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SU(2) quantum kinematics: Rotation-observable versus angular-momentum generalized commutation relations

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The canonical commutation relations of quantum mechanics are generalized to the case where appropriate dynamical variables are angular-momentum, rotation-angle, and rotation-axis observables. To this end, SU(2) is ''quantized'' on the compact group manifold, according to the standard procedure of non-Abelian quantum kinematics. Quantum-kinematic invariant operators are introduced, and their commutation relations with the rotation variables are found in an explicit manner. The quantum-kinematic invariants yield superselection rules in the form of eigenvalue equations of an isotopic structure (which one should solve in the applications, in order to get multiplets that carry the irreducible representations of the underlying quantum kinematic models). A wide range of applicability of SU(2) quantum kinematics is suggested.
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I. INTRODUCTION

The problem of extending the canonical commutation relations to non-Cartesian dynamical variables is a long-standing question in the general setting of quantum mechanics.[1] Most attempts to solve this problem have been focused on the search of new quantization methods for obtaining generalized minimum-uncertainty states of systems which are primarily described by non-Abelian dynamical variables.[2] However, important as it is, this particular purpose may be too narrow a motivation for such an ambitious endeavor.[3] Indeed, it is quite conceivable that some unknown general constraints should be taken into account for extending the conventional (i.e., Heisenberg) commutation relations to the appropriate ones. Kinematic constraints follow, for instance, from the superselection rules due to the symmetry group that characterizes a system.[4] Likewise, the search for generalized coherent states would be unable by itself to disclose such constraints. Other important motivations for having generalized canonical commutation relations in quantum mechanics appear in the current literature, as well.[5] As a matter of fact, this question has been already so amply discussed in the literature, for so many years, that it seems unnecessary to repeat here the sound physical motivation behind this issue.[6] This paper addresses this fundamental problem for the particular case of rotation configuration variables and angular momenta. Let us here remark that this is a problem in quantum kinematics.[7]

Much progress has been achieved during the last several years on the subject of group-*quantization* and non-Abelian quantum kinematics as used in this paper.[8] Here we shall follow a new group-theoretic approach to quantum kinematics which is quite consistent with the formalism of unitary symmetries used in ordinary quantum mechanics.[9] The new formalism, however, is enhanced by quantizing the parameters of the group (cf. below). In this way, quantum kinematics appears as a self-contained theory which can yield interesting models evolving on the arena of the group manifold. These abstract models can be projected on the configuration states of a physical system (on which the quantized group acts as a symmetry group), and thus a quantal model of the system arises (for details, see Ref. 8). Quantum kinematic theory yields a self-contained formalism of quantum mechanics, for it relinquished the use of ''prequantized'' classical analogs, and everything stems from the assumed symmetry group. Therefore, quantum kinematic models become

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interesting in themselves if the quantized group is a physically relevant symmetry group.[10,11]

The analysis of rotational symmetry, and the behavior of physical quantities under rotations, is one of the most common problems in the study of the symmetries of physical systems. In addition, the theory of angular momentum is the prototype of continuous symmetry groups of many types now found useful in the classification of the internal symmetries of elementary particle physics. So it was an important contribution of Wigner[12] to note that SU(2) is the group that enters quantum physics. Of course, the main consequences of this fact are well known and have been studied in the literature for a long time.[13] Nevertheless, some SU(2) quantal features will be considered here under the wide perspective offered by quantum kinematics. In fact, the main point dealt with in this paper is to obtain *generalized Heisenberg commutation relations* for *angular-momenta* and *rotation-angle* operators. (As far as we know, all the fundamental commutation relations presented in this paper are new.) To this end, we use the Euler–Rodrigues parameters[13] for describing the elements of the group, we then introduce the regular representation and we quantize SU(2) on the group manifold. This means that we replace the rotation-angle and the rotation-axis parameters (which are *c*-numbers that label the elements of the group) by a complete set of commuting Hermitian operators, acting as *generalized* position operators of the group manifold, which admit the parameters for spectra. In this approach, the generators of the representation afford the *generalized* momentum operators (i.e., angular momenta) of the model.[4] [The angular momenta and rotation variables mentioned here need not be taken literally, but may be understood in a wide sense. However, it will do just as well if the intuitive example of SO(3), acting in ordinary space, is kept in mind.] As we shall see, according to this formalism one is in a position to introduce a generalized SU(2) Weyl–Heisenberg algebra, and one obtains three quantum-kinematic invariant operators;[14] i.e., superselection rules follow, which reduce the model.

We wish to emphasize that this paper has only an introductory character, since the quantum kinematic theory of SU(2) discussed in the space allotted here is by no means complete. Given the relevance of this group, the endeavor of SU(2) quantum kinematics is so conspicuous, that each of its main features deserves a separate study by itself. Physical applications of SU(2) quantum kinematic theory are many. [Some applications will be given in some forthcoming papers. (For more comments, cf. Sec. IV, below.)] For instance, in the current literature (concerning ordinary rotations in quantum mechanics) the Euler-angle parametrization is predominantly used for describing the elements of SO(3), of which SU(2) is the universal covering group. Thus, in particular, we wish to remark that the introduction of SO(3) quantum kinematics (in terms of fundamental commutation relations for Euler-angle versus angular-momentum operators) follows from the theory presented in this paper, since (of course) the Euler-angles are related with the Euler–Rodrigues parameters in a well-defined way.[13] Strictly speaking, however, this subject belongs to the applications of the present theory, and thus it will be considered elsewhere. Here we are interested exclusively in the SU(2) quantum kinematic theory, for which the discussion in terms of the Euler–Rodrigues variables is the simplest one.[13]

The plan of the paper is the following. Section II deals with group quantization. For quantizing SU(2) we use an embedding approach that describes the group manifold as the surface $\mathcal{S}_3$ of the unit sphere in $\mathcal{E}_4$, in which SU(2) appears as a subgroup of the four-dimensional group of real quaternions.[13] [With this aim, in this paper we adopt the Euler–Rodrigues parameters,[13] which are best suited to this end (cf. Appendix A).] We then deduce the associated quantum kinematic commutation relations within the left regular representation, which is the basic working frame adopted in this article. [For the sake of completeness, both (left and right) regular representations of SU(2) are briefly dealt with in Appendix B.] Section III contains an overview of quantum kinematic invariant operators, which play a central role in this theory. Thus, in Secs. II and III, our discussion will be focused on the most general kinematic features of SU(2)-quantized-group models (although we do not solve for any concrete model in the present paper). Finally, Sec. IV contains our concluding remarks and some perspectives for future work.

II. QUANTUM KINEMATIC COMMUTATION RELATIONS

We begin our work by introducing generalized position operators in the group manifold $\mathcal{S}_3$ of SU(2). Here, $\mathcal{S}_3$ denotes the three-dimensional surface of the unit sphere in four-dimensional Euclidean space $\mathcal{E}_4$, defined by $q^\mu q^\mu = 1$. The four Cartesian variables $q^\mu = (\mathbf{q}, q^4) \in \mathcal{E}_4$, when

TABLE I. Synopsis of the left regular representation formalism on $\mathcal{S}_3$, within the carrier Hilbert space $\mathcal{H}[\mathrm{SU}(2)]$. (For details, see Appendix B.)

