PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE
FACULTAD DE MATEMÁTICAS
DEPARTAMENTO DE MATEMÁTICA

Lower Bounds for the Relative Regulator

POR
VICTOR CASTILLO GÁRATE.

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Director de Tesis:
Eduardo Friedman

Comisión informante:
1. Dr. Sebastián Herrero, Pontificia Universidad Católica de Valparaíso
2. Dr. Ricardo Menares, Pontificia Universidad Católica de Chile

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A mis padres, profesores y amigos.
Por su incondicional apoyo e inspiración.

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Contents

Resumen

El regulador relativo $\mathrm{Reg}(L/K)$ de una extensión de cuerpos de números L/K está estrechamente relacionada con el cuociente $\mathrm{Reg}(L)/\mathrm{Reg}(K)$ de reguladores clásicos de L y K. En 1999 Friedman y Skoruppa [FS99] demostraron que $\mathrm{Reg}(L/K)$ posee cotas inferiores que crecen exponencialmente con el grado absoluto $[L : \mathbb{Q}]$, siempre que el grado relativo $[L : K]$ sea suficientemente grande. Friedman y Skoruppa partieron de una desigualdad analítica que involucra $\mathrm{Reg}(L/K)$ y desarrollaron un análisis asintótico que funciona bien para grados relativos $[L : K] \geq 40$. En esta tesis, partimos de la misma desigualdad, pero para grados $[L : K] \leq 40$ usamos técnicas numéricas y asintóticas para demostrar el crecimiento exponencial de las cotas inferiores cuando $[L : K] \geq 12$. Imponiendo algunas hipótesis sobre la descomposición en L/K de los lugares arquimedianos, obtenemos también buenas cotas inferiores para $\mathrm{Reg}(L/K)$ para algunos grados $[L : K] < 12$. Por ejemplo, si K es totalmente complejo obtenemos buenas cotas inferiores para el regulador relativo si $[L : K] \geq 5$.

Abstract

The relative regulator $\mathrm{Reg}(L/K)$ of an extension of number fields L/K is closely related to the ratio $\mathrm{Reg}(L)/\mathrm{Reg}(K)$ of classical regulators of L and K. In 1999 Friedman and Skoruppa [FS99] showed that $\mathrm{Reg}(L/K)$ satisfies lower bounds that grow exponentially with the absolute degree $[L : \mathbb{Q}]$, provided the relative degree $[L : K]$ is large enough. They started from an analytical inequality involving $\mathrm{Reg}(L/K)$ and carried out an asymptotic analysis which was successful for relative degrees $[L : K] \geq 40$. For smaller $[L : K]$ we start from the same inequality, but use extensive numerical calculations and asymptotics in a different variable to prove exponentially growing lower bounds when $[L : K] \geq 12$. By making assumptions on the splitting in L/K of the Archimedean places, we also obtain good lower bounds on $\mathrm{Reg}(L/K)$ for some degrees $[L : K] < 12$. For example, if K is totally complex we obtain exponentially growing lower bounds for the relative regulator whenever $[L : K] \geq 5$.

Chapter 1

The relative regulator

1.1 Overview of the problem and results

The regulator $\mathrm{Reg}(L)$ of a number field L is one of its classical invariants, being essentially the co-volume of the lattice $\mathrm{LOG}(\mathcal{O}_L^*)$ under the usual logarithm map LOG taking the units $\mathcal{O}_L^*$ of L to the Euclidean space $\mathbb{R}^{\mathcal{A}_L}$. Here $\mathcal{A}_L$ denotes the set of Archimedean places of L, and for $a \in L^*,\ v \in \mathcal{A}_L,\ \big(\mathrm{LOG}(a)\big)_v = e_v \log|a|_v$, where $e_v = 1$ if v is real, $e_v = 2$ if v is complex, $|\ |_v$ being the absolute value corresponding to v extending the usual absolute value on $\mathbb{Q}$.

The regulator appears naturally in several contexts, both algebraic and analytical. In the analytical setting, the regulator occurs in the residue of the Dedekind zeta function, which leads to the inequality [Lou98a], [Lou98b], [Lou98c]

$$h_L \mathrm{Reg}(L) \leq \sqrt{|D_L|}\left(\frac{e\log(|D_L|)}{4n-4}\right)^{n-1},$$

where h_L, D_L and n are the class number, discriminant and degree of L, respectively. Thus, lower bounds for $\mathrm{Reg}(L)$ become important when we need upper bounds for the often elusive class number h_L.

The first lower bounds for the regulator were published in 1932 by Remak [Rem32]. Using the geometry of numbers he proved, for L totally real, that there exist absolute constants $b_0 > 0$ and $b_1 > 1$ such that

$$\mathrm{Reg}(L) \geq b_0 b_1^n. \tag{1.1}$$

Remak [Rem52], and later Pohst [Poh78], Silverman [Sil84], Friedman [Fri89] and others, applied geometric methods to obtain various lower bounds for the regulator in terms of the discriminant, the subfields of L or heights of elements of L.

Using a new analytic method, Zimmert [Zim81] in 1981 was able to remove the assumption that L is totally real in Remak's inequality (1.1). Zimmert gave explicit absolute constants $c_0 > 0$ and $c_1 > 1$ such that for any number field L we have

$$\mathrm{Reg}(L) \geq c_0 c_1^n. \tag{1.2}$$

In 1987 Bergé and Martinet [BM87] introduced a relative form $\mathrm{R}(L/K)$ of the classical regulator, associating it to an extension L/K of number fields. They defined $\mathrm{R}(L/K)$ essentially as the covolume of the lattice resulting from projecting the lattice $\mathrm{LOG}(\mathcal{O}_L^*)$ onto the subspace perpendicular to $\mathrm{LOG}(K^*)$. Costa and Friedman [CF91] modified Bergé and Martinet's definition by slightly changing the lattice, and proved that their relative regulator $\mathrm{Reg}(L/K)$ satisfied

$$\mathrm{Reg}(L/K) = \frac{1}{\left[\mathcal{O}_K^* : W_K N_{L/K}(\mathcal{O}_L^*)\right]} \frac{\mathrm{Reg}(L)}{\mathrm{Reg}(K)} \leq \frac{\mathrm{Reg}(L)}{\mathrm{Reg}(K)} \leq \mathrm{R}(L/K), \tag{1.3}$$

where W_K denotes the group of roots of unity in K. For simplicity, we will take the equality in (1.3) as our definition of the relative regulator $\mathrm{Reg}(L/K)$.[1] Relative regulators are closely related to the covolume of the lattice $\mathrm{LOG}\big(E(L/K)\big)$, where the relative units $E(L/K)$ are defined as

$$E(L/K) = \{\varepsilon \in \mathcal{O}_L^* : \ N_{L/K}(\varepsilon) \in W_K\}.$$

In fact,

$$\mathrm{covol}\big(\mathrm{LOG}(E(L/K))\big) = \mathrm{Reg}(L/K) \prod_{\omega \in \mathcal{A}_K} \sqrt{(\text{number of places of } L \text{ above } \omega)} \geq \mathrm{Reg}(L/K).$$

Of course, $\mathrm{Reg}(L/K)$ is a generalization of the usual regulator, in the sense that $\mathrm{Reg}(L/\mathbb{Q}) = \mathrm{Reg}(L)$. Note also that any lower bound for $\mathrm{Reg}(L/K)$ is also a lower bound for $\mathrm{Reg}(L)/\mathrm{Reg}(K)$. Thus a lower bound for relative regulators implies a lower bound for the classical regulator of L that incorporates the regulator of the subfield K.

Bergé and Martinet applied the geometry of numbers to generalize to $\mathrm{Reg}(L/K)$ some of the lower bounds known for $\mathrm{Reg}(L)$, and asked if results as strong as Zimmert's might hold for the relative regulator. Having introduced relative heights to prove some of their bounds, they asked if an analogue of Lehmer's conjecture [Leh33] on heights might hold.[2]

In 1999 Friedman and Skoruppa [FS99, p. 115] proved a lower bound for $\mathrm{Reg}(L/K)$ which is exponential in $[L:\mathbb{Q}]$ for $[L:K]$ large enough. Namely, for some absolute constants $d_0 > 0, d_1 > 1$, they showed for any extension L/K of number fields that

$$\mathrm{Reg}(L/K) \geq \left(d_0 d_1^{[L:K]}\right)^{[K:\mathbb{Q}]}. \tag{1.4}$$

For $[L:K] \geq 40$ they proved [FS99, Cor. 1][3]

$$\mathrm{Reg}(L/K) \geq 0.014 \cdot 1.15^{[L:\mathbb{Q}]}. \tag{1.5}$$

There is no *a priori* reason for the restriction $[L:K] \geq 40$ in (1.4). It is a consequence of Friedman and Skoruppa's method of making estimates which are sharp only as $[L:K] \to \infty$. In

[1] For relative class groups, relative units and relative regulators, see §7 of Henri Cohen's book *Advanced Topics in Computational Number Theory* [Coh00, pp. 347-387].

[2] For a discussion of Lehmer's still unproved conjecture, see §3.6 of Waldschmidt's book *Diophantine Approximation on Linear Algebraic Groups* [Wal00, pp. 86–105].

[3] We note that there is a slip in Friedman and Skoruppa's paper [FS99]. More precisely, in bounding J_1 in the proof of their Lemma 5.6, the real part of the error term ϱ in the exponential was neglected. This did not affect the proof of their Main Theorem (1.4), but it did affect the value of the numerical constant d_0. By sharpening the asymptotic estimates of [FS99] and using computer calculations for intermediate degrees (between 40 and 139), Sundstrom in his Ph. D. Thesis [Sun16] was able to verify (1.5) for $[L:K] \geq 40$. We will use different asymptotics and computations for $[L:K] \leq 40$.

fact, they conjectured [FS99, p. 118] that there are absolute constants $f_0 > 0$ and $f_1 > 1$ such that for any degree $[L : K]$ we have

$$\mathrm{Reg}(L/K) \geq f_0 f_1^{r_{L/K}}, \tag{1.6}$$

where $r_{L/K} = \mathrm{rank}\big(E(L/K)\big) = \#(\mathcal{A}_L) - \#(\mathcal{A}_K)$.

For simplicity we will use the absolute degree $[L : \mathbb{Q}]$ instead of the relative unit rank $r_{L/K}$. If $[L : K] \geq 3$, they are interchangeable in (1.6) since $[L : \mathbb{Q}]/6 \leq r_{L/K} \leq [L : \mathbb{Q}]$. Indeed,

$$[L : \mathbb{Q}] \geq r_{L/K} \geq \frac{[L : \mathbb{Q}]}{2} - [K : \mathbb{Q}] = \frac{[L : \mathbb{Q}]}{2} - \frac{[L : \mathbb{Q}]}{[L : K]} \geq \frac{[L : \mathbb{Q}]}{6} \qquad ([L : K] \geq 3).$$

The main aim of this thesis is to prove the following extension of Friedman and Skoruppa's inequality (1.5).

Main Theorem. *There are absolute constants $e_0 > 0$ and $e_1 > 1$ such that for any extension of number fields L/K of degree $[L : K] \geq 12$ we have*

$$\mathrm{Reg}(L/K) \geq e_0 e_1^{[L:\mathbb{Q}]} \qquad (e_0 > 0,\ e_1 > 1). \tag{1.7}$$

Moreover, (1.7) holds if $[L : K] = 10$, or if K is totally complex and $[L : K] \geq 5$. It also holds for $[L : K] = 8$ provided all real places of K have at least one real place of L above it.

We also get (1.7) for L totally real if $[L : K] \geq 3$, but this case was known by geometric methods even when $[L : K] = 2$. Our exponential growth constant e_1 for the totally real case is better than the geometric one when $[L : K] \geq 6$ (see Table 1.3 below).

Our work will concern only the case $[L : K] \leq 40$, as higher degrees are covered by inequality (1.5).[4] In fact, for $[L : K] \geq 12$, $[L : K] = 10$ and $[L : K] \geq 5$ if K is totally complex, the constants $e_0 = 0.02$ and $e_1 = 1.013$ in (1.7) are explicit (see Table 1.2), but we phrased our main result so as to emphasize that our aim was to obtain a lower bound on $\mathrm{Reg}(L/K)$ that increases exponentially with $[L : \mathbb{Q}]$.

Table 1.1: Values of C making $\mathrm{Reg}(L/K) \geq 0.02 \cdot C^{[L:\mathbb{Q}]}$ for $2 \leq d = [L : K] \leq 40$.

d	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
C	0.603	0.489	0.790	0.715	0.898	0.838	0.969	0.916	1.018	0.972
d	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**
C	1.054	1.013	1.082	1.045	1.105	1.071	1.123	1.092	1.138	1.110
d	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**	**31**
C	1.151	1.125	1.163	1.138	1.172	1.149	1.181	1.159	1.188	1.167
d	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**	
C	1.195	1.175	1.201	1.182	1.207	1.189	1.212	1.194	1.216	

[4] We re-do the known case $[L : K] = 40$ to show that inequalities can be improved by treating small degrees on their own (our C is 1.216 instead of 1.15).

On inspection of the table, we find an exponentially growing lower bound for $d \geq 12$, with C ranging from 1.013 to 1.216 in the range $12 \leq d \leq 40$, such that $\mathrm{Reg}(L/K) \geq 0.02 \cdot C^{[L:\mathbb{Q}]}$.

We now put our result in the greater context of Lehmer's conjecture on heights of algebraic numbers. Friedman and Skoruppa remarked [FS99, p. 118] that their conjectural inequality (1.6) implies the existence of a smallest Salem number, an important unproved special case of Lehmer's conjecture. They were unaware that in 1997 D. Bertrand [Ber97] had already studied units while framing a higher-dimensional Lehmer conjecture. Bertrand investigated covolumes of lattices $\mathrm{LOG}(E)$ for general subgroups $E \subset \mathcal{O}_L^*$, and conjectured an inequality of the type $\mathrm{covol}\big(\mathrm{LOG}(E)\big) \geq C_{r_E}$ for some positive C_{r_E} depending only on the rank $r_E \geq 2$ of E. Bertrand's conjecture was proved for $r_E \geq 3$ by Amoroso and David [AD99].

Friedman and Skoruppa's conjectural inequality (1.6) features a lower bound growing exponentially with the rank, and so is stronger than Bertrand conjectured. Recently, Rodriguez Villegas [Chi19, Appendix] and Amoroso and David [AD19] conjectured an exponentially increasing lower bound

$$\mathrm{covol}\big(\mathrm{LOG}(E)\big) \geq g_0 g_1^{r_E}, \tag{1.8}$$

for any subgroup E of units of rank $r_E \geq 2$. Here $g_0 > 0$ and $g_1 > 1$ should be absolute constants (BRVAD conjecture).[5]

Costa and Friedman [CF91, Thm. 4] showed that for subgroups $E \subset \mathcal{O}_L^*$ of any totally real field L, inequality (1.8) (with $g_1 = 1.4$) followed easily from work of Schinzel [Sch73] and Pohst [Poh78]. Thus, the BRVAD conjecture was long known in the totally real case. Our inequality (1.7) provides additional evidence in favor of the BRVAD conjecture.

1.2 Approach to the proof

The first steps of our proof of lower bounds for $\mathrm{Reg}(L/K)$ for $[L:K] < 40$ are the same as Friedman and Skoruppa's proof for $[L:K] \geq 40$. The difference will emerge in the way we analyze the functions appearing in the fundamental inequality below. They proved this inequality using a series $\Theta_{L/K}(t;\mathfrak{a})$ [FS99, p. 117] that they attached to a fractional ideal $\mathfrak{a}$ of L and $t > 0$. We note that $\Theta_{L/\mathbb{Q}}(t;\mathfrak{a})$ is the inverse Mellin transform of the partial zeta function attached to the ideal class of $\mathfrak{a}$ used by Zimmert to prove his regulator bounds [Zim81]. The importance of $\Theta_{L/K}(t;\mathfrak{a})$ for us is that its constant term is $\mathrm{Reg}(L/K)/w_L$, where $w_L = \#(W_L)$ is the number of roots of unity in L.

Friedman and Skoruppa observed that $\Theta_{L/K}(t;\mathfrak{a})$ is an obviously decreasing function of t and that the usual Θ-identity held, *i. e.* $\Theta_{L/K}(t^{-1},\mathfrak{a}) = t^{[L:\mathbb{Q}]/2}\Theta_{L/K}(t,(\mathfrak{a}\mathfrak{d})^{-1})$, with $\mathfrak{d}$ being the (absolute) different ideal of L [FS99, p. 120]. Thus $t \mapsto t^{\frac{[L:\mathbb{Q}]}{2}}\Theta_{L/K}(t,\mathfrak{a})$ is increasing. Taking $\frac{\mathrm{d}}{\mathrm{d}t}$, they obtained the inequality

$$\Theta_{L/K}(t,\mathfrak{a}) + \frac{2}{[L:\mathbb{Q}]} t\Theta'_{L/K}(t,\mathfrak{a}) \geq 0 \qquad\qquad (t > 0). \tag{1.9}$$

The definition of $\Theta_{L/K}$ is rather unwieldy as it involves an $r_{L/K}$-dimensional integral. Using Mellin

[5] For $r_E = 1$ one needs to use an L^1-norm of LOG of a unit, as in Lehmer's conjecture. Indeed, Rodriguez Villegas used an L^1-norm for any rank and so included $r_E = 1$, *i. e.* Lehmer's original conjecture. Amoroso and David conjectured that Rodriguez Villegas' inequality might be strengthened using the L^2-norm for $r_E \geq 2$.

transforms, Friedman and Skoruppa [FS99, Prop. 3.1] proved the more tractable expression

$$\Theta_{L/K}(t,\mathfrak{a}) = \frac{\mathrm{Reg}(L/K)}{w_L} + 2^{-r_{L/K}}\pi^{-r_2(L)/2} \sum_{\substack{a\in\mathfrak{a}/E(L/K)\\ a\neq 0}} \prod_{w\in\mathcal{A}_K} f_w(\log(tc_{\mathfrak{a}})+a_w), \tag{1.10}$$

$$f_w(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-se_w dy}\Gamma(s)^{p_w+q_w}\Gamma\big(s+\tfrac{1}{2}\big)^{q_w}\,ds \qquad (d=[L:K],\ y\in\mathbb{R},\ \text{any } c>0), \tag{1.11}$$

where p_w and q_w denote, respectively, the number of real and complex places of L extending the place $w\in\mathcal{A}_K$, $r_2(L)$ is the number of complex places of L,

$$c_{\mathfrak{a}} = \pi\Big(\sqrt{|D_L|}\,\mathrm{n}_{L/\mathbb{Q}}(\mathfrak{a})\Big)^{-2/[L:\mathbb{Q}]}, \qquad a_w = \frac{2}{d}\log|\mathrm{Norm}_{L/K}(a)|_w. \tag{1.12}$$

Using (1.9)-(1.11), and setting $y=\log(tc_{\mathfrak{a}})$, one has (for any $y\in\mathbb{R}$) the fundamental inequality

$$\frac{\mathrm{Reg}(L/K)}{w_L} \geq 2^{-r_{L/K}}\pi^{-r_2(L)/2} \sum_{\substack{a\in\mathfrak{a}/E_{L/K}\\ a\neq 0}} \Big\{-1-\frac{2}{[L:\mathbb{Q}]}\sum_{w\in\mathcal{A}_K}\frac{f_w'}{f_w}(a_w+y)\Big\}\prod_{w\in\mathcal{A}_K} f_w(a_w+y). \tag{1.13}$$

The function f_w and its logarithmic derivative will be the central analytic objects of study in this thesis. It is not hard to show that f_w is positive and log-concave, but we will need a higher convexity result, as well as inequalities comparing f_w's with different parameters.

If we take $\mathfrak{a}=\mathcal{O}_L$ in the sum over $a\in\mathfrak{a}/E(L/K)$ in (1.13), we see that the sum includes a term corresponding to $a=1$. Friedman and Skoruppa's strategy to prove (1.4) was to drop all terms with $a\neq 1$ in (1.13) and get a good lower bound for the term with $a=1$ (note that $1_w=0$). To follow this strategy, one first has to prove that the terms dropped are nonnegative. Unfortunately, we have very little control over a_w. On the favorable side, y is a free real parameter which can be set optimally.

It turns out that a convexity property of $-\frac{f_w'}{f_w}$ would ensure the desired nonnegativity of the term with brackets in (1.13). In [FS99] the authors found a convex and increasing function ρ, such that for $d=[L:K]\gg 0$

$$-\frac{f_w'}{f_w}(y) \geq e_w d\rho(y) \qquad (\forall w\in\mathcal{A}_K). \tag{1.14}$$

Note that $\sum_{w\in\mathcal{A}_K} e_w a_w = \frac{2}{d}\log|\mathrm{Norm}_{L/\mathbb{Q}}(a)| \geq 0$ since a is a nonzero algebraic integer. As ρ is convex and increasing, Friedman and Skoruppa [FS99, p.128] estimated from (1.14)

$$-\frac{2}{[L:\mathbb{Q}]}\sum_{w\in\mathcal{A}_K}\frac{f_w'}{f_w}(a_w+y) \geq \frac{2}{[K:\mathbb{Q}]}\sum_{w\in\mathcal{A}_K} e_w\rho(a_w+y) \geq 2\rho\big(y+\tfrac{2}{[L:\mathbb{Q}]}\log(N_{L/\mathbb{Q}}(a))\big) \geq 2\rho(y).$$

Thus the terms in the sum (1.13) will be positive if $2\rho(y)>1$. As with Friedman and Skoruppa, this gives our initial inequality

$$\frac{\mathrm{Reg}(L/K)}{w_L} \geq (2\rho(y)-1)2^{-r_{L/K}}\pi^{-r_2(L)/2}\prod_{w\in\mathcal{A}_K} f_w(y), \tag{1.15}$$

valid provided (1.14) holds for a convex increasing ρ. Thus, our main work will be to find a good ρ in (1.14), and a good lower bound for the product $\prod_{w\in\mathcal{A}_K} f_w(y)$ in (1.15). We will choose y so that $2\rho(y)-1=1/100$ (or any fixed small positive number).

Table 1.2: $C(y_0)$ and y_0 in (1.17) for K totally complex and $2\le d=[L:K]\le 40$.

d	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
$\mathbf{y_0}$	-0.933	-1.031	-1.083	-1.116	-1.138	-1.154	-1.166	-1.175	-1.183	-1.189
$\mathbf{C(y_0)}$	0.790	0.898	0.969	1.018	1.054	1.082	1.105	1.123	1.138	1.151
d	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**
$\mathbf{y_0}$	-1.195	-1.199	-1.203	-1.206	-1.209	-1.212	-1.214	-1.216	-1.218	-1.220
$\mathbf{C(y_0)}$	1.163	1.172	1.181	1.188	1.195	1.201	1.207	1.212	1.216	1.220
d	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**	**31**
$\mathbf{y_0}$	-1.221	-1.222	-1.224	-1.225	-1.226	-1.227	-1.228	-1.229	-1.230	-1.230
$\mathbf{C(y_0)}$	1.224	1.228	1.231	1.234	1.237	1.240	1.242	1.244	1.247	1.249
d	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**	
$\mathbf{y_0}$	-1.231	-1.232	-1.232	-1.233	-1.234	-1.234	-1.235	-1.235	-1.236	
$\mathbf{C(y_0)}$	1.251	1.252	1.254	1.256	1.257	1.259	1.260	1.262	1.263	

The simplest case for our purposes occurs when f_w is the same function for all $w\in\mathcal{A}_K$, as happens if K is totally complex or if L is totally real. *So we assume for now that K is totally complex.* Then $f_w=f_{(0,d)}$, a function depending only on $d=[L:K]\le 40$. Here $f_{(0,d)}(y)=f_w(y)$ in (1.11), with $e_w=2,\ p_w=0$ and $q_w=d$, as we must have if w is a complex place. As remarked above, $f_{(0,d)}(y)>0$, and $-\frac{f'_{(0,d)}}{f_{(0,d)}}(y)$ is known to be increasing in y. Graphing this logarithmic derivative suggests that $-\frac{f'_{(0,d)}}{f_{(0,d)}}$ is convex, as we will eventually prove using asymptotic and numerical calculations.[6] This means that we can choose the optimal lower bound in (1.14), namely

$$\rho(y)=\rho_d(y):=\frac{-1}{2d}\frac{f'_{(0,d)}}{f_{(0,d)}}(y). \tag{1.16}$$

We shall prove that $\lim_{y\to-\infty}\rho(y)=0$ and $\lim_{y\to\infty}\rho(y)=\infty$, so for each d there is a unique $y_0\in\mathbb{R}$ such that $2\rho(y_0)-1=1/100$.[7] Thus, from (1.15) we get

$$\frac{\operatorname{Reg}(L/K)}{w_L}\ge 0.01\cdot\frac{f_{(0,d)}(y_0)^{[K:\mathbb{Q}]/2}}{2^{\frac{[L:\mathbb{Q}]}{2}-\frac{[K:\mathbb{Q}]}{2}}\pi^{\frac{[L:\mathbb{Q}]}{4}}}=0.01\cdot\big(2^{-\frac{d-1}{2d}}\pi^{-\frac14}f_{(0,d)}(y_0)^{\frac{1}{2d}}\big)^{[L:\mathbb{Q}]}=:0.01\cdot C(y_0)^{[L:\mathbb{Q}]}. \tag{1.17}$$

Numerical calculations show that $C(y_0)<1$ for $d\le 4$, so we get exponentially decreasing bounds for $\operatorname{Reg}(L/K)$ if $d=[L:K]\le 4$. Fortunately, we get lower bounds that increase exponentially

[6] Numerical "evidence" suggests much more is true for all d: Not only the second derivative of $-\frac{f'_{(0,d)}}{f_{(0,d)}}$ is positive, its derivatives to all orders are positive. This is obvious for $d=1$, but already for $d=2$ the proof required delicate properties of the modified Bessel functions K_ν [KM18, Prop. 6].

[7] Since in practice ρ can only be evaluated to a given accuracy, in Table 1.2 we actually have $2\rho(y_0)-1>1/100$, but it is nearly an equality.

with $[L:\mathbb{Q}]$ if $5 \le [L:K] \le 40$, since $C(y_0) \ge 1.018$ in that range (see Table 1.2). For example, if $[L:K]=10$ and K is totally complex, using $w_L \ge 2$ and Table 1.2 we obtain

$$\mathrm{Reg}(L/K) \ge 0.02 \cdot 1.138^{[L:\mathbb{Q}]} \qquad (K \text{ totally complex, } [L:K]=10).$$

For $[L:K]=40$ we obtain

$$\mathrm{Reg}(L/K) \ge 0.02 \cdot 1.263^{[L:\mathbb{Q}]} \qquad (K \text{ totally complex, } [L:K]=40).$$

It is interesting to note that, for $[L:K] \gg 0$ and L totally complex (regardless of K), Friedman and Skoruppa [FS99, Corollary 2] proved $\mathrm{Reg}(L/K) \ge 1.33^{[L:\mathbb{Q}]}$. Our calculations show that for $[L:K]=40$ the lower bounds are not far from their asymptotics.

When L is totally real, the same line of reasoning (contingent on proving the convexity of $-\frac{f_w'}{f_w}(y)$ and numerical calculation of y_0 and of $f_w(y_0)$) gives the lower bound

$$\mathrm{Reg}(L/K) \ge 0.02 \cdot C_1(y_0)^{[L:\mathbb{Q}]}, \tag{1.18}$$

where $C_1(y_0)$ is given in Table 1.3.

Table 1.3: $C_1(y_0)$ and y_0 in (1.18) for L totally real and $2 \le d = [L:K] \le 40$.

d	**y_0**	**$C_1(y_0)$**
2	-1.196	0.878
3	-1.418	1.103
4	-1.539	1.270
5	-1.615	1.396
6	-1.667	1.493
7	-1.704	1.572
8	-1.733	1.636
9	-1.755	1.690
10	-1.773	1.735
11	-1.788	1.775
12	-1.800	1.809
13	-1.811	1.838
14	-1.820	1.865
15	-1.828	1.888
16	-1.834	1.910
17	-1.840	1.929
18	-1.846	1.946
19	-1.851	1.962
20	-1.855	1.977
21	-1.859	1.990
22	-1.863	2.002
23	-1.866	2.014
24	-1.869	2.024
25	-1.872	2.034
26	-1.874	2.043
27	-1.877	2.052
28	-1.879	2.060
29	-1.881	2.068
30	-1.883	2.075
31	-1.884	2.082
32	-1.886	2.088
33	-1.888	2.095
34	-1.889	2.100
35	-1.891	2.106
36	-1.892	2.111
37	-1.893	2.116
38	-1.894	2.121
39	-1.896	2.125
40	-1.897	2.130

As remarked above, in the totally real case an exponentially growing lower bound for $\mathrm{Reg}(L/K)$ has long been known whenever $L \ne K$. In our case, we get this only for $[L:K] \ge 3$, but Table 1.3 is still interesting because the geometrically obtained lower bound grows at best like $1.406^{[L:\mathbb{Q}]}$ [CF91, p. 290]. Table 1.3 shows that the analytic method beats this as soon as $[L:K] \ge 6$, reaching $2.13^{[L:\mathbb{Q}]}$ when $[L:K]=40$. Friedman and Skoruppa [FS99, Cor. 2] showed that the lower bound reaches $2.36^{[L:\mathbb{Q}]}$ for $[L:K] \gg 0$ and L totally real.

When the Archimedean places of K differ in their splitting behavior in L/K, they give rise to various functions f_w in the fundamental inequality, so we let for $y \in \mathbb{R}$,

$$f(y) = f_{(p,q)}(y) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-(p+2q)sy}\Gamma(s)^{p+q}\Gamma\left(s+\tfrac{1}{2}\right)^q \, ds \qquad (c,p,q \ge 0,\ p+q \ge 1). \tag{1.19}$$

Thus, in (1.11) we have $f_w(y) = f_{p_w,q_w}(y)$, and there is no longer an obvious ρ to choose in (1.14).

In Figure 1.1 we graph, for degrees $[L:K]$ going from 5 to 8, the logarithmic derivatives $-\frac{f'_w}{f_w}$ corresponding to all possible splittings in L/K of a real place w. We see that they are all convex and nested (*i. e.* they do not cross).

Figure 1.1: The functions $-\dfrac{f'_{(r_1,r_2)}}{f_{(r_1,r_2))}}$ for all signatures in degrees 5 to 8.

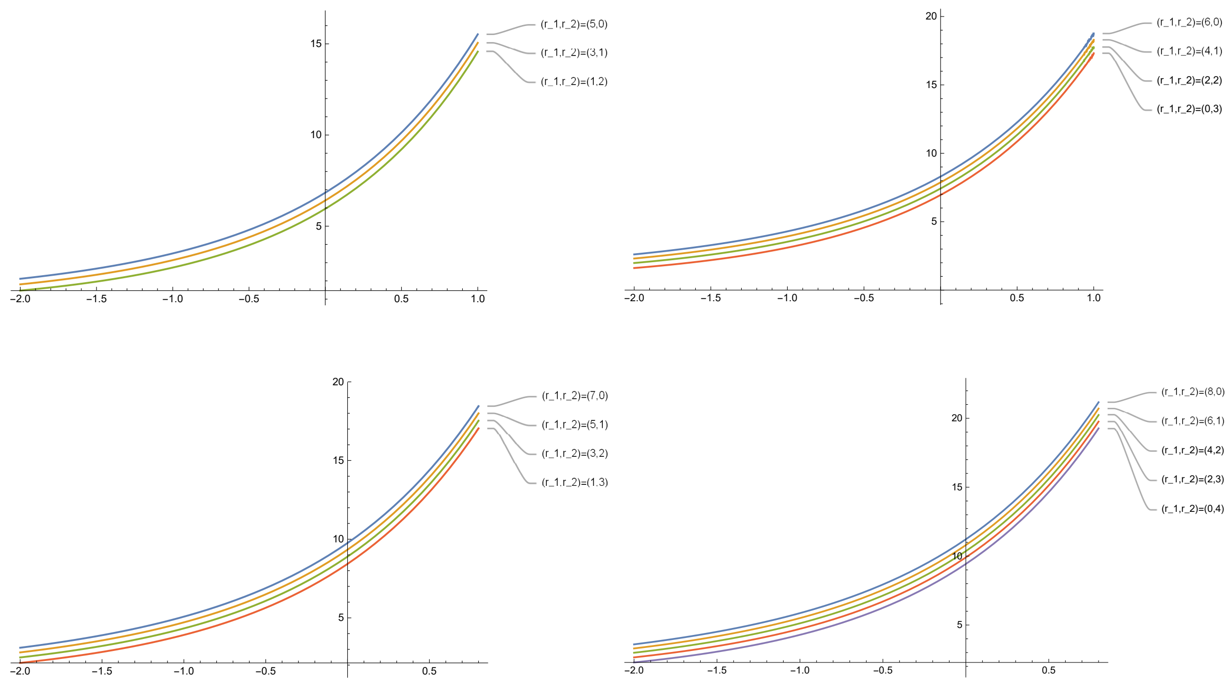


Figure 1.1 suggests we just take for ρ the one corresponding to the least number of real places above a real place w, *i. e.* none or one depending on the parity of $[L:K]$. A complex place w of K can only split completely in L/K, so we must also verify the previous choice of ρ satisfies $-\frac{f'_{(0,d)}}{f_{(0,d)}}(y) \geq 2d\rho(y)$. We are thus lead to the following strategy for proving our Main Theorem.

The Three Steps

Step 1 (Nesting). Prove for degrees $d \leq 40$ and all integers $r_1 \geq 0$, $r_2 \geq 1$, with $r_1 + 2r_2 = d$,

$$-\frac{f'_{(r_1,r_2)}}{f_{(r_1,r_2)}}(y) \leq -\frac{f'_{(r_1+2,r_2-1)}}{f_{(r_1+2,r_2-1)}}(y).$$

If d is odd, also prove $-\dfrac{f'_{(0,(d-1)/2)}}{f_{(0,(d-1)/2)}}(y) \leq -\dfrac{f'_{(1,(d-1)/2)}}{f_{(1,(d-1)/2)}}(y)$. These inequalities must hold for all $y \in \mathbb{R}$.

Step 2 (Complex place). For degrees $d \le 40$ prove[8]

$$-\frac{f'_{(0,d)}}{f_{(0,d)}}(y) \ge \begin{cases} -2\frac{f'_{(0,d/2)}}{f_{(0,d/2)}}(y) & \text{if } d \text{ is even,} \\ -2\frac{f'_{(0,(d-1)/2)}}{f_{(0,(d-1)/2)}}(y) & \text{if } d \text{ is odd} \end{cases} \qquad (\forall y \in \mathbb{R}). \tag{1.20}$$

Step 3 (Convexity). Prove, for $4 \le d \le 40$, that $-\frac{f'_{(0,d)}}{f_{(0,d)}}(y)$ is convex in y.[9]

Once the Three Steps have been accomplished, we will have for each $d \le 40$ a convex ρ in (1.14) for which we can find $y_0 \in \mathbb{R}$ such that $2\rho(y_0) - 1 = 1/100$. Then, in view of (1.15), for each d we will try to find a $B_d > 1$ such that for all K and all L/K with $[L : K] = d$ we have the lower bound

$$\Big(2^{-r_{L/K}}\pi^{-r_2(L)/2} \prod_{w \in \mathcal{A}_K} f_w(y_0)\Big)^{1/[K:\mathbb{Q}]} \ge B_d.$$

As this will be for only one value of y_0 and finitely many possible splitting patterns of w, this Step 4 will turn out to be easier that any of the previous three. As announced above, we will succeed in finding such a B_d for $[L : K] \ge 12$. This is carried out in detail in Chapter 5.

Taking the Three Steps is easy when $d = 1$ (since in that case $f_w(y) = e^{-e^y}$), and is possible using some properties of Bessel functions for $d = 2$ (see [KM18, Prop. 6]). Unfortunately, we have not been able to find a proof that works for a general d. Instead, the first part of the process will be to verify that the Three Steps hold for $|y| > D_{r_1,r_2}$, for some computable D_{r_1,r_2} and a fixed $d \le 40$. The second part of the process will be to implement a rigorous numerical verification of the Three Steps for $|y| \le D_{r_1,r_2}$. The first part works for a general r_1, r_2, and involves calculating explicit asymptotics as $y \to \pm\infty$, but the second part requires extensive computations for each signature.

The asymptotics of $f_{(p,q)}$, and of all its derivatives, were worked out by Braaksma in the general context of Meijer G-functions [Bra64], but without explicit inequalities. We carry out the explicit asymptotics in Chapter 2 with a view to making D_{r_1,r_2} as small as we can. Various computational problems arise in Chapters 3 and 4 where we verify Steps 1 to 3 for $|y| \le D_{r_1,r_2}$, the difficulties being considerably more pronounced for Step 3 (convexity). To delay some particularly boring proofs, we relegate them to the (long) Appendix, which also contains a short review of log-concave functions and our PARI/GP programs.

[8] A factor of 2 appears with $-f'/f$ in (1.20) because $e_w = 2$ appears in (1.14) when w is complex.

[9] We exclude $d \le 3$ because we will not make any claims for such small degrees and because it allows us to avoid some special cases in the asymptotic analysis.

Chapter 2

Asymptotics: Convexity of $-\frac{f'}{f}$ near $\pm\infty$

In this chapter we find the first N terms of the asymptotic approximations to

$$f(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-dsy}\Gamma(s)^{r_1+r_2}\Gamma\left(s+\tfrac{1}{2}\right)^{r_2}\, ds \qquad (y \in \mathbb{R})$$

and its derivatives. These asymptotics will be the main tool to carry out the Three Steps in §1.2 for large values of $|y|$. When $y \to -\infty$ it is easy to estimate the above integral: One simply shifts the contour to the left, acquiring the corresponding residues, and brutally estimates the integral over the new contour (see §2.2).

The asymptotics of f and its derivatives as $y \to +\infty$ are far more difficult and were worked out by Braaksma [Bra64]. We will follow him, making all his inequalities explicit (see §2.1 and the Appendix). Braaksma's strategy is easily lost amongst all the long formulas, so we now give a quick sketch of his main idea.

An integral over a vertical line L of the form $\frac{1}{2\pi i}\int_L e^{-dsy}\Gamma(As+B)\, ds$, where A, B are constants, is easily evaluated. However, we have to deal with (*i. e.* obtain the asymptotic expansion of order N for) the considerably harder integral $\frac{1}{2\pi i}\int_L e^{-dsy}\prod_{j=1}^{d}\Gamma(A_js+B_j)\, ds$. Braaksma approximates the complicated Γ-product by $C^s\Gamma(As+B)$ by writing

$$\begin{aligned}\prod_{j=1}^{d}\Gamma(A_js+B_j) &= C^s\Gamma(As+B)\left(\frac{\prod_{j=1}^{d}\Gamma(A_js+B_j)}{C^s\Gamma(As+B)}\right)\\ &= C^s\Gamma(As+B)\left(\frac{\prod_{j=1}^{d}\Gamma(A_js+B_j)}{C^s\Gamma(As+B)} - P_N(s)\right) + C^s\Gamma(As+B)P_N(s), \qquad (2.1)\end{aligned}$$

where $P_N(s)$ is a polynomial of degree N making $C^s\Gamma(As+B)P_N(s)$ an asymptotic approximation to $\prod_{j=1}^{d}\Gamma(A_js+B_j)$ on a convenient vertical line. Thus the term $C^s\Gamma(As+B)P_N(s)$ in (2.1), upon integration on a vertical line, will lead to the N-term asymptotic expansion. The previous term will lead to the remainder and our extensive efforts to estimate it explicitly.

2.1 Asymptotic expansion of $f^{(t)}(y)$ when $y \to +\infty$

The $t-$th derivative of f with respect to y is given by

$$\begin{aligned} f^{(t)}(y) := f^{(t)}_{(r_1,r_2)}(y) &= \left(\frac{1}{2\pi i}\int_{M-i\infty}^{M+i\infty} e^{-sdy}\Gamma(s)^{r_1+r_2}\Gamma(s+\tfrac{1}{2})^{r_2}\,ds\right)^{(t)} \\ &= \frac{(-d)^t}{2\pi i}\int_{M-i\infty}^{M+i\infty} s^t e^{-sdy}\Gamma(s)^{r_1+r_2}\Gamma(s+\tfrac{1}{2})^{r_2}\,ds \qquad (y\in\mathbb{R},\ \text{any } M>0), \end{aligned} \tag{2.2}$$

where for our purposes $t = 0, 1, 2$ or 3, and $d := r_1 + 2r_2$. To avoid special cases that we will not need in the end, we will assume that the non-negative integers r_1 and r_2 satisfy $r_1 + r_2 \geq 3$, as we will use $r_1 + r_2 \geq t$. Applying the change of variable $s \mapsto -s$ in the integral (2.2), and using $\Gamma(-s+1) = (-s)\Gamma(-s)$,

$$f^{(t)}(y) = \frac{(-d)^t}{2\pi i}\int_{-M-i\infty}^{-M+\infty} e^{sdy}(-s)^t\Gamma(-s)^{r_1+r_2}\Gamma(-s+\tfrac{1}{2})^{r_2}\,ds = \frac{(-d)^t}{2\pi i}\int_{-M-i\infty}^{-M+i\infty} z^s h(s)\,ds,$$

where

$$z := e^{dy}, \qquad h(s) := \Gamma(-s)^{r_1+r_2-t}\Gamma(-s+1)^t\Gamma(-s+\tfrac{1}{2})^{r_2}. \tag{2.3}$$

The variable z will prove convenient to avoid writing double exponentials like $\exp(-e^y)$. Note that $z \to +\infty$ as $y \to +\infty$.

For $z > 0$ define

$$H(z) := \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} h(s)z^s\,ds \qquad (\omega := -M < 0). \tag{2.4}$$

Thus,

$$f^{(t)}(y) = (-d)^t H(e^{dy}). \tag{2.5}$$

Following Braaksma [Bra64], using $(x+j)\Gamma(x+j) = \Gamma(x+j+1)$ for $x = -N - ds - \frac{r_1+1}{2} + t - \frac{r_2}{2}$ and $j = 0, 1, \ldots, N$, we obtain

$$\begin{aligned} H(d^{-d}z) = {}& (2\pi)^{d-1}\sum_{k=0}^{N}(-1)^k A_k \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\Gamma(1-k-ds-\tfrac{r_1+1}{2}+t-\tfrac{r_2}{2})z^s\,ds \\ & -i(2\pi)^{d-2}\int_{\omega-i\infty}^{\omega+i\infty}\varrho(s)\Gamma(-N-ds-\tfrac{r_1+1}{2}+t-\tfrac{r_2}{2})z^s\,ds, \end{aligned} \tag{2.6}$$

in the form *main term* + *error term*, where

$$\begin{aligned} A_0 &:= (2\pi)^{\frac{1-d}{2}} d^{\frac{r_1+r_2}{2}-t}, \\ \varrho(s) &:= \frac{(2\pi)^{1-d}d^{-ds}h(s)}{\Gamma\left(-N-ds-\frac{r_1+1}{2}+t-\frac{r_2}{2}\right)} + \sum_{k=0}^{N}(-1)^{k+1}A_k\prod_{j=0}^{N-k}(-N-ds-\tfrac{r_1+1}{2}+t-\tfrac{r_2}{2}+j), \end{aligned} \tag{2.7}$$

h is as in (2.3) and $A_1, \ldots, A_N$ are explicitly computable in terms of r_1, r_2 and t (see §6.4.2 in the Appendix).

