}^{-1} B_{K}^{-\top}$ to obtain the integral over $K$.

For the second integral from (2.9), since only one coordinate of $\psi_{j}$ is nonzero, namely $\psi_{j}$, this expression can either be $\left(\phi_{i}, \partial_{x} \psi_{j}\right)_{K}$ or $\left(\phi_{i}, \partial_{y} \psi_{j}\right)_{K}$. Following the same procedure as before, we can see that

$$
\begin{aligned}
& \left(\phi_{i}, \partial_{x} \psi_{j}\right)_{K}=\left|\operatorname{det} B_{K}\right| \cdot\left(\hat{\phi}_{i}, B_{K(1 \bullet)}^{-\top} \hat{\nabla} \hat{\psi}_{j}\right)_{\hat{K}} \\
& \left(\phi_{i}, \partial_{y} \psi_{j}\right)_{K}=\left|\operatorname{det} B_{K}\right| \cdot\left(\hat{\phi}_{i}, B_{K(2 \bullet)}^{-\top} \hat{\nabla} \hat{\psi}_{j}\right)_{\hat{K}}
\end{aligned}
$$

where $B_{K(\ell)}^{-\top}$ represents the $\ell$-th row of $B_{K}^{-\top}$. Similar as before, we only need to compute $\left(\hat{\phi}_{i}, \partial_{\hat{x}} \hat{\psi}_{j}\right)_{\hat{K}}$ and $\left(\hat{\phi}_{i}, \partial_{\hat{y}} \hat{\psi}_{j}\right)_{\hat{K}}$, and then multiply their values by the corresponding coordinates of $B_{K}^{-\top}$ to obtain the integral over $K$. For the last integral from (2.9), by making the same change of variables we see that

$$
\left(\phi_{i}, 1\right)_{K}=\left|\operatorname{det}\left(B_{K}\right)\right| \cdot\left(\hat{\phi}_{i}, 1\right)_{\hat{K}}=\frac{\left|\operatorname{det}\left(B_{K}\right)\right|}{6} \quad i=1,2,3 .
$$

Thus, we only need to compute a few integrals over the reference element $\hat{K}$ and then use these values to compute the integrals over an arbitrary element $K$.

### 2.4. Two-dimensional Fourier interpolation

Even though for the steady-state interface Stokes problem in Section 3 we assume that we know the parametric form of the interface, this is not necessarily true for the moving interface problem that we solve in Section 4, since the position of the interface keeps changing at every iteration. We will now show how to approximate the parametrization of the interface by using a two-dimensional Fourier interpolation of the curve.

In order to approximate the interface $\boldsymbol{X}$ at a given time $t_{n}$, which is denoted by $\boldsymbol{X}^{n}$, we take $M$ points along the interface equally spaced in the parametric space, $\boldsymbol{X}_{0}^{n}, \ldots, \boldsymbol{X}_{M-1}^{n}$, i.e., we take samples $\boldsymbol{X}_{m}^{n}=\boldsymbol{X}^{n}\left(s_{m}\right)$ such that $s_{m}=2 \pi m / M$ for all $m=0, \ldots, M-1$.

We approximate the interface using the $M$ points with the Fast Fourier Transform. For a fixed time $t_{n}$ we write each point $\boldsymbol{X}_{m}^{n}=\left(x_{m}^{n}, y_{m}^{n}\right)^{\top}$ as a point in the complex plane, $z_{m}^{n}=x_{m}^{n}+i \cdot y_{m}^{n}$. Any parametrization of the interface $\boldsymbol{X}^{n}(s)=\left(x^{n}(s), y^{n}(s)\right)^{\top}$ can be interpreted as a complex function $f^{n}(s)=x^{n}(s)+i \cdot y^{n}(s)$. Since we assume that the interface is closed and smooth, then we can think of $f^{n}$ as a smooth periodic function with
period $2 \pi$. Thus, $f^{n}$ (and its derivatives) can be expanded as a Fourier series:

$$
\begin{aligned}
f^{n}(s) & =\sum_{k=-\infty}^{\infty} Z_{k}^{n} \cdot \mathrm{e}^{i k s} \\
\frac{\partial^{\ell}}{\partial s^{\ell}} f^{n}(s) & =\sum_{k=-\infty}^{\infty} Z_{k}^{n}(i k)^{\ell} \mathrm{e}^{i k s} .
\end{aligned}
$$

Following the procedure outlined in (Johnson, 2011), we approximate $f^{n}$ using the discrete Fourier transform (DFT). Considering that the points $\boldsymbol{X}_{m}^{n}$ are taken in counterclockwise direction, consider that we have sample points equally spaced from 0 to $2 \pi$ independent of time, $z_{m}^{n}=f^{n}\left(s_{m}^{n}\right)=f^{n}(2 \pi m / M)$ for $m=0, \ldots, M-1$, such that

$$
Z_{k}^{n}=\frac{1}{M} \sum_{m=0}^{M-1} z_{m}^{n} \mathrm{e}^{-\frac{2 \pi i}{M} m k}
$$

Using the inverse DFT (IDFT), we retrieve the points $z_{m}^{n}$ :

$$
z_{m}^{n}=\sum_{k=0}^{M-1} Z_{k}^{n} \mathrm{e}^{+\frac{2 \pi i}{M} m k}
$$

Our goal is to approximate the function $f^{n}(s)$. It is easy to see that even with fixed values for $Z_{k}^{n}$, any function of the form

$$
f^{n}(s)=\sum_{k=0}^{M-1} Z_{k}^{n} \mathrm{e}^{i\left(k+\ell_{k} M\right) s}
$$

will yield the exact same samples $z_{m}^{n}$ for any values of $\ell_{k} \in \mathbb{Z}$. If we choose $\ell_{k}=0$ for $k<M / 2, \ell_{k}=-1$ for $k>M / 2$ and split the $Z_{M / 2}^{n}$ term between the frequencies $\pm i M / 2$ (giving a $\cos (s M / 2)$ term), we obtain the unique minimal-oscillation trigonometric interpolation of order $M$, in the sense that it minimizes $\left\|\partial_{s} f^{n}\right\|_{L^{2}\left(\mathcal{S}^{1}\right)}$. Thus, the resulting interpolator is given by:

$$
f^{n}(s)=Z_{0}^{n}+\sum_{0<k<M / 2}\left(Z_{k}^{n} \mathrm{e}^{i k s}+Z_{M-k}^{n} \mathrm{e}^{-i k s}\right)+Z_{M / 2}^{n} \cos (s M / 2)
$$

where the $M / 2$ term (Nyquist) vanishes for odd $M$. Finally, in order to recover the interface $\boldsymbol{X}^{n}(s)$ and its derivatives, we simply take the real and imaginary part of the trigonometric interpolation function $f^{n}(s)$ and its corresponding derivatives.

### 2.5. Numerical methods for ODEs

As we will see in Section 4, the interface $\Gamma$ immersed in the domain $\Omega$ moves according to the velocity field $\boldsymbol{u}$ by following a no-slip condition. If we represent the points on the interface at a time $t$ by the parametrization $\boldsymbol{X}(s, t), s \in[0,2 \pi]$, the no-slip condition between the interface and the fluid is represented by $\partial_{t} \boldsymbol{X}=\boldsymbol{u}(\boldsymbol{X}, t)$. This corresponds to a first order ODE, and we can use different numerical methods to approximate its solution.

Suppose we want to solve a problem of the form $\boldsymbol{y}^{\prime}(t)=f(t, \boldsymbol{y})$ with an initial condition $y\left(t_{0}\right)=y_{0}$. Consider the sequence $t_{0}, t_{1}, t_{2}, \ldots$ with $t_{n}=t_{0}+n h(h=\Delta t>0)$ and that we have an approximation of the first values of $\boldsymbol{y}$, with $y\left(t_{k}\right) \approx y_{k}$ for all $k$. Timestepping numerical methods for approximating the solution of an ODE can be expressed as

$$
\boldsymbol{y}_{n+1}=\mathcal{Y}_{n}\left(\boldsymbol{f}, h, \boldsymbol{y}_{0}, \boldsymbol{y}_{1}, \ldots, \boldsymbol{y}_{n}\right)
$$

The method is said to be of order $p$ if

$$
\boldsymbol{y}\left(t_{n+1}\right)-\mathcal{Y}_{n}\left(\boldsymbol{f}, h, \boldsymbol{y}\left(t_{0}\right), \boldsymbol{y}\left(t_{1}\right), \ldots, \boldsymbol{y}\left(t_{n}\right)\right)=\mathcal{O}\left(h^{p+1}\right)
$$

We will now show different methods for approximating the solution of such ODEs, taken from (Iserles, 2008).

### 2.5.1. Euler method

Euler method is an explicit ODE method where, starting from a point $\left(t_{n}, \boldsymbol{y}_{n}\right)$ with $\boldsymbol{y}_{n} \approx \boldsymbol{y}\left(t_{n}\right)$ we approximate the new value $\boldsymbol{y}\left(t_{n+1}\right)$ from the formula

$$
\boldsymbol{y}\left(t_{n+1}\right) \approx \boldsymbol{y}_{n+1}=\boldsymbol{y}_{n}+h \cdot \boldsymbol{f}\left(t_{n}, \boldsymbol{y}_{n}\right)
$$

Using the Taylor expansion for $\boldsymbol{y}\left(t_{n+1}\right)$ and the fact that $\boldsymbol{y}^{\prime}=\boldsymbol{f}(t, \boldsymbol{y})$ we can easily verify that this method is of order 1 . This method is also convergent, in the sense that for any fixed $t^{*}>t_{0}$ with a time step $h$ we have that

$$
\lim _{h \rightarrow 0}\left\|\boldsymbol{y}\left(t_{0}+n h\right)-\boldsymbol{y}_{n, h}\right\|=0, \quad n=0,1, \ldots,\left\lfloor\left(t^{*}-t_{0}\right) / h\right\rfloor,
$$

where $y_{n, h}$ is the approximation of the $n$-th point obtained by using the time step $h$.
This method can be slightly modified into an implicit method by taking

$$
\boldsymbol{y}_{n+1}=\boldsymbol{y}_{n}+h \cdot \boldsymbol{f}\left(t_{n+1}, \boldsymbol{y}_{n+1}\right) .
$$

This is known as the backward Euler method, and even though it is also convergent and of order 1 , it is generally more stable than the explicit Euler method.

Geometrically, the Euler method approximates the new point $\boldsymbol{y}_{n+1}$ by considering the slope of the solution at the left-most point of an interval, $\boldsymbol{y}^{\prime}\left(t_{n}\right) \approx \boldsymbol{f}\left(t_{n}, \boldsymbol{y}_{n}\right)$, while its implicit variant considers the slope at the right-most point of the interval, $\boldsymbol{y}^{\prime}\left(t_{n+1}\right) \approx$ $\boldsymbol{f}\left(t_{n+1}, \boldsymbol{y}_{n+1}\right)$.

### 2.5.2. Runge-Kutta schemes

We can obtain an expression for $\boldsymbol{y}\left(t_{n+1}\right)$ from $\boldsymbol{y}\left(t_{n}\right)$ with $t_{n+1}=t_{n}+h$ by using the integral formula,

$$
\boldsymbol{y}\left(t_{n+1}\right)=\boldsymbol{y}\left(t_{n}\right)+\int_{t_{n}}^{t_{n+1}} \boldsymbol{f}(t, \boldsymbol{y}(t)) \mathrm{d} t=\boldsymbol{y}\left(t_{n}\right)+h \int_{0}^{1} \boldsymbol{f}\left(t_{n}+h \tau, \boldsymbol{y}\left(t_{n}+h \tau\right)\right) \mathrm{d} \tau
$$

and then obtain an approximation by replacing the second integral by a quadrature. This results in

$$
\boldsymbol{y}_{n+1}=\boldsymbol{y}_{n}+h \sum_{j=1}^{\nu} b_{j} \boldsymbol{f}\left(t_{n}+c_{j} h, \boldsymbol{y}\left(t_{n}+c_{j} h\right)\right)
$$

However, we do not know the value of $\boldsymbol{y}$ at every node $t_{n}+c_{j} h$. The Runge-Kutta methods focus in approximating $\boldsymbol{y}\left(t_{n}+c_{j} h\right)$ by new variables $\boldsymbol{\xi}_{j}$, and then using these values to update $\boldsymbol{y}_{n}$ into $\boldsymbol{y}_{n+1}$. Specifically, we solve the equations

$$
\begin{aligned}
\boldsymbol{\xi}_{1} & =\boldsymbol{y}_{n}+h \sum_{j=1}^{\nu} a_{1, j} \boldsymbol{f}\left(t_{n}+c_{j} h, \boldsymbol{\xi}_{j}\right), \\
\boldsymbol{\xi}_{2} & =\boldsymbol{y}_{n}+h \sum_{j=1}^{\nu} a_{2, j} \boldsymbol{f}\left(t_{n}+c_{j} h, \boldsymbol{\xi}_{j}\right), \\
& \vdots \\
\boldsymbol{\xi}_{\nu} & =\boldsymbol{y}_{n}+h \sum_{j=1}^{\nu} a_{\nu, j} \boldsymbol{f}\left(t_{n}+c_{j} h, \boldsymbol{\xi}_{j}\right),
\end{aligned}
$$

And use the solution to obtain the next point

$$
\boldsymbol{y}_{n+1}=\boldsymbol{y}_{n}+h \sum_{i=1}^{\nu} b_{i} \boldsymbol{f}\left(t_{n}+c_{i} h, \boldsymbol{\xi}_{i}\right) .
$$