$\langle q' \vert q\rangle = \mu_0^{-} L(q)\delta^{(3)}(q'-q), \quad \oint_{\mathcal{S}_3} d\mu(q) \vert q\rangle\langle q \vert = I$	(I.1)
$U(q')\vert q\rangle = \vert g(q';q)\rangle, \quad U(q) = \oint_{\mathcal{S}_3} d\mu(q') \vert g(q;q')\rangle\langle q' \vert$	(I.2)
$U(q')U(q) = U[(g(q';q)], \quad U^{\dagger}(q) = U^{-1}(q) = U(\overline{q})$	(I.3)
$U(e+\delta q) = I - \frac{i}{h}\,\delta q^j L_j, \quad U^{\dagger}(q) L_j U(q) = A_{jk}(q) L_k$	(I.4)
$L_j \vert q\rangle = i\hbar X_j(q)\vert q\rangle, \quad [L_j, L_k] = -2i\hbar \epsilon_{jk\ell} L_\ell$	(I.5)

restricted to $q \in \mathcal{S}_3 \subset \mathcal{E}_4$, play the role of the Euler–Rodrigues parameters which label the elements of the group in a faithful way.[13] (See Appendix A for some useful details.) We then focus our attention on the regular representation,[15] since this is the paramount structure in the kinematic quantization method. [The regular representation of SU(2) is reviewed in Appendix B.]

Now, given the resolution of the identity stated in Eq. (I.1) (cf. Table I), we define a set of operators Q^{μ} by means of the following spectral integrals:

$$Q^{\mu} = \oint_{\mathcal{S}_3} d\mu(q) |q\rangle q^{\mu} \langle q|, \tag{2.1}$$

where $d\mu(q)$ denotes the Hurwitz invariant measure on $\mathcal{S}_3$ defined in Eq. (A17). This is elementary, but fundamental. In general, this means that in Lie group theory one can consider integrals over the group manifold, like (2.1), as a basic geometric definition of quantization. One borrows this definition from standard quantum mechanics. As we will see through the chosen example, in this way one achieves a consistent and fruitful generalized setting for quantum theory. The Q's are generalized position operators on the group manifold $\mathcal{S}_3$, since they satisfy the following demands for such qualification:

$$Q^{\mu\dagger} = Q^{\mu}, \quad [Q^{\mu}, Q^{\nu}] = 0, \quad Q^{\mu}|q\rangle = q^{\mu}|q\rangle. \tag{2.2}$$

In particular, we observe that these operators satisfy the required constraint:

$$Q^{\mu} Q^{\mu} = I. \tag{2.3}$$

So we quantize the group.

In this article we adopt the left-regular representation as the working frame for developing the theory (cf. Appendix B); namely, we here consider exclusively the left unitary group-operators $\mathrm{U}(q)$ defined in Eq. (B7). (See also Table I.)

We next show that the Q's are interesting mathematical objects. We first look for the kinematic law [under SU(2) transformations] obeyed by the position operators of the group. According to their definition, it turns out that the position operators defined in Eq. (2.1) obey the same quaternion kinematic law (A2) that characterizes the parameters of the group SU(2) on $\mathcal{S}_3$:

$$U^{\dagger}(q) Q^{\mu} U(q) = g^{\mu}(q;Q) = \sigma^{\mu}_{\nu\lambda} q^{\nu} Q^{\lambda}. \tag{2.4}$$

$U(q)$ denotes the group operators [cf. Table I, Eq. (I.2)], and the group coefficients $\sigma^{\mu}_{\nu\lambda}$ are those used in Eq. (A2). It must be borne in mind that, in the present theory, Eq. (2.4) is *not* an assumption. As a matter of fact, this analytic identity is a general result of group-theoretic quantum kinematics,[6] whose proof does not demand an explicit realization of the operators involved. One proves Eq. (2.4) in a few lines. Since one adopts the regular representation of the quantized group as the working frame of the formalism, from definition (2.1), using Eqs. (I.2a) and (I.3b) of Table I, and recalling that $d\mu(q)$ is an invariant SU(2) measure on $\mathcal{S}_3$, it follows:

$$
\begin{aligned}
U^\dagger(q)Q^\mu U(q) &= \oint_{\mathcal{S}_3} d\mu(q')U^\dagger(q)|q'\rangle q'^\mu\langle q'|U(q) \\
&= \oint_{\mathcal{S}_3} d\mu(q')|g(\overline{q};q')\rangle q'^\mu\langle g(\overline{q};q')| \\
&= \oint_{\mathcal{S}_3} d\mu[g(q;q'')]|q''\rangle g^\mu(q;q'')\langle q''| \\
&= \oint_{\mathcal{S}_3} d\mu(q'')|q''\rangle g^\mu(q;q'')\langle q''| = g^\mu(q;Q),
\end{aligned}
\tag{2.5}
$$

which according to (A2) is precisely Eq. (2.4). This is one of the most important results provided by the Lie group quantization technique.

If we now consider the infinitesimal transformation $U^\dagger(e+\delta q)Q^\mu U(e+\delta q)$, Eq. (I.4) (Table I) and Eq. (2.4) lead us immediately to a set of generalized Heisenberg commutation relations for the position operators on $\mathcal{S}_3$ with the non-Abelian momenta represented by the SU(2) generators. Indeed, a typical group-theoretic calculation yields the desired commutation rules, which read

$$
[Q^\mu,L_j]=i\hbar R_j^\mu(Q)=i\hbar\sigma^\mu_{j\nu}Q^\nu=i\hbar\begin{bmatrix} Q^4 & Q^3 & -Q^2 & -Q^1 \\ -Q^3 & Q^4 & Q^1 & -Q^2 \\ Q^2 & -Q^1 & Q^2 & -Q^3 \end{bmatrix}. \tag{2.6}
$$

Here we have written L_j $(j=1,2,3)$ to denote the generators of the left regular representation [see Appendix B, Eqs. (B10) and (B11)]. The entries $R_j^\mu(q)$ of the right transport matrix are given in Eq. (A4.1). It is important to note that the σ-coefficients satisfy

$$
\sigma^\mu_{j\nu}\sigma^\nu_{k\lambda}-\sigma^\mu_{k\nu}\sigma^\nu_{j\lambda}+2\epsilon_{jk\ell}\sigma^\mu_{\ell\lambda}=0, \tag{2.7}
$$

for these imply that the commutation relations (2.6) are consistent with the Lie algebra (B11), in the sense of the Jacobi identity. Furthermore, Eq. (2.6) is also consistent with the constraint $Q^\mu Q^\mu=I$, because on $\mathcal{S}_3$ one has: $\sigma^\mu_{j\nu}q^\mu q^\nu=0$. (These features are not trivial at all.)