2.1.1 Error estimates.

Define the error function for $z > 0$

$$\sigma(z) = \int_{\omega-i\infty}^{\omega+i\infty} \varrho(s)\ \Gamma(-N - ds - \tfrac{r_1+1}{2} + t - \tfrac{r_2}{2}) z^s\,\mathbf{ds}. \tag{2.8}$$

Since $\Gamma(s)$ is the Mellin transform of e^z, we have [MH08, pp. 15-20]

$$\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} \Gamma(T - ds) z^s\,\mathbf{ds} = \frac{z^{\frac{T}{d}}}{d} e^{-z^{\frac{1}{d}}} \qquad (z > 0, \quad T - d\omega > 0). \tag{2.9}$$

Using this formula with $T = 1 - \frac{r_1+1}{2} + t - \frac{r_2}{2} - k$ and ω such that $T - d\omega > 0$, we find that the term in blue in (2.6) is given by

$$\begin{aligned}(2\pi)^{d-1}\sum_{k=0}^{N}(-1)^k A_k \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} \Gamma(1 - k - ds - \tfrac{r_1+1}{2} + t - \tfrac{r_2}{2}) z^s\,\mathbf{ds} \\ = \frac{(2\pi)^{d-1}}{d} e^{-z^{\frac{1}{d}}} z^{-\frac{r_1+r_2+1}{2d}+\frac{t}{d}} \sum_{k=0}^{N}(-1)^k A_k z^{\frac{1-k}{d}}.\end{aligned} \tag{2.10}$$

We want an upper bound for the error term $\sigma(z)$ with $z > 0$, so we first bound ϱ in (2.7).

Lemma 2.1.1. *The constants A_k in (2.7) can be chosen in such a way that the following hold. There exist constants $K_1, K_2 > 0$, given explicitly in terms of r_1, r_2 and t, such that in the region*

$$\Theta(K_1) := \{s \in \mathbb{C} : \mathrm{Re}(-s) \geq K_1\},$$

the function ϱ defined in (2.7), is bounded by

$$|\varrho(s)| \leq K_2. \tag{2.11}$$

Also, there is an explicit formula that calculates A_k from $A_0, A_1, \ldots, A_{k-1}$ $(k = 1, \ldots, N)$.

Proof. See §6.4.2 of the Appendix. We will only use the case $N = 2$. □

Lemma 2.1.2. *Let $\gamma := \frac{r_1+1}{2} + \frac{r_2}{2} - t + N > 0$. There exists constants K_1' and $K_2' > 0$ such that in the region $\Theta(K_1')$, we have $|\varrho(s)| \leq K_2'$ and*

$$|\Gamma(-ds - \gamma)| \leq K_2'|s|^{-\frac{1}{2}-\gamma} e^{-d\mathrm{Re}(s)\log|ds| + d\mathrm{Re}(s) + d(\arg(-s))\mathrm{Im}(s)}.$$

Here $\arg(z)$ denotes the principal argument, *i. e.* $\arg(z) = 0$ for $z > 0$.

Proof. The proof uses Stirling's formula in an explicit form (see §6.4.3 of the Appendix). □

Using the previous two lemmas, we can obtain an upper bound for the function $\sigma(z)$, so we can write an asymptotic expansion for the function H with an explicitly controlled error term. More concretely, for the function $H\left(d^{-d}e^{dy}\right)$ we have the following estimate as $y \to +\infty$.

Lemma 2.1.3. *Let* $\gamma := \frac{r_1+1}{2} + \frac{r_2}{2} - t + N > 0$, $K_1 > \max\left(\frac{1}{2}, \frac{\gamma+\frac{1}{2}}{d}\right)$ *and* $y \geq \log(dK_1) =: S$. *Then the function* H *defined in (2.4) has the following* N^{th} *order asymptotic expansion when* $y \to +\infty$

$$H(d^{-d}e^{dy}) = \frac{(2\pi)^{d-1}}{d} e^{-y(\frac{r_1+r_2-1}{2}-t)} e^{-e^y} \sum_{k=0}^{N} (-1)^k A_k e^{-ky} \; - \; i(2\pi)^{d-2} E(y),$$

where the error term $E(y)$ *is explicitly bounded as follows*

$$|E(y)| = |\sigma(e^{dy})| \leq C e^{-y(\frac{r_1+r_2-1}{2}-t+N+1)} e^{-e^y},$$

for some explicitly computable constant $C = C(K_1, \gamma, d)$.

Proof. See §6.4.4 of the Appendix. $\square$

2.1.2 Asymptotic expansion of $\left(-\frac{f'}{f}\right)''(y)$ when $y \to +\infty$.

To obtain the asymptotic expansion of the function $\left(-\frac{f'(y)}{f(y)}\right)''$ we will use the identity

$$\left(-\frac{f'(y)}{f(y)}\right)'' = -\frac{f'''(y)}{f(y)} + 3\frac{f''(y)f'(y)}{f(y)^2} - 2\frac{f'(y)^3}{f(y)^3} \tag{2.12}$$

together with the asymptotic expansion found in Lemma 2.1.3. That lemma easily yields an asymptotic expansion for $f^{(t)}(y)$ $(t = 0, 1, 2, 3)$, when $y \to +\infty$, as follows

$$\begin{aligned} f^{(t)}_{(r_1,r_2)}(y) &= (-d)^t H(e^{dy}) \\ &= (-d)^t H(d^{-d}e^{d(y+\log d)}) \\ &= (-d)^t \frac{(2\pi)^{d-1}}{d} d^{-\frac{r_1+r_2+1}{2}+t} \, e^{y\left(-\frac{r_1+r_2+1}{2}+t\right)} e^{-de^y} \left(\sum_{k=0}^{N} (-1)^k A_k d^{1-k} e^{(1-k)y}\right) \\ &\quad -(-d)^t i(2\pi)^{d-2} F(y), \end{aligned} \tag{2.13}$$

where the constants $A_k = A_k(t, r_1, r_2)$ are explicit, and $F(y) := E(y + \log(d))$ is bounded by

$$|F(y)| = |E(y + \log(d))| \leq \frac{C}{d^{\frac{r_1+r_2+1}{2}-t+N}} e^{-y\left(\frac{r_1+r_2-1}{2}-t+N+1\right)} e^{-de^y}, \qquad \text{for } y \geq S.$$

We can rewrite equation (2.13) in the more standard form

$$\begin{aligned} f^{(t)}_{(r_1,r_2)}(y) = A_0 d(-1)^t (2\pi)^{d-1} d^{-\frac{r_1+r_2+1}{2}+2t-1} \, e^{-y\left(\frac{r_1+r_2+1}{2}-t-1\right)} e^{-de^y} \\ \cdot \left(\sum_{k=0}^{N} (-1)^k \tilde{A}_k e^{-ky} + F_{r_1,r_2,t}(y) e^{-(N+1)y}\right), \end{aligned} \tag{2.14}$$

where (omitting the dependence on r_1, r_2),

$$\tilde{A}_k = \tilde{A}_k(t) \quad := \quad \frac{A_k(t)}{A_0(t)d^k} \qquad\qquad (k = 0, 1, \ldots, N), \tag{2.15}$$

and for $y \geq S$, the error function

$$F_{r_1,r_2,t}(y) := \frac{-iF(y)e^{de^y}}{2\pi A_0 d^{-\frac{r_1+r_2+1}{2}+t}\, e^{-y\left(\frac{r_1+r_2+1}{2}-t+N\right)}},$$

has the following upper bound

$$|F_{r_1,r_2,t}(y)| \leq \frac{C}{d^{\frac{r_1+r_2}{2}-t+N}(2\pi)^{\frac{1-d}{2}+1}} =: M_t. \tag{2.16}$$

For simplicity, from now on we will write simply $F_t := F_{r_1,r_2,t}(y)$.

Replacing (2.14) and then in (2.12) we get

$$\begin{aligned}
\left(-\frac{f^{(1)}_{(r_1,r_2)}(y)}{f_{(r_1,r_2)}(y)}\right)'' &= -\frac{f^{(3)}_{(r_1,r_2)}(y)}{f^{(0)}_{(r_1,r_2)}(y)} + 3\frac{f^{(2)}_{(r_1,r_2)}(y)f^{(1)}_{(r_1,r_2)}(y)}{\left(f^{(0)}_{(r_1,r_2)}(y)\right)^2} - 2\frac{\left(f^{(1)}_{(r_1,r_2)}(y)\right)^3}{\left(f^{(0)}_{(r_1,r_2)}(y)\right)^3} \\
&= d^3e^{3y}\left[\frac{\sum_{k=0}^N(-1)^k\tilde{A}_k(3)e^{-ky}+F_3e^{-(N+1)y}}{\sum_{k=0}^N(-1)^k\tilde{A}_k(0)e^{-ky}+F_0e^{-(N+1)y}}\right. \\
&\quad - 3\frac{\left(\sum_{k=0}^N(-1)^k\tilde{A}_k(2)e^{-ky}+F_2e^{-(N+1)y}\right)\left(\sum_{k=0}^N(-1)^k\tilde{A}_k(1)e^{-ky}+F_1e^{-(N+1)y}\right)}{\left(\sum_{k=0}^N(-1)^k\tilde{A}_k(0)e^{-ky}+F_0e^{-(N+1)y}\right)^2} \\
&\quad \left. + 2\frac{\left(\sum_{k=0}^N(-1)^k\tilde{A}_k(1)e^{-ky}+F_1e^{-(N+1)y}\right)^3}{\left(\sum_{k=0}^N(-1)^k\tilde{A}_k(0)e^{-ky}+F_0e^{-(N+1)y}\right)^3}\right].
\end{aligned} \tag{2.17}$$

For instance, if $N = 1$, then from (6.47) we get

$$\tilde{A}_0(t) = 1, \qquad\qquad \tilde{A}_1(t) = \frac{1}{24d}\left(r_1^2 + r_1r_2 - 12r_1t + r_2^2 - 12r_2t + 12t^2 - 1\right).$$

If $N = 2$ we get (and will need) the far more complicated rational numbers

$$\begin{aligned}
\tilde{A}_2(t) = &\frac{1}{1152d^2}\Big(r_1^4 + 2r_1^3r_2 - 24r_1^3t + 3r_1^2r_2^2 - 48r_1^2r_2t + 168r_1^2t^2 + 2r_1r_2^3 - 48r_1r_2^2t + 312r_1r_2t^2 \\
&- 288r_1t^3 + r_2^4 - 24r_2^3t + 168r_2^2t^2 - 288r_2t^3 + 144t^4 - 24r_1^2r_2 - 192r_1^2t - 24r_1r_2^2 - 336r_1r_2t \\
&+ 576r_1t^2 - 192r_2^2t + 576r_2t^2 - 384t^3 + 22r_1^2 + 22r_1r_2 - 264r_1t + 22r_2^2 - 264r_2t + 264t^2 - 23\Big).
\end{aligned} \tag{2.18}$$

Expanding the right hand side of (2.17) and grouping all the terms that depend on the functions

F_0, F_1, F_2, F_3 we get

$$\Big(- \frac{f^{(1)}_{(r_1,r_2)}(y)}{f_{(r_1,r_2)}(y)}\Big)'' = \frac{d^3 e^{3y}(\sum_{j=0}^{N} \alpha_j e^{-jy} + (\sum_{j=N+1}^{3N+3} \alpha_j(y, F_0(y), F_1(y), F_2(y), F_3(y)))e^{-(N+1)y})}{(\sum_{k=0}^{N} (-1)^k \tilde{A}_k(0) e^{-ky} + F_0(y) e^{-(N+1)y})^3}, \tag{2.19}$$

where $y \mapsto \alpha_j(y, F_0(y), F_1(y), F_2(y), F_3(y))$ are bounded functions for $N+1 \leq j \leq 3N+3$, but constant (functions) for $1 \leq j \leq N$. To simplify notation, we shall denote $\alpha_j(y, F_0(y), F_1(y), F_2(y), F_3(y))$ (for $N+1 \leq j \leq 3N+3$) simply by $\alpha_j(y)$. Note that the $\alpha_1, \ldots, \alpha_N$ depend only on $\tilde{A}_1, \ldots, \tilde{A}_N$ defined in (2.15).

2.1.3 Bound for the error term.

We need a bound for the error term (in red) in the asymptotic expansion (2.19). Although the constants α_i are huge rational expressions in terms of r_1, r_2 and t, it turns out that the constants α_0, α_1 and α_2 are surprisingly simple. Also the functions $\alpha_i(y)$ are polynomials in the error functions $F_0(y), F_1(y), F_2(y), F_3(y)$. This is the content of the next lemma.

Lemma 2.1.4. *Using the previous notation, we have $\alpha_0 = \alpha_1 = 0$, $\alpha_2 = \frac{1}{d^2}$. Also the functions α_i, with $N+1 \leq i \leq 3N+3$ are of the form*

$$\alpha_i(y) = e^{(N+1-i)y} \sum_{j=0}^{3} \sum_{\substack{I=(i_0,i_1,i_2,i_3) \\ i_0+i_1+i_2+i_3=j}} h_{I,i}(r_1, r_2)\ F_0(y)^{i_0} F_1(y)^{i_1} F_2(y)^{i_2} F_3(y)^{i_3}, \tag{2.20}$$

where $h_{I,i}(x, y) \in \mathbb{Q}(x, y)$ are rational functions with rational coefficients.

Proof. Expanding the right hand side of (2.17) and calculating the constant coefficient of e^{-jy}, $(j = 0, 1)$, we see that

$$\begin{aligned}
\alpha_0 &= 2\tilde{A}_0(1)^3 - 3\tilde{A}_0(2)\tilde{A}_0(1)\tilde{A}_0(0) + \tilde{A}_0(3)\tilde{A}_0(0)^2 \\
\alpha_1 &= 3\tilde{A}_0(2)\tilde{A}_0(1)\tilde{A}_1(0) + 3\tilde{A}_0(2)\tilde{A}_1(1)\tilde{A}_0(0) + 3\tilde{A}_1(2)\tilde{A}_0(1)\tilde{A}_0(0) - 2\tilde{A}_0(3)\tilde{A}_0(0)\tilde{A}_1(0) \\
&\quad -6\tilde{A}_0(1)^2\tilde{A}_1(1) - \tilde{A}_1(3)\tilde{A}_0(0)^2.
\end{aligned}$$

A direct calculation using the explicit definition of the constants $\tilde{A}_0, \tilde{A}_1, \tilde{A}_2$ in (2.18) shows that $\alpha_0 = \alpha_1 = 0$. A similar direct computation shows that $\alpha_2 = \frac{1}{d^2}$.

Finally, from (2.17) and the definition of the constants $\tilde{A}_k \in \mathbb{Q}$, it's clear that the functions α_i are of the form (2.20). $\square$

We consider the special case $N = 2$, as we will need this later. Factoring out by e^{-2y} in (2.19) and using Lemma 2.1.4 along with (2.14) we get

$$\begin{aligned}
\left(- \frac{f^{(1)}_{(r_1,r_2)}(y)}{f_{(r_1,r_2)}(y)} \right)'' &= \frac{d^3 e^{y} \left(\frac{1}{d^2} + (\alpha_3(y) + \alpha_4(y) + \cdots + \alpha_9(y))\, e^{-y} \right)}{(\sum_{k=0}^{2} (-1)^k \tilde{A}_k(0) e^{-ky} + F_0 e^{-3y})^3} \\
&= \frac{d^3 e^{y} \left(\frac{1}{d^2} + (\alpha_3(y) + \alpha_4(y) + \cdots + \alpha_9(y))\, e^{-y} \right)}{\left(f_{(r_1,r_2)}(y) e^{y\left(\frac{r_1+r_2+1}{2} - 1\right)} e^{de^y} \left(A_0 d (2\pi)^{d-1} d^{-\frac{r_1+r_2+1}{2} - 1} \right)^{-1} \right)^3},
\end{aligned} \tag{2.21}$$

where the functions $\alpha_i(y)$ are bounded by

$$|\alpha_i(y)| \leq e^{(3-i)y} \sum_{j=0}^{3} \sum_{\substack{I=(i_0,i_1,i_2,i_3) \\ i_0+i_1+i_2+i_3=j}} |h_{I,i}(r_1,r_2)|\; M_0^{i_0} M_1^{i_1} M_2^{i_2} M_3^{i_3} =: e^{(3-i)y}\Omega_i \qquad (i = 3,4,\ldots,9).$$

Note that $\Omega_i \geq 0$.

2.1.4 Positivity of $\left(-\frac{f'}{f}\right)''(y)$ for $y \gg 0$

We look for an explicit $y_0 \in \mathbb{R}$ such that $\left(-\frac{f'(y)}{f(y)}\right)'' > 0$ for all $y \geq y_0$. From log-concavity (see the line preceding (6.5) in the Appendix), we know that $f_{(r_1,r_2)}(y) > 0$ for all $y \in \mathbb{R}$. Therefore, from the expression (2.21) we see that

$$\left(-\frac{f^{(1)}_{(r_1,r_2)}(y)}{f_{(r_1,r_2)}(y)}\right)'' > 0 \Longleftrightarrow \left(\frac{1}{d^2} + (\alpha_3(y) + \alpha_4(y) + \cdots + \alpha_9(y))\, e^{-y}\right) > 0.$$

Since $|\alpha_i(y)| \leq e^{(3-i)y}\Omega_i$, $\; i = 3,4,\ldots,9$, we have

$$\frac{1}{d^2} + (\alpha_3 + \alpha_4 + \cdots + \alpha_9)\, e^{-y} \geq \frac{1}{d^2} - \kappa(y),$$

where $\kappa(y) := \sum_{i=3}^{9} \Omega_i e^{(2-i)y}$. Since $\kappa(y)$ is decreasing, we see that if

$$\frac{1}{d^2} - \kappa(y^*) = 0, \tag{2.22}$$

then $\frac{1}{d^2} - \kappa(y) > 0$ for $y > y^*$.

Table 2.1: The value of y^* in (2.22) for signatures $(0, r_2)$ with $3 \leq r_2 \leq 40$.

d	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
y^*	-0.461	0.169	0.655	1.048	1.378	1.662	1.911	2.133	2.333	2.514
d	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**
y^*	2.681	2.835	2.977	3.111	3.236	3.354	3.465	3.570	3.670	3.766
d	**23**	**24**	**25**	**26**	**27**	**28**	**28**	**30**	**31**	**32**
y^*	3.857	3.944	4.027	4.107	4.184	4.258	4.330	4.399	4.465	4.530
d	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**		
y^*	4.593	4.653	4.712	4.769	4.825	4.879	4.932	4.983		

Note that such $y^* \in \mathbb{R}$ satisfying $\frac{1}{d^2} - \kappa(y^*) = 0$ always exists and is unique since $\kappa(y)$ is strictly decreasing, $\lim_{y\to+\infty} \kappa(y) = 0$ and $\lim_{y\to-\infty} \kappa(y) = +\infty$. Also note that, after a change of variable $x = e^{-y}$, the equation (2.22) becomes a polynomial equation in the variable x, thus it can be solved (to a given desired accuracy, for instance, using the command `polrootsreal` of the program `PARI-GP`). Table 2.1 shows the value of y^* solving (2.22),[1] using the program `PARI-GP`, for signatures $(0, r_2)$ for $3 \le r_2 \le 40$.

2.2 Asymptotic expansion of $f^{(t)}(y)$ when $y \to -\infty$

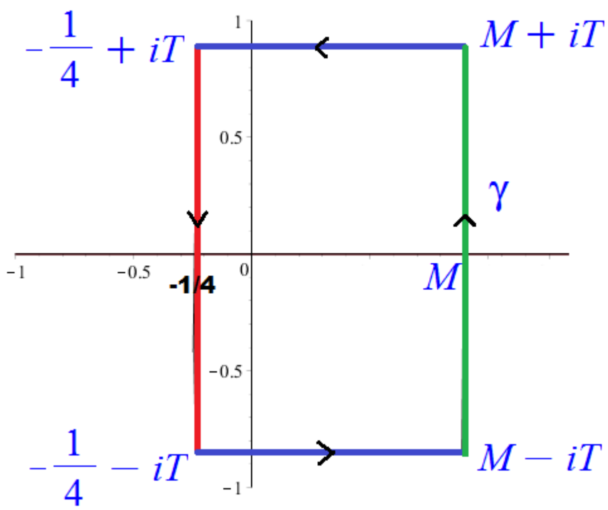
In this section we construct an asymptotic expansion for $f^{(t)}(y)$ when $y \to -\infty$. This will prove much easier than the analysis as $y \to \infty$ because it can be accomplished by a shift of the integration contour, acquiring residues, followed by a straight-forward upper bound for the integrand along the new contour. In §3.3, we will adjust the number of residues according to the required accuracy.

For $M > 0, z > 0$ and $d := r_1 + 2r_2$, let

$$G_z(s) := s^t\, z^s\, \Gamma(s)^{r_1+r_2}\, \Gamma(s + \tfrac{1}{2})^{r_2}, \qquad F^{(t)}(z) := \frac{(-d)^t}{2\pi i} \int_{M-i\infty}^{M+i\infty} G_z(s)\, ds \tag{2.23}$$

where the principal branch of log is used to define z^s. Note that $f^{(t)}_{(r_1,r_2)}(y) = F^{(t)}(z)$ if $z := \mathrm{e}^{-dy}$. We are interested in the asymptotic behaviour of $F^{(t)}(z)$ when $z \to +\infty$, or equivalently as $y \to -\infty$ in $f^{(t)}(y)$. To avoid special cases below, we will suppose that $r_1 + r_2 \ge 4$.[2]

Figure 2.1: The complex contour $\mathcal{C}$ in (2.24).



Let $\mathcal{C}$ be the complex contour shown in Figure 2.1. Thus, by Cauchy's residue theorem

$$\frac{1}{2\pi i} \oint_{\mathcal{C}} G_z(s)\, ds = \mathrm{Res}_{s=0}\left(G_z(s)\right). \tag{2.24}$$

When $T \to +\infty$ the integral over the green line in Figure 2.1 corresponds to $f^{(t)}(y)$, whereas the integrals over the horizontal blue lines go to zero. Also, the integral over the red line corresponds

[1] In practice, we cannot fix y^* with infinite accuracy, so we just choose a y_0 such that $d^{-2} - \kappa(y^*)$ is guaranteed to be a very small positive number.

[2] This is why in Step 3 on page 9 we only consider $d \ge 4$.

to the error term.

Lemma 2.2.1. *The residue of $G_z(s)$ at $s=0$ is a polynomial in $\log(z)$, more explicitly*

$$\mathrm{Res}_{s=0}\left(G_z(s)\right) = \sum_{j=1+t}^{r_1+r_2} e_j \frac{(\log z)^{j-1-t}}{(j-1-t)!}, \tag{2.25}$$

where $e_j = e_j(r_1, r_2, t)$ are explicitly calculable real constants of alternating signs. The last two of them are given by

$$\begin{aligned} e_{r_1+r_2} &= \pi^{r_2/2} \\ e_{r_1+r_2-1} &= -\pi^{\frac{r_2}{2}}\left(\gamma(r_1+2r_2)+2r_2\log 2\right) \neq 0, \end{aligned}$$

where γ is Euler's constant.

Proof. The Laurent series of the Gamma function $\Gamma(s)$ around the points $s=0$ and $s=1/2$ are given by

$$\Gamma(s) = \frac{1}{s} - \gamma + \left(\frac{\pi^2}{12}+\frac{\gamma^2}{2}\right)s + O(s^2) = \sum_{k=-1}^{\infty} a_k s^k, \quad |s|<1, \tag{2.26}$$

$$\Gamma(s+\tfrac{1}{2}) = \sqrt{\pi} - (\gamma+2\log 2)\sqrt{\pi}s + O(s^2) = \sum_{k=0}^{\infty} b_k s^k, \quad |s| < \frac{1}{2},$$

where γ is Euler's constant and the coefficients $a_k \in \mathbb{C}$ are given by the following recurrence relations (see [GR07, page 894] for the formulas and Chapters 40-41 of [Nie06] for the corresponding proofs)

$$a_{-1}=1, \quad a_0=-\gamma, \quad a_k = \frac{\sum_{m=0}^{k}(-1)^{m+1}s_{m+1}a_{k-m-1}}{k+1}, \quad s_1=\gamma, \quad s_n=\zeta(n),\, n\geq 2. \tag{2.27}$$

The coefficients $b_k \in \mathbb{C}$ can be calculated in the following way. By the duplication formula for the Gamma function [Art15, page 24], we know that

$$\Gamma(2s) = \frac{2^{2s-1}}{\sqrt{\pi}}\Gamma(s)\Gamma(s+\tfrac{1}{2}). \tag{2.28}$$

The entire function $\frac{1}{\Gamma(s)}$ has Taylor series near $s=0$ given by

$$\frac{1}{\Gamma(s)} =: \sum_{k=0}^{\infty} f_k s^{k+1}, \quad s\in\mathbb{C}, \tag{2.29}$$

where the coefficients $f_k \in \mathbb{C}$ are given by the recurrence relations [GR07]

$$f_0 = 1, \quad f_{n+1} = \frac{\sum_{k=0}^{n}(-1)^k s_{k+1} f_{n-k}}{n+1}, \quad s_1=\gamma, \quad s_n=\zeta(n), \quad n\geq 2.$$

Therefore, from (2.26), (2.28) and (2.29), we see that the coefficients b_k can be obtained multiplying term by term the following three Laurent series:

$$\begin{aligned}\Gamma(s+\tfrac{1}{2}) &= \frac{1}{\Gamma(s)}\Gamma(2s)\sqrt{\pi}2^{1-2s} = 2\frac{1}{\Gamma(s)}\Gamma(2s)\sqrt{\pi}\exp(-2s\log(2)) \\ &= 2\sqrt{\pi}\sum_{m=0}^{\infty} f_m s^{m+1}\ \sum_{l=-1}^{\infty} 2^l a_l s^l\ \sum_{j=0}^{\infty}\frac{(-2\log 2)^j}{j!}s^j = \sum_{k=0}^{\infty} b_k s^k.\end{aligned}$$

Thus the coefficients b_k are given explicitly by the formula

$$b_k = \sum_{\substack{j+l+m=k \\ j\geq 0,\, l\geq -1,\, m\geq 1}} f_{m-1}2^l a_l \frac{(-2\log 2)^j}{j!}. \tag{2.30}$$

In conclusion we get

$$\Gamma(s)^{r_1+r_2} = \frac{1}{s^{r_1+r_2}} - (r_1+r_2)\frac{\gamma}{s^{r_1+r_2-1}} + \cdots =: \sum_{k=-r_1-r_2}^{\infty} c_k s^k, \tag{2.31}$$

$$\Gamma(s+\tfrac{1}{2})^{r_2} = \pi^{\frac{r_2}{2}} - r_2\,\pi^{\frac{r_2}{2}}(\gamma+2\log 2)s + \cdots =: \sum_{k=0}^{\infty} d_k s^k,$$

where

$$\begin{aligned} c_k &= \sum_{l_1+\cdots+l_{r_1+r_2}=k} a_{l_1}a_{l_2}\cdots a_{l_{r_1+r_2}} \\ d_k &= \sum_{l_1+\cdots+l_{r_2}=k} b_{l_1}b_{l_2}\cdots b_{l_{r_2}}, \end{aligned}$$

and the coefficients a_k, b_k can be calculated explicitly using the formulas (2.27) and (2.30).

Using $z^s = e^{s\log(z)} = \sum_{m=0}^{\infty}\frac{(\log(z))^m}{m!}s^m$ and replacing (2.31) in the definition of G_z, we get

$$\begin{aligned}\mathrm{Res}_{s=0}\left(G_z(s)\right) &= \mathrm{Coeff}_{s^{-1}} s^t\left(\sum_{m=0}^{\infty}\frac{(\log(z))^m}{m!}s^m\right)\left(\Gamma(s)^{r_1+r_2}\Gamma(s+\tfrac{1}{2})^{r_2}\right) \\ &= \sum_{a=-r_1-r_2}^{-1-t}\left(\mathrm{Coeff}_{s^a}\left(\Gamma(s)^{r_1+r_2}\Gamma(s+\tfrac{1}{2})^{r_2}\right)\right)\frac{(\log z)^{-a-1-t}}{(-a-1-t)!} \\ &= \sum_{a=-r_1-r_2}^{-1-t}\left(\sum_{l=-r_1-r_2}^{a}\mathrm{Coeff}_{s^l}\Gamma(s)^{r_1+r_2}\cdot\mathrm{Coeff}_{s^{a-l}}\Gamma(s+\tfrac{1}{2})^{r_2}\right)\frac{(\log z)^{-a-1-t}}{(-a-1-t)!} \\ &= \sum_{a=-r_1-r_2}^{-1-t}\left(\sum_{l=-r_1-r_2}^{a} c_l\cdot d_{a-l}\right)\frac{(\log z)^{-a-1-t}}{(-a-1-t)!} = \sum_{j=1+t}^{r_1+r_2}\left(\sum_{l=j}^{r_1+r_2} c_{-l}\cdot d_{l-j}\right)\frac{(\log z)^{j-1-t}}{(j-1-t)!} \\ &= \sum_{j=1+t}^{r_1+r_2} e_j\frac{(\log z)^{j-1-t}}{(j-1-t)!}, \quad \text{where } e_j := \sum_{l=j}^{r_1+r_2} c_{-l}\cdot d_{l-j}.\end{aligned}$$

Using the recurrence relations for the coefficients a_k and f_k, it's easy to see that the signs of the coefficients c_k and d_k are given by $\text{sgn}(c_k) = (-1)^{k+r_1+r_2}$, $\text{sgn}(d_k) = (-1)^k$. Hence the sign of the coefficient e_j is given by

$$\text{sgn}(e_j) = (-1)^{r_1+r_2-j}. \tag{2.32}$$

This shows that the expression

$$\text{Res}_{s=0}\left(G_z(s)\right) = \sum_{j=1+t}^{r_1+r_2} e_j \frac{(\log z)^{j-1-t}}{(j-1-t)!}, \tag{2.33}$$

is a polynomial in the variable $\log(z)$ with coefficients of alternating signs.

Note also that the last and the second to last coefficients e_j are given by

$$\begin{aligned} e_{r_1+r_2} &= \pi^{r_2/2} && (2.34)\\ e_{r_1+r_2-1} &= c_{-r_1-r_2+1}d_0 + c_{-r_1-r_2}d_1 = (-\gamma(r_1+r_2))\pi^{\frac{r2}{2}} - 1\cdot(r_2\pi^{\frac{r2}{2}}(\gamma+2\log 2)) \\ &= -\pi^{\frac{r2}{2}}(\gamma(r_1+2r_2)+2r_2\log 2) \neq 0. \quad \square && (2.35)\end{aligned}$$

2.2.1 Error estimate in the residue formula for $f^{(t)}(y)$.

In this section we estimate the approximation error when the finite sum $(-d)^t \sum_{j=1+t}^{r_1+r_2} e_j \frac{(\log z)^{j-1-t}}{(j-1-t)!}$ is used to approximate the function $F^{(t)}(z)$. Such error function comes from a contour integral over a line of the complex plane. In fact, by Cauchy's residue formula and the form of the contour $\mathcal{C}$ depicted in Figure 2.1, we can split the contour integral $\oint_{\mathcal{C}} G_z(s)\,ds$ in the form

$$\begin{aligned}\text{Res}_{s=0}\left(G_z(s)\right) &= \frac{1}{2\pi i}\oint_{\mathcal{C}} G_z(s)\,ds = \frac{1}{2\pi}\int_{-T}^{T} G_z(M+iu)\,du - \frac{1}{2\pi}\int_{-T}^{T} G_z(-1/4+iu)\,du \\ &\quad + \underbrace{\frac{1}{2\pi i}\int_{M}^{-1/4} G_z(x+iT)\,dx + \frac{1}{2\pi i}\int_{-1/4}^{M} G_z(x-iT)\,dx.}_{\to 0 \text{ as } T\to+\infty} \qquad (2.36)\end{aligned}$$

By Stirling's formula, we have that

$$|\Gamma(x \pm iT)| = \sqrt{2\pi}\, T^{x-1/2} e^{-\pi T/2}\left(1+O(1/T)\right)$$

and

$$\left|\Gamma\left(x \pm iT + \frac{1}{2}\right)\right| = \sqrt{2\pi}\, T^{x} e^{-\pi T/2}\left(1+O(1/T)\right)$$

as $T \to +\infty$. Thus we conclude that

$$|G_z(x \pm iT)| = e^{x\log(z)}(x^2+T^2)^{t/2}\left|\Gamma(x\pm iT)^{r_1+r_2}\,\Gamma\left(x \pm iT + \frac{1}{2}\right)^{r_2}\right| \longrightarrow 0,$$

exponentially fast as $T \to +\infty$, for all x in a compact real interval. Thus, taking $\lim_{T\to+\infty}$ in

(2.36), we get

$$\begin{aligned}
\mathrm{Res}_{s=0}\left(G_z(s)\right) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G_z(M+iu)\,du - \frac{1}{2\pi}\int_{-\infty}^{\infty} G_z(-1/4+iu)\,du, \\
&= \frac{F^{(t)}(z)}{(-d)^t} - \frac{1}{2\pi}\int_{-\infty}^{\infty} G_z(-1/4+iu)\,du \qquad \text{(using (2.23))}.
\end{aligned}$$

Replacing (2.25) in the previous equality and using (2.34), we obtain

$$\begin{aligned}
&\frac{F^{(t)}(z)}{(-d)^t} = \sum_{j=1+t}^{r_1+r_2} e_j \frac{(\log z)^{j-1-t}}{(j-1-t)!} + \frac{1}{2\pi}\int_{-\infty}^{\infty} G_z(-1/4+iu)\,du = \sum_{j=1+t}^{r_1+r_2} e_j \frac{(\log z)^{j-1-t}}{(j-1-t)!} + E(z) \\
&= \pi^{r_2/2}\frac{(\log z)^{r_1+r_2-1-t}}{(r_1+r_2-1-t)!}\Big(1+\sum_{j=1+t}^{r_1+r_2-1}\frac{e_j}{\pi^{r_2/2}}(r_1+r_2-1-t)!\frac{(\log z)^{j-r_1-r_2}}{(j-1-t)!}+\frac{(r_1+r_2-1-t)!}{\pi^{r_2/2}(\log z)^{r_1+r_2-1-t}}\,E(z)\Big) \\
&= \pi^{r_2/2}\frac{(\log z)^{r_1+r_2-1-t}}{(r_1+r_2-1-t)!}\Big(1+S(r_1,r_2,t;z)+H(r_1,r_2,t;z)\Big), \qquad (2.37)
\end{aligned}$$

where

$$\begin{aligned}
E(z) &:= \frac{1}{2\pi}\int_{-\infty}^{\infty} G_z(-1/4+iu)\,du, \\
S(r_1,r_2,t;z) &:= \sum_{j=1+t}^{r_1+r_2-1}\frac{e_j}{\pi^{r_2/2}}(r_1+r_2-1-t)!\frac{(\log z)^{j-r_1-r_2}}{(j-1-t)!}, \\
H(r_1,r_2,t;z) &:= \frac{(r_1+r_2-1-t)!}{\pi^{r_2/2}\log(z)^{r_1+r_2-1-t}}\,E(z) \qquad \text{(= the error term)}.
\end{aligned}$$

Now we estimate $H(r_1,r_2,t;z)$. First we estimate $E(z)$ as follows.

$$\begin{aligned}
\left|\int_{-\infty}^{\infty} G_z(-\tfrac{1}{4}+iu)\,du\right| &\leq \int_{-\infty}^{\infty} |G_z(-\tfrac{1}{4}+iu)|\,du \\
&= \int_{-\infty}^{\infty} \left|(-\tfrac{1}{4}+iu)^t\, z^{-\frac{1}{4}+iu}\,\Gamma(-\tfrac{1}{4}+iu)^{r_1+r_2}\,\Gamma\left(-\tfrac{1}{4}+iu+\tfrac{1}{2}\right)^{r_2}\right| du \\
&= e^{-\frac{1}{4}\log z}\int_{-\infty}^{\infty}\left(\tfrac{1}{16}+u^2\right)^{t/2}\left|\Gamma\left(-\tfrac{1}{4}+iu\right)\right|^{r_1+r_2}\left|\Gamma\left(\tfrac{1}{4}+iu\right)\right|^{r_2}\,du.
\end{aligned}$$

Thus $E(z)$ is defined by a convergent integral and satisfies the upper bound

$$|E(z)| \leq C(r_1,r_2,t)\cdot\frac{1}{\sqrt[4]{z}},$$

where

$$C(r_1,r_2,t) := \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\tfrac{1}{16}+u^2\right)^{t/2}\left|\Gamma\left(-\tfrac{1}{4}+iu\right)\right|^{r_1+r_2}\left|\Gamma\left(\tfrac{1}{4}+iu\right)\right|^{r_2}\,du. \qquad (2.38)$$

Therefore,

$$|H(r_1,r_2,t;z)| \leq \frac{(r_1+r_2-1-t)!\,C(r_1,r_2,t)}{\sqrt[4]{z}\,\pi^{r_2/2}\,|\log(z)|^{r_1+r_2-1-t}}. \qquad (2.39)$$

We proceed to find an easily calculated upper bound for $C(r_1, r_2, t)$.

Using $\Gamma(s+1) = s\Gamma(s)$, we obtain

$$s^t \Gamma(s)^{r_1+r_2} \Gamma(s+\tfrac{1}{2})^{r_2} = \Gamma(s+1)^t \Gamma(s)^{r_1+r_2-t} \Gamma(s+\tfrac{1}{2})^{r_2}. \tag{2.40}$$

Therefore, for $s = -1/4 + iu$, using the inequality $|\Gamma(x+iy)| \leq \Gamma(x) \quad (x, y \in \mathbb{R},\, x > 0)$, from (2.40) we get

$$\left| s^t \Gamma(s)^{r_1+r_2} \Gamma(s+\tfrac{1}{2})^{r_2} \right| \leq \Gamma(3/4)^t \left| \Gamma\left(-\frac{1}{4} + iu\right) \right|^{r_1+r_2-t} \Gamma\left(\frac{1}{4}\right)^{r_2}. \tag{2.41}$$

Using the inequality

$$\left| \Gamma\left(\frac{3}{4} + iu\right) \right| \leq \Gamma\left(\frac{3}{4}\right), \quad \forall u \in \mathbb{R},$$

we see that, on abbreviating $p := r_1 + r_2 - t - 2 \geq 0$, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \left| \Gamma\left(-\tfrac{1}{4} + iu\right) \right|^{p+2} du &= \int_{-\infty}^{\infty} \frac{\left| \Gamma\left(\frac{3}{4} + iu\right) \right|^{p+2}}{\left| -\frac{1}{4} + iu \right|^{p+2}} \, du \\
&\leq \int_{-\infty}^{\infty} \frac{\left| \Gamma\left(\frac{3}{4} + iu\right) \right|^{p}}{\left(\frac{1}{4}\right)^{p+2}} \cdot \left| \Gamma\left(\tfrac{3}{4} + iu\right) \right|^2 du \\
&\leq 4^{p+2} \Gamma(\tfrac{3}{4})^p \int_{-\infty}^{\infty} \left| \Gamma\left(\tfrac{3}{4} + iu\right) \right|^2 du.
\end{aligned} \tag{2.42}$$

To estimate $\int_{-\infty}^{\infty} \left| \Gamma\left(\frac{3}{4} + iu\right) \right|^2 du$, we use [GR07, p. 895]

$$\left| \frac{\Gamma(x+iu)}{\Gamma(x)} \right|^2 = \prod_{k=0}^{\infty} \left(1 + \frac{u^2}{(x+k)^2} \right)^{-1} \qquad (x, u \in \mathbb{R},\ x \notin \mathbb{Z}_{\leq 0}).$$

Therefore $x \mapsto \left| \frac{\Gamma(x+iu)}{\Gamma(x)} \right|^2$ is decreasing for $x > 0$. In particular, for all $u > 0$ we have

$$\frac{\left| \Gamma\left(\frac{3}{4} + iu\right) \right|^2}{\left| \Gamma\left(\frac{3}{4}\right) \right|^2} \leq \frac{\left| \Gamma\left(1 + iu\right) \right|^2}{\left| \Gamma\left(1\right) \right|^2} = \left| \Gamma\left(1 + iu\right) \right|^2.$$

Thus,

$$\left| \Gamma\left(\tfrac{3}{4} + iu\right) \right|^2 \leq \Gamma\left(\tfrac{3}{4}\right)^2 \left| \Gamma\left(1 + iu\right) \right|^2. \tag{2.43}$$

Since $\left| \Gamma\left(1 + iu\right) \right|^2 = \frac{\pi u}{\sinh(\pi u)}$ (see [MO48, Chapter 3]), we get

$$\int_{-\infty}^{\infty} \left| \Gamma\left(1 + iu\right) \right|^2 du = \int_{-\infty}^{\infty} \frac{\pi u}{\sinh(\pi u)} \, du = \frac{\pi}{2}. \tag{2.44}$$

From (2.43) and (2.44) we obtain

$$\int_{-\infty}^{\infty} \left| \Gamma\left(\tfrac{3}{4} + iu\right) \right|^2 du \leq \Gamma(3/4)^2 \frac{\pi}{2}. \tag{2.45}$$

Replacing (2.41), (2.42), (2.45) in (2.38), calculating Γ-values numerically, and using $d = r_1 + 2r_2,\ 0 \le t \le 3$, we obtain

$$C(r_1, r_2, t) < 2.8905 \cdot (17.7716)^d. \tag{2.46}$$

In conclusion, replacing all the previous inequalities in (2.37), we get a residue approximation for $f^{(t)}(y)$ in the form principal term + error term

$$f^{(t)}(y) = (-d)^t \pi^{r_2/2} c_t (-dy)^{r_1+r_2-1-t} \left(1 + S_t(y) + H_t(y)\right), \tag{2.47}$$

where

$$c_t \ := \ c(r_1, r_2; t) = \frac{1}{(r_1 + r_2 - 1 - t)!}, \tag{2.48}$$

$$S_t(y) \ := \ S(r_1, r_2, t; e^{-dy}) = \frac{(r_1 + r_2 - 1 - t)!}{\pi^{r_2/2}} \sum_{j=1+t}^{r_1+r_2-1} e_j \frac{(-dy)^{j-r_1-r_2}}{(j-1-t)!}, \tag{2.49}$$

$$|H_t(y)| \ := \ |H(r_1, r_2, t; e^{-dy})| \le \frac{(r_1 + r_2 - 1 - t)!\, 2.8905 \cdot (17.7716)^d}{\pi^{r_2/2}\, |dy|^{r_1+r_2-1-t}} e^{\frac{dy}{4}}, \tag{2.50}$$

with H as in (2.39). Note that the error term $H_t(y) \to 0$ as $y \to -\infty$. For simplicity of the notation, sometimes we denote simply $S_t = S_t(y)$ and $H_t = H_t(y)$ $(t = 0, 1, 2, 3)$.

2.2.2 Asymptotic expansion of $\left(-\frac{f'}{f}\right)''(y)$ when $y \to -\infty$.

Using (2.47) we can find an approximation by residues for $\left(-\frac{f'(y)}{f(y)}\right)''$.

$$\left(-\frac{f'(y)}{f(y)}\right)'' = \frac{-f'''(y)f(y)^2 + 3f''(y)f'(y)f(y) - 2f'(y)^3}{f(y)^3} = \frac{\Psi}{y^3 \cdot c_0^3\, (1 + S_0 + H_0)^3}, \tag{2.51}$$

$$\begin{aligned}\Psi = &-c_3\, c_0^2\, (1 + S_3 + H_3)(1 + S_0 + H_0)^2 \\ &+ 3c_2\, c_1\, c_0 (1 + S_2 + H_2)(1 + S_1 + H_1)(1 + S_0 + H_0) - 2c_1^3\, (1 + S_1 + H_1)^3,\end{aligned}$$

where the c_t are as in (2.48) and we used (2.47) for $t = 0, 1, 2, 3$, to replace each appearance of f, f', f'' and f''' by its approximation.

2.2.3 Bound for the error term.

To have a valid asymptotic expansion of $\left(-\frac{f'(y)}{f(y)}\right)''$ we need to isolate the main term from the error term in (2.51) and then obtain an upper bound for the error.

Lemma 2.2.2. *With S_i as in* (2.49)*, let*

$$T_0 = 1 + 2S_0 + S_3 + S_0^2 + 2S_0S_3 + S_3S_0^2, \qquad T_2 = 1 + 3S_1 + 3S_1^2 + S_1^3,$$
$$T_1 = 1 + S_0 + S_1 + S_2 + S_0S_1 + S_1S_2 + S_0S_2 + S_0S_1S_2.$$

Then $\left(-\frac{f'(y)}{f(y)}\right)''$ *can be written in the asymptotic form*

$$\left(-\frac{f'(y)}{f(y)}\right)'' = \frac{-c_3\, c_0^2\, [T_0+\varepsilon_1] + 3c_0\, c_1\, c_2\, [T_1+\varepsilon_2] - 2c_1^3\, [T_2+\varepsilon_3]}{y^3\cdot c_0^3\,(1+S_0+H_0)^3},$$

with explicit upper bounds $|\varepsilon_i| \le \Lambda_i(y)$, $i=1,2,3,$ *where* $\Lambda_i(y)\to 0$ *monotonically as* $y\to-\infty$.