The matrix $A=\left[a_{i, j}\right]_{i, j=1}^{n}$ is called the RK matrix, while the vectors $\boldsymbol{b}=\left(b_{1}, \ldots, b_{\nu}\right)^{\top}$ and $\boldsymbol{c}=\left(c_{1}, \ldots, c_{\nu}\right)^{\top}$ are the RK weights and RK nodes respectively. This data is usually arranged in a Butcher tableau, which has the form

$$
\begin{array}{c|c}
\boldsymbol{c} & A \\
\hline & \boldsymbol{b}^{\top}
\end{array}
$$

When the RK matrix is lower-triangular and has zero diagonal terms, the method corresponds to an explicit RK scheme (ERK). In any other case the method corresponds to an implicit RK scheme (IRK). An example of an ERK method is given by

| 0 |  |  |  |
| :---: | :---: | :---: | :---: |
| $\frac{1}{2}$ | $\frac{1}{2}$ |  |  |
| 1 | -1 | 2 |  |
|  | $\frac{1}{6}$ | $\frac{2}{3}$ | $\frac{1}{6}$ |,

which is an explicit third-order three-stage RK method. This tableau is equivalent to solving the equations

$$
\begin{aligned}
& \boldsymbol{\xi}_{1}=\boldsymbol{y}_{n} \\
& \boldsymbol{\xi}_{2}=\boldsymbol{y}_{n}+h \cdot \frac{1}{2} \boldsymbol{f}\left(t_{n}, \boldsymbol{\xi}_{1}\right), \\
& \boldsymbol{\xi}_{3}=\boldsymbol{y}_{n}+h\left(-1 \cdot \boldsymbol{f}\left(t_{n}, \boldsymbol{\xi}_{1}\right)+2 \cdot \boldsymbol{f}\left(t_{n}+\frac{h}{2}, \boldsymbol{\xi}_{2}\right)\right),
\end{aligned}
$$

and then using this values to obtain the next point,

$$
\boldsymbol{y}_{n+1}=\boldsymbol{y}_{n}+h\left(\frac{1}{6} \boldsymbol{f}\left(t_{n}, \boldsymbol{\xi}_{1}\right)+\frac{2}{3} \boldsymbol{f}\left(t_{n}+\frac{h}{2}, \boldsymbol{\xi}_{2}\right)+\frac{1}{6} \boldsymbol{f}\left(t_{n}+h, \boldsymbol{\xi}_{3}\right)\right) .
$$

Generally, we can obtain $\nu$-stages ERK schemes of order $\nu$ only for $\nu \leq 4$, and to obtain higher order methods we need to consider even more stages.

## 3. STEADY-STATE INTERFACE PROBLEM

The solution to the steady-state Stokes interface problem is proven to be discontinuous. We show that ignoring this discontinuities can lead to sub-optimal convergence rates when using a standard finite element method. We show how to precalculate functions, which we call correction functions, satisfying these discontinuities across $\Gamma$. Such functions can be included into the problem to obtain a corrected finite element method. We show that the convergence rate for any stable discretization spaces $V_{h}$ and $Q_{h}$ in the corrected finite element method is the same as it would be in standard finite element methods for the non-interface standard Stokes problem.

### 3.1. Strong formulation

We now show how to obtain the equations describing a steady, incompressible Stokes flow in presence of a massless, impermeable, elastic interface $\Gamma$. Since the fluid cannot go through the interface, we impose that the movement of the interface is given by the fluid velocity at that point. Thus, we impose continuity of the velocity across $\Gamma$,

$$
\begin{equation*}
\llbracket u \rrbracket=\mathbf{0} \tag{3.1}
\end{equation*}
$$

Here, for any quantity $\phi$ defined in $\Omega$, we write $\phi^{ \pm}=\left.\phi\right|_{\Omega^{ \pm}}$and define $\llbracket \phi \rrbracket$ as the jump of $\phi$ across the interface, i.e. $\llbracket \phi \rrbracket=\left.\left(\phi^{+}-\phi^{-}\right)\right|_{\Gamma}$. As shown in Section 1.3, the presence of an interface $\Gamma=\{\boldsymbol{X}(s), s \in[0,2 \pi]\}$ can be considered in the Stokes flow equations by adding a local force $\boldsymbol{B}$ defined on $\Gamma$ (Mori, 2008),

$$
\begin{align*}
-\mu \Delta \boldsymbol{u}+\nabla p & =\boldsymbol{f}+\boldsymbol{B}  \tag{3.2a}\\
\boldsymbol{B} & =\int_{\Gamma} \boldsymbol{\beta} \cdot \delta(\boldsymbol{x}-\boldsymbol{X}) \mathrm{d} S \tag{3.2b}
\end{align*}
$$

where $\boldsymbol{\beta}$ is a density force acting on $\Gamma$. Alternatively, it is possible to consider the local force $\boldsymbol{B}$ only as a discontinuity of the solution across the interface. Multiplying (3.2) by a smooth function $\phi$ such that $\phi=0$ on $\partial \Omega$ and integrating, we obtain

$$
\begin{equation*}
-\int_{\Omega} \nabla \cdot(\mu \nabla \boldsymbol{u}-p I) \cdot \boldsymbol{\phi} \mathrm{d} \boldsymbol{x}=\int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{\phi} \mathrm{d} \boldsymbol{x}+\int_{\Gamma} \boldsymbol{\beta} \cdot \boldsymbol{\phi} \mathrm{d} S . \tag{3.3}
\end{equation*}
$$

Integrating by parts the left-hand side, we find that

$$
-\int_{\Omega} \nabla \cdot(\mu \nabla \boldsymbol{u}-p I) \cdot \boldsymbol{\phi} \mathrm{d} \boldsymbol{x}=\int_{\Gamma} \llbracket(\mu \nabla \boldsymbol{u}-p I) \boldsymbol{n} \rrbracket \cdot \boldsymbol{\phi} \mathrm{d} S+\int_{\Omega}(\mu \nabla \boldsymbol{u}-p I): \nabla \boldsymbol{\phi} \mathrm{d} \boldsymbol{x},
$$

where $\boldsymbol{n}$ denotes the unit normal vector to the interface $\Gamma$ pointing outwards $\Omega^{-}$. By choosing test functions $\phi$ defined on $\Gamma$ which decay rapidly to 0 , the integrals over $\Omega$ vanish and therefore equation (3.3) becomes

$$
\int_{\Gamma} \llbracket(\mu \nabla \boldsymbol{u}-p I) \boldsymbol{n} \rrbracket \cdot \boldsymbol{\phi} \mathrm{d} S=\int_{\Gamma} \boldsymbol{\beta} \cdot \boldsymbol{\phi} \mathrm{d} S .
$$

Since this holds for any such functions $\phi$, we conclude that the local density force $\boldsymbol{\beta}$ can be expressed as a discontinuity condition across $\Gamma$

$$
\begin{equation*}
\llbracket(\mu \nabla \boldsymbol{u}-p I) \boldsymbol{n} \rrbracket=\boldsymbol{\beta} . \tag{3.4}
\end{equation*}
$$

By strongly imposing conditions (3.1) and (3.4), considering a Dirichlet boundary condition $\boldsymbol{u}=0$ on $\partial \Omega$ and including the constraint $\int_{\Omega} p \mathrm{~d} \boldsymbol{x}=0$ to ensure uniqueness, we
obtain the strong formulation of the steady-state problem

$$
\begin{align*}
-\mu \Delta \boldsymbol{u}+\nabla p & =\boldsymbol{f} & & \text { in } \Omega \backslash \Gamma,  \tag{3.5a}\\
\nabla \cdot \boldsymbol{u} & =0 & & \text { in } \Omega \backslash \Gamma,  \tag{3.5b}\\
\boldsymbol{u} & =0 & & \text { on } \partial \Omega,  \tag{3.5c}\\
\llbracket \boldsymbol{u} \rrbracket & =0 & & \text { on } \Gamma,  \tag{3.5d}\\
\llbracket(\mu \nabla u-p I) \boldsymbol{n} \rrbracket & =\boldsymbol{\beta} & & \text { on } \Gamma,  \tag{3.5e}\\
\int_{\Omega} p \mathrm{~d} \boldsymbol{x} & =0 . & & \tag{3.5f}
\end{align*}
$$

### 3.2. Weak Formulation

As we did in Section 2.2, to obtain the weak formulation of problem (3.5) we multiply the first equation by a vector function $\boldsymbol{v} \in\left[H_{0}^{1}(\Omega)\right]^{2}=V$ and the second equation by a function $q \in L_{0}^{2}(\Omega)=Q$ and then integrate in $\Omega$. The first equation becomes

$$
\begin{aligned}
(\boldsymbol{f}, \boldsymbol{v})_{\Omega}= & -(\nabla \cdot(\mu \nabla \boldsymbol{u}-p I), \boldsymbol{v})_{\Omega}, \\
= & -(\nabla \cdot(\mu \nabla \boldsymbol{u}-p I), \boldsymbol{v})_{\Omega^{+}}-(\nabla \cdot(\mu \nabla \boldsymbol{u}-p I), \boldsymbol{v})_{\Omega^{-}}, \\
= & (\mu \nabla \boldsymbol{u}-p I, \nabla \boldsymbol{v})_{\Omega^{+}}+(\mu \nabla \boldsymbol{u}-p I, \nabla \boldsymbol{v})_{\Omega^{-}} \\
& +\langle(\mu \nabla \boldsymbol{u}-p I) \boldsymbol{n}, \boldsymbol{v}\rangle_{\Gamma^{+}}-\langle(\mu \nabla \boldsymbol{u}-p I) \boldsymbol{n}, \boldsymbol{v}\rangle_{\Gamma^{-}}, \\
= & (\mu \nabla \boldsymbol{u}-p I, \nabla \boldsymbol{v})_{\Omega}+\langle\llbracket(\mu \nabla \boldsymbol{u}-p I) \boldsymbol{n} \rrbracket, \boldsymbol{v}\rangle_{\Gamma}, \\
= & (\mu \nabla \boldsymbol{u}, \nabla \boldsymbol{v})_{\Omega}-(p, \nabla \cdot \boldsymbol{v})_{\Omega}+\langle\mathbb{L}(\mu \nabla \boldsymbol{u}-p I) \boldsymbol{n} \rrbracket, \boldsymbol{v}\rangle_{\Gamma^{\prime}},
\end{aligned}
$$

where $\boldsymbol{n}$ is the unit normal vector of $\Gamma$ pointing outwards $\Omega^{-}$. Thus, the first equation can be written as

$$
(\mu \nabla \boldsymbol{u}, \nabla \boldsymbol{v})_{\Omega}-(p, \nabla \cdot \boldsymbol{v})_{\Omega}=(\boldsymbol{f}, \boldsymbol{v})_{\Omega}-\langle\boldsymbol{\beta}, \boldsymbol{v}\rangle_{\Gamma} .
$$

Following the same procedure we did in Section 2.2, we can consider a Lagrange multiplier in the second equation for the $\int_{\Omega} q \mathrm{~d} \boldsymbol{x}$ condition, thus resulting in the weak formulation of the steady-state interface problem: find $(\boldsymbol{u}, p, \lambda) \in V \times Q \times \mathbb{R}$ such that

$$
\begin{array}{rlrl}
(\mu \nabla \boldsymbol{u}, \nabla \boldsymbol{v})_{\Omega}-(p, \nabla \cdot \boldsymbol{v})_{\Omega} & =(\boldsymbol{f}, \boldsymbol{v})_{\Omega}-\langle\boldsymbol{\beta}, \boldsymbol{v}\rangle_{\Gamma} & \forall \boldsymbol{v} \in\left[H_{0}^{1}(\Omega)\right]^{n}, \\
-(q, \nabla \cdot \boldsymbol{u})_{\Omega}+\lambda(q, 1)_{\Omega} & =0 & \forall q \in L^{2}(\Omega), \\
(p, 1)_{\Omega} & =0, & & \\
\llbracket \boldsymbol{u} \rrbracket & =\mathbf{0}, & &
\end{array}
$$

where the constraint $\llbracket u \rrbracket=\mathbf{0}$ is strongly imposed on the solution.

Just like in the non-interface problem, we may think that to approximate the solution we could use the same spaces $V_{h} \subset V$ and $Q_{h} \subset Q$ of continuous, piecewise polynomial functions (Taylor-Hood elements). However, as shown in appendix A, the only function that should be always continuous across the interface is $\boldsymbol{u}$, and we must also consider possible discontinuities for the derivatives of $\boldsymbol{u}$, as well as discontinuities of $p$ and its derivatives across $\Gamma$.

### 3.3. Correction functions

Suppose we look for a function which is discontinuous across $\Gamma$, and that we know the discontinuity that this function holds across $\Gamma$. In this section we will show how to obtain a discontinuous function $w$, which we call correction function, that approximates such discontinuity. The idea is that any discontinuous function $\tilde{\varphi}$ can be split into an approximation of its continuous and discontinuous part, $\tilde{\varphi}=\varphi+w^{\varphi}$. For a given triangulation $\mathcal{T}_{h}$ define the set of all triangles which the interface intersects as

$$
\mathcal{T}_{h}^{\Gamma}=\{K \in \mathcal{T}: K \cap \Gamma \neq \phi\}
$$

We now show how to obtain correction functions on each element $K \in \mathcal{T}_{h}^{\Gamma}$. The method presented here gives us a discontinuous function $w$ defined on each of these triangles, assuming that $w$ has known jump functions $J_{w, 0}=\llbracket w \rrbracket$ and $J_{w, 1}=\llbracket \partial_{n} w \rrbracket$ defined over $\Gamma_{K}=\Gamma \cap K$. We look for functions $w^{ \pm}$defined on $K^{ \pm}=K \cap \Omega^{ \pm}$such that

$$
w(x, y)=\left\{\begin{array}{l}
w^{-}(x, y) \text { in } K^{-} \\
w^{+}(x, y) \text { in } K^{+}
\end{array}\right.
$$


(a) Points considered for linear correction functions.