This is the point where the quantum principle begins to emerge from the traditional (i.e., classical) theory of SU(2). Notice that, since $\mu=1,2,3,4$ and $j=1,2,3$, the generalized closed commutation relations shown in Eq. (2.6) do not correspond to a quantized symplectic structure, as do the canonical commutation relations (in the traditional Heisenberg form) of ordinary quantum mechanics. In this sense, considering future applications, let us here remark that it not necessary to redefine Eq. (2.6) in order to get some more conventional 3×3 matrix forms, instead of the 3×4 matrix form obtained here, because *all* four operators Q^μ stand on a same footing (see, below) and satisfy the constraint $Q^\mu Q^\mu=I$. Indeed, as we have discussed in our previous work, non-Abelian quantum kinematics leads us beyond the traditional canonical quantization methods.[8] Much of what follows is a consequence of the generalized commutation relations (2.6).

We now have the kinematic model. Let us proceed to examine the Q's more closely. In general, we define functions of the commuting position operators by means of their spectral integrals:

$$
F(Q)=\oint_{\mathcal{S}_3} d\mu(q)|q\rangle F(q)\langle q| \tag{2.8}
$$

(see Table I). Then it is worth noting that Eq. (2.6) yields the general commutation rule

$$
[F(Q),L_j]=i\hbar R_j^\mu(Q)F_{,\mu}(Q)=i\hbar X_j(Q)F(Q), \tag{2.9}
$$

where $F_{,\mu}(q)$ is a vector field in $\mathcal{E}_4$, defined by $F_{,\mu}(q)=(\partial/\partial q^\mu)F(q)$, and where $X_j(q)$ denotes Lie's right vector fields on $\mathcal{S}_3$ [defined in Eq. (A10)]. [Certainly, the fact that the q's are not independent variables on $\mathcal{S}_3$ does not preclude the existence of $F_{,\mu}(q)$ when $q\in\mathcal{S}_3\subset\mathcal{E}_4$. See also the discussion in Appendix A, concerning Eqs. (A11) and (A12).]

Equation (2.9) is a very useful expression. For instance, let us define the rotation axis as follows: $F^j(q)=q^j/\sin\phi=q^j/\sqrt{1-(q^4)^2}\equiv\hat{n}^j$. In quantum kinematics, the rotation axis $\hat{\mathbf{n}}$ becomes a self-adjoint operator $\hat{\mathbf{N}}$: $\hat{\mathbf{n}}\rightarrow\hat{\mathbf{N}}$. Using Eq. (2.9), we easily find the generalized commutation relations obeyed by the Cartesian components $\hat{N}^j$ of the rotation-axis-operator with the generators L_j, which read

$$[\hat{N}^j,L_k]=i\hbar[\cot\Phi(\delta^j_k-\hat{N}^j\hat{N}^k)-\epsilon_{jk\ell}\hat{N}^\ell]. \tag{2.10}$$

[In fact, one has $X_k(q)F^j(q)=\cot\phi(\delta^j_k-\hat{n}^j\hat{n}^k)-\epsilon_{jk\ell}\hat{n}^\ell$. If one defines instead $G^j(q)=q^j/\sqrt{q^kq^k}\equiv\hat{n}^j$ (say) one obtains the same result (2.10). Indeed, Eq. (2.10) does not depend on how one defines the unit vector $\hat{\mathbf{n}}$ in terms of the q's. For the operator $\cot\Phi$, see Eq. (2.12), below.] The constraint $\hat{N}^j\hat{N}^j=I$ follows, since the rotation-axis operator $\hat{\mathbf{N}}=(\hat{N}^1,\hat{N}^2,\hat{N}^3)$ is a bounded operator given by

$$\hat{\mathbf{N}}=\mu_0\oint_{\mathcal{S}_3}d\Omega(\phi,\hat{\mathbf{n}})|\phi,\hat{\mathbf{n}}\rangle\hat{\mathbf{n}}\langle\phi,\hat{\mathbf{n}}|, \tag{2.11}$$

so that $\hat{\mathbf{N}}|\phi,\hat{\mathbf{n}}\rangle=\hat{\mathbf{n}}|\phi,\hat{\mathbf{n}}\rangle$ holds. [Concerning the $(\phi,\hat{\mathbf{n}})$ notation, see Appendix B].

In the same manner, we define two bounded operators $\cos\Phi$ and $\sin\Phi$ on $\mathcal{S}_3$:

$$\cos\Phi=\mu_0\oint_{\mathcal{S}_3}d\Omega(\phi,\hat{\mathbf{n}})|\phi,\hat{\mathbf{n}}\rangle\cos\phi\langle\phi,\hat{\mathbf{n}}|, \tag{2.12a}$$

$$\sin\Phi=\mu_0\oint_{\mathcal{S}_3}d\Omega(\phi,\hat{\mathbf{n}})|\phi\cdot\hat{\mathbf{n}}\rangle\sin\phi\langle\phi,\hat{\mathbf{n}}|, \tag{2.12b}$$

which satisfy the rather natural constraint $\cos^2\Phi+\sin^2\Phi=I$ and do not cause difficulties related to periodicity. As the reader can prove, for these operators we obtain the following commutation relations with the L's:

$$[\cos\Phi,L_j]=-i\hbar\hat{N}^j\sin\Phi,\quad[\sin\Phi,L_j]=i\hbar\hat{N}^j\cos\Phi, \tag{2.13}$$

which have a very suggestive form. [See also Eq. (2.17), below.] Similar commutation relations are well known for the (trivial) case of U(1) Abelian quantum kinematics.[1,2] Indeed, from Eq. (2.13) one obtains

$$[\cos\Phi,L]=-i\hbar\sin\Phi,\quad[\sin\Phi,L]=i\hbar\cos\Phi, \tag{2.14}$$

where the Hermitian operator $L=\frac{1}{2}(\hat{\mathbf{N}}\cdot\mathbf{L}+\mathbf{L}\cdot\hat{\mathbf{N}})$ may be interpreted as the U(1) angular-momentum generator in the direction of $\hat{\mathbf{N}}$. [Compare with Ref. 1, and with the references given in Ref. 2.] Equation (2.14) is quite familiar. However, to our best knowledge, Eqs. (2.10) and (2.13) do not figure in the literature, because in order to obtain the SU(2) position operators $(\Phi,\hat{\mathbf{N}})$ one has to quantize the group.

If now one considers the Euler–Rodrigues operators, i.e.,

$$Q^j=\hat{N}^j\sin\Phi,\quad Q^4=\cos\Phi \tag{2.15}$$

(cf. Appendix A), recalling the definition of the σ-coefficients, one can use Eq. (2.6) directly to obtain other basic commutation relations. For example, in this way, the commutators

$$[\hat{N}^j\sin\Phi,L_k]=i\hbar(\delta^j_k\cos\Phi-\epsilon_{jk\ell}\hat{N}^\ell\sin\Phi), \tag{2.16}$$

follow immediately from (2.6). [Of course, these commutators are consistent with Eqs. (2.10) and (2.13).]

Of particular interest are the following commutation relations:

$$[\cos 2\Phi,J_j]=-2i\hbar\hat{N}^j\sin 2\Phi,\quad[\sin 2\Phi,J_j]=2i\hbar\hat{N}^j\cos 2\Phi, \tag{2.17}$$

where we have written $2J_j = L_j$. [The reader will have no difficulty verifying Eq. (2.17), after a short calculation.] These commutation relations are called on to play an important role in the quantum kinematic theory of proper rotations, since the operators J_j are the standard angular-momentum operators in ordinary space, and 2Φ is the SO(3) rotation-angle operator about the axis-operator $\hat{\mathbf{N}}$. [See also Eq. (A9).]