Proof. By direct calculation, we can write the numerator of (2.51) in the form

$$\begin{aligned}
&- c_3\, c_0^2\,(1+S_3+H_3)(1+S_0+H_0)^2 + 3c_2\, c_1\, c_0(1+S_2+H_2)(1+S_1+H_1)(1+S_0+H_0)\\
&\quad - 2c_1^3\,(1+S_1+H_1)^3 = -c_3\, c_0^2\, \left[1+2S_0+S_3+S_0^2+2S_0S_3+S_3S_0^2+\varepsilon_1\right] \qquad (2.52)\\
&\quad\quad + 3c_0\, c_1\, c_2\,[1+S_0+S_1+S_2+S_0S_1+S_1S_2+S_0S_2+S_0S_1S_2+\varepsilon_2] - 2c_1^3\, \left[1+3S_1+3S_1^2+S_1^3+\varepsilon_3\right],
\end{aligned}$$

where the terms ε_i are the functions involving the error terms H_j, and are given by

$$\begin{aligned}
\varepsilon_1 &:= (1+S_3+H_3)(2H_0+2S_0H_0+H_0^2) + H_3(1+2S_0+S_0^2)\\
\varepsilon_2 &:= (1+S_1+S_2+H_1+H_2+S_1S_2+S_2H_1+H_2S_1+H_2H_1)H_0 \qquad (2.53)\\
&\quad +(H_1+H_2+S_2H_1+H_2S_1+H_2H_1)(1+S_0)\\
\varepsilon_3 &:= 3H_1+3H_1^2+6H_1S_1+3H_1^2S_1+3H_1S_1^2+H_1^3.
\end{aligned}$$

The terms with no $\varepsilon_i's$ (the blue terms) in (2.52) form the principal term when $y\to-\infty$.

Recall from (2.34) that $e_{r_1+r_2-1} = -\pi^{\frac{r_2}{2}}(\gamma(r_1+2r_2)+2r_2\log 2)\neq 0$. Thus each function $S_t(y)$ defined in (2.49) is a Laurent polynomial of the form

$$S_t(y) = \frac{(r_1+r_2-1-t)!}{\pi^{r_2/2}}\sum_{j=1+t}^{r_1+r_2-1} e_j\,\frac{(-dy)^{j-r_1-r_2}}{(j-1-t)!} = \sum_{i=1}^{r_1+r_2-1-t}\frac{\alpha_{i,t}}{y^i} \qquad (\alpha_{i,t}\in\mathbb{C},\ \alpha_{1,t}\neq 0). \quad (2.54)$$

The constants $\alpha_{i,t}$ are given explicitly by

$$\alpha_{i,t} := (-1)^i\frac{(r_1+r_2-1-t)!}{\pi^{\frac{r_2}{2}}}\cdot\frac{e_{r_1+r_2-i}}{d^i(r_1+r_2-i-1-t)!}. \qquad (2.55)$$

Note that, by (2.32), S_t is a polynomial in $1/y$ with positive coefficients and vanishing constant term. In particular, $S_t(y)\to 0$ and $S_t(|y|)$ tends monotonely to 0 as $y\to-\infty$. Also, by (2.50), $\lim_{y\to-\infty} H_t(y)=0$. Thus, as $y\to-\infty$, the numerator Ψ in (2.51) tends to

$$-c_3c_0^2+3c_0c_1c_2-2c_1^3 = -\frac{2(r_1+r_2-1)}{((r_1+r_2-1)!)^3} < 0,$$

using (2.48). Therefore, using (2.55) we see that, as $y\to-\infty$, the principal term in (2.52) is of the form

$$-\frac{2(r_1+r_2-1)}{((r_1+r_2-1)!)^3} + \sum_{j=1}^{3r_1+3r_2-6}\frac{p_j}{y^j},$$

where the coefficient of the term $\frac{1}{y}$ is given by

$$\begin{aligned} p_1 &= -c_3c_0^2(2\alpha_{1,0}+\alpha_{1,3})+3c_0c_1c_2(\alpha_{1,0}+\alpha_{1,1}+\alpha_{1,2})-2c_1^3(3\alpha_{1,1}) \\ &= -6\,\frac{(\gamma(r_1+2r_2)+2\,r_2\,\log(2))\,(r_1+r_2-2)\,(r_1+r_2-1)}{(r_1+2\,r_2)\,((r_1+r_2-1)!)^3}<0. \end{aligned}$$

To find an upper bound for the error term in (2.51), we first need a bound for the ε_i. Let us write the functions $H_t(y)$ in the form

$$H_t(y)=h_t(y)\cdot\frac{e^{\frac{dy}{4}}}{|y|^{r_1+r_2-1-t}} \qquad (t=0,1,2,3), \tag{2.56}$$

where, by (2.50),

$$|h_t(y)|=\left|H_t(y)y^{r_1+r_2-1-t}e^{-\frac{dy}{4}}\right|\leq\frac{(r_1+r_2-1-t)!\,C(r_1,r_2,t)}{\pi^{r_2/2}\,d^{r_1+r_2-1-t}}=:K_t.$$

Note that, using (2.46), we have

$$K_t\leq\frac{(r_1+r_2-1-t)!\,\left(2.8905\cdot(17.7716)^d\right)}{\left(\sqrt{\pi}\right)^{r_2}\,d^{r_1+r_2-1-t}}.$$

From (2.32) we know that the signs of the constants e_j are given by $\operatorname{sgn}(e_{r_1+r_2-i})=(-1)^i$, thus from (2.55) we get $\alpha_{i,t}\geq 0$ for all i. This implies that $|S_t(y)|\leq S_t(|y|)$. Hence from (2.53) and (2.56) we get

$$|\varepsilon_1|\leq\left(1+S_3(|y|)+\widetilde{H_3}\right)\left(2\widetilde{H_0}+2S_0(|y|)\widetilde{H_0}+\widetilde{H_0}^2\right)+\widetilde{H_3}\left(1+2S_0(|y|)+S_0(|y|)^2\right)=:\Lambda_1(y) \tag{2.57}$$

$$\begin{aligned} |\varepsilon_2|\leq{}&\left(1+S_1(|y|)+S_2(|y|)+\widetilde{H_1}+\widetilde{H_2}+S_1(|y|)S_2(|y|)+S_2(|y|)\widetilde{H_1}+\widetilde{H_2}S_1(|y|)+\widetilde{H_2}\widetilde{H_1}\right)\widetilde{H_0} \\ &+\left(\widetilde{H_1}+\widetilde{H_2}+S_2(|y|)\widetilde{H_1}+\widetilde{H_2}S_1(|y|)+\widetilde{H_2}\widetilde{H_1}\right)(1+S_0(|y|))=:\Lambda_2(y) \end{aligned} \tag{2.58}$$

$$|\varepsilon_3|\leq 3\widetilde{H_1}+3\widetilde{H_1}^2+6\widetilde{H_1}S_1(|y|)+3\widetilde{H_1}^2S_1(|y|)+3\widetilde{H_1}S_1(|y|)^2+\widetilde{H_1}^3=:\Lambda_3(y), \tag{2.59}$$

where $\widetilde{H_t}:=K_t\frac{e^{\frac{dy}{4}}}{|y|^{r_1+r_2-1-t}}$ $(t=0,1,2,3)$. Note that if $a<b<0$, then $0<\Lambda_i(a)\leq\Lambda_i(b)$ since the same holds for each S_t and for $\widetilde{H}_t$. $\square$

2.2.4 Positivity of $\left(-\frac{f'}{f}\right)''(y)$ for $y\ll 0$

We need a $y_*<0$ such that $\left(-\frac{f'(y)}{f(y)}\right)''>0$ for all $y\leq y_*$. From the logarithmic concavity of f (for a proof, see the line preceding (6.5) in §6.1 of the Appendix), we know that the function $f(y)$ is positive. Thus from (2.47) we get that

$$f(y)=\pi^{r_2/2}c_0(-dy)^{r_1+r_2-1}\left(1+S_0(y)+H_0(y)\right)>0,$$

for all $y\in\mathbb{R}$. Therefore, formula (2.51) says that it suffices to find $y_*<0$ such that the numerator given in (2.52) is negative for all $y\leq y_*$.

Using the bounds (2.57), (2.58) and (2.59) for the errors ε_i, we see that it is enough to find $y_* < 0$ such that

$$-c_3\, c_0^2 \left[1 + 2S_0 + S_3 + S_0^2 + 2S_0S_3 + S_3S_0^2\right] + 3c_0\, c_1\, c_2 \big[1 + S_0 + S_1 + S_2 + S_0S_1 + S_1S_2 + S_0S_2 \\ + S_0S_1S_2\big] - 2c_1^3 \left[1 + 3S_1 + 3S_1^2 + S_1^3\right] + E < 0 \qquad (\forall y \le y_*), \tag{2.60}$$

where each $S_t = S_t(y)$ is given by (2.54) and E is defined as

$$E = E(y) := c_3c_0^2\Lambda_1(y) + 3c_0c_1c_2\Lambda_2(y) + 2c_1^3\Lambda_3(y).$$

Note that $E(y)$ decreases monotonically to zero as $y \searrow -\infty$, since each $\Lambda_i(y)$ does. Also the blue term in (2.60) is a polynomial P in $1/y$. Note that $P(1/y)$ tends to $-\frac{2(r_1+r_2-1)}{((r_1+r_2-1)!)^3} < 0$ as $y \to -\infty$. Therefore, such $y_* < 0$ in (2.60) exists and can be found in practice by making sure that

$$P\left(\frac{1}{y_*}\right) < 0, \qquad \left|P\left(\frac{1}{y_*}\right)\right| > E(y_*), \tag{2.61}$$

and that y_* lies to the left of any smallest critical point of $y \mapsto P(1/y)$. Using the program `PARI-GP` we find the values of y_* listed in Table 2.2 for signatures $(0, r_2)$, for $4 \le r_2 \le 40$.

Table 2.2: The value of y_* solving (2.61).

d	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
y_*	$*$	-11.62	-10.28	-9.45	-8.87	-8.45	-8.14	-7.89	-7.69	-7.52	-7.38
d	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**
y_*	-7.26	-7.16	-7.07	-6.99	-6.92	-6.86	-6.81	-6.76	-6.71	-6.67	-6.63
d	**25**	**26**	**27**	**28**	**29**	**30**	**31**	**32**	**33**	**34**	**35**
y_*	-6.59	-6.56	-6.53	-6.50	-6.48	-6.45	-6.43	-6.41	-6.39	-6.37	-6.35
d	**36**	**37**	**38**	**39**	**40**						
y_*	-6.33	-6.32	-6.30	-6.29	-6.28						

Chapter 3

Convexity of $-\frac{f'}{f}$ on an interval

In §2.1.4 and §2.2.4 we proved $\left(-\frac{f'}{f}\right)''(y) > 0$ for $f = f_{(0,d)}$ $(4 \le d \le 40)$ and $y \ge y^*$ or $y \le y_*$ (see Tables 2.1 and 2.2). In this chapter we prove the validity of a numerical subdivision algorithm to prove the same on the compact interval $[y_*, y^*]$, finishing the proof of convexity (Step 3, p. 9).

3.1 Idea of the method

Although our aim is to prove the convexity of $-\frac{f'}{f}(y)$ for $f = f_{(0,d)}$ and y in a compact interval, it is convenient to use the identity

$$-\frac{f'_{(0,d)}}{f_{(0,d)}}(y) = -\frac{f'_{(d,0)}}{f_{(d,0)}}(y + \log 2) \tag{3.1}$$

which follows form Gauss' duplication formula. We prefer to use $f = f_{(d,0)}$ since the integral defining $f^{(t)}_{(0,d)}$ has $2d$ Γ-factors, while $f^{(t)}_{(d,0)}$ only involves d of them. This will make the numerical calculations in this chapter far easier for large d.

For simplicity, in this section we write $f := f_{(d,0)}$ (with $d \le 40$ fixed). First year calculus shows

$$\left(-\frac{f'}{f}\right)'' = f_1 - f_2, \qquad f_1 := 3\frac{f''f'}{f^2}, \qquad f_2 := \frac{f'''}{f} + 2\frac{(f')^3}{f^3}. \tag{3.2}$$

More interestingly, log-concavity shows that f_1 and f_2 are decreasing (and negative) functions (for the proof, see (6.6) in the Appendix). Therefore, to check positivity of $\left(-\frac{f'}{f}\right)''$ over a compact interval of the form $[L, R]$ it is enough to find a finite sequence of points

$$L = y_0 < y_1 < \cdots < y_k = R \qquad \text{satisfying} \qquad f_1(y_j) > f_2(y_{j-1}) \qquad (j = 1, \ldots, k). \tag{3.3}$$

It turns out that for $(-\infty, 0] \cap [L, R]$, the function f_1 stays sufficiently far above f_2 for this to work very simply as follows.

`Algorithm for` y_i `negative in` (3.3)

`Constructing the sequence` $y_0,\dots,y_k$ `on` $(-\infty,0]\cap[L,R]$ `(if` $L<0$`).`

1. Choose $\epsilon>0$ small enough[1] and construct $\epsilon-$numerical approximations $f_{1,\mathrm{num}}$ and $f_{2,\mathrm{num}}$ of f_1 and f_2 (see §3.3), *i. e.*

$$|f_1(y)-f_{1,\mathrm{num}}(y)|\le\epsilon,\qquad |f_2(y)-f_{2,\mathrm{num}}(y)|\le\epsilon,$$

   for all $y\in(-\infty,0]\cap[L,R]$. Set $i=0,\ y_0:=L$ and go to step 2.

2. Set $\delta=1/10$.
   (a) If $f_{1,\mathrm{num}}(y_i+\delta)-\epsilon>f_{2,\mathrm{num}}(y_i)+\epsilon$, then set $y_{i+1}:=y_i+\delta$ and go to step 3.
   (b) Otherwise, set $\delta\leftarrow\delta/2$ and repeat the previous step.[2]

3. If $y_{i+1}<0$ and $y_{i+1}<R$, go to step 2 with i replaced by $i+1$. Otherwise output the list of y_i's.

The details of the residue method approximation are given in §3.3. The `PARI-GP` code can be found in §6.6.4 of the Appendix.

Unfortunately, this method is not practical for the positive part of the interval $[y_*,y^*]$, as even for moderate positive values of y the functions f_1 and f_2 are too close. This makes the value of δ in step 2 of the previous algorithm becomes of the order 10^{-8} for $y\approx5$ and $d\ge10$. So to cover an interval of the form $[0,10]$ we would need approximately 10^9 iterations. Unfortunately, since $\epsilon>0$ has to be very small, the residue approximation of f_1 and f_2 requires thousands of residues, making the numerical evaluation of $f_{1,\mathrm{num}}$ and $f_{2,\mathrm{num}}$ too slow. Instead, to ensure that $f_1>f_2$ on $[0,+\infty)\cap[y_*,y^*]$ we do the following.

`Algorithm for` y_i `positive in` (3.3)
`Constructing the sequence` $y_0,\dots,y_k$ `on` $[0,+\infty)\cap[L,R]$`.`

1. We make an asymptotic normalization of the functions f_1 and f_2. Essentially we divide (and multiply) f_1 and f_2 by their asymptotic approximations when $y\to+\infty$ and take logarithms in order to separate them away from each other as much as possible. Such asymptotic approximations of f_1 and f_2 are based on the results of §2.1. For details see §3.2. Call $f_{1,\mathrm{n}}$ and $f_{2,\mathrm{n}}$ such normalized functions.

   The problem of checking $f_1>f_2$ turns out to be equivalent to checking that

$$F_1:=f_{2,\mathrm{n}}-f_{1,\mathrm{n}}>F_2,$$

   where F_2 is the logarithm of the ratio of the asymptotic expansions, an easily evaluated convex decreasing function.

[1] In practice, we chose $\epsilon=\min\big(1/100,\big(f_{1,\mathrm{num}}(y_0)-f_{2,\mathrm{num}}(y_0)\big)/10\big)$. If this eventually proved too large to advance after some y_i, we would replace y_0 $(=L)$ by y_i and start again.

[2] Since f_1 and f_2 are decreasing, this ensures that for $y_i\le y\le y_{i+1}$ we have

$$f_1(y)\ge f_1(y_{i+1})\ge f_{1,\mathrm{num}}(y_{i+1})-\epsilon>f_{2,\mathrm{num}}(y_i)+\epsilon\ge f_2(y_i)\ge f_2(y),$$

which is the desired inequality for the interval $[y_i,y_{i+1}]$.

2. Divide the interval $[0, R]$ in, say, 1000 subintervals $[R_k, R_{k+1}]$ (the number of subintervals will depend on d). For each of them do the following.

   2a) Find a bound $|F_1'(y)| \leq M_k$ ($\forall y \in [R_k, R_{k+1}]$) for the derivative of F_1. Details are given in Lemma 6.5.1.

   2b) Given $\epsilon > 0$ small, we can construct an $\epsilon-$numerical approximation $F_{1,\mathrm{Num}}$ of F_1 using, for instance, the Double Exponential Method for numerical integration described in §6.2 of the Appendix, in such a way that $|F_1(y) - F_{1,\mathrm{Num}}(y)| < \epsilon$, for all $y \in [R_k, R_{k+1}]$.

   2c) Set $y_0 = R_k$. Check that $F_1(y_0) > F_2(y_0)$ and suppose that $y_0, \ldots, y_i \in [R_k, R_{k+1}]$ have already been constructed. Choose $\epsilon > 0$ so that $F_2(y_i) + \epsilon < F_{1,\mathrm{Num}}(y_i)$.

   2d) Using the properties of the function F_2, we can prove that the point $y_{i+1} := y_i + \delta$, with

$$\delta := \frac{F_{1,\mathrm{Num}}(y_i) - F_2(y_i) - \epsilon}{\frac{F_2(y_i) - F_2(R_{k+1})}{y_i - R_{k+1}} + M_k} > 0, \tag{3.4}$$

   satisfies $F_1(y) > F_2(y)$, for all $y \in [y_i, y_{i+1}]$.

   2e) Set $i \leftarrow i + 1$ and repeat the last two steps until $y_i \geq R_{k+1}$ for some i.

The details of the normalization method in step 1 are given in §3.2 and the analysis of the parameters we need for numerical integration in step 2 are given in §6.2. The `PARI-GP` codes used for steps 1 and 2 can be seen in §6.6.5 of the Appendix.

3.2 The asymptotic corrections of f_1 and f_2.

In this section we describe a numerically efficient method to check the validity of the inequality $\left(-\frac{f'}{f}\right)'' = f_1 - f_2 > 0$ on the interval $[0, +\infty) \cap [L, R]$. More precisely, using the asymptotic expansion of f and its derivatives given in (2.14), we perform a normalization of the functions f_1 and f_2 such that a simple Newton-type algorithm applied to those new normalized functions is reasonably fast verifying $\left(-\frac{f'}{f}\right)'' > 0$ on $[0, +\infty) \cap [L, R]$ (see §3.2.1).

Recall $f_1 := 3\frac{f''f'}{f^2}$ and $f_2 := \frac{f'''}{f} + 2\left(\frac{f'}{f}\right)^3$, where $f = f_{(r_1,r_2)}$. Although we are mainly interested in the case $(r_1, r_2) = (d, 0)$, we work in general. Since, by log-concavity (see (6.5) in the Appendix), we know that $f > 0, f' < 0, f'' > 0, f''' < 0$, we obtain that $f_1, f_2 < 0$. Recall from (2.14) that the asymptotic part of order 2 for the function $f^{(t)}(y)$, $(t = 0, 1, 2, 3)$ when $y \to +\infty$, is given by

$$f_{\mathrm{a}}^{(t)}(y) := A_0(-1)^t(2\pi)^{d-1}d^{-\frac{r_1+r_2+1}{2}+2t}e^{-y\left(\frac{r_1+r_2-1}{2}-t\right)}e^{-d\exp(y)}\sum_{k=0}^{2}(-1)^k\tilde{A}_k\exp(-ky), \tag{3.5}$$

with $\tilde{A}_k = \tilde{A}_k(t)$ as in (2.15). Define the asymptotic parts $f_{1\mathrm{a}}$ and $f_{2\mathrm{a}}$ of f_1 and f_2, respectively, as the result of replacing each appearance of $f^{(t)}$ in f_1 and f_2 by its asymptotic approximation,

$$f_{1\mathrm{a}} := 3\frac{f_{\mathrm{a}}^{(2)}f_{\mathrm{a}}^{(1)}}{(f_{\mathrm{a}}^{(0)})^2}, \qquad f_{2\mathrm{a}} := \frac{f_{\mathrm{a}}^{(3)}}{f_{\mathrm{a}}^{(0)}} + 2\left(\frac{f_{\mathrm{a}}^{(1)}}{f_{\mathrm{a}}^{(0)}}\right)^3. \tag{3.6}$$

Using the fact that $f_1 < 0$ and $f_2 < 0$, and after verifying the same for $f_{1\mathrm{a}}$ and $f_{2\mathrm{a}}$, we get

$$\left(-\frac{f'}{f}\right)'' > 0 \iff f_1 > f_2 \iff \frac{f_2}{f_1} > 1 \iff \frac{\frac{f_2}{f_{2\mathrm{a}}}}{\frac{f_1}{f_{1\mathrm{a}}}} > \frac{f_{1\mathrm{a}}}{f_{2\mathrm{a}}} \iff \log\left(\frac{\frac{f_2}{f_{2\mathrm{a}}}}{\frac{f_1}{f_{1\mathrm{a}}}}\right) > \log\left(\frac{f_{1\mathrm{a}}}{f_{2\mathrm{a}}}\right)$$
$$\iff \log\left(\frac{f_2}{f_{2\mathrm{a}}}\right) - \log\left(\frac{f_1}{f_{1\mathrm{a}}}\right) > \log\left(\frac{f_{1\mathrm{a}}}{f_{2\mathrm{a}}}\right).$$

So, we consider the *logarithmically normalized* functions

$$f_{1,\mathrm{n}} := \log\left(\frac{f_1}{f_{1\mathrm{a}}}\right) \quad \text{and} \quad f_{2,\mathrm{n}} := \log\left(\frac{f_2}{f_{2\mathrm{a}}}\right).$$

Thus an equivalent way to verify that $f_1 > f_2$ over a compact interval, is to prove that the asymptotic corrections

$$F_1 := f_{2,\mathrm{n}} - f_{1,\mathrm{n}} = \log\left(\frac{f_2}{f_{2\mathrm{a}}}\right) - \log\left(\frac{f_1}{f_{1\mathrm{a}}}\right), \qquad F_2 := \log\left(\frac{f_{1\mathrm{a}}}{f_{2\mathrm{a}}}\right), \tag{3.7}$$

satisfy $F_1 > F_2$ over such interval. Note from (2.18) that $\widetilde{A}_k \in \mathbb{Q}$ for $k = 0, 1, 2$, and so $\frac{f_{1\mathrm{a}}}{f_{2\mathrm{a}}}(y)$ is a rational function of e^{-y} with rational coefficients. We can therefore prove the convexity of $F_2(y)$ by writing F_2'' as a quotient of rational polynomials in e^{-y}, and then verifying numerically that such rational function has no positive roots. For this we use the `PARI-GP` function `polrootsreal` which gives rigorous intervals bounding the location of real roots of polynomials with rational coefficients.

As Figure 3.1 and Table 3.1 show, F_1 and F_2 become reasonably separated where f_1 and f_2 are not, so one can readily prove $F_1 > F_2$ numerically. The details of the method are given in §3.2.1.

Table 3.1: Some values of the functions f_1 and f_2 in signature $(r_1, r_2) = (5, 3)$.

y	$\mathbf{f_1}$	$\mathbf{f_2}$	**y**	$\mathbf{f_1}$	$\mathbf{f_2}$
0.05	-9386.54	-9397.97	0.56	-33160.78	-33179.85
0.10	-10600.82	-10612.85	0.61	-37813.79	-37833.86
0.15	-11985.10	-11997.77	0.66	-43155.18	-43176.30
0.20	-13564.44	-13577.77	0.71	-49290.29	-49312.51
0.25	-15367.71	-15381.75	0.76	-56340.99	-56364.37
0.30	-17428.21	-17442.99	0.81	-64448.23	-64472.83
0.35	-19784.35	-19799.90	0.86	-73775.12	-73801.00
0.40	-22480.44	-22496.81	0.91	-84510.47	-84537.70
0.45	-25567.64	-25584.86	0.96	-96872.84	-96901.48
0.51	-29105.01	-29123.13	1.01	-111115.33	-111145.46

3.2.1 Newton-type subdivision method.

To check that $F_1 > F_2$ over a compact interval $[a, b]$, we will approximate F_1 from below by a line, and (the convex) F_2 from above by a secant line. Suppose we have a uniform bound for the derivative $|F_1'| \leq M$ over $[a, b]$. Then if we can check that $F_1(a) > F_2(a)$, we can ensure that

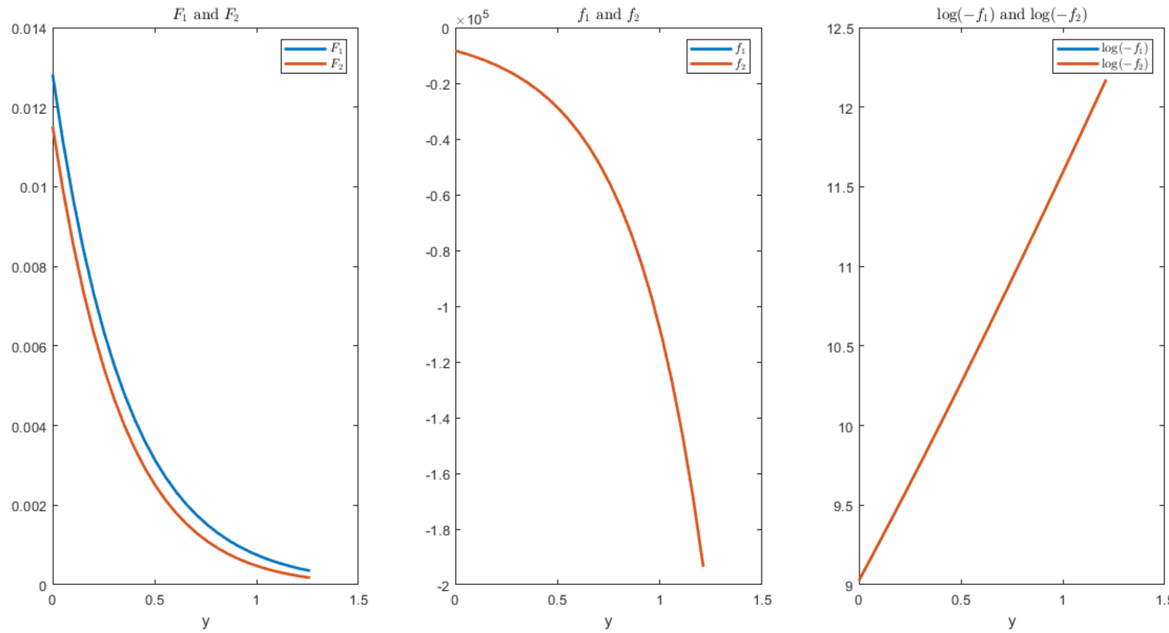


Figure 3.1: f_1 and f_2 (with the graph of f_2 completely covering that of f_1), along with their asymptotic corrections F_1 and F_2 for $(r_1, r_2) = (5, 3)$.

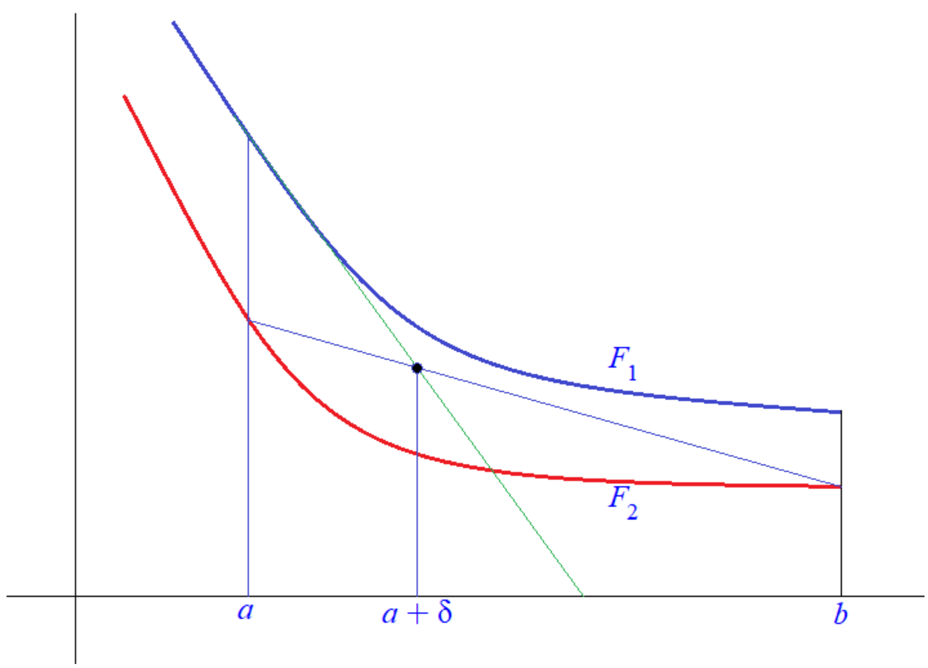


Figure 3.2: F_1, F_2 and the positivity subinterval $[a, a + \delta]$.

$F_1 > F_2$ over the subinterval $[a, a+\delta]$, where

$$\delta := \begin{cases} \min\Big(b-a,\ \frac{F_1(a)-F_2(a)}{\frac{F_2(a)-F_2(b)}{a-b}+M}\Big) & \text{if } \frac{F_2(a)-F_2(b)}{a-b}+M>0, \\ b-a & \text{otherwise.} \end{cases}$$

Geometrically δ is obtained intersecting the line through $\big(a, F_2(a)\big)$ and $\big(b, F_2(b)\big)$ with the lower estimate for the tangent line to F_1 given by $y \mapsto -M(y-a)+F_1(a)$ (see Figure 3.2). We can iterate this method replacing a by $a+\delta$, to check $F_1 > F_2$ over all of $[a,b]$. An explicit value of M is given in Lemma 6.5.1 in the Appendix.

3.3 Approximating $f^{(t)}$

In approximating $f^{(t)}$, it is illustrative to start with a trivial case. Namely, if $(r_1, r_2) = (1, 0)$, then $f_{(r_1,r_2)}(y) = \exp(-e^y)$. If we try to compute f using the power series for the exponential, we are very successful for $y < 0$ or even for small positive values of y, but we get a computational disaster already for moderately large y due to the oscillating character of the series.

The same phenomenon happens in general if we approximate $f^{(t)} = f^{(t)}_{(r_1,r_2)}(y)$ by the (convergent) power series expressing f as a sum of residues (obtained by shifting the vertical line of integration far to the left). Therefore, for positive values of y we compute $f^{(t)}(y)$ directly from its definition as an integral. To make this fast and accurate, we use Double Exponential numerical integration. Since this technique is not so widely known, we devote §6.2 of the Appendix to this method and its application to the computation of $f^{(t)}$. Here we will show how to use the truncated power series.

3.3.1 Computing $f^{(t)}$ using residues.

In this subsection we improve the approximation using the residue at $s = 0$ of §2.2. Since we need a stricter control over the error term, we use a variable number of residues. In practice, six residues will be enough for our purposes. For $z = e^{-dy}$ with $y \in \mathbb{R}$, let G_z be the meromorphic function defined in (2.23) by

$$G_z(s) = G_z(s; r_1, r_2, t) = s^t z^s \Gamma(s)^{r_1+r_2} \Gamma\left(s+\frac{1}{2}\right)^{r_2}. \tag{3.8}$$

We note that G_z has poles in the set $\mathcal{P} = \big\{s_j = \frac{1-j}{2}\ (j = 1, 2, \dots)\big\}$. For $k \in \mathbb{N}$ even, let γ be the complex contour shown in Figure 3.3. Denote by $\mathcal{P}_k$ the set of poles of the function G_z that are

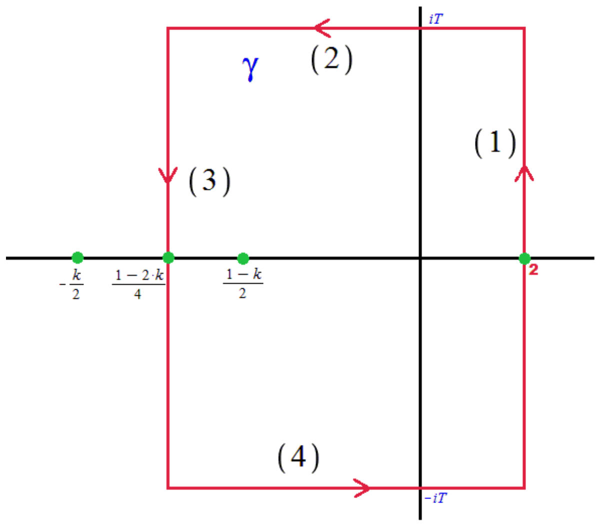


Figure 3.3: The contour γ in (3.9).

inside the curve γ. Then $\mathcal{P}_k = \left\{s_j = \frac{1-j}{2}\ (j = 1, \ldots, k)\right\}$. So by Cauchy's residue theorem

$$\sum_{j=1}^{k} \operatorname{Res}\left(G_z, \frac{1-j}{2}\right) = \frac{1}{2\pi i} \oint_\gamma G_z(s)\, ds \tag{3.9}$$

$$= \frac{1}{2\pi i}\Bigg[\overbrace{\int_{-T}^{T} G_z(2+iu) i\, du}^{\text{integral over line (1)}} + \overbrace{\int_{2}^{-\frac{2k-1}{4}} G_z(x+iT)\, dx}^{\text{integral over line (2)}} + \underbrace{\int_{T}^{-T} G_z\left(-\frac{2k-1}{4} + iu\right) i\, du}_{\text{integral over line (3)}} + \underbrace{\int_{-\frac{2k-1}{4}}^{2} G_z(x-iT)\, dx}_{\text{integral over line (4)}}\Bigg]. \tag{3.10}$$

Since $|G_z(x \pm iT)| \xrightarrow{T\to+\infty} 0$ exponentially fast, we have that

$$\lim_{T\to+\infty} \int_{-\frac{2k-1}{4}}^{2} G_z(x-iT)\, dx = 0 = \lim_{T\to+\infty} \int_{2}^{-\frac{2k-1}{4}} G_z(x+iT)\, dx.$$

Therefore, taking $\lim_{T\to+\infty}$ in (3.10), we get

$$\begin{aligned}\sum_{j=1}^{k} \operatorname{Res}\left(G_z, \frac{1-j}{2}\right) &= \frac{1}{2\pi i}\left[\int_{-\infty}^{\infty} G_z(2+iu) i\, du - \int_{-\infty}^{\infty} G_z\left(-\frac{2k-1}{4} + iu\right) i\, du\right] \\ &= \frac{1}{2\pi i}\left[\int_{2-i\infty}^{2+i\infty} G_z(s)\, ds - i\int_{-\infty}^{\infty} G_z\left(-\frac{2k-1}{4} + iu\right) du\right] \\ &= \frac{f^{(t)}(y)}{(-d)^t} - \frac{1}{2\pi}\int_{-\infty}^{\infty} G_z\left(-\frac{2k-1}{4} + iu\right) du =: \frac{f^{(t)}(y)}{(-d)^t} - \frac{E_k(y)}{(-d)^t}.\end{aligned} \tag{3.11}$$

Lemma 3.3.1. *Let $f^{(t)}(y)$ and $G_z(s)$ be as in* (2.2) *and* (3.8) *and let $k \in \mathbb{N}$ be even. Then*

$$E(y) = E_k(y) = f^{(t)}(y) - (-d)^t \sum_{j=1}^{k} \operatorname{Res}\big(G_z, \tfrac{1-j}{2}\big) \tag{3.12}$$

satisfies

$$|E(y)| \le d^t \exp\!\big(dy(2k-1)/4\big) \frac{C(k)}{2\pi} \frac{\sqrt{\pi} 4^{d-1} \Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \qquad (\forall y \in \mathbb{R}), \tag{3.13}$$

where $C(k) = O\big(k^{t-\frac{k}{2}(r_1+2r_2)}\big)$ as $k \to \infty$ is given by

$$C(k) := \frac{\Big(\Gamma\big(\frac{5}{4}\big)\Big)^{r_1+r_2} \Big(\Gamma\big(\frac{3}{4}\big)\Big)^{r_2}}{\left(\frac{\pi\sqrt{2}}{\Gamma\left(\frac{3}{4}\right)\left|\Gamma\left(\frac{1}{4}-\frac{k}{2}\right)\right|}\right)^{r_1+r_2-t} \left(\frac{\pi\sqrt{2}}{\Gamma\left(\frac{3}{4}\right)\left|\Gamma\left(\frac{5}{4}-\frac{k}{2}\right)\right|}\right)^{t} \left(\frac{4\Gamma\left(\frac{3}{4}\right)}{\left|\Gamma\left(\frac{3}{4}-\frac{k}{2}\right)\right|}\right)^{r_2}}. \tag{3.14}$$

Figure 3.4 compares $|E(y)|$ to the upper bound in the Lemma in a sample case.

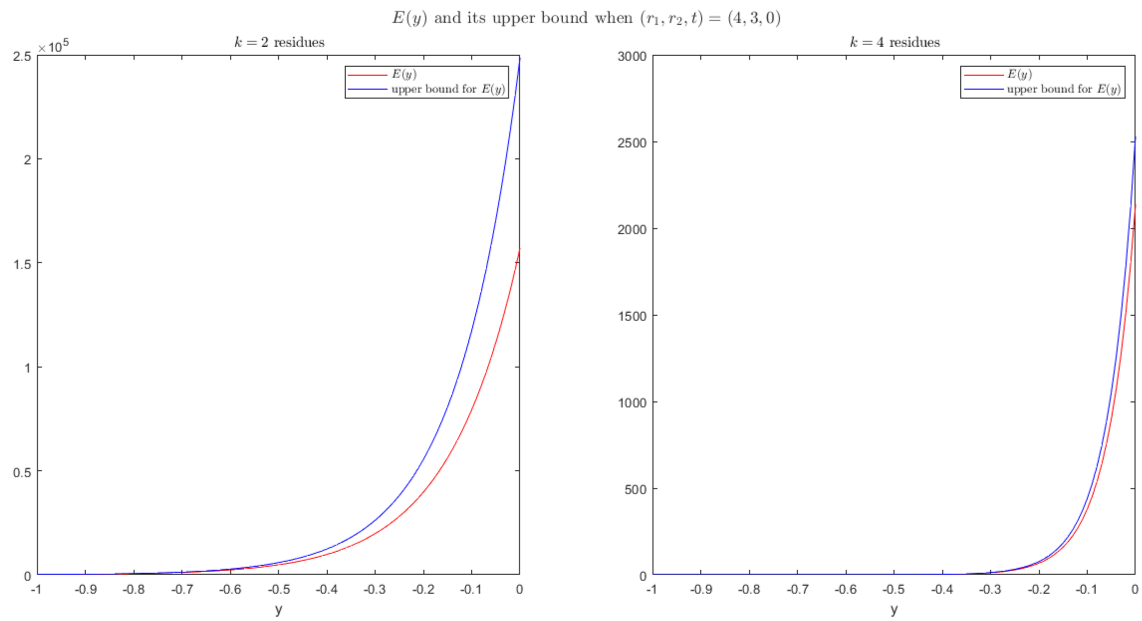


Figure 3.4: The error $|E(y)|$ and its upper bound (3.13) for $r_1 = 4$, $r_2 = 3$ and $t = 0$.

Proof. We first estimate $G(s) = G_z(s)$ along vertical lines. Abbreviating $u_k := -\frac{2k-1}{4} + iu$,

$$|G(u_k)| \le \left| e^{-dyu_k} \Gamma(u_k)^{r_1+r_2-t}\, \Gamma(u_k+1)^t\, \Gamma(u_k + \tfrac{1}{2})^{r_2} \right|. \tag{3.15}$$

Applying $k/2$ times the identity $w\Gamma(w) = \Gamma(w+1)$ to the Γ-factors in (3.15), we get

$$\Gamma(u_k) = \frac{\Gamma\left(\frac{5}{4}+iu\right)}{\prod_{l=0}^{k/2}\left(l-\left(\frac{2k-1}{4}\right)+iu\right)}, \qquad \Gamma(u_k+1) = \frac{\Gamma\left(\frac{5}{4}+iu\right)}{\prod_{l=0}^{k/2-1}\left(l+1-\left(\frac{2k-1}{4}\right)+iu\right)},$$
$$\Gamma(u_k+\tfrac{1}{2}) = \frac{\Gamma\left(\frac{3}{4}+iu\right)}{\prod_{l=0}^{k/2-1}\left(l+\frac{1}{2}-\left(\frac{2k-1}{4}\right)+iu\right)}. \tag{3.16}$$

Since $|\Gamma(x+iu)| \leq \Gamma(x)$ for $x > 0$ and $u \in \mathbb{R}$, we just need a lower bound for the elementary products in (3.16). Separating the last factor in the product we get

$$\left|\prod_{l=0}^{k/2}\left(l-\left(\tfrac{2k-1}{4}\right)+iu\right)\right|^2 = \prod_{l=0}^{k/2}\left(\left(l-\tfrac{k}{2}+\tfrac{1}{4}\right)^2+u^2\right) \geq \left(\tfrac{1}{16}+u^2\right)\prod_{l=0}^{k/2-1}\left(l-\tfrac{k}{2}+\tfrac{1}{4}\right)^2.$$

Writing the last product in terms of the Γ-function and using the reflection formula, we obtain

$$\left|\prod_{l=0}^{k/2}\left(l-\left(\frac{2k-1}{4}\right)+iu\right)\right| \geq \left(\frac{1}{16}+u^2\right)^{\frac{1}{2}}\left(\frac{\pi\sqrt{2}}{\Gamma\left(\frac{3}{4}\right)\left|\Gamma\left(\frac{1}{4}-\frac{k}{2}\right)\right|}\right). \tag{3.17}$$

Proceeding in analogous way for the other factors we get

$$\left|\prod_{l=0}^{k/2-1}\left(l+1-\left(\frac{2k-1}{4}\right)+iu\right)\right| \geq \left(\frac{1}{16}+u^2\right)^{\frac{1}{2}}\left(\frac{\pi\sqrt{2}}{\Gamma\left(\frac{3}{4}\right)\left|\Gamma\left(\frac{5}{4}-\frac{k}{2}\right)\right|}\right),$$
$$\left|\prod_{l=0}^{k/2-1}\left(l+\frac{1}{2}-\left(\frac{2k-1}{4}\right)+iu\right)\right| \geq \left(\frac{1}{16}+u^2\right)^{\frac{1}{2}}\left(\frac{4\Gamma\left(\frac{3}{4}\right)}{\left|\Gamma\left(\frac{3}{4}-\frac{k}{2}\right)\right|}\right). \tag{3.18}$$

Replacing (3.16)-(3.18) in (3.15) we get the estimate

$$\left|G\left(-\left(\frac{2k-1}{4}\right)+iu\right)\right| \leq \exp\left(dy\left(\frac{2k-1}{4}\right)\right)\frac{(\Gamma(5/4))^{r_1+r_2-t}}{\left(\left(\frac{1}{16}+u^2\right)^{\frac{1}{2}}\left(\frac{\pi\sqrt{2}}{\Gamma\left(\frac{3}{4}\right)\left|\Gamma\left(\frac{1}{4}-\frac{k}{2}\right)\right|}\right)\right)^{r_1+r_2-t}}$$
$$\cdot\frac{(\Gamma(5/4))^t}{\left(\left(\frac{1}{16}+u^2\right)^{\frac{1}{2}}\left(\frac{\pi\sqrt{2}}{\Gamma\left(\frac{3}{4}\right)\left|\Gamma\left(\frac{5}{4}-\frac{k}{2}\right)\right|}\right)\right)^t}\frac{(\Gamma(3/4))^{r_2}}{\left(\left(\frac{1}{16}+u^2\right)^{\frac{1}{2}}\left(\frac{4\Gamma\left(\frac{3}{4}\right)}{\left|\Gamma\left(\frac{3}{4}-\frac{k}{2}\right)\right|}\right)\right)^{r_2}}.$$

In other words,

$$\left|G\left(-\left(\frac{2k-1}{4}\right)+iu\right)\right| \leq \frac{\exp\left(dy\left(\frac{2k-1}{4}\right)\right)C(k)}{\left(\frac{1}{16}+u^2\right)^{\frac{r_1+2r_2}{2}}}, \tag{3.19}$$

with C_k as in (3.14). Note that $C(k) = O\left(k^{t-\frac{k}{2}(r_1+2r_2)}\right) \longrightarrow 0$, as $k \to \infty$, by Stirling's formula coupled with the reflection formula $\Gamma(w)\Gamma(1-w) = \pi/\sin(\pi w)$.