(b) Points considered for quadratic correction functions.

Figure 3.1. Intersection between the interface $\Gamma$ and an arbitrary element $K$ and points considered for linear and quadratic (first and second order, respectively) correction functions on the interface.

If we assume that both $w^{+}$and $w^{-}$are linear functions, we need 6 equations to obtain them. As shown in Figure 3.1a, for linear functions $w^{ \pm}$we consider the points $v_{i}$ with $i=1,2,3$, where $v_{1}, v_{2}, v_{3}$ are the nodes of $K$, and the points $z_{\ell}^{k}, \ell=0, \ldots, k$, defined as the Lagrange nodes on $\Gamma_{K}$ for polynomials of order $k=0,1$. This function $w$ should satisfy the jump conditions across the interface and avoid causing discontinuities in points
of the domain other than $\Gamma$. The linear system that gives such discontinuous function $w$ is given by

$$
\left\{\begin{align*}
w\left(v_{i}\right) & =0, & & i=1,2,3  \tag{3.6}\\
w^{+}\left(z_{\ell}^{1}\right)-w^{-}\left(z_{\ell}^{1}\right) & =J_{w, 0}\left(z_{\ell}^{1}\right), & & \ell=0,1 \\
\partial_{n}\left(w^{+}\left(z_{0}^{0}\right)-w^{-}\left(z_{0}^{0}\right)\right) & =J_{w, 1}\left(z_{0}^{0}\right) & &
\end{align*}\right.
$$

The first equation states that the piecewise linear Lagrange interpolator of $w$ is zero, such that if every jump condition is equal to zero (i.e., there is no jump across $\Gamma$ ), the resulting correction function $w$ should also be zero. This system must be solved for every element $K \in \mathcal{T}_{h}^{\Gamma}$, resulting in a piecewise linear correction function $w$ with non-zero values only in these elements. Since the functions are linear, we will only be able to represent the jump conditions exactly if the interface parametrization in $K$ and the jump functions $J_{w, 0}$ and $J_{w, 1}$ are linear. However, it is possible to obtain higher order correction functions using higher order jump conditions, which result in better approximations of the solution.

Suppose we also know the second order jump condition $J_{w, 2}=\llbracket \partial_{n}^{2} w \rrbracket$. It is possible to find quadratic correction functions solving a $12 \times 12$ linear system on each element $K \in \mathcal{T}_{h}$ that $\Gamma$ intersects. Similar to the linear case, we consider the points $v_{i}, i=1, \ldots, 6$, where $v_{1}, v_{2}, v_{3}$ are the nodes of $K$ and $v_{4}, v_{5}, v_{6}$ are the middle points of the edges of $K$. For $k=0,1,2$, points $z_{\ell}^{k}, \ell=0, \ldots, k$, are the Lagrange nodes for polynomials of order $k$ on $\Gamma_{K}$. The linear system we need to solve in this case is given by

$$
\left\{\begin{align*}
w\left(v_{i}\right) & =0, & & i=1, \ldots, 6  \tag{3.7}\\
w^{+}\left(z_{\ell}^{2}\right)-w^{-}\left(z_{\ell}^{2}\right) & =J_{w, 0}\left(z_{\ell}^{2}\right), & & \ell=0,1,2 \\
\partial_{n}\left(w^{+}\left(z_{\ell}^{1}\right)-w^{-}\left(z_{\ell}^{1}\right)\right) & =J_{w, 1}\left(z_{\ell}^{1}\right), & & \ell=0,1 \\
\partial_{n}^{2}\left(w^{+}\left(z_{0}^{0}\right)-w^{-}\left(z_{0}^{0}\right)\right) & =J_{w, 2}\left(z_{0}^{0}\right) & &
\end{align*}\right.
$$

As in the linear case, the first equation states that the piecewise quadratic Lagrange interpolator of $w$ is zero.

Generally, to obtain correction functions of order $k^{\prime}$, we need to know the jumps up to order $k^{\prime}$ across $\Gamma, \llbracket \partial_{n}^{k} w \rrbracket=J_{w, k}$ for $k=0, \ldots, k^{\prime}$. Taking the the points $v_{i}$ for $i=$ $1, \ldots, \frac{(k+1)(k+2)}{2}$ as the Lagrange nodes in $K$ of order $k^{\prime}$ and points $z_{\ell}^{k}$ for $\ell=0, \ldots, k$ as the Lagrange nodes in $\Gamma_{K}$ of order $k=0, \ldots, k^{\prime}$, the equations we need to solve to obtain the corresponding correction function is given by solving the $\frac{(k+1)(k+2)}{2} \times \frac{(k+1)(k+2)}{2}$ linear system

$$
\left\{\begin{aligned}
w\left(v_{i}\right) & =0, & & i=1, \ldots, \frac{(k+1)(k+2)}{2} \\
\partial_{n}^{k^{\prime}-k}\left(w^{+}\left(z_{\ell}^{k}\right)-w^{-}\left(z_{\ell}^{k}\right)\right) & =J_{w, k^{\prime}-k}\left(z_{\ell}^{k}\right), & & \ell=0, \ldots, k, \quad k=0, \ldots, k^{\prime}
\end{aligned}\right.
$$

To obtain correction functions for the steady-state interface problem (3.5) in Appendix A. 1 we use the fact that the jump condition $\llbracket(\nabla \boldsymbol{u}-p I) \boldsymbol{n} \rrbracket=-\boldsymbol{\beta}$ can be split into jump conditions for $\boldsymbol{u}$ and $p$ separately (Guzmán et al., 2016b). Defining the rotation matrix $R=[\boldsymbol{n} \boldsymbol{\tau}]$, we express the normal and tangential components of $\boldsymbol{\beta}$ as $\hat{\boldsymbol{\beta}}=(\boldsymbol{\beta} \cdot \boldsymbol{n}, \boldsymbol{\beta} \cdot \boldsymbol{\tau})=$ $R^{\top} \boldsymbol{\beta}$, where $\boldsymbol{\tau}=\partial_{s} \boldsymbol{X} /\left\|\partial_{s} \boldsymbol{X}\right\|$ is the unit tangential vector to $\Gamma$ and $\boldsymbol{n}=\left(\tau_{2},-\tau_{1}\right)$ is the unit normal vector pointing outwards $\Omega^{-}$. The jump conditions are given by

$$
\begin{align*}
\llbracket p \rrbracket & =-\hat{\beta}_{1}  \tag{3.8a}\\
\llbracket \partial_{n} p \rrbracket & =\llbracket \boldsymbol{f} \cdot \boldsymbol{n} \rrbracket-\partial_{s} \hat{\beta}_{2}  \tag{3.8b}\\
\llbracket \boldsymbol{u} \rrbracket & =0  \tag{3.8c}\\
\llbracket \mu \partial_{n} \boldsymbol{u} \rrbracket & =\boldsymbol{\beta}-\hat{\beta}_{1} \boldsymbol{n} \tag{3.8d}
\end{align*}
$$

Higher order jump conditions can be achieved by obtaining derivatives of the equations (3.5) and the jump conditions (3.8) across the interface (Guzmán et al., 2016b). For
example, as shown in appendix A.2, the second normal jump condition for $\boldsymbol{u}$ is given by

$$
\llbracket \mu \partial_{n}^{2} \boldsymbol{u} \rrbracket=R\binom{\llbracket \partial_{n} p \rrbracket}{\partial_{s} \llbracket p \rrbracket /\left|\partial_{s} \boldsymbol{X}\right|}-k \llbracket \mu \partial_{n} \boldsymbol{u} \rrbracket-\llbracket \boldsymbol{f} \rrbracket,
$$

where $k$ is the curvature of the interface $\Gamma$. Replacing the expressions (3.8), we obtain

$$
\llbracket \mu \partial_{n}^{2} \boldsymbol{u} \rrbracket=R\binom{\llbracket \boldsymbol{f} \cdot \boldsymbol{n} \rrbracket-\partial_{s} \hat{\beta}_{2}}{-\partial_{s} \hat{\beta}_{1} /\left|\partial_{s} \boldsymbol{X}\right|}-k\left(\boldsymbol{\beta}-\hat{\beta}_{1} \boldsymbol{n}\right)-\llbracket \boldsymbol{f} \rrbracket .
$$

Thus obtaining the second order jump condition for $\boldsymbol{u}$. It is possible to obtain correction functions of an arbitrary order for both $\boldsymbol{u}$ and $p$. Taking divergence of equation $-\mu \Delta \boldsymbol{u}+\nabla p=\boldsymbol{f}$ we get $\Delta p=\nabla \cdot \boldsymbol{f}=\tilde{f}$, with known jumps $\llbracket p \rrbracket$ and $\llbracket \partial_{n} p \rrbracket$. Since this is a Poisson problem, we can obtain arbitrary order jump conditions for $p$ (Guzmán et al., 2016b). Similarly for $\boldsymbol{u}$, we can consider the problem $-\mu \Delta \boldsymbol{u}=\boldsymbol{f}-\nabla p$, and since we can obtain all jumps for the right-hand side and it is stated as a Poisson problem, it is also possible to obtain arbitrary order correction functions for $\boldsymbol{u}$.

### 3.4. Corrected Finite Element Methods

Using the explicit expressions for the jump conditions of $p$ and $\boldsymbol{u}$ and their normal derivatives, we can obtain correction functions $w_{h}^{p}$ and $\boldsymbol{w}_{h}^{u}$ defined on all elements $K \in$ $\mathcal{T}_{h}^{\Gamma}$. We define the approximate solution $\left(\tilde{\boldsymbol{u}}_{h}, \tilde{p}_{h}\right)$ of the Stokes problem as $\tilde{\boldsymbol{u}}_{h}=\boldsymbol{u}_{h}+$ $\boldsymbol{w}_{h}^{u}$ and $\tilde{p}_{h}=p_{h}+w_{h}^{p}$, such that $\boldsymbol{u}_{h}$ and $p_{h}$ and their derivatives up to the order of their respective correction functions used are continuous across the interface, while the discontinuities are approximated by correction functions $\boldsymbol{w}_{h}^{u}$ and $w_{h}^{p}$.

We use the same Taylor-Hood spaces $V_{h}$ and $Q_{h}$ from (2.8) of continuous piecewise polynomial functions of order 2 and 1 respectively. The corrected problem reads as follows. Find $\left(\boldsymbol{u}_{h}, p_{h}, \lambda_{h}\right) \in V_{h} \times Q_{h} \times \mathbb{R}$ such that

$$
\begin{align*}
\left(\mu \nabla \boldsymbol{u}_{h}, \nabla \boldsymbol{v}_{h}\right)_{\Omega}-\left(p_{h}, \nabla \cdot \boldsymbol{v}_{h}\right)_{\Omega}= & \left(\boldsymbol{f}, \boldsymbol{v}_{h}\right)_{\Omega}-\left\langle\boldsymbol{\beta}, \boldsymbol{v}_{h}\right\rangle_{\Gamma} \\
& -\left(\mu \nabla \boldsymbol{w}_{h}^{u}, \nabla \boldsymbol{v}_{h}\right)_{\Omega}+\left(w_{h}^{p}, \nabla \cdot \boldsymbol{v}_{h}\right)_{\Omega},  \tag{3.9a}\\
-\left(q_{h}, \nabla \cdot \boldsymbol{u}_{h}\right)_{\Omega}+\lambda_{h}\left(q_{h}, 1\right)_{\Omega}= & \left(q_{h}, \nabla \cdot \boldsymbol{w}_{h}^{u}\right)_{\Omega},  \tag{3.9b}\\
\left(p_{h}, 1\right)_{\Omega}= & -\left(w_{h}^{p}, 1\right)_{\Omega}, \tag{3.9c}
\end{align*}
$$

for all $\boldsymbol{v}_{h} \in V_{h}$ and $q_{h} \in Q_{h}$.

If we wanted to approximate the solution without considering correction functions, for a given triangulation $\mathcal{T}_{h}$ we would be approximating a non-smooth velocity $\boldsymbol{u}$ and a discontinuous pressure $p$ by continuous piecewise smooth functions $\boldsymbol{u}_{h}$ and $p_{h}$. When correction functions are used, we approximate the continuous and discontinuous part of the solution simultaneously, and as we will see in Sections 3.5 and 3.6 this allows us to recover optimal convergence rates for the finite element spaces used.

The main advantage of this approach is that the left-hand side of equations (3.9) stays the same for a fixed domain $\Omega$, independently of the function $\beta$ or the position of the interface $\Gamma$. This allows us to work efficiently with moving-interface problems, because the only terms we need to recalculate are the ones containing $\boldsymbol{\beta}$ or correction functions, which are non-zero only in the elements $K \in \mathcal{T}_{h}^{\Gamma}$ and are not involved in the matrix of the linear system.