All required generalized commutation relations can be obtained in this fashion, one way or another, from Eqs. (2.6) and (2.9). We have been unable to find such generalized Heisenberg commutation relations in the extensive literature devoted to SU(2).

III. QUANTUM KINEMATIC INVARIANT OPERATORS

We next take advantage of this technique, from a very general point of view. Let us first introduce the following matrix-operator [cf. Eq. (A7)]:

$$A_{jk}(Q) = Q^\mu Q_\mu \delta_{jk} + 2Q^j Q^k - 2Q^4 \epsilon_{jk\ell} Q^\ell, \tag{3.1}$$

which will be referred to as the antiadjoint operator. Since $U^\dagger(q)F(Q)U(q) = F[g(q;Q)]$ holds in general, from the discussion presented in Appendix A we conclude that the antiadjoint operator obeys the kinematic law given by the unitary transformation

$$U^\dagger(q)\overline{A}_{jk}(Q)U(q) = \overline{A}_{j\ell}(Q)\overline{A}_{\ell k}(q). \tag{3.2}$$

Hence a closed algebra follows immediately:

$$[\overline{A}_{jk}(Q), L_\ell] = -2i\hbar \epsilon_{k\ell m} \overline{A}_{jm}(Q). \tag{3.3}$$

Equation (3.3), together with the Lie algebra (B11) and the fact $[\overline{A}_{jk}(Q), \overline{A}_{\ell m}(Q)] = 0$, define the generalized Weyl–Heisenberg algebra of SU(2) quantum kinematics, which is analogous to the Weyl–Heisenberg algebra of the canonical commutation relations of ordinary quantum mechanics. This algebra is interesting, because it produces new invariant operators of the theory, as we shall see presently.

The main reason for introducing the antiadjoint operator of a Lie group, in general, is that it allows the definition of the following operators:[14]

$$\begin{aligned} R_j(Q;L) &= \overline{A}_{jk}(Q)L_k \\ &= (\cos^2\Phi - \sin^2\Phi)L_j + 2\sin^2\Phi \hat{N}^j \hat{N}^k L_k + 2\cos\Phi \sin\Phi \epsilon_{jk\ell} \hat{N}^k L_\ell, \end{aligned} \tag{3.4}$$

for which Eq. (3.2) together with (B12) imply the SU(2) invariant property:

$$U^\dagger(q) R_j(Q;L) U(q) = R_j(Q;L). \tag{3.5}$$

Thus one has

$$[R_j(Q;L), L_k] = 0, \tag{3.6}$$

for $j,k = 1,2,3$. Moreover, a rather lengthy but straightforward calculation, using the antiadjoint representation, yields $X_k(q)\overline{A}_{jk}(q) = 0$, which means that the invariant operators defined in (3.4) are Hermitian: $R_j^\dagger(Q;L) = R_j(Q;L)$, notwithstanding the fact that the Q's and the L's do not commute. Hence, in the kinematic theory of SU(2), besides the familiar Casimir operator $\mathbf{L}^2$, the group acquires three basic Hermitian invariant operators: R_1, R_2, R_3, which stem from the group-quantization procedure.[16]

Furthermore, using well-known properties of the adjoint representation, one proves that the basic invariant operators (3.4) satisfy the right SU(2) Lie algebra:

$$[R_j(Q;L), R_k(Q;L)] = 2i\hbar \epsilon_{jk\ell} R_\ell(Q;L) \tag{3.7}$$

TABLE II. Synopsis of the right regular representation formalism on $\mathcal{S}_3$, within the carrier Hilbert space $\mathcal{H}[\mathrm{SU}(2)]$.

$\langle q'\|q\rangle=\mu_0^{-1}R(q)\delta^{(3)}(q'-q),\quad \oint_{\mathcal{S}_3} d\mu(q)\|q\rangle\langle q\|=I$	(II.1)
$V(q')\|q\rangle=\|g(q;q')\rangle,\quad V(q)=\oint_{\mathcal{S}_3} d\mu(q')\|g(q';q)\rangle\langle q'\|$	(II.2)
$V(q')V(q)=V[g(q;q')],\quad V^{\dagger}(q)=V^{-1}(q)=V(\overline{q})$	(II.3)
$V(e+\delta q)=I-\frac{i}{\hbar}\,\delta q^j R_j,\quad V^{\dagger}(q)R_jV(q)=\overline{A}_{jk}(q)R_k$	(II.4)
$R_j\|q\rangle=i\hbar Y_j(q)\|q\rangle,\quad [R_j,R_k]=2i\hbar\epsilon_{jk\ell}R_\ell$	(II.5)

[see Table II, Eq. (II.5)]. For the Casimir operator one has $\mathbf{L}^2=\mathbf{R}^2$. Therefore, we identify the operators $R_j(Q;L)$ as the generators of the right regular representation acting as invariant operators within the left representation of the group.[14] [See also, Eq. (A15).]

To end up this brief study, we consider the following commutation rules:

$$[F(Q),R_j]=i\hbar L_j^{\mu}(Q)F_{,\mu}(Q)=i\hbar Y_j(Q)F(Q), \tag{3.8}$$

where $Y_j(q)$ denotes Lie's left vector fields on $\mathcal{S}_3$ [defined in Eq. (A10)]. These rules yield, for instance,

$$[N^j \sin\Phi,R_k]=i\hbar(\delta_k^j\cos\Phi+\epsilon_{jk\ell}N^\ell\sin\Phi), \tag{3.9}$$

$$[\cos\Phi,R_j]=-i\hbar N^j\sin\Phi,\quad [\sin\Phi,R_j]=i\hbar N_j\cos\Phi. \tag{3.10}$$

Indeed, the kinematic calculus of generalized Heisenberg commutation relations for SU(2) can be applied easily to the kinematic invariant operators.

We here omit the calculations leading to these results. It is important to note, however, that one does not arrive at these results if one does not quantize the group.

IV. CONCLUDING REMARKS AND PERSPECTIVES

In our opinion, the quantum-kinematic operator calculus obtained in this paper bears a great potential value for physics, in general, because it stems from a standard procedure for handling symmetries in quantum theory. In our previous work, a group-quantization program has been developed for the case of noncompact non-Abelian Lie groups (cf. Ref. 8 and references quoted therein). The attained formalism has been applied to some models for which the Schrödinger equation, as well as the corresponding propagation kernel, have been deduced on a strict group-theoretic fashion.[10,11,17,18] It is rather clear, however, that further progress in quantum kinematics requires the consideration of symmetries described by compact Lie groups.

In particular, SU(2) symmetry must be examined upon this new perspective, as we have done in this introductory paper. Quantization of this fundamental symmetry is needed, for instance, for the development of the quantum kinematic theory of the complete Kepler group (recently introduced in the literature[19]), of the Poincaré group and the Galileo group in four-dimensional space–time, and of many other relevant Lie groups which contain SU(2) as a proper subgroup. Work is in progress concerning a companion paper to the present one, devoted to the quantum kinematic theory of the three-dimensional isotropic harmonic oscillator, in which SU(2) quantum kinematics (as introduced here) plays a central role. In fact, the piece of work presented in this paper is ripe for many physical applications.