We can now turn to bounding $E = E_k$ in (3.12). From (3.11)

$$E(y) = \frac{(-d)^t}{2\pi} \int_{-\infty}^{\infty} G_z \left(-\frac{2k-1}{4} + iu \right) du,$$

and (3.19) we see that to bound $|E(y)|$ we need the classical integral [Nie06, page 158]

$$\int_{-\infty}^{\infty} \frac{dx}{(A^2+x^2)^M} = \frac{\sqrt{\pi} A^{1-2M} \Gamma\left(M - \frac{1}{2}\right)}{\Gamma(M)} \qquad (A > 0,\ M > \tfrac{1}{2}).$$

Applying this with $A = \frac{1}{4},\ M = \frac{r_1+2r_2}{2} = \frac{d}{2}$, we get the desired upper bound (3.13). $\square$

To evaluate $f^{(t)}(y)$ using Lemma 3.3.1, we need the following fast method for calculating residues, whose proof is found in §6.5.2 of the Appendix.

Proposition 3.3.2. *There exist explicitly calculable coefficients* $c_{p,i}$ $(1 \le p \le r_1 + r_2,\ 1 \le i \le k)$ *such that*

$$\sum_{i=1}^{k} \operatorname{Res}\left(G_z, \tfrac{1-i}{2}\right) = \sum_{i=1}^{k} z^{\frac{1-i}{2}} \sum_{p=1}^{r_1+r_2} c_{p,i} \frac{\left(\log(z)\right)^{p-1}}{(p-1)!}.$$

3.4 Approximating $-\left(\frac{f'}{f}\right)''$ from the approximation of $f^{(t)}$.

In the previous sections we have described how to approximate $f^{(t)}(y)$ by a computable $f_{\text{ap}}^{(t)}(y)$ within an error bounded by ϵ_t, that is

$$f^{(t)}(y) = f_{\text{ap}}^{(t)}(y) + E_t(y), \qquad |E_t(y)| < \epsilon_t \qquad (t = 0, 1, 2, 3)$$

where $f_{\text{ap}}^{(t)}(y)$ is given by (3.5) if $y \gg 0$ or for the blue part in (2.47) if $y \ll 0$. Elementary calculus shows that

$$\left(\frac{-f'}{f}\right)'' = \frac{3ff'f'' - 2(f')^3 - f^2 f'''}{f^3} = \Delta(f, f', f'', f'''),$$

where Δ is the rational function

$$\Delta(A, B, C, D) := \frac{3ABC - 2B^3 - A^2 D}{A^3}. \tag{3.20}$$

By bounding the propagation of errors in such a rational expression, we can control the accuracy of our calculation of $\left(\frac{-f'}{f}\right)''$.

We start with the following elementary lemma, where we interpret H_1 and H_0 as small perturbations of X and Y, respectively.

Lemma 3.4.1. *Suppose the real numbers* X, Y, H_1, H_0, M_0, M_1 *satisfy* $Y \neq 0$, $|M_0/Y| \le \frac{1}{2}$, $|H_1| \le M_1$ *and* $|H_0| \le M_0$*. Then*

$$\left| \frac{X + H_1}{Y + H_0} - \frac{X}{Y} \right| \le 2M_0 \frac{|X|}{Y^2} + 2M_1 \frac{1}{|Y|}.$$

The lemma implies a simple upper bound for the error committed on estimating f'/f by $f'_{\mathrm{ap}}/f_{\mathrm{ap}}$. Higher logarithmic derivatives are messier, as the following shows.

Lemma 3.4.2. *Suppose for $t = 0, 1, 2, 3$ we are given real numbers $f^{(t)}, f_{\mathrm{ap}}^{(t)}$ and $\epsilon_t > 0$, satisfying $|f^{(t)} - f_{\mathrm{ap}}^{(t)}| \le \epsilon_t$ and $\epsilon_0/|f_{\mathrm{ap}}^{(0)}| \le 1/\sqrt[3]{2}$. Then*

$$\left|\Delta(f^{(0)}, f^{(1)}, f^{(2)}, f^{(3)}) - \Delta(f_{\mathrm{ap}}^{(0)}, f_{\mathrm{ap}}^{(1)}, f_{\mathrm{ap}}^{(2)}, f_{\mathrm{ap}}^{(3)})\right| \le \frac{2M_1 + 2M_0\Delta(f_{\mathrm{ap}}^{(0)}, f_{\mathrm{ap}}^{(1)}, f_{\mathrm{ap}}^{(2)}, f_{\mathrm{ap}}^{(3)})}{\left(f_{\mathrm{ap}}^{(0)}\right)^3},$$

where

$$\begin{aligned} M_0 :=& 3(f_{\mathrm{ap}}^{(0)})^2\epsilon_0 + 3|f_{\mathrm{ap}}^{(0)}|\epsilon_0^2 + \epsilon_0^3, \\ M_1 :=& |f_{\mathrm{ap}}^{(3)}|\big(2|f_{\mathrm{ap}}^{(2)}|\epsilon_2 + \epsilon_2^2\big) + \epsilon_3\big(|f_{\mathrm{ap}}^{(2)}| + \epsilon_2\big)^2 + 2\big(3(f_{\mathrm{ap}}^{(1)})^2\epsilon_1 + 3|f_{\mathrm{ap}}^{(1)}|\epsilon_1^2 + \epsilon_1^3\big) \\ &+ 3\Big(|f_{\mathrm{ap}}^{(2)}|\big(|f_{\mathrm{ap}}^{(1)}|\epsilon_0 + |f_{\mathrm{ap}}^{(0)}|\epsilon_1 + \epsilon_0\epsilon_1\big) + \epsilon_2\big(|f_{\mathrm{ap}}^{(1)}| + \epsilon_1\big)\big(|f_{\mathrm{ap}}^{(0)}| + \epsilon_0\big)\Big) \end{aligned}$$

Proof. Write $f^{(t)} = f_{\mathrm{ap}}^{(t)} + E_t$, where $|E_t| \le \epsilon_t$, $t = 0, 1, 2, 3$. Then

$$\begin{aligned} \Delta(f^{(0)}, f^{(1)}, f^{(2)}, f^{(3)}) &= \Delta(f_{\mathrm{ap}}^{(0)} + E_0, f_{\mathrm{ap}}^{(1)} + E_1, f_{\mathrm{ap}}^{(2)} + E_2, f_{\mathrm{ap}}^{(3)} + E_3) \\ &= \frac{-f_{\mathrm{ap}}^{(3)}(f_{\mathrm{ap}}^{(2)})^2 + 3f_{\mathrm{ap}}^{(2)}f_{\mathrm{ap}}^{(1)}f_{\mathrm{ap}}^{(0)} - 2(f_{\mathrm{ap}}^{(1)})^3 + H_1}{(f_{\mathrm{ap}}^{(0)})^3 + H_0}, \end{aligned}$$

where

$$\begin{aligned} H_0 &:= 3(f_{\mathrm{ap}}^{(0)})^2E_0 + 3f_{\mathrm{ap}}^{(0)}E_0^2 + E_0^3, \\ H_1 &:= -f_{\mathrm{ap}}^{(3)}\left(2f_{\mathrm{ap}}^{(2)}E_2 + E_2^2\right) - E_3\left(f_{\mathrm{ap}}^{(2)} + E_2\right)^2 - 2\left(3(f_{\mathrm{ap}}^{(1)})^2E_1 + 3f_{\mathrm{ap}}^{(1)}E_1^2 + E_1^3\right) \\ &\quad + 3\Big(f_{\mathrm{ap}}^{(2)}\big(f_{\mathrm{ap}}^{(1)}E_0 + f_{\mathrm{ap}}^{(0)}E_1 + E_0E_1\big) + E_2\big(f_{\mathrm{ap}}^{(1)} + E_1\big)\big(f_{\mathrm{ap}}^{(0)} + E_0\big)\Big). \end{aligned}$$

Since $|E_t| \le \epsilon_t$, we have $|H_0| \le M_0$ and $|H_1| \le M_1$. Lemma 3.4.1 with

$$X := -f_{\mathrm{ap}}^{(3)}(f_{\mathrm{ap}}^{(2)})^2 + 3f_{\mathrm{ap}}^{(2)}f_{\mathrm{ap}}^{(1)}f_{\mathrm{ap}}^{(0)} - 2(f_{\mathrm{ap}}^{(1)})^3, \qquad Y := \left(f_{\mathrm{ap}}^{(0)}\right)^3,$$

concludes the proof. ∎

3.5 Numerical verification of convexity in $[y_*, y^*]$.

After programming in `PARI/GP` the algorithms described in pages 28 and 28 we get a finite (but long) sequence of points $\{y_i\}$ satisfying (3.3), ensuring the convexity of $-f'_{(d,0)}/f_{(d,0)}$ on $[y_*, y^*]$ for $4 \le d \le 40$.

In the case of the algorithm for y_i negative, the corresponding numerical output is summarized in Table 3.2. As Figure 3.5 shows, the length of the sequence and the corresponding time to perform the algorithm increase nonlinearly with the degree d, making it impractical to apply this algorithm for large degrees, say for $d \ge 50$.[3]

[3] For all the numerical calculations in this thesis we used PARI/GP on a Linux Ubuntu 20.04 PC platform with

Something similar happens for y_i positive. The `PARI/GP` program for the algorithm on page 28 gives the numerical parameters shown in Table 3.3. As can be seen in Figure 3.6, the number n of terms required by the Double Exponential method to evaluate $F_{1,\mathrm{Num}}$ increases linearly with the degree d. However, since the precision needed to perform the computations also increases linearly with d, the final time required by the method increases in a nonlinear way.

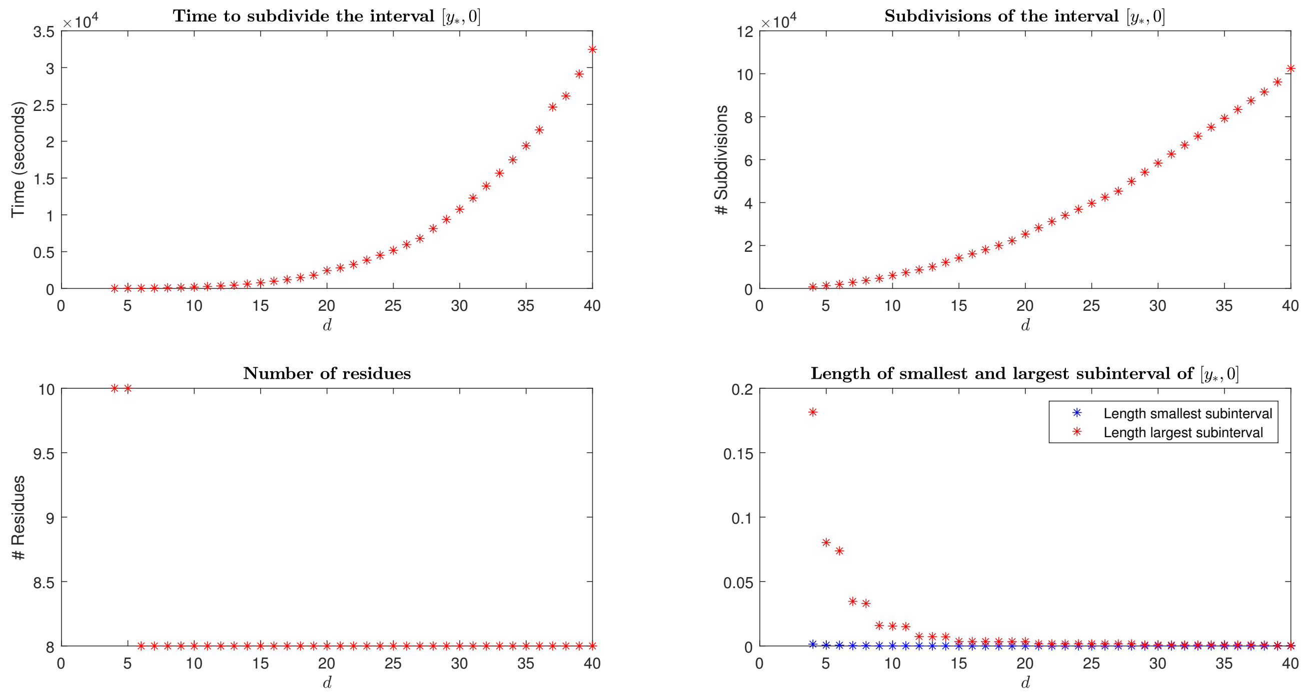


Figure 3.5: Time, number of subdivisions, Number of residues and length of shortest/largest subinterval in $[y_*, 0]$ as functions of the degree d.

an Intel Core i5-9400F CPU and 8 GB of RAM.

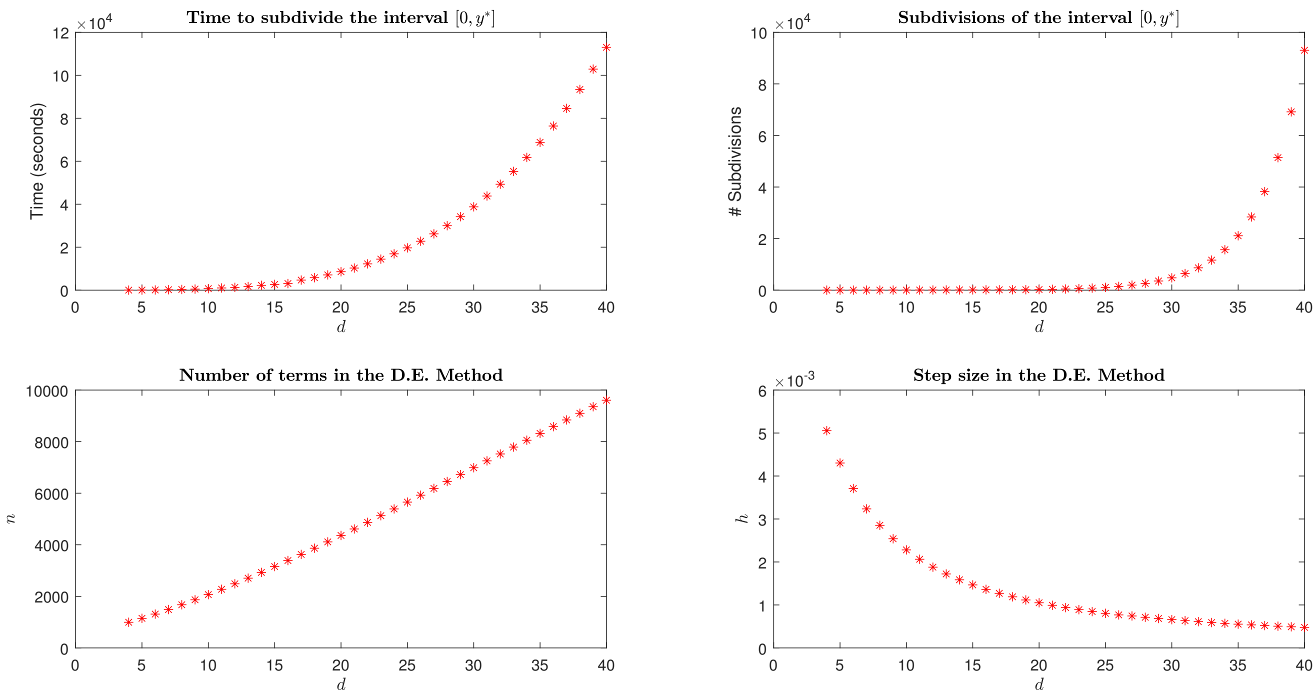


Figure 3.6: Time, number of subdivisions in $[0, y^*]$ and values of n, h as functions of the degree d.

d	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
Number of subdivisions	$*$	744	1326	1895	2814	3700	4696	6045	7406	8702	10095
Time (seconds)	$*$	12	24	36	59	89	129	189	263	346	448
Length shortest subint.	$*$	$1.4\ 10^{-3}$	$6.2\ 10^{-4}$	$5.7\ 10^{-4}$	$2.7\ 10^{-4}$	$2.5\ 10^{-4}$	$1.2\ 10^{-4}$	$1.2\ 10^{-4}$	$1.1\ 10^{-4}$	$1.1\ 10^{-4}$	$5.6\ 10^{-5}$
Length largest subint.	$*$	0.1815	0.0803	0.0738	0.0346	0.0330	0.0158	0.0154	0.0150	0.0073	0.0072

d	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**
Number of subdivisions	12155	14211	16142	18068	19992	22241	25330	28297	31163	34021	36860
Time (seconds)	600	781	987	1201	1478	1797	2431	2802	3256	3851	4513
Length shortest subint.	$5.5\ 10^{-5}$	$5.4\ 10^{-5}$	$5.3\ 10^{-5}$	$5.3\ 10^{-5}$	$5.2\ 10^{-5}$	$2.6\ 10^{-5}$	$2.5\ 10^{-5}$	$2.5\ 10^{-5}$	$2.5\ 10^{-5}$	$2.5\ 10^{-5}$	$2.5\ 10^{-5}$
Length largest subint.	0.0070	0.0034	0.0034	0.0034	0.0033	0.0033	0.0033	0.0016	0.0016	0.0016	0.0016

d	**25**	**26**	**27**	**28**	**29**	**30**	**31**	**32**	**33**	**34**	**35**
Number of subdivisions	39690	42512	45330	49839	54163	58396	62616	66804	70966	75108	79234
Time (seconds)	5179	5958	6784	8133	9371	10752	12288	13914	15666	17483	19387
Length shortest subint.	$2.5\ 10^{-5}$	$2.5\ 10^{-5}$	$2.4\ 10^{-5}$	$1.2\ 10^{-5}$	$1.2\ 10^{-5}$	$1.2\ 10^{-5}$	$1.2\ 10^{-5}$	$1.2\ 10^{-5}$	$1.2\ 10^{-5}$	$1.2\ 10^{-5}$	$1.2\ 10^{-5}$
Length largest subint.	0.0016	0.0016	0.0015	0.0015	0.0007	0.0007	0.0007	0.0007	0.0007	0.0007	0.0007

d	**36**	**37**	**38**	**39**	**40**						
Number of subdivisions	83346	87442	91534	96175	102477						
Time (seconds)	21545	24644	26150	29131	32472						
Length shortest subint.	$1.2\ 10^{-5}$	$1.2\ 10^{-5}$	$1.2\ 10^{-5}$	$5.9\ 10^{-6}$	$5.9\ 10^{-6}$						
Length largest subint.	0.0007	0.0007	0.0007	0.0003	0.0003						

Table 3.2: Overview of the subdivision task on $[y_*, 0]$.

d	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
n	$*$	995	1147	1312	1488	1674	1867	2067	2274	2487	2705
Number of subdivisions	$*$	2	4	7	13	19	27	36	49	60	76
Time (seconds)	$*$	19	43	95	205	327	504	708	1023	1291	1694
Length shortest subint.	$*$	0.472	0.232	0.125	0.062	0.031	0.031	0.015	0.015	0.015	0.007
Length largest subint.	$*$	0.472	0.500	0.500	1.000	0.990	0.980	0.970	0.960	0.950	0.941

d	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**
n	2928	3156	3389	3626	3867	4112	4361	4613	4869	5128	5390
Number of subdivisions	95	110	128	148	166	174	201	200	225	237	249
Time (seconds)	2261	2689	3084	3656	4199	4418	5183	5219	5959	6367	6780
Length shortest subint.	0.007	0.007	0.003	0.003	0.003	0.003	0.003	0.002	0.001	0.001	0.001
Length largest subint.	0.932	0.922	0.913	0.904	0.895	0.886	0.877	0.868	0.860	0.851	0.842

d	**25**	**26**	**27**	**28**	**29**	**30**	**31**	**32**	**33**	**34**	**35**
n	5655	5923	6186	6453	6720	6988	7255	7522	7789	8054	8318
Number of subdivisions	256	271	275	279	280	285	290	304	310	330	347
Time (seconds)	6997	7473	7698	7923	8090	8194	8230	8290	8570	8890	9070
Length shortest subint.	0.001953	0.001953	$9.76\ 10^{-4}$	$4.8\ 10^{-4}$	$2.4\ 10^{-4}$	$1.2\ 10^{-4}$	$6.10\ 10^{-5}$	$3.05\ 10^{-5}$	$1.52\ 10^{-5}$	$7.6\ 10^{-6}$	$3.8\ 10^{-6}$
Length largest subint.	0.834	0.826	0.743	0.669	0.602	0.542	0.487	0.439	0.395	0.355	0.320

d	**36**	**37**	**38**	**39**	**40**						
n	8581	8841	9099	9354	9606						
Number of subdivisions	355	360	377	398	430						
Time (seconds)	9169	9669	10059	10334	10488						
Length shortest subint.	$1.9\ 10^{-6}$	$9.5\ 10^{-7}$	$4.7\ 10^{-7}$	$2.3\ 10^{-7}$	$1.1\ 10^{-7}$						
Length largest subint.	0.288	0.259	0.233	0.210	0.189						

Table 3.3: Overview of the subdivision task on $[0, y^*]$.

Chapter 4

Behavior of $-\frac{f'_{(r_1,r_2)}}{f_{(r_1,r_2)}}$ as (r_1, r_2) varies

In this chapter we prove Steps 1 and 2 described on page 8. Namely, for $d = r_1 + 2r_2 \leq 40$, we prove the following inequalities for all $y \in \mathbb{R}$.

$$-\frac{f'_{(r_1,r_2)}(y)}{f_{(r_1,r_2)}(y)} \leq -\frac{f'_{(r_1+2,r_2-1)}(y)}{f_{(r_1+2,r_2-1)}(y)} \qquad (r_1 \geq 0,\ r_2 \geq 1), \tag{4.1}$$

$$-\frac{f'_{(0,(d-1)/2)}(y)}{f_{(0,(d-1)/2)}(y)} \leq -\frac{f'_{(1,(d-1)/2)}(y)}{f_{(1,(d-1)/2)}(y)} \qquad (d \text{ odd}), \tag{4.2}$$

$$-\frac{f'_{(0,d)}(y)}{f_{(0,d)}(y)} \geq -2\frac{f'_{(0,d/2)}(y)}{f_{(0,d/2)}(y)} \qquad (d \text{ even}), \tag{4.3}$$

$$-\frac{f'_{(0,d)}(y)}{f_{(0,d)}(y)} \geq -2\frac{f'_{(0,(d-1)/2)}(y)}{f_{(0,(d-1)/2)}(y)} \qquad (d \text{ odd}). \tag{4.4}$$

For empirical evidence of inequalities (4.1)-(4.4) in degree 20 and 21, see Figures 4.2 and 4.1. As Figure 4.2 suggests, the numerical part of the proof of (4.3) will be the most delicate of the four.

We will proceed as in Chapters 2 and 3, where we showed $\left(-\frac{f'}{f}\right)'' > 0$. Namely, we will use the asymptotic expansions of $f^{(t)}_{(r_1,r_2)}(y)$ as $y \to \pm\infty$ to first find a compact interval $[L, R] \subseteq \mathbb{R}$ such that (4.1)-(4.4) hold for $y \notin [L, R]$. Then we shall use a subdivision argument for $y \in [L, R]$. This last step will be a lot easier than for $\left(-\frac{f'}{f}\right)''$ because $y \mapsto -\frac{f'_{(r_1,r_2)}(y)}{f_{(r_1,r_2)}(y)}$ is known to be increasing by the theory of log-concavity (see (6.5) in the Appendix).

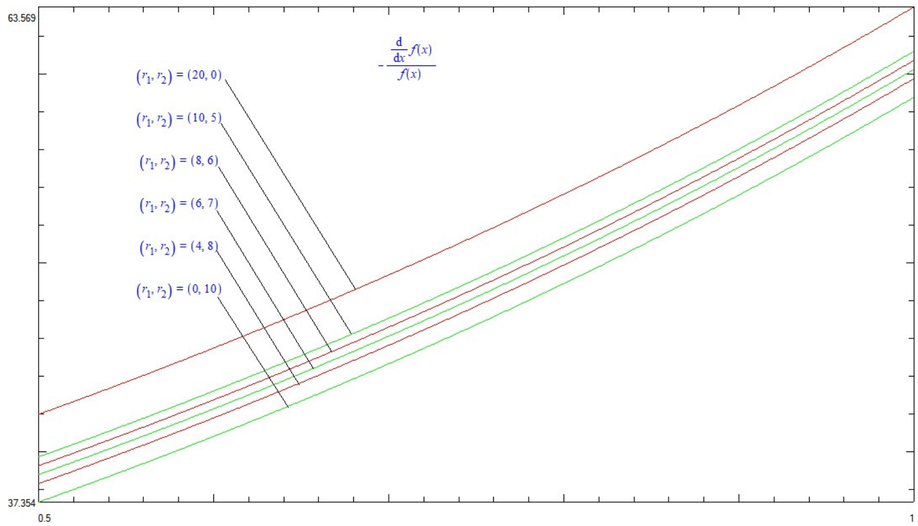


Figure 4.1: Graphs of $-\dfrac{f'_{(r_1,r_2)}(y)}{f_{(r_1,r_2)}(y)}$ for several signatures (r_1, r_2) with degree $d = r_1 + 2r_2 = 20$.

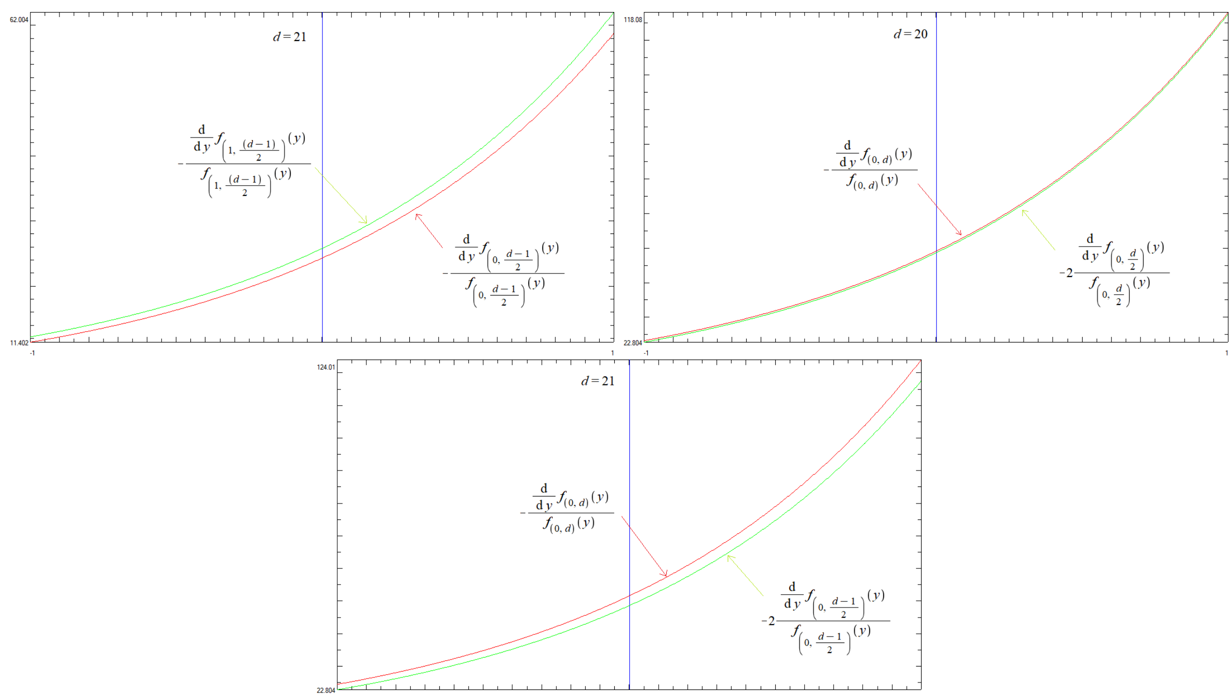


Figure 4.2: Graphs of the functions appearing in (4.2), (4.3) and (4.4) for $d = 20$ and $d = 21$.

4.1 Asymptotics: Numerical verification of inequalities (4.1)-(4.4) for $|y| \gg 0$.

We start by verifying that inequalities (4.1), (4.2), (4.3) and (4.4) hold for $y \to \pm\infty$. From formulas (2.14) (with $N = 2$) and (2.16) in Chapter 2, we know that for $y \to +\infty$:

$$f^{(t)}(y) = \kappa_t e^{-y\left(\frac{r_1+r_2+1}{2}-t-1\right)} e^{-de^y}\left(\widetilde{A_0}(t) - \widetilde{A_1}(t)e^{-y} + \widetilde{A_2}(t)e^{-2y} + F_t(y)e^{-3y}\right), \tag{4.5}$$

where $\kappa_t = (-1)^t A_0(t) d(2\pi)^{d-1} d^{-\frac{r_1+r_2+1}{2}+2t-1}$, $|F_t(y)| \le M_t$ are bounded and $\widetilde{A_k}(t)$ are explicit rational constants depending only on r_1, r_2 and t. Since we later use the PARI program `polrootsreal`, we round M_t up to a rational number in applications. For $y \to -\infty$ we have

$$f^{(t)}(y) = \frac{(-d)^t \pi^{r_2/2}(-dy)^{r_1+r_2-1-t}}{(r_1+r_2-1-t)!}\Big(1 + \frac{(r_1+r_2-1-t)!}{\pi^{r_2/2}} \sum_{j=1+t}^{r_1+r_2-1} e_j \frac{(-dy)^{j-r_1-r_2}}{(j-1-t)!} + H_t(y)\Big), \tag{4.6}$$

where by (2.50) and (2.47),

$$|H_t(y)| \le 2.8905 \cdot \frac{(r_1+r_2-1-t)! \cdot (17.7716)^d}{\pi^{r_2/2}|dy|^{r_1+r_2-1-t}} e^{\frac{dy}{4}}, \tag{4.7}$$

and the e_j are constants depending only on r_1 and r_2 (given explicitly in the proof of Lemma 2.2.1).

We will use the asymptotic expansions to convert each inequality (4.1), (4.2), (4.3) and (4.4) into a polynomial inequality. That will give us a compact interval $[L, R]$ outside of which the corresponding inequality holds.

To find R so that (4.1) holds for $y > R$, we use the asymptotic expansion (4.5) for both (r_1, r_2) and $(r_1 + 2, r_2 - 1)$. Substituting them in (4.1) we get the equivalent inequality

$$\frac{\widetilde{A_0}(1) - \widetilde{A_1}(1)e^{-y} + \widetilde{A_2}(1)e^{-2y} + F_1(y)e^{-3y}}{\widetilde{A_0}(0) - \widetilde{A_1}(0)e^{-y} + \widetilde{A_2}(0)e^{-2y} + F_0(y)e^{-3y}} \le \frac{\widetilde{B_0}(1) - \widetilde{B_1}(1)e^{-y} + \widetilde{B_2}(1)e^{-2y} + G_1(y)e^{-3y}}{\widetilde{B_0}(0) - \widetilde{B_1}(0)e^{-y} + \widetilde{B_2}(0)e^{-2y} + G_0(y)e^{-3y}}, \tag{4.8}$$

where the constants $\widetilde{A_k}$ and the error functions $F_0(y), F_1(y)$ are associated with the signature (r_1, r_2), while $\widetilde{B_k}$ and $G_0(y), G_1(y)$ correspond to $(r_1 + 2, r_2 - 1)$.

Using $f > 0$ we see that the denominators in (4.8) are positive, therefore we can cross multiply and rearrange terms to get that (4.8) is equivalent to

$$\begin{aligned}
&\Big(\sum_{k=0}^{2}(-1)^k\widetilde{A_k}(1)e^{-ky}\Big)\Big(\sum_{k=0}^{2}(-1)^k\widetilde{B_k}(0)e^{-ky}\Big) - \Big(\sum_{k=0}^{2}(-1)^k\widetilde{B_k}(1)e^{-ky}\Big)\Big(\sum_{k=0}^{2}(-1)^k\widetilde{A_k}(0)e^{-ky}\Big)\\
&\quad\le G_1(y)\Big(\sum_{k=0}^{2}(-1)^k\widetilde{A_k}(0)e^{-ky}e^{-3y} + F_0(y)e^{-6y}\Big) + F_0(y)\sum_{k=0}^{2}(-1)^k\widetilde{B_k}(1)e^{-ky}e^{-3y}\\
&\qquad - F_1(y)\Big(\sum_{k=0}^{2}(-1)^k\widetilde{B_k}(0)e^{-ky}e^{-3y} + G_0(y)e^{-6y}\Big) - G_0(y)\sum_{k=0}^{2}(-1)^k\widetilde{A_k}(1)e^{-ky}e^{-3y}.
\end{aligned}$$

Since $|F_t(y)| \le M_t$ and $|G_t(y)| \le N_t$ are bounded functions, and for $y \gg 0$ the functions $\widetilde{A_0}(t) -$

$\widetilde{A_1}(t)e^{-y} + \widetilde{A_2}(t)e^{-2y}$ and $\widetilde{B_0}(t) - \widetilde{B_1}(t)e^{-y} + \widetilde{B_2}(t)e^{-2y}$ are positive, we see that to ensure the validity of (4.1) for $y \geq R$, it is enough that for all $y \geq R$ we have

$$\left(\sum_{k=0}^{2}(-1)^k\widetilde{A_k}(1)e^{-ky}\right)\left(\sum_{k=0}^{2}(-1)^k\widetilde{B_k}(0)e^{-ky}\right) - \left(\sum_{k=0}^{2}(-1)^k\widetilde{B_k}(1)e^{-ky}\right)\left(\sum_{k=0}^{2}(-1)^k\widetilde{A_k}(0)e^{-ky}\right) \leq$$
$$- N_1\left(\sum_{k=0}^{2}|\widetilde{A_k}(0)|e^{-(k+3)y} + M_0e^{-6y}\right) - M_0\sum_{k=0}^{2}|\widetilde{B_k}(1)|e^{-(k+3)y}$$
$$- M_1\left(\sum_{k=0}^{2}|\widetilde{B_k}(0)|e^{-(k+3)y} + N_0e^{-6y}\right) - N_0\sum_{k=0}^{2}|\widetilde{A_k}(1)|e^{-(k+3)y}.$$

Making the change of variable $x = e^{-y}$ we convert this inequality into a polynomial inequality in the variable $x > 0$. The inequality can be solved numerically using the command `polrootsreal` of the `PARI-GP` program.

Likewise, to obtain L so that (4.1) holds for $y < L < 0$, we use the asymptotic expansion (4.6) for both (r_1, r_2) and $(r_1 + 2, r_2 - 1)$. Replacing them in (4.1) we get the equivalent inequality

$$\frac{1 - r_1 - r_2}{y} \cdot \frac{1 + \frac{(r_1+r_2-1-1)!}{\pi^{r_2/2}}\sum_{j=2}^{r_1+r_2-1} e_j\frac{(-dy)^{j-r_1-r_2}}{(j-1-1)!} + H_1(y)}{1 + \frac{(r_1+r_2-1)!}{\pi^{r_2/2}}\sum_{j=1}^{r_1+r_2-1} e_j\frac{(-dy)^{j-r_1-r_2}}{(j-1)!} + H_0(y)}$$
$$\leq \frac{1 - r_1 - r_2 - 1}{y} \cdot \frac{1 + \frac{(r_1+r_2+1-1-1)!}{\pi^{(r_2-1)/2}}\sum_{j=2}^{r_1+r_2+1-1} e'_j\frac{(-dy)^{j-r_1-r_2-1}}{(j-1-1)!} + I_1(y)}{1 + \frac{(r_1+r_2+1-1)!}{\pi^{(r_2-1)/2}}\sum_{j=1}^{r_1+r_2+1-1} e'_j\frac{(-dy)^{j-r_1-r_2-1}}{(j-1)!} + I_0(y)},$$

where the constants e_j and error functions $H_0(y), H_1(y)$ are associated with (r_1, r_2), whereas e'_j and $I_0(y), I_1(y)$ are associated with $(r_1 + 2, r_2 - 1)$. Again by log-concavity, the sums involving H_t and I_t are positive. After cross multiplying (remembering that now $y < 0$), the inequality above becomes

$$\frac{r_1 + r_2 - 1}{r_1 + r_2}\big(1 + S_{A,1}(y) + H_1(y)\big)\big(1 + S_{B,0}(y) + I_0(y)\big) \leq \big(1 + S_{A,0}(y) + H_0(y)\big)\big(1 + S_{B,1}(y) + I_1(y)\big), \quad (4.9)$$

where the S are the polynomials in $\frac{1}{y}$

$$S_{A,0}(y) := \frac{(r_1 + r_2 - 1)!}{\pi^{r_2/2}}\sum_{j=1}^{r_1+r_2-1} e_j\frac{(-dy)^{j-r_1-r_2}}{(j-1)!},$$
$$S_{A,1}(y) := \frac{(r_1 + r_2 - 2)!}{\pi^{r_2/2}}\sum_{j=2}^{r_1+r_2-1} e_j\frac{(-dy)^{j-r_1-r_2}}{(j-2)!},$$
$$S_{B,0}(y) := \frac{(r_1 + r_2)!}{\pi^{(r_2-1)/2}}\sum_{j=1}^{r_1+r_2} e'_j\frac{(-dy)^{j-r_1-r_2-1}}{(j-1)!},$$
$$S_{B,1}(y) := \frac{(r_1 + r_2 - 1)!}{\pi^{(r_2-1)/2}}\sum_{j=2}^{r_1+r_2} e'_j\frac{(-dy)^{j-r_1-r_2-1}}{(j-2)!}.$$

From (2.50) and (2.46), we get the following error bounds, valid for $y < 0$

$$
\begin{aligned}
|H_0(y)| &\leq 2.8905 \cdot \frac{(r_1+r_2-1)! \cdot (17.7716)^d}{\pi^{r_2/2}|dy|^{r_1+r_2-1}} e^{\frac{dy}{4}} \leq 2.8905 \cdot \frac{(r_1+r_2-1)! \cdot (17.7716)^d}{\pi^{r_2/2}(-dy)^{r_1+r_2-1}} =: w_0(y) \\
|H_1(y)| &\leq 2.8905 \cdot \frac{(r_1+r_2-2)! \cdot (17.7716)^d}{\pi^{r_2/2}|dy|^{r_1+r_2-2}} e^{\frac{dy}{4}} \leq 2.8905 \cdot \frac{(r_1+r_2-2)! \cdot (17.7716)^d}{\pi^{r_2/2}(-dy)^{r_1+r_2-2}} =: w_1(y) \\
|I_0(y)| &\leq 2.8905 \cdot \frac{(r_1+r_2)! \cdot (17.7716)^d}{\pi^{(r_2-1)/2}|dy|^{r_1+r_2}} e^{\frac{dy}{4}} \leq 2.8905 \cdot \frac{(r_1+r_2)! \cdot (17.7716)^d}{\pi^{(r_2-1)/2}(-dy)^{r_1+r_2}} =: i_0(y) \\
|I_1(y)| &\leq 2.8905 \cdot \frac{(r_1+r_2-1)! \cdot (17.7716)^d}{\pi^{(r_2-1)/2}|dy|^{r_1+r_2-1}} e^{\frac{dy}{4}} \leq 2.8905 \cdot \frac{(r_1+r_2-1)! \cdot (17.7716)^d}{\pi^{(r_2-1)/2}(-dy)^{r_1+r_2-1}} =: i_1(y).
\end{aligned}
$$

Expanding inequality (4.9) and grouping all the error terms on the right we obtain

$$
\begin{aligned}
&q\big(1+S_{A,1}(y)\big)\big(1+S_{B,0}(y)\big) - \big(1+S_{A,0}(y)\big)\big(1+S_{B,1}(y)\big) \leq -q\big(1+S_{A,1}(y)\big)I_0(y) \\
&\quad - q\big(1+S_{B,0}(y)\big)H_1(y) - qH_1(y)I_0(y) + \big(1+S_{A,0}(y)\big)I_1(y) + \big(1+S_{B,1}(y)\big)H_0(y) + H_0(y)I_1(y),
\end{aligned} \tag{4.10}
$$

where $q = \frac{r_1+r_2-1}{r_1+r_2}$. Using the bounds for the error terms, the triangle inequality and the fact that the functions $S_{A,j}, S_{B,j}$ are positive, we see that to ensure the validity of (4.1) for $y \leq L$, it is enough that for all $y \leq L$ we have

$$
\begin{aligned}
&q\left(1+S_{A,1}(y)\right)\left(1+S_{B,0}(y)\right) - \left(1+S_{A,0}(y)\right)\left(1+S_{B,1}(y)\right) \leq \\
&\qquad - q\left(1+S_{A,1}^{\text{abs}}(y)\right)i_0(y) - q\left(1+S_{B,0}^{\text{abs}}(y)\right)w_1(y) - qw_1(y)i_0(y) \\
&\qquad\qquad - \left(1+S_{A,0}^{\text{abs}}(y)\right)i_1(y) - \left(1+S_{B,1}^{\text{abs}}(y)\right)w_0(y) - w_0(y)i_1(y),
\end{aligned} \tag{4.11}
$$

where

$$
\begin{aligned}
S_{A,0}^{\text{abs}}(y) &:= \frac{(r_1+r_2-1)!}{\pi^{r_2/2}} \sum_{j=1}^{r_1+r_2-1} |e_j| \frac{(-dy)^{j-r_1-r_2}}{(j-1)!}, \\
S_{A,1}^{\text{abs}}(y) &:= \frac{(r_1+r_2-2)!}{\pi^{r_2/2}} \sum_{j=2}^{r_1+r_2-1} |e_j| \frac{(-dy)^{j-r_1-r_2}}{(j-2)!}, \\
S_{B,0}^{\text{abs}}(y) &:= \frac{(r_1+r_2)!}{\pi^{(r_2-1)/2}} \sum_{j=1}^{r_1+r_2} |e'_j| \frac{(-dy)^{j-r_1-r_2-1}}{(j-1)!}, \\
S_{B,1}^{\text{abs}}(y) &:= \frac{(r_1+r_2-1)!}{\pi^{(r_2-1)/2}} \sum_{j=2}^{r_1+r_2} |e'_j| \frac{(-dy)^{j-r_1-r_2-1}}{(j-2)!}.
\end{aligned}
$$

As before, (4.11) is a polynomial inequality (this time in the variable $x = \frac{1}{y}$). It can again be solved numerically using the command `polrootsreal` of the `PARI-GP` program. Table 4.1 shows the intervals $[L, R]$ found for inequality (4.1) using the above estimates.