### 3.5. Analysis of the corrected method

The correction functions for finite elements were first introduced in (Guzmán et al., 2016a) for the Poisson interface problem, using continuous piecewise linear elements
and introducing corrections based on computation on edges intersecting the interface. In (Guzmán et al., 2016b) the authors generalized this idea to higher order polynomial spaces for Poisson problems and Stokes interface problems. More specifically, the authors developed the correction function under the following assumptions on the space:

A1. $\boldsymbol{V}_{h}$ and $Q_{h}$ are a pair of inf-sup stables sub-spaces, with $V_{h} \subset\left[H_{0}^{1}(\Omega)\right]^{2}$
A2. We let $k \geq 1$ as the maximum integer such that

$$
\boldsymbol{V}_{h}^{k}:=\left\{\boldsymbol{v} \in C(\Omega) \cap\left[H_{0}^{1}(\Omega)\right]^{2}:\left.\boldsymbol{v}\right|_{K} \in \mathcal{P}^{k}(K), \forall K \in \mathcal{T}_{h}\right\} \subseteq \boldsymbol{V}_{h}
$$

and, if $Q_{h}$ contains the discontinuous pressure space of degree $k-1$ we let

$$
Q_{h}^{k-1}:=\left\{q \in L_{0}^{2}(\Omega):\left.q\right|_{K} \in \mathcal{P}^{k-1}(K), \forall K \in \mathcal{T}_{h}\right\} \subset Q_{h}
$$

otherwise

$$
Q_{h}^{k-1}:=\left\{q \in L_{0}^{2}(\Omega) \cap C(\Omega):\left.q\right|_{K} \in \mathcal{P}^{k-1}(K), \forall K \in \mathcal{T}_{h}\right\} \subset Q_{h}
$$

THEOREM 3. Let $(\boldsymbol{u}, p)$ be solution of (3.5) and assume that $\boldsymbol{u}^{ \pm} \in\left[C^{k+1}\left(\Omega^{ \pm}\right)\right]^{2}$ and $p^{ \pm} \in C^{k}\left(\Omega^{ \pm}\right)$. Let $\boldsymbol{V}_{h}$ and $M_{h}$ be the finite element spaces satisfying assumptions $A 1$ and $A 2$ and consider the definitions above for $k, I_{h}$ and $J_{h}$. Let $\left(\boldsymbol{u}_{h}, p_{h}\right) \in \boldsymbol{V}_{h} \times M_{h}$ be solution of (3.9). Then, there exists a constant $C>0$, such that

$$
\begin{aligned}
&\left\|\nabla\left(I_{h}(\boldsymbol{u})-\boldsymbol{u}_{h}\right)\right\|_{L^{\infty}(\Omega)}+\left\|J_{h}(p)-p_{h}\right\|_{L^{\infty}(\Omega)} \leq C h^{k}\left(\left\|\boldsymbol{u}^{+}\right\|_{C^{k+1}\left(\Omega^{+}\right)}+\left\|\boldsymbol{u}^{-}\right\|_{C^{k+1}\left(\Omega^{-}\right)}\right. \\
&\left.+\left\|p^{+}\right\|_{C^{k}\left(\Omega^{+}\right)}+\left\|p^{-}\right\|_{C^{k}\left(\Omega^{-}\right)}\right)
\end{aligned}
$$

and

$$
\left\|I_{h}(\boldsymbol{u})-\boldsymbol{u}_{h}\right\|_{L^{\infty}(\Omega)} \leq C h^{k+1} \log (1 / k)\left(\left\|\boldsymbol{u}^{+}\right\|_{C^{k+1}\left(\Omega^{+}\right)}+\left\|\boldsymbol{u}^{-}\right\|_{C^{k+1}\left(\Omega^{-}\right)}\right)
$$

where $I_{h}$ and $J_{h}$ are the piecewise polynomial Lagrange interpolant operator onto $\boldsymbol{V}_{h}^{k}$ and $Q_{h}^{k-1}$ (or the $L^{2}$ projection operator in the discontinuous case), $\|\cdot\|_{C^{k}\left(\Omega^{ \pm}\right)}$is the
maximum of the $L^{\infty}\left(\Omega^{ \pm}\right)$norms of all derivatives of order $k$, and the constant $C$ depends on the shape-regularity of the triangulation, $k$, and the regularity of $\Gamma$.

### 3.6. Numerical experiments

In this section we will show numerical estimates of the convergence rate of the corrected finite element method (3.9). We will work on $\Omega=(-1,1)^{2}$ and we will consider a circular interface $\Gamma=\left\{\frac{1}{3}(\cos s, \sin s)^{\top}, s \in[0,2 \pi]\right\}$. As we can see in Figure 3.2, for the triangulation of the domain we divide $\Omega$ in $N \times N$ squares, and then divide each of them into two triangles. The top-right and bottom-left squares are divided into four triangles in order to fulfill the conditions of Lemma 1. We also see that from the parametrization of $\Gamma$ we are able to find all intersections between the interface and the triangulation. This allows us to obtain correction functions in every element $K \in \mathcal{T}_{h}^{\Gamma}$.

Consider an exact solution of problem (3.5) with $\mu=1$ given by

$$
\begin{gather*}
\boldsymbol{u}=\binom{u_{1}}{u_{2}}, \quad u_{1}=\left\{\begin{array}{rr}
3 y, & r \leq 1 / 3 \\
\frac{4 y}{3 r}-y, & r>1 / 3
\end{array}, \quad u_{2}=\left\{\begin{array}{rr}
-3 x, & r \leq 1 / 3 \\
x-\frac{4 x}{3 r}, & r>1 / 3
\end{array}\right.\right.  \tag{3.10}\\
p=\left\{\begin{array}{rr}
4 r^{2}\left(4-\frac{\pi}{9}\right), & r \leq 1 / 3 \\
-\frac{2 \pi}{81}, & r>1 / 3
\end{array}\right. \tag{3.11}
\end{gather*}
$$

where $r=\sqrt{x^{2}+y^{2}}$. We can easily check that $\nabla \cdot \boldsymbol{u}=0$ and $p$ integrates 0 . The jump conditions are given by

$$
\begin{array}{cc}
\llbracket u_{1} \rrbracket=0, \quad \llbracket \partial_{n} u_{1} \rrbracket=-4 \sin s, \quad \llbracket \partial_{n}^{2} u_{1} \rrbracket=0, \\
\llbracket u_{2} \rrbracket=0, \quad \llbracket \partial_{n} u_{2} \rrbracket=4 \cos s, \quad \llbracket \partial_{n}^{2} u_{2} \rrbracket=0, \\
\llbracket p \rrbracket=-\frac{2 \pi}{81}-\left(4-\frac{\pi}{9}\right) \frac{4}{9}, \quad \llbracket \partial_{n} p \rrbracket=-\left(4-\frac{\pi}{9}\right) \frac{8}{3} .
\end{array}
$$



Figure 3.2. Example of the triangulation of the domain $\Omega=(-1,1)^{2}$ with $N=8$ and the intersections with a circular interface $\Gamma$.

The right-hand side is given by

$$
\boldsymbol{f}=\binom{f_{1}}{f_{2}}, \quad f_{1}=\left\{\begin{array}{rr}
8(4-\pi / 9) x, & r \leq 1 / 3 \\
4 y / 3 r^{3}, & r>1 / 3
\end{array}, \quad f_{2}=\left\{\begin{array}{rr}
8(4-\pi / 9) y, & r \leq 1 / 3 \\
-4 x / 3 r^{3}, & r>1 / 3
\end{array}\right.\right.
$$

The density force $\boldsymbol{\beta}$ is determined by the jumps of $\partial_{n} \boldsymbol{u}$ and $p$ :

$$
\boldsymbol{\beta}=\llbracket(\nabla \boldsymbol{u}-p I) \boldsymbol{n} \rrbracket=\llbracket \partial_{n} \boldsymbol{u} \rrbracket-\llbracket p \rrbracket \boldsymbol{n}=\binom{-4 \sin s}{4 \cos s}+\left(\frac{2 \pi}{81}+\left(4-\frac{\pi}{9}\right) \frac{4}{9}\right)\binom{\cos s}{\sin s} .
$$

Table 3.1. $L^{2}$ error for $\boldsymbol{u}, \nabla \boldsymbol{u}$ and $p$ and their estimated order of convergence (e.o.c.) without using correcting functions in a circular interface with $h=2 \sqrt{2} / N$.

| $N$ | $\boldsymbol{u}_{h}$ |  | $\nabla \boldsymbol{u}_{h}$ |  | $p_{h}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | error | e.o.c. | error | e.o.c. | error | e.o.c. |
| 4 | $3.07 \mathrm{e}-02$ | - | $4.56 \mathrm{e}-01$ | - | $1.07 \mathrm{e}-01$ | - |
| 8 | $9.77 \mathrm{e}-03$ | 1.65 | $2.10 \mathrm{e}-01$ | 1.12 | $1.47 \mathrm{e}-01$ | -0.45 |
| 16 | 3.28e-03 | 1.57 | $1.12 \mathrm{e}-01$ | 0.91 | $9.50 \mathrm{e}-02$ | 0.63 |
| 32 | $1.27 \mathrm{e}-03$ | 1.37 | 8.15e-02 | 0.46 | 7.26e-02 | 0.39 |
| 64 | $4.35 \mathrm{e}-04$ | 1.55 | 5.76e-02 | 0.50 | $5.19 \mathrm{e}-02$ | 0.48 |
| 128 | $1.53 \mathrm{e}-04$ | 1.50 | $4.30 \mathrm{e}-02$ | 0.42 | $3.89 \mathrm{e}-02$ | 0.42 |

Table 3.2. $L^{2}$ error for $\boldsymbol{u}, \nabla \boldsymbol{u}$ and $p$ and their estimated order of convergence (e.o.c.) using correcting functions in a circular interface with $h=2 \sqrt{2} / N$.

| $N$ | $\boldsymbol{u}_{h}$ |  | $\nabla \boldsymbol{u}_{h}$ |  | $p_{h}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | error | e.o.c. | error | e.o.c. | error | e.o.c. |
| 4 | $3.44 \mathrm{e}-02$ | - | $4.72 \mathrm{e}-01$ | - | $1.62 \mathrm{e}-01$ | - |
| 8 | 4.97e-03 | 2.79 | $1.37 \mathrm{e}-01$ | 1.78 | $4.97 \mathrm{e}-02$ | 1.71 |
| 16 | 6.82e-04 | 2.86 | 4.26e-02 | 1.69 | 1.17e-02 | 2.09 |
| 32 | $9.60 \mathrm{e}-05$ | 2.83 | $1.18 \mathrm{e}-02$ | 1.85 | $3.31 \mathrm{e}-03$ | 1.82 |
| 64 | 1.25e-05 | 2.95 | $3.21 \mathrm{e}-03$ | 1.88 | 8.56e-04 | 1.95 |
| 128 | 1.62e-06 | 2.94 | $8.39 \mathrm{e}-04$ | 1.94 | $2.19 \mathrm{e}-04$ | 1.97 |

We now compute the solution of problem (3.9). We can see from Tables 3.1 and 3.2 that the presence of correction functions $\boldsymbol{w}_{h}^{u}$ and $w_{h}^{p}$ not only give a much better approximation of the solution, but it also recovers the optimal convergence rate of the $\mathcal{P}^{2}-\mathcal{P}^{1}$ elements for a Stokes flow without an interface $\Gamma$, convergence rate of 3 for $\boldsymbol{u}$ and 2 for $\nabla \boldsymbol{u}$ and $p$.

In Figure 3.3 we show the approximation of the exact velocity (3.10) by using the corrected finite elements method. We can see that inside the circle the velocity changes


Figure 3.3. Magnitude and vector field of $\boldsymbol{u}_{h}$, an approximation of the velocity (3.10) of the corrected finite element method using $N=128$.
linearly and that it decays with $r$ when far from the interface. It is also possible to see that the solution is continuous across $\Gamma$.

To understand how the correction functions change the behavior of the solution, it is easier to show how the pressure changes since it is discontinuous across the interface. In Figure 3.4 we can see that both corrected and non-corrected numerical schemes seem to give a good approximation of the presssure (3.11), which is quadratic in $\Omega^{-}$and constant in $\Omega^{+}$. We can see that when using correction functions we are able to reproduce the discontinuities on the solution. However, because the solution given by the non-corrected scheme is continuous, the only way that this method can adapt to the local force $\boldsymbol{\beta}$ is by considering more triangles near $\Gamma$. We also show the absolute error of the pressure for each scheme, and we can clearly see that the non-corrected scheme has a bigger error near the interface because it is not able to work with discontinuous solutions.


Figure 3.4. Approximation and absolute error of the pressure of the problem with $N=128$, for corrected and non-corrected schemes.

Looking at the error of the corrected method in Figure 3.4, we can see that even though we are using correction functions for $p$, there still is more error across the interface $\Gamma$ than in other parts of the domain. This is because we are only considering linear correction functions for $p$, but the actual jump of the solution is quadratic.

## 4. MOVING INTERFACE PROBLEM

Because the corrected finite element method does not need the interface to match the triangulation of the domain, it is easy to extend our method to moving-interface problems. We provide energy estimates for the standard and corrected method and numerically test the stability of our method for different initial settings of $\Gamma$.

### 4.1. Strong formulation

In this section we consider a moving interface Stokes problem. For a given position of the interface, from Section 3 we already know how to approximate the fluid's velocity field and pressure by considering an external force caused by the interface. Since the fluid cannot pass through the interface, for the steady-state problem we imposed a continuity condition of $\boldsymbol{u}$ across the interface, $\llbracket u \rrbracket=\mathbf{0}$. In the moving interface problem we must also consider a no-slip condition

$$
\begin{equation*}
\partial_{t} \boldsymbol{X}(s, t)=\boldsymbol{u}(\boldsymbol{X}(s, t), t), \tag{4.1}
\end{equation*}
$$

such that the movement of a point in the interface at a time $t$ is given by the velocity field at that point. For this problem we will not only look for the velocity and pressure, but also we are interested in how the interface changes in time, i.e., we also look for a function $\boldsymbol{X}(s, t)$ which describes the interface at any time $t$. Physically, for any $s \in[0,2 \pi]$ we can interpret the point $\boldsymbol{X}(s, t)$ as the position of the original material point $\boldsymbol{X}(s, 0)$ at a given time $t$.