In light of the previous results, the first point we want to note is that several uncertainty correlations (of the mean-square deviations between rotation observables and angular momentum), which appear in quantum kinematics, are worthy of a detailed study in every application of SU(2) in quantum theory. Maybe there are problems in which new physical predictions can be found by considering SU(2) quantum kinematic models in this sense.

We also have to remark that SU(2) quantum kinematics is quite different from the quantum theory of rotation angles of Barnett and Pegg.[20] As we have shown, the group-quantization approach goes much further into the very roots of the problem set by angle observables in quantum mechanics, than any other attempt considered previously.

We are now in position to use a maximal set of superselection rule operators, as given for instance by $\{\mathbf{R}^2,R_3\}$, in order to obtain full reduction (i.e., maximal diagonalization) of the regular representation. In other words, we can diagonalize the incoherent Hilbert space $\mathcal{H}[\mathrm{SU}(2)]$ into invariant subspaces $\mathcal{H}_{JM}[\mathrm{SU}(2)]$, each carrying an irreducible representation of the group, which corresponds to a SU(2) elementary system.[21] To this end, one must search for a complete orthogonal basis $\{|JMN\rangle\}$ within the left regular representation, associated with the complete set of commuting operators $\{\mathbf{R}^2,R_3,L_3\}$. That is, one looks for vectors satisfying the basic conditions

$$\langle JMN|J'M'N'\rangle=\delta_{JJ'}\delta_{MM'}\delta_{NN'}, \tag{4.1.1}$$

$$\sum_{J=0}^{\infty}\sum_{M=-J}^{J}\sum_{N=-J}^{J}|JMN\rangle\langle JMN|=I, \tag{4.1.2}$$

where the multiplets $|JMN\rangle$ are solutions to the following system of simultaneous eigenvalue equations:

$$\mathbf{R}^2|JMN\rangle=\hbar^2J(J+2)|JMN\rangle. \tag{4.2.1}$$

$$R_3|JMN\rangle=\hbar M|JMN\rangle, \tag{4.2.2}$$

$$L_3|JMN\rangle=\hbar N|JMN\rangle. \tag{4.2.3}$$

Here: $J=0,1,2,...$ and $M,N=-J,-J+2,...,J-2,J$. This problem will be examined in a forthcoming paper.

Here we wish to remark only that, if one considers SU(2) [i.e., the universal covering group of SO(3)] acting as an external symmetry group of a system, then Eqs. (4.2.1) and (4.2.2) define automatically an isotopic angular momentum problem for that system, because both $\mathbf{R}^2$ and R_3 are invariant operators of SU(2). (N is not a good quantum number because L_3 is not an invariant operator.) This result is peculiar to quantum kinematics. It is a very reassuring feature, for it means that quantum kinematics brings into the fore an isotopic structure as a necessary theoretical construct, in a rather natural way indeed.

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APPENDIX A: THE REAL-QUATERNION GROUP

It is well known that there is a one-to-one relation between points on the three-dimensional surface $\mathcal{S}_3$ (of the unit sphere in four-dimensional Euclidean space $\mathcal{E}_4$) and elements of SU(2): $\mathcal{S}_3\ni q\leftrightarrow u(q)\in\mathrm{SU}(2)$, which is preserved by the group law;[13] briefly, the group manifold of SU(2) is the unit spherical surface $\mathcal{S}_3$. Since this is the basic framework adopted in this paper, a compact geometric notation for handling SU(2) on $\mathcal{S}_3$ will be introduced in this appendix, which is a very useful tool for quantum kinematics.

For simplicity and definiteness we label the group elements by means of the Euler–Rodrigues parameters:[13] $\mathbf{q}=\hat{\mathbf{n}}\sin\phi$ and $q^4=\cos\phi$, with $0\leqslant\phi\leqslant\pi$ and $\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}=1$. Thus we write the 2×2 matrices

$$u(q)=u(\mathbf{q},q^4)=q^4\sigma_4+iq^k\sigma_k=e^{i\phi\hat{\mathbf{n}}\cdot\sigma}\in\mathrm{SU}(2), \tag{A1}$$

which meaning is clear. One obtains the group law for SU(2) from the so-called quaternionic composition rule,[13] which can be written in a compact manner as follows:

$$q''^{\mu}=g^{\mu}(q';q)=\sigma^{\mu}_{\nu\lambda}q'^{\nu}q^{\lambda}. \tag{A2}$$

One defines the group coefficients $\sigma^{\mu}_{\nu\lambda}$ using well-known properties of the Pauli matrices. Thus one sets: $\sigma^{j}_{k\ell}=-\epsilon_{jk\ell}$, $\sigma^{\mu}_{4\nu}=\sigma^{\mu}_{\nu 4}=\delta^{\mu}_{\nu}$, and $\sigma^{4}_{\mu\nu}=\eta_{\mu\nu}$, where the matrix $\eta_{\mu\nu}=\mathrm{diag}(---+)$ denotes the Minkowski metric. Using this notation, the inversion law for the Euler–Rodrigues parameters reads $\overline{q^{\mu}}=\eta_{\mu}=\eta_{\mu\nu}q^{\nu}$, since the identity point is at the ''north pole;'' i.e., $e^{\mu}=\delta^{\mu}_{4}=(0,0,0,1)\in\mathcal{S}_3$.

Useful relations obeyed by the group-coefficients follow from the group property. Especially, we note a remarkable feature of the quaternionic composition law:

$$q''^{\mu}q''^{\mu}=(q'^{\nu}q'^{\nu})(q^{\lambda}q^{\lambda}), \tag{A3}$$

from which the implication $(q';q)\in\mathcal{S}_3\times\mathcal{S}_3\Rightarrow q''\in\mathcal{S}_3$ follows. The Euler–Rodrigues variables are not essential parameters of the group, to be sure, for they do not correspond to independent variables. They are faithful parameters, however, because $q=(\mathbf{q},q^4)=(\hat{\mathbf{n}}\sin\phi,\cos\phi)$ covers the whole group manifold $\mathcal{S}_3$ in a strictly one-to-one way. [So we see that there is indeed a one-to-one relation between points on $\mathcal{S}_3$ and elements of SU(2), which is preserved by the group law.]

By means of the same group-coefficients $\sigma^{\mu}_{\nu\lambda}$ one defines another four-dimensional Lie group in $\mathcal{E}_4$. With this aim, all one needs to modify is the inversion law of the parameters, to read $\overline{q^{\mu}}=\rho^{-2}\eta_{\mu\nu}q^{\nu}$, where $\rho=\sqrt{q^{\mu}q^{\mu}}$, and one identifies the points $\overline{0}=\infty$ and $\overline{\infty}=0$. This group is the real-quaternion group;[13] we denote it by $\Sigma(4)$. The noncompact group $\Sigma(4)$ is isomorphic to $\mathcal{R}_x\otimes\mathrm{SU}(2)$, has the whole Euclidean space $\mathcal{E}_4$ for group manifold, and corresponds to the analytic continuation of SU(2) along the radius $0<p<\infty$. In fact, in this construct, SU(2) appears simply as that subgroup which arises from restricting the action of the quaternion group $\Sigma(4)$ to the locus $\mathcal{S}_3\subset\mathcal{E}_4$.