Inequalities (4.2), (4.3) and (4.4) are handled just as we have done for inequality (4.1). The intervals $[L, R]$ found for inequalities (4.2), (4.3) and (4.4) are shown in Tables 4.1 and 4.2.

r_1 \ r_2	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
0	**	[-11.91,-0.85]	[-9.52,-0.72]	[-8.39,-0.59]	[-7.73,-0.46]	[-7.31,-0.33]	[-7.01,-0.20]	[-6.78,-0.08]	[-6.61,0.03]	[-6.47,0.13]
1	[-12.70,-0.44]	[-9.63,-0.47]	[-8.36,-0.39]	[-7.67,-0.30]	[-7.23,-0.21]	[-6.93,-0.11]	[-6.71,-0.01]	[-6.54,0.08]	[-6.40,0.17]	[-6.29, 0.25]
2	[-9.79,-0.15]	[-8.33,-0.16]	[-7.60,-0.11]	[-7.15,-0.04]	[-6.85,0.02]	[-6.63,0.10]	[-6.46,0.17]	[-6.33,0.25]	[-6.23,0.33]	[-6.14,0.40]
3	[-8.29,0.06]	[-7.51,0.07]	[-7.06,0.11]	[-6.76,0.16]	[-6.54,0.22]	[-6.38,0.28]	[-6.26,0.34]	[-6.16,0.41]	[-6.07,0.47]	[-6.01,0.53]
4	[-7.4,0.26]	[-6.94,0.27]	[-6.66,0.3]	[-6.45,0.34]	[-6.3,0.4]	[-6.18,0.44]	[-6.08,0.49]	[-6,0.55]	[-5.94,0.6]	[-5.88,0.66]
5	[-6.80,0.41]	[-6.53,0.43]	[-6.34,0.46]	[-6.20,0.49]	[-6.09,0.54]	[-6.01,0.58]	[-5.93,0.63]	[-5.87,0.68]	[-5.82, 0.72]	[-5.78, 0.77]
6	[-6.38,0.56]	[-6.21,0.57]	[-6.09,0.60]	[-5.99,0.63]	[-5.91,0.67]	[-5.85,0.71]	[-5.80,0.75]	[-5.75,0.79]	[-5.72,0.84]	[-5.68,0.88]
7	[-6.06, 0.69]	[-5.96, 0.69]	[-5.88,0.72]	[-5.82,0.75]	[-5.77,0.79]	[-5.72,0.82]	[-5.68,0.86]	[-5.65,0.90]	[-5.62,0.94]	[-5.60,0.98]
8	[-5.82,0.80]	[-5.76,0.80]	[-5.71,0.83]	[-5.67,0.86]	[-5.64,0.89]	[-5.61,0.93]	[-5.58,0.96]	[-5.56,1.00]	[-5.54,1.03]	[-5.52,1.07]
9	[-5.62,0.91]	[-5.59,0.90]	[-5.57,0.93]	[-5.55,0.96]	[-5.52,0.99]	[-5.51,1.03]	[-5.49,1.05]	[-5.47,1.09]	[-5.46,1.12]	[-5.44,1.15]
10	[-5.46,1.01]	[-5.45,0.99]	[-5.45,1.02]	[-5.44,1.05]	[-5.43,1.08]	[-5.42, 1.11]	[-5.41,1.14]	[-5.40,1.17]	[-5.39, 1.20]	[-5.38,1.23]
11	[-5.33,1.09]	[-5.34, 1.06]	[-5.34,1.10]	[-5.34,1.13]	[-5.34,1.16]	[-5.33,1.19]	[-5.33,1.22]	[-5.33,1.24]	[-5.32,1.27]	[-5.32,1.30]
12	[-5.21,1.18]	[-5.23,1.15]	[-5.25,1.17]	[-5.25,1.20]	[-5.26,1.23]	[-5.26,1.26]	[-5.26,1.29]	[-5.26,1.31]	[-5.26,1.34]	[-5.26,1.37]
13	[-5.12,1.25]	[-5.14,1.23]	[-5.16,1.24]	[-5.18,1.27]	[-5.19,1.30]	[-5.20,1.32]	[-5.20,1.35]	[-5.20,1.38]	[-5.21,1.40]	[-5.21,1.43]
14	[-5.03,1.33]	[-5.07,1.30]	[-5.09,1.29]	[-5.11,1.33]	[-5.12,1.36]	[-5.13,1.38]	[-5.14,1.41]	[-5.15,1.43]	[-5.16,1.46]	[-5.16,1.48]
15	[-4.96, 1.41]	[-5.00,1.37]	[-5.03,1.36]	[-5.05,1.38]	[-5.07,1.41]	[-5.08,1.44]	[-5.09,1.46]	[-5.10,1.49]	[-5.11,1.51]	[-5.12,1.53]
16	[-4.89,1.48]	[-4.93, 1.43]	[-4.97,1.43]	[-4.99,1.43]	[-5.01,1.46]	[-5.03,1.49]	[-5.04,1.51]	[-5.06,1.54]	[-5.07, 1.56]	[-5.07,1.58]
17	[-4.83,1.56]	[-4.88,1.51]	[-4.91,1.49]	[-4.94,1.49]	[-4.96,1.50]	[-4.98,1.53]	[-5.00,1.56]	[-5.01,1.58]	[-5.02,1.60]	[-5.03,1.62]
18	[-4.78,1.62]	[-4.83, 1.58]	[-4.86,1.54]	[-4.89,1.54]	[-4.92,1.55]	[-4.94,1.57]	[-4.96, 1.60]	[-4.97,1.62]	[-4.99,1.64]	[-5.00,1.66]
19	[-4.73,1.69]	[-4.78,1.64]	[-4.82,1.61]	[-4.85,1.60]	[-4.88,1.60]	[-4.90,1.61]	[-4.92,1.63]	[-4.93, 1.65]	[-4.95,1.68]	[-4.96,1.70]
20	[-4.69,1.75]	[-4.74,1.71]	[-4.78,1.68]	[-4.81,1.65]	[-4.84,1.65]	[-4.86,1.66]	[-4.88,1.66]	[-4.90,1.68]	[-4.92,1.71]	[-4.93,1.73]
21	[-4.65,1.81]	[-4.70,1.77]	[-4.74,1.74]	[-4.77,1.72]	[-4.80, 1.70]	[-4.83,1.70]	[-4.85,1.71]	[-4.87,1.72]	[-4.88,1.73]	∗
22	[-4.62,1.86]	[-4.67,1.83]	[-4.71,1.80]	[-4.74,1.78]	[-4.77,1.76]	[-4.79,1.75]	[-4.82,1.75]	[-4.84,1.76]	[-4.85,1.77]	∗
23	[-4.59,1.92]	[-4.63,1.88]	[-4.67,1.86]	[-4.71,1.84]	[-4.74,1.82]	[-4.76, 1.81]	[-4.79,1.80]	[-4.81,1.80]	∗	∗
24	[-4.56,1.97]	[-4.60,1.94]	[-4.64,1.91]	[-4.68,1.89]	[-4.71,1.88]	[-4.73,1.87]	[-4.76,1.86]	[-4.78,1.86]	∗	∗
25	[-4.53,2.02]	[-4.57,1.99]	[-4.61,1.97]	[-4.65,1.95]	[-4.68,1.93]	[-4.71,1.92]	[-4.73,1.91]	∗	∗	∗
26	[-4.50, 2.07]	[-4.55, 2.04]	[-4.59, 2.02]	[-4.62, 2.00]	[-4.65, 1.98]	[-4.68, 1.97]	[-4.71, 1.97]	∗	∗	∗
27	[-4.48, 2.12]	[-4.52, 2.09]	[-4.56, 2.07]	[-4.60, 2.05]	[-4.63, 2.03]	[-4.66, 2.02]	∗	∗	∗	∗
28	[-4.45, 2.17]	[-4.50, 2.14]	[-4.54, 2.12]	[-4.58, 2.10]	[-4.61, 2.08]	[-4.63, 2.07]	∗	∗	∗	∗
29	[-4.43, 2.21]	[-4.48, 2.18]	[-4.52, 2.16]	[-4.55, 2.15]	[-4.58, 2.13]	∗	∗	∗	∗	∗
30	[-4.41, 2.25]	[-4.46, 2.23]	[-4.50, 2.21]	[-4.53, 2.19]	[-4.56, 2.18]	∗	∗	∗	∗	∗
31	[-4.40, 2.30]	[-4.44, 2.27]	[-4.48, 2.25]	[-4.51, 2.24]	∗	∗	∗	∗	∗	∗
32	[-4.38, 2.34]	[-4.42, 2.31]	[-4.46, 2.29]	[-4.49, 2.28]	∗	∗	∗	∗	∗	∗
33	[-4.36, 2.38]	[-4.40, 2.35]	[-4.44, 2.34]	∗	∗	∗	∗	∗	∗	∗
34	[-4.35, 2.42]	[-4.39, 2.39]	[-4.43, 2.38]	∗	∗	∗	∗	∗	∗	∗
35	[-4.33, 2.45]	[-4.37, 2.43]	∗	∗	∗	∗	∗	∗	∗	∗
36	[-4.32, 2.49]	[-4.36, 2.47]	∗	∗	∗	∗	∗	∗	∗	∗
37	[-4.30, 2.53]	∗	∗	∗	∗	∗	∗	∗	∗	∗
38	[-4.29,2.56]	∗	∗	∗	∗	∗	∗	∗	∗	∗

Table 4.1: The intervals $[L, R]$ for inequality (4.1).

$\mathbf{r_1}$ \ $\mathbf{r_2}$	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
0	[-6.35,0.23]	[-6.26,0.32]	[-6.18,0.40]	[-6.11,0.48]	[-6.05,0.55]	[-6.00,0.63]	[-5.95,0.71]	[-5.91, 0.78]	[-5.87,0.86]	[-5.84,0.92]
1	[-6.20,0.33]	[-6.12,0.40]	[-6.06,0.48]	[-6.00, 0.55]	[-5.95,0.62]	[-5.90,0.68]	[-5.87,0.74]	[-5.83, 0.82]	[-5.80,0.89]	$*$
2	[-6.07, 0.46]	[-6.00, 0.53]	[-5.95,0.59]	[-5.90,0.64]	[-5.86, 0.70]	[-5.82,0.76]	[-5.79, 0.81]	[-5.76, 0.86]	[-5.73,0.92]	$*$
3	[-5.94, 0.59]	[-5.89,0.65]	[-5.85, 0.70]	[-5.81,0.75]	[-5.78, 0.79]	[-5.74,0.83]	[-5.72,0.88]	[-5.69, 0.93]	$*$	$*$
4	[-5.84,0.71]	[-5.80, 0.76]	[-5.76,0.81]	[-5.73, 0.85]	[-5.70, 0.89]	[-5.67,0.93]	[-5.65, 0.96]	[-5.63,1.00]	$*$	$*$
5	[-5.74, 0.82]	[-5.71, 0.87]	[-5.68, 0.91]	[-5.65,0.95]	[-5.63,0.99]	[-5.61,1.02]	[-5.59,1.05]	$*$	$*$	$*$
6	[-5.65,0.92]	[-5.63,0.96]	[-5.60, 1.00]	[-5.58, 1.04]	[-5.56,1.07]	[-5.55,1.10]	[-5.53,1.13]	$*$	$*$	$*$
7	[-5.57, 1.02]	[-5.55,1.05]	[-5.53, 1.09]	[-5.52,1.12]	[-5.50, 1.15]	[-5.49,1.18]	$*$	$*$	$*$	$*$
8	[-5.50,1.10]	[-5.48,1.14]	[-5.47,1.17]	[-5.46, 1.20]	[-5.45,1.23]	[-5.44,1.25]	$*$	$*$	$*$	$*$
9	[-5.43,1.18]	[-5.42,1.21]	[-5.41, 1.24]	[-5.40, 1.27]	[-5.39,1.30]	$*$	$*$	$*$	$*$	$*$
10	[-5.37, 1.26]	[-5.36, 1.29]	[-5.36,1.31]	[-5.35,1.34]	[-5.34, 1.36]	$*$	$*$	$*$	$*$	$*$
11	[-5.31, 1.33]	[-5.31,1.35]	[-5.31, 1.38]	[-5.30, 1.40]	$*$	$*$	$*$	$*$	$*$	$*$
12	[-5.26,1.39]	[-5.26,1.41]	[-5.26,1.44]	[-5.26, 1.46]	$*$	$*$	$*$	$*$	$*$	$*$
13	[-5.21,1.45]	[-5.21,1.47]	[-5.21,1.49]	$*$	$*$	$*$	$*$	$*$	$*$	$*$
14	[-5.17, 1.50]	[-5.17,1.52]	[-5.17, 1.54]	$*$	$*$	$*$	$*$	$*$	$*$	$*$
15	[-5.12, 1.55]	[-5.13,1.57]	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
16	[-5.08, 1.60]	[-5.09,1.62]	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
17	[-5.04,1.64]	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
18	[-5.01, 1.68]	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$

Table 4.2: The intervals $[L, R]$ for inequality (4.1) (continuation).

	Inequality (4.2)		Inequality (4.3)		Inequality (4.4)	
d	**L**	**R**	**L**	**R**	**L**	**R**
2	**	**	**	**	**	**
3	**	**	**	**	**	**
4	**	**	-12.7612	-0.8428	**	**
5	-11.9082	-1.2821	**	**	-12.3027	-1.1787
6	**	**	-10.2123	-0.5096	**	**
7	-9.5231	-1.0928	**	**	-9.9584	-0.9066
8	**	**	-9.0268	-0.2280	**	**
9	-8.4007	-0.9271	**	**	-8.8585	-0.6823
10	**	**	-8.3442	0.0157	**	**
11	-7.7485	-0.7777	**	**	-8.2208	-0.4926
12	**	**	-7.9003	0.2285	**	**
13	-7.3216	-0.6438	**	**	-7.8039	-0.3298
14	**	**	-7.5880	0.4160	**	**
15	-7.0197	-0.5242	**	**	-7.5095	-0.1886
16	**	**	-7.3560	0.5830	**	**
17	-6.7945	-0.4170	**	**	-7.2900	-0.0644
18	**	**	-7.1767	0.7333	**	**
19	-6.6199	-0.3206	**	**	-7.1199	0.0460
20	**	**	-7.0337	0.8699	**	**
21	-6.4803	-0.2335	**	**	-6.9840	0.1455
22	**	**	-6.9170	0.9948	**	**
23	-6.3660	-0.1544	**	**	-6.8728	0.2358
24	**	**	-6.8197	1.1114	**	**
25	-6.2707	-0.0763	**	**	-6.7800	0.3204
26	**	**	-6.7374	1.2196	**	**
27	-6.1898	0.0008	**	**	-6.7014	0.3987
28	**	**	-6.6668	1.3205	**	**
29	-6.1204	0.0726	**	**	-6.6339	0.4714
30	**	**	-6.6056	1.4150	**	**
31	-6.0601	0.1398	**	**	-6.5752	0.5392
32	**	**	-6.5519	1.5040	**	**
33	-6.0071	0.2028	**	**	-6.5238	0.6029
34	**	**	-6.5044	1.5879	**	**
35	-5.9603	0.2621	**	**	-6.4783	0.6627
36	**	**	-6.4622	1.6674	**	**
37	-5.9185	0.3181	**	**	-6.4377	0.7193
38	**	**	-6.4243	1.7428	**	**
39	-5.8810	0.3711	**	**	-6.4013	0.7728
40	**	**	-6.3901	1.8146	**	**

Table 4.3: Intervals $[L, R]$ for inequalities (4.2)-(4.4).

4.2 Subdivision algorithm for y in a compact interval.

In the previous section we constructed a compact interval $[L, R] \subseteq \mathbb{R}$ such that inequalities (4.1), (4.2), (4.3) and (4.4) hold for $y \in \mathbb{R} \setminus [L, R]$. To prove that those inequalities also hold for $y \in [L, R]$ we use the fact that, for any signature (r_1, r_2), the functions $y \mapsto -\frac{f'_{(r_1,r_2)}(y)}{f_{(r_1,r_2)}(y)}$ are increasing (see (6.5)).

Let us start with inequality (4.1) and suppose for a moment that we can evaluate our function with infinite precision. As in §3.1, to prove that

$$g(y) := -\frac{f'_{(r_1,r_2)}(y)}{f_{(r_1,r_2)}(y)} \leq -\frac{f'_{(r_1+2,r_2-1)}(y)}{f_{(r_1+2,r_2-1)}(y)} =: h(y),$$

for $y \in [L, R]$, it suffices to construct a finite sequence of points $L = y_1 < \cdots < y_n = R$ such that $g(y_{i+1}) \leq h(y_i)$ $(i = 1, \ldots, n-1)$. We proceed similarly for inequalities (4.2), (4.3) and (4.4).

As can be seen in Tables 4.1 - 4.3, the values of y we have to consider in the subdivision algorithm are always smaller than 3, so in practice to numerically evaluate the functions g and h inside the interval $[L, R]$, we can use the residue method. More precisely, we use the results of §3.2 to construct approximations g_{approx} and h_{approx} of g and h respectively, such that

$$\begin{aligned} |g(y) - g_{\text{approx}}(y)| &\leq \epsilon \\ |h(y) - h_{\text{approx}}(y)| &\leq \epsilon, \end{aligned}$$

for a small suitably chosen $\epsilon > 0$, and construct a finite sequence of points $L = y_1 < \cdots < y_n = R$ recursively, in such a way that

$$h_{\text{approx}}(y_i) - \epsilon \geq g_{\text{approx}}(y_{i+1}) + \epsilon \qquad \forall i = 1, \ldots, n-1.$$

This readily implies that $g(y) \leq h(y)$ for all y in $[L, R]$.

The number of subdivisions needed for inequality (4.1) is shown in Table 4.5. Only the signatures with degree less than 40 and $r_2 \geq 1$ are shown, as these are the only ones needed. Using the same method, the number of subdivisions needed for inequalities (4.2), (4.3) and (4.4) is shown in Table 4.4. Observe that the number of subdivisions needed for inequality (4.3) is considerably bigger than for inequalities (4.2) and (4.4), as we expected from looking at the graphs of the functions involved. In all cases the subdivision points tend to accumulate near the right endpoint of the interval we are subdividing, due to the fact that the curves involved come very close together as the variable y increases.

d	Ineq. (4.2)	Ineq. (4.3)	Ineq. (4.4)	**d**	Ineq. (4.2)	Ineq. (4.3)	Ineq. (4.4)
2	$*$	$*$	$*$	**22**	$*$	391	$*$
3	$*$	$*$	$*$	**23**	54	$*$	59
4	$*$	20	$*$	**24**	$*$	481	$*$
5	9	$*$	10	**25**	59	$*$	66
6	$*$	36	$*$	**26**	$*$	581	$*$
7	13	$*$	13	**27**	65	$*$	77
8	$*$	55	$*$	**28**	$*$	676	$*$
9	16	$*$	18	**29**	69	$*$	87
10	$*$	84	$*$	**30**	$*$	761	$*$
11	21	$*$	23	**31**	76	$*$	95
12	$*$	113	$*$	**32**	$*$	908	$*$
13	25	$*$	28	**33**	85	$*$	103
14	$*$	161	$*$	**34**	$*$	1059	$*$
15	30	$*$	34	**35**	95	$*$	110
16	$*$	202	$*$	**36**	$*$	1203	$*$
17	34	$*$	41	**37**	102	$*$	116
18	$*$	262	$*$	**38**	$*$	1337	$*$
19	40	$*$	49	**39**	110	$*$	122
20	$*$	329	$*$	**40**	$*$	1382	$*$
21	48	$*$	54	$*$	$*$	$*$	$*$

Table 4.4: Number of subdivisions in $[L, R]$ needed for inequalities (4.2)-(4.4) (see §4.2).

r_1 \ r_2	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
0	*	12	17	25	33	41	54	66	75	95	112	128	144	169	193	219	243	272	314	354
1	13	18	27	35	42	54	66	76	94	112	127	140	167	190	212	234	255	294	335	*
2	21	29	37	46	58	68	79	99	114	129	144	169	191	212	233	254	286	321	357	*
3	31	37	50	61	69	85	102	117	132	149	173	196	218	238	256	284	319	353	*	*
4	39	52	62	72	89	104	120	135	154	178	201	223	242	262	295	326	356	384	*	*
5	52	64	76	93	107	124	138	160	183	205	227	248	271	305	337	367	392	*	*	*
6	66	77	94	111	127	140	164	188	210	231	253	278	313	346	375	403	429	*	*	*
7	80	97	113	129	146	169	193	214	235	256	286	319	354	384	411	438	*	*	*	*
8	99	114	132	148	174	196	218	240	259	291	328	359	391	419	445	470	*	*	*	*
9	118	132	149	174	199	222	243	263	298	333	367	396	424	451	477	*	*	*	*	*
10	134	149	176	201	223	245	265	303	337	371	403	430	457	483	522	*	*	*	*	*
11	155	175	202	225	247	269	308	342	375	406	434	462	487	532	*	*	*	*	*	*
12	182	201	225	248	271	308	345	378	409	439	466	492	540	586	*	*	*	*	*	*
13	209	226	248	269	307	345	380	411	441	469	495	545	593	*	*	*	*	*	*	*
14	234	250	268	305	344	381	412	443	470	500	550	598	643	*	*	*	*	*	*	*
15	259	271	302	340	377	411	443	473	499	552	602	647	*	*	*	*	*	*	*	*
16	291	312	341	370	408	441	471	498	552	601	649	694	*	*	*	*	*	*	*	*
17	335	353	377	406	436	469	497	547	598	647	696	*	*	*	*	*	*	*	*	*
18	376	393	412	438	462	493	537	592	644	692	737	*	*	*	*	*	*	*	*	*
19	415	430	448	469	495	524	581	635	686	733	*	*	*	*	*	*	*	*	*	*
20	452	467	484	501	532	580	625	676	725	771	*	*	*	*	*	*	*	*	*	*
21	487	502	522	555	588	632	676	718	762	*	*	*	*	*	*	*	*	*	*	*
22	527	556	587	617	651	684	724	766	804	*	*	*	*	*	*	*	*	*	*	*
23	591	619	648	679	710	742	775	811	*	*	*	*	*	*	*	*	*	*	*	*
24	653	680	709	738	768	798	830	861	*	*	*	*	*	*	*	*	*	*	*	*
25	711	738	766	793	823	852	882	*	*	*	*	*	*	*	*	*	*	*	*	*
26	769	794	820	850	876	905	934	*	*	*	*	*	*	*	*	*	*	*	*	*
27	823	848	872	899	927	955	*	*	*	*	*	*	*	*	*	*	*	*	*	*
28	875	899	924	951	994	1045	*	*	*	*	*	*	*	*	*	*	*	*	*	*
29	925	950	989	1039	1088	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
30	989	1038	1084	1129	1177	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
31	1086	1129	1173	1219	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
32	1175	1216	1262	1305	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
33	1261	1303	1345	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
34	1346	1386	1426	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
35	1428	1467	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
36	1507	1544	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
37	1583	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
38	1658	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*

Table 4.5: Number of subdivisions in $[L, R]$ needed for inequality (4.1).

Chapter 5

Numerical lower bounds for Reg(L/K)

In the previous chapter we concluded the proof of the Three Steps (see page 8), establishing that

$$\rho(y) = \rho_d(y) := \begin{cases} -\frac{1}{d}\frac{f'_{0,d/2}}{f_{0,d/2}}(y), & \text{if } d \text{ is even,} \\ -\frac{1}{d}\frac{f'_{0,(d-1)/2}}{f_{0,(d-1)/2}}(y), & \text{if } d \text{ is odd,} \end{cases} \tag{5.1}$$

is a convex increasing function affording the lower bound (1.14) for the terms $-f'_w/f_w$ appearing in the fundamental inequality (1.13) for $8 \leq d \leq 40$.[1] Recall that $f_w = f_{(p_w,q_w)}$, where $f_{(p,q)}$ was defined in (1.19), and p_w (resp. q_w) is the number of real (resp. complex) places of L lying above the Archimedean place w of K.

In this chapter we take the step still remaining, *i. e.* we prove lower bounds for the right-hand side of

$$\frac{\mathrm{Reg}(L/K)}{w_L} \geq (0.01) \cdot 2^{-(\#\mathcal{A}_L - \#\mathcal{A}_K)} \pi^{-r_2(L)/2} \prod_{w \in \mathcal{A}_K} f_w(y_0). \tag{5.2}$$

This is the practical version (1.15) of (1.13). *Throughout this chapter we fix $y_0 = y_0(d)$ as given in Table 1.2, so that it satisfies $2\rho_d(y_0) - 1 > 0.01$ nearly with equality.* In the different sections of this chapter we deduce lower bounds for the relative regulator under varying hypotheses on the degree d or on the splitting of the real places in L/K.

5.1 Exponentially growing lower bounds for $[L : K] \geq 12$

Using

$$r_1(L) = \sum_{w \in \mathcal{A}_K} p_w, \qquad r_2(L) = \sum_{w \in \mathcal{A}_K} q_w, \qquad r_1(K) + r_2(K) = \#\mathcal{A}_K, \qquad r_1(L) + r_2(L) = \#\mathcal{A}_L,$$

[1] We did assume (for convenience) that $r_1 + r_2 \geq 4$ in proving Steps 1 to 3. This is not a problem since we will make no claims to new exponentially increasing lower bounds for the relative regulator when $d = [L : K] \leq 8$, except when K is totally complex and $d = r_2 \geq 5$ or when L is totally real and $d = r_1 \geq 6$.

we can rewrite (5.2) in the form

$$\frac{\operatorname{Reg}(L/K)}{w_L} \geq 0.01 \cdot 2^{-(\sum_w p_w + \sum_w q_w) + \#\mathcal{A}_K} \pi^{-\frac{1}{2}\sum_w q_w} \prod_w f_w(y_0) = 0.01 \prod_{w \in \mathcal{A}_K} \tilde{f}_{(p_w,q_w)}(y_0), \tag{5.3}$$

where $\tilde{f}_{(p_w,q_w)} := 2^{-(p_w+q_w-1)}\pi^{-\frac{1}{2}q_w} f_{(p_w,q_w)}$.

To find a lower bound for $\prod_w \tilde{f}_{p_w,q_w}(y_0)$ we estimate

$$\begin{aligned}\prod_{w\in\mathcal{A}_K} \tilde{f}_{(p_w,q_w)}(y_0) &= \prod_{\substack{w\in\mathcal{A}_K \\ w \text{ real}}} \tilde{f}_{(p_w,q_w)}(y_0) \prod_{\substack{w\in\mathcal{A}_K \\ w \text{ complex}}} \tilde{f}_{(0,d)}(y_0) \\ &= \Big(\prod_{\substack{w\in\mathcal{A}_K \\ w \text{ real}}} \tilde{f}_{(p_w,q_w)}(y_0)\Big) \cdot \Big(\tilde{f}_{(0,d)}(y_0)\Big)^{r_2(K)} \geq B_r^{dr_1(K)} A_c^{2dr_2(K)} \geq C^{[L:\mathbb{Q}]},\end{aligned} \tag{5.4}$$

where

$$A_c := \big(\tilde{f}_{(0,d)}(y_0)\big)^{\frac{1}{2d}}, \qquad B_r := \min_{\substack{p,q \\ p+2q=d}} \Big\{\big(\tilde{f}_{(p,q)}(y_0)\big)^{\frac{1}{d}}\Big\}, \qquad C := \min(B_r, A_c). \tag{5.5}$$

Replacing this in (5.3), we get

$$\frac{\operatorname{Reg}(L/K)}{w_L} \geq 0.01 \cdot C^{[L:\mathbb{Q}]}. \tag{5.6}$$

Table 5.1 shows the values of A_c, B_r, C and y_0 for $2 \leq d \leq 40$. Observe that for d in the table, the minimum over p, q in the definition (5.5) of B_r is always attained at the signature with the smallest possible p, *i. e.* at $(p,q) = (0, d/2)$ for d even and at $(p,q) = (1,(d-1)/2)$ for d odd. Thus, for $d = [L : K]$ fixed, the constant C in (5.6) will have the smallest value when K is totally real and each real place has as many complex places above it in L as possible. This also implies that in Table 5.1 the entries corresponding to d even are the same as the entries in Table 1.2 corresponding to degree $d/2$ for K totally complex. We also note that $C > 1$ as long as $d \geq 10$, $d \neq 11$. Thus we have proved the first claim in our Main Theorem.

For general L/K we can say nothing more about $\operatorname{Reg}(L/K)$. However, if we keep track of how much we give up in our estimates by assuming the worst case, we can improve our lower bounds if we have additional information on the splitting of the Archimedean places in L/K. The following notation will keep track of these losses. For $p = 0, 1, \ldots, d = [L : K]$ with $p \equiv d \pmod 2$, let

$$r_p = r_p(L/K) := \#\{w \in \mathcal{A}_K : \ w \text{ real and } p_w = p\}, \qquad c_p := \begin{cases} \dfrac{\tilde{f}_{\left(p,\frac{d-p}{2}\right)}}{\tilde{f}_{\left(0,\frac{d}{2}\right)}}(y_0), & \text{if } d \text{ is even,} \\[2ex] \dfrac{\tilde{f}_{\left(p,\frac{d-p}{2}\right)}}{\tilde{f}_{\left(1,\frac{d-1}{2}\right)}}(y_0), & \text{if } d \text{ is odd.}\end{cases} \tag{5.7}$$

From $r_1(L) = \sum_{w\in\mathcal{A}_K} p_w$, we get

$$d \sum_{\substack{p=0 \\ p\equiv d \,(\mathrm{mod}\ 2)}}^{d} r_p \geq \sum_{\substack{p=0 \\ p\equiv d \,(\mathrm{mod}\ 2)}}^{d} p r_p = r_1(L). \tag{5.8}$$

d	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
C	0.603	0.489	0.790	0.715	0.898	0.838	0.969	0.916	1.018	0.972	1.054	1.013	1.082
$\mathbf{y_0}$	-0.683	-0.277	-0.933	-0.659	-1.031	-0.825	-1.083	-0.918	-1.116	-0.978	-1.138	-1.020	-1.154
$\mathbf{A_c}$	0.686	0.536	0.894	0.775	0.997	0.898	1.058	0.975	1.100	1.027	1.129	1.065	1.152
$\mathbf{B_r}$	0.603	0.489	0.790	0.715	0.898	0.838	0.969	0.916	1.018	0.972	1.054	1.013	1.082
minimum in B_r attained for $[\mathbf{p}, \mathbf{q}] =$	[0, 1]	[1, 1]	[0, 2]	[1, 2]	[0, 3]	[1, 3]	[0, 4]	[1, 4]	[0, 5]	[1, 5]	[0, 6]	[1, 6]	[0, 7]

d	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**
C	1.045	1.105	1.071	1.123	1.092	1.138	1.110	1.151	1.125	1.163	1.138	1.172	1.149
$\mathbf{y_0}$	-1.051	-1.166	-1.074	-1.175	-1.093	-1.183	-1.108	-1.189	-1.121	-1.195	-1.131	-1.199	-1.140
$\mathbf{A_c}$	1.094	1.169	1.117	1.183	1.136	1.195	1.151	1.204	1.164	1.213	1.175	1.220	1.185
$\mathbf{B_r}$	1.045	1.105	1.071	1.123	1.092	1.138	1.110	1.151	1.125	1.163	1.138	1.172	1.149
minimum in B_r attained for $[\mathbf{p}, \mathbf{q}] =$	[1, 7]	[0, 8]	[1, 8]	[0, 9]	[1, 9]	[0, 10]	[1, 10]	[0, 11]	[1, 11]	[0, 12]	[1, 12]	[0, 13]	[1, 13]

d	**28**	**29**	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**
C	1.181	1.159	1.188	1.167	1.195	1.175	1.201	1.182	1.207	1.189	1.212	1.194	1.216
$\mathbf{y_0}$	-1.203	-1.148	-1.206	-1.155	-1.209	-1.161	-1.212	-1.166	-1.214	-1.171	-1.216	-1.175	-1.218
$\mathbf{A_c}$	1.226	1.193	1.232	1.201	1.237	1.207	1.241	1.213	1.245	1.218	1.248	1.223	1.252
$\mathbf{B_r}$	1.181	1.159	1.188	1.167	1.195	1.175	1.201	1.182	1.207	1.189	1.212	1.194	1.216
minimum in B_r attained for $[\mathbf{p}, \mathbf{q}] =$	[0, 14]	[1, 14]	[0, 15]	[1, 15]	[0, 16]	[1, 16]	[0, 17]	[1, 17]	[0, 18]	[1, 18]	[0, 19]	[1, 19]	[0, 20]

Table 5.1: The value of C in (5.5) and (5.6).

If $\tilde{f}_{(0,\frac{d}{2})}(y_0) > 1$ and d is even, using (5.8) we get

$$\prod_{\substack{w\in\mathcal{A}_K\\ w \text{ real}}} \tilde{f}_{(p_w,q_w)}(y_0) = \prod_{\substack{p=0\\ p \text{ even}}}^{d} \left(\tilde{f}_{(p,\frac{d-p}{2})}(y_0)\right)^{r_p} = \prod_p \left(c_p \tilde{f}_{(0,\frac{d}{2})}(y_0)\right)^{r_p} = \left(\tilde{f}_{(0,\frac{d}{2})}(y_0)\right)^{\sum_p r_p} \prod_p c_p^{r_p}$$
$$\geq \left(\tilde{f}_{(0,\frac{d}{2})}(y_0)\right)^{\frac{r_1(L)}{d}} \prod_p c_p^{r_p} = \left(\left(\prod_{\substack{p=0\\ p \text{ even}}}^{d} c_p^{r_p} \right)^{\frac{1}{dr_1(K)}} \left(\tilde{f}_{(0,\frac{d}{2})}(y_0)\right)^{\frac{r_1(L)}{d^2 r_1(K)}} \right)^{dr_1(K)}.$$

Similarly, if d is odd and $\tilde{f}_{(1,\frac{d-1}{2})}(y_0) > 1$ we obtain

$$\prod_{\substack{w\in\mathcal{A}_K\\ w \text{ real}}} \tilde{f}_{(p_w,q_w)}(y_0) \geq \left(\left(\prod_{\substack{p=1\\ p \text{ odd}}}^{d} c_p^{r_p} \right)^{\frac{1}{dr_1(K)}} \left(\tilde{f}_{(1,\frac{d-1}{2})}(y_0)\right)^{\frac{r_1(L)}{d^2 r_1(K)}} \right)^{dr_1(K)}.$$

Therefore, if we have information on $r_p, r_1(L)$ and $r_1(K)$ we can replace B_r in (5.5) and (5.6) by

$$\widetilde{B}_r := \left(\prod_{\substack{p=0\\ p\equiv d \,(\mathrm{mod}\ 2)}}^{d} c_p^{r_p} \right)^{\frac{1}{dr_1(K)}} \cdot \begin{cases} \left(\tilde{f}_{(0,\frac{d}{2})}(y_0)\right)^{\frac{r_1(L)}{d^2 r_1(K)}} & (\text{if } d \text{ is even and } \tilde{f}_{(0,\frac{d}{2})}(y_0) > 1), \\ \left(\tilde{f}_{(1,\frac{d-1}{2})}(y_0)\right)^{\frac{r_1(L)}{d^2 r_1(K)}} & (\text{if } d \text{ is odd and } \tilde{f}_{(1,\frac{d-1}{2})}(y_0) > 1). \end{cases} \tag{5.9}$$

In the following Table 5.2, we calculate $c_p = c_p(d)$ and $\tilde{f}_{(0,\frac{d}{2})}(y_0)$ and $\tilde{f}_{(1,\frac{d-1}{2})}(y_0)$ for $d \leq 40$. The corresponding `PARI-GP` codes can be found in §6.6.1 of the Appendix.

5.2 Lower bounds for $\mathrm{Reg}(L/K)$ for some splitting types.

As we saw in the previous section, for relative degrees $d = [L : K] < 12$ we do not in general obtain lower bounds for $\mathrm{Reg}(L/K)$ growing exponentially with $[L : \mathbb{Q}]$. Here we make some assumptions on L/K that allow us to still get such bounds. We can, for instance, assume a lower bound

$$p_w \geq p_0 \qquad\qquad (\text{for all real } w \in \mathcal{A}_K), \tag{5.10}$$

for the number p_w of real places of L above any real place w of K. In terms of r_p in (5.7), this means $r_p = 0$ for $p < p_0$. We conclude, as in (5.3) and (5.4), that assuming (5.10) implies

$$\frac{\mathrm{Reg}(L/K)}{w_L} \geq 0.01 \left(\tilde{f}_{(0,d)}(y_0)\right)^{r_2(K)} \prod_{\substack{p=p_0\\ p\equiv d \,(\mathrm{mod}\ 2)}}^{d} \left(\tilde{f}_{(p,\frac{d-p}{2})}(y_0)\right)^{r_p} = 0.01 \left(\tilde{f}_{(0,d)}(y_0)\right)^{r_2(K)} \prod_{\substack{p=p_0\\ p\equiv d \,(\mathrm{mod}\ 2)}}^{d} t_p^{\alpha_p},$$

$$t_p = t_p(d) := \left(\tilde{f}_{(p,\frac{d-p}{2})}(y_0)\right)^{1/p}, \qquad \alpha_p = \alpha_p(d) := p \cdot r_p, \qquad \sum_{\substack{p=p_0\\ p\equiv d \,(\mathrm{mod}\ 2)}}^{d} \alpha_p = r_1(L). \tag{5.11}$$

d	vector $\mathbf{c_p}$ with $\mathbf{p = 0, 1, \ldots, d}$ and $\mathbf{p \equiv d \mod 2}$	$\tilde{f}_{(0,\frac{d}{2})}(y_0)$ for d even $\tilde{f}_{\left(1,\frac{d-1}{2}\right)}(y_0)$ for d odd
2	$[1.00, 1.13]$	0.36
3	$[1.00, 0.93]$	0.11
4	$[1.00, 1.30, 1.56]$	0.38
5	$[1.00, 1.13, 1.20]$	0.18
6	$[1.00, 1.38, 1.78, 2.17]$	0.52
7	$[1.00, 1.23, 1.44, 1.63]$	0.29
8	$[1.00, 1.42, 1.91, 2.45, 3.02]$	0.77
9	$[1.00, 1.29, 1.60, 1.92, 2.23]$	0.45
10	$[1.00, 1.44, 1.99, 2.65, 3.39, 4.21]$	1.19
11	$[1.00, 1.33, 1.72, 2.15, 2.60, 3.07]$	0.73
12	$[1.00, 1.46, 2.06, 2.79, 3.68, 4.70, 5.85]$	1.89
13	$[1.00, 1.36, 1.81, 2.33, 2.91, 3.56, 4.24]$	1.19
14	$[1.00, 1.47, 2.10, 2.91, 3.91, 5.11, 6.53, 8.14]$	3.045
15	$[1.00, 1.39, 1.88, 2.47, 3.17, 3.98, 4.88, 5.87]$	1.959
16	$[1.00, 1.48, 2.14, 3.00, 4.09, 5.45, 7.11, 9.06, 11.33]$	4.95
17	$[1.00, 1.41, 1.93, 2.59, 3.39, 4.35, 5.46, 6.72, 8.143]$	3.23
18	$[1.00, 1.49, 2.17, 3.07, 4.25, 5.75, 7.61, 9.88, 12.58, 15.75]$	8.15
19	$[1.00, 1.42, 1.98, 2.69, 3.58, 4.67, 5.97, 7.51, 9.286, 11.29]$	5.38
20	$[1.00, 1.50, 2.19, 3.13, 4.38, 6.00, 8.05, 10.6, 13.73, 17.48, 21.91]$	13.49
21	$[1.00, 1.43, 2.01, 2.77, 3.74, 4.95, 6.44, 8.23, 10.35, 12.83, 15.67]$	8.995
22	$[1.00, 1.50, 2.21, 3.18, 4.49, 6.21, 8.44, 11.2, 14.78, 19.09, 24.30, 30.48]$	22.47
23	$[1.00, 1.44, 2.05, 2.84, 3.88, 5.20, 6.85, 8.89, 11.36, 14.30, 17.75, 21.75]$	15.09
24	$[1.00, 1.51, 2.23, 3.23, 4.59, 6.40, 8.78, 11.8, 15.73, 20.58, 26.54, 33.77, 42.39]$	37.60
25	$[1.00, 1.45, 2.07, 2.91, 4.00, 5.42, 7.23, 9.50, 12.29, 15.69, 19.77, 24.58, 30.20]$	25.40
26	$[1.00, 1.51, 2.24, 3.27, 4.67, 6.57, 9.09, 12.3, 16.60, 21.96, 28.66, 36.90, 46.93, 58.97]$	63.15
27	$[1.00, 1.46, 2.10, 2.96, 4.11, 5.62, 7.57, 10.0, 13.17, 17.01, 21.70, 27.35, 34.07, 41.95]$	42.89
28	$[1.00, 1.51, 2.25, 3.30, 4.75, 6.72, 9.36, 12.8, 17.40, 23.24, 30.64, 39.89, 51.31, 65.24, 82.02]$	106.42
29	$[1.00, 1.46, 2.12, 3.01, 4.21, 5.81, 7.89, 10.57, 13.98, 18.26, 23.56, 30.04, 37.88, 47.23, 58.28]$	72.60
30	$[1.00, 1.52, 2.27, 3.33, 4.81, 6.85, 9.61, 13.29, 18.14, 24.44, 32.51, 42.74, 55.53, 71.35, 90.69, 114.07]$	179.83
31	$[1.00, 1.47, 2.14, 3.05, 4.30, 5.97, 8.17, 11.05, 14.75, 19.44, 25.34, 32.65, 41.61, 52.48, 65.51, 80.97]$	123.15
32	$[1.00, 1.52, 2.27, 3.35, 4.87, 6.97, 9.84, 13.69, 18.82, 25.55, 34.27, 45.44, 59.58, 77.29, 99.22, 126.08, 158.66]$	304.62
33	$[1.00, 1.47, 2.15, 3.09, 4.38, 6.12, 8.44, 11.49, 15.46, 20.56, 27.03, 35.16, 45.26, 57.67, 72.75, 90.90, 112.52]$	209.33
34	$[1.00, 1.52, 2.28, 3.38, 4.92, 7.08, 10.04, 14.06, 19.45, 26.58, 35.92, 48.00, 63.47, 83.04, 107.56, 137.97, 175.29, 220.67]$	517.12
35	$[1.00, 1.48, 2.17, 3.13, 4.45, 6.25, 8.68, 11.90, 16.13, 21.61, 28.65, 37.59, 48.81, 62.77, 79.95, 100.88, 126.15, 156.37]$	356.43
36	[1.0, 1.5, 2.2, 3.4, 4.9, 7.1, 10.2, 14.4, 20.0, 27.5, 37.4, 50.4, 67.1, 88.6, 115.7, 149.6, 191.8, 243.7, 306.9]	879.5
37	[1.0, 1.4, 2.1, 3.1, 4.5, 6.3, 8.9, 12.2, 16.7, 22.6, 30.2, 39.9, 52.2, 67.7, 87.0, 110.8, 139.9, 175.1, 217.3]	607.8
38	[1.0, 1.5, 2.3, 3.4, 5.0, 7.2, 10.4, 14.7, 20.5, 28.4, 38.9, 52.7, 70.7, 93.9, 123.6, 161.1, 208.2, 266.7, 338.8, 426.8]	1498.3
39	[1.0, 1.4, 2.1, 3.1, 4.5, 6.4, 9.1, 12.6, 17.3, 23.5, 31.6, 42.1, 55.6, 72.6, 94.1, 120.8, 153.8, 194.1, 243.1, 302.0]	1037.9
40	[1.0, 1.5, 2.3, 3.4, 5.0, 7.3, 10.5, 15.0, 21.0, 29.3, 40.3, 54.9, 74.1, 99.1, 131.3, 172.4, 224.5, 289.8, 371.0, 471.1, 593.7]	2556.3

Table 5.2: The vector c_p and $\tilde{f}_{(0,\frac{d}{2})}(y_0)$ or $\tilde{f}_{\left(1,\frac{d-1}{2}\right)}(y_0)$ for degrees $d \leq 40$.