Because the interface is moving constantly, the local force caused by the interface is also changing. As we saw in Section 1.3, by considering a generalized Hooke's law to model the boundary tension $T=\sigma\left(\left\|\partial_{s} \boldsymbol{X}\right\|\right)$ and considering the unit tanget vector to the
interface $\boldsymbol{\tau}=\partial_{s} \boldsymbol{X} /\left\|\partial_{s} \boldsymbol{X}\right\|$, then

$$
\boldsymbol{\beta}(s, t)=\frac{\partial_{s}\left(\sigma\left(\left\|\partial_{s} \boldsymbol{X}\right\|\right) \boldsymbol{\tau}\right)}{\left\|\partial_{s} \boldsymbol{X}\right\|}
$$

If we assume $\sigma$ to be proportional to $\left\|\partial_{s} \boldsymbol{X}\right\|$, then the density force on the interface would be given by

$$
\boldsymbol{\beta}(s, t)=-\kappa \cdot \frac{\partial_{s}^{2} \boldsymbol{X}}{\left\|\partial_{s} \boldsymbol{X}\right\|},
$$

where $\kappa>0$ and the negative sign appears such that the energy of the system dissipates.
We want to solve for all $t \geq 0$ the velocity $\boldsymbol{u}(\cdot, t)$, pressure $p(\cdot, t)$ and the position of the interface $\boldsymbol{X}(\cdot, t)$ of the moving interface problem

$$
\begin{align*}
-\mu \Delta \boldsymbol{u}+\nabla p & =\boldsymbol{f} & & \text { in } \Omega \backslash \Gamma,  \tag{4.2a}\\
\nabla \cdot \boldsymbol{u} & =0 & & \text { in } \Omega \backslash \Gamma,  \tag{4.2b}\\
\boldsymbol{u} & =0 & & \text { on } \partial \Omega,  \tag{4.2c}\\
\llbracket \boldsymbol{u} \rrbracket & =0 & & \text { on } \Gamma,  \tag{4.2d}\\
\llbracket(\nabla u-p I) \boldsymbol{n} \rrbracket & =-\kappa \cdot \frac{\partial_{s}^{2} \boldsymbol{X}}{\left\|\partial_{s} \boldsymbol{X}\right\|}=\boldsymbol{\beta} & & \text { on } \Gamma,  \tag{4.2e}\\
\int_{\Omega} p \mathrm{~d} x & =0, & &  \tag{4.2f}\\
\partial_{t} \boldsymbol{X} & =\boldsymbol{u}(\boldsymbol{X}, t), & &  \tag{4.2~g}\\
\boldsymbol{X}(\cdot, 0) & =\boldsymbol{X}^{0}(\cdot) . & & \text { on } \Gamma, \tag{4.2h}
\end{align*}
$$

where the initial position of the interface $\boldsymbol{X}^{0}(\cdot)$ is known. This is a simplified model of the time-dependant Stokes problem where we omitted the $\partial_{t} \boldsymbol{u}$ term in the first equation. We discuss the additional challenges that the time derivative presents in Section 5.

Following the same procedure as in the steady-state problem, we can write the weak formulation of problem (4.2) as follows. For all $t \geq 0$ find $(\boldsymbol{u}(\cdot, t), p(\cdot, t), \lambda(t)) \in$
$\left[H_{0}^{1}(\Omega)\right]^{2} \times L^{2}(\Omega) \times \mathbb{R}$ such that

$$
\begin{align*}
(\mu \nabla \boldsymbol{u}, \nabla \boldsymbol{v})_{\Omega}-(p, \nabla \cdot \boldsymbol{v})_{\Omega} & =(\boldsymbol{f}, \boldsymbol{v})_{\Omega}-\langle\boldsymbol{\beta}, \boldsymbol{v}\rangle_{\Gamma} & & \forall \boldsymbol{v} \in\left[H_{0}^{1}(\Omega)\right]^{2},  \tag{4.3a}\\
-(q, \nabla \cdot \boldsymbol{u})_{\Omega}+\lambda(q, 1)_{\Omega} & =0 & & \forall q \in L^{2}(\Omega),  \tag{4.3b}\\
\tau(p, 1)_{\Omega} & =0, & & \forall \tau \in \mathbb{R}  \tag{4.3c}\\
\llbracket \boldsymbol{u} \rrbracket & =\mathbf{0}, & &  \tag{4.3d}\\
\partial_{t} \boldsymbol{X} & =\boldsymbol{u}(\boldsymbol{X}, t), & &  \tag{4.3e}\\
\boldsymbol{X}(\cdot, 0) & =\boldsymbol{X}^{0}(\cdot) . & & \tag{4.3f}
\end{align*}
$$

We introduce the elastic energy of the system

$$
\mathcal{E}(t)=\frac{\kappa}{2}\left\|\partial_{s} \boldsymbol{X}\right\|_{L^{2}(\Gamma)}^{2}
$$

Taking $\boldsymbol{f}=\mathbf{0}$ and testing equations (4.3) with $\boldsymbol{v}=\boldsymbol{u}$ and $q=p$, we can see that

$$
\begin{aligned}
-(\mu \nabla \boldsymbol{u}, \nabla \boldsymbol{u})_{\Omega} & =\int_{\Gamma} \boldsymbol{\beta} \cdot \boldsymbol{u} \mathrm{d} S=-\kappa \int_{0}^{2 \pi} \partial_{s}^{2} \boldsymbol{X} \cdot \partial_{t} \boldsymbol{X} \mathrm{~d} s \\
& =\kappa \int_{0}^{2 \pi} \partial_{s} \boldsymbol{X} \cdot \partial_{t} \partial_{s} \boldsymbol{X} \mathrm{~d} s=\frac{\mathrm{d}}{\mathrm{~d} t}\left(\frac{\kappa}{2} \int_{0}^{2 \pi}\left\|\partial_{s} \boldsymbol{X}\right\|^{2} \mathrm{~d} s\right)=\frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{E}
\end{aligned}
$$

and thus we obtain the dissipation of the energy of the system

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{E}=-\mu\|\nabla \boldsymbol{u}\|_{\Omega}^{2} \tag{4.4}
\end{equation*}
$$

Using that $\nabla \cdot \boldsymbol{u}=0$ and that the interface is closed, we also note that there is conservation of area:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} t}\left|\Omega^{-}\right| & =\frac{1}{2} \frac{\mathrm{~d}}{\mathrm{~d} t} \int_{0}^{2 \pi} \boldsymbol{X} \cdot \boldsymbol{n}\left\|\partial_{s} \boldsymbol{X}\right\| \mathrm{d} s=\frac{1}{2} \frac{\mathrm{~d}}{\mathrm{~d} t} \int_{0}^{2 \pi}\binom{x}{y} \cdot \partial_{s}\binom{y}{-x} \mathrm{~d} s \\
& =\frac{1}{2}\left(\int_{0}^{2 \pi} \partial_{t}\binom{x}{y} \cdot \partial_{s}\binom{y}{-x} \mathrm{~d} s+\int_{0}^{2 \pi}\binom{x}{y} \cdot \partial_{t} \partial_{s}\binom{y}{-x} \mathrm{~d} s\right) \\
& =\frac{1}{2}\left(\int_{0}^{2 \pi} \partial_{t}\binom{x}{y} \cdot \partial_{s}\binom{y}{-x} \mathrm{~d} s-\int_{0}^{2 \pi} \partial_{s}\binom{x}{y} \cdot \partial_{t}\binom{y}{-x} \mathrm{~d} s\right) \\
& =\int_{0}^{2 \pi} \partial_{t}\binom{x}{y} \cdot \partial_{s}\binom{y}{-x} \mathrm{~d} s=\int_{\Gamma} \boldsymbol{u} \cdot \boldsymbol{n} \mathrm{d} S=\int_{\Omega^{-}} \nabla \cdot \boldsymbol{u} \mathrm{d} \boldsymbol{x} \\
& =0
\end{aligned}
$$

### 4.2. Semi-discrete scheme

To solve this problem we may consider a semi-discrete scheme, where we have a triangulation $\mathcal{T}_{h}$ of the domain $\Omega$. Using the same spaces as in the steady-state problem, we multiply the equations (4.2) by test functions $\boldsymbol{v}_{h} \in V_{h}$ and $q_{h} \in Q_{h}$. For every $t \geq 0$ we look for functions $\boldsymbol{u}_{h}(\cdot, t) \times q_{h}(\cdot, t) \times \lambda(t) \in V_{h} \times Q_{h} \times \mathbb{R}$ and a parametrization of the interface $\boldsymbol{X}(\cdot, t)$ such that

$$
\begin{array}{rlrl}
\left(\mu \nabla \boldsymbol{u}_{h}, \nabla \boldsymbol{v}_{h}\right)_{\Omega}-\left(p_{h}, \nabla \cdot \boldsymbol{v}_{h}\right)_{\Omega} & =\left(\boldsymbol{f}, \boldsymbol{v}_{h}\right)_{\Omega}-\left\langle\boldsymbol{\beta}, \boldsymbol{v}_{h}\right\rangle_{\Gamma} & & \forall \boldsymbol{v}_{h} \in V_{h} \\
-\left(q_{h}, \nabla \cdot \boldsymbol{u}_{h}\right)_{\Omega}+\lambda\left(q_{h}, 1\right)_{\Omega} & =0 & & \forall q_{h} \in Q_{h} \\
\left(p_{h}, 1\right)_{\Omega} & =0 \\
\partial_{t} \boldsymbol{X}(\cdot, t) & =\boldsymbol{u}_{h}(\boldsymbol{X}(\cdot, t), t), & & \\
\boldsymbol{X}(\cdot, 0) & =\boldsymbol{X}^{0}(\cdot) & \tag{4.5e}
\end{array}
$$

If we also consider correction functions for $\boldsymbol{u}$ and $p$, the corrected semi-discrete scheme is given by

$$
\begin{align*}
\left(\mu \nabla \boldsymbol{u}_{h}, \nabla \boldsymbol{v}_{h}\right)_{\Omega}-\left(p_{h}, \nabla \cdot \boldsymbol{v}_{h}\right)_{\Omega}= & \left(\boldsymbol{f}, \boldsymbol{v}_{h}\right)_{\Omega}-\left\langle\boldsymbol{\beta}, \boldsymbol{v}_{h}\right\rangle_{\Gamma} \\
& -\left(\mu \nabla \boldsymbol{w}_{h}^{u}, \nabla \boldsymbol{v}_{h}\right)_{\Omega}+\left(w_{h}^{p}, \nabla \cdot \boldsymbol{v}_{h}\right)_{\Omega},  \tag{4.6a}\\
-\left(q_{h}, \nabla \cdot \boldsymbol{u}_{h}\right)_{\Omega}+\lambda_{h}\left(q_{h}, 1\right)_{\Omega}= & \left(q_{h}, \nabla \cdot \boldsymbol{w}_{h}^{u}\right)_{\Omega},  \tag{4.6b}\\
\left(p_{h}, 1\right)_{\Omega}= & -\left(w_{h}^{p}, 1\right)_{\Omega},  \tag{4.6c}\\
\partial_{t} \boldsymbol{X}(\cdot, t)= & \boldsymbol{u}_{h}(\boldsymbol{X}(\cdot, t), t)+\left\{\left\{\boldsymbol{w}_{h}^{u}(\boldsymbol{X}(\cdot, t), t)\right\},\right.  \tag{4.6d}\\
\boldsymbol{X}(\cdot, 0)= & \boldsymbol{X}^{0}(\cdot), \tag{4.6e}
\end{align*}
$$

for all $\boldsymbol{v}_{h} \in V_{h}$ and $q_{h} \in Q_{h}$. Here, $\{\varphi \varphi\}=\frac{\left.\left(\varphi^{+}+\varphi^{-}\right)\right|_{\Gamma}}{2}$ is the average value of $\varphi$ across $\Gamma$. We use the average value of the solution across $\Gamma$ to evolve the interface because even though the constraint $\llbracket \boldsymbol{u} \rrbracket=\mathbf{0}$ is considered for the correction functions, we can only guarantee that it holds for a few amount of points depending of the linear system that is solved for obtaining them, e.g. at points $v_{4}$ and $v_{5}$ from equations (3.6) or at points $v_{7}, v_{8}$ and $v_{9}$ from equations (3.7).

### 4.3. Fully-discrete scheme

For the discrete domain and time problem we will use a quasi-static method to approximate the solution of (4.2). We consider evenly separated time intervals $t_{n}=n \cdot \Delta t$. At a given time $t_{n}$ we may assume that we know the parametrization of the interface $\boldsymbol{X}\left(s, t_{n}\right)=$ $\boldsymbol{X}^{n}(s)$. Using this, at a given time $t_{n}$ we are looking for $\left(\boldsymbol{u}_{h}^{n}, p_{h}^{n}, \lambda_{h}^{n}\right) \in V_{h} \times Q_{h} \times \mathbb{R}$ such that

$$
\begin{array}{rlrl}
\left(\mu \nabla \boldsymbol{u}_{h}^{n}, \nabla \boldsymbol{v}_{h}\right)_{\Omega}-\left(p_{h}^{n}, \nabla \cdot \boldsymbol{v}_{h}\right)_{\Omega} & =\left(\boldsymbol{f}, \boldsymbol{v}_{h}\right)_{\Omega}-\left\langle\boldsymbol{\beta}^{n}, \boldsymbol{v}_{h}\right\rangle_{\Gamma} & & \forall \boldsymbol{v}_{h} \in V_{h}, \\
-\left(q_{h}, \nabla \cdot \boldsymbol{u}_{h}^{n}\right)_{\Omega}+\lambda_{h}^{n}\left(q_{h}, 1\right)_{\Omega} & =0 & \forall q_{h} \in Q_{h}, \\
\left(p_{h}^{n}, 1\right)_{\Omega} & =0, & & \tag{4.7c}
\end{array}
$$

where $\boldsymbol{\beta}^{n}=\kappa \cdot \partial_{s s} \boldsymbol{X}^{n} /\left\|\partial_{s} \boldsymbol{X}^{n}\right\|$. After solving these equations, we evolve the interface by using explicit ODE methods for the equation (4.1), for example, by using an explicit forward Euler method

$$
\frac{\boldsymbol{X}^{n+1}-\boldsymbol{X}^{n}}{\Delta t}=\boldsymbol{u}_{h}^{n}\left(\boldsymbol{X}^{n}\right)
$$

thus obtaining the new parametrization $X^{n+1}$ which we then use to solve the Stokes equations for the next time step.