Since $\Sigma(4)$ is a noncompact (connected and simply connected) Lie group, right and left transport matrices are defined in the usual manner: $R^{\mu}_{\nu}(q)=\lim_{q'\to e}(\partial/\partial q'^{\nu})g^{\mu}(q';q)$ and $L^{\mu}_{\nu}(q)=\lim_{q'\to e}(\partial/\partial q'^{\nu})g^{\mu}(q;q')$, respectively. Expressions for these transport matrices in $\mathcal{E}_4$ are then obtained as follows:

$$R^{\mu}_{\nu}(q)=\sigma^{\mu}_{\nu\lambda}q^{\lambda}=\begin{bmatrix}\delta^{k}_{j}q^4+\epsilon_{jk\ell}q^{\ell} & -q^{j}\\ q^{k} & q^4\end{bmatrix}, \tag{A4.1}$$

$$L^{\mu}_{\nu}(q)=\sigma^{\mu}_{\lambda\nu}q^{\lambda}=\begin{bmatrix}\delta^{k}_{j}q^4-\epsilon_{jk\ell}q^{\ell} & -q^{j}\\ q^{k} & q^4\end{bmatrix}. \tag{A4.2}$$

By the same token, one gets the corresponding inverse transport matrices of $\Sigma(4)$, which are given by $\rho^{-2}\overline{R}^{\mu}_{\nu}(q)$ and $\rho^{-2}\overline{L}^{\mu}_{\nu}(q)$, where

$$\overline{R}^{\mu}_{\nu}(q)=\sigma^{\mu}_{\nu\lambda}\overline{q^{\lambda}}=\begin{bmatrix}\delta^{k}_{j}q^4-\epsilon_{jk\ell}q^{\ell} & q^{j}\\ -q^{k} & q^4\end{bmatrix}, \tag{A5.1}$$

$$\overline{L}^{\mu}_{\nu}(q)=\sigma^{\mu}_{\lambda\nu}\overline{q^{\lambda}}=\begin{bmatrix}\delta^{k}_{j}q^4+\epsilon_{jk\ell}q^{\ell} & q^{j}\\ -q^{k} & q^4\end{bmatrix}. \tag{A5.2}$$

[Here, the superscripts (μ) label the columns, and the subscripts (ν) label the rows.] One checks these matrices against the required property: $R^{\lambda}_{\mu}(q)\overline{R}^{\nu}_{\lambda}(q)=L^{\lambda}_{\mu}(q)\overline{L}^{\nu}_{\lambda}(q)=\rho^2\delta^{\nu}_{\mu}$, for all $q\in\mathcal{E}_4$. Also $R(q)=\det[R^{\mu}_{\nu}(q)]=L(q)=\det[L^{\mu}_{\nu}(q)]=(q^{\mu}q^{\mu})^2=\rho^4$ can be shown rather easily. Moreover, a detailed analysis shows that, if $q\in\mathcal{S}_3$, these matrices transport vectors which are tangent to $\mathcal{S}_3$ into tangent vectors to $\mathcal{S}_3$. Therefore, fixing $q^{\mu}q^{\mu}=1$, Eqs. (A4) and (A5) yield precisely the transport matrices of SU(2) on $\mathcal{S}_3$.

The transport matrices are affine connections on the group manifold. Furthermore, for the noncompact Lie group $\Sigma(4)$ it follows that the mixed transport matrix given by $\rho^{-2}R^{\lambda}_{\nu}(q)\bar{L}^{\mu}_{\lambda}(q)$ carries the *adjoint representation* of the group.[14] Hence, in the present instance, using Eqs. (A4.1) and (A5.2), for the action of SU(2) on $\mathcal{S}_3$ we get

$$A^{\mu}_{\nu}(q)=\begin{bmatrix} A_{jk}(q) & 0 \\ 0 & 1 \end{bmatrix}. \tag{A6}$$

The 3×3 matrix $A_{jk}(q)$ in Eq. (A6), has entries given by

$$A_{jk}(q)=q^{\mu}q_{\mu}\delta_{jk}+2q^{j}q^{k}+2q^{4}\epsilon_{jk\ell}q^{\ell}, \tag{A7}$$

and carries the adjoint representation of SU(2). In fact, one easily finds

$$u^{\dagger}(q)\sigma_{j}u(q)=A_{jk}(q)\sigma_{k}, \tag{A8}$$

and therefore the A's satisfy the group law $A_{j\ell}(q')A_{\ell k}(q)=A_{jk}[g(q';q)]$. We can further analyze this construct recalling that proper rotations in ordinary space $\mathcal{E}_3$ are characterized by an angle of rotation $\vartheta(0\leqslant\vartheta\leqslant 2\pi)$ about an axis of rotation $\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}=1)$, and are represented by the group SO(3) of real orthogonal matrices of the form:

$$R_{jk}(\vartheta,\hat{\mathbf{n}})=\delta_{jk}\cos\vartheta+\hat{n}^{j}\hat{n}^{k}(1-\cos\vartheta)+\epsilon_{jk\ell}\hat{n}^{\ell}\sin\vartheta. \tag{A9}$$

One identifies $2\phi=\vartheta$ and one obtains: $A_{jk}(q)=R_{jk}(\vartheta,\hat{\mathbf{n}})\in\mathrm{SO}(3)$. Also, defining $\mathbf{x}=x^{k}\sigma_{k}$, the unitary transformation $\mathbf{x}'=u^{\dagger}(q)\mathbf{x}u(q)$ yields $x'^{j}=x^{k}R_{jk}(\vartheta,\hat{\mathbf{n}})$. Hence, the double covering of SU(2) onto SO(3) becomes manifest. (These things are, of course, well known.[13])

We next define right and left vector fields for $\Sigma(4)$ in $\mathcal{E}_4$ by means of the transport matrices, as usual: $X_{\mu}(q)=R^{\nu}_{\mu}(q)\partial_{\nu}=\sigma^{\nu}_{\mu\lambda}q^{\lambda}\partial_{\nu}$, and $Y_{\mu}(q)=L^{\nu}_{\mu}(q)\partial_{\nu}=\sigma^{\nu}_{\lambda\mu}q^{\lambda}\partial_{\nu}$. Writing them more explicitly, one sees that these operators correspond to SU(2) right and left generators acting on $\mathcal{S}_3$, which in the Euler–Rodrigues parametrization read:

$$X_{j}(q)=q^{4}\partial_{j}-q^{j}\partial_{4}-\epsilon_{jk\ell}q^{k}\partial_{\ell},\quad Y_{j}(q)=q^{4}\partial_{j}-q^{j}\partial_{4}+\epsilon_{jk\ell}q^{k}\partial_{\ell}, \tag{A10}$$

respectively, as well as to the dilation generator of $\Sigma(4)$ acting along the radius ρ in $\mathcal{E}_4$, which is given by $X_4(q)=Y_4(q)=q^{\mu}\partial_{\mu}=\rho(\partial/\partial\rho)$.