In view of the restriction $\alpha_p \geq 0$, $\sum_p \alpha_p = r_1(L)$, the product in (5.11) is minimized when all but one of the α_p vanish, the exception $\hat{p} = \hat{p}(d, p_0)$ minimizing t_p for $p \geq p_0$. From (5.11) we obtain

$$\frac{\mathrm{Reg}(L/K)}{w_L} \geq 0.01 \left(\tilde{f}_{(0,d)}(y_0)\right)^{r_2(K)} t_{\hat{p}}^{r_1(L)}.$$

If $\tilde{f}_{(0,d)}(y_0) \leq 1$ or $t_{\hat{p}} \leq 1$ we cannot deduce exponentially increasing lower bounds, but if both of these inequalities are reversed we have, using $r_1(L) \geq p_0 r_1(K)$,

$$\frac{\mathrm{Reg}(L/K)}{w_L} \geq 0.01 A^{2dr_2(K)} \tilde{B}^{dr_1(K)} \geq 0.01 \tilde{C}^{[L:\mathbb{Q}]} \qquad (\tilde{f}_{(0,d)}(y_0) > 1,\ t_{\hat{p}} > 1,\ p_w \geq p_0 \text{ for } w \text{ real}), \tag{5.12}$$

where

$$A := \left(\tilde{f}_{(0,d)}(y_0)\right)^{\frac{1}{2d}}, \qquad \tilde{B} := t_{\hat{p}}^{p_0/d}, \qquad \tilde{C} := \min(\tilde{B}, A). \tag{5.13}$$

Table 5.3 gives the values of t_p for $d \leq 12$.

d	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$	$p=9$	$p=10$	$p=11$	$p=12$	$\tilde{f}_{(0,d)}(y_0)$
2	*	0.6442	*	*	*	*	*	*	*	*	*	*	0.2222
3	0.1170	*	0.4779	*	*	*	*	*	*	*	*	*	0.0237
4	*	0.7137	*	0.8842	*	*	*	*	*	*	*	*	0.4101
5	0.1870	*	0.5958	*	0.7427	*	*	*	*	*	*	*	0.0788
6	*	0.8532	*	0.9847	*	1.0234	*	*	*	*	*	*	0.9691
7	0.2902	*	0.7099	*	0.8408	*	0.8987	*	*	*	*	*	0.2251
8	*	1.0511	*	1.1042	*	1.1138	*	1.1130	*	*	*	*	2.5003
9	0.4583	*	0.8404	*	0.9409	*	0.9824	*	1.0025	*	*	*	0.6376
10	*	1.3161	*	1.2438	*	1.2124	*	1.1917	*	1.1756	*	*	6.7429
11	0.7353	*	0.9945	*	1.0487	*	1.0678	*	1.0750	*	1.0769	*	1.8176
12	*	1.6648	*	1.4053	*	1.3204	*	1.2747	*	1.2445	*	1.2220	18.671

Table 5.3: Values of t_p in (5.11) for $d \leq 12$ and $1 \leq p \leq d$, $p \equiv d \,(\mathrm{mod}\, 2)$, and $\tilde{f}_{(0,d)}(y_0)$.

We can now prove some exponentially lower bounds for degrees 8 and 11, finishing the proof of the results in section 1.1.

Proposition 5.2.1. *Suppose L/K is an extension of degree 8 (resp. 11) such that every real place of K has at least 2 (resp. 5) real places of L above it. Then the relative regulator satisfies* $\mathrm{Reg}(L/K) \geq (0.02) \cdot 1.01^{[L:\mathbb{Q}]}$ *(resp.* $\mathrm{Reg}(L/K) \geq (0.02) \cdot 1.02^{[L:\mathbb{Q}]}$*).*

Proof. From Table 5.3 we see that the assumptions of (5.12) are fulfilled with $p_0 = 2$ if $d = 8$, and $p_0 = 5$ if $d = 11$. If $d = 8$, the same table shows that $\hat{p} = 2, t_{\hat{p}} = 1.0511$ and

$$\tilde{C} = \min(2.5003^{1/16}, 1.0511^{2/8}) = 1.0125\cdots.$$

For $d = 11$, the table gives $\hat{p} = 5, t_{\hat{p}} = 1.0487$ and

$$\tilde{C} = \min(1.8176^{1/22}, 1.0487^{5/11}) = 1.0218\cdots.$$

The Proposition now follows from (5.12) and $w_L \geq 2$. $\square$

Chapter 6

Appendix

In this chapter we present some additional information, both numerical and analytic, about the main results of this thesis.

6.1 Log-concave functions

We shall say that a function $f : \mathbb{R} \to \mathbb{R}$ is logarithmically concave (abbreviated log-concave) if it is positive and $\log(f)$ is concave. For (positive) smooth functions this is equivalent to $-f'/f$ being an increasing function. The product gh and the convolution product

$$(g * h)(y) = \int_{-\infty}^{\infty} g(x)h(y-x)\,dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-ys}G(s)H(s)\,ds \tag{6.1}$$

of log-concave functions g and h is again log-concave.[1] Here all integrals are assumed finite and the two-sided Laplace transforms $G := L(g)$ and $H := L(h)$, where $L(f)(s) := \int_{-\infty}^{\infty} e^{sx} f(x)\,dx$, are assumed to converge at $s = c \in \mathbb{R}$.[2]

In this thesis we deal extensively with convolutions of the following three log-concave functions.

$$g_0(y) := \exp(-e^y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-sy}\Gamma(s)\,ds, \tag{6.2}$$

$$g_1(y) = e^y \exp(-e^y) = -\frac{d}{dy}\exp(-e^y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-sy}\Gamma(s+1)\,ds, \tag{6.3}$$

$$g_{\frac{1}{2}}(y) := e^{y/2}\exp(-e^y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-sy}\Gamma(s+\tfrac{1}{2})\,ds, \tag{6.4}$$

where $y \in \mathbb{R}$ and the integrals are independent of c as long as $c > 0$. Indeed, letting g^{*n} denote the n-fold convolution of g with itself, from (2.2) and (6.1) we have

$$(-1)^t f^{(t)}_{(p,q)}(y) = d^t \cdot (g_1^{*t} * g_0^{*(p+q-t)} * g_{\frac{1}{2}}^{*q})(dy) \qquad (p+q \geq t,\ d := p+2q).$$

[1] For the product gh this is obvious, but for $g * h$ see [Sim11, p. 203].

[2] To make the comparison easier with the Mellin transform conventions more familiar to number theorists, we have chosen the exponent e^{sx} in the Laplace transform instead of the usual e^{-sx}. Since $y \to G(y)$ is log-concave if and only if $y \to G(-y)$ is log-concave, this distinction is not important here.

Hence $(-1)^t f_{(p,q)}^{(t)}$ is log-concave $(0 \le t \le p+q)$. In particular, $f := f_{(p,q)}^{(0)}$ is positive and

$$-\frac{f'}{f}(y), \qquad -\frac{f''}{f'}(y), \qquad -\frac{f'''}{f''}(y) \tag{6.5}$$

are all positive increasing functions of $y \in \mathbb{R}$ if $p+q \ge 3$.

We now calculate

$$\begin{aligned} &\Big(-\frac{f'}{f}\Big)'' = 3\frac{f''f'}{f^2} - \Big(\frac{f'''}{f} + 2\frac{(f')^3}{f^3}\Big) = f_1 - f_2, \qquad (6.6)\\ &f_1 := 3\frac{f''f'}{f^2} = -3\Big(-\frac{f''}{f'}\Big)\Big(-\frac{f'}{f}\Big)^2, \qquad f_2 := -2\Big(-\frac{f'}{f}\Big)^3 - \Big(-\frac{f'''}{f''}\Big)\Big(-\frac{f''}{f'}\Big)\Big(-\frac{f'}{f}\Big), \end{aligned}$$

which writes $(-f'/f)''$ as a difference of two decreasing negative functions, as claimed in (3.2).

6.2 Numerical integration by the double exponential method.

In this section we describe an application of a numerical integration technique called the Double Exponential Method. We need this tool to evaluate the function f in (1.19) and its derivatives accurately in regions where the method by residues is too slow or useless. The double exponential transformation is an optimal variable transformation invented by Takahasi and Mori [TM74] which has proved immensely useful for useful for high-accuracy numerical integration of analytic functions. Its advantage over traditional methods is that its cost grows nearly linearly with the desired accuracy, *i. e.* it takes $\mathrm{O}(D \log D)$ operations to achieve D digits of accuracy. In practice it can quickly give 10000 provably accurate digits, a feat far beyond the capacities of other methods.

Although Takahasi and Mori published a (non-rigorous) account of their method in 1974, it was hardly noticed by number theorists for more than 20 years. We follow the account in Pascal Molin's (unfortunately unpublished) doctoral dissertation [Mol], where he gave a detailed and rigorous version with explicit error bounds.

Let us consider the numerical evaluation of an integral over the real line

$$I = \int_{-\infty}^{+\infty} g(x)\, dx, \tag{6.7}$$

in which $g(x)$ is analytic in an open neighborhood of $\mathbb{R} \subset \mathbb{C}$.[3] A change of variables $x = \phi(u)$ such that $\phi(-\infty) = -\infty$, $\phi(+\infty) = +\infty$ and $\phi'(u) > 0$ transforms (6.7) into

$$I = \int_{-\infty}^{+\infty} g\big(\phi(u)\big)\phi'(u)\, du.$$

Applying an approximation by the simplest Riemann sum, we get the quadrature formula

$$I_h = h \sum_{k=-\infty}^{\infty} g\big(\phi(kh)\big)\phi'(kh) \approx I \qquad\qquad (h > 0).$$

[3] Double exponential numerical integration also applies well to finite intervals, but we do not treat this case here.

For computational purposes we have to truncate this series to a finite sum

$$I_h^{(N)} = h \sum_{k=-N_-}^{N_+} g\big(\phi(kh)\big)\phi'(kh) \qquad\qquad (N_-, N_+ \geq 0), \tag{6.8}$$

where $N = N_- + N_+ + 1$ measures the cost of the approximation $I \approx I_h^{(N)}$. The overall error of (6.8) is

$$\Delta I_h^{(N)} := I - I_h^{(N)} = I - I_h + I_h - I_h^{(N)} = \Delta I_h + \epsilon_h^N,$$

where

$$\Delta I_h := I - I_h = \int_{-\infty}^{+\infty} g\big(\phi(u)\big)\phi'(u)\,du - h\sum_{k=-\infty}^{\infty} g\big(\phi(kh)\big)\phi'(kh),$$

is the discretization error, and

$$\epsilon_h^{(N)} := I_h - I_h^{(N)} = h\sum_{k=-\infty}^{-1-N_-} g\big(\phi(kh)\big)\phi'(kh) + h\sum_{k=1+N_+}^{\infty} g\big(\phi(kh)\big)\phi'(kh),$$

is the truncation error (which depends on h, N_+, N_-, ϕ and g).

In general, for a fixed h, if $g(\phi(u))\phi'(u)$ decays rapidly as $u \to \pm\infty$, then ΔI_h becomes large if h is relatively big compared with the efffective support of $g(\phi(u))\phi'(u)$. On the other hand, if $g(\phi(u))\phi'(u)$ decays slowly as $u \to \pm\infty$, then $\epsilon_h^{(N)}N$ becomes large. Therefore $|\Delta I_h|$ and $|\epsilon_h^N|$ cannot be made small at the same time and there should be an optimal decay rate of $|g\big(\phi(u)\big)\phi'(u)|$ as $u \to \pm\infty$. Takahasi and Mori [TM74] suggested making the decay of $|g\big(\phi(u)\big)\phi'(u)|$ be doubly exponential, *i. e.* for some $c > 0$ we have $|g\big(\phi(u)\big)\phi'(u)| \sim \exp(-ce^{|u|})$ as $|u| \to \infty$. Quadrature based on this idea are called double exponential numerical integration formulas. Of course, each behavior of g at infinity requires a different change of variable ϕ to ensure that $\phi' \cdot g \circ \phi$ decays doubly exponentially at infinity. Thus, Takahari and Mori created a method rather that a single formula.

Double exponential integration can be proved to give excellent results for analytic functions g [Mol], provided we have some prior information on how g decays in a (complex) neighborhood of infinity. Here we apply it to calculate the functions $f^{(t)}(y)$ in (2.2), which are defined by an integral and play a central role in this thesis. Their crucial property is that they are analytic in y and decrease exponentially along the integration contour. Pascal Molin [Mol, Th. 2.3] proved the following double exponential integration formula with explicit error bounds.

Theorem (Takahasi-Mori, Molin). *For $\tau \in (0, \pi/2)$, $\sigma \in (0, \tau)$ and $D \geq 1$, define*

$$Z_\tau := \big\{z = \rho + it : \rho \in \mathbb{C},\ |\arg(\rho)| \leq \tau \text{ or } |\arg(-\rho)| \leq \tau \text{ or } \rho = 0,\ t \in \mathbb{R},\ t \in [-\sin(\tau), \sin(\tau)]\big\},$$

so that $\mathbb{R} \subset Z_\tau \subset \mathbb{C}$. Assume $F : Z_\tau \to \mathbb{C}$ is analytic in the interior of Z_τ and satisfies

$$\begin{aligned} &|F(x)| \leq P_1 \exp(-\alpha|x|^\beta) && (x \in \mathbb{R},\ P_1, \alpha, \beta > 0,\ \beta\tau < \pi/2), \\ &|F(z)| \leq P_2 \exp(Ae^{\gamma|z|}) && (z \in Z_\tau,\ P_2, A \geq 0,\ \gamma < \beta). \end{aligned} \tag{6.9}$$

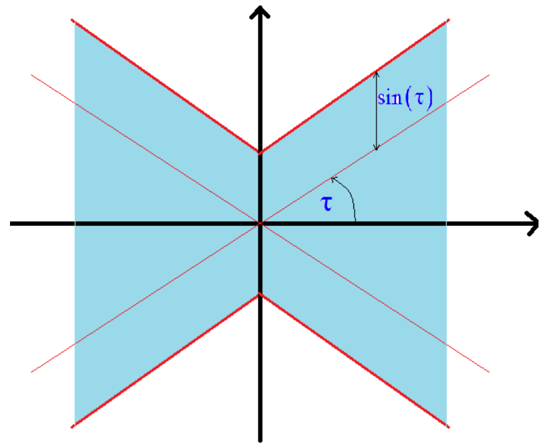


Figure 6.1: The complex region Z_τ.

Define

$$\delta := \frac{1}{\tan(\beta\tau)}, \qquad \alpha_\sigma := \alpha\big(\cos(\beta\sigma) - \delta\sin(\beta\sigma)\big), \qquad A_\sigma := \frac{A\cos(\gamma\sigma)}{\cos(\gamma\tau)},$$
$$C_\sigma := A_\sigma\left(\frac{A_\sigma\gamma+1}{\alpha_\sigma\beta}\right)^{\frac{\gamma}{\beta-\gamma}} - \alpha_\sigma\left(\frac{A_\sigma\gamma}{\alpha_\sigma\beta}\right)^{\frac{\beta}{\beta-\gamma}} + \frac{1}{\beta-\gamma}\log\left(\frac{A_\sigma\gamma+1}{\alpha_\sigma\beta}\right),$$
$$L_{\alpha,\beta}(r) := \inf\left\{X\in\mathbb{R} : \int_X^\infty \exp(-\alpha x^\beta)\,dx \le \exp(-r)\right\} \qquad (r\in\mathbb{R}),$$

and assume

$$0 < h \le \frac{2\pi\sigma}{D + C_\sigma + \log(4P_2 + 2e^{-D-C_\sigma})}, \qquad n \ge \frac{\operatorname{arcsinh}\big(L_{\alpha,\beta}(D+\log(2P_1))\big)}{h} \qquad (n\in\mathbb{N}).$$

Then

$$\left|\int_{-\infty}^\infty F(x)\,dx - h\sum_{k=-n}^{n} F\big(\sinh(kh)\big)\cosh(kh)\right| < e^{-D}.$$

Moreover, as $D\to\infty$, using the largest $h\in\mathbb{R}$ and smallest $n\in\mathbb{N}$ satisfying the above conditions, we have $n < \kappa D\log D$ for some $\kappa = \kappa(\alpha,\beta,\gamma,\tau,P_1,P_2,A,\sigma)$.

We will apply the above theorem to

$$f^{(t)}(y) = \frac{(-d)^t}{2\pi i}\int_{M-i\infty}^{M+i\infty} s^t \exp(-sdy)\,\Gamma(s)^{r_1+r_2}\,\Gamma(s+\tfrac12)^{r_2}\,ds \qquad (t=0,1,2,3,\ y\in\mathbb{R},\ M>0),$$

which we regard ias an integral over the real line by writing

$$f^{(t)}(y) = f^{(t)}_{(r_1,r_2)}(y) \quad = \quad \frac{(-d)^t}{2\pi}\exp(-dMy)\int_{-\infty}^{+\infty} g(x)\,dx, \tag{6.10}$$

where

$$g(x) := (M+ix)^t\ \exp\left(-dixy\right)\Gamma(M+ix)^{r_1+r_2}\ \Gamma(M+\tfrac{1}{2}+ix)^{r_2}. \tag{6.11}$$

We first bound g on the real line.

6.2.1 Upper bound for $|g|$ over $\mathbb{R}$

Write the the factor $(M+ix)^t\ \Gamma(M+ix)^{r_1+r_2}$ of g in the form

$$(M+ix)^t\ \Gamma(M+ix)^{r_1+r_2} = \Big((M+ix)^{\frac{t}{r_1+r_2}}\ \Gamma(M+ix)\Big)^{r_1+r_2}.$$

Stieltjes 1889 version of Stirling's formula with an error bound reads [Sti89][p. 433] [SS90][p. 456]

$$\Big|\log\Gamma(z) - \big((z-\tfrac{1}{2})\log z - z + \tfrac{1}{2}\log(2\pi)\big)\Big| \leq \frac{1}{12|z|\cos^2(\arg(z)/2)} \qquad (|\arg(z)|<\pi). \tag{6.12}$$

Applying this to the half-plane where $|\arg(z)|<\pi/2$ we get

$$\big|(M+ix)^{\frac{t}{r_1+r_2}}\Gamma(M+ix)\big| = \exp\big(\tfrac{t\log(M^2+x^2)}{2(r_1+r_2)}\big)|\Gamma(M+ix)| \leq \sqrt{2\pi}\exp\big(\tfrac{1}{6M}-M\big)\exp(\phi(x)) \qquad (x\in\mathbb{R}),$$

where

$$\phi(x) = \frac{1}{2}\big(M-\tfrac{1}{2}\big)\log(M^2+x^2) - x\arctan(\tfrac{x}{M}) + \tfrac{t}{2(r_1+r_2)}\log(M^2+x^2).$$

Note that ϕ is an even function. Suppose we find $x_0>0$ such that $\phi(x)\leq -\pi x/4$ for $x\geq x_0$. Then,

$$\big|(M+ix)^{\frac{t}{r_1+r_2}}\ \Gamma(M+ix)\big| \leq \widehat{M_1}\exp(-\tfrac{\pi}{4}|x|) \qquad (x\in\mathbb{R}),$$

where

$$\begin{aligned}\widehat{M_1} &:= \sqrt{2\pi}\exp(\tfrac{1}{6M}-M)\max\Big(1, \sup_{x\in[0,x_0]}\big\{\exp(\phi(x)+\tfrac{\pi}{4}x)\big\}\Big)\\ &\leq \sqrt{2\pi}\exp\big(\tfrac{1}{6M}-M\big)\max\Big(1,\exp\big(\tfrac{1}{2}\big(M-\tfrac{1}{2}\big)\log\big(M^2+x_0^2\big)+x_0\tfrac{\pi}{4}+\tfrac{t}{2(r_1+r_2)}\log(M^2+x_0^2)\big)\Big).\end{aligned}$$

In conclusion, we get the bound for $x\in\mathbb{R}$

$$\big|(M+ix)^t\ \Gamma(M+ix)^{r_1+r_2}\big| = \Big|\Big((M+ix)^{\frac{t}{r_1+r_2}}\ \Gamma(M+ix)\Big)^{r_1+r_2}\Big| \leq M_1\exp(-\alpha_1|x|), \tag{6.13}$$

with $\alpha_1 = \frac{\pi}{4}(r_1+r_2)$ and

$$M_1 = \Big(\sqrt{2\pi}\exp\big(\tfrac{1}{6M}-M\big)\max\Big(1,\exp\big(\tfrac{1}{2}(M-\tfrac{1}{2})\log(M^2+x_0^2)+x_0\tfrac{\pi}{4}+\tfrac{t\log(M^2+x_0^2)}{2(r_1+r_2)}\big)\Big)\Big)^{r_1+r_2}. \tag{6.14}$$

Similarly,

$$\big|\Gamma\big(M+\tfrac{1}{2}+ix\big)\big| \leq \widehat{M_2}\exp\big(-\tfrac{\pi}{4}|x|\big),$$

where

$$\begin{aligned}\widehat{M_2} \quad &:= \quad \sqrt{2\pi}\exp\left(\tfrac{1}{6M}-M\right)\max\Big(1, \sup_{x\in[0,\widetilde{x}_0]}\big\{\exp(\widetilde{\phi}(x)+\tfrac{\pi}{4}x)\big\}\Big),\\ &\leq \quad \sqrt{2\pi}\exp\left(\tfrac{1}{6M}-M\right)\max\left(1,\exp\left(\tfrac{M}{2}\log((M+\tfrac{1}{2})^2+\widetilde{x}_0^2)+\tfrac{\pi}{4}\widetilde{x}_0\right)\right)\end{aligned}$$

and

$$\widetilde{\phi}(x) := \frac{M}{2}\log\Big(\left(M+\tfrac{1}{2}\right)^2+x^2\Big) - x\arctan\left(\tfrac{x}{M+\frac{1}{2}}\right),$$

and $\widetilde{x}_0 > 0$ is a point satisfying

$$\widetilde{\phi}(x) \leq -\frac{\pi}{4}x, \quad \forall x \geq \widetilde{x}_0.$$

In conclusion, we get the bound

$$\left|\Gamma(M+\tfrac{1}{2}+ix)^{r_2}\right| \leq M_2\exp(-\alpha_2|x|) \qquad (x\in\mathbb{R}), \tag{6.15}$$

with $\alpha_2 = \frac{\pi}{4}r_2$ and

$$M_2 = \Big(\sqrt{2\pi}\exp\left(\tfrac{1}{6M}-M\right)\max\left(1,\exp(\tfrac{M}{2}\log((M+\tfrac{1}{2})^2+\widetilde{x}_0^2)+\tfrac{\pi}{4}\widetilde{x}_0)\right)\Big)^{r_2}. \tag{6.16}$$

Combining (6.13) and (6.15) we obtain for $x\in\mathbb{R}$,

$$|g(x)| = \left|(M+ix)^t\ \exp\left(-dixy\right)\Gamma(M+ix)^{r_1+r_2}\ \Gamma(M+1/2+ix)^{r_2}\right| \leq P_1\exp\left(-\alpha|x|^{\beta}\right),$$

where $\alpha = -\frac{\pi}{4}\left(r_1+2r_2\right)$, $\beta=1$, $P_1 = M_1M_2$, and M_1, M_2 are defined in (6.14) and (6.16).

6.2.1.1 Finding x_0 and $\widetilde{x}_0$.

To find x_0, note that dividing by M the inequality $\phi(x) \leq -\frac{\pi}{4}x$, we get with $u := \frac{x}{M}$,

$$\tfrac{1}{2}\left(1-\tfrac{1}{2M}\right)\left(2\log M+\log(1+u^2)\right) - u\arctan(u) + \frac{t}{2M(r_1+r_2)}\left(2\log M+\log(1+u^2)\right) \leq -\frac{\pi}{4}u.$$

Since we are considering $\frac{1}{2}\leq M\leq 4$, $t\leq 4$, $r_1+r_2\geq 2$, it suffices to find $u_0 = \frac{x_0}{M} > 0$ such that

$$\lambda(u) := \frac{1}{2}\left(2\log 4+\log(1+u^2)\right) - u\arctan(u) + 2(2\log 4+\log(1+u^2)) + \frac{\pi}{4}u \leq 0, \quad \forall u\geq u_0.$$

Using the graph of the function $u\mapsto\lambda(u)$ (Figure 6.2) and some basic calculus, we see that $u_0 = 32$ works. Therefore we can take $x_0 = 32M$.

To find $\widetilde{x}_0$, note that dividing by $M+\frac{1}{2}$ the inequality $\widetilde{\phi}(x)\leq -\frac{\pi}{4}x$, we get

$$\frac{M}{2}(M+\tfrac{1}{2})\left(2\log(M+\tfrac{1}{2})+\log(1+v^2)\right) - v\arctan(v) \leq -\tfrac{\pi}{4}v \qquad \left(v := \tfrac{x}{M+\frac{1}{2}}\right).$$

Since $\frac{1}{2}\leq M\leq 4$, it is enough to find $v_0>0$ such that for $v\geq v_0$ we have

$$\eta(v) := \frac{1}{2}\left(2\log(4+\tfrac{1}{2})+\log(1+v^2)\right) - v\arctan(v) + \frac{\pi}{4}v \leq 0.$$

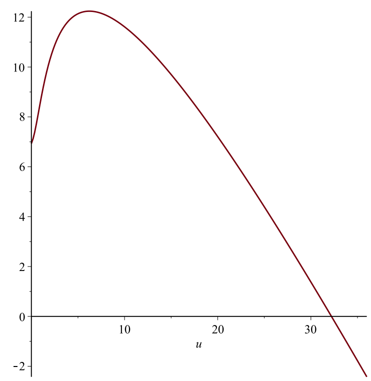


Figure 6.2: The function $\lambda(u) = \frac{1}{2}\left(2\log 4 + \log(1+u^2)\right) - u\arctan(u) + 2(2\log 4 + \log(1+u^2)) + \frac{\pi}{4}u$.

Using the graph of the function $v \mapsto \eta(v)$ (Figure 6.3) and finding the roots of its derivative, we see that $v_0 = 7$ works. Therefore we can take $\widetilde{x}_0 = 7\left(M + \frac{1}{2}\right)$.

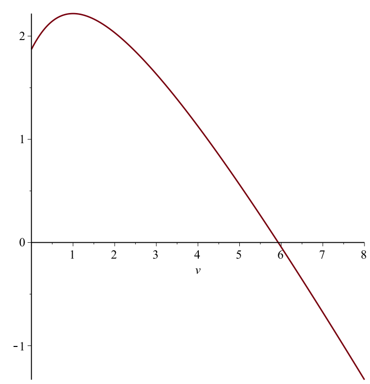


Figure 6.3: The function $\eta(v) = \frac{1}{2}\left(2\log\left(4+\frac{1}{2}\right) + \log(1+v^2)\right) - v\arctan v + \frac{\pi}{4}v$.

6.2.2 Upper bound for $|g|$ over Z_τ.

From now on we fix $\tau = \pi/4$. The function g defined in (6.11) has a holomorphic extension to Z_τ if we require $M > 1$, as we assume from now on. Set $z = x_1 + ix_2 \in Z_\tau$. Then

$$|g(z)| = \left((M - x_2)^2 + x_1^2\right)^{\frac{t}{2}} e^{dx_2 y} |\Gamma\left(M - x_2 + ix_1\right)|^{r_1 + r_2} \left|\Gamma\left(M + \tfrac{1}{2} - x_2 + ix_1\right)\right|^{r_2}. \tag{6.17}$$

Using the Stieltjes-Stirling formula (6.12), noting that $|\arg(M + iz)| < 3\pi/4$ for $z \in \mathbb{Z}_\tau$, we get

$$|\Gamma\left(M - x_2 + ix_1\right)| \leq \sqrt{2\pi}\exp\left(\tfrac{6.83}{12\Delta_M} - M\right)\exp\left(\phi(M; x_1, x_2)\right), \tag{6.18}$$

where

$$\phi(M; x_1, x_2) := \frac{1}{2}\left(M - x_2 - \tfrac{1}{2}\right)\log\left((M - x_2)^2 + x_1^2\right) - x_1\arctan\left(\frac{x_1}{M - x_2}\right) + x_2,$$

and

$$\Delta_M := \inf_{x_1+ix_2\in Z_\tau} \left\{|M-x_2+ix_1|\right\} = \Big|\frac{M-\frac{\sqrt{2}}{2}}{\sqrt{2}}\Big|.$$

Similarly, for the other Γ-function appearing in (6.11),

$$\left|\Gamma\left(M+\tfrac{1}{2}-x_2+ix_1\right)\right| \leq \sqrt{2\pi}\exp\left(\tfrac{6.83}{12\Delta_{M+1/2}}-M-\tfrac{1}{2}\right)\exp\left(\phi(M+\tfrac{1}{2};x_1,x_2)\right). \tag{6.19}$$

Replacing (6.18) and (6.19) in (6.17) we get

$$|g(z)| \leq C\exp\Big(\tfrac{t}{2}\log\Big((M-x_2)^2+x_1^2\Big)+dx_2y+(r_1+r_2)\,\phi(M;x_1,x_2)+r_2\,\phi\left(M+\tfrac{1}{2};x_1,x_2\right)\Big),$$

where

$$P_2 := \left(\sqrt{2\pi}\exp\left(\tfrac{6.83}{12\Delta_M}-M\right)\right)^{r_1+r_2}\left(\sqrt{2\pi}\exp\left(\tfrac{6.83}{12\Delta_{M+1/2}}-M-\tfrac{1}{2}\right)\right)^{r_2}. \tag{6.20}$$

Using

$$\log\left((M-x_2)^2+x_1^2\right) \leq \log\left((M+|z|)^2\right) = 2\log(M+|z|) < M+|z|+\log(2),$$

$$\begin{aligned}|g(z)| &\leq P_2\exp\left(\tfrac{t}{2}\log\Big((M-x_2)^2+x_1^2\Big)+dx_2y+(r_1+r_2)\,\phi(M;x_1,x_2)+r_2\,\phi\left(M+\tfrac{1}{2};x_1,x_2\right)\right)\\ &\leq P_2\exp\big(\widetilde{\phi}(|z|)\big),\end{aligned}$$

where

$$\begin{aligned}\widetilde{\phi}(|z|) := \tfrac{t}{2}\left(\log(2)+M+|z|\right)+d|z|y+(r_1+r_2)\left[\tfrac{1}{2}\left(M+|z|+\tfrac{1}{2}\right)\left(\log 2+M+|z|\right)+|z|\left(\tfrac{\pi}{2}+1\right)\right]\\ +r_2\left[\tfrac{1}{2}\left(M+1+|z|\right)\left(\log 2+M+\tfrac{1}{2}+|z|\right)+|z|\left(\tfrac{\pi}{2}+1\right)\right].\end{aligned} \tag{6.21}$$

Let us fix $\gamma=\frac{1}{2}$. Then (6.9) is satisfied with

$$A := \sup_{|z|>0}\left(\widetilde{\phi}(|z|)\exp(-\gamma|z|)\right). \tag{6.22}$$

To find this supremum we can use elementary calculus. The critical points of $u\mapsto\widetilde{\phi}(u)\exp(-\gamma u)$ are the solutions of

$$\widetilde{\phi}'(u) = \gamma\widetilde{\phi}(u). \tag{6.23}$$

Using (6.21), we see that (6.23) is a quadratic equation in u whose only positive real solution is

$$\begin{aligned}u^\star = |z| = \frac{-b+\sqrt{b^2-4ac}}{2a},\qquad a := \gamma r_1/2+\gamma r_2,\qquad b := \gamma r_2\pi+\gamma r_2\log 2+2\gamma r_2 M\\ +\gamma r_1 M+\tfrac{1}{2}\gamma r_1\log 2+\tfrac{1}{2}\gamma r_1\pi+\gamma dy+\tfrac{5}{4})\gamma r_1+3\gamma r_2+\tfrac{1}{2}\gamma t-r_1-2r_2,\\ c := -t/2-dy-(r_1+r_2)\left(\tfrac{1}{2}\log(2)+M+\tfrac{5}{4}+\tfrac{\pi}{2}\right)-r_2\left(\tfrac{\pi}{2}\log(2)+M+\tfrac{7}{4}+\tfrac{\pi}{2}\right)\\ +\gamma\Big(\tfrac{1}{2}t\left(\log(2)+M\right)+(r_1+r_2)\left(\tfrac{M}{2}+\tfrac{1}{4}\right)\left(\log(2)+M\right)+r_2\left(\tfrac{M}{2}+\tfrac{1}{2}\right)\left(\log(2)+M+\tfrac{1}{2}\right)\Big).\end{aligned} \tag{6.24}$$

Hence, the supremum A in (6.22) is

$$A = \widetilde{\phi}(u^\star)\exp(-\gamma u^\star), \tag{6.25}$$

which yields the estimate for $z \in Z_\tau$, $|g(z)| \leq P_2 \exp\left(A \exp(\gamma|z|)\right)$, where $\gamma = \frac{1}{2}$, P_2 is given by (6.20), $\widetilde{\phi}$ by (6.21), $u^\star$ by (6.24) and A by (6.25).

Given $D > 1$ and the choices $\frac{\pi}{8} = \sigma < \tau = \frac{\pi}{4}$, we now have all the parameters needed to apply the Takahasi-Mori-Molin theorem to $f^{(t)}$. Thus, letting

$$\mathrm{fs}_t(y) := \frac{(-d)^t}{2\pi} e^{-dMy} \sum_{k=-n}^{n} g(\sinh(kh)) \cosh(kh), \tag{6.26}$$

we have

$$|f^{(t)}(y) - \mathrm{fs}_t(y)| = |E_t(y)| \leq \varepsilon_t(y, D) := \frac{(-d)^t}{2\pi} e^{-dMy-D} \qquad (t = 0, 1, 2, 3). \tag{6.27}$$

6.3 Error bound for our numerical approximation to F_1 in (3.7).

We now deduce an upper bound for the error $\left| F_1 - \left(\log(f_{2\mathrm{approx}}/f_{2\mathrm{a}}) - \log(f_{1\mathrm{approx}}/f_{1\mathrm{a}}) \right) \right|$, where $f_{1\mathrm{a}}, f_{2\mathrm{a}}$ are defined in (3.6) and, using the notation of (6.26),

$$f_{1\mathrm{approx}} := 3 \frac{\mathrm{fs}_2 \cdot \mathrm{fs}_1}{(\mathrm{fs}_0)^2}, \qquad f_{2\mathrm{approx}} := \frac{\mathrm{fs}_3}{\mathrm{fs}_0} + 2 \left(\frac{\mathrm{fs}_1}{\mathrm{fs}_0} \right)^3.$$

Recall from (6.27), that we have an approximation $f^{(t)}(y) = \mathrm{fs}_t(y) + E_t(y) =$ computable+error with an explicit upper bound for the error term $|E_t(y)| \leq \varepsilon_t(y, D)$. Thus,

$$\begin{aligned} f_1 &:= 3 \frac{f^{(2)} f'}{f^2} = 3 \frac{(\mathrm{fs}_2(y) + E_2(y))(\mathrm{fs}_1(y) + E_1(y))}{(\mathrm{fs}_0(y) + E_0(y))^2} = 3 \frac{\alpha_1 + a_1}{\beta_1 + b_1} \\ f_2 &:= \frac{f^{(3)}}{f} + 2 \left(\frac{f'}{f} \right)^3 = \frac{\mathrm{fs}_3(y) + E_3(y)}{\mathrm{fs}_0(y) + E_0(y)} + 2 \left(\frac{\mathrm{fs}_1(y) + E_1(y)}{\mathrm{fs}_0(y) + E_0(y)} \right)^3 = \frac{\alpha_2 + a_2}{\beta_2 + b_2} + 2 \frac{\mu_2 + u_2}{\lambda_2 + l_2}, \end{aligned}$$

where

$$\alpha_1 := \mathrm{fs}_1(y)\mathrm{fs}_2(y), \quad \beta_1 := (\mathrm{fs}_0(y))^2, \quad \alpha_2 := \mathrm{fs}_3(y), \quad \beta_2 := \mathrm{fs}_0(y, \quad \mu_2 := (\mathrm{fs}_1(y))^3, \quad \lambda_2 := (\mathrm{fs}_0(y))^3,$$

are the main parts, and the corresponding perturbations are given by

$$\begin{aligned} &a_1 := \mathrm{fs}_1(y)E_2(y) + \mathrm{fs}_2(y)E_1(y) + E_1(y)E_2(y), \quad b_1 := 2\mathrm{fs}_0(y)E_0(y) + E_0(y)^2, \quad a_2 := E_3(y), \\ &b_2 := E_0(y), \quad u_2 := 3(\mathrm{fs}_1(y))^2 E_1(y) + 3\mathrm{fs}_1(y)(E_1(y))^2 + (E_1(y))^3, \\ &l_2 := 3(\mathrm{fs}_0(y))^2 E_0(y) + 3\mathrm{fs}_0(y)(E_0(y))^2 + (E_0(y))^3. \end{aligned}$$

Using Lemma 3.4.1 for the expressions $\frac{\alpha_1+a_1}{\beta_1+b_1}$, $\frac{\alpha_2+a_2}{\beta_2+b_2}$ and $\frac{\mu_2+u_2}{\lambda_2+l_2}$, along with the upper bounds for the errors $E_t(y)$, we get

$$\begin{aligned}\big|f_1 - 3\frac{\mathrm{fs}_1(y)\cdot \mathrm{fs}_2(y)}{(\mathrm{fs}_0(y))^2}\big| = \big|3\frac{\alpha_1+a_1}{\beta_1+b_1} - 3\frac{\alpha_1}{\beta_1}\big| \le 3\bigg(2\left(2|\mathrm{fs}_0(y)|\varepsilon_0(y,D) + (\varepsilon_0(y,D))^2\right)\frac{|\mathrm{fs}_1(y)\mathrm{fs}_2(y)|}{(\mathrm{fs}_0(y))^4}\\ + 2\big(|\mathrm{fs}_1(y)|\varepsilon_2(y,D) + |\mathrm{fs}_2(y)|\varepsilon_1(y,D) + \varepsilon_1(y,D)\varepsilon_2(y,D)\big)\frac{1}{(\mathrm{fs}_0(y))^2}\bigg) =: \mathrm{error}_1,\end{aligned}$$

and

$$\begin{aligned}\big|f_2 - \frac{\mathrm{fs}_3(y)}{\mathrm{fs}_0(y)} - 2\frac{(\mathrm{fs}_1(y))^3}{(\mathrm{fs}_0(y))^3}\big| = \big|\frac{\alpha_2+a_2}{\beta_2+b_2} + 2\frac{\mu_2+u_2}{\lambda_2+l_2} - \frac{\alpha_2}{\beta_2} - 2\frac{\mu_2}{\lambda_2}\big| \le \big|\frac{\alpha_2+a_2}{\beta_2+b_2} - \frac{\alpha_2}{\beta_2}\big|\\ + 2\big|\frac{\mu_2+u_2}{\lambda_2+l_2} - \frac{\mu_2}{\lambda_2}\big| \le 2\big(\varepsilon_0(y,D)\big)\frac{|\mathrm{fs}_3(y)|}{(\mathrm{fs}_0(y))^2} + 2\big(\varepsilon_3(y,D)\big)\frac{1}{|\mathrm{fs}_0(y)|}\\ + 2\Big(2\big(3(\mathrm{fs}_0(y))^2\varepsilon_0(y,D) + 3|\mathrm{fs}_0(y)|(\varepsilon_0(y,D))^2 + (\varepsilon_0(y,D))^3\big)\frac{|(\mathrm{fs}_1(y))^3|}{(\mathrm{fs}_0(y))^6}\\ + 2\big(3(\mathrm{fs}_1(y))^2\varepsilon_1(y,D) + 3|\mathrm{fs}_1(y)|(\varepsilon_1(y,D))^2 + (\varepsilon_1(y,D))^3\big)\frac{1}{|(\mathrm{fs}_0(y))^3|}\Big) =: \mathrm{error}_2.\end{aligned}$$

Therefore, using the definitions of $f_{1\mathrm{approx}}$ and $f_{2\mathrm{approx}}$, we get

$$\big|f_1 - f_{1\mathrm{approx}}\big| \le \frac{\mathrm{error}_1}{f_{1\mathrm{approx}}}, \qquad \big|f_2 - f_{2\mathrm{approx}}\big| \le \frac{\mathrm{error}_2}{f_{2\mathrm{approx}}}. \tag{6.28}$$

Using (6.28) and $\log(1+\epsilon) \le \sqrt{\epsilon}$ for $\epsilon \ge 0$, we obtain

$$\begin{aligned}\left|F_1 - \left(\log\left(\frac{f_{2\mathrm{approx}}}{f_{2\mathrm{a}}}\right) - \log\left(\frac{f_{1\mathrm{approx}}}{f_{1\mathrm{a}}}\right)\right)\right| &\le \left|\log\left(\frac{f_2}{f_{2\mathrm{approx}}}\right)\right| + \left|\log\left(\frac{f_1}{f_{1\mathrm{approx}}}\right)\right|\\ &\le \sqrt{\frac{\mathrm{error}_2}{f_{2\mathrm{approx}}}} + \sqrt{\frac{\mathrm{error}_1}{f_{1\mathrm{approx}}}} =: \mathrm{error}_{F_1}.\end{aligned}$$

6.4 Proofs of claims from Chapter 2.

The proof of Lemma 2.1.1 will hinge on computing an explicit asymptotic expansion, so we turn to this subject first.

6.4.1 Explicit asymptotic expansions

For $K > 0$ let $\Omega(K) := \{s \in \mathbb{C} | \operatorname{Re}(s) > K\}$. Fix an integer $N \ge 0$ and suppose for some $\alpha_j \in \mathbb{C}$ $(0 \le j \le N)$ and $E, \kappa \in \mathbb{R}$ with $\kappa > K$, the function $g : \Omega(K) \to \mathbb{C}$ satisfies for all $s \in \Omega(\kappa)$

$$\Big|g(s) - \sum_{j=0}^{N} \frac{\alpha_j}{s^j}\Big| \le \frac{E}{|s^{N+1}|}. \tag{6.29}$$

Then we shall write

$$g(s) \sim \Big(\sum_{j=0}^{N} \alpha_j s^{-j}; \kappa; E\Big) \qquad\qquad (\kappa > K),$$

and call $\big(\sum_{j=0}^{N} \alpha_j s^{-j}; \kappa; E\big)$ an (explicit) asymptotic expansion of g of order N. Of course, $\alpha_j = \alpha_{j,g}$ depends only on g, but K, κ and E have some freedom. In fact, usually E will depend on κ. Even K plays a role since often estimates will hold only for $\kappa > K$. To operate with these expansions it is convenient to rewrite the above definition as

$$g(s) = \sum_{j=0}^{N} \frac{\alpha_j}{s^j} \quad + \quad \frac{G(s)}{s^{N+1}} \qquad\qquad \big(\alpha_j \in \mathbb{C},\ \kappa > K, \quad |G(s)| \le E \ \ \forall s \in \Omega(\kappa)\big), \tag{6.30}$$

where $G : \Omega(K) \to \mathbb{C}$ is defined by the above equality.