Even though the no-slip condition (4.1) has the form of a simple ODE equation where implicit methods can be used, we decided to use explicit methods for evolving the interface because we do not have direct access to compute the right-hand side for any values of $t$. This is because when solving equations (4.7) we only obtain an approximation of $\boldsymbol{u}\left(\boldsymbol{x}, t_{n}\right)$. If we want to use an implicit method we would need to know how $\boldsymbol{u}_{h}^{n+1}$ looks, which is not possible to obtain with precision before knowing the density force $\boldsymbol{\beta}^{n+1}$, for which we would need the parametrization of the interface. However, it is possible to approximate a solution by carrying out an iterative scheme to represent the interface at a time $t_{n+1}$ ( Tu \& Peskin, 1992).

If we consider correction functions for $\boldsymbol{u}_{h}^{n}$ and $p_{h}^{n}$ in every time step, the corrected fully-discrete problem is given by

$$
\begin{align*}
\left(\mu \nabla \boldsymbol{u}_{h}^{n}, \nabla \boldsymbol{v}_{h}\right)_{\Omega}-\left(p_{h}^{n}, \nabla \cdot \boldsymbol{v}_{h}\right)_{\Omega}= & \left(\boldsymbol{f}, \boldsymbol{v}_{h}\right)_{\Omega}-\left\langle\boldsymbol{\beta}^{n}, \boldsymbol{v}_{h}\right\rangle_{\Gamma} \\
& -\left(\mu \nabla \boldsymbol{w}_{h}^{\boldsymbol{u}, n}, \nabla \boldsymbol{v}_{h}\right)_{\Omega}+\left(w_{h}^{p, n}, \nabla \cdot \boldsymbol{v}_{h}\right)_{\Omega}  \tag{4.8a}\\
-\left(q_{h}, \nabla \cdot \boldsymbol{u}_{h}^{n}\right)_{\Omega}+\lambda_{h}^{n}\left(q_{h}, 1\right)_{\Omega}= & \left(q_{h}, \nabla \cdot \boldsymbol{w}_{h}^{\boldsymbol{u}, n}\right)_{\Omega},  \tag{4.8b}\\
\left(p_{h}^{n}, 1\right)_{\Omega}= & -\left(w_{h}^{p, n}, 1\right)_{\Omega}, \tag{4.8c}
\end{align*}
$$

for all $\boldsymbol{v}_{h} \in V_{h}$ and $q_{h} \in Q_{h}$, and as in the semi-discrete scheme, the interface at a time $t_{n+1}$ is obtained by using an explicit ODE method like forward Euler,

$$
\frac{\boldsymbol{X}^{n+1}-\boldsymbol{X}^{n}}{\Delta t}=\boldsymbol{u}_{h}^{n}\left(\boldsymbol{X}^{n}\right)+\left\{\left\{\boldsymbol{w}_{h}^{\boldsymbol{u}, n}\left(\boldsymbol{X}^{n}\right)\right\}\right\},
$$

thus obtaining the new parametrization $X^{n+1}$ which we then use to solve the corrected problem for the next time step. Similarly as before, we decided to use explicit methods for evolving the interface since after solving equations (4.8) we only obtain $\boldsymbol{u}_{h}^{n}$ and we don't know how the right-hand side looks like at a time $t_{n+1}$, i.e. we don't know how to compute the correction terms or the density force $\boldsymbol{\beta}^{n+1}$, without knowing the parametrization $\boldsymbol{X}^{n+1}$.

Note that the left-hand side of equations (4.7) and (4.8) has always the same form, while the right-hand side only changes depending on the position of the interface $\Gamma$ at a specific time. Because of this, the linear system that results from both problems always has the same matrix, which allows us to obtain the solution at every time step much faster than in other methods where the mesh or the finite element spaces $V_{h}$ and $Q_{h}$ are modified. It is possible to obtain a factorization of the matrix only once (e.g. LU factorization) and use it to efficiently solve the linear system for a particular right-hand side at every time step.

### 4.4. Energy estimates

Throughout this section we will assume $f=\mathbf{0}$ to find energy estimates for each method, corrected and non corrected semi- and fully-discrete schemes, to determine theoretically the stability of each method. Recall that for the continuous problem, we have the dissipation condition (4.4).

### 4.4.1. Non corrected schemes

For the non-corrected semi-discrete scheme, we can obtain the dissipation of the energy by following the same procedure as in the continuous case by testing equations (4.5)
with functions $\boldsymbol{v}_{h}=\boldsymbol{u}_{h}$ and $q_{h}=p_{h}$, obtaining

$$
\begin{aligned}
-\left(\mu \nabla \boldsymbol{u}_{h}, \nabla \boldsymbol{u}_{h}\right)_{\Omega} & =\int_{\Gamma} \boldsymbol{\beta} \cdot \boldsymbol{u}_{h} \mathrm{~d} S=-\kappa \int_{0}^{2 \pi} \partial_{s}^{2} \boldsymbol{X} \cdot \partial_{t} \boldsymbol{X} \mathrm{~d} s \\
& =\kappa \int_{0}^{2 \pi} \partial_{s} \boldsymbol{X} \cdot \partial_{t} \partial_{s} \boldsymbol{X} \mathrm{~d} s=\partial_{t}\left(\frac{\kappa}{2} \int_{0}^{2 \pi}\left\|\partial_{s} \boldsymbol{X}\right\|^{2} \mathrm{~d} s\right)=\partial_{t} \mathcal{E}
\end{aligned}
$$

thus obtaining the energy estimate

$$
\partial_{t} \mathcal{E}=-\mu\left\|\nabla \boldsymbol{u}_{h}\right\|_{\Omega}^{2}
$$

Similarly for the non-corrected fully-discrete problem, testing equations (4.7) with $\boldsymbol{v}_{h}=$ $\boldsymbol{u}_{h}^{n}$ and $q_{h}=p_{h}^{n}$ we obtain

$$
\begin{aligned}
-\left(\mu \nabla \boldsymbol{u}_{h}^{n}, \nabla \boldsymbol{u}_{h}^{n}\right)_{\Omega} & =\int_{\Gamma} \boldsymbol{\beta}^{n} \cdot \boldsymbol{u}_{h}^{n} \mathrm{~d} S=-\kappa \int_{0}^{2 \pi} \partial_{s}^{2} \boldsymbol{X}^{n} \cdot\left(\frac{\boldsymbol{X}^{n+1}-\boldsymbol{X}^{n}}{\Delta t}\right) \mathrm{d} s \\
& =\frac{\kappa}{\Delta t} \int_{0}^{2 \pi} \partial_{s} \boldsymbol{X}^{n} \cdot\left(\partial_{s} \boldsymbol{X}^{n+1}-\partial_{s} \boldsymbol{X}^{n}\right) \mathrm{d} s \\
& =\frac{\kappa}{2 \Delta t}\left(\left\|\partial_{s} \boldsymbol{X}^{n+1}\right\|_{L^{2}(\Gamma)}^{2}-\left\|\partial_{s} \boldsymbol{X}^{n}\right\|_{L^{2}(\Gamma)}^{2}-\left\|\partial_{s} \boldsymbol{X}^{n+1}-\partial_{s} \boldsymbol{X}^{n}\right\|_{L^{2}(\Gamma)}^{2}\right)
\end{aligned}
$$

thus obtaining the energy estimate

$$
\mathcal{E}^{n+1}=\mathcal{E}^{n}-\mu \Delta t\left\|\nabla \boldsymbol{u}_{h}^{n}\right\|_{\Omega}^{2}+\frac{\kappa}{2}\left\|\partial_{s} \boldsymbol{X}^{n+1}-\partial_{s} \boldsymbol{X}^{n}\right\|_{L^{2}(\Gamma)}^{2}
$$

Because of the last term, the non-corrected fully-discrete scheme is not necessarily stable. If a backward Euler method (implicit) was used to evolve the interface, i.e. $\frac{\boldsymbol{X}^{n+1}-\boldsymbol{X}^{n}}{\Delta t}=\boldsymbol{u}_{h}^{n+1}(\boldsymbol{X})$, then it is easy to prove that in this case

$$
\mathcal{E}^{n+1}=\mathcal{E}^{n}-\mu \Delta t\left\|\nabla \boldsymbol{u}_{h}^{n+1}\right\|_{\Omega}^{2}-\frac{\kappa}{2}\left\|\partial_{s} \boldsymbol{X}^{n+1}-\partial_{s} \boldsymbol{X}^{n}\right\|_{L^{2}(\Gamma)}^{2},
$$

which proves to be more stable. However we will only show numerical experiments for explicit methods.

### 4.4.2. Corrected schemes

We now test the corrected semi-discrete equations (4.6) with $\boldsymbol{v}_{h}=\boldsymbol{u}_{h}$ and $q_{h}=p_{h}$. From the first equation we obtain

$$
\begin{align*}
\left(\mu \nabla\left(\boldsymbol{u}_{h}+\boldsymbol{w}_{h}^{u}\right), \nabla \boldsymbol{u}_{h}\right)_{\Omega}-\left(p_{h}+w_{h}^{p}, \nabla \cdot \boldsymbol{u}_{h}\right)_{\Omega}= & -\left\langle\boldsymbol{\beta}, \boldsymbol{u}_{h}\right\rangle_{\Gamma}, \\
= & -\left\langle\boldsymbol{\beta}, \boldsymbol{u}_{h}+\left\{\left\{\boldsymbol{w}_{h}^{u}\right\}\right\}\right\rangle_{\Gamma} \\
& +\left\langle\boldsymbol{\beta},\left\{\left\{\boldsymbol{w}_{h}^{u}\right\}\right\}\right\rangle_{\Gamma} . \tag{4.9}
\end{align*}
$$

Recalling that the solution must give $\lambda_{h}=0$, the second equation becomes $\left(p_{h}, \nabla \cdot \boldsymbol{u}_{h}\right)_{\Omega}=$ $-\left(p_{h}, \nabla \cdot \boldsymbol{w}_{h}^{u}\right)$. Also note that

$$
\left(\mu \nabla\left(\boldsymbol{u}_{h}+\boldsymbol{w}_{h}^{\boldsymbol{u}}\right), \nabla \boldsymbol{u}_{h}\right)_{\Omega}=\frac{\mu}{2}\left(\left\|\nabla\left(\boldsymbol{u}_{h}+\boldsymbol{w}_{h}^{\boldsymbol{u}}\right)\right\|_{\Omega}^{2}+\left\|\nabla \boldsymbol{u}_{h}\right\|_{\Omega}^{2}-\left\|\nabla \boldsymbol{w}_{h}^{u}\right\|_{\Omega}^{2}\right) .
$$

Using this, we can write (4.9) as

$$
\begin{aligned}
\left\langle\boldsymbol{\beta}, \boldsymbol{u}_{h}+\left\{\boldsymbol{w}_{h}^{u}\right\}\right\rangle_{\Gamma}= & \mu\left(\nabla\left(\boldsymbol{u}_{h}+\boldsymbol{w}_{h}^{\boldsymbol{u}}\right), \nabla \boldsymbol{u}_{h}\right)_{\Omega} \\
& +\left(p_{h}, \nabla \cdot \boldsymbol{u}_{h}\right)_{\Omega}+\left(w_{h}^{p}, \nabla \cdot \boldsymbol{u}_{h}\right)+\left\langle\boldsymbol{\beta},\left\{\boldsymbol{w}_{h}^{u}\right\}\right\rangle_{\Gamma} \\
= & -\frac{\mu}{2}\left(\left\|\nabla\left(\boldsymbol{u}_{h}+\boldsymbol{w}_{h}^{u}\right)\right\|_{\Omega}^{2}+\left\|\nabla \boldsymbol{u}_{h}\right\|_{\Omega}^{2}\right)+\frac{\mu}{2}\left\|\nabla \boldsymbol{w}_{h}^{u}\right\|_{\Omega}^{2} \\
& -\left(p_{h}, \nabla \cdot \boldsymbol{w}_{h}^{u}\right)_{\Omega}+\left(w_{h}^{p}, \nabla \cdot \boldsymbol{u}_{h}\right)+\left\langle\boldsymbol{\beta},\left\{\left\{\boldsymbol{w}_{h}^{u}\right\}\right\}\right\rangle_{\Gamma} .
\end{aligned}
$$

The same result can be obtained for the fully-discrete scheme, where the only thing that changes is that functions have an additional superindex $(\cdot)^{n}$ depending on the time $t_{n}$ that is being solved.