Using (θ,φ,ϕ) as the independent variables of the SU(2) theory [where $\hat{\mathbf{n}}$ is a unit 3-vector corresponding to $\hat{\mathbf{n}}=(\theta,\varphi)$], given any function $\psi(q)$ defined on $\mathcal{S}_3$, the displacement $q\to(q+dq)\in\mathcal{S}_3$ produces the differential

$$d\psi(q)=\frac{\partial\psi}{\partial q^{j}}\sin\phi\, d\hat{n}^{j}+\left(\frac{\partial\psi}{\partial q^{j}}\hat{n}^{j}\cos\phi-\frac{\partial\psi}{\partial q^{4}}\sin\phi\right)d\phi, \tag{A11}$$

with $d\hat{n}^{j}=(\partial\hat{n}^{j}/\partial\theta)d\theta+(\partial\hat{n}^{j}/\partial\varphi)d\varphi$. On the other hand, defining $\delta q^{\mu}_{R}=\bar{R}^{\mu}_{\nu}(q)dq^{\nu}$, a straightforward calculation shows that $d\psi(q)=\delta q^{j}_{R}X_{j}(q)\psi(q)$ is given by

$$d\psi(q)=\delta\phi\hat{n}^{j}_{R}\left(\cos\phi\,\frac{\partial\psi}{\partial q^{j}}-\sin\phi\hat{n}^{j}\,\frac{\partial\psi}{\partial q^{4}}-\sin\phi\epsilon_{jk\ell}\hat{n}^{k}\,\frac{\partial\psi}{\partial q^{\ell}}\right). \tag{A12}$$

Hence, consistency demands: $d\phi\hat{n}^{j}=\delta\phi\hat{n}^{j}_{R}$ and $d\hat{n}^{j}=\delta\phi\epsilon_{jk\ell}\hat{n}^{k}\hat{n}^{\ell}_{R}$ [which follow from the transport formula $dq^{\mu}=R^{\mu}_{\nu}(q)\delta q^{\nu}_{R}$], as the reader can prove in a few lines. Furthermore, one also proves that $X_4(q)\psi(q)=0$ follows if $q\in\mathcal{S}_3$. (We left this as an exercise for the reader.) In this fashion we conclude that the action of the four operators [either $X_{\mu}(q)$ or $Y_{\mu}(q)$], on functions $\psi(q)$ defined on the three-dimensional surface $\mathcal{S}_3$, is consistent with the three degrees of freedom of the group SU(2).

As operators in $\mathcal{E}_4$, the generators of $\Sigma(4)$ satisfy the following Lie algebra:

$$[X_{j}(q),X_{k}(q)]=2\epsilon_{jk\ell}X_{\ell}(q),\quad [Y_{j}(q),Y_{k}(q)]=-2\epsilon_{jk\ell}Y_{\ell}(q), \tag{A13}$$

$$[X_j(q),X_4(q)]=0, \quad [Y_j(q),Y_4(q)]=0, \tag{A14}$$

and moreover

$$[X_\mu(q),Y_\nu(q)]=0. \tag{A15}$$

Equation (A13) corresponds to the (right and left) realizations of the SU(2) Lie algebra, whereas Eq. (A14) assures a consistent result (for our purposes here). Equation (A15) is of fundamental importance in quantum kinematics [i.e., see Eq. (3.6)].

Finally, since $\Sigma(4)$ is a noncompact Lie group, an invariant measure in $\mathcal{E}_4$ follows immediately:

$$d\mu(q)=\mu_0\rho^{-4}\ dq^1\ dq^2\ dq^3\ dq^4=\mu_0\ \frac{\sin\ \theta\ \sin^2\ \phi}{\rho}\ d\rho\ d\theta\ d\varphi\ d\phi=\mu_0\ \frac{d\rho}{\rho}\ d\Omega(\phi,\hat{\mathbf{n}}), \tag{A16}$$

where μ_0 denotes an arbitrary normalization constant and $d\Omega(\phi,\hat{\mathbf{n}})$ is the element of solid angle in $\mathcal{E}_4$. The direct-product structure of the quaternion group $\Sigma(4)\sim\mathcal{R}\times\otimes\mathrm{SU}(2)$ means that (if we fix $\rho=1$) we obtain an invariant measure on $\mathcal{S}_3$ as well, which is given by

$$d\mu(q)=\mu_0\ \sin\ \theta\ \sin^2\ \phi\ d\theta\ d\varphi\ d\phi=\mu_0\ d\Omega(\phi,\hat{\mathbf{n}}). \tag{A17}$$

This well-known measure is indeed (right and left) SU(2)-invariant.

APPENDIX B: SU(2) REGULAR REPRESENTATION REVISITED

We here append a brief description of the regular representation of SU(2) as it follows from the embedding scheme $\mathcal{S}_3\subset\mathcal{E}_4$ presented in Appendix A. Henceforth we use a shorthand notation for handling the Euler–Rodrigues parameters $q=(\mathbf{q},q^4)$ in terms of the angle parameters $(\phi,\hat{\mathbf{n}})$. Thus we set: $q=(\hat{\mathbf{n}}\ \sin\ \phi,\cos\ \phi)\equiv(\phi,\hat{\mathbf{n}})\in\mathcal{S}_3$.

As one knows, the regular representation[15] of SU(2) is carried by functions $\psi(q)=\psi(\phi,\hat{\mathbf{n}})$, which have a finite norm with respect to the invariant measure on $\mathcal{S}_3$ defined in Eq. (A17):

$$\langle\psi|\psi\rangle=\mu_0\oint_{\mathcal{S}_3}d\Omega(\phi,\hat{\mathbf{n}})|\psi(\phi,\hat{\mathbf{n}})|^2<\infty. \tag{B1}$$

To proceed, one introduces a Hilbert space $\mathcal{H}[\mathrm{SU}(2)]$ such that $|\psi\rangle\in\mathcal{H}[\mathrm{SU}(2)]\Leftrightarrow\langle\psi|\psi\rangle<\infty$. In order to have a one-to-one mapping between wave functions $\psi(q)$ and vectors $|\psi\rangle$, one considers the associated rigged Hilbert space $\tilde{\mathcal{H}}[\mathrm{SU}(2)]$, which is spanned by continuous basic vectors $|q\rangle$ satisfying the one-to-one correspondence $\mathcal{S}_3\ni q\leftrightarrow|q\rangle\in\tilde{\mathcal{H}}[\mathrm{SU}(2)]$. The basic vectors obey the orthogonal relation:

$$\langle\phi',\hat{\mathbf{n}}'|\phi,\hat{\mathbf{n}}\rangle=\mu_0^{-1}\ \frac{\delta(\phi'-\phi)\,\delta(\theta'-\theta)\,\delta(\varphi'-\varphi)}{\sin^2\ \phi\ \sin\ \theta}, \tag{B2}$$

and yield the following resolution of the identity:

$$\mu_0\oint_{\mathcal{S}_3}d\Omega(\phi,\hat{\mathbf{n}})|\phi,\hat{\mathbf{n}}\rangle\langle\phi,\hat{\mathbf{n}}|=I. \tag{B3}$$

Wave functions are then defined on the group manifold in the usual manner; namely, one writes $\psi(q)=\langle q|\psi\rangle\equiv\psi(\phi,\hat{\mathbf{n}})=\langle\phi,\hat{\mathbf{n}}|\psi\rangle$, whenever $|\psi\rangle\in\mathcal{H}[\mathrm{SU}(2)]$ and $q=(\phi,\hat{\mathbf{n}})\in\mathcal{S}_3$.