For a fixed $a \in \mathbb{C}$, the expansion of $ag(s)$ is clearly $ag(s) \sim \big(\sum_{j=0}^{N} a\alpha_j s^{-j}; \kappa; |a|E\big)$. If $N \le M$, reduction from degree M to degree N expansions is given by

$$g(s) \sim \Big(\sum_{j=0}^{M} \alpha_j s^{-j}; \kappa; E\Big) \Longrightarrow g(s) \sim \Big(\sum_{j=0}^{N} \alpha_j s^{-j}; \kappa; \frac{E}{\kappa^{M-N}} + \sum_{j=N+1}^{M} |\alpha_j| \kappa^{-j+N+1}\Big). \tag{6.31}$$

Sums and products of expansions of the same order are easily defined. Namely,

$$g(s) \sim \Big(\sum_{j=0}^{N} \alpha_j s^{-j}; \kappa; E\Big), \qquad \tilde{g}(s) \sim \Big(\sum_{j=0}^{N} \tilde{\alpha}_j s^{-j}; \tilde{\kappa}; \tilde{E}\Big) \implies$$

$$(g + \tilde{g})(s) \sim \Big(\sum_{j=0}^{N} (\alpha_j + \tilde{\alpha}_j) s^{-j}; \hat{\kappa}; E + \tilde{E}\Big), \qquad g\tilde{g}(s) \sim \Big(\sum_{j=0}^{N} \big(\sum_{i=0}^{j} \alpha_i \tilde{\alpha}_{j-i}\big) s^{-j}; \hat{\kappa}; \hat{E}\Big),$$

$$\hat{\kappa} := \max(\kappa, \tilde{\kappa}), \qquad \hat{E} := \sum_{j=1}^{N} \sum_{i=N+1-j}^{N} \frac{|\alpha_j \tilde{\alpha}_i|}{\hat{\kappa}^{i+j-N-1}} \ + \ \sum_{j=0}^{N} \frac{\tilde{E}|\alpha_j| + E|\tilde{\alpha}_j|}{\hat{\kappa}^j}.$$

Suppose now that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is a Maclaurin series with real coefficients $a_k \ge 0$, convergent for $|z| < \gamma$ for some $\gamma > 0$, and we wish to compute the asymptotic expansion of $f(b/s^k)$ for some $b \in \mathbb{C}$ and $k \in \mathbb{N}$. For (real) $x \in [0, \gamma)$, the Lagrange form of the remainder gives $f(x) = \sum_{k=0}^{N} a_k x^k + R_N(x)$, where $R_N(x) := \sum_{k=N+1}^{\infty} a_k x^k = \frac{x^{N+1} f^{(N+1)}(\xi)}{(N+1)!}$ for some $\xi \in (0, x)$. Assuming $\mathrm{Re}(s) > \kappa$ and $|b|/\kappa^j < \gamma$ we have

$$f(b/s^j) = \sum_{k=0}^{N} a_k b^k s^{-jk} + \sum_{k=N+1}^{\infty} a_k b^k s^{-jk} = \sum_{k=0}^{N} a_k b^k s^{-jk} + R_N(b/s^j).$$

However, since we are assuming $a_k \ge 0$, we find for some $\xi \in (0, |b/s^j|) \subset (0, |b|/\kappa^j)$,

$$|R_N(b/s^j)| \le \sum_{k=N+1}^{\infty} a_k |b/s^j|^k = R_N(|b/s^j|) = \frac{|b|^{N+1}}{(N+1)!|s|^{j(N+1)}} |f^{(N+1)}(\xi)| \le \frac{|b|^{N+1} \mu_N(\kappa)}{(N+1)!|s|^{j(N+1)}},$$

where $\mu_N(\kappa) := \sup_{\xi\in[0,|b|/\kappa^j]} \big\{|f^{(N+1)}(\xi)|\big\}$. Hence we have the asymptotics of order $j(N+1)-1$

$$f(b/s^j) \sim \Big(\sum_{k=0}^{N} a_k b^k s^{-jk} + 0\cdot s^{-(j(N+1)-1)}; \kappa; |b|^{N+1}\mu_N(\kappa)/(N+1)!\Big) \qquad \big(\kappa > (|b|/\gamma)^{1/j}\big). \quad (6.32)$$

If $j>1$ the estimate of $R_N(b/s^j)$ shows that this is not naturally an expansion of order N, but rather of order $j(N+1)-1$ with coefficients of $s^{-\ell}$ vanishing for $jN<\ell<j(N+1)$, but we will always apply the reduction operation (6.31) to make it of order N. An example, which we will need, is (the reduction to order N of)

$$\exp(b/s^j) \sim \Big(\sum_{k=0}^{N} \frac{b^k}{k!} s^{-kj}; \kappa; \frac{|b|^{N+1}\exp(|b|/\kappa^j)}{(N+1)!}\Big) \qquad (\kappa>0).$$

We now compute a few other expansions. Let

$$g(s) := \frac{1}{s+a} = \frac{1/s}{1-\frac{-a}{s}} = \frac{1}{s}\Big(\sum_{j=0}^{N-1}\frac{(-a)^j}{s^j} + \frac{(-a)^N}{s^N}\frac{1}{1+\frac{a}{s}}\Big) = \sum_{j=1}^{N}\frac{(-a)^{j-1}}{s^j} + \frac{(-a)^N}{s^{N+1}}\frac{1}{1+\frac{a}{s}}. \quad (6.33)$$

Since we are assuming $\mathrm{Re}(s)>\kappa>0$, we obtain for real a,

$$\frac{1}{s+a} \sim \Big(\sum_{j=1}^{N}\frac{(-a)^{j-1}}{s^j}; \kappa; \frac{|a|^N}{\min(1,1+\frac{a}{\kappa})}\Big) \qquad \big(\kappa>\max(0,-a)\big). \quad (6.34)$$

Similarly, for $a\in\mathbb{R}$,

$$\frac{1}{1+\frac{a}{s}} \sim \Big(\sum_{j=0}^{N}\frac{(-a)^{j}}{s^j}; \kappa; \frac{|a|^{N+1}}{\min(1,1+\frac{a}{\kappa})}\Big) \qquad \big(\kappa>\max(0,-a)\big). \quad (6.35)$$

Suppose now that for some $c\in\mathbb{R}$ and $b>0$ we want to obtain the asymptotic expansion of $g(bs+c)$ from that of $g\sim\big(\sum_{j=0}^{N}\alpha_j s^{-j};\kappa;E\big)$. If $\mathrm{Re}(s)>\hat\kappa := \max\big(0,(\kappa-c)/b\big)$, so that $\mathrm{Re}(bs+c)>\kappa$, we have

$$\Big|g(bs+c)-\sum_{j=0}^{N}\frac{\alpha_j}{(bs+c)^j}\Big| \le \frac{E}{|bs+c|^{N+1}} = \frac{E(b|s|)^{-(N+1)}}{|1+\frac{c/b}{s}|^{N+1}} \le \frac{Eb^{-(N+1)}|s|^{-(N+1)}}{\big(\min(1,1+\frac{c}{b\hat\kappa})\big)^{(N+1)}} \quad (6.36)$$

Since $(bs+c)^{-j} = b^{-j}\frac{1}{(s+\frac{c}{b})^j}$, its asymptotic expansion can be computed as a scalar multiple of the j-th power of the expansion (6.34). Summing multiples of these and adding the error estimate in (6.36), we obtain the desired expansion of $g(bs+c)$.

We will need to find the asymptotic expansion of $\exp\big(g(s)\big)$ from that of $g(s)\sim\big(\sum_{j=0}^{N}\frac{\alpha_j}{s^j};\kappa;E\big)$. Most of this work is already done since for $\mathrm{Re}(s)>\kappa$

$$e^{g(s)} = \exp\big(G(s)/s^{N+1}\big)e^{\alpha_0}\prod_{j=1}^{N} e^{\alpha_j/s^j} \qquad (|G(s)|\le E). \quad (6.37)$$

We have already seen how to compute the expansion of e^{α_j/s^j}, and hence of the product above. Since for any $z \in \mathbb{C}$, we have $|e^z - 1| \le |z|e^{|z|}$, we conclude

$$\exp\bigl(G(s)/s^{N+1}\bigr) = 1 + \frac{F(s)}{s^{N+1}}, \qquad |F(s)| \le |G(s)| \exp(|G(s)/s^{N+1}|) \le E\exp(E/\kappa^{N+1}).$$

Thus we have asymptotics of order N for $\exp(G(s)/s^{N+1}) \sim \bigl(1; \kappa; E\exp(E/\kappa^{N+1})\bigr)$, and so (6.37) yields an expansion of order N for $\exp\bigl(g(s)\bigr)$.

We will also need the asymptotics of $\log\bigl(1+\frac{a}{s}\bigr)$ for $a \in \mathbb{R}$. Since $\frac{d^{N+1}}{dt^{N+1}}\bigl(-\log(1-t)\bigr) = N!/(1-t)^{N+1}$, we obtain from (6.32)

$$\log\bigl(1+\tfrac{a}{s}\bigr) \sim \Bigl(-\sum_{j=1}^{N} \frac{(-a)^j}{js^j}; \kappa; \frac{|a|^{N+1}}{(N+1)\bigl(1-\frac{|a|}{\kappa}\bigr)^{N+1}}\Bigr) \qquad (\kappa > |a|). \tag{6.38}$$

Next we obtain the expansion of $Q(s)/(12s)$, defined by

$$\Gamma(s) = \sqrt{2\pi} s^{s-1/2} e^{-s} e^{\frac{Q(s)}{12s}} \qquad \bigl(s \notin (-\infty, 0]\bigr), \tag{6.39}$$

and so closely related to Stirling's formula. The explicit asymptotics of $Q(s)/(12s)$ were worked out by Stieltjes [Sti89] [SS90] for $s \notin (-\infty, 0]$ in terms of Bernoulli numbers as

$$\frac{Q(s)}{12s} = \sum_{k=1}^{n-1} \frac{B_{2k}}{2k(2k-1)s^{2k-1}} + R_n(s), \qquad |R_n(s)| < \frac{|B_{2n}|}{2n(2n-1)|s|^{2n-1}\cos^{2n}(\frac{1}{2}\arg(s))} \qquad (n \ge 1).$$

Since $B_j = 0$ for odd integers $j > 1$ and $|\arg(s)/2| < \pi/4$ if $\mathrm{Re}(s) > 0$, we obtain for any $\kappa > 0$

$$\frac{Q(s)}{12s} \sim \Bigl(\sum_{j=1}^{N} \frac{B_{j+1}}{(j+1)js^j}; E_N; \kappa\Bigr), \qquad E_N := \begin{cases} \frac{\sqrt{2}^{N+2}|B_{N+2}|}{(N+2)(N+1)} & \text{if } N \ge 0 \text{ is even,} \\ \frac{\sqrt{2}^{N+3}|B_{N+3}|}{\kappa(N+3)(N+2)} & \text{if } N \ge 1 \text{ is odd.} \end{cases} \tag{6.40}$$

6.4.2 Proof of Lemma 2.1.1.

We begin by recalling the notation used in Lemma 2.1.1. For a fixed integer $N \ge 0$, let

$$\psi(s) := \varrho(-s) = \frac{(2\pi)^{1-d} d^{ds} \Gamma(s)^{r_1+r_2-t} \Gamma(s+1)^t \Gamma(s+\frac{1}{2})^{r_2}}{\Gamma(-N+s_t)} + \sum_{k=0}^{N} (-1)^{k+1} A_k \prod_{j=0}^{N-k} (-N+s_t+j)\,,$$
$$s_t := ds - \frac{r_1+r_2+1}{2} + t = t + \frac{r_2}{2} - \frac{1}{2} + d(s-\tfrac{1}{2}) \qquad (d := r_1+2r_2,\ t = 0,1,2 \text{ or } 3). \tag{6.41}$$

Lemma 2.1.1 states that there exist A_k and $K_1, K_2 \in \mathbb{R}$ such that $\mathrm{Re}(s) > K_1$ implies $|\psi(s)| \le K_2$. More precisely, $A_0 := (2\pi)^{\frac{1-d}{2}} d^{\frac{r_1+r_2}{2}-t}$ and for any $K_1 > 1/2$, we shall give an algorithm calculating $A_k \in \mathbb{R}$ $(1 \le k \le N)$ and $K_2 = K_2(K_1)$ such that $\mathrm{Re}(s) > K_1 > 1/2$ implies $|\psi(s)| \le K_2$.[4] The

[4] It would appear at first sight that the A_k depend on N, and should therefore be written $A_{k,N}$ $(1 \le k \le N)$. However, in the end we will prove that the $A_{k,N}$ determine the k-th term of the asymptotic expansion of $f^{(t)}(y)$ (see (2.3) and Lemma 2.1.3), and hence turn out not to depend on N. We therefore always write A_k for $A_{k,N}$.

algorithm will turn out to be just the computation of a certain explicit asymptotic expansion of order N using the results of §6.4.1.

Since we always assume $K_1 > \frac{1}{2}$ and $s \in \Omega(K_1)$, we have $\mathrm{Re}(s_t+1) > 1/2$. Using $\Gamma(s_t - j + 1) = (s_t - j)\Gamma(s_t - j)$ for $j = 0, 1, \ldots, N$, we find

$$\begin{aligned}\psi(s) = (s_t - N)\, &\frac{(2\pi)^{1-d} d^{ds} s^t \Gamma(s)^{r_1+r_2} \Gamma(s+\frac{1}{2})^{r_2}}{\Gamma(s_t+1)} \prod_{j=0}^{N-1} (s_t - j) \\ &- (s_t - N) \sum_{k=0}^{N} (-1)^k A_k \prod_{j=1}^{N-k} (s_t - N + j) = (s_t - N)\big(\xi(s) - \hat{\xi}(s)\big), \end{aligned} \tag{6.42}$$

$$\xi(s) := \frac{(2\pi)^{1-d} d^{ds} s^t \Gamma(s)^{r_1+r_2} \Gamma(s+\frac{1}{2})^{r_2}}{\Gamma(s_t+1)} \prod_{j=0}^{N-1} (s_t - j), \qquad \hat{\xi}(s) := \sum_{k=0}^{N} (-1)^k A_k \prod_{j=1}^{N-k} (s_t - N + j).$$

Denoting by $Q(s)/(12s)$ the error committed in replacing $\log\Gamma$ by Stirling's approximation (see (6.39)), we get

$$\begin{aligned}\xi(s) = \Big(\prod_{j=0}^{N-1} (s_t - j)\Big) &(2\pi)^{1-d} e^{ds\log d} e^{t\log s} (2\pi)^{\frac{r_1+r_2}{2}} s^{(s-\frac{1}{2})(r_1+r_2)} e^{-(r_1+r_2)s} e^{\frac{(r_1+r_2)Q(s)}{12s}} \\ &\cdot (2\pi)^{\frac{r_2}{2}-\frac{1}{2}} \exp\Big(r_2 s\log(s+\tfrac{1}{2}) - r_2(s+\tfrac{1}{2}) + r_2\frac{Q(s+\frac{1}{2})}{12(s+\frac{1}{2})}\Big) \Big/ \Big((s_t+1)^{s_t+\frac{1}{2}} e^{-s_t-1+\frac{Q(s_t+1)}{12(s_t+1)}}\Big),\end{aligned} \tag{6.43}$$

where log denotes the principal branch of log on $\mathbb{C} - (-\infty, 0]$. Note that this branch satisfies $\log(ab^{\pm 1}) = \log(a) \pm \log(b)$ if $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(b) > 0$. Letting $A := t + 1 - \frac{r_1+r_2+1}{2} > -d/2$ and noting $s_t + 1 = ds + A$, we have

$$\log(s+\tfrac{1}{2}) = \log s + \log(1+\tfrac{1}{2s}), \qquad \log(s_t+1) = \log(ds+A) = \log s + \log(d+\tfrac{A}{s}) \qquad (\,\mathrm{Re}(s) > \tfrac{1}{2}),$$

since

$$\mathrm{Re}(d + \tfrac{A}{s}) = d + A\,\mathrm{Re}(1/s) = d + A\,\mathrm{Re}(s)/|s|^2 > d - (d/2)\mathrm{Re}(s)/|s|^2 \geq d - (d/2)/\mathrm{Re}(s) > 0.$$

Letting

$$h(s) := q(s) + A - \tfrac{r_2}{2} + r_2 s\log(1+\tfrac{1}{2s}) + ds\log\big(\tfrac{d}{d+\frac{A}{s}}\big) + \big(\tfrac{r_1+r_2}{2} + 1 - t\big)\log(d + \tfrac{A}{s}),$$

$$q(s) := (r_1+r_2)\tfrac{Q(s)}{12s} + r_2\tfrac{Q(s+\frac{1}{2})}{12(s+\frac{1}{2})} - \tfrac{Q(ds+A)}{12(ds+A)}, \qquad C(s) := (2\pi)^{\frac{1-d}{2}} \frac{\prod_{j=0}^{N-1}(s_t - j)}{ds + A},$$

we obtain

$$\begin{aligned}
\xi(s) &= C(s)\exp\Big(q(s) + A + ds\log d + \big(t + (s-\tfrac{1}{2})(r_1+r_2)\big)\log s \\
&\qquad\qquad - \tfrac{r_2}{2} + r_2 s\log(s+\tfrac{1}{2}) - (ds + A - \tfrac{3}{2})\big(\log s + \log(d+\tfrac{A}{s})\big)\Big) \\
&= C(s)\exp\Big(q(s) + A + ds\log d + (t + (s-\tfrac{1}{2})(r_1+r_2))\log s + r_2 s\big(\log s + \log(1+\tfrac{1}{2s})\big) \\
&\quad - \tfrac{r_2}{2} - (ds + A - \tfrac{3}{2})\big(\log s + \log(d+\tfrac{A}{s})\big)\Big) = C(s)\exp(h(s) + \log s) = C(s)s\exp(h(s)).
\end{aligned}$$

Since $A_0 := (2\pi)^{\frac{1-d}{2}} d^{\frac{r_1+r_2}{2}-t}$, ψ in (6.42) can be written in the form

$$\begin{aligned}
\psi(s) &= (s_t - N)(\xi(s) - \hat{\xi}(s)) = (s_t - N)\Big(C(s)se^{h(s)} - \sum_{k=0}^{N}(-1)^k A_k \prod_{j=1}^{N-k}(s_t - N + j)\Big) \\
&= A_0(ds)^{N+1}\big(1 + \frac{A-1-N}{ds}\big)\Big(\frac{C(s)s}{A_0(ds)^N}e^{h(s)} - \sum_{k=0}^{N}\frac{(-1)^k A_k}{A_0(ds)^k}\prod_{j=1}^{N-k}\big(1 + \frac{A-1-N+j}{ds}\big)\Big) \\
&= A_0(ds)^{N+1}\big(1 + \frac{A-1-N}{ds}\big)\Big(\tilde{C}(s)e^{\tilde{h}(s)} - \sum_{k=0}^{N}\frac{(-1)^k A_k}{A_0(ds)^k}\prod_{j=1}^{N-k}\big(1 + \frac{A-1-N+j}{ds}\big)\Big), \quad (6.44)
\end{aligned}$$

where

$$\tilde{C}(s) := \frac{C(s)}{(2\pi)^{\frac{1-d}{2}}(ds)^{N-1}} = \big(1 + \tfrac{dk_1+1}{ds}\big)^{-1}\prod_{j=0}^{N-1}(1 + \frac{dk_1 - j}{ds}) \qquad \big(k_1 := \frac{A-1}{d}\big), \quad (6.45)$$

$$\begin{aligned}
\tilde{h}(s) &:= h(s) - (\frac{r_1+r_2}{2} - t + 1)\log(d) = (r_1+r_2)\tfrac{Q(s)}{12s} + r_2\tfrac{Q(s+\frac{1}{2})}{12(s+\frac{1}{2})} - \tfrac{Q(ds+A)}{12(ds+A)} \\
&\quad + \big(r_2 s\log(1+\frac{1}{2s}) - \frac{r_2}{2}\big) + (\frac{r_1+r_2}{2} + 1 - t)\log(1+\frac{A}{ds}) + A - ds\log(1+\tfrac{A}{ds}). \quad (6.46)
\end{aligned}$$

The proof of Lemma 2.1.1 is now clear. It suffices to prove that we can compute an explicit asymptotic expansion $g(s) \sim \big(\sum_{j=0}^{N}\alpha_j s^{-j};\kappa;E\big)$ for $g(s) := \tilde{C}(s)e^{\tilde{h}(s)}$, valid for any $\kappa > 1/2$.[5] Indeed, induction on k shows that for any polynomial $p(x) \in \mathbb{C}[x]$ of degree N with $p(0) = 1$, and in particular for $p(x) := \sum_{j=0}^{N}\alpha_j x^j$, it is possible to find $A_k \in \mathbb{C}$ $(0 \le k \le N)$ so that $p(x) = \sum_{k=0}^{N}\frac{(-1)^k A_k x^k}{A_0 d^k}\prod_{j=1}^{N-k}\big(1 + \frac{(A-1-N+j)x}{d}\big)$. Then, by definition (6.29) of asymptotic expansion of order N, $\big|s^{N+1}\big(\tilde{C}(s)e^{\tilde{h}(s)} - p(1/s)\big)\big| \le E$ for $\mathrm{Re}(s) > \kappa$.

We now show how to compute an expansion for $\tilde{C}(s)e^{\tilde{h}(s)}$. We saw in §6.4.1 how to compute a product, sum and exponential of explicit asymptotic expansions, so most of our work is done. Indeed, the product $\prod_{j=0}^{N-1}(1 + \frac{dk_1-j}{ds})$ in (6.45) is already of the form $p(1/s)$, with p a polynomial of degree N, and so its asymptotic expansion is just $(p(1/s);\kappa;0)$ for any $\kappa > 0$. The expansion for

[5] We must also show that $\alpha_0 = 1$, since we claimed (and used) a value for A_0. This is clear since $\lim_{\mathrm{Re}(s)\to\infty}\tilde{C}(s) = 1$ and $\lim_{\mathrm{Re}(s)\to\infty}\tilde{h}(s) = 0$, as follows directly from the definitions of $\tilde{C}$ and $\tilde{h}$.

the other factor in (6.45) is given by (6.35). Namely,

$$\frac{1}{1+\frac{dk_1+1}{ds}} = \frac{1}{1+\frac{A}{ds}} \sim \Big(\sum_{j=0}^{N} \big(\tfrac{-A}{d}\big)^j s^{-j}; \kappa; \frac{|A/d|^{N+1}}{\min(1, 1+\frac{A}{d\kappa})}\Big) \qquad \big(\kappa > \max(0, -A/d)\big).$$

Since $\frac{1}{2} > \max(0, -A/d)$, we have an asymptotic expansion of $\tilde{C}(s)$ for any $\kappa > \frac{1}{2}$. The function $\tilde{h}(s)$ in (6.46) contains a sum of terms of the kind $Q(b_j s + c_j)/12(b_j s + c_j)$, with $b_j = 1$ or d and $c_j = 0, \frac{1}{2}$ or A. By (6.40) and the paragraph following (6.35) we obtain an explicit asymptotic expansion for $\mathrm{Re}(s) > \frac{1}{2}$ valid for the three Q-terms (using $-A/d < 1/2$, already noted above). The other terms in (6.46) have an expansion given by (6.38), valid again for $\mathrm{Re}(s) > \frac{1}{2}$ (but we have to expand one of the logarithms to order $N+1$ since s multiplies it). This concludes the proof of Lemma 2.1.1.

For example, if $N = 2$ (the case used in this thesis) the coefficients A_1 and A_2 are given by

$$A_1 = A_0\big(r_1^2 + r_1 r_2 - 12 r_1 t + r_2^2 - 12 r_2 t + 12t^2 - 1\big)/24, \tag{6.47}$$

$$\begin{aligned} A_2 = A_0\Big(&r_1^4 + 2r_1^3 r_2 - 24 r_1^3 t + 3r_1^2 r_2^2 - 48 r_1^2 r_2 t + 168 r_1^2 t^2 + 2r_1 r_2^3 - 48 r_1 r_2^2 t + 312 r_1 r_2 t^2 - 288 r_1 t^3 \\ &+ r_2^4 - 24 r_2^3 t + 168 r_2^2 t^2 - 288 r_2 t^3 + 144t^4 - 24 r_1^2 r_2 - 192 r_1^2 t - 24 r_1 r_2^2 - 336 r_1 r_2 t + 576 r_1 t^2 \\ &- 192 r_2^2 t + 576 r_2 t^2 - 384 t^3 + 22 r_1^2 + 22 r_1 r_2 - 264 r_1 t + 22 r_2^2 - 264 r_2 t + 264 t^2 - 23\Big)/1152. \end{aligned}$$

6.4.3 Proof of Lemma 2.1.2.

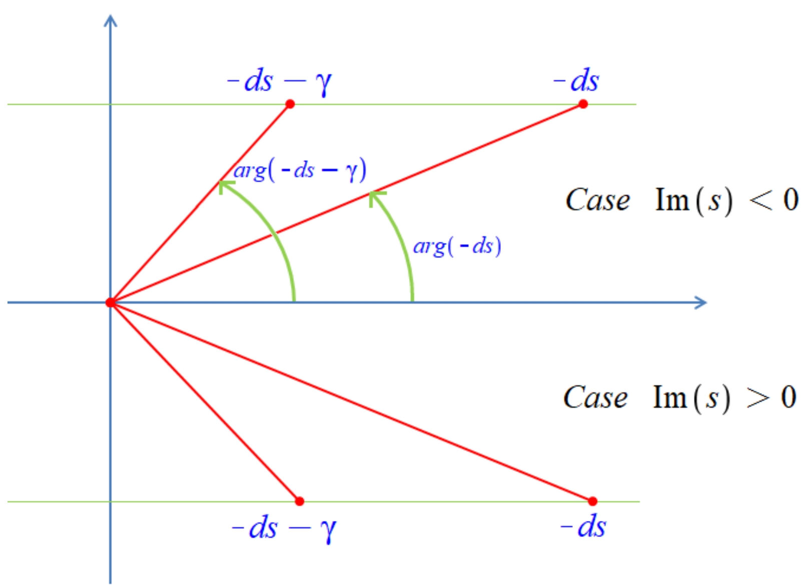


Figure 6.4: Real parts and arguments of the complex points $-ds$ and $-ds - \gamma$.

We prove a more precise version of Lemma 2.1.2 that goes as follows.

Assume $\gamma := \frac{r_1+1}{2} + \frac{r_2}{2} - t + N > 0$ *and* $K_1 > \max(\frac{1}{2}, \frac{\gamma + \frac{1}{2}}{d})$ *where* $d := r_1 + 2r_2$. *Then for*

$\mathrm{Re}(s) < -K_1$ *we have the inequalities* $|\varrho(s)| \le K_2$ *and*

$$|\Gamma(-ds-\gamma)| \le K_2'|s|^{-\frac{1}{2}-\gamma} e^{-d\,\mathrm{Re}(s)\log|ds| + d\,\mathrm{Re}(s) + d\arg(-s)\mathrm{Im}(s)}, \qquad K_2' := \sqrt{2\pi} e^{\gamma+\frac{1}{6}} d^{-\gamma-1/2}, \quad (6.48)$$

where $\arg(s)$ *is the branch of the argument which is real for* $s > 0$*, and* $K_2 = K_2(K_1)$ *is as in the lemma proved in the previous subsection.*

The bound on ϱ was already proved for any $K_1 > 1/2$, so we now prove the other bound. Note that $\mathrm{Re}(-ds-\gamma) > 1/2$ since we are assuming $\mathrm{Re}(s) < -K_1 < \frac{-\gamma-\frac{1}{2}}{d}$. Using (6.39), and (6.40) with $N = 0$, we have $|Q(s)/(12s)| \le 1/6$ for $\mathrm{Re}(s) > 0$. Hence for $\mathrm{Re}(s) < -K_1'$ we have

$$\begin{aligned}
|\Gamma(-ds-\gamma)| &= \left| \sqrt{2\pi}(-ds-\gamma)^{-ds-\gamma-1/2} e^{ds+\gamma} e^{\frac{Q(-ds-\gamma)}{12(-ds-\gamma)}} \right| \\
&= \sqrt{2\pi} e^{\gamma+\frac{1}{6}} e^{d\,\mathrm{Re}(s)} e^{(-d\,\mathrm{Re}(s)-\gamma-1/2)\log|ds+\gamma|} e^{d\,\mathrm{Im}(s)\ \arg(-ds-\gamma)} \\
&\le \sqrt{2\pi} e^{\gamma+\frac{1}{6}} e^{d\,\mathrm{Re}(s)} e^{(-d\,\mathrm{Re}(s)-\gamma-1/2)\log|ds|} e^{d\,\mathrm{Im}(s)\ \arg(-ds-\gamma)} \\
&\le \sqrt{2\pi} e^{\gamma+\frac{1}{6}} e^{d\,\mathrm{Re}(s)} e^{(-d\,\mathrm{Re}(s)-\gamma-1/2)\log|ds|} e^{d\,\mathrm{Im}(s)\ \arg(-s)} \qquad (6.49) \\
&= K_2'|s|^{-\frac{1}{2}-\gamma} e^{-d\,\mathrm{Re}(s)\log|ds| + d\,\mathrm{Re}(s) + d\arg(-s)\mathrm{Im}(s)}.
\end{aligned}$$

For (6.49) use $\mathrm{Im}(s)\ \arg(-ds-\gamma) \le \mathrm{Im}(s)\ \arg(-ds)$ (see Figure 6.4) and $\arg(-ds) = \arg(-s)$. □

6.4.4 Proof of Lemma 2.1.3.

Recall that for $z > 0$ and any $\omega < 0$ we defined in (2.4)

$$H(z) := \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} z^s \Gamma(-s)^{r_1+r_2-t}\Gamma(-s+1)^t \Gamma(-s+\tfrac{1}{2})^{r_2}\, ds$$

We prove a more precise version of Lemma 2.1.3 that goes as follows.
Assume $\gamma := \frac{r_1+1}{2} + \frac{r_2}{2} - t + N > 0$*,* $K_1 > \max(\frac{1}{2}, \frac{\gamma+\frac{1}{2}}{d})$ *and* $y \ge \log(dK_1)$*. Then*

$$H(d^{-d}e^{dy}) = \frac{(2\pi)^{d-1}}{d} e^{-y(\frac{r_1+r_2-1}{2}-t)} e^{-e^y} \sum_{k=0}^{N} (-1)^k A_k e^{-ky} \ - \ i(2\pi)^{d-2}E(y), \qquad (6.50)$$

where

$$|E(y)| \le \frac{K_2 e^{\gamma+\frac{1}{6}}\pi\sqrt{8}}{d}\Big(1 + \frac{8e^{-.438dK_1}}{\pi^{3/2}\sqrt{dK_1}}\Big) e^{-y(\frac{r_1+r_2-1}{2}-t+N+1)} e^{-e^y}, \qquad (6.51)$$

where $K_2 = K_2(K_1)$*, as in §6.4.2.*
Aside from the explicit constants, the main point of course is that $E(y)$ is bound by e^{-y} times the smallest term in the sum, *i. e.* that corresponding to $k = N$.

Proof. Set $z := e^{dy}$. Combining (2.6) and (2.9), we see that (6.50) is just notation if we define $E(y) = \sigma(z) := \int_{\omega-i\infty}^{\omega+i\infty} \varrho(s)\Gamma(-ds-\gamma)z^s\, ds$. We now turn to the proof of the estimate (6.51). Using the previous lemma and parameterizing the contour integral by $s = \omega(1+ix)$ $(x \in \mathbb{R})$, we get for any $\omega < -K_1$,

$$
\begin{aligned}
|\sigma(z)| &\le \int_{\omega-i\infty}^{\omega+i\infty} \left|\varrho(s)\Gamma(-ds-\gamma)z^s\,ds\right| \le K_2|z|^{\omega}\int_{\omega-i\infty}^{\omega+i\infty}\left|\Gamma(-ds-\gamma)\right|\,|ds| \\
&\le K_2K_2'z^{\omega}(-\omega)^{-\frac{1}{2}-\gamma}\int_{\omega-i\infty}^{\omega+i\infty} e^{-d\,\mathrm{Re}(s)\log|ds|+d\,\mathrm{Re}(s)+d\arg(-s)\,\mathrm{Im}(s)}\,|ds| \\
&= K_2K_2'z^{\omega}(-\omega)^{-\frac{1}{2}-\gamma}\int_{-\infty}^{\infty} e^{-d\omega\log(-d\omega\sqrt{1+x^2})+d\omega+d\omega x\arctan(x)}(-\omega)\,dx \quad \big(\text{we set } s=\omega(1+ix)\big) \\
&= K_2K_2'(-\omega)^{\frac{1}{2}-\gamma}z^{\omega}(-d\omega)^{-d\omega}\int_{-\infty}^{\infty} e^{d\omega f(x)}\,dx = K_2K_2'(-\omega)^{\frac{1}{2}-\gamma}\int_{-\infty}^{\infty}e^{d\omega f(x)}\,dx, \qquad (6.52)
\end{aligned}
$$

where $f(x) := -\frac{1}{2}\log(1+x^2)+1+x\arctan(x)$ and we have chosen $\omega := -z^{1/d}/d = -e^y/d < -K_1$ by our assumption on y.

We now find lower bound for f. Note that f is even, $f(0)=1$ and its first two derivatives are given by $f'(x)=\arctan(x)$, $f''(x)=1/(1+x^2)>0$. Therefore, for $0\le x\le 1$ we have $\frac{\partial^2 f}{\partial x^2}\ge\frac{1}{2}$. By Taylor's Theorem with remainder we conclude that $f(x)\ge 1+\frac{x^2}{4}$ for $x\in[0,1]$. Hence

$$
\int_0^1 e^{d\omega f(x)}\,dx \le \int_0^1 e^{d\omega\left(1+\frac{x^2}{4}\right)}\,dx \le e^{d\omega}\int_0^{\infty} e^{d\omega\frac{x^2}{4}}\,dx = e^{d\omega}\sqrt{\frac{\pi}{-d\omega}}. \qquad (6.53)
$$

If $x\ge 1$, the convexity of f yields $f(x)\ge f(1)+(x-1)f'(1)=\frac{\pi}{4}x+1-\frac{1}{2}\log 2$. Hence

$$
\int_1^{\infty} e^{d\omega f(x)}\,dx \le \int_1^{\infty} e^{d\omega(\frac{\pi}{4}x+1-1/2\log 2)}\,dx = \frac{4}{d\pi}\cdot\frac{1}{-\omega}e^{\left(\frac{\pi}{4}+1-\frac{1}{2}\log 2\right)d\omega} \le \frac{4e^{-.438dK_1}}{\pi d}\frac{e^{d\omega}}{-\omega}, \qquad (6.54)
$$

where we used $\omega < -K_1$ and $\frac{\pi}{4}-\frac{1}{2}\log 2 = 0.4388\cdots$. Combining (6.52)-(6.54), recalling $\omega := -e^y/d$, the assumption $y\ge\log(dK_1)$ and the value of K_2' in (6.48), we get (6.51). □

6.5 Proofs of claims from Chapter 3

6.5.1 Bound for F_1'

We now prove a rather complicated, but algorithmically useful, bound for the derivative F_1' that was needed in Chapter 3 to check inequalities in compact intervals.

Lemma 6.5.1. *Fix $N=2$ and assume $\gamma := \frac{r_1+1}{2}+\frac{r_2}{2}-t+N>0$, $K_1>\max(\frac{1}{2},\frac{\gamma+\frac{1}{2}}{d})$, $y>\log(K_1)$,* *and let* $F_1(y) := \log\big(\frac{f_2}{f_{2\mathrm{a}}}(y)\big) - \log\big(\frac{f_1}{f_{1\mathrm{a}}}(y)\big)$ *as in (3.7). Then its derivative* $F_1' = F_1'(y)$ *satisfies*

$$
\begin{aligned}
|F_1'| \le\ & |P_{2\mathrm{a}} - R_2| + |P_{1\mathrm{a}} - R_1| + 2\,|P_{0\mathrm{a}} - R_0| + \frac{|P_{0\mathrm{a}} - R_0| + |R_3 - P_{3\mathrm{a}}|}{\left|1 + 2\frac{R_0^2}{R_2R_1}\right|} \\
&+ \frac{2\,|P_{3\mathrm{a}} - P_{0\mathrm{a}}|\left(|R_{0\mathrm{a}} - R_0|\,\frac{|R_{0\mathrm{a}}+R_0|}{|R_{2\mathrm{a}}R_{1\mathrm{a}}|} + \frac{R_0^2}{|R_2R_{2\mathrm{a}}R_{1\mathrm{a}}|}\,|R_2 - R_{2\mathrm{a}}| + \frac{R_0^2}{|R_2R_1R_{1\mathrm{a}}|}\,|R_1 - R_{1\mathrm{a}}|\right)}{\left|1 + 2\frac{R_0^2}{R_2R_1}\right|\,\left|1 + 2\frac{R_{0\mathrm{a}}^2}{R_{2\mathrm{a}}R_{1\mathrm{a}}}\right|} \\
&+ \quad 6\,\frac{|P_{0\mathrm{a}} - R_0| + |R_1 - P_{1\mathrm{a}}|}{\left|2 + \frac{R_2R_1}{R_0^2}\right|} \qquad (6.55) \\
&+ \frac{|P_{1\mathrm{a}} - P_{0\mathrm{a}}|\left(|R_{2\mathrm{a}} - R_2|\left|\frac{R_{1\mathrm{a}}}{R_{0\mathrm{a}}^2}\right| + \left|\frac{R_2}{R_0^2}\right|\left|\frac{R_{1\mathrm{a}}}{R_{0\mathrm{a}}^2}\right|\,|R_0 + R_{0\mathrm{a}}|\,|R_0 - R_{0\mathrm{a}}| + \left|\frac{R_2}{R_0^2}\right|\,|R_1 - R_{1\mathrm{a}}|\right)}{\left|2 + \frac{R_2R_1}{R_0^2}\right|\,\left|2 + \frac{R_{2\mathrm{a}}R_{1\mathrm{a}}}{R_{0\mathrm{a}}^2}\right|},
\end{aligned}
$$

where (again omitting y*)*

$$
R_i := \frac{f^{(i+1)}}{f^{(i)}}, \quad R_{i\mathrm{a}} := \frac{f_{\mathrm{a}}^{(i+1)}}{f_{\mathrm{a}}^{(i)}}, \quad P_{i\mathrm{a}} := \frac{\frac{d}{dy}\left(f_{\mathrm{a}}^{(i)}\right)}{f_{\mathrm{a}}^{(i)}}, \quad |R_i - R_{i\mathrm{a}}| \le C_i, \quad |R_i - P_{i\mathrm{a}}| \le \tilde{C}_i \quad (0 \le i \le 3),
$$

and

$$
\begin{aligned}
C_i(y) &:= \frac{|R_{i\mathrm{a}}(y)| M_{i+1} e^{-3y}}{N_{i+1}(y)} + \frac{|R_i(y)| M_i e^{-3y}}{N_i(y)}, \qquad N_i(y) := \sum_{j=0}^{2} (-1)^j \widetilde{A_j}(i) e^{-jy}, \\
\tilde{C}_i(y) &:= |P_{i\mathrm{a}}(y)|\, \frac{\sum_{j=0}^{2} |\kappa_{j,i-1} + (-1)^j d\widetilde{A_j}(i)| e^{-jy} \ + \left(|\kappa_{3,i-1}| + dM_i\right) e^{-3y}}{\left|\sum_{j=0}^{3} \kappa_{j,i-1} e^{-jy}\right|} + \frac{|R_i(y)| M_i e^{-3y}}{N_i(y)}, \\
\kappa_{0,i-1} &:= -d\widetilde{A_0}(i-1), \qquad \kappa_{1,i-1} := \left(i - \frac{r_1 + r_2 + 1}{2}\right)\widetilde{A_0}(i-1) + d\widetilde{A_1}(i-1), \\
\kappa_{2,i-1} &:= -\left(i - \frac{r_1 + r_2 + 1}{2}\right)\widetilde{A_1}(i-1) - d\widetilde{A_2}(i-1) + \widetilde{A_1}(i-1), \qquad (6.56) \\
\kappa_{3,i-1} &:= \left(i - \frac{r_1 + r_2 + 1}{2}\right)\widetilde{A_2}(i-1) - 2\widetilde{A_2}(i-1). \\
M_i &= M_i(K_1) := (2\pi)^{\frac{d-3}{2}} (e/d)^{\gamma} \pi e^{\frac{1}{6}} \sqrt{8/d}\Big(1 + \frac{8e^{-.438dK_1}}{\pi^{3/2}\sqrt{dK_1}}\Big) K_2(K_1),
\end{aligned}
$$

with the $\widetilde{A_0}(i)$ *as in (2.15) and* $K_2(K_1)$ *as in (6.50). Moreover,* $R_{i\mathrm{a}}(y)$*,* $N_i(y)$ *and* $P_{i\mathrm{a}}$ *are explicit rational functions in* e^{-y}*, and* $|R_i(y)|$ *is positive and monotone increasing in* y*.*

Before giving the proof, we explain how the lemma gives an algorithm for finding an upper bound on $|F_1'(y)|$ valid for all y in any given interval $[a, b]$, provided $a > \log(K_1)$. Namely, first we replace by C_i or $\tilde{C}_i$ every difference $|R_i - R_{i\mathrm{a}}|$ or $|R_i - P_{i\mathrm{a}}|$ appearing in (6.55). In the new inequality we replace every function by its minimum or maximum value in $[a, b]$, choosing always the one that guarantees the inequality. The last sentence in the lemma tells allows us to find these extrema. For the R_i they are just the values $R_i(a), R_i(b)$. For the other functions we locate their critical points in $[a,b]$ (this amounts to finding the roots in $[e^{-b},e^{-a}]$ of a polynomial in $x=e^{-y}$), evaluate them there and at the endpoints to obtain the extremal values. This may seem and is very coarse, but since we always work with very short intervals $[a,b]$, we do not really weaken the inequality significantly.

Proof. It follows from log concavity that R_i is strictly negative and monotone. From their definition in the lemma and (3.5), it is clear that $R_{i\mathrm{a}}$, $P_{i\mathrm{a}}$ and N_i are explicit rational functions of e^{-y}.

For $x\in\mathbb{R}$, let $H^*(x):=H(e^{dx}/d^d)$, so that (6.50) gives an asymptotic expansion of H^* for $x>\log(dK_1)$. By (2.5) we have $f^{(t)}(y)=(-d)^tH(e^{yd})=(-d)^tH^*(y+\log d)$, so that for $y>\log K_1$ we obtain the asymptotic expansion (in the form (2.14) for $N=2$),

$$f^{(t)}(y)=f_{\mathrm{a}}^{(t)}(y)+CF_t(y)\mathrm{e}^{-y(\frac{r_1+r_2-1}{2}-t)}e^{-de^y}e^{-3y},\quad C:=A_0(-1)^t(2\pi)^{d-1}d^{-\frac{r_1+r_2+1}{2}+2t},\tag{6.57}$$
$$|F_t(y)|\le(2\pi)^{\frac{d-3}{2}}(e/d)^{\gamma}\pi e^{\frac{1}{6}}\sqrt{8/d}\Big(1+\frac{8e^{-.438dK_1}}{\pi^{3/2}\sqrt{dK_1}}\Big)K_2(K_1)=:M_t.$$

We now apply this to
$$F_1'=\frac{{f_1}_{\mathrm{a}}'}{{f_1}_{\mathrm{a}}}-\frac{f_1'}{f_1}+\frac{f_2'}{f_2}-\frac{{f_2}_{\mathrm{a}}'}{{f_2}_{\mathrm{a}}},\tag{6.58}$$
as follows from the definition of F_1 in (3.7). The problem is that so far we only control the difference $|f^{(t)}(y)-f_{\mathrm{a}}^{(t)}(y)|$ of two very small functions, as can be seen from the double exponential factor e^{-de^y} in (6.57) and (3.5). In contrast, the four ratios on the right hand side of (6.58) are far larger functions, as the double exponentials cancel out.

The main idea for bounding F_1' is to write it as a function of the numerically well-behaved differences $\frac{f^{(i+1)}}{f^{(i)}}-\frac{f_{\mathrm{a}}^{(i+1)}}{f_{\mathrm{a}}^{(i)}}$, and then find an upper bound for each of them using (3.5) and (6.57).[6] This explicit re-writing of all functions leads to the awkward and complicated expressions in the lemma, but to no deeper waters.

Using the definitions (3.2) and (3.6) we obtain

$$\frac{f_1'}{f_1}=\frac{f^{(3)}}{f''}+\frac{f''}{f'}-2\frac{f'}{f},\qquad\qquad \frac{{f_1}_{\mathrm{a}}'}{{f_1}_{\mathrm{a}}}=\frac{\frac{d}{dy}\left(f_{\mathrm{a}}^{(2)}\right)}{f_{\mathrm{a}}^{(2)}}+\frac{\frac{d}{dy}\left(f_{\mathrm{a}}^{(1)}\right)}{f_{\mathrm{a}}^{(1)}}-2\frac{\frac{d}{dy}\left(f_{\mathrm{a}}\right)}{f_{\mathrm{a}}},\tag{6.59}$$

and

$$\frac{f_2'}{f_2}=\frac{\frac{ff^{(4)}-f^{(3)}f'}{f^2}+6\left(\frac{f'}{f}\right)^2\cdot\frac{ff''-(f')^2}{f^2}}{\frac{f^{(3)}}{f}+2\left(\frac{f'}{f}\right)^3},$$
$$\frac{{f_2}_{\mathrm{a}}'}{{f_2}_{\mathrm{a}}}=\frac{f_{\mathrm{a}}\left(f_{\mathrm{a}}\frac{d}{dy}f_{\mathrm{a}}^{(3)}-f_{\mathrm{a}}^{(3)}\frac{d}{dy}\left(f_{\mathrm{a}}\right)\right)}{f_{\mathrm{a}}^2f_{\mathrm{a}}^{(3)}+2\left(f_{\mathrm{a}}'\right)^3}+\frac{6\left(f_{\mathrm{a}}^{(1)}\right)^2\left(f_{\mathrm{a}}\frac{d}{dy}(f_{\mathrm{a}}^{(1)})-f_{\mathrm{a}}^{(1)}\frac{d}{dy}\left(f_{\mathrm{a}}\right)\right)}{f_{\mathrm{a}}^3f_{\mathrm{a}}^{(3)}+2f_{\mathrm{a}}\left(f_{\mathrm{a}}'\right)^3}.$$

[6] We must warn the reader that $f_{\mathrm{a}}^{(i)}$ is the 2-order asymptotic aproximation to $f^{(i)}$, so $f_{\mathrm{a}}^{(i+1)}\neq\frac{d}{dy}f_{\mathrm{a}}^{(i)}$ (cf. (6.59)).