Following the same procedure as in the non-corrected energy estimates, we have that for the semi-discrete scheme

$$
\begin{aligned}
\partial_{t} \mathcal{E}= & -\frac{\mu}{2}\left(\left\|\nabla\left(\boldsymbol{u}_{h}+\boldsymbol{w}_{h}^{u}\right)\right\|_{\Omega}^{2}+\left\|\nabla \boldsymbol{u}_{h}\right\|_{\Omega}^{2}\right)+\frac{\mu}{2}\left\|\nabla \boldsymbol{w}_{h}^{\boldsymbol{u}}\right\|_{\Omega}^{2} \\
& \left.-\left(p_{h}, \nabla \cdot \boldsymbol{w}_{h}^{\boldsymbol{u}}\right)_{\Omega}+\left(w_{h}^{p}, \nabla \cdot \boldsymbol{u}_{h}\right)+\left\langle\boldsymbol{\beta},\left\{\boldsymbol{w}_{h}^{u}\right\}\right\}\right\rangle_{\Gamma},
\end{aligned}
$$

while for the fully discrete scheme

$$
\begin{aligned}
\mathcal{E}^{n+1}= & \mathcal{E}^{n}+\Delta t\left(-\frac{\mu}{2}\left(\left\|\nabla\left(\boldsymbol{u}_{h}+\boldsymbol{w}_{h}^{\boldsymbol{u}}\right)\right\|_{\Omega}^{2}+\left\|\nabla \boldsymbol{u}_{h}\right\|_{\Omega}^{2}\right)+\frac{\mu}{2}\left\|\nabla \boldsymbol{w}_{h}^{u}\right\|_{\Omega}^{2}\right. \\
& -\left(p_{h}, \nabla \cdot \boldsymbol{w}_{h}^{u}\right)_{\Omega}+\left(w_{h}^{p}, \nabla \cdot \boldsymbol{u}_{h}\right)+\left\langle\boldsymbol{\beta},\left\{\left\{\boldsymbol{w}_{h}^{u}\right\}\right\rangle_{\Gamma}\right) \\
& +\frac{\kappa}{2}\left\|\partial_{s} \boldsymbol{X}^{n+1}-\partial_{s} \boldsymbol{X}^{n}\right\|_{L^{2}(\Gamma)}^{2}
\end{aligned}
$$

### 4.5. Numerical experiments

In this section we will show how the fully-discrete schemes evolve the interface for different initial settings $\boldsymbol{X}^{0}=\boldsymbol{X}(\cdot, 0)$ and a fixed time step $\Delta t$. To obtain a parametrization of the interface at a time $t_{n}=n \Delta t$ for each $n \geq 0$, we take $M$ sample points across the initial interface, $\boldsymbol{X}_{0}^{n}, \ldots, \boldsymbol{X}_{M-1}^{n}$, and approximate the interface by its Fourier interpolation as shown in Section 2.4. We then use explicit methods for evolving the interface, thus obtaining the new position of the sample points $\boldsymbol{X}_{0}^{n+1}, \ldots, \boldsymbol{X}_{M-1}^{n+1}$ which allows us to obtain a parametrization of the interface in $t_{n+1}$. Since we want to test numerically the stability of our method we will assume $f=0$. Throughout this section we consider $\Omega=(-1,1)^{2}$ with the same triangulation as in Section 3.6, $\kappa=50$ and $\mu=1$. We take $M=16$ sample points on the interface at any time $t_{n}$.

### 4.5.1. Validation in circle

In the particular case where $\boldsymbol{X}^{0}(s)=r(\cos s, \sin s)^{\top}$, we can easily verify that the solution of the problem is $\boldsymbol{u}=\mathbf{0}$ and $p=$ constant in $\Omega^{ \pm}$with jump $\llbracket p \rrbracket=-\kappa$ and $\int_{\Omega} p \mathrm{~d} \boldsymbol{x}=0$. Since $\boldsymbol{u}=\mathbf{0}$, the interface does not move and thus the energy of the system should be constant. In Figure 4.1 we see that in this case the corrected method seems to be much more stable than the non-corrected one. When considering correction functions, the corrected method is able to assign all discontinuities to the pressure, thus obtaining a much better approximation of the solution $\boldsymbol{u}$ and maintaining a steady interface. We

(a) Evolution of the energy of the system for a circular interface.

(b) Evolution of the area of $\Omega^{-}$for a circular interface.

Figure 4.1. Evolution of area and energy of the homogeneous problem for an initial circular interface with $N=32$ and $\Delta t=0.01$.
can see that the energy of the system and area of the interior region for the non-corrected scheme constantly decreases.

As the circle is the stability point of the Stokes equations, when the initial configuration of the interface is not a circle then any homogeneous problem should give a solution $(\boldsymbol{u}, p)$ that moves the interface towards a circle. We will now see how such solution looks like for two different initial configurations of $\Gamma$.

### 4.5.2. Ellipse

For this example we consider an initial interface given by the ellipse equation

$$
\boldsymbol{X}^{0}(s)=\binom{\frac{2}{3} \cos s}{\frac{1}{3} \sin s}, \quad s \in[0,2 \pi] .
$$

In Figure 4.2 we see how the energy and the area of $\Omega^{-}$change in time in both corrected and non-corrected schemes. We can see that the energy of both systems decay in the first iterations, which makes sense since the initial interface is not a circle. However,
the corrected method seems to converge to a solution since the quantities $\mathcal{E}$ and $\left|\Omega^{-}\right|$tend to stabilize, while the non-corrected scheme does not.

(a) Evolution of the energy of the system for an elliptic interface.

(b) Evolution of the area of $\Omega^{-}$for an elliptic interface.

Figure 4.2. Evolution of area and energy of the homogeneous problem for an initial elliptic interface with $N=32$ and $\Delta t=0.01$.

We also see that a small area $\left|\Omega^{-}\right|$is lost during the first iterations. This is probably due to the fact that to integrate the correction functions in a triangle $K$ we use an isoparametric interpolation of the interface, where we approximate the segment $\Gamma \cap K$ by a polynomial of the same order of $\boldsymbol{u}_{h}$, in this case, by a quadratic polynomial. Thus we are only maintaining the area of a piecewise-quadratic interpolation of the interface, which can be improved by considering a higher order representation of the interface in each triangle.

In Figure 4.3 we see how the elliptic interface moves according to the velocity field while slowly transforming into a circle. The fact that the interface converges to a circle matches with the information shown in Figures 4.2 and 4.1 for corrected schemes, since the the energy of the system and the area of $\Omega^{-}$converge to a constant value.


Figure 4.3. Evolution of the velocity field and the interface for elliptic shaped initial interface problem, fully-discrete scheme. Approximation given by a triangulation with $N=32$ and $\Delta t=0.01$.

### 4.5.3. Heart

For this example we consider an initial interface given by the heart equation

$$
\boldsymbol{X}^{0}(s)=\frac{1}{20}\binom{(7(1-\sin s)+3(1-\cos s)) \cos s}{(3(1-\sin s)+7(1-\cos s)) \sin s}+\binom{0.12}{0.12}, \quad s \in[0,2 \pi] .
$$

In Figure 4.4 we see how the energy and the area of $\Omega^{-}$change in time in both corrected and non-corrected schemes. We can see that the energy of both systems decay in the first iterations, which makes sense since the initial interface is not a circle. However, the corrected method seems to converge to a solution since the quantities $\mathcal{E}$ and $\left|\Omega^{-}\right|$tend to stabilize, while the non-corrected scheme does not.


Figure 4.4. Evolution of area and energy of the homogeneous problem for an initial heart-shaped interface with $N=32$ and $\Delta t=0.01$.

We can see that a small area $\left|\Omega^{-}\right|$is gained during the first iterations. Similarly as before, this is probably due to the fact that to integrate the correction functions in a triangle $K$ we are approximating the parametrization of the segment $\Gamma \cap K$ by using an isoparametric interpolation, where the order of the interpolation curve matches the order of $\boldsymbol{u}_{h}$, a piecewise quadratic polynomial. The fact that some area is gained instead of lost like in
the elliptic interface case is probably caused by the non-convexity of the interior domain $\Omega^{-}$.

In Figure 4.5 we see how the heart shaped interface moves according to the velocity field while slowly transforming into a circle. Similar as the elliptic case, the fact that the interface converges to a circle matches with the information shown in Figures 4.4 and 4.1 for corrected schemes, since the the energy of the system and the area of $\Omega^{-}$converge to a constant value. This proves that the corrected method is stable for non-convex interior domains $\Omega^{-}$.


Figure 4.5. Evolution of the velocity field and the interface for heart shaped initial interface problem, fully-discrete scheme. Approximation given by a triangulation with $N=32$ and $\Delta t=0.01$.

## 5. CONCLUSIONS AND FUTURE WORK

In Section 3 we showed a numerical method to solve the steady-state Stokes equations with an elastic interface immersed in the domain $\Omega$. The approximate solutions $\boldsymbol{u}_{h} \in V_{h}$ and $p_{h} \in Q_{h}$ were considered to be piecewise polynomial in $\mathcal{P}^{2}$ and $\mathcal{P}^{1}$ respectively, with polynomial correction functions of the same order. We obtained that in this case, the approximate solution $\boldsymbol{u}_{h}$ converges with order 3 , while $\nabla \boldsymbol{u}_{h}$ and $p_{h}$ converge with order 2. This matches the optimal convergence rate of the method for the standard steady-state Stokes problem (2.5) on the same spaces $V_{h}$ and $Q_{h}$.

In Section 4 we presented a fully-discrete scheme to solve a moving interface Stokes problem. The corrected method proved to be more stable than the non-corrected method by maintaining some physical quantities like the energy of the system and the conservation of mass at each side of the interface for the steady-state problem (circular interface), and giving better estimations of the evolution of those quantities for different initial configurations of the interface (convex and non-convex domains $\Omega^{-}$enclosed by $\Gamma$ ).

As we can see from the numerical experiments of moving interface problems in Figures 4.3 and 4.5 , the points across the interface are constantly moving. This movement is not restricted to a few triangles, and the interface can pass through any triangle $K$ in every iteration. If we were restricted to use meshes aligned with the interface, it would be necessary to obtain a new triangulation of the domain and recalculate the matrix of the linear system at every time step. Since our method considers a fixed triangulation the matrix of the linear system stays always the same for every time step, so we only need to obtain its inverse once (or any factorization that allows us to efficiently solve the linear system) independent of the time step we are trying to solve.

As we can see from Section 3.3, the correction function for $\boldsymbol{u}_{h}$ depends on the known jump condititons $\llbracket \boldsymbol{u} \rrbracket, \llbracket \mu \partial_{n} \boldsymbol{u} \rrbracket$ and $\llbracket \mu \partial_{n n} \boldsymbol{u} \rrbracket$. Since we only worked with constant values of $\mu$, dividing by $\mu$ at both sides of the jump conditions result in the jump conditions $\llbracket \partial_{n} \boldsymbol{u} \rrbracket$
and $\llbracket \partial_{n n} \boldsymbol{u} \rrbracket$, which we can use to obtain the correction function for $\boldsymbol{u}$. If different fluids were considered on each side of the interface, then the viscosity $\mu$ would not be constant across the interface, and different methods should be used to obtain precisely the jump conditions for the derivative of $\boldsymbol{u}$. To illustrate this, consider the case where all jump conditions $\llbracket \boldsymbol{u} \rrbracket, \llbracket \mu \partial_{n} \boldsymbol{u} \rrbracket$ and $\llbracket \mu \partial_{n n} \boldsymbol{u} \rrbracket$ are equal to zero. Using the proposed method to obtain the correction function for $\boldsymbol{u}$ would result in a homogeneous linear system, which can only result in a correction function equal to zero. However, since $\llbracket \mu \partial_{n} \boldsymbol{u} \rrbracket=0$ and $\llbracket \mu \rrbracket \neq 0$, then $\partial_{n} \boldsymbol{\mu}$ should be discontinuous across the interface. Thus, additional methods need to be used in order to obtain the correction functions in this case where $\mu$ is discontinuous across $\Gamma$.

To evolve the interface we only provided energy estimates and showed numerical experiments from using Euler's method. The stability of the method might be improved by using higher order ODE methods like explicit Runge-Kutta schemes from Section 2.5.

Three-dimensional problems may be studied using the same approach as we did on this thesis. All convergence proofs and energy estimates are valid in higher dimensions, however, the fact that in $\mathbb{R}^{3}$ the interface is represented by a surface suggests that different interpolation methods must be used to represent it, and finding the intersection between the interface and the faces of each element becomes a nontrivial task. Using level-set representations of the curve may be a better approach for this problem.

In the future, we would like to extend this method for a non-stationary Stokes flow $\rho \partial_{t} \boldsymbol{u}-\mu \Delta \boldsymbol{u}+\nabla p=\boldsymbol{f}$. This presents an additional challenge, because for any approximation of the time derivative, for example $\partial_{t} \boldsymbol{u}\left(\cdot, t_{n}\right) \approx \frac{\boldsymbol{u}\left(\cdot, t_{n+1}\right)-\boldsymbol{u}\left(\cdot, t_{n}\right)}{\Delta t}$, correction functions must be considered for the velocity at different time steps simultaneously. The Stokes problem is still linear in $\boldsymbol{u}$ and $p$, and thus the correction functions method can still be used to efficiently solve the resulting linear system.

Finally, we would like to use our method to approximate the solution of the incompressible Navier-Stokes equations $\rho\left(\partial_{t} \boldsymbol{u}+\boldsymbol{u} \cdot \nabla \boldsymbol{u}\right)-\mu \Delta \boldsymbol{u}+\nabla p=\boldsymbol{f}$. Because of the non linearity of the Navier-Stokes problem, the matrix of the linear system will inevitably change for different configurations of the interface but only a small number of coordinates would need to be updated.

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APPENDIX

## A. FIRST AND SECOND ORDER JUMP CONDITIONS

We show how to obtain jump conditions for the solution $(p, \boldsymbol{u})$ of the Stokes problem across an elastic interface $\Gamma=\left\{\boldsymbol{X}(s)=(x(s), y(s))^{\top}, s \in[0,2 \pi]\right\}$. We assume that the parametrization of $\Gamma$ and all its derivatives are periodic and continuous on $[0,2 \pi]$.

## A.1. First order jumps

To split the jump condition $\llbracket(\mu \nabla \boldsymbol{u}-p I) \boldsymbol{n} \rrbracket=\boldsymbol{\beta}$ into separate jump conditions $\llbracket \mu \partial_{n} \boldsymbol{u} \rrbracket$, $\llbracket p \rrbracket$ and $\llbracket \partial_{n} p \rrbracket$, we follow the procedure shown in (Guzmán et al., 2016b).