Using this notation, the quaternion group law (A2) can be cast in terms of the angle variables $(\phi,\hat{\mathbf{n}})$. In fact, (A2) corresponds to the well-known combination formulas:[13]

$$\hat{\mathbf{n}}''\ \sin\ \phi''=\hat{\mathbf{n}}'\ \sin\ \phi'\ \cos\ \phi+\hat{\mathbf{n}}\ \sin\ \phi\ \cos\ \phi'-\hat{\mathbf{n}}\times\hat{\mathbf{n}}\ \sin\ \phi'\ \sin\ \phi, \tag{B4a}$$

$$\cos\ \phi''=\cos\ \phi'\ \cos\ \phi-\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}\ \sin\ \phi'\ \sin\ \phi. \tag{B4b}$$

At times it is useful to write this law in a compact symbolic way:

$$(\phi'',\hat{\mathbf{n}}'')=(\phi',\hat{\mathbf{n}}')\circ(\phi,\hat{\mathbf{n}}). \tag{B5}$$

For instance, if we consider a point $q=(\phi,\hat{\mathbf{n}})\in\mathcal{S}_3$ which is left-transported by means of $e+\delta q'=(\delta\phi,\hat{\mathbf{n}}')\in\mathcal{S}_3$, Eq. (B4) allows us to express Lie's left-differentials as follows:

$$\delta_L(\hat{\mathbf{n}}\sin\phi)=\delta\phi(\hat{\mathbf{n}}'\cos\phi-\hat{\mathbf{n}}'\times\hat{\mathbf{n}}\sin\phi),\quad \delta_L(\cos\phi)=-\delta\phi\hat{\mathbf{n}}'\cdot\hat{\mathbf{n}}\sin\phi, \tag{B6}$$

where $\hat{\mathbf{n}}''\sin\phi''=\hat{\mathbf{n}}\sin\phi+\delta_L(\hat{\mathbf{n}}\sin\phi)$ and $\cos\phi''=\cos\phi+\delta_L(\cos\phi)$ give us the angle coordinates of the image point, i.e., $(\phi'',\hat{\mathbf{n}})=(\delta\phi,\hat{\mathbf{n}}')\circ(\phi,\hat{\mathbf{n}})$.

A sensible definition of the unitary group-operators $U(q)\equiv U(\phi,\hat{\mathbf{n}})$, of the left regular representation, can then be given in the following manner:

$$U(\phi,\hat{\mathbf{n}})=e^{-(i/\hbar)\phi\hat{\mathbf{n}}\cdot\mathbf{L}}=\mu_0\int_0^\pi \mathbf{d}\phi'\ \sin^2\phi'\ \oint_{\mathcal{S}_2}d\Omega(\hat{\mathbf{n}}')|(\phi,\hat{\mathbf{n}})\circ(\phi',\hat{\mathbf{n}}')\rangle\langle\phi',\hat{\mathbf{n}}'|. \tag{B7}$$

$\mathbf{L}$ denotes the generators of this representation: $\mathbf{L}=(L_1,L_2,L_3)$. In fact, a straightforward calculation yields the unitary representation property obeyed by these operators:

$$U(\phi',\hat{\mathbf{n}}')U(\phi,\hat{\mathbf{n}})=U[(\phi',\hat{\mathbf{n}}')\circ(\phi,\hat{\mathbf{n}})],\quad U^\dagger(\phi,\hat{\mathbf{n}})=U^{-1}(\phi,\hat{\mathbf{n}}), \tag{B8}$$

as well as

$$U(\phi',\hat{\mathbf{n}}')|\phi,\hat{\mathbf{n}}\rangle=|(\phi',\hat{\mathbf{n}}')\circ(\phi,\hat{\mathbf{n}})\rangle, \tag{B9}$$

As we see, the operators (B7) realize the left regular representation of SU(2) on $\mathcal{S}_3$. The infinitesimal operators are given by $U(\delta\phi,\hat{\mathbf{n}})=I-(i/\hbar)\delta\phi\hat{\mathbf{n}}\cdot\mathbf{L}$. Therefore, using Eq. (A10), we obtain

$$L_j|\phi,\hat{\mathbf{n}}\rangle=i\hbar X_j(q)|q\rangle=i\hbar(\cos\phi\partial_j-\sin\phi\hat{n}^j\partial_4-\sin\phi\epsilon_{jk\ell}\hat{n}^k\partial_\ell)|\phi,\hat{\mathbf{n}}\rangle, \tag{B10}$$

for $q=(\phi,\hat{\mathbf{n}})\in\mathcal{S}_3$, as consistency demands. This expression yields the familiar Lie Algebra in terms of the left-regular representation's generators [cf., Eq. (A13)]:

$$[L_j,L_k]=-2i\hbar\epsilon_{jk\ell}L_\ell. \tag{B11}$$

[Notice that the generators L_j equal twice the standard physical operators of angular momentum J_j. See, for instance, Eq. (2.17). The reader can change to the conventional scale if he likes.]

Within the left regular representation, the adjoint representation is obtained from $U^\dagger(\phi,\hat{\mathbf{n}})U(\delta\phi,\hat{\mathbf{n}}')U(\phi,\hat{\mathbf{n}})$. This yields [see Eq. (A7)]

$$U^\dagger(\phi,\hat{\mathbf{n}})L_jU(\phi,\hat{\mathbf{n}})=[(\cos^2\phi-\sin^2\phi)\delta_{jk}+2\sin^2\phi\hat{n}^j\hat{n}^k+2\cos\phi\sin\phi\epsilon_{jk\ell}\hat{n}^\ell]L_k \tag{B12}$$

[which leads also to the Lie Algebra (B11)].

To end up this brief revision, let us finally observe that $X_4(q)\psi^*(q)=\langle\psi|X_4(q)|q\rangle=0$ holds for all $|\psi>\in\mathcal{H}[\mathrm{SU}(2)]$ and $q\in\mathcal{S}_3$. So we have

$$q\in\mathcal{S}_3\Rightarrow X_4(q)|q\rangle=0. \tag{B13}$$

In fact, the dilation operator $X_4(q)=Y_4(q)$ is the sole generator of the embedding quaternion group $\Sigma(4)$ not represented in SU(2) quantum kinematics.

Table I summarizes these features. One introduces the right regular representation of SU(2) in a similar manner, defining appropriate group-operators $V(q)$, which act "from the right" in the following sense $V(q')|q\rangle=|g(q;q')\rangle$. For the sake of completeness, Table II presents a synoptic setting of the right regular representation. Let us here recall that in the left quantum kinematic formalism, as considered in this article, the right generators R_j presented in Table II play an important role, because they correspond to the basic invariant operators of the theory (cf., Sec. III).

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