We can express $\frac{f_1'}{f_1}$, $\frac{{f_1}_\mathrm{a}'}{{f_1}_\mathrm{a}}$, $\frac{f_2'}{f_2}$ and $\frac{{f_2}_\mathrm{a}'}{{f_2}_\mathrm{a}}$ in terms of numerically well-behaved functions as follows.

$$\frac{f_1'}{f_1} = R_2 + R_1 - 2R_0, \qquad \frac{{f_1}_\mathrm{a}'}{{f_1}_\mathrm{a}} = P_{2\mathrm{a}} + P_{1\mathrm{a}} - 2P_{0\mathrm{a}}$$

$$\frac{f_2'}{f_2} = \frac{R_3 - R_0}{1 + 2\frac{R_0^2}{R_2 R_1}} + \frac{6\,(R_1 - R_0)}{2 + \frac{R_2 R_1}{R_0^2}}, \qquad \frac{{f_2}_\mathrm{a}'}{{f_2}_\mathrm{a}} = \frac{P_{3\mathrm{a}} - P_{0\mathrm{a}}}{1 + 2\frac{{R_0}_\mathrm{a}^2}{R_{2\mathrm{a}} R_{1\mathrm{a}}}} + \frac{6\,(P_{1\mathrm{a}} - P_{0\mathrm{a}})}{2 + \frac{R_{2\mathrm{a}} R_{1\mathrm{a}}}{{R_0}_\mathrm{a}^2}}.$$

Replacing these expressions in (6.58) we get

$$F_1' = (P_{2\mathrm{a}} - R_2) + (P_{1\mathrm{a}} - R_1) - 2\,(P_{0\mathrm{a}} - R_0) + \left(\frac{R_3 - R_0}{1 + 2\frac{R_0^2}{R_2 R_1}} - \frac{P_{3\mathrm{a}} - P_{0\mathrm{a}}}{1 + 2\frac{{R_0}_\mathrm{a}^2}{R_{2\mathrm{a}} R_{1\mathrm{a}}}} \right) + 6 \left(\frac{R_1 - R_0}{2 + \frac{R_2 R_1}{R_0^2}} - \frac{P_{1\mathrm{a}} - P_{0\mathrm{a}}}{2 + \frac{R_{2\mathrm{a}} R_{1\mathrm{a}}}{{R_0}_\mathrm{a}^2}} \right). \tag{6.60}$$

Now, we proceed to find an upper bound for each term inside parentheses in (6.60). Recall that the only tools at our disposal are (6.57) and the monotonicity of the functions R_i. The first task is to find an upper bound for $|R_i - R_{i\mathrm{a}}|$ and $|R_i - P_{i\mathrm{a}}|$. This will follow easily once we have a bound for the relative errors

$$E_i = \left| \frac{f^{(i)} - f_\mathrm{a}^{(i)}}{f_\mathrm{a}^{(i)}} \right|, \qquad \tilde{E}_i = \left| \frac{\frac{d}{dy} f_\mathrm{a}^{(i-1)} - f^{(i)}}{\frac{d}{dy} f_\mathrm{a}^{(i-1)}} \right|. \tag{6.61}$$

Adding and subtracting $\frac{f'}{f_\mathrm{a}}$ we get

$$|R_0 - R_{0\mathrm{a}}| = \left| \frac{f'}{f} - \frac{f_\mathrm{a}^{(1)}}{f_\mathrm{a}} \right| \le \left| \frac{f_\mathrm{a}^{(1)}}{f_\mathrm{a}} \right| \left| \frac{f_\mathrm{a}^{(1)} - f'}{f_\mathrm{a}^{(1)}} \right| + \left| \frac{f'}{f} \right| \left| \frac{f_\mathrm{a} - f}{f_\mathrm{a}} \right| = |R_{0\mathrm{a}}| E_1 + |R_0| E_0.$$

Using the same argument for each $|R_i - R_{i\mathrm{a}}|$ we get $|R_i - R_{i\mathrm{a}}| \le |R_{i\mathrm{a}}| E_{i+1} + |R_i| E_i$. Similarly,

$$|R_i - P_{i\mathrm{a}}| \le \left| \frac{\frac{d}{dy}\left(f_\mathrm{a}^{(i)}\right)}{f_\mathrm{a}^{(i)}} \right| \left| \frac{\frac{d}{dy}\left(f_\mathrm{a}^{(i)}\right) - f^{(i+1)}}{\frac{d}{dy}\left(f_\mathrm{a}^{(i)}\right)} \right| + \left| \frac{f^{(i+1)}}{f^{(i)}} \right| \left| \frac{f_\mathrm{a}^{(i)} - f^{(i)}}{f_\mathrm{a}^{(i)}} \right| = |P_{i\mathrm{a}}|\, \tilde{E}_{i+1} + |R_i| E_i.$$

As noted above, we can estimate $R_i, R_{i\mathrm{a}}$ and $P_{i\mathrm{a}}$ in an interval by the greater of their values at the endpoints of the interval.

Next we bound E_i and $\tilde{E}_i$. Using (6.57) we get, with $\widetilde{A_0}(i)$ as in (3.5) and (2.15),

$$E_i(y) = \left| \frac{f^{(i)}(y) - f_\mathrm{a}^{(i)}(y)}{f_\mathrm{a}^{(i)}(y)} \right| \le \frac{M_i \exp(-3y)}{\widetilde{A_0}(i) - \widetilde{A_1}(i)\exp(-y) + \widetilde{A_2}(i)\exp(-2y)}.$$

Now, to do the same for $\tilde{E}_i$, we first need an explicit expression for $\frac{d}{dy} f_\mathrm{a}^{(i-1)}$. Taking derivatives in

(3.5) we get, after some basic calculations,

$$\begin{aligned}\frac{d}{dy} f_{\mathrm{a}}^{(t)}(y) &= (-1)^t (2\pi)^{d-1} A_0(t) d^{-\frac{r_1+r_2+1}{2}+2t} \exp\big(-y(\tfrac{r_1+r_2+1}{2}-t-2)\big)\exp(-de^y)\\ &\cdot \Big[-d\widetilde{A_0}(t) + ((t+1-\tfrac{r_1+r_2+1}{2})\widetilde{A_0}(t) + d\widetilde{A_1}(t))\exp(-y) + (-(t+1-\tfrac{r_1+r_2+1}{2})\widetilde{A_1}(t)\\ &\quad - d\widetilde{A_2}(t) + \widetilde{A_1}(t))\exp(-2y) + ((t+1-\tfrac{r_1+r_2+1}{2})\widetilde{A_2}(t) - 2\widetilde{A_2}(t))\exp(-3y)\Big].\end{aligned}$$

Using this equation with $t = i-1$ and replacing it in (6.61), we get

$$\tilde{E}_i(y) = \left| \frac{\sum_{j=0}^{3} \kappa_{j,i-1} e^{-jy} \ + \ \frac{d^2 A_0(i)}{A_0(i-1)} \left(\sum_{j=0}^{2} (-1)^j \widetilde{A_j}(i) e^{-jy} \ + \ F_i(y) e^{-3y} \right)}{\sum_{j=0}^{3} \kappa_{j,i-1} e^{-jy}} \right|, \tag{6.62}$$

which leads to the values given in (6.56). Replacing $A_0(t) = (2\pi)^{\frac{1-d}{2}} d^{\frac{r_1+r_2}{2}-t}$ and using $|F_{t+1}| \leq M_{t+1}$, we obtain

$$\tilde{E}_i(y) \leq \frac{\sum_{j=0}^{2} |\kappa_{j,i-1} + (-1)^j d\widetilde{A_j}(i)| e^{-jy} \ + \ (|\kappa_{3,i-1}| + dM_i) e^{-3y}}{\left| \sum_{j=0}^{3} \kappa_{j,i-1} e^{-jy} \right|}. \tag{6.63}$$

In summary, we have proved so far that

$$|R_i - R_{i\mathrm{a}}| \leq |R_{i\mathrm{a}}| E_{i+1} + |R_i| E_i \qquad (i = 0,1,2), \tag{6.64}$$

$$|R_i - P_{i\mathrm{a}}| \leq |P_{i\mathrm{a}}|\, \tilde{E}_{i+1} + |R_i| E_i \qquad (i = 0,1,2,3), \tag{6.65}$$

where, due to their monotonicity, the terms $|R_{i\mathrm{a}}|$, $|P_{i\mathrm{a}}|$ and $|R_i|$ are evaluated at the endpoint $y = R$ of the interval $[L, R]$, $E_i(y)$ is bounded by (6.62) and $\tilde{E}_i(y))$ by (6.63). These are bounded by rational functions in e^{-y}, so their critical points (and therefore their extreme values) in an interval can be computed. Thus they can be bounded. These estimates give an upper bound for the first three terms in the right hand side of (6.60).

To find a similar upper bound for the other two terms, namely for

$$\frac{R_3 - R_0}{1 + 2\frac{R_0^2}{R_2 R_1}} - \frac{P_{3\mathrm{a}} - P_{0\mathrm{a}}}{1 + 2\frac{R_{0\mathrm{a}}^2}{R_{2\mathrm{a}} R_{1\mathrm{a}}}} \qquad \text{and} \qquad \frac{R_1 - R_0}{2 + \frac{R_2 R_1}{R_0^2}} - \frac{P_{1\mathrm{a}} - P_{0\mathrm{a}}}{2 + \frac{R_{2\mathrm{a}} R_{1\mathrm{a}}}{R_{0\mathrm{a}}^2}}. \tag{6.66}$$

We note that the first one can be written in the form

$$\frac{(P_{0\mathrm{a}} - R_0)\left(1 + 2\frac{R_{0\mathrm{a}}^2}{R_{2\mathrm{a}} R_{1\mathrm{a}}}\right) + (R_3 - P_{3\mathrm{a}})\left(1 + 2\frac{R_{0\mathrm{a}}^2}{R_{2\mathrm{a}} R_{1\mathrm{a}}}\right) + 2\,(P_{3\mathrm{a}} - P_{0\mathrm{a}})\left(\frac{R_{0\mathrm{a}}^2}{R_{2\mathrm{a}} R_{1\mathrm{a}}} - \frac{R_0^2}{R_2 R_1}\right)}{\left(1 + 2\frac{R_0^2}{R_2 R_1}\right)\left(1 + 2\frac{R_{0\mathrm{a}}^2}{R_{2\mathrm{a}} R_{1\mathrm{a}}}\right)}. \tag{6.67}$$

Ther first two terms in the numerator are easily bounded using (6.64) or (6.65), but the third term shows that we need upper bounds for $\left| \frac{R_{0\mathrm{a}}^2}{R_{2\mathrm{a}} R_{1\mathrm{a}}} - \frac{R_0^2}{R_2 R_1} \right|$. Since there is second term in (6.66), we also need to bound $\left|\frac{R_{2\mathrm{a}}R_{1\mathrm{a}}}{R_{0\mathrm{a}}^2} - \frac{R_2R_1}{R_0^2}\right|$. After some easy calculations we get

$$\begin{aligned}\left|\frac{R_{2\mathrm{a}}R_{1\mathrm{a}}}{R_{0\mathrm{a}}^2} - \frac{R_2R_1}{R_0^2}\right| &= \left|(R_{2\mathrm{a}} - R_2)\frac{R_{1\mathrm{a}}}{R_{0\mathrm{a}}^2} + \frac{R_2}{R_0^2}\cdot\frac{R_{1\mathrm{a}}}{R_{0\mathrm{a}}^2}(R_0 + R_{0\mathrm{a}})(R_0 - R_{0\mathrm{a}}) - \frac{R_2}{R_0^2}(R_1 - R_{1\mathrm{a}})\right| \\ &\le |R_{2\mathrm{a}} - R_2|\left|\frac{R_{1\mathrm{a}}}{R_{0\mathrm{a}}^2}\right| + \left|\frac{R_2}{R_0^2}\right|\left|\frac{R_{1\mathrm{a}}}{R_{0\mathrm{a}}^2}\right||R_0 + R_{0\mathrm{a}}||R_0 - R_{0\mathrm{a}}| + \left|\frac{R_2}{R_0^2}\right||R_1 - R_{1\mathrm{a}}|, \end{aligned} \tag{6.68}$$

$$\begin{aligned}\left|\frac{R_{0\mathrm{a}}^2}{R_{2\mathrm{a}}R_{1\mathrm{a}}} - \frac{R_0^2}{R_2R_1}\right| &= \left|(R_{0\mathrm{a}} - R_0)\frac{(R_{0\mathrm{a}} + R_0)}{R_{2\mathrm{a}}R_{1\mathrm{a}}} + \frac{R_0^2}{R_2R_{2\mathrm{a}}R_{1\mathrm{a}}}(R_2 - R_{2\mathrm{a}}) + \frac{R_0^2}{R_2R_1R_{1\mathrm{a}}}(R_1 - R_{1\mathrm{a}})\right| \\ &\le |R_{0\mathrm{a}} - R_0|\frac{|R_{0\mathrm{a}} + R_0|}{|R_{2\mathrm{a}}R_{1\mathrm{a}}|} + \frac{R_0^2}{|R_2R_{2\mathrm{a}}R_{1\mathrm{a}}|}|R_2 - R_{2\mathrm{a}}| + \frac{R_0^2}{|R_2R_1R_{1\mathrm{a}}|}|R_1 - R_{1\mathrm{a}}|.\end{aligned}$$

Replacing (6.68) in (6.60), using (6.66) and (6.67), we get finally get the bound (6.55) on F_1'. □

6.5.2 Proof of Proposition 3.3.2

Proposition 3.3.2 follows directly from the following residue calculation.

Proposition 6.5.2. *Let $d := r_1 + 2r_2$, $g_y(s) := s^t e^{-dys}\,\Gamma(s)^{r_1+r_2}\,\Gamma(s+\frac{1}{2})^{r_2}$, and let $i \in \mathbb{N}$. Then there exist explicitly calculable coefficients $c_p = c_{p,i}$ $(1 \le p \le r_1 + r_2)$ such that*

$$\operatorname{Res}_{s=\frac{1-i}{2}}\big(g_y(s)\big) = e^{\frac{i-1}{2}dy}\sum_{p=1}^{r_1+r_2} c_p\frac{(-dy)^{p-1}}{(p-1)!}.$$

Proof. Define the c_p's by the truncated Laurent expansion

$$s^t\Gamma(s)^{r_1+r_2}\Gamma(s+\tfrac{1}{2})^{r_2} = \sum_{p=1}^{r_1+r_2}\frac{c_p}{(s-\frac{1-i}{2})^p} \;+\; \big(\text{a function analytic at } s = \tfrac{1-i}{2}\big),$$

where $c_p = 0$ for $p > r_2$ if i is even. The Proposition follows from multiplying the above with the MacLaurin expansion of e^{-dys}, provided we can show that the c_p's are explicitly calculable. The (finite) Taylor expansion around $s = \frac{1-i}{2}$ of $s^t = \big((s - \frac{1-i}{2}) + \frac{1-i}{2}\big)^t$ is obvious, so we just need to find the Laurent expansion of $\Gamma(s)^{r_1+r_2}\Gamma(s+\frac{1}{2})^{r_2}$, which reduces to finding that of $\Gamma(s)$ around any integer or half-integer. The MacLaurin expansion of $\Gamma(s+1)$ is given by [GR07, p. 903]) [Nie06, p. 40]

$$\Gamma(s+1) = \sum_{k=0}^{\infty} c_k s^k, \qquad c_0 = 1, \qquad c_{n+1} = \frac{1}{n+1}\sum_{k=0}^{n}(-1)^{k+1}\zeta(k+1)c_{n-k} \qquad (|s| < 1), \tag{6.69}$$

where ζ is the Riemann zeta function, except that $\zeta(1) :=$ Euler's constant. Using $\Gamma(s-1) = \Gamma(s)/(s-1)$ we can obtain the Laurent expansion of $\Gamma(s)$ around any integer. For half-integers we use the duplication formula in the form

$$\Gamma(s+\tfrac{1}{2}) = \frac{2^{1-2s}\sqrt{\pi}\,\Gamma(2s)}{\Gamma(s)},$$

which reduces the problem to finding the MacLaurin expansion of $1/\Gamma(s)$. This follows from the classical expansion [GR07, p. 903]) [Nie06, p. 41],

$$\frac{1}{\Gamma(u+1)} = \sum_{k=0}^{\infty} d_k u^k, \qquad d_0 = 1, \qquad d_{n+1} = \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \zeta(k+1) d_{n-k}. \tag{6.70}$$

□

6.6 PARI-GP Codes

In this section we provide the main PARI-GP programs used to obtain the numerical results of this thesis. For all the numerical calculations we used PARI-GP version 2.13.1 on a Linux Ubuntu 20.04 PC platform.

6.6.1 Exponential growth rate of $\mathrm{Reg}(L/K)$.

The following PARI-GP program calculates the exponential growth rates shown in Tables 1.1, 1.2, 1.3 and 5.1 for $\mathrm{Reg}(L/K)$.

```
\allocatemem(1500000000)
\p 100;
\ps 50;
r1=0;
r2=de;
d=r1+2*r2;
t0=0;
ceros0=vector(r1+r2-t0);
dos0=vector(t0);
unos0=vector(r2);
for(j=1,t0,dos0[j]=2);
for(j=1,r2,unos0[j]=1);
v210=concat(dos0,unos0);
A0=concat(ceros0,v210);
Cte0=(-d)^t0/(2*(Pi)^(-(2*t0+r2)/2));
G0 = gammamellininvinit(A0,0);
h0(y) =Cte0*gammamellininv(G0,sqrt((exp(d*(y)))/(Pi^d)));
t1=1;
ceros1=vector(r1+r2-t1);
dos1=vector(t1);unos1=vector(r2);
for(j=1,t1,dos1[j]=2);
for(j=1,r2,unos1[j]=1);
v211=concat(dos1,unos1);
A1=concat(ceros1,v211);
Cte1=(-d)^t1/(2*(Pi)^(-(2*t1+r2)/2));
G1 = gammamellininvinit(A1,0);
```

```
h1(y) =Cte1*gammamellininv(G1,sqrt((exp(d*(y)))/(Pi^d)));
F1ex(y)=-h1(y)/h0(y);
M(y)=(2^(-(de-1)/(2*de)))*(Pi^(-1/4))*(h0(y))^(1/(2*de));
ystar=solve(y=-5,0,(1/de)*F1ex(y)-1-0.01);
print("[d,y_star,M(y_star)]=",precision([de,ystar,M(ystar)],5));
write(valor_de_cte_con_K_tot_complejov2,"[d,y_star,M(y_star)]="
precision([de,ystar,M(ystar)],5));
```

6.6.2 Right positivity interval $[y^*, +\infty)$ for $\left(-\frac{f'}{f}\right)''$.

The following PARI-GP program finds a point $y^* > 0$ such that $\left(-\frac{f'}{f}\right)'' > 0$ for $y \geq y^*$, as shown in Table 2.1.

```
/*This program generates the values of y^*>0 such that ...
    (-f'/f)''>0 for y>y^* */
\p 300
allocatemem(1000000000);
\ps 100
for(rr2=2,100,r1=0;r2=rr2;read(Value_of_R));
/*This program (Value_of_R_Thesis_Appendix) find the value of ...
    R>0 such that the function (-f'/f)'' is positive for y>R */
/*The signature */
d = r1+2*r2;
/* Mi is the bound for Fi(y), i=0,1,2,3. */
/*The constants A0tilde, A1tilde, A2tilde as functions of ...
    t=0,1,2,3.  */
A0(t)= 1;
A1(t)=  (r1^2+r1*r2-12*r1*t+r2^2-12*r2*t+12*t^2-1)/(24*r1+48*r2);
A2(t)= ...
    (1/1152)*(-23+22*r1*r2+144*t^4-384*t^3+r1^4+r2^4+2*r1^3*r2-...
-24*r1^2*r2+168*r1^2*t^2+2*r1*r2^3-24*r1*r2^2-288*r1*t^3-...
-288*r2*t^3-192*r1^2*t+576*r1*t^2-192*r2^2*t+576*r2*t^2
-48*r1*r2^2*t+312*r1*r2*t^2-48*r1^2*r2*t-...
/* Given a polynomial P and n0,n1,n2,n3 natural numbers, the ...
    function "coeficientes" calculate the
coefficient of (x^n0)*(y^n1)*(z^n2)*(w^n3)  in the polynomial ...
    P(x,y,z,w). */
coeficientes(P,n0,n1,n2,n3)=
{my(P0);
my(P1);
my(P2);
my(P3);
my(salida);
```

```
P0=P;
for(j=1,n0,P0=deriv(P0,MM0));
P1=P0;
for(j=1,n1,P1=deriv(P1,MM1));
P2=P1;
for(j=1,n2,P2=deriv(P2,MM2));
P3=P2;
for(j=1,n3,P3=deriv(P3,MM3));
salida=(polcoeff(polcoeff(polcoeff(polcoeff(P3,0),0),0),0))/...
salida;
};
/*In what follows x=exp(-y).
expr2(x) is the numerator in the asymptotic expansion of ...
   %(-f'/f)'' when y->+oo (except for the factor d^3/(x^3)).
Here MM0, MM1, MM2, MM3 are variables that represent the ...
   bounds for the error functions F0(y), F1(y), F2(y), F3(y).*/

expr2(x) = (A0(3)-A1(3)*x+A2(3)*x^2+MM3*x^3)*(A0(0)
p0=expr2(0);
p1=polcoeff(expr2(x), 1,{x});
p2=polcoeff(expr2(x), 2,{x});

/*We have p0=p1=0,  p2=1/d^2.*/
/*The following are cubic polynomials in the variables MM0, ...
   MM1, MM2, MM3.*/

p3=polcoeff(expr2(x), 3,{x});
p4=polcoeff(expr2(x), 4,{x});
p5=polcoeff(expr2(x), 5,{x});
p6=polcoeff(expr2(x), 6,{x});
p7=polcoeff(expr2(x), 7,{x});
p8=polcoeff(expr2(x), 8,{x});
p9=polcoeff(expr2(x), 9,{x});

p3abs = {abs(p3_1110)*M0*M1*M2+abs(p3_1101)*M0*M1*M3+
abs(p3_1011)*M0*M2*M3+abs(p3_0111)*M1*M2*M3
+abs(p3_0012)*M2*M3^2+abs(p3_0102)*M1*M3^2+abs(p3_0120)*M1*M2^2+
abs(p3_1002)*M0*M3^2+abs(p3_0000)
+abs(p3_3000)*M0^3+abs(p3_0300)*M1^3+abs(p3_0030)*M2^3+
abs(p3_0003)*M3^3+abs(p3_1000)*M0
+abs(p3_0100)*M1+abs(p3_0010)*M2+abs(p3_0001)*M3+abs(p3_2000)
*M0^2+abs(p3_0200)*M1^2+abs(p3_0020)*M2^2
+abs(p3_0002)*M3^2+abs(p3_2010)*M0^2*M2+abs(p3_1200)*M0*M1^2
+abs(p3_0021)*M2^2*M3+abs(p3_0201)*M1^2*M3
+abs(p3_1020)*M0*M2^2+abs(p3_0110)*M1*M2+abs(p3_2001)*M0^2*M3
```

```
+abs(p3_0210)*M1^2*M2+abs(p3_0011)*M2*M3
+abs(p3_1010)*M0*M2+abs(p3_0101)*M1*M3+abs(p3_2100)*M0^2*M1
+abs(p3_1100)*M0*M1+abs(p3_1001)*M0*M3};

soluc2=polrootsreal(p3abs*x+p4abs*x^2+p5abs*x^3+p6abs*x^4+
p7abs*x^5+p8abs*x^6+p9abs*x^7 - 1/d^2,[0,10^1000]);
```

6.6.3 Left positivity interval $(-\infty, y_*]$ for $\left(-\frac{f'}{f}\right)''$.

The following PARI-GP program finds a point $y_* < 0$ such that $\left(-\frac{f'}{f}\right)'' > 0$ for $y \le y_*$, as shown in Table 2.2.

```
allocatemem(10^9);
\ps 200;
\p 100;
for(j=3,40,for(k=5,5,r1=j;r2=k;read(Valor_de_L)));
/*This program find L<0 such that (-f'/f)''>0 for all y<L. */

/*The signature (r1,r2)*/
de=r1+2*r2;
/*c(k) are the Laurent coefficients of Gamma(s)^(r1+r2)
d(k) are the Laurent coefficients of Gamma(s+1/2)^(r2)
*/
a1=gamma(s)^(r1+r2);
c(k)=polcoeff(a1,k)*1.0;
a2=gamma(s+1/2)^(r2);
d(k)=polcoeff(a2,k)*1.0;
/*e(j) are the Laurent coefficients of Res_{s=0}(Gz(s))*/
e(j)={
my(suma);
suma=0;
for (l=j,r1+r2,suma = suma+c(-l)*d(l-j));
suma;
};
/*The constants c_t */
ce(t)=1/((r1+r2-1-t)!);
/*The functions S_t(y) are the principal parts in the ...
    asymptotic expansion of
f^{(t)}(y) as y->-oo */
S(t,y)={
my(suma);
suma=0;
for (j=1+t,r1+r2-1,suma = ...
```

```
suma+(-1)^(j-r1-r2)*((r1+r2-1-t)!)/(Pi^(r2/2))*e(j)*
(de*y)^(j-r1-r2)/((j-1-t)!));
suma;
};
/*The bound for the constant C */
C=(2.89048)*(17.7715)^de;
/*Upper bounds for the functions H_t(y) */
H(t,y)=  ((r1+r2-1-t)!*C/(Pi^(r2/2)*de^(r1+r2-1-t)))*
exp((1/4)*de*y)/((abs(y))^(r1+r2-1-t));

/*Upper bounds Lambda_i(y) for the error terms epsilon_i, ...
   i=1,2,3. Here H(t,y) are the functions \tilde{H}(t,y).*/

epsilon1(y)= ...
   (1+S(3,abs(y))+H(3,y))*(2*H(0,y)+2*S(0,abs(y))*H(0,y)
+H(0,y)^2)+H(3,y)*(1+2*S(0,abs(y))+S(0,abs(y))^2);
epsilon2(y)=(1+S(1,abs(y))+S(2,abs(y))+H(1,y)+
H(2,y)+S(1,abs(y))*S(2,abs(y))+S(2,abs(y))*H(1,y)+H(2,y)
*S(1,abs(y))+H(2,y)*H(1,y))*H(0,y)+(H(1,y)+H(2,y)+
S(2,abs(y))*H(1,y)+H(2,y)*S(1,abs(y))+H(2,y)*H(1,y))
*(1+S(0,abs(y)));
epsilon3(y)=  ...
   3*H(1,y)+3*H(1,y)^2+6*H(1,y)*S(1,abs(y))+3*H(1,y)^2*S(1,abs(y))
+3*H(1,y)*S(1,abs(y))^2+H(1,y)^3;

/*L1(x) is the main term in the asymp. expansion of the ...
   numerator of (-f'/f)'' as y->-oo.
Here x=1/y, so L1(x) is a Laurent polynomial in x of the form ...
   c_3*x^3+...+c_{-3r1-3r2+6}/x^(3r1+3r2-6) */

L1(x)=((-ce(3)*ce(0)^2*(1+2*S(0, x)+S(3, x)+S(0, x)^2+2*S(0, ...
   x)*S(3, x)+S(3, x)*S(0, x)^2)+3*ce(0)*ce(1)*ce(2)*(1+S(0, ...
   x)+S(1, x)+S(2, x)+S(0, x)*S(1, x)+S(1, x)*S(2, x)+S(0, ...
   x)*S(2, x)+S(0, x)*S(1, x)*S(2, x))-2*ce(1)^3*(1+3*S(1, ...
   x)+3*S(1, x)^2+S(1, x)^3)));
L1derivada=L1';

/* yymin is a vector with all the roots of the Laurent ...
   polynomial L1'(x). To convert L1'(x)
into a true polynomial, we multiply it by  x^(3*r1+3*r2-5).*/

yymin =polroots(L1(x)'*x^(3*r1+3*r2-5));
largo=length(yymin);
/*The number of (complex) roots of L1'(x) */
minimo=10^100;
```

```
for(kk=1,largo, if(abs(imag(yymin[kk]))<10^(-10),
minimo=min(real(yymin[kk]),minimo),));
ymin=minimo;
/*ymin is the point z1. The principal term decreases ...
   monotonically to its limit for y≤z1. */
/* E(y) is the upper bound for the error term. */
E(y)=ce(3)*ce(0)^2*epsilon1(y)+3*ce(0)*ce(1)*ce(2)*epsilon2(y)
+2*ce(1)^3*epsilon3(y);
/*"desig" find a point ≤ -mini such that h(point)<0.*/
desig(h, mini)={
my(N1);
my(j);
my(x0);
my(sol);
 j = 0;
 x0 = ceil(abs(mini));
 N1 = 10^100;
 while(j ≤ N1,if(0 ≤ h(-x0-(1/100)*j),j = j+1,sol = ...
    -x0-(1/100)*j; j = N1+1;));
 sol;
};
y1 = desig(L1, ymin)*1.0;
/*y1 is a point ≤z1 such that L1(y)<0 */
diferencia(y)= E(y)-abs(L1(y));
y2 = desig(diferencia, ymin)*1.0;
/*y2 is a point ≤z1 such that |L1(y)|>E(y) */
y0 = min(y1, y2)*1.0;
/*y0 is the point L */
write(valor_de_L_convex,"[r1,r2,L]=",[r1,r2,precision(y0,2)]);
```

6.6.4 Evaluation of $\left(-\frac{f'}{f}\right)''$ using a variable number of residues.

The following PARI-GP program evaluates the function $\left(-\frac{f'}{f}\right)'' = f_1 - f_2$ using a number of residues depending on the desired precision. It also subdivides the interval $[y_*, 0]$ to check that $f_1 > f_2$.

```
allocatemem(1000000000);
\p 80;
\ps 50;
/* This program evaluates the function F=numerator of ...
   (-f'/f)'' (F=3*f*f'*f''-2*(f')^3-f^2*f''') within an specified
error = epsilon. The number of residues depends on epsilon. */
/*The interval we consider is [y_*,0] */
d=r1+2*r2;
```

```
/*The Riemann Zeta function. If k=1 it is equal to the Euler's ...
    constant.*/
zetariem(k)={
my(salida);
if(k==1,salida=Euler(),salida=zeta(k));
salida;};
/*This function gives the values c(0),..,c(n), (n=>0) the ...
    Taylor coefficients of Gamma(s+1). */
c(n)={
my(resultado);
my(vector1);
vector1=vector(n+1);
if(n==0,resultado=[1,-zetariem(1)],resultado=vector(n+2);
resultado[1]=1; resultado[2]=-zetariem(1);for(j=1,n,
resultado[j+1+1]=sum(k=0,j,(-1)^(k+1)...
vector1};
/*This function gives the values d(0),..,d(n), (n=>0) the ...
    Taylor coefficients of 1/Gamma(u+1). */
de(n)={
my(resultado);
my(vector1);
vector1=vector(n+1);
if(n==0,resultado=[1,zetariem(1)],resultado=vector(n+2);
resultado[1]=1;resultado[2]=zetariem(1);for(j=1,n,
resultado[j+1+1]=sum(k=0,j,(-1)^(k)*...
for(j=1,n+1,vector1[j]=resultado[j]);
vector1;};
/*The function "alpha(j,i,t)" gives the jth-Taylor coefficient ...
    centered at (1-i)/2 of s^t.*/
alpha(j,i,t)={
t!/(j!*(t-j)!)*((1-i)/2)^(t-j);};
/*The function "beta(j,i,t)" gives the jth-Taylor coefficient ...
    centered at (1-i)/2 of exp(-sdy)
except the exponential term q=exp(1/2*d*y) which we will keep ...
    as an extra variable for a moment.*/
beta(j,i,y)={
(-1)^j*d^j*y^j/(j!)};
/*The functions gamaiimpar and gamaipar calculate the m ...
    th-Taylor coefficients gama(m,i) of
Gamma(s) centered at (1-i)/2, depending whether i is odd or ...
    even respectively*/
gamaiimpar(m,i)={
my(vectorc);
vectorc=c(2*d+1); /*saves the vector [c(0),...,c(2*d+1)] */
my(coeffi);
```

```
coeffi=polcoeff(Ser(sum(w=0,2*d+1,vectorc[w+1]*x^(w-1))/...
coeffi;};
gamaipar(m,i)={
my(coeffi);
my(vectorc);
vectorc=c(2*d+1); /*saves the vector [c(0),...,c(2*d+1)] */
my(vectorde);
vectorde=de(2*d+1); /*saves the vector [d(0),...,d(2*d+1)] */
coeffi=sqrt(Pi)*polcoeff(Ser(sum(j=0,2*d+1,(-2*log(2))^j/(j!)
*x^j)*sum(w=0,2*d+1,vectorc[w+1]*2^w*x^w)*sum(p=0,2*d+1,
vectorde[p+1]*x^p)...
coeffi;};
gama(m,i)={
my(coeffi);
if(frac(i/2)==0,coeffi=gamaipar(m,i),coeffi=gamaiimpar(m,i));
coeffi;};
/* △(m,i) is the m th-Taylor coefficient of Gamma(s+1/2) ...
    centered at (1-i)/2. */
△(m,i)={
gama(m,i-1);};
/*Only the coefficients beta(j,i,y) depend on y. We ...
    pre-calculate and store for each signature (r1,r2)
the coefficients alpha, gama and △.*/
/*The sum from p=0 to d-1 of (the p th- Taylor coefficient of ...
    R(u,i,t))*beta(d-p-1,i,y)
is equal to the sum of the residues of Gz*/
R(u,i,t)=(sum(j=0,t,alpha(j,i,t)*u^j))*((sum(s=0,d,gama(s-1,i)*
u^s))^(r1+r2)+O(u^d))*((sum(m=0,d,△(m-1,i)*u^m))^r2+O(u^d))
+O(u^d);
/*The function matrizM(i,r,t) calculates the r th-Taylor ...
    coefficient of s^t*Gamma(s)^(r1+r2)*/
```

6.6.5 Evaluation of $\log\left(\frac{f_2}{f_{2_a}}\right) - \log\left(\frac{f_1}{f_{1_a}}\right)$ by the Double Exponential method.

The following PARI-GP code was used to evaluate the normalized function $F_1 := \log\left(\frac{f_2}{f_{2_a}}\right) - \log\left(\frac{f_1}{f_{1_a}}\right)$ on the subinterval $[0, y^*]$ using the Double Exponential method. This program also subdivides that interval to ensure that $F_1 \geq F_2$.

```
\p 190;
\ps 190;
allocatemem(1000000000);
for(migrado=3,20,r1=migrado;r2=0;
read(Right_Subdiv_Value_of_R);read(Double_Expo_and_Newton_Subd));
```

```
ln(h)=log(h); /*Natural Logarithm */
/*The signature */
d = r1+2*r2;
M9=2; /*Vertical integration line. */
/*Double Exponential Method */
/*The function eneyh gives the values of n and h given the ...
   inputs y=point, D=De=precision */
eneyh(y,De,te)={
my(t);
my(n);
my(M);
my(ustar);
my(A);
my(Delta1a);
my(Delta1b);
my(M2);
my(gama);
my(x0);
my(Mprima);
my(M1);
my(alpha);
my(beta);
my(alpha1);
my(A1);
my(L);
my(Ct);
my(h);
my(tau);
my(ustar);
M=M9; /*Vertical integration line */
/* Computation of constants M2, A, gama such that |f(z)|≤ ...
   M2*exp(A*exp(gama|z|))  */
tau= Pi/4;
t=tau/2;
gama=1/2;
ustar= -(1/4)*(4*gama*r1*M+8*gama*r2*M+2*Pi*gama*r1+4*Pi*gama*r2+
2*gama*ln(2)*r1+...
A=  (ustar*((r1+2.*r2)*y+2.917369917*r1+2.917369917*r2)+
.6931471806*(.50000...
Delta1a= abs(M-(1/2)*sqrt(2))/sqrt(2) ;
Delta1b= abs(M+1/2-(1/2)*sqrt(2))/sqrt(2);
M2=(sqrt(2*Pi)*exp(1.1/(12*Delta1a)-M))^(r1+r2)*
(sqrt(2*Pi)*exp(1.1/...
/*Computation of constants M1, alpha, beta such that |f(x)|≤ ...
   M1*exp(-alpha|x|^beta) */
```

```
x0=30*M; /*this value works if M is approx. 2*/
Mprima = sqrt(2*Pi)*exp(1.1/(12*M)-M+(1/2*(M-1/2+te/(r1+r2)))
*log(M^2+x0^2)+(1/4)*x0*Pi);
M1 = ((max(sqrt(2*Pi)*exp(1.1/(12*(M+1/2))-M-1/2), ...
   sqrt(2*Pi)*exp(1.1/(12*(M+1/2))-M-1/2+1/2*M*log((M+1/2)^2+13^2)
+13*Pi/4) ))^r2)*max((sqrt(2*Pi)*exp(1.1/(12*M)-M))^(r1+r2), ...
   Mprima^(r1+r2));
alpha = (r1+r2)*Pi*(1/4)+r2*Pi/4;
beta = 1;
/*inputs of the double exponential method: tau, M1, alpha, ...
   beta, M2, A, gama.   */
alpha1 = alpha*(cos(beta*t)-(1/(tan(beta*tau)))*sin(beta*t));
A1 = A*cos((gama)*t)/cos(gama*tau);
L = (log(alpha)-De-log(2*M1))/(-alpha);
Ct = A1*(((gama)*A1+1+1)/(alpha1*beta))^(gama/(beta-gama))
-alpha1*(A1*gama...
h = 2*Pi*t/(De+Ct+log(4*M2+2*exp(-De-Ct)));
n = ceil((asinh(L)/h));
[n,h];};
```

6.6.5.1 *C* code used with the gp2c compiler.

The following code was used to compile the dot product function appearing in the Double Exponential method of section 6.6.5 using the `gp2c` compiler of `PARI-GP`.

```
/*The function to be compiled. */
dotp(v,w)={
my(salida);
salida=w*mattranspose(v);
}

/*The C-code for the above function. */
#include "pari.h"

void init_dotpvectores23(void);
GEN dotpvectores23(GEN v, GEN w);
/*End of prototype*/

void
init_dotpvectores23(void)     /* void */
{
  return;
}

GEN
```

```
dotpvectores23(GEN v, GEN w)
{
  long i, l = lg(v);
  double s = 0.;
  for (i = 1; i < l; i++) s += rtodbl(gel(v,i)) * ...
     rtodbl(gel(w,i));
  return dbltor(s);
}

/*Compilation of the C-code */
install("init_prodpunto","v","init_prodpunto",
"./prodpunto.gp.so");
install("prodpunto","D0,G,D0,G,","prodpunto",
"./prodpunto.gp.so");
```

6.6.6 Programs used in the proof of (4.1), (4.2), (4.3) and (4.4).

The following PARI-GP programs were used to prove the monotonicity of the functions $-\frac{f'_{(r_1,r_2)}}{f_{(r_1,r_2)}}$ with respect to the signature (r_1, r_2), more specifically, to prove inequalities (4.1), (4.2), (4.3) and (4.4) for degrees $d \leq 40$.

```
/*This program checks that, for a fixed degree deg, the ...
   function  -f'/f for signature (r1,r2)
is the smallest when the number r2 is the maximum possible, ...
   i.e., when r2=deg/2 if deg is even
or r2=(deg-1)/2 if deg is odd. */
allocatemem(1500000000)
\p 150;
\ps 50;
serie_Gamma= gamma(u);
serie_Gamma_en_un_medio= gamma(u+1/2);
deg1=40;
deg2=40;
/* From degree=deg1 to deg2,
if degree is odd, decrease the signature from (deg-2,1) up to ...
   (1,(deg-1)/2) checking positivity in each step.
if degree is even, decrease the signature from (deg-2,1) up to ...
   (0,deg/2) checking positivity in each step.
*/
for(deg=deg1,deg2,write("subdivision_vector_signature_ordering"
,"-------...
print("--------------");print("DEGREE=",deg);
print("--------------");
if(frac(deg/2)!=0,for(j=1,(deg-1)/2, r=deg-2*j;s=j; ...
```

```
    read(signature_ordering)),for(j=1,(deg)/2,
r=deg-2*j;s=j; read(signature_ordering)));
write("subdivision_vector_signature...
read(recursive_signature_ordering);
/* This program verifies that F1=(-f'/f)>F2=(-g'/g), where f ...
    is the function associated with
the signature (1,(d-1)/2) and g is associated with the ...
    signature (0,(d-1)/2).
The input is the degree d=degr odd.
The output is a finite interval [L,R] and a vector of ...
    verification points v.*/
print("signature ",[1,(degr-1)/2]);
print("versus signature ",[0,(degr-1)/2]);
/*Read the asymptotic approximations */
read(asymp_plus_infinity); /*Get the value of R=valordeR */
read(asymp_minus_infinity); /*Get the value of L=valordeL */
L=valordeL_diff_degs;
R=valordeR_diff_degs;
print("[L,R]=",precision([L,R],2)); /*The finite interval to ...
    consider */
/*From now on, we verify that F1>F2 on the interval [L,R]. */
read(residue_approx_of_ft); /*Using residues, this program
generates two functions F1, F2 approximating -f'/f
in signatures (1,(degr-1)/2) and (0,(degr-1)/2) on the ...
    interval [L,0]*/
FFsub(y)={
my(a1,a2);
a1=F2diffdeg(y);
a2=F1diffdeg(y);
[a1,a2];
};
vsub=precision([L,0],20);
f1sub(x)=FFsub(x)[1]; /*Only need f1sub and f2sub to make ...
    graphs. */
f2sub(x)=FFsub(x)[2];
print("Subdividing the interval ",[L,-1]);
read(interval_subdivision_function); /*Loads the function ...
    subdiv(FF,v) for subdivision. */
v1=subdiv(FFsub,vsub);  /*The subdivision vector for [L,-1] */
v1_low_precision=precision(v1,2);
n1=length(v1);
print("# subdivisions needed (to the left)= ",n1);
read(double_exp_approx_of_ft);/*Using the Double Exponential ...
    Method, this program generates two functions F11, F12 ...
    approximating -f'/f in signatures (1,(degr-1)/2) and ...
```

```
    (0,(degr-1)/2) on the interval [0,R]*/
FFsubb(y)={
my(a1,a2);
a1=F12diffdeg(y);
a2=F11diffdeg(y);
[a1,a2];
};
vsubb=precision([-1,R],100);
f1subb(x)=FFsubb(x)[1]; /*Only need f1sub and f2sub to make ...
    graphs. */
f2subb(x)=FFsubb(x)[2];
print("Subdividing the interval ",precision([0,R],2));
/*Loads the function subdiv(FF,v) that subdivides [0,R] */
read(interval_subdivision_function);
/*The subdivision vector for [0,R] */
v1a=subdiv(FFsubb,vsubb);
v1_low_precisiona=precision(v1a,2);
n1a=length(v1a);
print("# subdivisions needed (to the right)= ",n1a);
write("subdivision_vector_signature_ordering_diff_degs",[[1,(...
concat(v1_low_precision,v1_low_precisiona)],";");
print("Subdivision vector written on file ...
    subdivision_vector_signature_ordering_diff_degs with format
 [[1,(degree-1)/2],[0,(degree-1)/2],vector];");
print("----------------------------------");
```

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