We define for $\epsilon>0$ the set $\Omega_{\epsilon}=\{\boldsymbol{x} \in \Omega: \operatorname{dist}(\boldsymbol{x}, \Gamma)<\epsilon\}$, where dist is the standard distance function. We also denote $\Gamma_{\epsilon}^{ \pm}=\partial \Omega_{\epsilon} \cap \Omega^{ \pm}$.

We consider the integral form of the elastic interface Stokes Flow, $-\Delta \boldsymbol{u}+\nabla p=\boldsymbol{f}+\boldsymbol{B}$, where $\boldsymbol{B}(\boldsymbol{x})=\int_{\Gamma} \boldsymbol{\beta}(s) \delta(\boldsymbol{x}-\boldsymbol{X}(s)) \mathrm{d} S$. Taking divergence, multiplying by an arbitrary $\phi \in C^{2}$ of compact support in $\Omega_{\epsilon}$ and integrating we obtain

$$
\begin{aligned}
\int_{\Omega_{\epsilon}} \nabla \cdot \nabla p \phi \mathrm{~d} \boldsymbol{x} & =\int_{\Omega_{\epsilon}} \nabla \cdot \boldsymbol{f} \phi \mathrm{d} \boldsymbol{x}+\int_{\Omega_{\epsilon}} \nabla \cdot \boldsymbol{B} \phi \mathrm{d} \boldsymbol{x} \\
& =\int_{\Omega_{\epsilon}} \nabla \cdot \boldsymbol{f} \phi \mathrm{d} \boldsymbol{x}-\int_{\Gamma} \boldsymbol{\beta} \cdot \nabla \phi \mathrm{d} S
\end{aligned}
$$

For the left-hand side we integrate by parts twice obtaining

$$
\int_{\Omega_{\epsilon}} \nabla \cdot \nabla p \phi \mathrm{~d} \boldsymbol{x}=\int_{\Gamma_{\epsilon}^{ \pm}} \nabla p \cdot \boldsymbol{n} \phi \mathrm{~d} S-\int_{\Gamma_{\epsilon}^{ \pm}} \nabla \phi \cdot \boldsymbol{n} p \mathrm{~d} S+\int_{\Omega_{\epsilon}} p \nabla \cdot \nabla \phi \mathrm{~d} \boldsymbol{x} .
$$

By the definitions of $\Gamma_{\epsilon}^{ \pm}$we can fix the normal vectors to be the normal vector to $\Gamma$ pointing outwards $\Omega^{-}$. Taking limit as $\epsilon \rightarrow 0$ we obtain

$$
\lim _{\epsilon \rightarrow 0} \int_{\Omega_{\epsilon}} \nabla \cdot \nabla p \phi \mathrm{~d} \boldsymbol{x}=-\int_{\Gamma} \llbracket \partial_{n} p \rrbracket \phi \mathrm{~d} S+\int_{\Gamma} \partial_{n} \phi \llbracket p \rrbracket \mathrm{~d} S .
$$

Similarly for the right-hand side, integrating the first integral by parts and taking limit we see that

$$
\lim _{\epsilon \rightarrow 0} \int_{\Omega_{\epsilon}} \nabla \cdot \boldsymbol{f} \phi \mathrm{d} \boldsymbol{x}=-\int_{\Gamma} \llbracket \boldsymbol{f} \cdot \boldsymbol{n} \rrbracket \phi \mathrm{d} S
$$

On the other hand, we need to write the second integral in terms of the normal and tangential derivatives. We do this by using the rotation matrix $R=[\boldsymbol{n} \boldsymbol{\tau}]$, where $\boldsymbol{n}$ and $\boldsymbol{\tau}$ are the unit normal and tangent vectors, respectively. Using this matrix we can easily see that $\nabla_{x, y} \varphi=R \nabla_{n, \tau} \varphi$ for any differentiable function $\varphi$ (A.3). Defining $\hat{\boldsymbol{\beta}}=R^{\top} \boldsymbol{\beta}$ as the normal and tangential components of $\beta$, we obtain

$$
\begin{aligned}
\int_{\Gamma} \boldsymbol{\beta} \cdot \nabla \phi \mathrm{d} S & =\int_{\Gamma} \boldsymbol{\beta} \cdot\left(R \nabla_{n, \tau} \phi\right) \mathrm{d} S \\
& =\int_{\Gamma} \hat{\boldsymbol{\beta}} \cdot \nabla_{n, \tau} \phi \mathrm{~d} S \\
& =\int_{\Gamma} \hat{\beta}_{1} \partial_{n} \phi \mathrm{~d} S+\int_{\Gamma} \hat{\beta}_{2} \partial_{\tau} \phi \mathrm{d} S \\
& =\int_{\Gamma} \hat{\beta}_{1} \partial_{n} \phi \mathrm{~d} S-\int_{\Gamma} \partial_{s} \hat{\beta}_{2} \phi \mathrm{~d} S
\end{aligned}
$$

where on the last equality we integrated by parts and used the fact that $\Gamma$ is closed and smooth. Putting all together, we have shown that

$$
-\int_{\Gamma} \llbracket \partial_{n} p \rrbracket \phi \mathrm{~d} S+\int_{\Gamma} \llbracket p \rrbracket \partial_{n} \phi \mathrm{~d} S=-\int_{\Gamma} \llbracket \boldsymbol{f} \cdot \boldsymbol{n} \rrbracket \phi \mathrm{d} S-\int_{\Gamma} \hat{\beta}_{1} \partial_{n} \phi \mathrm{~d} S+\int_{\Gamma} \partial_{s} \hat{\beta}_{2} \phi \mathrm{~d} S .
$$

Testing with adequate functions $\phi$, we conclude that

$$
\llbracket p \rrbracket=-\hat{\beta}_{1}, \quad \llbracket \partial_{n} p \rrbracket=\llbracket \boldsymbol{f} \cdot \boldsymbol{n} \rrbracket-\partial_{s} \hat{\beta}_{2} .
$$

Replacing this in $\llbracket(\nabla \boldsymbol{u}-p I) \boldsymbol{n} \rrbracket=\boldsymbol{\beta}$, we obtain the normal jump condition for $\boldsymbol{u}$,

$$
\llbracket \mu \partial_{n} \boldsymbol{u} \rrbracket=\boldsymbol{\beta}+\llbracket p \rrbracket \boldsymbol{n}=\boldsymbol{\beta}-\hat{\beta}_{1} \boldsymbol{n} .
$$

## A.2. Second order jumps

Let $(p, \boldsymbol{u})$ be a solution of $-\mu \Delta \boldsymbol{u}+\nabla p=\boldsymbol{f}$ on $\Omega^{ \pm}$with known first order jump conditions, $\llbracket p \rrbracket=J_{p, 0}, \llbracket \partial_{n} p \rrbracket=J_{p, 1}, \llbracket \boldsymbol{u} \rrbracket=0$ and $\llbracket \mu \partial_{n} \boldsymbol{u} \rrbracket=\boldsymbol{J}_{\boldsymbol{u}, 1}$, and let $\theta$ be the angle from the positive $X$ axis to the unit normal vector $\boldsymbol{n}$. First we note that for any function $\varphi$ defined on $\Omega$ with continuous second order partial derivatives,

$$
\begin{align*}
\partial_{x} \varphi & =\partial_{n} \varphi \cos \theta-\partial_{\tau} \varphi \sin \theta,  \tag{A.1a}\\
\partial_{y} \varphi & =\partial_{n} \varphi \sin \theta+\partial_{\tau} \varphi \cos \theta,  \tag{A.1b}\\
\partial_{x}^{2} \varphi & =\partial_{n}^{2} \varphi \cos ^{2} \theta-2 \partial_{n} \partial_{\tau} \varphi \cos \theta \sin \theta+\partial_{\tau}^{2} \varphi \sin ^{2} \theta,  \tag{A.1c}\\
\partial_{y}^{2} \varphi & =\partial_{n}^{2} \varphi \sin ^{2} \theta+2 \partial_{n} \partial_{\tau} \varphi \cos \theta \sin \theta+\partial_{\tau}^{2} \varphi \cos ^{2} \theta,  \tag{A.1d}\\
\partial_{x} \partial_{y} \varphi & =\left(\partial_{n}^{2} \varphi-\partial_{\tau}^{2} \varphi\right) \cos \theta \sin \theta+\partial_{n} \partial_{\tau} \varphi\left(\cos ^{2} \theta-\sin ^{2} \theta\right) . \tag{A.1e}
\end{align*}
$$

A tangent vector to $\Gamma$ is given by $\boldsymbol{\tau}=\left(x^{\prime}, y^{\prime}\right)^{\top}$, so the normal vector is given by $\boldsymbol{n}=\left(y^{\prime},-x^{\prime}\right)$. This gives a relation between $\theta$ and the parametrization of the interface:

$$
\begin{equation*}
\cos \theta=\frac{y^{\prime}}{\left\|\partial_{s} \boldsymbol{X}\right\|}, \quad-\sin \theta=\frac{x^{\prime}}{\left\|\partial_{s} \boldsymbol{X}\right\|} . \tag{A.2}
\end{equation*}
$$

Using this, the rotation matrix $R$ can be written as

$$
R=[\boldsymbol{n} \boldsymbol{\tau}]=\frac{1}{\left\|\partial_{s} \boldsymbol{X}\right\|}\left[\begin{array}{cc}
y^{\prime} & x^{\prime}  \tag{A.3}\\
-x^{\prime} & y^{\prime}
\end{array}\right]=\left[\begin{array}{cc}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{array}\right]
$$

such that $\nabla_{x, y}(\cdot)=R \nabla_{n, \tau}(\cdot)$.

To obtain second order jump conditions consider first the continuity of the velocity and obtain its first and second derivative on the interface using the chain rule:

$$
\begin{align*}
\mathbf{0} & =\llbracket \boldsymbol{u} \rrbracket  \tag{A.4a}\\
\partial_{s}() \Longrightarrow \mathbf{0} & =\llbracket \partial_{x} \boldsymbol{u} \rrbracket x^{\prime}+\llbracket \partial_{y} \boldsymbol{u} \rrbracket y^{\prime},  \tag{A.4b}\\
\partial_{s}() \Longrightarrow \mathbf{0} & =\llbracket \partial_{x}^{2} \boldsymbol{u} \rrbracket\left(x^{\prime}\right)^{2}+2 \llbracket \partial_{x} \partial_{y} \boldsymbol{u} \rrbracket x^{\prime} y^{\prime}+\llbracket \partial_{y}^{2} \boldsymbol{u} \rrbracket\left(y^{\prime}\right)^{2}+\llbracket \partial_{x} \boldsymbol{u} \rrbracket x^{\prime \prime}+\llbracket \partial_{y} \boldsymbol{u} \rrbracket y^{\prime \prime}, \tag{A.4c}
\end{align*}
$$

where all partial derivatives were taken on each coordinate separately. Replacing all derivatives on $x$ and $y$ by normal and tangential derivatives using expressions (A.1) and changing the trigonometric functions for the identities (A.2), we can (A.4b) as

$$
\begin{aligned}
\mathbf{0} & =\left(\llbracket \partial_{n} \boldsymbol{u} \rrbracket \cos \theta-\llbracket \partial_{\tau} \boldsymbol{u} \rrbracket \sin \theta\right)(-\sin \theta)+\left(\llbracket \partial_{n} \boldsymbol{u} \rrbracket \sin \theta+\llbracket \partial_{\tau} \boldsymbol{u} \rrbracket \cos \theta\right)(\cos \theta), \\
& =\llbracket \partial_{\tau} \boldsymbol{u} \rrbracket .
\end{aligned}
$$

As for (A.4c), dividing by $\left\|\partial_{s} \boldsymbol{X}\right\|^{2}$ and replacing $x^{\prime}$ and $y^{\prime}$ using (A.2) we obtain that

$$
-\frac{\llbracket \partial_{x} \boldsymbol{u} \rrbracket x^{\prime \prime}+\llbracket \partial_{y} \boldsymbol{u} \rrbracket y^{\prime \prime}}{\left\|\partial_{s} \boldsymbol{X}\right\|^{2}}=\llbracket \partial_{x}^{2} \boldsymbol{u} \rrbracket \sin ^{2} \theta-2 \llbracket \partial_{x} \partial_{y} \boldsymbol{u} \rrbracket \cos \theta \sin \theta+\llbracket \partial_{y}^{2} \boldsymbol{u} \rrbracket \cos ^{2} \theta
$$

Using the equations (A.1), it is easy to show that the right-hand side is equal to $\llbracket \partial_{\tau}^{2} \boldsymbol{u} \rrbracket$. For the left-hand side, we can also use the fact that $\llbracket \partial_{n} \boldsymbol{u} \rrbracket=\boldsymbol{J}_{\boldsymbol{u}, 1}$ and $\llbracket \partial_{\tau} \boldsymbol{u} \rrbracket=\mathbf{0}$, obtaining

$$
\begin{aligned}
-\frac{\llbracket \partial_{x} \boldsymbol{u} \rrbracket x^{\prime \prime}+\llbracket \partial_{y} \boldsymbol{u} \rrbracket y^{\prime \prime}}{\left\|\partial_{s} \boldsymbol{X}\right\|^{2}} & =-\frac{\left(\boldsymbol{J}_{u, 1} \cos \theta\right) x^{\prime \prime}+\left(\boldsymbol{J}_{u, 1} \sin \theta\right) y^{\prime \prime}}{\left\|\partial_{s} \boldsymbol{X}\right\|^{2}} \\
& =\frac{x^{\prime} y^{\prime \prime}-y^{\prime} x^{\prime \prime}}{\left\|\partial_{s} \boldsymbol{X}\right\|^{3 / 2}} \boldsymbol{J}_{\boldsymbol{u}, 1} \\
& =k \boldsymbol{J}_{u, 1}
\end{aligned}
$$

